Double pendulum, behavior and applications in human motion

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Introduction

The double pendulum is a dynamical system where one pendulum is coupled at the end of another one, as shown in fig.1. their movement is described by a pair of differential equations which are normally difficult to resolve.

Although it has already been shown using various methods that it is a chaotic system, this paper aims to demonstrate that fact numerically using a computer as main tool.

The capital feature of chaotic systems is the dramatic change between trajectories with very close initial conditions, in fact was demonstrated that the trajectories diverge exponentially during the course of time. The system proposed in this work is a well known example of that, but this approach and the experiments performed by the program developed show in a simpler way the consequences of that property.

As was said this system is widely known, here the importance of studying this system as it is useful mainly by sports researchers as a mathematical model for the movements of the upper and lower limbs in activities like kicking a ball, swinging a baseball bat or a tennis racket, running and throwing.

This paper develops differential equations first , then describes the process that was programmed to simulate the movement of the system which consist basically in calculate the position of pendulum through an integrator routine and finally write down some comments on the results and its repercussion in everyday life.

Carrying out the experiment

Consider a system similar to the one shown in fig.1, but for simplification consider that the pendulums have the same mass m and length l. Let's solve for the pendulum's angles θ_1 and θ_2 using the Euler-Lagrange equations. We'll begin by finding the Lagrangian of the system

$$L = T - V \tag{1}$$

Now, let's express the Cartesian position of the pendulums in terms of the angles, that is

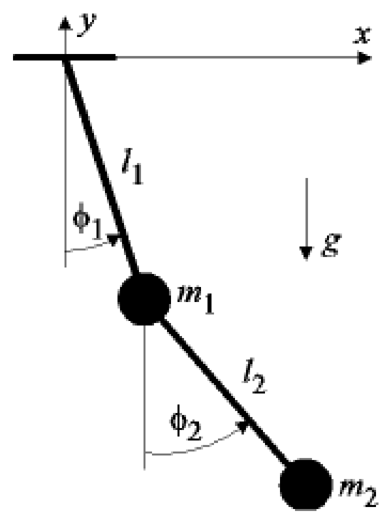

Figure 1. Double pendulum example

$$\begin{aligned} x_1 &= l\sin\theta_1 \\ y_1 &= -l\cos\theta_1 \\ x_2 &= l\sin\theta_2 \\ y_2 &= -l\cos\theta_2 \end{aligned}$$

Note that the only potential energy in this system is the gravitational potential energy, so $V = V_g$. If we consider the ceiling the level where $V_g = 0$ then

$$\begin{aligned} V &= -mgh_1 - mgh_2 \\ &= -mgy_1 - mg(y_1 + y_2) \\ &= -2mgl\cos\theta_1 - mgl\cos\theta_2 \end{aligned}$$

Now for the kinetic energy, remember that

$$T = \frac{1}{2}m\left(v_1^2 + v_2^2\right)$$

The speeds are found as follows

$$v_1^2 = \dot{x_1}^2 + \dot{y_1}^2 = l^2\theta_1^2$$

$$
\begin{aligned}
\vec{v_2} &= \vec{v_1} + \vec{v_{2/1}} \\
v_2^2 &= (\vec{v_1} + \vec{v_{2/1}}) \cdot (\vec{v_1} + \vec{v_{2/1}}) \\
&= v_1^2 + 2\vec{v_1} \cdot \vec{v_{2/1}} + v_{2/1}^2 \\
&= v_1^2 + 2v_1 v_{2/1} \cos(\theta_1 - \theta_2) + v_{2/1}^2 \\
&= l^2\theta_1^2 + 2l^2\theta_1\theta_2 \cos(\theta_1 - \theta_2) + l^2\theta_2^2
\end{aligned}
$$

Where $\vec{v_{2/1}}$ is the speed of pendulum 2 measured by with pendulum 1. With this, the kinetic energy is

$$T = \frac{1}{2} m \left(2l^2\theta_1^2 + 2l^2\theta_1\theta_2 \cos(\theta_1 - \theta_2) + l^2\theta_2^2\right)$$

So the Lagrangian is

$$
\begin{aligned}
L = &\frac{1}{2} m \left(2l^2\theta_1^2 + 2l^2\theta_1\theta_2 \cos(\theta_1 - \theta_2) + l^2\theta_2^2\right) + \\
&2mgl \cos\theta_1 + mgl \cos\theta_2
\end{aligned}
$$

With the Lagrangian, we can propose the two differential equations for this system, given by the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \tag{2}$$

Using (2) we get

$$2\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta_2}^2 \sin(\theta_1 - \theta_2) + 2\frac{g}{l} \sin\theta_1 = 0$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta_1}^2 \sin(\theta_1 - \theta_2) + \frac{g}{l} \sin\theta_2 = 0$$

Now, solving for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in terms of the other variables

$$
\begin{aligned}
\ddot{\theta}_1 = &\frac{-1}{3 - \cos(2\theta_1 - 2\theta_2)} \Big(3\frac{g}{l} \sin\theta_1 + \frac{g}{l} \sin(\theta_1 - 2\theta_2) + \\
&2 \sin(\theta_1 - \theta_2) \left(\dot{\theta_2}^2 + \dot{\theta_1}^2 \cos(\theta_1 - \theta_2)\right)\Big)
\end{aligned}
$$

$$\ddot{\theta}_2 = \frac{2\sin(\theta_1 - \theta_2)}{3 - \cos(2\theta_1 - 2\theta_2)} \left(2\dot{\theta_1}^2 + 2\frac{g}{l} \cos\theta_1 + \dot{\theta_2}^2 \cos(\theta_1 - \theta_2)\right)$$

This ordinary differential equations system is too complicated to solve analytically; but because we have the highest derivatives isolated, we can use the “integrators" numerical method for solving ODE systems. For this we use a Python script with visualization tools, the code is available at `https://github.com/sebasvega95/Double-pendulum`.

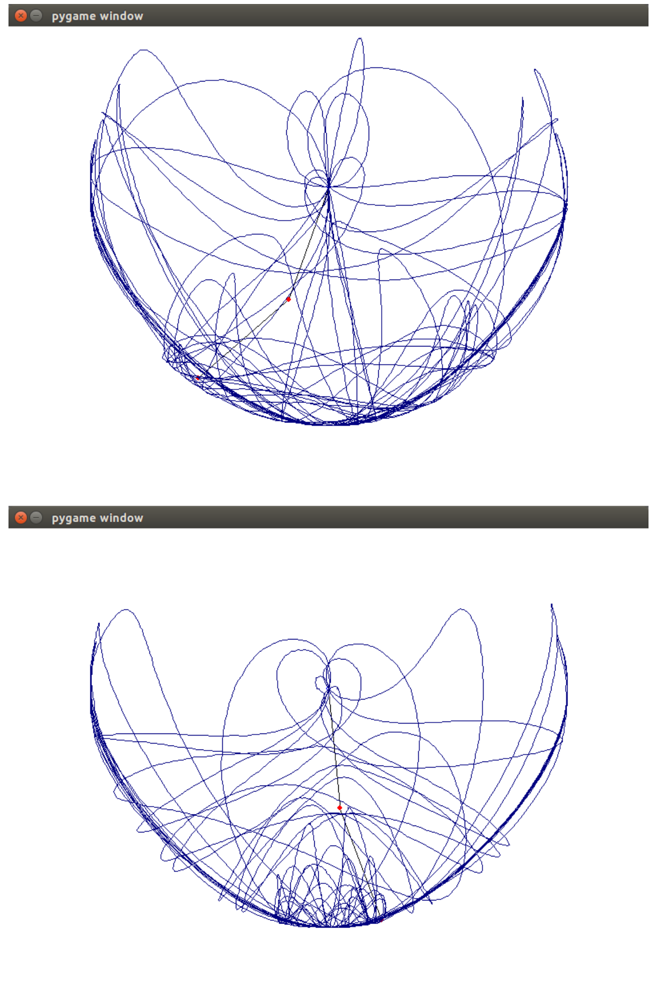

Figure 2. Comparison between two similar initial condition systems

Findings

In an attempt to demonstrate the chaotic behavior of the system, we ran the simulation using two initial conditions very similar, as shown in fig.2. The system on the top has as initial conditions $\theta_1 = \theta_2 = 90°$ and the system on the bottom $\theta_1 = \theta_2 = 90.1°$ (A difference of $0.1°$ is really a difference of about 0.0017 since the angles are measured in radians) and both are at rest.

It is clear that there are a lot of difference between the two systems after just 1 minute, and also knowing the stable behavior of other coupled pendulum systems is natural to come to the conclusion that this system is highly unstable, and presents chaotic behavior.

Even though this system seems too unpredictable to be useful, if one limits the motion to the first cycle of the pendulum, it bares a lot of similarities with the motion of some parts of the human body —such as legs and arms— when

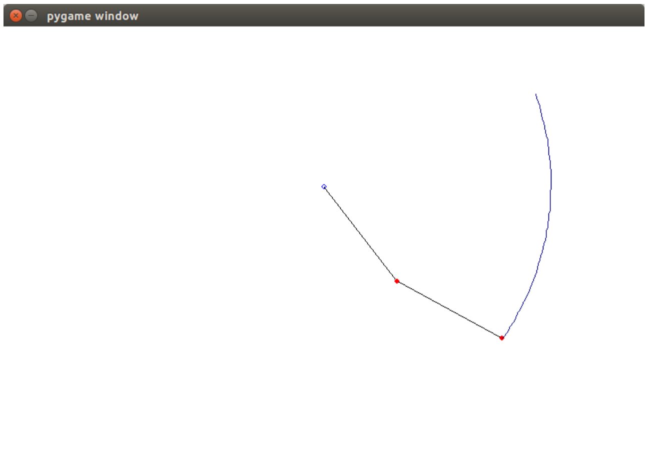

Figure 3. Kick illustrated using the double pendulum

performing actions like throwing, kicking or using a baseball bat. Fig.3 shows that, picking adequately the initial conditions, one can very easily simulate a kick of for example a football player.

Conclusions

The chaotic behavior of the double pendulum has been proved. However unlike the simple pendulum, double pendulum allows us to model more complex movements; It allows to model movements of the upper and lower extremities of the human body and the swing of various implements used in sport because it is similar to the motion of the first cycle of double pendulum. So in summary, we can model similarity between limbs and double pendulum, so we can describe the classic sport movements like golf by the equations described above and finally we can analyze simulation results of the first cycle of the program developed. These analysis could have impact on the way the plays run.

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