3 Efficient Sorting – The Power of Divide & Conquer

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COMP526 (Fall 2022)
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Learning Outcomes

1. Know principles and implementation of *mergesort* and *quicksort*.
2. Know properties and *performance characteristics* of mergesort and quicksort.
3. Know the comparison model and understand the corresponding *lower bound*.
4. Understand *counting sort* and how it circumvents the comparison lower bound.
5. Know ways how to exploit *presorted* inputs.

Unit 3: *Efficient Sorting*

Outline

3 Efficient Sorting

Why study sorting?

- fundamental problem of computer science that is still not solved
  Algorithm with optimal #comparisons in worst case?
- building brick of many more advanced algorithms
  - for preprocessing
  - as subroutine
- playground of manageable complexity
  to practice algorithmic techniques

Here:

- "classic" fast sorting method
- exploit **partially sorted** inputs
- ~~**parallel** sorting~~ → Unit 5

Part I

The Basics

Rules of the game

- **Given:**
  - array $A[0..n) = A[0..n-1]$ of n objects
  - a total order relation $\leq$ among $A[0], \ldots, A[n-1]$
    (a comparison function)
    Python: elements support <= operator (`__le__()`)
    Java: `Comparable` class (`x.compareTo(y) <= 0`)
- **Goal:** rearrange (i. e., permute) elements within A,
  so that A is *sorted*, i. e., $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

- for now: A stored in main memory (*internal sorting*)
  single processor (*sequential sorting*)

Clicker Question

What is the complexity of sorting? Type you answer, e. g., as "`Theta(sqrt(n))`"

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3.1 Mergesort

Clicker Question

How does mergesort work?

- **A** Split elements around median, then recurse on small / large elements.
- **B** Recurse on left / right half, then combine sorted halves.
- **C** Grow sorted part on left, repeatedly add next element to sorted range.
- **D** Repeatedly choose 2 elements and swap them if they are out of order.
- **E** Don't know.

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How does mergesort work?

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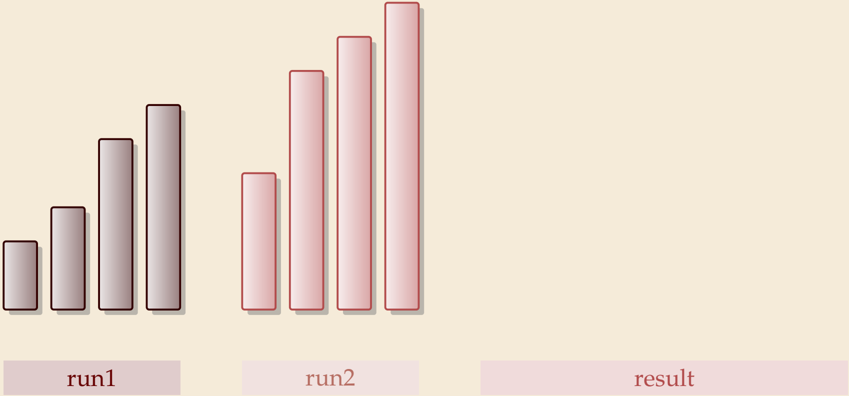
Merging sorted lists

Merging sorted lists



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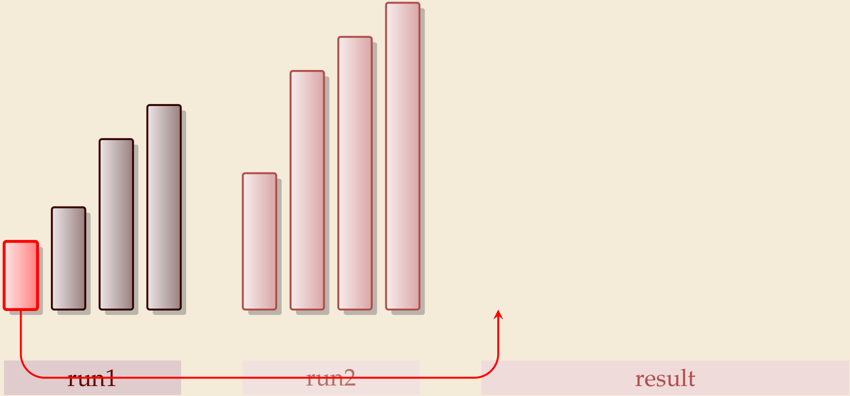

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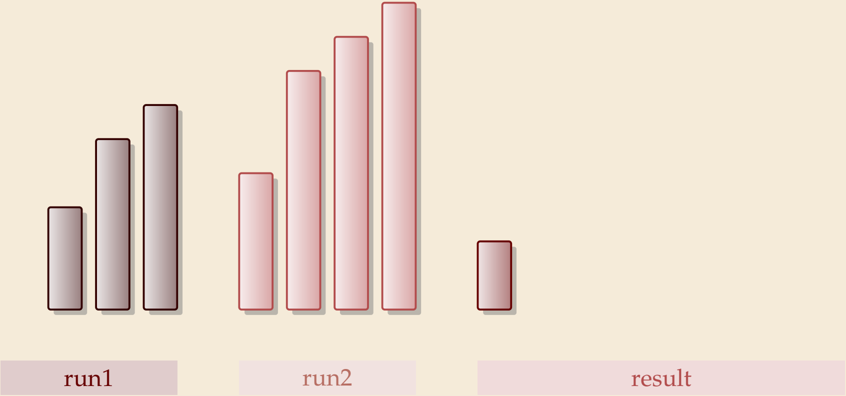

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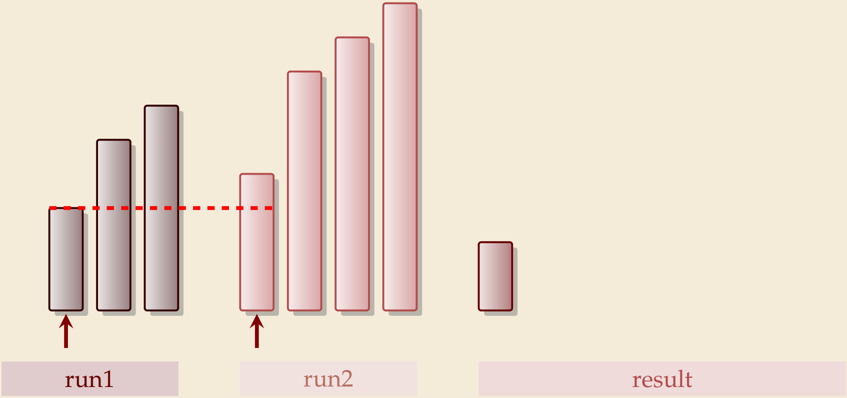

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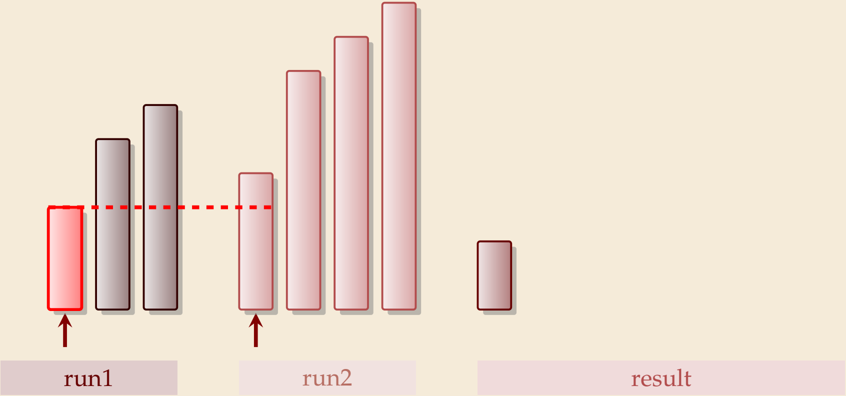

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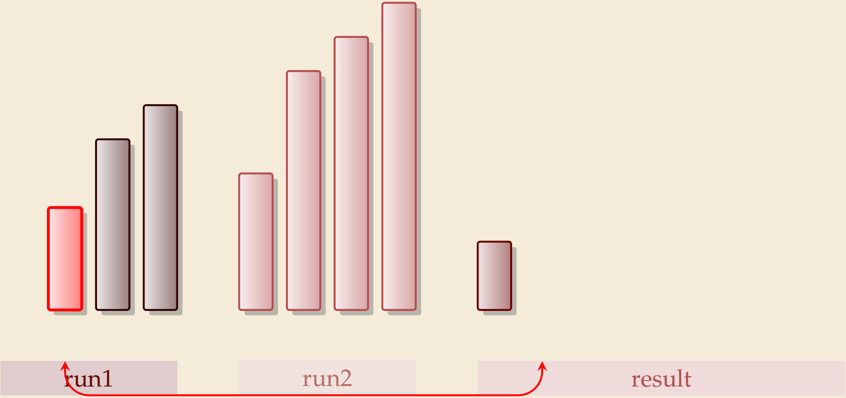

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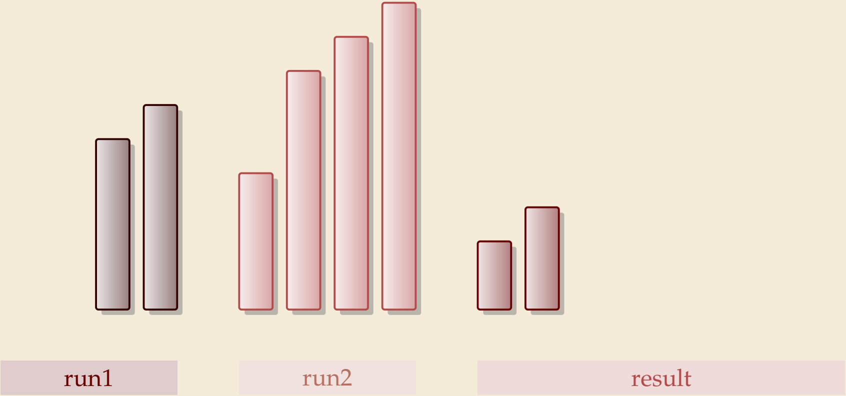

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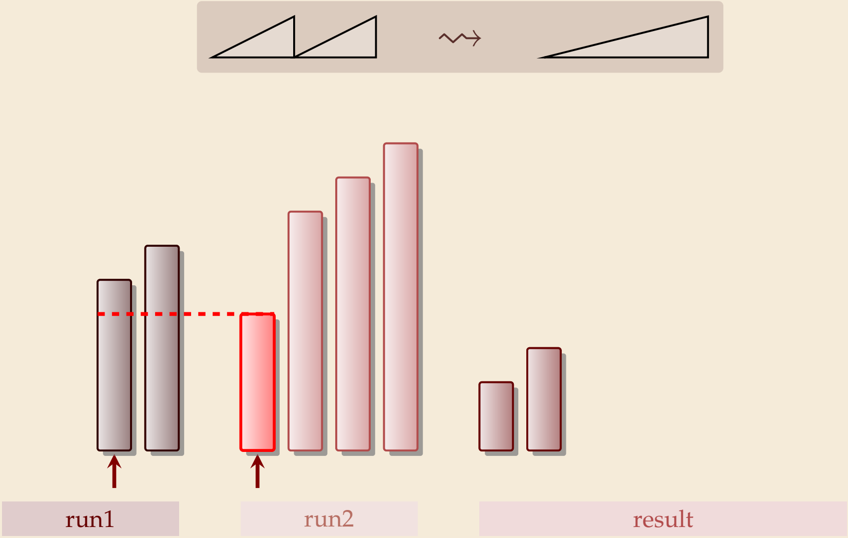

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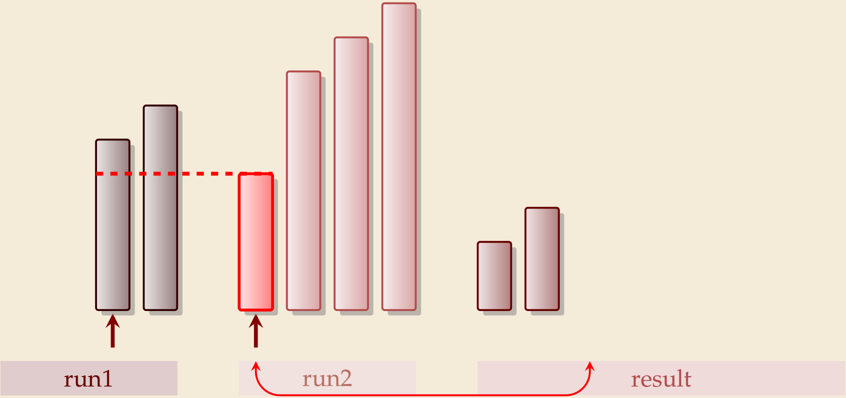

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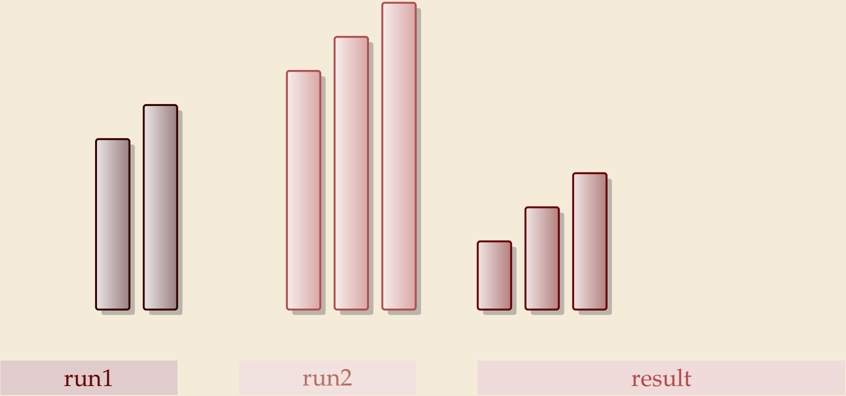

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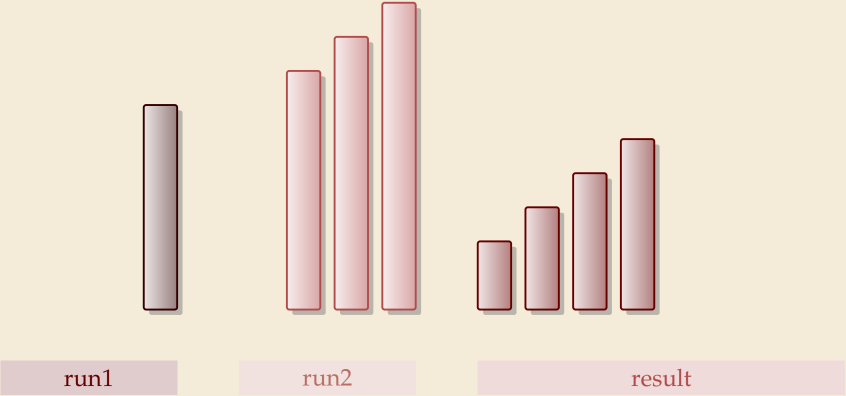

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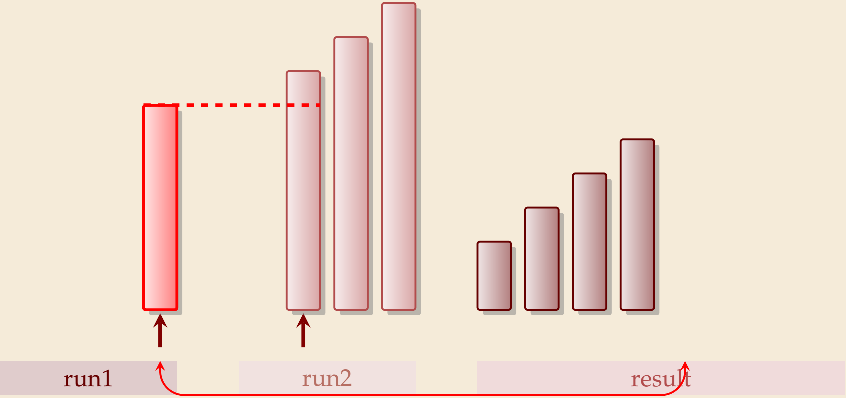

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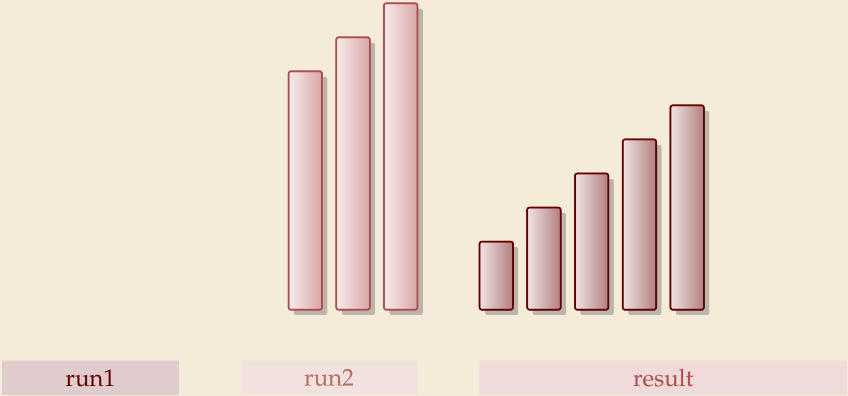

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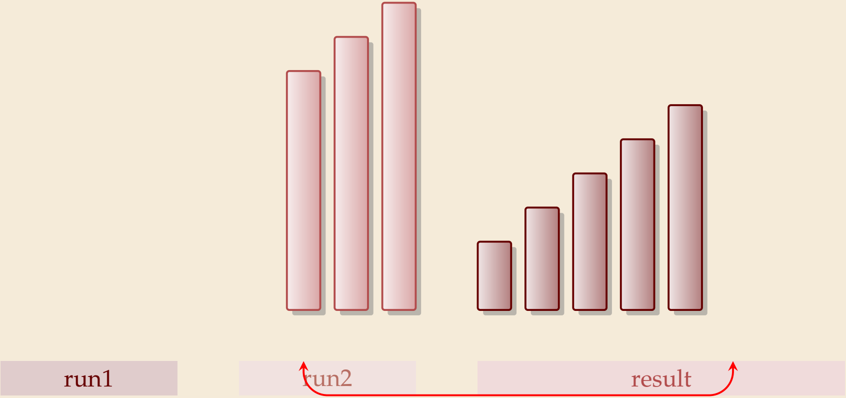

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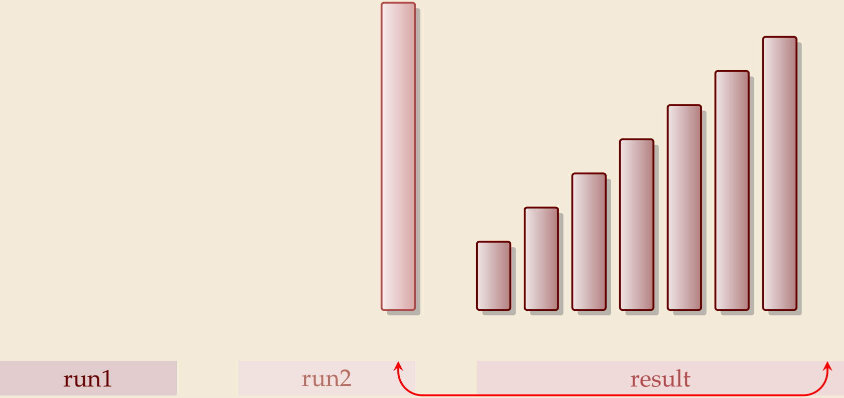

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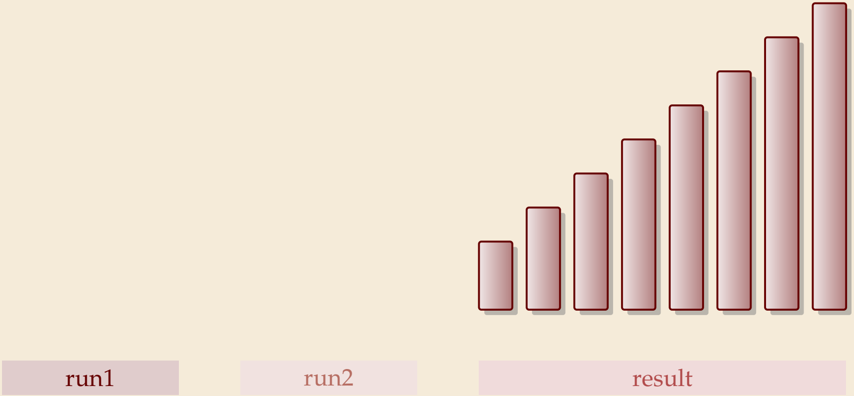

Clicker Question

What is the worst-case running time of mergesort?

A	$\Theta(1)$	**G**	$\Theta(n \log n)$
B	$\Theta(\log n)$	**H**	$\Theta(n \log^2 n)$
C	$\Theta(\log \log n)$	**I**	$\Theta(n^{1+\epsilon})$
D	$\Theta(\sqrt{n})$	**J**	$\Theta(n^2)$
E	$\Theta(n)$	**K**	$\Theta(n^3)$
F	$\Theta(n \log \log n)$	**L**	$\Theta(2^n)$

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Clicker Question

What is the worst-case running time of mergesort?

A $\cancel{\Theta(1)}$

B $\cancel{\Theta(\log n)}$

C $\cancel{\Theta(\log \log n)}$

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Mergesort

procedure mergesort($A[l..r)$)
    $n := r - l$
    if $n \le 1$ **return**
    $m := l + \left\lfloor \frac{n}{2} \right\rfloor$
    mergesort($A[l..m)$)
    mergesort($A[m..r)$)
    merge($A[l..m)$, $A[m..r)$, buf)
    copy buf to $A[l..r)$

proc sort($A[0..n)$)
    mergesort($A[0..n)$)

- recursive procedure
- merging needs
  - temporary storage *buf* for result (of same size as merged runs)
  - to read and write each element twice (once for merging, once for copying back)

Mergesort

```
procedure mergesort(A[l..r])
    n := r − l
    if n ≤ 1 return
    m := l + ⌊n/2⌋
    mergesort(A[l..m])
    mergesort(A[m..r])
    merge(A[l..m], A[m..r], buf)
    copy buf to A[l..r]
```

- recursive procedure
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array accesses

Analysis: count *"element visits"* (read and/or write)

$C(n) \sim \boxed{2n \lg n}$

$$C(n) \;=\; \begin{cases} 0 & n \le 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + \underline{2n} & n \ge 2 \end{cases}$$

same for best and worst case!

Simplification $\boxed{n = 2^k}$

$$C(2^k) \;=\; \begin{cases} 0 & k \le 0 \\ 2 \cdot C(2^{k-1}) + 2 \cdot 2^k & k \ge 1 \end{cases} \;=\; 2 \cdot 2^k + 2^2 \cdot 2^{k-1} + 2^3 \cdot 2^{k-2} + \cdots + 2^k \cdot 2^1 \;=\; 2k \cdot 2^k$$

$$C(n) \;=\; 2n \lg(n) \;=\; \Theta(n \log n)$$

$$C(2^k) \underset{k \geq 2}{=} 2\,C(2^{k-1}) + 2 \cdot 2^k$$

$$= 2\left(2 \cdot C(2^{k-2}) + 2 \cdot 2^{k-1}\right) + 2 \cdot 2^k$$

$$= 2^2 \cdot C(2^{k-2}) + \underbrace{2^2 \cdot 2^{k-1}}_{= 2^{k+1}} + \underbrace{2 \cdot 2^k}_{= 2^{k+1}}$$

Mergesort – Discussion

- optimal time complexity of $\Theta(n \log n)$ in the worst case
- *stable* sorting method   i. e., retains relative order of equal-key items
- memory access is sequential (scans over arrays)

- requires $\Theta(n)$ extra space

  there are in-place merging methods,
  but they are substantially more complicated
  and not (widely) used

3.2 Quicksort

Clicker Question

How does quicksort work?

A split elements around median, then recurse on small / large elements.

B recurse on left / right half, then combine sorted halves.

C grow sorted part on left, repeatedly add next element to sorted range.

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E Don't know.

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Partitioning around a pivot

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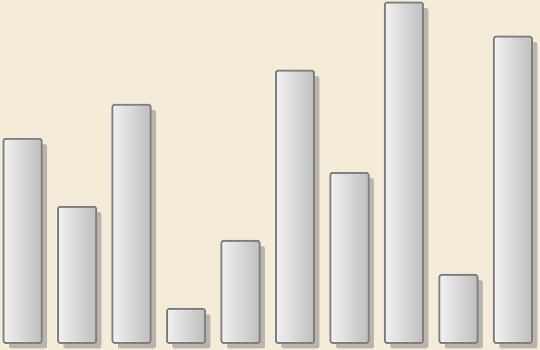

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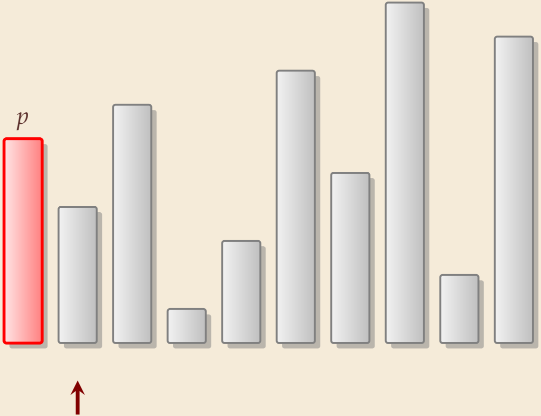

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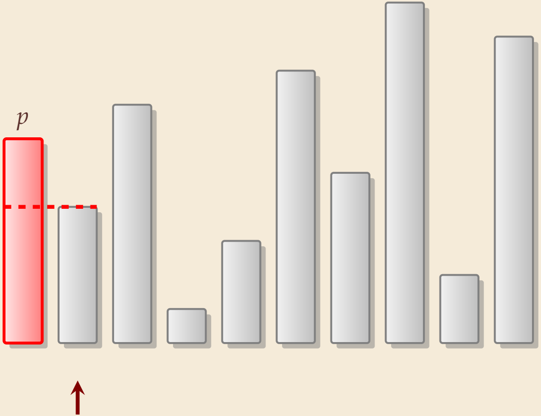

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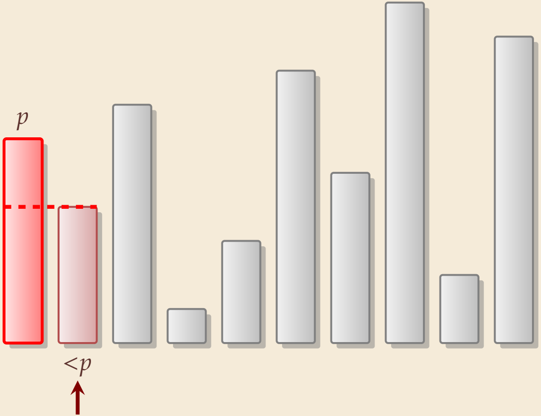

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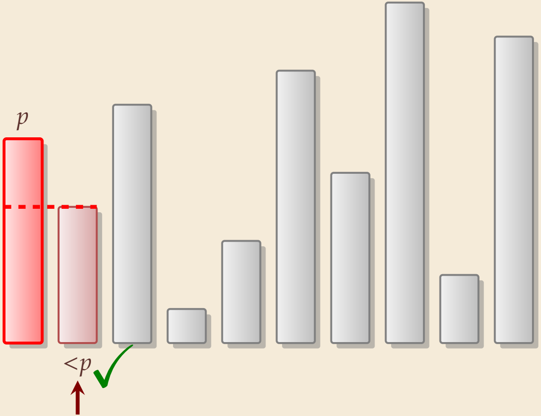

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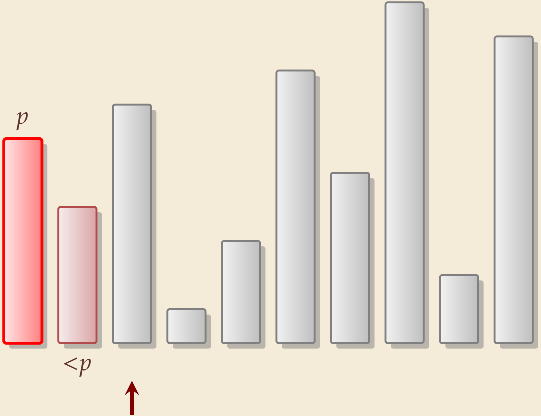

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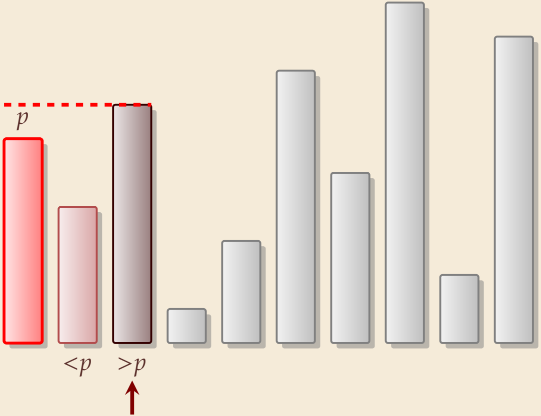

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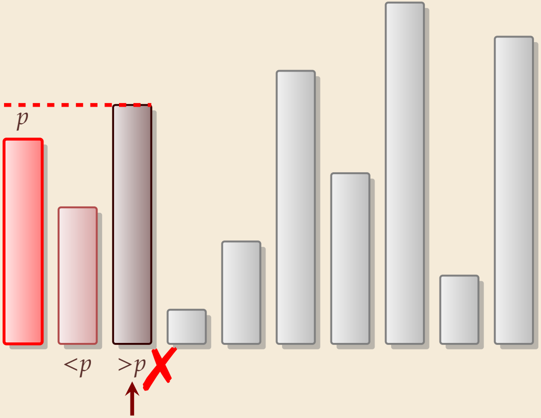

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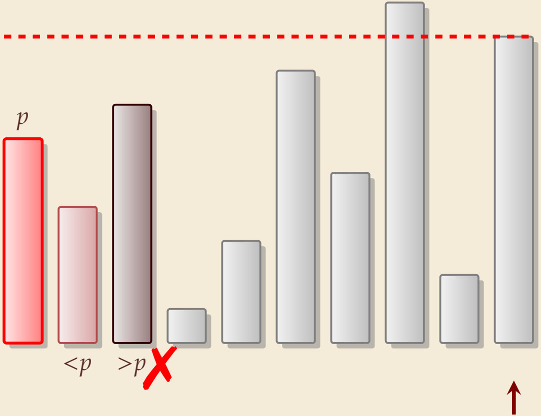

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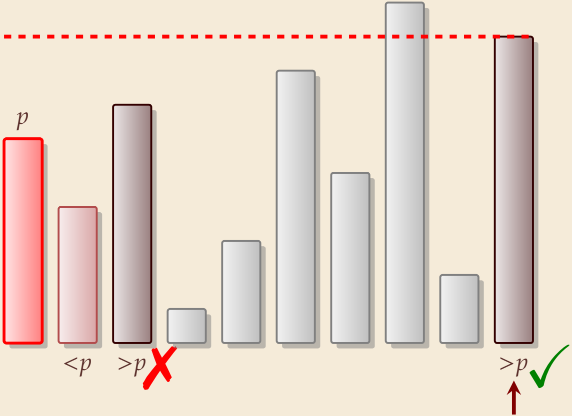

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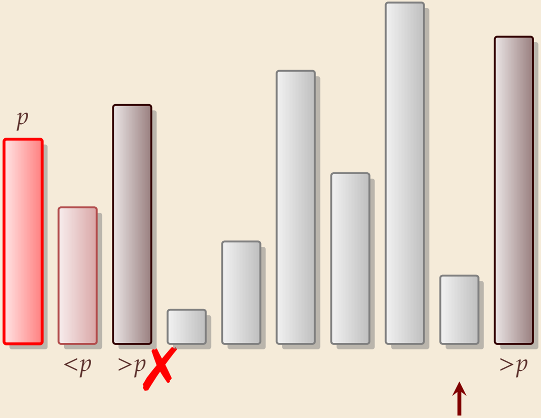

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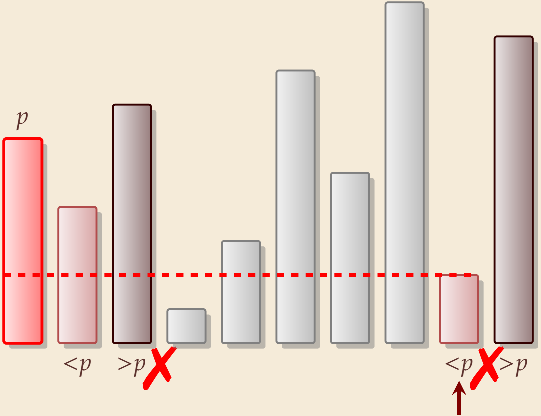

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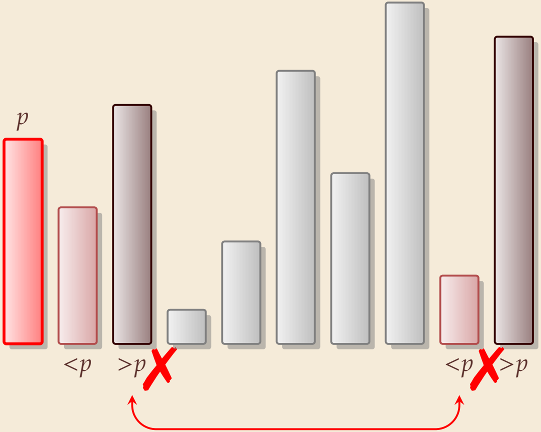

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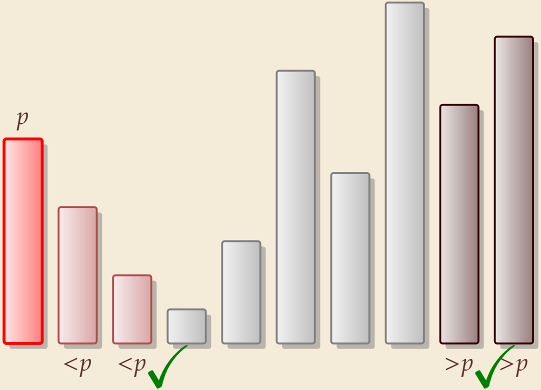

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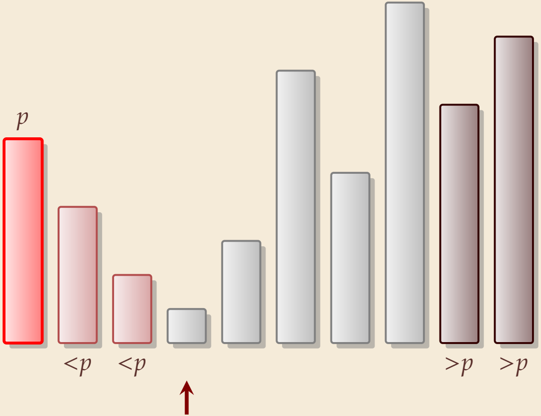

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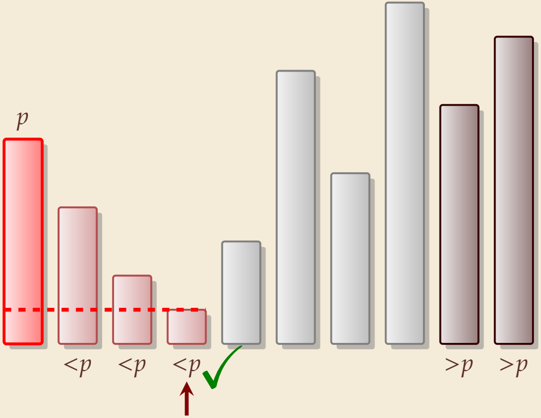

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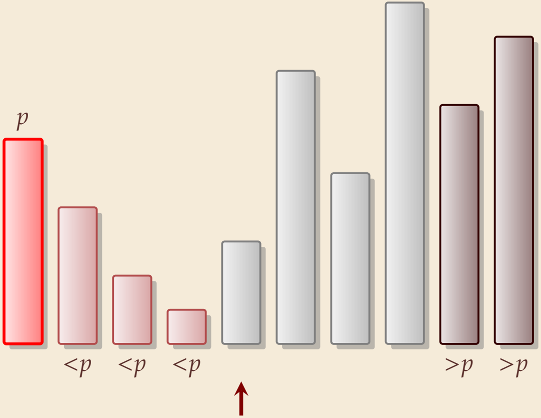

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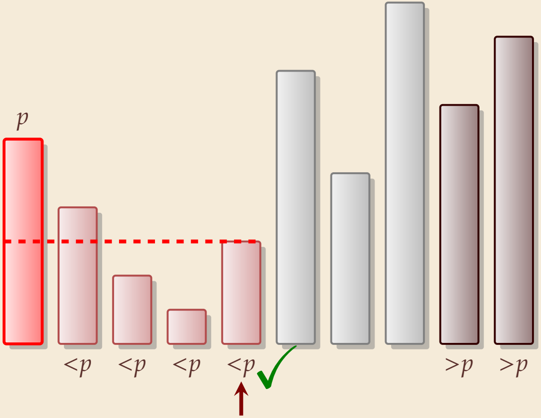

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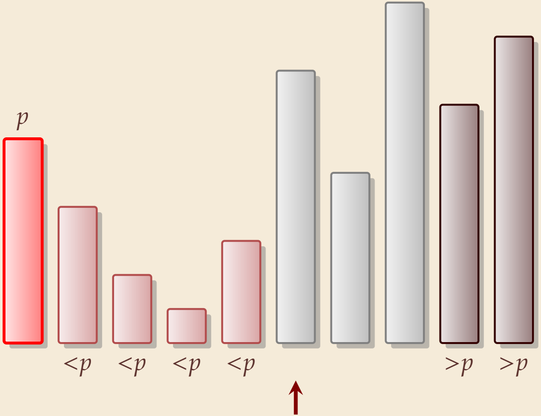

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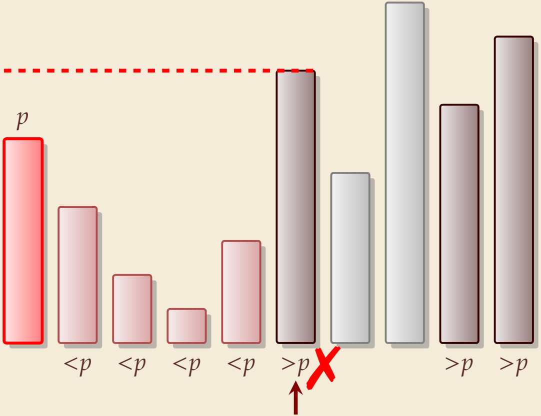

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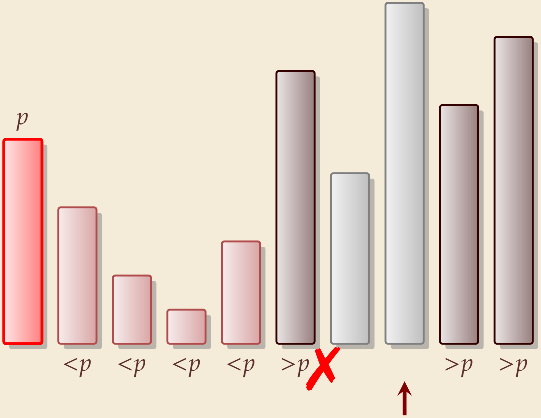

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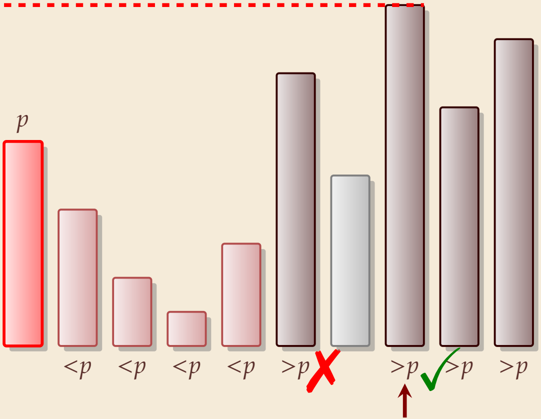

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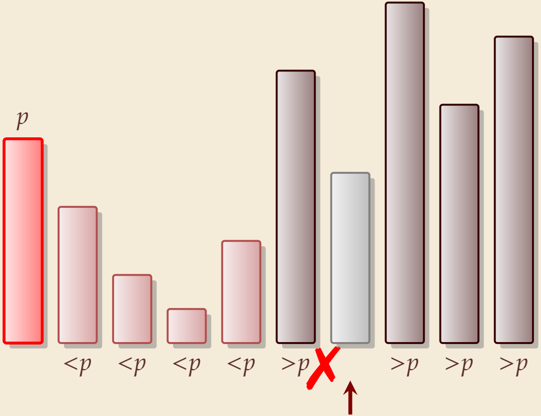

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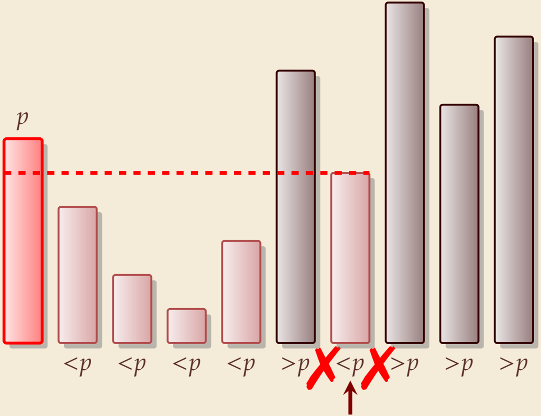

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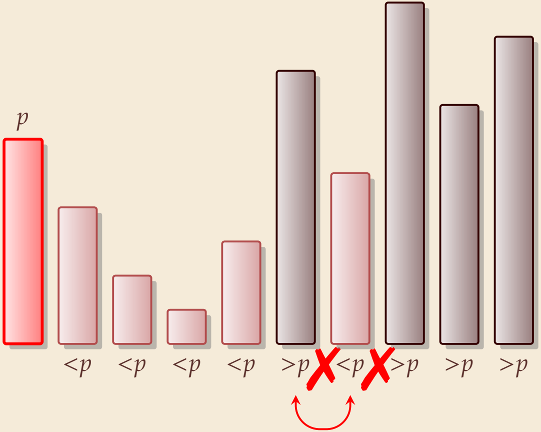

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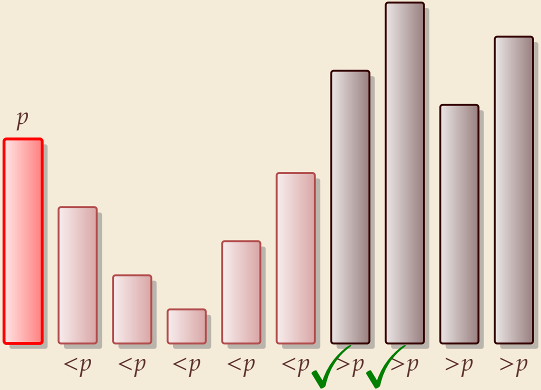

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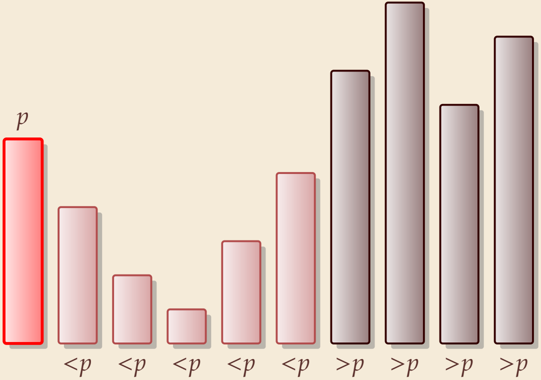

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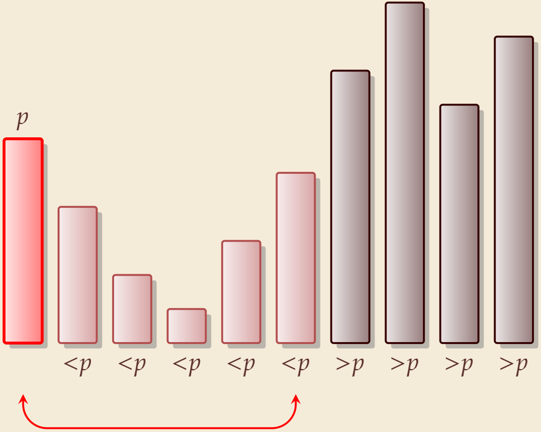

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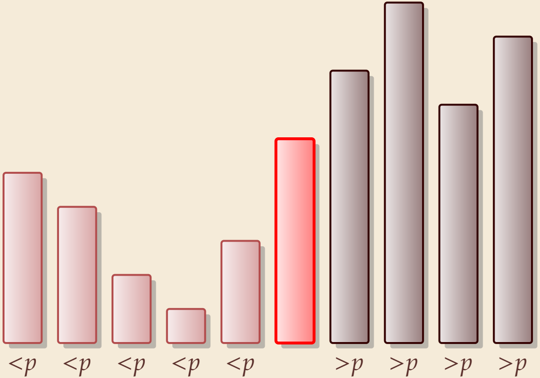

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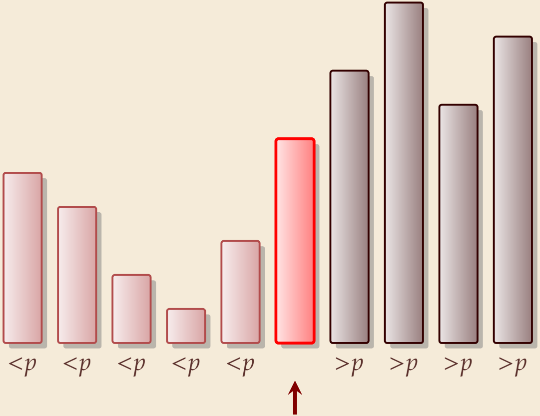

- no extra space needed
- visits each element once
- returns rank/position of pivot

Partitioning – Detailed code

Beware: details easy to get wrong; use this code! (if you ever have to)

```
1  procedure partition(A, b)
2      // input: array A[0..n), position of pivot b ∈ [0..n)
3      swap(A[0], A[b])
4      i := 0,   j := n
5      while true do
6          do i := i + 1 while i < n and A[i] < A[0]
7          do j := j − 1 while j ≥ 1 and A[j] > A[0]
8          if i ≥ j then break    (goto 11)
9          else swap(A[i], A[j])
10     end while
11     swap(A[0], A[j])
12     return j
```

Loop invariant (5–10): A

p	$\leq p$	?	$\geq p$
		i	j

Quicksort

```
procedure quicksort(A[l..r))
    if r − ℓ ≤ 1 then return
    b := choosePivot(A[l..r))
    j := partition(A[l..r), b)
    quicksort(A[l..j))
    quicksort(A[j + 1..r))
```

- recursive procedure
- choice of pivot can be
  - fixed position $\rightsquigarrow$ dangerous!
  - random
  - more sophisticated, e. g., median of 3

Clicker Question

What is the worst-case running time of quicksort?

A $\Theta(1)$

B $\Theta(\log n)$

C $\Theta(\log \log n)$

D $\Theta(\sqrt{n})$

E $\Theta(n)$

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G $\Theta(n \log n)$

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I ~~$\Theta(n^{1+\epsilon})$~~

J $\Theta(n^2)$ ✓

K ~~$\Theta(n^3)$~~

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Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

Quicksort & Binary Search Trees

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7	4	2	9	1	3	8	5	6

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6

9	8

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6	7	9	8

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6	7	9	8

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6

7

9	8

2	1	3

4

5	6

Quicksort & Binary Search Trees

Quicksort

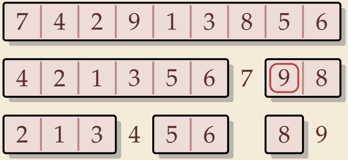
7	4	2	9	1	3	8	5	6

[4 | 2 | 1 | 3 | 5 | 6] 7 [9 | 8]

[2 | 1 | 3] 4 [5 | 6]

Quicksort & Binary Search Trees

Quicksort



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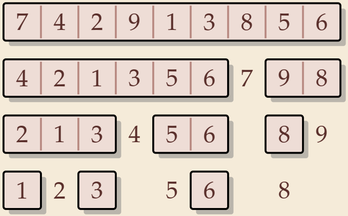

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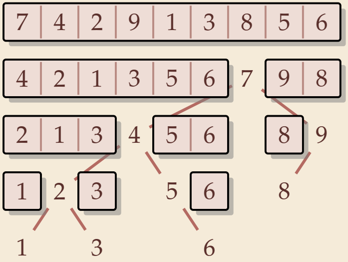

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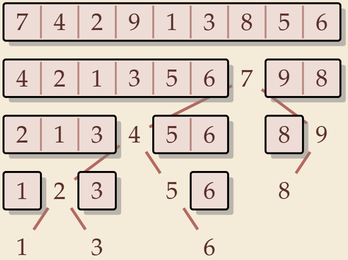
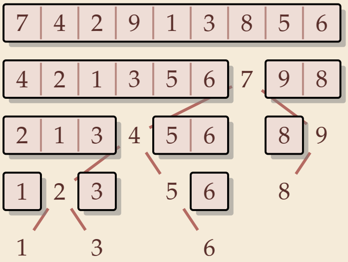


Binary Search Tree (BST)

7 4 2 9 1 3 8 5 6

Quicksort & Binary Search Trees

Quicksort



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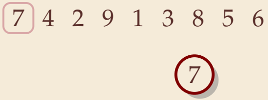

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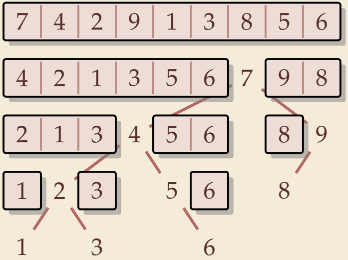

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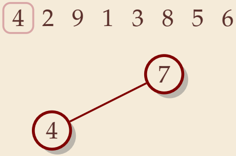

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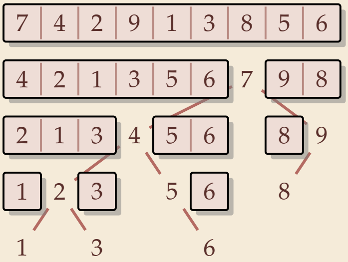

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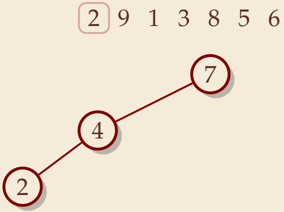

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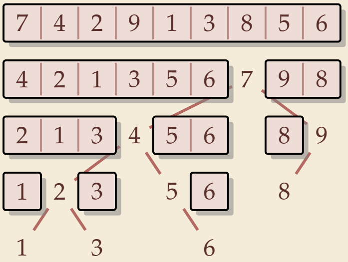

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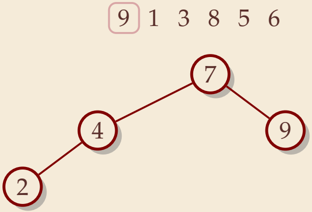

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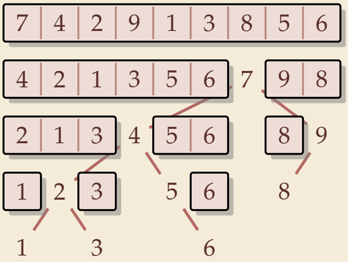

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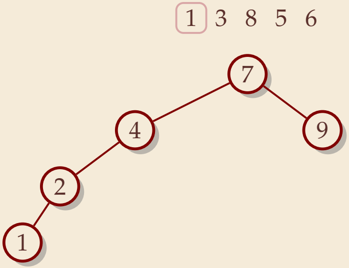

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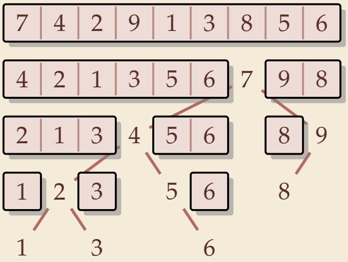

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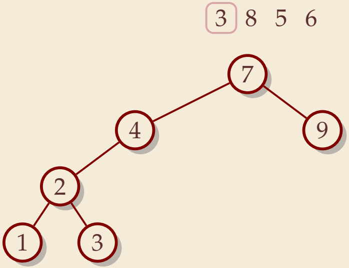

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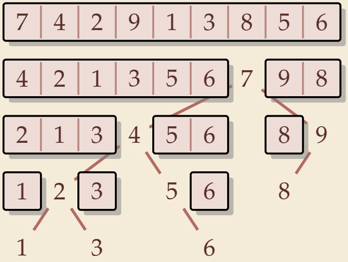

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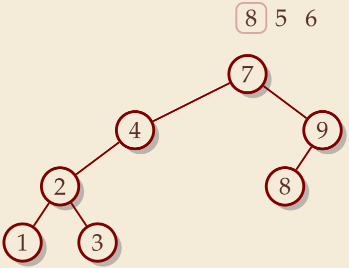

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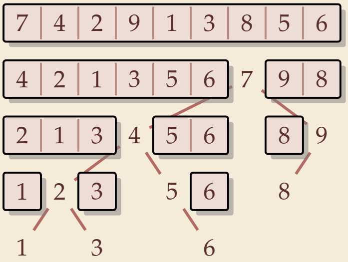

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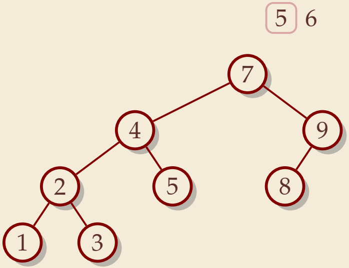

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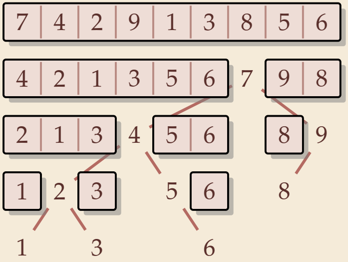

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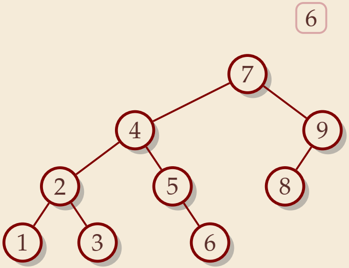

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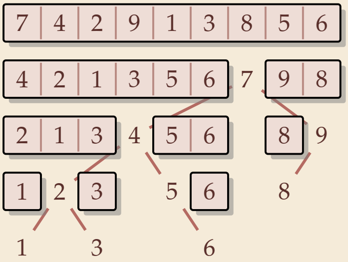

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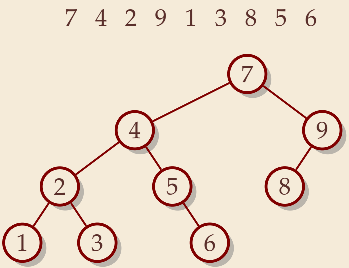
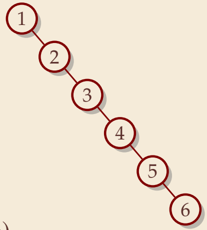


- recursion tree of quicksort $=$ binary search tree from successive insertion
- comparisons in quicksort $=$ comparisons to built BST
- comparisons in quicksort $\approx$ comparisons to search each element in BST

Quicksort – Worst Case

- Problem: BSTs can degenerate
- Cost to search for k is $k - 1$

$\leadsto$ Total cost $\displaystyle\sum_{k=1}^{n}(k-1) = \frac{n(n-1)}{2} \sim \frac{1}{2}n^2$



$\leadsto$ quicksort worst-case running time is in $\Theta(n^2)$

terribly slow!

But, we can fix this:

Randomized quicksort:

- choose a *random pivot* in each step

$\leadsto$ same as randomly *shuffling* input before sorting

Randomized Quicksort – Analysis

- $C(n)$ = element visits (as for mergesort)

$\rightsquigarrow$ quicksort needs $\sim 2\ln(2) \cdot n \lg n \approx 1.39 n \lg n$ *in expectation*

- also: very unlikely to be much worse:
  e. g., one can prove: $\Pr[\text{cost} > 10n \lg n] = O(n^{-2.5})$
  distribution of costs is "concentrated around mean"
- intuition: have to be *constantly* unlucky with pivot choice

Quicksort – Discussion

- 👍 fastest general-purpose method
- 👍 $\Theta(n \log n)$ average case
- 👍 works *in-place* (no extra space required)
- 👍 memory access is sequential (scans over arrays)
- 👎 $\Theta(n^2)$ worst case (although extremely unlikely)
- 👎 not a *stable* sorting method

Open problem: Simple algorithm that is fast, stable and in-place.