4 Efficient Sorting

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Prof. Dr. Sebastian Wild

CS566 (Wintersemester 2025–26)
Philipps-Universität Marburg

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Learning Outcomes

Unit 4: ***Efficient Sorting***

1. Know principles and implementation of *mergesort* and *quicksort*.
2. Know properties and *performance characteristics* of mergesort and quicksort.
3. Know the comparison model and understand the corresponding *lower bound*.
4. Understand *counting sort* and how it circumvents the comparison lower bound.
5. Know ways how to exploit *presorted* inputs.

Outline

4 Efficient Sorting

Why study sorting?

- fundamental problem of computer science that is still not solved
  Algorithm with optimal #comparisons in worst case?
- building brick of many more advanced algorithms
  - for preprocessing
  - as subroutine
- playground of manageable complexity
  to practice algorithmic techniques

Here:

- "classic" fast sorting method
- exploit **partially sorted** inputs
- **parallel** sorting   later

Part I

The Basics

Rules of the game

- **Given:**
  - array $A[0..n) = A[0..n-1]$ of n objects
  - a total order relation $\leq$ among $A[0], \ldots, A[n-1]$
    (a comparison function)
    Python: elements support <= operator (`__le__()`)
    Java: `Comparable` class (`x.compareTo(y) <= 0`)
- **Goal:** rearrange (i. e., permute) elements within A,
  so that A is *sorted*, i. e., $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

- for now: A stored in main memory (*internal sorting*)
  single processor (*sequential sorting*)

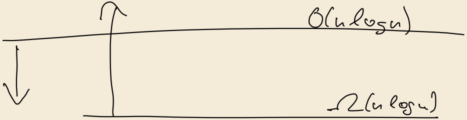
Clicker Question

$\Theta(n \log n)$

What is the complexity of sorting? Type you answer, e. g., as "`Theta(sqrt(n))`"

(a) $O(n \log n)$ algorithm solving the problem

(b) lower bound for problem $\Omega(n \log n)$



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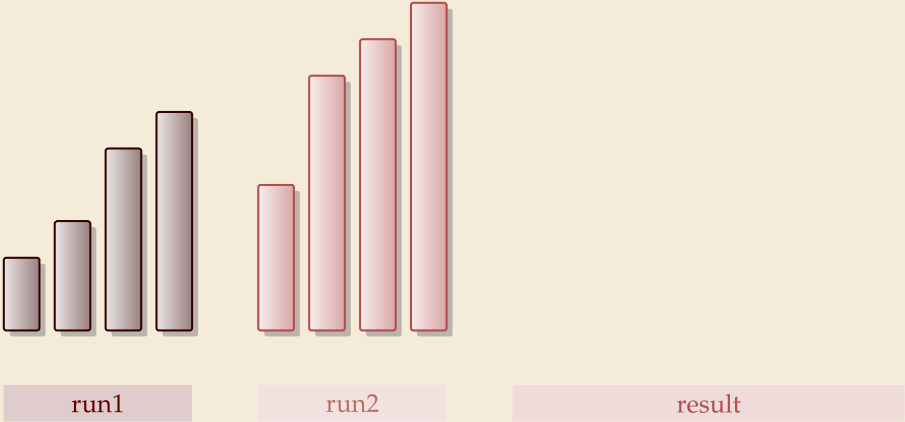
4.1 Mergesort

Merging sorted lists

Merging sorted lists



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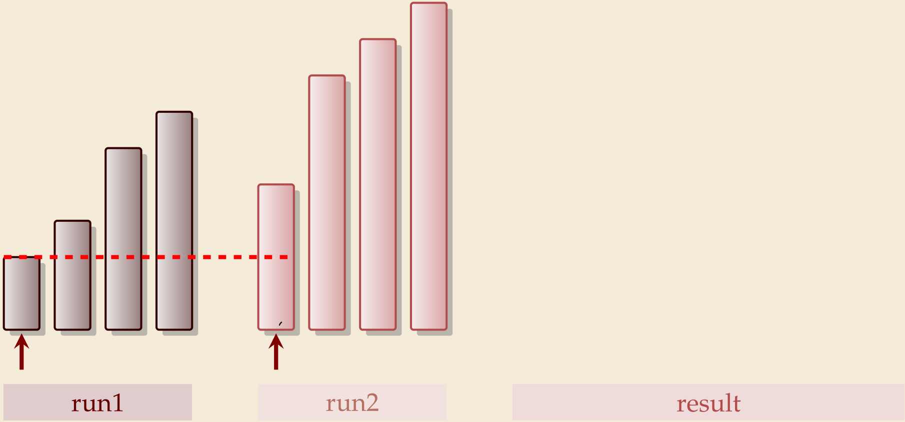

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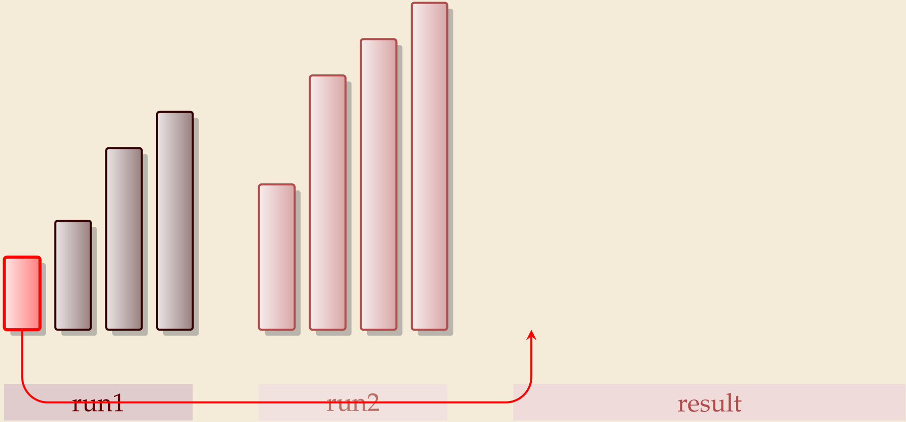

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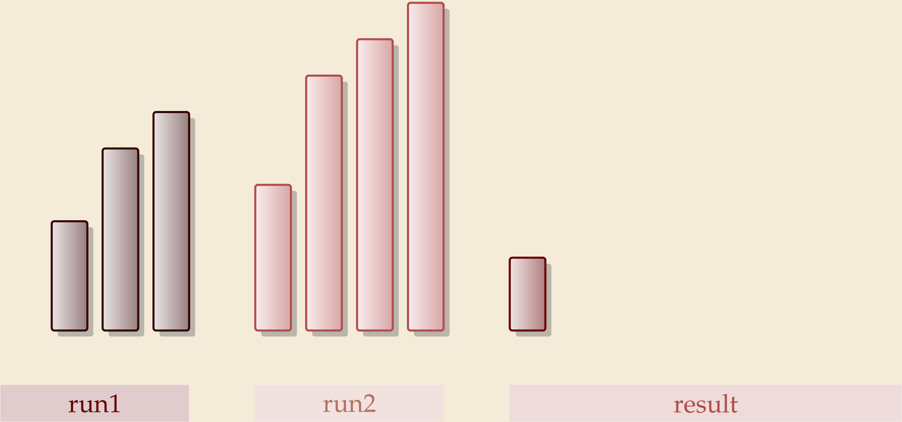

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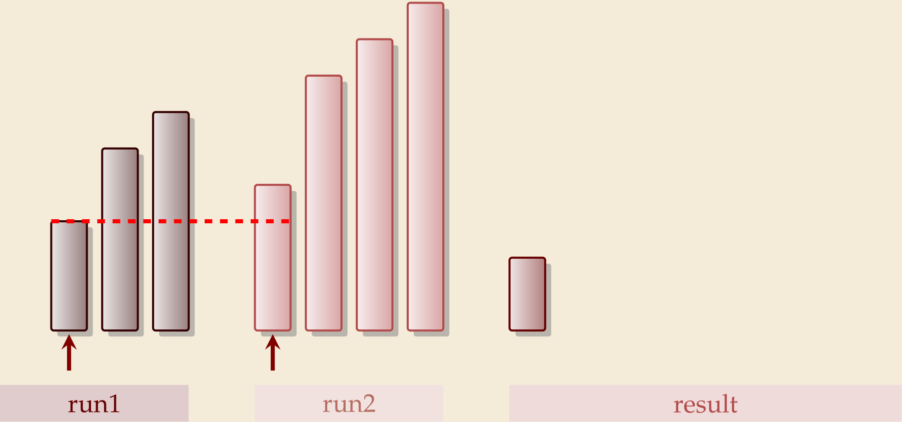

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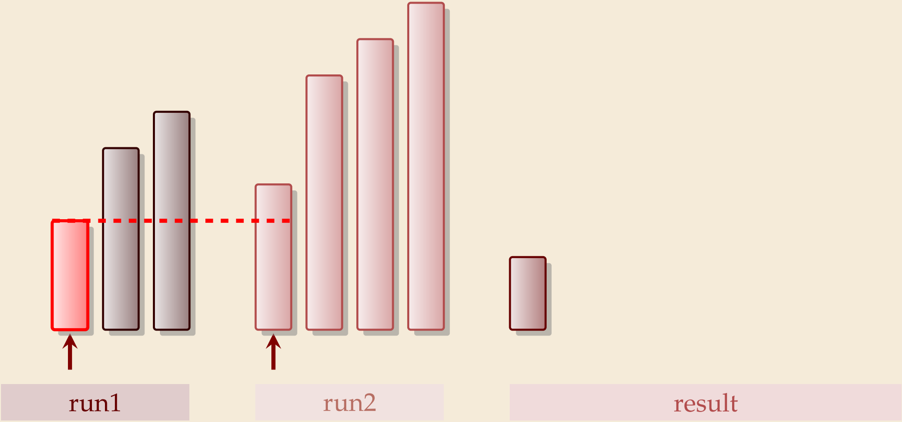

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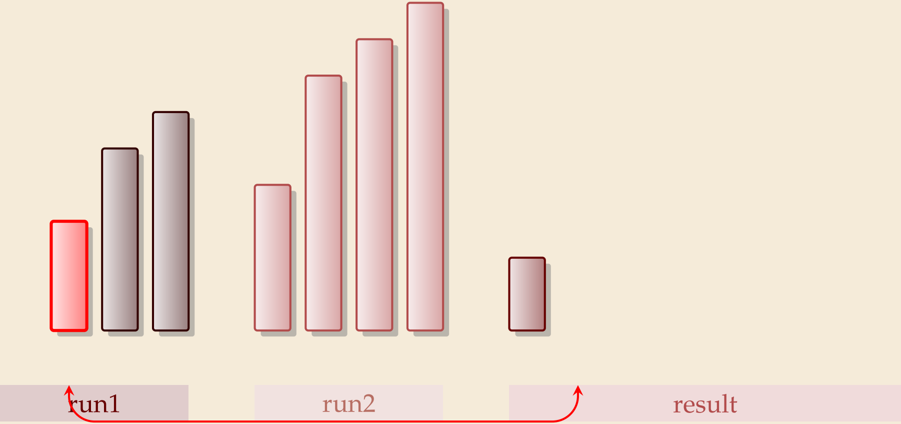

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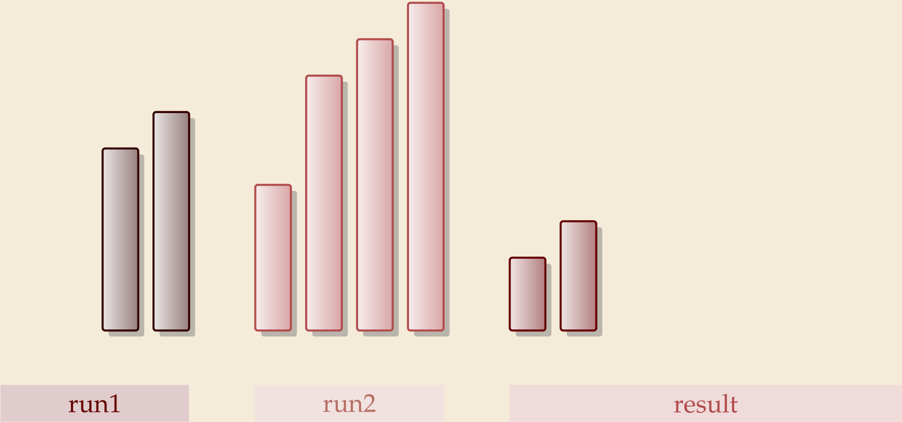

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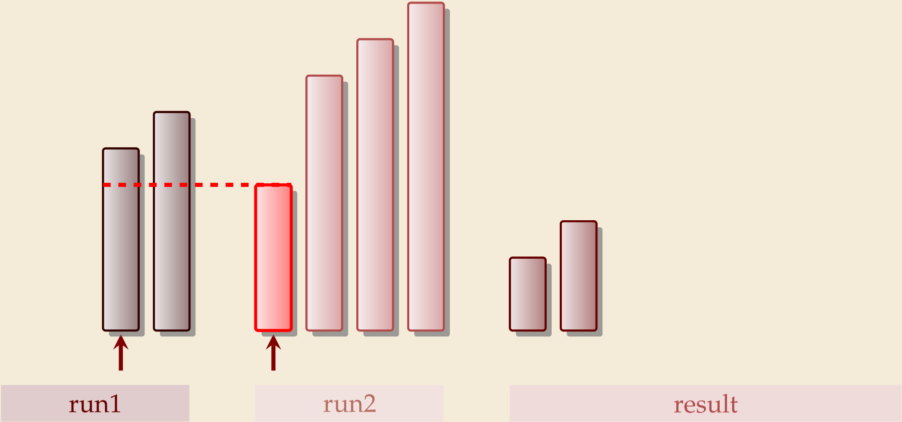

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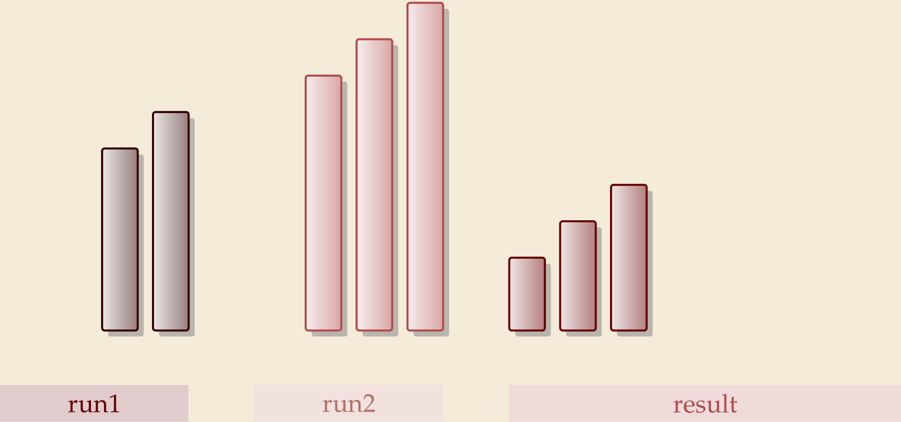

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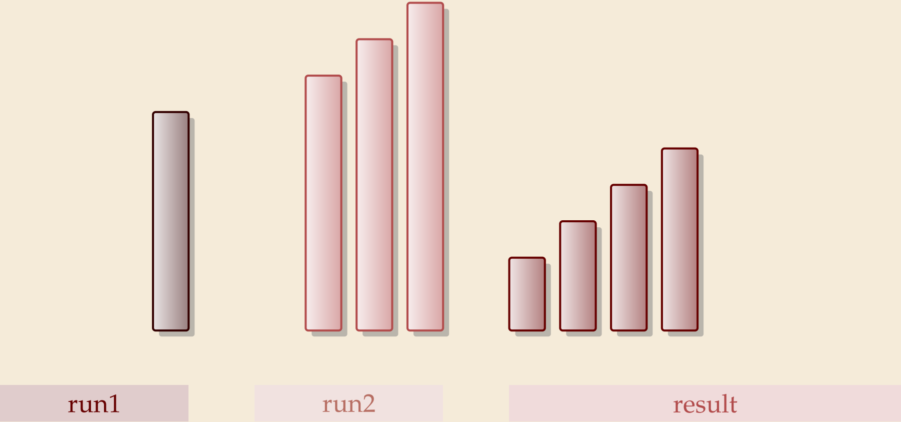

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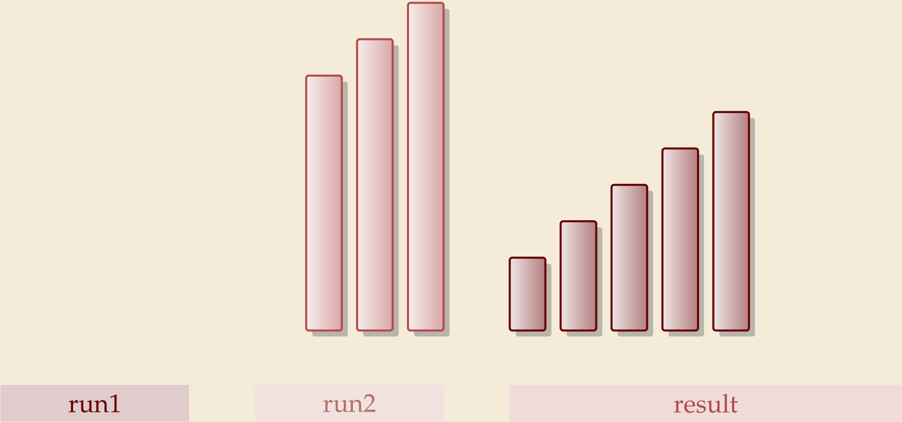

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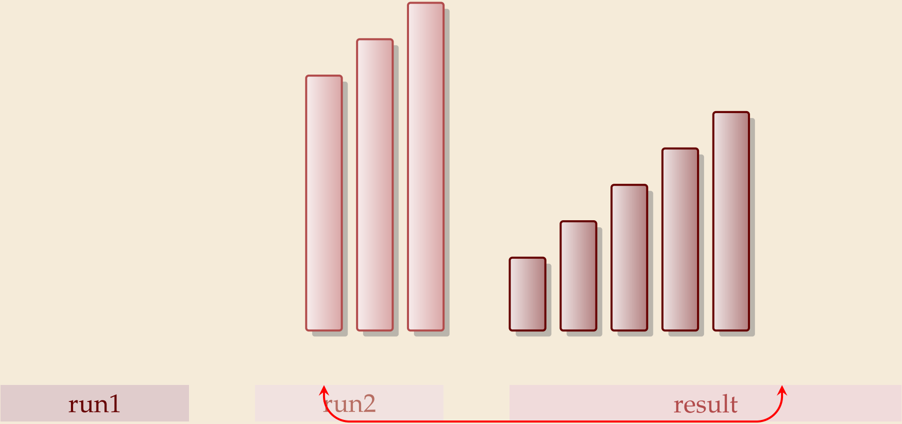

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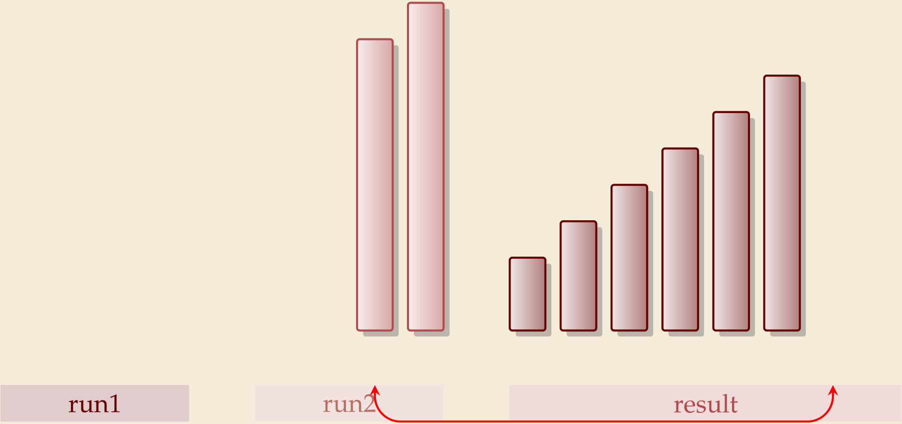

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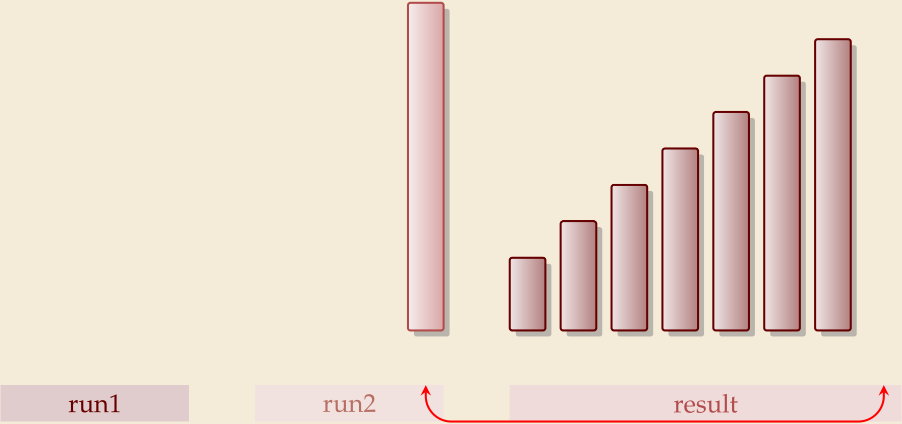

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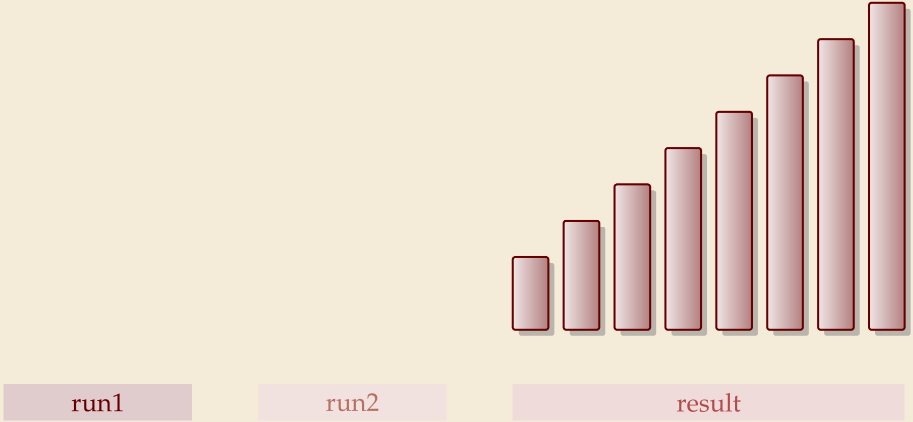

Clicker Question

What is the worst-case running time of mergesort?

A $\Theta(1)$

B $\Theta(\log n)$

C $\Theta(\log \log n)$

D $\Theta(\sqrt{n})$

E $\Theta(n)$

F $\Theta(n \log \log n)$

G $\Theta(n \log n)$

H $\Theta(n \log^2 n)$

I $\Theta(n^{1+\epsilon})$

J $\Theta(n^2)$

K $\Theta(n^3)$

L $\Theta(2^n)$

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- **L** $\cancel{\Theta(2^n)}$

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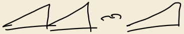
Mergesort

```
procedure mergesort(A[l..r]):
    n := r − l
    if n ≤ 1 return
    m := l + ⌊n/2⌋
    mergesort(A[l..m])
    mergesort(A[m..r])
    merge(A[l..m), A[m..r), buf)
    copy buf to A[l..r)
```



- recursive procedure
- merging needs
  - temporary storage *buf* for result (of same size as merged runs)
  - to read and write each element twice (once for merging, once for copying back)

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Analysis: count *"element visits"* (read and/or write)

$$C(n) \;=\; \begin{cases} 0 & n \le 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \ge 2 \end{cases}$$

same for best and worst case!

Simplification $\boxed{n = 2^k}$

$$C(2^k) \;=\; \begin{cases} 0 & k \le 0 \\ 2 \cdot C(2^{k-1}) + 2 \cdot 2^k & k \ge 1 \end{cases} \;=\; 2 \cdot 2^k + 2^2 \cdot 2^{k-1} + 2^3 \cdot 2^{k-2} + \cdots + 2^k \cdot 2^1 \;=\; 2k \cdot 2^k$$

$$C(n) \;=\; 2n\lg(n) \;=\; \Theta(n \log n)$$

(arbitrary n: $C(n) \;\le\; C(\text{next larger power of 2}) \;\le\; 4n\lg(n) + 2n \;=\; \Theta(n \log n)$)

Mergesort

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same for best and worst case!

(precisely(!) solvable *without* assumption $n = 2^k$: $C(n) = 2n\lg(n) + \big(2 - \{\lg(n)\} - 2^{1-\{\lg(n)\}}\big)2n$ with $\{x\} := x - \lfloor x \rfloor$)

Simplification $\boxed{n = 2^k}$

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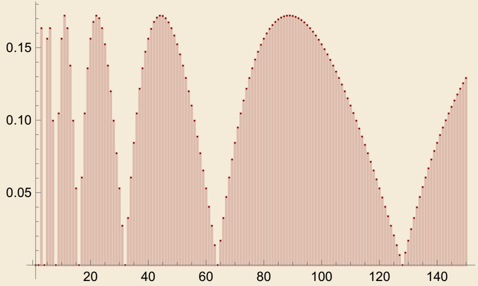
Linear Term of $C(n)$

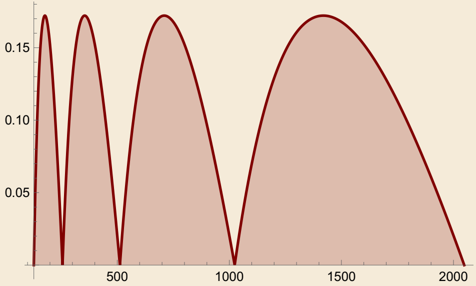
$\notin$ exam

Recall:

$C(n) \;=\; 2n\lg(n) + \left(2 - \{\lg(n)\} - 2^{1-\{\lg(n)\}}\right)2n$ with $\{x\} := x - \lfloor x \rfloor$

Plot of $2\left(2 - \{\lg(n)\} - 2^{1-\{\lg(n)\}}\right)$





$\rightsquigarrow$ Can prove: $\boxed{C(n) \;\le\; 2n\lg n + 0.172n}$

Mergesort – Discussion

- optimal time complexity of $\Theta(n \log n)$ in the worst case
- *stable* sorting method i. e., retains relative order of equal-key items
- memory access is sequential (scans over arrays)
- requires $\Theta(n)$ extra space

there are in-place merging methods,
but they are substantially more complicated
and not (widely) used