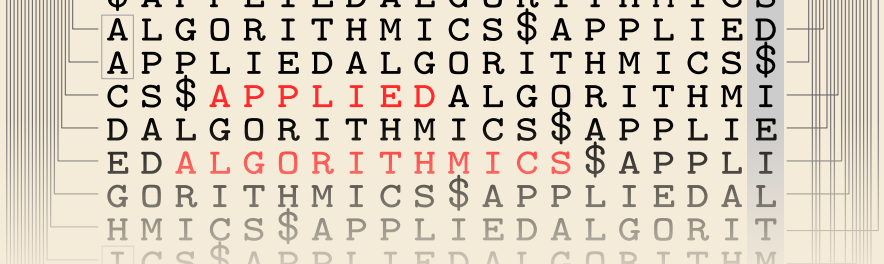

2 Fundamental Data Structures

17 February 2021

Sebastian Wild

version 2021-02-18 13:22

Outline

2 Fundamental Data Structures

2.1 Stacks & Queues

Abstract Data Types

abstract data type (ADT)

- list of supported operations
- **what** should happen
- **not:** how to do it
- **not:** how to store data

≈ Java `interface`
(with Javadoc comments)

VS.

data structures

- specify exactly
  how data is represented
- algorithms for operations
- has concrete costs
  (space and running time)

≈ Java `class` (implementing interfaces)
(non abstract)

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Why separate?

- Can swap out implementations ⇝ "drop-in replacements")

⇝ reusable code!

- (Often) better abstractions
- Prove generic lower bounds (⇝ Unit 3)

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Clicker Question

Which of the following are examples of abstract data types?

- A ADT
- B Stack
- C Deque
- D Linked list
- E binary search tree
- F Queue
- G resizable array
- H heap
- I priority queue
- J dictionary/symbol table
- K hash table

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Click on "Polls" tab

Clicker Question

Which of the following are examples of abstract data types?

- A ~~ADT~~
- B Stack ✓
- C Deque ✓
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- H ~~heap~~
- I priority queue ✓
- J dictionary/symbol table ✓
- K ~~hash table~~

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Click on "Polls" tab

Stacks

Stack ADT

- `top()`
  Return the topmost item on the stack
  Does not modify the stack.
- `push(`x`)`
  Add x onto the top of the stack.
- `pop()`
  Remove the topmost item from the stack
  (and return it).
- `isEmpty()`
  Returns `true` iff stack is empty.
- `create()`
  Create and return an new empty stack.

Clicker Question

Suppose a stack initially contains the numbers 1, 2, 3, 4, 5 with 1 at the top.
What is the content of the stack after the following operations:

```
pop(); pop(); push(1);
```

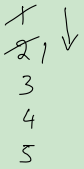
A 1,2,3,1

B 3,4,5,1

C 1,3,4,5

D empty

E 1,2,3,4,5



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Clicker Question

Suppose a stack initially contains the numbers $1, 2, 3, 4, 5$ with 1 at the top.
What is the content of the stack after the following operations:

```
pop(); pop(); push(1);
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C 1,3,4,5 ✓

D ~~empty~~

E ~~1,2,3,4,5~~

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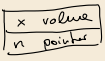
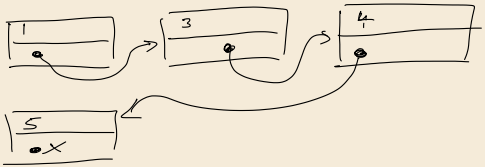
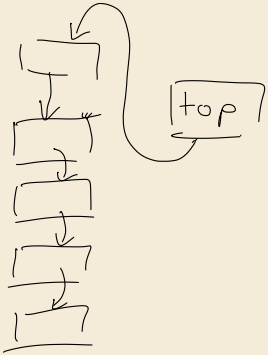
Linked-list implementation for Stack

Invariants:

- maintain `top` pointer to topmost element
- each element points to the element below it (or `null` if bottommost)







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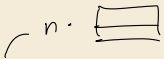
Linked stacks:



- require $\Theta(n)$ space when n elements on stack
- All operations take $O(1)$ time

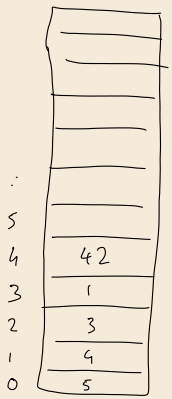
Array-based implementation for Stack

Can we avoid extra space for pointers?

⇝ array-based implementation

Invariants:

- maintain array `S` of elements, from bottommost to topmost
- maintain index `top` of position of topmost element in `S`.



top: 3 4

Array-based implementation for Stack

Can we avoid extra space for pointers?

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Invariants:

- maintain array `S` of elements, from bottommost to topmost
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What to do if stack is full upon `pop`?

Array stacks:

- require *fixed capacity* C (known at creation time)!
- require $\Theta(C)$ space for a capacity of C elements
- all operations take $O(1)$ time

2.2 Resizable Arrays

Digression – Arrays as ADT

Arrays can also be seen as an ADT!

Array operations:

- `create(`n`)` *Java*: `A = new int[`n`];`
  Create a new array with n cells, with positions $0, 1, \ldots, n-1$
- `get(`i`)` *Java*: `A[`i`]`
  Return the content of cell i
- `set(`i`,`x`)` *Java*: `A[`i`] =` x`;`
  Set the content of cell i to x.

⇝ Arrays have fixed size (supplied at creation).

Digression – Arrays as ADT

Arrays can also be seen as an ADT! . . . but are commonly seen as specific data structure

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⇝ Arrays have fixed size (supplied at creation).

Usually directly implemented by compiler + operating system / virtual machine.

Difference to others ADTs: ***Implementation usually fixed***
to "a contiguous chunk of memory".

Doubling trick

Can we have unbounded stacks based on arrays? Yes!

Doubling trick

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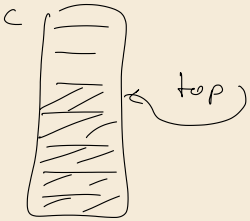
Invariants:

- maintain array `S` of elements, from bottommost to topmost
- maintain index `top` of position of topmost element in `S`
- maintain capacity C = `S.length` so that $\frac{1}{4}C \leq n \leq C$

⇝ can always push more elements!



Doubling trick

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⇝ can always push more elements!

How to maintain the last invariant?

- before `push`
  If $n = C$, allocate new array of size $2n$, copy all elements.
- after `pop`
  If $n < \frac{1}{4}C$, allocate new array of size $2n$, copy all elements.

⇝ ***"Resizing Arrays"***

an implementation technique, not an ADT!

Clicker Question

Which of the following statements about resizable array that currently stores n elements is correct?

- **A** The elements are stored in an array of size $2n$.
- **B** Adding or deleting an element at the end takes constant time.
- **C** A sequence of m insertions or deletions at the end of the array takes time $O(n + m)$.
- **D** Inserting and deleting any element takes $O(1)$ amortized time.

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Amortized Analysis

- Any individual operation `push` / `pop` can be expensive!
  $\Theta(n)$ time to copy all elements to new array.
- **But:** An one expensive operation of cost T means $\Omega(T)$ next operations are cheap!

Amortized Analysis

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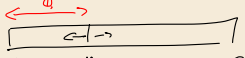
#array accesses

distance to boundary

Formally: consider "credits/potential" $\Phi = \min\{n - \frac{1}{4}C, C - n\} \in [0, \frac{1}{2}n]$



- amortized cost of an operation = actual cost − $\frac{1}{4}$ · change in Φ
  - cheap `push`/`pop`: actual cost 1 array access, consumes ≤ 1 credits ⇝ amortized cost ≤ 5
  - copying `push`: actual cost $2n+1$ array accesses, creates n credits ⇝ amortized cost 5
  - copying `pop`: actual cost $n+1$ array accesses, creates $\frac{1}{2}n$ credits ⇝ amortized cost 5

⇝ **sequence** of m operations: total actual cost ≤ total amortized cost + ~~initial~~ final credits

here: $\leq 5m + \leq \frac{1}{4} \cdot \frac{1}{2}n = 5m + 2n = \Theta(m+n)$

$a_i = c_i - 4(\Phi_i - \Phi_{i-1}) \leq 5$

$\sum_{i=1}^{m} a_i \leq 5m \geq \sum_{i=1}^{m} a_i = \sum_{i=1}^{m} c_i - 4\sum_{i=1}^{m}(\Phi_i - \Phi_{i-1}) = \sum_{i=1}^{m} c_i - 4(\Phi_m - \Phi_0)$

$\sum_{i=1}^{m} c_i \leq 5m + 4\Phi_m - 4\Phi_0 \leq 5m + 4\Phi_m$

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- **D** ~~Inserting and deleting any element takes $O(1)$ amortized time.~~

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Click on "Polls" tab

Queues

Operations:

- `enqueue(`x`)`
  Add x at the end of the queue.
- `dequeue()`
  Remove item at the front of the queue and return it.

Implementations similar to stacks.

Bags

What do Stack and Queue have in common?

Bags

What do Stack and Queue have in common?

They are special cases of a ***Bag***!

Operations:

- `insert(`x`)`
  Add x to the items in the bag.
- `delAny()`
  Remove any one item from the bag and return it.
  (Not specified which; any choice is fine.)
- roughly similar to Java's `Collection`

Sometimes it is useful to state that order is irrelevant $\rightsquigarrow$ Bag
Implementation of Bag usually just a Stack or a Queue

2.3 Priority Queues

Clicker Question

What is a heap-ordered tree?

A A tree in which every node has exactly 2 children.

B A tree where all keys in the left subtree are smaller than the key at the root and all keys in the right subtree are bigger than the key at the root.

C A tree where all keys in the left subtree and right subtree are bigger than the key at the root.

D An tree that is stored in the heap-area of the memory.

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Click on “Polls” tab

Priority Queue ADT

Now: elements in the bag have different *priorities*.

(Max-oriented) Priority Queue (MaxPQ):

- `construct(`A`)`
  Construct from from elements in array A.
- `insert(`x`,`p`)`
  Insert item x with priority p into PQ.
- `max()`
  Return item with largest priority. (Does not modify the PQ.)
- `delMax()`
  Remove the item with largest priority and return it.
- `changeKey(`x`,`p'`)`
  Update x's priority to p'.
  Sometimes restricted to *increasing* priority.
- `isEmpty()`

Fundamental building block in many applications.

Priority Queue ADT – min-oriented version

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- `~~max~~()`
  Return item with ~~largest~~ priority. (Does not modify the PQ.)
- `del~~Max~~()`
  Remove the item with ~~largest~~ priority and return it.
- `changeKey(`x`,`p'`)`
  Update x's priority to p'.
  Sometimes restricted to ~~in~~*creasing* priority.
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Fundamental building block in many applications.

Clicker Question

Suppose we start with an empty priority queue and insert the numbers $7, 2, 4, 1, 9$ in that order. What is the result of `delMin()`?

A $-\infty$

B 1

C 2

D 4

E 7

F 9

G not allowed

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Clicker Question

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A ~~$-\infty$~~

B 1 ✓

C ~~2~~

D ~~4~~

E ~~7~~

F ~~9~~

G ~~not allowed~~

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Click on "Polls" tab

PQ implementations

Elementary implementations

- unordered list $\rightsquigarrow$ $\Theta(1)$ `insert`, but $\Theta(n)$ `delMax`
- sorted list $\rightsquigarrow$ $\Theta(1)$ `delMax`, but $\Theta(n)$ `insert`

PQ implementations

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Can we get something between these extremes? Like a "slightly sorted" list?

PQ implementations

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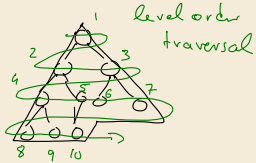


Can we get something between these extremes? Like a "slightly sorted" list?

Yes! *Binary heaps.*

Array view

Heap = array A with
$\forall i \in [n] : A[\lfloor i/2 \rfloor] \geq A[i]$

$\equiv$

store nodes
in level order
in $A[1..n]$

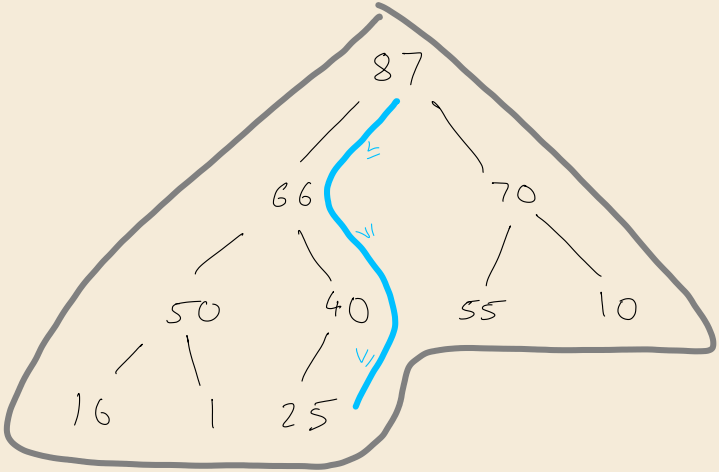
Tree view

Heap = tree that is
(i) a complete binary tree
(ii) heap ordered

all but last level full
last level flush left

parent
~~father~~ $\geq$ children

Binary heap example



Why heap-shaped trees?

Why complete binary tree shape?

- only one possible tree shape $\rightsquigarrow$ keep it simple!
- complete binary trees have minimal height among all binary trees
- simple formulas for moving from a node to parent or children:

  For a node at index k in A
  - parent at $\lfloor k/2 \rfloor$
  - left child at $2k$
  - right child at $2k + 1$

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Why heap ordered?

- Maximum must be at root! $\rightsquigarrow$ `max()` is trivial!
- But: Sorted only along paths of the tree; leaves lots of leeway for fast inserts

how? . . . stay tuned

Clicker Question

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B ~~A tree where all keys in the left subtree are smaller than the key at the root and all keys in the right subtree are bigger than the key at the root.~~

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D ~~An tree that is stored in the heap area of the memory.~~

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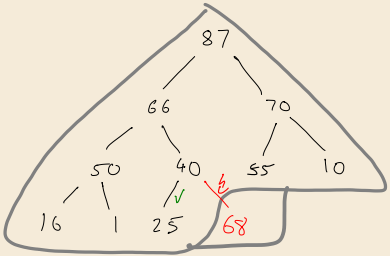
Insert

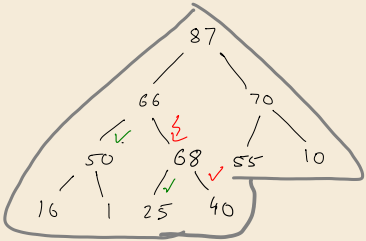


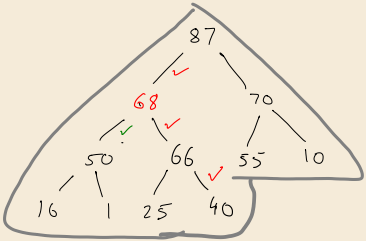
68

shape : only 1 possible position

BUT heap-order ↯

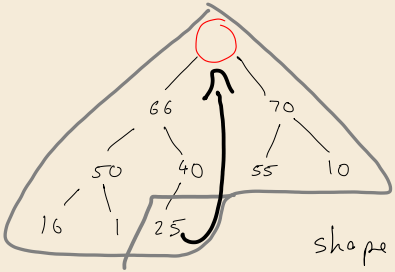
=> swim up the heap





✓

Delete ~~Min~~ Max



87 find max is easy

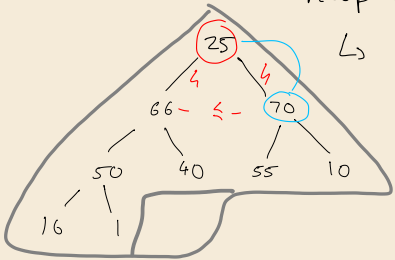
after removing it

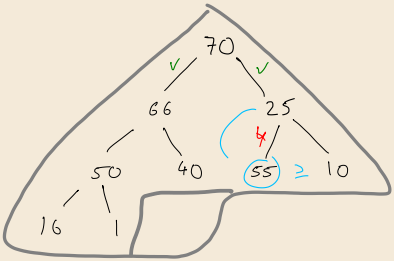
↳ complete binary tree

shape ✓

heap-order ↯

↳ let 25 sink in heap

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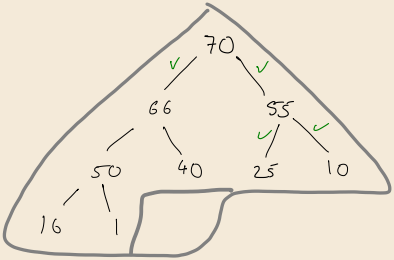
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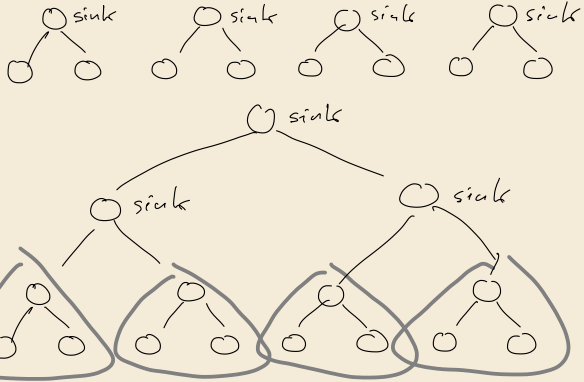
Heap construction

n · insert $\Rightarrow$ $\Theta(n \log n)$

can do better:



$\frac{n}{4}$ · sink in heap of size ≤ 3

$\frac{n}{8}$ · " ≤ 7

$\frac{n}{2^k}$ $\leq 2^k - 1$

$k = \lg n$

$\Theta(n)$ total time for heap construction.

Analysis

Running times:

- `insert` $O(\log n)$
- `delMax` $O(\log n)$
- `construct` $O(n)$