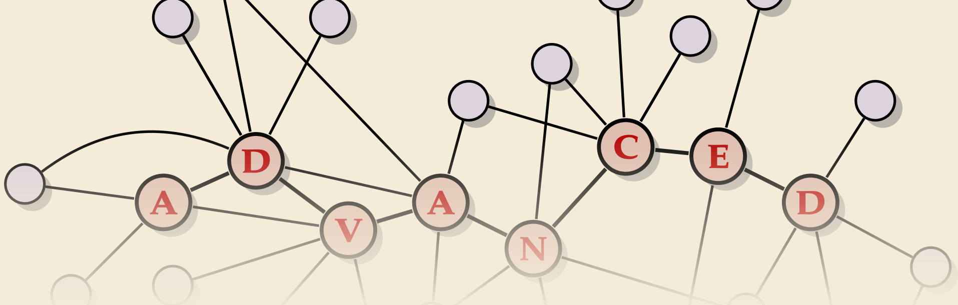

6 Advanced Parameterized Ideas

3 June 2025

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CS627 (Summer 2025)
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Outline

6 Advanced Parameterized Ideas

6.1 Linear Programs – A Mighty Blackbox Tool

Linear Programs

- ***Linear programs (LPs)*** are a class of optimization problems of **continuous** (numerical) variables
- can be exactly solved in worst case polytime (LinearProgramming $\in$ P)
  - interior-point methods, Ellipsoid method
- routinely solved in practice to optimality with millions of variables and constraints
  - Simplex algorithm, interior-point methods
  - many existing solvers, commercial and open source (e. g., HiGHS)

Hessy James's Apple Farm

- Hessy tries to maximize the profit of his apple farm
  - He is committed to promote regional Hessian heirloom varieties, so he only grows *"Sossenheimer Roter"* and *"Korbacher Edelrenette"*
  - each tree of *"Sossenheimer Roter"* yields apples worth € 195 per year
  - each tree of *"Korbacher Edelrenette"* yields applies worth € 255 per year
  - He has an orchard of 5 000 m^2
  - each tree needs 4 m^2 of orchard space
  - each tree of *"Sossenheimer Roter"* needs 6 kg of organic fertilizer and 1 h harvest effort per year
  - each tree of *"Korbacher Edelrenette"* needs 4.5 kg of organic fertilizer and 3 h harvest effort per year
  - Hessy can only afford 3000 kg of fertilizer and 1700 h of harvester time per year

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⇝ How many trees of each variety should Hessy plant?
  - What will constrain us most? Space? Fertilizer? Harvest hours?
  - What profit can Hessy expect?

Formal Linear Program for Hessy James's Apple Farm

- Classic application of linear programming in *operations research (OR)*
- We formally write LPs as follows:

optimization goal

objective function

constraint

$$
\begin{array}{llcll}
\textbf{Maximize:} & 195s + 255k & & & \\
\textbf{Subject to:} & 4s + 4k & \leq & 5000 & \text{(Orchard constraint)} \\
 & 6s + 4.5k & \leq & 3000 & \text{(Fertilizer constraint)} \\
 & 1s + 3k & \leq & 1700 & \text{(Harvest constraint)} \\
 & s & \geq & 0 & \text{(Non-negativity)} \\
 & k & \geq & 0 & \text{(Non-negativity)}
\end{array}
\qquad (P)
$$

name of the LP

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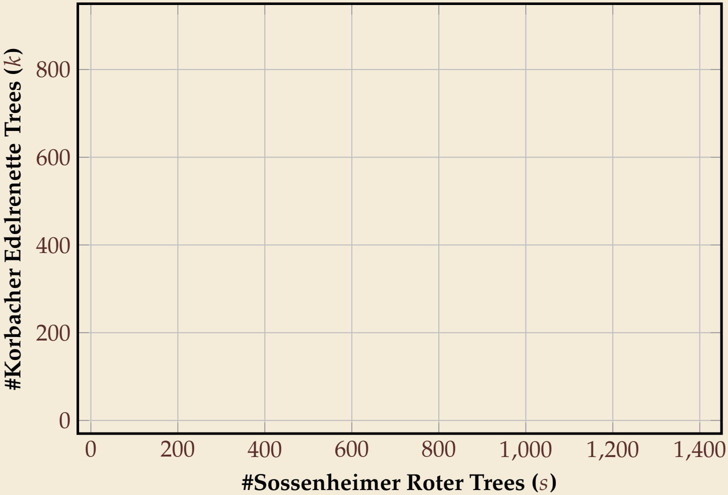
 & 1s + 3k \leq 1700 & \text{(Harvest constraint)} & (P) \\
 & s \geq 0 & \text{(Non-negativity)} & \\
 & k \geq 0 & \text{(Non-negativity)} &
\end{array}$$

- **Terminology:**
  - s and k are the two *variables* of the problem; these are always real numbers.
  - A vector $(s, k) \in \mathbb{R}^2$ is a *feasible solution* for the LP if it satisfied all constraints.
  - The largest value of the objective function (over all feasible solutions) is the *(optimal) value* z^* of the LP
  - A feasible solution $(s^*, k^*) \in R^2$ with optimal objective value z^* is called an *optimal solution*

2D LPs – Graphical Solution

LPs with **two** variables can be solved graphically



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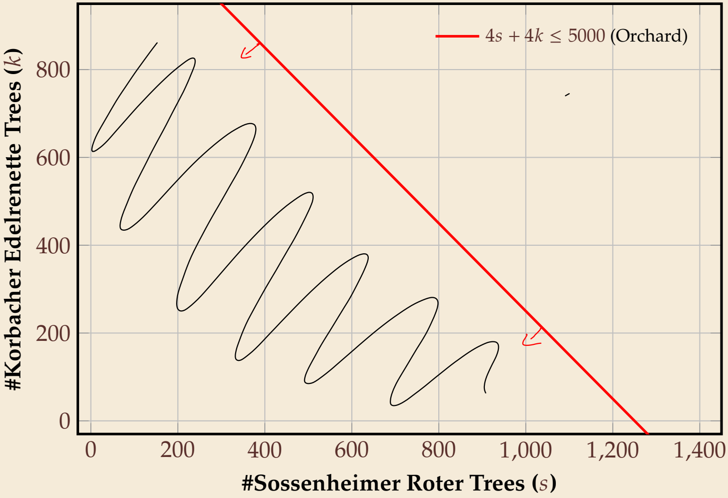

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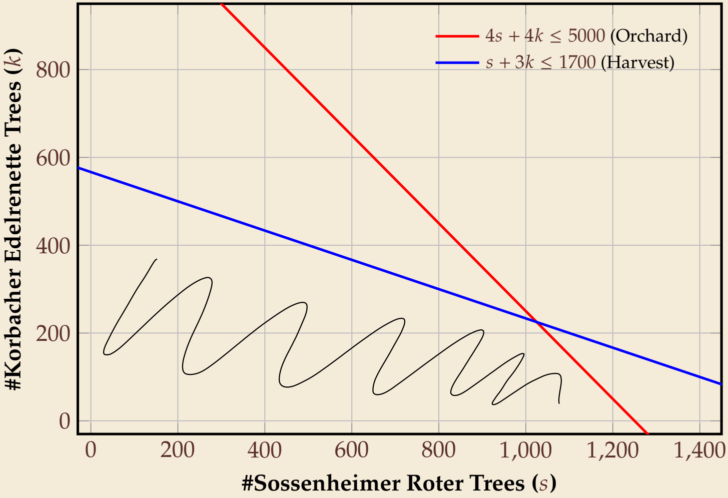

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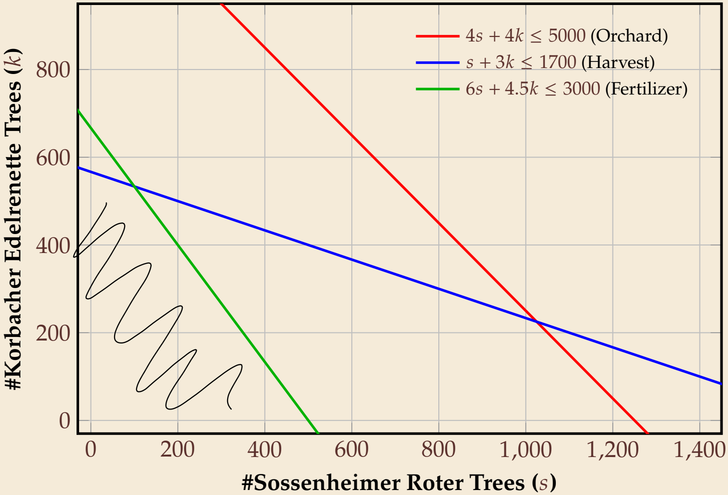

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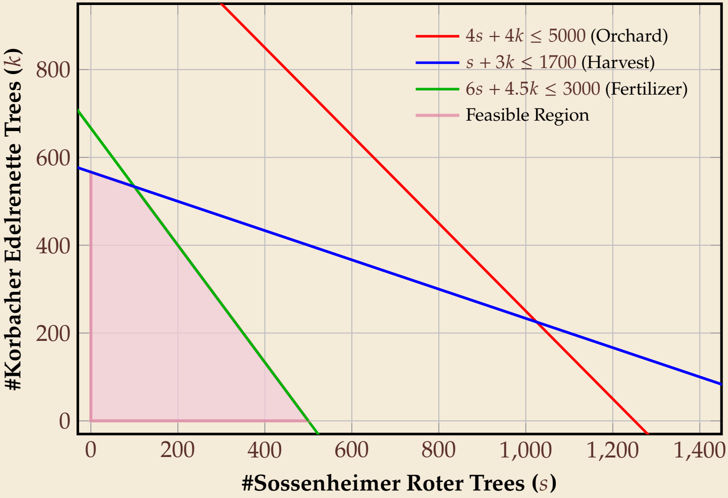

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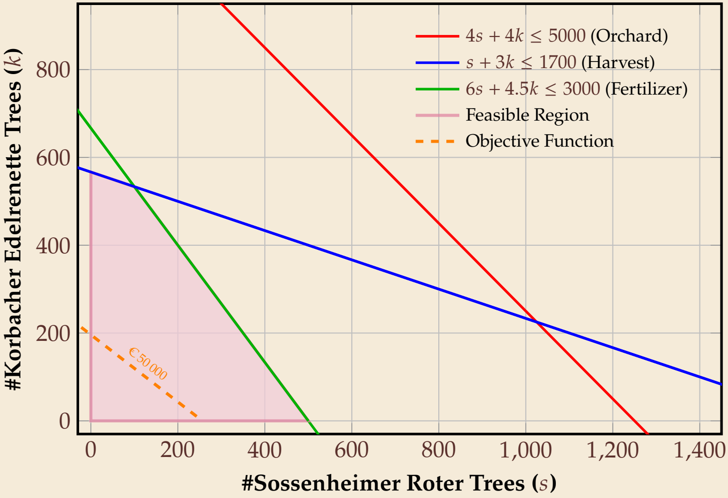

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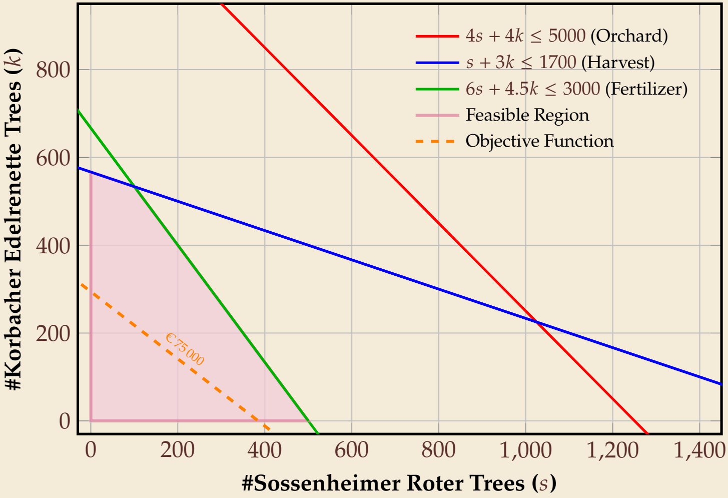

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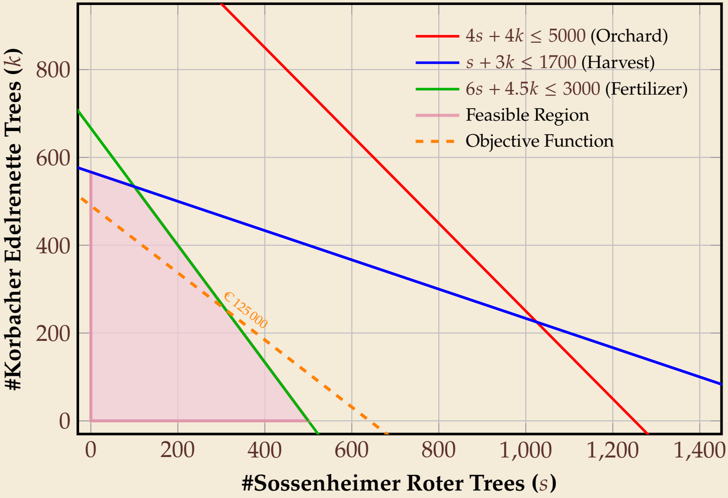

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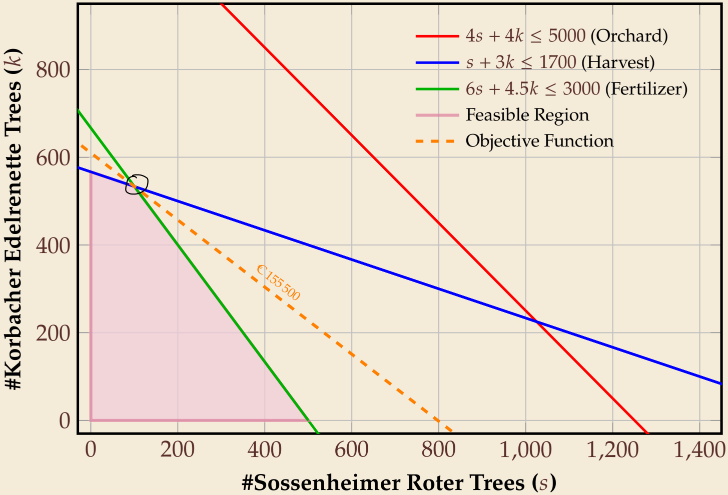

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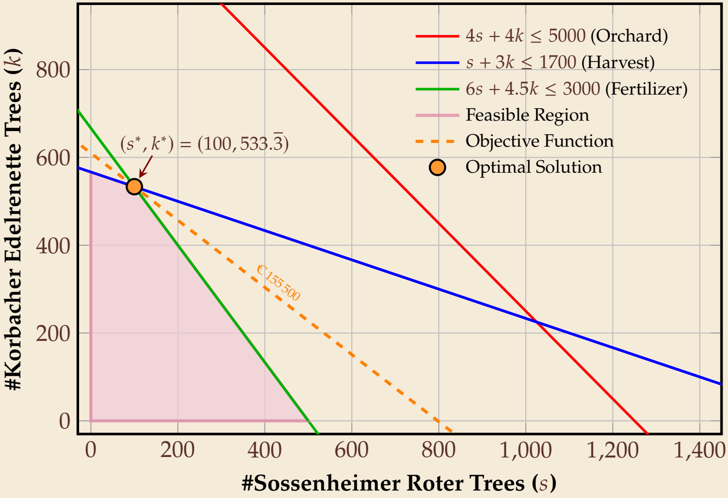

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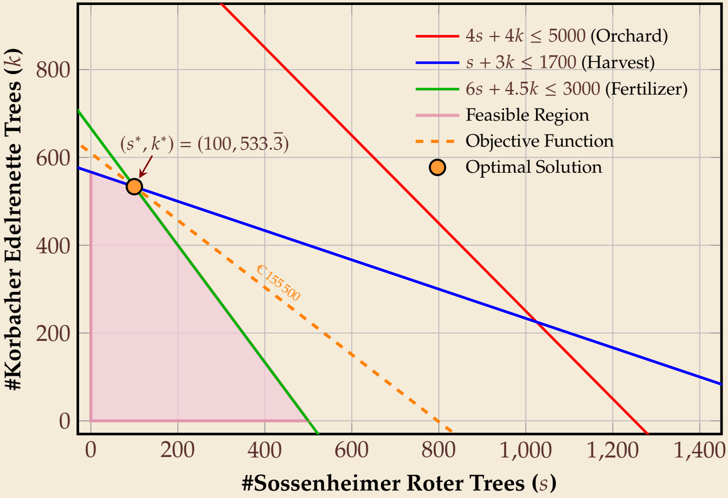

⇝ Hessy should plant

- 100 *Sossenheimer Roter* trees and
- hmm . . .
- $533+\frac{1}{3}$ *Korbacher Edelrenette* trees
- Harvest **and** fertilizer *tight*
- orchard space isn't

⇝ know what to change

LPs – The General Case

- General LP:

$$\begin{array}{rrcll} \min & c_1x_1 + \cdots + \ c_nx_n & & & \\ \text{s.t.} & a_{i,1}x_1 + \cdots + a_{i,n}x_n & = & b_i & (\text{for } i = 1, \ldots, p) \\ & a_{i,1}x_1 + \cdots + a_{i,n}x_n & \leq & b_i & (\text{for } i = p+1, \ldots, q) \\ & a_{i,1}x_1 + \cdots + a_{i,n}x_n & \geq & b_i & (\text{for } i = q+1, \ldots, m) \\ & x_j & \geq & 0 & (\text{for } j = 1 \ldots, r) \\ & x_j & \lesseqgtr & 0 & (\text{for } j = r+1 \ldots, n) \end{array}$$

"don't care" (just to make it explicit)

  - arbitrary **linear** objective function
  - arbitrary **linear** constraints, of type "=", "≤" or "≥"
  - variables with non-negativity constraint and unconstrained variables

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  - arbitrary **linear** objective function
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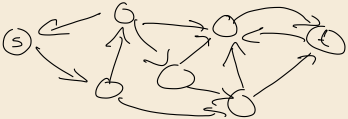
  - variables with non-negativity constraint and unconstrained variables

- In general, an LP can
  (a) have a *finite* optimal *objective value*
  (b) be *infeasible* (contradictory constraints / empty feasibility region), or
  (c) be *unbounded* (allow arbitrarily small objective values "$-\infty$")

⇝ in polytime, can detect which case applies **and** compute optimal solution in case (a)

Classic Modeling Example – Max Flow

- The maximum-s-t-flow problem in a graph $G = (V, E)$ can be reduced to an LP (Flow)
  - variable f_e for each edge $e \in E$
  - maximize flow value F = flow out of s
  - constraint for edge capacity $C(e)$ at each edge
  - constraint for flow conservation at each vertex v (except s and t)



$v \to w \quad (v, w)$

$$\begin{aligned}
\max \quad & F & & & \\
\text{s. t.} \quad & F &=& \sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} & \\
& f_{vw} &\le& C(vw) & (\text{for } vw \in E) \qquad \text{(Flow)} \\
& \sum_{w\in V} f_{wv} &=& \sum_{w\in V} f_{vw} & (\text{for } v \in V \setminus \{s,t\}) \\
& f_e &\ge& 0 & (\text{for } e \in E)
\end{aligned}$$

6.2 Linear Programs – Reformulation Tricks

How to solve an LP?

- Our focus will be on using LPs as a tool
  - in theory: reducing problem to an LP means polytime solvable
  - in practice: call good solver!

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  - in theory: reducing problem to an LP means polytime solvable
  - in practice: call good solver!

- *But as with any good tool, it helps to gave an idea of* ***how*** *it works to effectively use it*

$\rightsquigarrow$ We will briefly visit the conceptual ideas of the simplex algorithm

Recall: General Form of LPs

- General LP:

$$
\begin{array}{rrcll}
\min & c_1 x_1 + \cdots + \ c_n x_n & & & \\
\text{s.t.} & a_{i,1} x_1 + \cdots + a_{i,n} x_n & = & b_i & (\text{for } i = 1, \ldots, p) \\
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 & x_j & \lesseqgtr & 0 & (\text{for } j = r+1 \ldots, n)
\end{array}
$$

  - linear objective function and constraints ("$=$", "$\leq$", or "$\geq$")
  - variables with non-negativity constraint and unconstrained variables

- **Conventions:**
  - n variables (always called x_j)
  - m constraints (coefficients always called $a_{i,j}$, right-hand sides b_i)
  - minimize objective ("<u>c</u>ost"), coefficients c_j; objective value $z = c_1 x_1 + \cdots c_n x_n$

Enter Linear Algebra

$$
\begin{array}{llcll}
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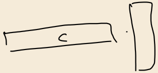

- Spelling out all those linear combinations is cumbersome

⇝ Concise notation via **matrix and vector products**

- We write
  - **variables** $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ **cost** coefficients $\boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$ ⇝ **objective**: $\min \boldsymbol{c}^T \cdot \boldsymbol{x}$

bold ⇝ vector/matrix

Enter Linear Algebra

$$\begin{array}{lrcll} \min & c_1x_1 + \cdots + \ c_nx_n & & & \\ \text{s.t.} & a_{i,1}x_1 + \cdots + a_{i,n}x_n & = & b_i & (\text{for } i = 1, \ldots, p) \\ & a_{i,1}x_1 + \cdots + a_{i,n}x_n & \leq & b_i & (\text{for } i = p+1, \ldots, q) \\ & a_{i,1}x_1 + \cdots + a_{i,n}x_n & \geq & b_i & (\text{for } i = q+1, \ldots, m) \\ & x_j & \geq & 0 & (\text{for } j = 1 \ldots, r) \\ & x_j & \lesseqgtr & 0 & (\text{for } j = r+1 \ldots, n) \end{array}$$

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transpose

dot product / scalar product

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    bold $\rightsquigarrow$ vector/matrix

    transpose

    dot product / scalar product
  - **"="-constraints**

$$\boldsymbol{A}^{(=)} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p,1} & a_{p,2} & \cdots & a_{p,n} \end{pmatrix} \in \mathbb{R}^{p \times n} \qquad \boldsymbol{b}^{(=)} = \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix} \in \mathbb{R}^p \qquad \rightsquigarrow \quad \boldsymbol{A}^{(=)} \cdot \boldsymbol{x} = \boldsymbol{b}^{(=)}$$

Enter Linear Algebra

$$\begin{array}{llcll} \min & c_1x_1 + \cdots + \ c_nx_n & & & \\ \text{s.t.} & a_{i,1}x_1 + \cdots + a_{i,n}x_n & = & b_i & (\text{for } i = 1, \ldots, p) \\ & a_{i,1}x_1 + \cdots + a_{i,n}x_n & \le & b_i & (\text{for } i = p+1, \ldots, q) \\ & a_{i,1}x_1 + \cdots + a_{i,n}x_n & \ge & b_i & (\text{for } i = q+1, \ldots, m) \\ & x_j & \ge & 0 & (\text{for } j = 1 \ldots, r) \\ & x_j & \lesseqgtr & 0 & (\text{for } j = r+1 \ldots, n) \end{array}$$

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  - similarly for "$\le$" and "$\ge$" constraints: $\boldsymbol{A}^{(\le)}\boldsymbol{x} \le \boldsymbol{b}^{(\le)}$ (elementwise $\le$) and $\boldsymbol{A}^{(\ge)}\boldsymbol{x} \ge \boldsymbol{b}^{(\ge)}$

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  $\rightsquigarrow$ a **single** constraint i can be written as $\quad \boldsymbol{A}_{i,\bullet}\,\boldsymbol{x} = b_i$ A[i, :]

  (generally write $\boldsymbol{A}_{i,\bullet}$ for the ith row of $\boldsymbol{A}$ and $\boldsymbol{A}_{\bullet,j}$ for the jth column)

Reformulations

Tricks of the Trade for working with LPs:

- min suffices: $\max \boldsymbol{c}^T \boldsymbol{x} \; = \; -\min(-\boldsymbol{c})^T \boldsymbol{x}$
- "≥"-constraints: $A_{i,\bullet} \boldsymbol{x} \; \geq \; b_i \quad \iff \quad (-A)_{i,\bullet} \boldsymbol{x} \; \leq \; -b_i$

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- *slack variables*: $\boldsymbol{A}_{i,\bullet}\boldsymbol{x} \;\leq\; b_i \quad\Longleftrightarrow\quad \boldsymbol{A}_{i,\bullet}\boldsymbol{x} + \boldsymbol{x_{s_i}} \;=\; b_i \quad \text{and} \quad x_{s_i} \;\geq\; 0$

  (x_{s_i} is a new additional variable)

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  (x_{s_i} is a new additional variable)
- *nonnegative*: variable $x_j \lesseqgtr 0 \;\iff\; x_j \;=\; x_{j,+} - x_{j,-}$ and $x_{j,+}, x_{j,-} \;\geq\; 0$

  ($x_{j,+}$ and $x_{j,-}$ are new additional variables)

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- *slack variables*: $A_{i,\bullet}\boldsymbol{x} \leq b_i \iff A_{i,\bullet}\boldsymbol{x} + \boldsymbol{x_{s_i}} = b_i$ and $x_{s_i} \geq 0$

  (x_{s_i} is a new additional variable)
- *nonnegative*: variable $x_j \lesseqgtr 0 \iff x_j = x_{j,+} - x_{j,-}$ and $x_{j,+}, x_{j,-} \geq 0$

  ($x_{j,+}$ and $x_{j,-}$ are new additional variables)

$\rightsquigarrow$ To solve LPs, can assume one of the following **normal forms**

$$\boxed{\begin{aligned} &\min\ \boldsymbol{c}^T\boldsymbol{x} \\ &\text{s.t.}\ \ A\boldsymbol{x} \leq \boldsymbol{b} \\ &\qquad\ \ \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} &\min\ \boldsymbol{c}^T\boldsymbol{x} \\ &\text{s.t.}\ \ A\boldsymbol{x} = \boldsymbol{b} \\ &\qquad\ \ \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}} \quad \text{with } A \in \mathbb{R}^{m\times n},\ \boldsymbol{b} \in \mathbb{R}^m, \text{ and } \boldsymbol{c} \in \mathbb{R}^n$$

$m \geq n$ $\qquad\qquad$ $m \lesseqgtr n$

6.3 Linear Programs – The Simplex Algorithm

Simplex – Geometric Intuition

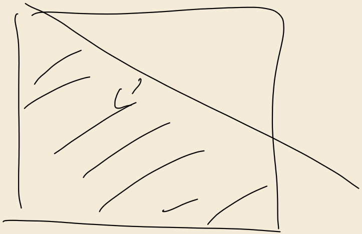
$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax \le b \\ & x \ge 0 \end{aligned}$$

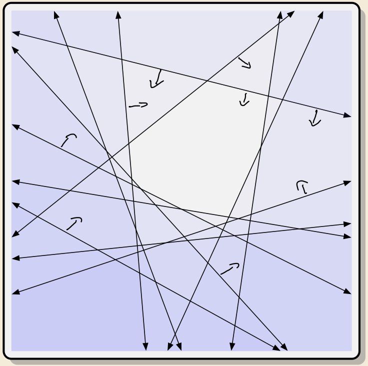
+ *nondegeneracy*

- constraint $A_{i,\bullet}x \le b_i$ defines a *hyperplane*/*halfspace*

$\boxed{n = 2, m = 12}$

$\rightsquigarrow$ $H_i^= = \{x \in \mathbb{R}^n : A_{i,\bullet}x = b_i\}$
$H_i = \{x \in \mathbb{R}^n : A_{i,\bullet}x \le b_i\}$





Simplex – Geometric Intuition

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s. t.} \quad & A\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

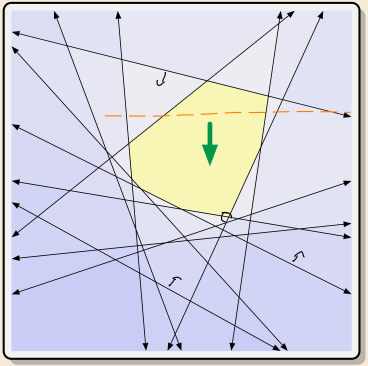
+ *nondegeneracy*

- constraint $A_{i,\bullet}\boldsymbol{x} \le b_i$ defines a *hyperplane*/*halfspace*

$\boxed{n = 2, m = 12}$

$\rightsquigarrow$ $H_i^{=} = \{\boldsymbol{x} \in \mathbb{R}^n : A_{i,\bullet}\boldsymbol{x} = b_i\}$

$H_i = \{\boldsymbol{x} \in \mathbb{R}^n : A_{i,\bullet}\boldsymbol{x} \le b_i\}$



- $\boldsymbol{c}$ = **direction** of improvement in $\mathbb{R}^n$ (*normal vector* for hyperplane $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{c}^T\boldsymbol{x} = 0\}$)
  - *"Roll a ball downhill inside feasible region"*

Simplex – Geometric Intuition

$$\begin{aligned} \min \;\; & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \;\; & A\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

+ *nondegeneracy*

- constraint $A_{i,\bullet}\boldsymbol{x} \le b_i$ defines a *hyperplane*/*halfspace*

$\boxed{n = 2, m = 12}$

⇝ $H_i^{=} = \{\boldsymbol{x} \in \mathbb{R}^n : A_{i,\bullet}\boldsymbol{x} = b_i\}$
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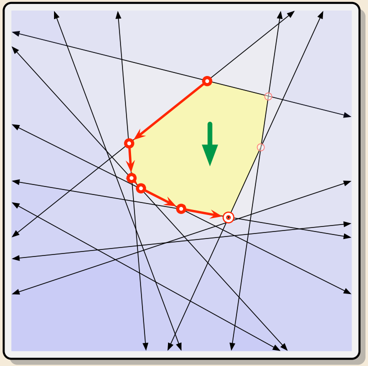

- $\boldsymbol{c}$ = **direction** of improvement in $\mathbb{R}^n$
  (*normal vector* for hyperplane $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{c}^T\boldsymbol{x} = 0\}$)
  - *"Roll a ball downhill inside feasible region"*

  ⇝ Optimal point $\boldsymbol{x}^*$ must lie on boundary!

  (assuming finite optimal objective value z^*)

Simplex – Geometric Intuition

$$\begin{aligned} \min\ & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t. }\ & A\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$
+ *nondegeneracy*

- constraint $A_{i,\bullet}\boldsymbol{x} \le b_i$ $\boxed{n = 2, m = 12}$
  defines a *hyperplane*/*halfspace*

$\rightsquigarrow$ $H_i^{=} = \{\boldsymbol{x} \in \mathbb{R}^n : A_{i,\bullet}\boldsymbol{x} = b_i\}$
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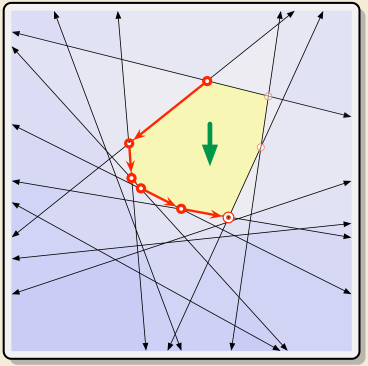

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  - *"Roll a ball downhill inside feasible region"*

  $\rightsquigarrow$ Optimal point $\boldsymbol{x}^*$ must lie on boundary!

  (assuming finite optimal objective value z^*)

assuming nondegeneracy

- intersection of n hyperplanes $H_i^{=}$ is unique point

Simplex – Geometric Intuition

$$\boxed{\begin{array}{rrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \le & b \\ & x & \ge & 0 \end{array}}$$
+ *nondegeneracy*

- constraint $A_{i,\bullet}x \le b_i$ defines a *hyperplane*/*halfspace*

$\boxed{n = 2, m = 12}$

$\rightsquigarrow$ $H_i^{=} = \{x \in \mathbb{R}^n : A_{i,\bullet}x = b_i\}$
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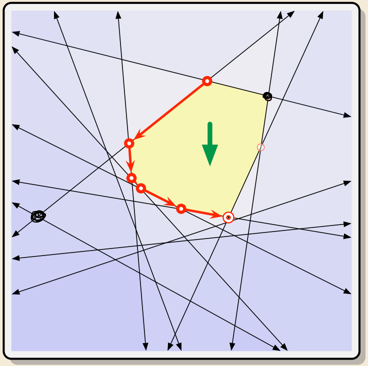

- c = **direction** of improvement in $\mathbb{R}^n$ (*normal vector* for hyperplane $\{x \in \mathbb{R}^n : c^T x = 0\}$)
  - *"Roll a ball downhill inside feasible region"*

  $\rightsquigarrow$ Optimal point x^* must lie on boundary!

  (assuming finite optimal objective value z^*)

assuming nondegeneracy

- intersection of n hyperplanes $H_i^{=}$ is unique point

  $\rightsquigarrow$ ***vertex*** $\{x_I\} = \bigcap_{i \in I} H_i^{=}$ (for $I \subset [m]$, $|I| = n$)

Simplex – Geometric Intuition

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \end{array}$$
+ *nondegeneracy*

- constraint $A_{i,\bullet}x \le b_i$ defines a *hyperplane*/*halfspace* $\boxed{n = 2, m = 12}$

$\rightsquigarrow$ $H_i^{=} = \{x \in \mathbb{R}^n : A_{i,\bullet}x = b_i\}$
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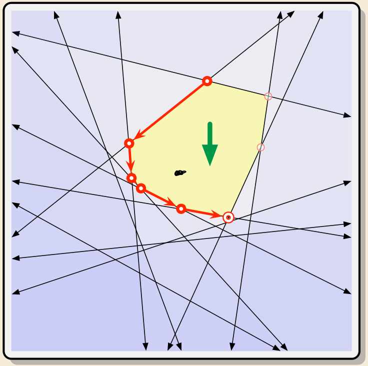

- c = **direction** of improvement in $\mathbb{R}^n$ (*normal vector* for hyperplane $\{x \in \mathbb{R}^n : c^T x = 0\}$)
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  (assuming finite optimal objective value z^*)

assuming nondegeneracy

- intersection of n hyperplanes $H_i^{=}$ is unique point

  $\rightsquigarrow$ ***vertex*** $\{x_I\} = \bigcap_{i \in I} H_i^{=}$ (for $I \subset [m]$, $|I| = n$)

- always have $c^T x^* = c^T x_{I^*}$ for a **vertex** x_{I^*}

Simplex – Geometric Intuition

$$\min\ c^T x \quad \text{s.t.}\ Ax \leq b,\ x \geq 0$$
+ *nondegeneracy*

- constraint $A_{i,\bullet} x \leq b_i$ defines a *hyperplane*/*halfspace* $\boxed{n = 2, m = 12}$

  ⇝ $H_i^= = \{x \in \mathbb{R}^n : A_{i,\bullet} x = b_i\}$

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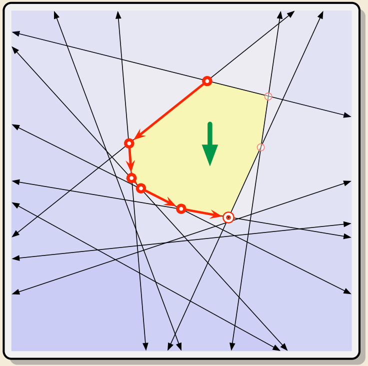

- c = **direction** of improvement in $\mathbb{R}^n$ (*normal vector* for hyperplane $\{x \in \mathbb{R}^n : c^T x = 0\}$)
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  ⇝ Optimal point x^* must lie on boundary!

  (assuming finite optimal objective value z^*)

assuming nondegeneracy

- intersection of n hyperplanes $H_i^=$ is unique point

  ⇝ ***vertex*** $\{x_I\} = \bigcap_{i \in I} H_i^=$ (for $I \subset [m]$, $|I| = n$)

- always have $c^T x^* = c^T x_{I^*}$ for a **vertex** x_{I^*}
  - "only" $\binom{m}{n}$ vertices x_I (all n-subsets of $[m]$)

(n const ⇝ polytime brute force)

Simplex – Geometric Intuition

$$\begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{array}$$
+ *nondegeneracy*

- constraint $A_{i,\bullet}\boldsymbol{x} \le b_i$ defines a *hyperplane*/*halfspace* $\boxed{n = 2, m = 12}$

$\rightsquigarrow$ $H_i^= = \{\boldsymbol{x} \in \mathbb{R}^n : A_{i,\bullet}\boldsymbol{x} = b_i\}$
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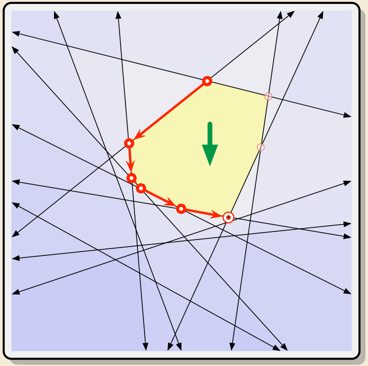

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  - *"Roll a ball downhill inside feasible region"*

  $\rightsquigarrow$ Optimal point $\boldsymbol{x}^*$ must lie on boundary!

  (assuming finite optimal objective value z^*)

assuming nondegeneracy

- intersection of n hyperplanes $H_i^=$ is unique point

  $\rightsquigarrow$ ***vertex*** $\{\boldsymbol{x}_I\} = \bigcap_{i\in I} H_i^=$ (for $I \subset [m]$, $|I| = n$)

- always have $\boldsymbol{c}^T\boldsymbol{x}^* = \boldsymbol{c}^T\boldsymbol{x}_{I^*}$ for a **vertex** $\boldsymbol{x}_{I^*}$
  - "only" $\binom{m}{n}$ vertices $\boldsymbol{x}_I$ (all n-subsets of $[m]$)

  $\rightsquigarrow$ *Simplex algorithm*:

  Move to better neighbor until optimal.
  - $\boldsymbol{x}_I$ and $\boldsymbol{x}_{I'}$ neighbors if $|I \cap I'| = n - 1$

Simplex – Geometric Intuition

$$\begin{array}{rl} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{array}$$
+ *nondegeneracy*

- constraint $A_{i,\bullet}\boldsymbol{x} \le b_i$ defines a *hyperplane*/*halfspace* $\boxed{n = 2, m = 12}$

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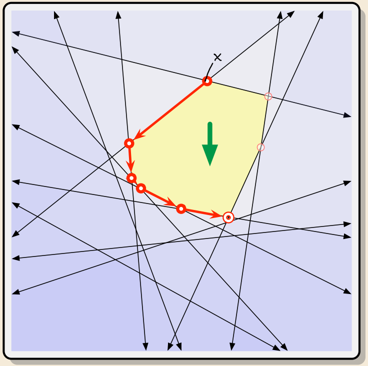

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assuming nondegeneracy

- intersection of n hyperplanes $H_i^=$ is unique point

  $\rightsquigarrow$ ***vertex*** $\{\boldsymbol{x}_I\} = \bigcap_{i \in I} H_i^=$ (for $I \subset [m]$, $|I| = n$)

- always have $\boldsymbol{c}^T\boldsymbol{x}^* = \boldsymbol{c}^T\boldsymbol{x}_{I^*}$ for a **vertex** $\boldsymbol{x}_{I^*}$
  - "only" $\binom{m}{n}$ vertices $\boldsymbol{x}_I$ (all n-subsets of $[m]$)

  $\rightsquigarrow$ *Simplex algorithm*:
  Move to better neighbor until optimal.
  - $\boldsymbol{x}_I$ and $\boldsymbol{x}_{I'}$ neighbors if $|I \cap I'| = n - 1$

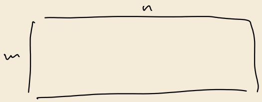
```
procedure simplexIteration(H = {H_1, ..., H_m}):
    if ⋂H == ∅ return INFEASIBLE
    x := any feasible vertex
    while x is not locally optimal // c "against wall"
        // pivot towards better objective function
        if ∀ feasible neighbor vertex x' : c^T x' > c^T x
            return UNBOUNDED
        else
            x := some feasible lower neighbor of x
    return x
```

Simplex – Linear Algebra Realization

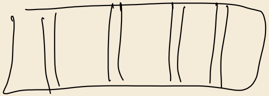
$$\boxed{\begin{aligned} \min\ & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.}\ & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \\ & + \textit{nondegeneracy} \end{aligned}}$$

- Here use equality constraints $\rightsquigarrow$ $\underline{m \le n}$
- Assume $\operatorname{rank}(A) = m$ (nondegeneracy)
- every $J = \{j_1, \ldots, j_m\} \subseteq [n]$ corresponds to *basis* of A: $\{A_{\bullet,j_1}, \ldots, A_{\bullet,j_m}\}$

assuming nondegeneracy



Simplex – Linear Algebra Realization

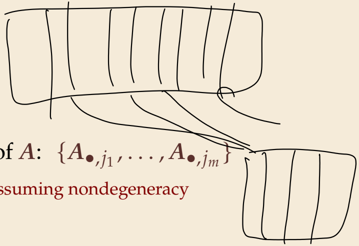


$$\begin{array}{rl} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{array}$$
+ *nondegeneracy*

- Here use equality constraints $\rightsquigarrow$ $m \le n$
- Assume $\operatorname{rank}(A) = m$ (nondegeneracy)
- every $J = \{j_1, \ldots, j_m\} \subseteq [n]$ corresponds to *basis* of A: $\{A_{\bullet,j_1}, \ldots, A_{\bullet,j_m}\}$
  assuming nondegeneracy

- **Notation:**
  - $\boldsymbol{x}_J = (x_{j_1}, \ldots, x_{j_m})^T$ vector of *basis variables*
  - $\boldsymbol{x}_{\bar{J}} = (x_{\bar{j}_1}, \ldots, x_{\bar{j}_{n-m}})^T$ vector of *non-basis variables* for $\bar{J} = [n] \setminus J = \{\bar{j}_1, \ldots, \bar{j}_{n-m}\}$

Simplex – Linear Algebra Realization



$$\begin{array}{rl} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s. t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$
+ *nondegeneracy*

- Here use equality constraints $\rightsquigarrow$ $m \leq n$
- Assume $\operatorname{rank}(A) = m$ (nondegeneracy)
- every $J = \{j_1, \ldots, j_m\} \subseteq [n]$ corresponds to *basis* of A: $\{\boldsymbol{A}_{\bullet,j_1}, \ldots, \boldsymbol{A}_{\bullet,j_m}\}$

  assuming nondegeneracy

- **Notation:**
  - $\boldsymbol{x}_J = (x_{j_1}, \ldots, x_{j_m})^T$ vector of *basis variables*
  - $\boldsymbol{x}_{\bar{J}} = (x_{\bar{j}_1}, \ldots, x_{\bar{j}_{n-m}})^T$ vector of *non-basis variables* for $\bar{J} = [n] \setminus J = \{\bar{j}_1, \ldots, \bar{j}_{n-m}\}$
  - $\boldsymbol{A}_J = (\boldsymbol{A}_{\bullet,j_1}, \ldots, \boldsymbol{A}_{\bullet,j_m}) \in \mathbb{R}^{m \times m}$; similarly $\boldsymbol{A}_{\bar{J}} = (\boldsymbol{A}_{\bullet,\bar{j}_1}, \ldots, \boldsymbol{A}_{\bullet,\bar{j}_{n-m}}) \in \mathbb{R}^{(n-m)\times m}$
  - $\boldsymbol{c}_J$ and $\boldsymbol{c}_{\bar{J}}$ defined similarly

Simplex – Linear Algebra Realization

$$\begin{array}{|rl|} \hline \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \\ \multicolumn{2}{|l|}{+\ \textit{nondegeneracy}} \\ \hline \end{array}$$

- Here use equality constraints $\rightsquigarrow$ $m \leq n$
- Assume $\operatorname{rank}(A) = m$ (nondegeneracy)
- every $J = \{j_1, \ldots, j_m\} \subseteq [n]$ corresponds to *basis* of A: $\{A_{\bullet,j_1}, \ldots, A_{\bullet,j_m}\}$
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- **Notation:**
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  - $A_J = (A_{\bullet,j_1}, \ldots, A_{\bullet,j_m}) \in \mathbb{R}^{m \times m}$; similarly $A_{\bar{J}} = (A_{\bullet,\bar{j}_1}, \ldots, A_{\bullet,\bar{j}_{n-m}}) \in \mathbb{R}^{(n-m)\times m}$
  - $\boldsymbol{c}_J$ and $\boldsymbol{c}_{\bar{J}}$ defined similarly

$\rightsquigarrow$ We have $A\boldsymbol{x} = \boldsymbol{b} \iff \underbrace{A_J}_{\text{square \& full rank}}\boldsymbol{x}_J + A_{\bar{J}}\boldsymbol{x}_{\bar{J}} = \boldsymbol{b} \iff \boxed{\boldsymbol{x}_J = A_J^{-1}\boldsymbol{b} - A_J^{-1}A_{\bar{J}}\boldsymbol{x}_{\bar{J}}}$

$\boldsymbol{x}_J$ is uniquely determined by choosing $\boldsymbol{x}_{\bar{J}}$

Simplex – Linear Algebra Realization

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & x \ge 0 \end{aligned}$$
$+$ *nondegeneracy*

- Here use equality constraints $\rightsquigarrow$ $m \le n$
- Assume $\operatorname{rank}(A) = m$ (nondegeneracy)
- every $J = \{j_1, \ldots, j_m\} \subseteq [n]$ corresponds to *basis* of A: $\{A_{\bullet,j_1}, \ldots, A_{\bullet,j_m}\}$
  assuming nondegeneracy

- **Notation:**
  - $x_J = (x_{j_1}, \ldots, x_{j_m})^T$ vector of *basis variables*
  - $x_{\bar J} = (x_{\bar j_1}, \ldots, x_{\bar j_{n-m}})^T$ vector of *non-basis variables* for $\bar J = [n] \setminus J = \{\bar j_1, \ldots, \bar j_{n-m}\}$
  - $A_J = (A_{\bullet,j_1}, \ldots, A_{\bullet,j_m}) \in \mathbb{R}^{m\times m}$; similarly $A_{\bar J} = (A_{\bullet,\bar j_1}, \ldots, A_{\bullet,\bar j_{n-m}}) \in \mathbb{R}^{(n-m)\times m}$
  - c_J and $c_{\bar J}$ defined similarly

square & full rank

$\rightsquigarrow$ We have $Ax = b \iff A_J x_J + A_{\bar J} x_{\bar J} = b \iff \boxed{x_J = A_J^{-1} b - A_J^{-1} A_{\bar J} x_{\bar J}}$

x_J is uniquely determined by choosing $x_{\bar J}$

- *basic solution* setting $x_{\bar J} = 0$ gives $x_J = A_J^{-1} b$ $\rightsquigarrow$ correspond to *vertices* from before
  - may or may not be a *feasible basic solution*: $x_J \ge 0$?

$\rightsquigarrow$ given J, can easily compute basic solution and check feasibility

Simplex – Local Optimality Test

- basic solution: $\boxed{x_J = A_J^{-1}b - A_J^{-1}A_{\bar{J}}x_{\bar{J}}}$ and $x_{\bar{J}} = \mathbf{0}$

$$\begin{array}{rrcl} \min & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & \mathbf{0} \end{array}$$
+ *nondegeneracy*

Simplex – Local Optimality Test

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s. t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

$+$ *nondegeneracy*

- basic solution: $\boxed{\boldsymbol{x}_J = A_J^{-1}\boldsymbol{b} - A_J^{-1}A_{\bar{J}}\boldsymbol{x}_{\bar{J}}}$ and $\boldsymbol{x}_{\bar{J}} = \boldsymbol{0}$
- How to locally modify basic solution without violating constraints?
  - can't change x_{j_k} for $j_k \in J$ (equality constraint);
  - can't *decrease* $x_{\bar{j}_k}$ for $\bar{j}_k \in \bar{J}$ (nonnegativity);

  $\rightsquigarrow$ can only increase $x_{\bar{j}_k}$ by small $\delta > 0$

Simplex – Local Optimality Test

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

+ *nondegeneracy*

- basic solution: $\boxed{\boldsymbol{x}_J \;=\; A_J^{-1}\boldsymbol{b} - A_J^{-1}A_{\bar{J}}\boldsymbol{x}_{\bar{J}}}$ and $\boldsymbol{x}_{\bar{J}} = \boldsymbol{0}$
- How to locally modify basic solution without violating constraints?
  - can't change x_{j_k} for $j_k \in J$ (equality constraint);
  - can't *decrease* $x_{\bar{j}_k}$ for $\bar{j}_k \in \bar{J}$ (nonnegativity);
  - $\rightsquigarrow$ can only increase $x_{\bar{j}_k}$ by small $\delta > 0$
- rewrite cost: $\boldsymbol{c}^T\boldsymbol{x} \;=\; \boldsymbol{c}_J\boldsymbol{x}_J + \boldsymbol{c}_{\bar{J}}^T\boldsymbol{x}_{\bar{J}}$

Simplex – Local Optimality Test

$$\begin{aligned} \min \;\; & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \;\; & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

+ *nondegeneracy*

- basic solution: $\boxed{\boldsymbol{x}_J = A_J^{-1}\boldsymbol{b} - A_J^{-1}A_{\bar{J}}\boldsymbol{x}_{\bar{J}}}$ and $\boldsymbol{x}_{\bar{J}} = \boldsymbol{0}$
- How to locally modify basic solution without violating constraints?
  - can't change x_{j_k} for $j_k \in J$ (equality constraint);
  - can't *decrease* $x_{\bar{j}_k}$ for $\bar{j}_k \in \bar{J}$ (nonnegativity);
  - ⇝ can only increase $x_{\bar{j}_k}$ by small $\delta > 0$
- rewrite cost:

$$\begin{aligned} \boldsymbol{c}^T\boldsymbol{x} &= \boldsymbol{c}_J\boldsymbol{x}_J + \boldsymbol{c}_{\bar{J}}^T\boldsymbol{x}_{\bar{J}} \\ &= \boldsymbol{c}_J\left(A_J^{-1}\boldsymbol{b} - A_J^{-1}A_{\bar{J}}\boldsymbol{x}_{\bar{J}}\right) + \boldsymbol{c}_{\bar{J}}^T\boldsymbol{x}_{\bar{J}} \end{aligned}$$

Simplex – Local Optimality Test

$$\min \ \boldsymbol{c}^T\boldsymbol{x} \quad \text{s.t.} \quad A\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}$$
+ *nondegeneracy*

- basic solution: $\boxed{\boldsymbol{x}_J = A_J^{-1}\boldsymbol{b} - A_J^{-1}A_{\bar{J}}\boldsymbol{x}_{\bar{J}}}$ and $\boldsymbol{x}_{\bar{J}} = \boldsymbol{0}$
- How to locally modify basic solution without violating constraints?
  - can't change x_{j_k} for $j_k \in J$ (equality constraint);
  - can't *decrease* $x_{\bar{j}_k}$ for $\bar{j}_k \in \bar{J}$ (nonnegativity);
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Convex function over a convex domain
⇝ local opt $\Longrightarrow$ global opt

⇝ **No** (local) improvement possible $\iff$ $\tilde{\boldsymbol{c}}_{\bar{J}} \ge 0$ $\iff$ current basic solution **optimal**

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Convex function over a convex domain
$\rightsquigarrow$ local opt $\implies$ global opt

$\rightsquigarrow$ **No** (local) improvement possible $\iff$ $\tilde{\boldsymbol{c}}_{\bar{J}} \geq 0$ $\iff$ current basic solution **optimal**

- Otherwise: Bring $\bar{j}_k$ with $\tilde{\boldsymbol{c}}_{\bar{j}_k} < 0$ into basis
  - This means we increase $x_{\bar{j}_k}$ as much as possible until some x_{j_k} becomes 0
  - $\rightsquigarrow$ corresponds to moving to neighbor vertex

Summary LP Algorithms

- **Simplex Algorithm**
  - 👍 simple and mostly combinatorial algorithm
  - 👍 easy to implement
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    but no rule known that guarantees polytime
    - 👍 but *smoothed analysis* proves: random perturbations of input yield expected polytime on any input

Summary LP Algorithms

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- **Alternative methods**
  - **ellipsoid method** (separation-oracle based)
  - **interior-point methods** (numeric algorithms)
  - worst case polytime
  - interior-point method fastest in practice
  - more complicated, harder to implement well

6.4 Integer Linear Programs

When LPs Are Too Smooth

- Many natural optimization problems have linear objective and constraints
  - Example: **The Knapsack Problem**

    Given: items $1, \ldots, n$ with weights $\boldsymbol{w} \in \mathbb{N}^n$ and values $\boldsymbol{v} \in \mathbb{N}^n$
    knapsack weight capacity $b \in \mathbb{N}$

    Goal: Select subset of items of maximal total value, subject to fitting in the knapsack

  $\rightsquigarrow$ Introduce variable x_i, such that "item included" iff $x_1 = 1$

$$
\begin{aligned}
\max \quad & \boldsymbol{v}^T \boldsymbol{x} \\
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  These problems are NP-hard; so there must be something wrong?

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  These problems are NP-hard; so there must be something wrong?

ϟ Integrality! Optimal fractional Knapsack $\boldsymbol{x}^*$ can be nonsensical:
Could have $x_i = \frac{1}{2}$ for a single high-value item of weight $2b$, etc.

Integer Linear Programs

- A *(mixed)* ***integer*** *linear program* (ILP/IP resp. MILP) is a linear program, where (some) variables are constrained to integers, $x_i \in \mathbb{Z}$.
  - focus here on the case that all variables are integral: $\boldsymbol{x} \in \mathbb{Z}^n$

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intersection of halfspaces

$\rightsquigarrow$ feasibility region of an LP is a *polyhedron* $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0}\}$
feasibility region of an ILP is the intersection of P with the integer lattice:
$P_{\mathbb{Z}} = P \cap \mathbb{Z}^n \subset P$

Solving ILP is NP-hard. (0/1 - INTEGER PROGRAMMING)

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feasibility region of an ILP is the intersection of P with the integer lattice:
$\underline{P_{\mathbb{Z}} \;=\; P \cap \mathbb{Z}^n \;\subset\; P}$

$\rightsquigarrow$ Still get a lower bound on objective value

$\boxed{\text{optimal objective value of LP} \;\leq\; \text{optimal objective value of ILP}}$

LP Relaxations

- Given a combinatorial optimization problem as ILP,
  its *LP relaxation* is the LP obtained by dropping all integrality constraints.

Idea: cheap way to get lower bounds for optimal value

LP Relaxations

- Given a combinatorial optimization problem as ILP,
  its *LP relaxation* is the LP obtained by dropping all integrality constraints.

- **Example:** Independent Set
  - Given: $G = (V, E)$
    Goal: Maximum-cardinality independent set
  - Introduce variable $x_v \in \{0, 1\}$ for $v \in V$

$$
\begin{aligned}
\max \quad & \sum_{v \in V} x_v \\
\text{s.t.} \quad & x_v + x_w \le 1 && (\forall vw \in E) \qquad \text{(IS-ILP)} \\
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Integrality Gap

- The ratio $\frac{z^*_{\text{ILP}}}{z^*_{\text{LP}}}$ is called the *integrality gap* of an LP relaxation.

(can also refer to integrality gap of a problem

Integrality Gap

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  - Hessy James's apple trees: use 533 instead of $533.33\ldots$ trees

  $\rightsquigarrow$ actual profit € 155 415 instead of € 155 500 $\rightsquigarrow$ minuscule difference

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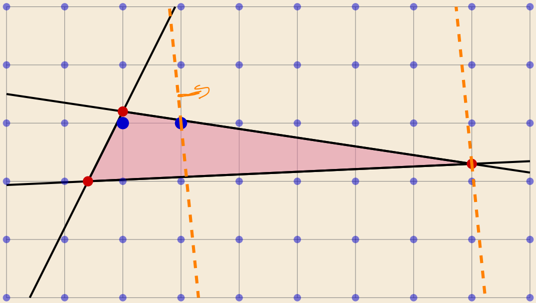
- in the worst case, integrality gap can be bad



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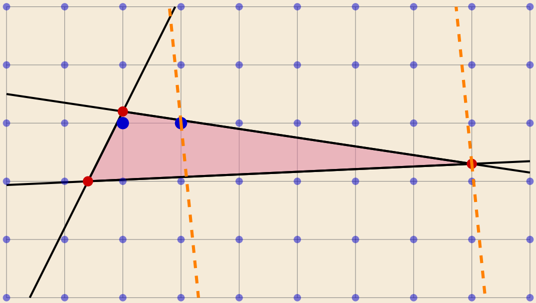

- actual example: Independent Set
  - Consider complete graph $G = K_n$
  - Largest independent set is single vertex $\leadsto$ $z^*_{\text{ILP}} = 1$
  - Fractional solution possible with $z^*_{\text{LP}} = n/2$ by setting all $x_v = \frac{1}{2}$
  - $\leadsto$ unbounded integrality gap

6.5 LP-Based Kernelization

Vertex Cover as (Integer) Linear Program

Consider optimization version of VertexCover:
Given: Graph $G = (V, E)$
Goal: Vertex cover of G with minimal cardinality.

solvable in $O(1.3^k n^c)$

$O(k^2)$ kernel

Vertex Cover as (Integer) Linear Program

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$\rightsquigarrow$ equivalent to the following integer linear program

$$\begin{aligned} &\min \textstyle\sum_{v \in V} x_v \\ &\text{s.t. } x_u + x_v \geq 1 \quad \text{for all } \{u, v\} \in E \\ &\qquad x_v \in \{0, 1\} \quad \text{for all } v \in V \end{aligned}$$

indp. set

max $\sum x_v$

s.t. $x_u + x_v \leq 1$

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Consider *relaxation* to $x_v \in \mathbb{R}$, $x_v \geq 0$.
$\rightsquigarrow$ LP that can by solved in polytime.

"trick" for LPs:
grow the feasibility region
in a way that never changes
z^* or $\{x^* : c^T x^* = z^*\}$

Vertex Cover as (Integer) Linear Program

Consider optimization version of VERTEXCOVER:
Given: Graph $G = (V, E)$
Goal: Vertex cover of G with minimal cardinality.

$\rightsquigarrow$ equivalent to the following integer linear program

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Consider *relaxation* to $x_v \in \mathbb{R}$, $x_v \geq 0$.
$\rightsquigarrow$ LP that can by solved in polytime.

any x^* will satisfy $0 \leq x^* \leq 1$

For an *optimal* solution $\vec{x}$ of the *relaxation*, we define

$$\begin{aligned} I_0 &= \{v \in V : x_v < \tfrac{1}{2}\} \\ V_0 &= \{v \in V : x_v = \tfrac{1}{2}\} \\ C_0 &= \{v \in V : x_v > \tfrac{1}{2}\} \end{aligned}$$

Kernel for VC

Theorem 6.1 (Kernel for Vertex Cover)
Let $(G = (V, E), k)$ an instance of p-VERTEX-COVER.

1. There exists a minimal vertex cover S with $C_0 \subseteq S$ and $S \cap I_0 = \emptyset$.
2. V_0 implies a problem kernel $(G[V_0], k - |C_0|)$ with $|V_0| \leq 2k$.

Here $G[V_0]$ is the induced subgraph of V_0 in G.

Proof:

ad (1) Let S^* be optimal VC for G

Claim: $S := (S^* \setminus I_0) \cup C_0$ is also optimal VC

$= (S^* \setminus S_I) \dot\cup \bar{S}_C \qquad S_I = S^* \cap I_0, \quad \bar{S}_C = C_0 \setminus S^*$

"S VC" only edges with endpoints in I_0 could remain uncovered

$e = vw \qquad v \in I_0 \quad \Rightarrow \quad x^*_v < \frac{1}{2} \quad \Rightarrow \quad x^*_w > \frac{1}{2} \quad \Rightarrow \quad w \in C_0$ ✓

(all other edges unchanged

Kernel for VC [2]

Proof (cont.):

"$|S| = |S^*|$" $S_I \subseteq S^*$, $\overline{S_c} \cap S^* = \emptyset$

$$\Rightarrow |S| = |S^*| - |S_I| + |\overline{S_c}|$$

suffices to show that $|\overline{S_c}| \leq |S_I|$

$\varepsilon := \min\{x_v - \frac{1}{2} : v \in C_0\} > 0$

$x' = x^*$ except for

- $u \in S_I$ $x'_u := x^*_u + \varepsilon$
- $u \in \overline{S_c}$ $x'_u := x^*_u - \varepsilon \geq \frac{1}{2}$ $(*)$

Claim: x' fulfills constraints of LP

$x'_v + x'_w \overset{!}{\geq} 1$ for $vw \in E$ could only be violated for vw with $v \in \overline{S_c}$

 $= C_0 \setminus S^*$

Kernel for VC [3]

(1) $w \in I_0 \setminus S^*$ $\quad v \notin S^*$ $\quad w \notin S^*$ $\quad ↯ \; S^*$ VC

Proof (cont.):

(2) $w \in S_I \rightsquigarrow x'_w = x^*_w + \varepsilon$ $\qquad x^*$ feasible

$$x'_w + x'_v = x^*_w + x^*_v \geq 1$$

(3) $w \notin I_0 \Rightarrow x'_w \geq \frac{1}{2}, \quad x'_v \geq \frac{1}{2}$ (*)

$$\Rightarrow x'_w + x'_v \geq 1$$

$\square$ claim

x optimal $\quad \sum_v x'_v \geq \sum_v x_v$

$$\sum_v x'_v = \sum_v x_v + \underbrace{\varepsilon}_{>0} \underbrace{\left(|S_I| - |\overline{S_C}|\right)}_{\geq 0} \Rightarrow |\overline{S_C}| \leq |S_I|$$

$\square$ (1)

ad (2) LP objective $\sum_v x_v^*$ $\leq$ ILP objective for any input

$$\Rightarrow \quad |S^*| \geq \sum_v x_v^*$$

apply reduction rule until only V_0 vertices left:

solve LP relaxation of VC $\rightsquigarrow x^* \rightsquigarrow I_0, V_0, C_0$

delete vertices in I_0, include in solution from C_0 and delete

$$\Rightarrow \quad x_v^* = \tfrac{1}{2} \quad \forall v \in V_0$$

$$\Rightarrow \quad |S^*| \geq \sum_v x_v^* = \tfrac{1}{2} |V_0|$$

If $|V_0| > 2k$ $\Rightarrow$ no VC for size $\leq k$ (output No instance)

otherwise $|V_0| \leq 2k$ $\rightsquigarrow$ output as kernel

$\square$.

6.6 Lower Bounds by ETH

so far: W[1]-hardness to show "probably $\notin$ FPT"

how about problems in FPT

can we get lower bounds for $f(k)$ in fpt runtime?

Example: VC allows fpt algorithm

with time $O(\underline{1.4^k} n^2)$

can we do better?

unlikely $f(k) = O(k^c)$ for c const

but could be subexponential

$2^{\sqrt{n}}$ $\quad 2^{\log^5(n)} = n^{\log^4 n}$

The Exponential Time Hypothesis

Definition 6.2 (Exponential-Time Hypothesis)

The *Exponential-Time Hypothesis (ETH)* asserts that there is a constant $\varepsilon > 0$ so that every algorithm for p-3SAT requires $\Omega(2^{\varepsilon k})$ time, where k is the number of variables. ◀

variables

$(1+\delta)^k \quad \delta > 0$

The Exponential Time Hypothesis

Definition 6.3 (Exponential-Time Hypothesis)

The *Exponential-Time Hypothesis (ETH)* asserts that there is a constant $\varepsilon > 0$ so that every algorithm for p-3SAT requires $\Omega(2^{\varepsilon k})$ time, where k is the number of variables. ◀

Equivalent formulations:

- There is a $\delta > 0$ so that every 3-SAT algorithm needs $\Omega((1+\delta)^k)$ time.
- There is no $O(2^{o(k)} n^c)$-time algorithm for 3-SAT.
- There is no subexponential-time algorithm for 3-SAT.

Lower Bounds Conditional on ETH

- **Idea:** Show that solving X in time $f(k, n)$
  implies a $O(2^{\varepsilon k} n^c)$ algorithm for 3SAT *for all* $\varepsilon > 0$.

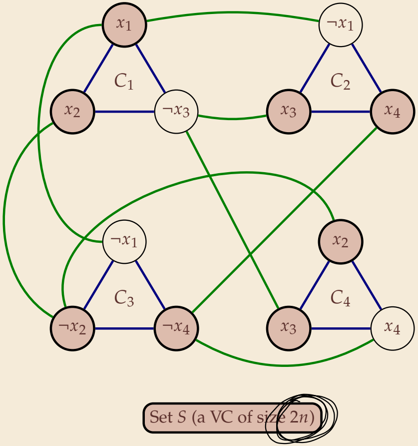
$\rightsquigarrow$ unless ETH false, no such $f(k, n)$-time algorithm for X exists.

- That needs a 3SAT-reduction that preserves parameter k tightly.

Recall: Classical Reduction from 3SAT to Vertex Cover

(ii) 3SAT $\leq_p$ VertexCover – Example

$\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_4) \wedge (x_2 \vee x_3 \vee x_4)$



- **Idea:** Vertices ***not in*** vertex cover S define a variable assignment.
  - Cannot be contradictory, otherwise "negation"-edge not covered.
  - Must take ≥ 2 vertices per clause into S (otherwise triangle not covered)
  - $\rightsquigarrow$ $|S| \geq 2n$ for every vertex cover.
- In the example:
  - Fat vertices form a vertex cover for G
  - corresponding assignment: $V = \{x_1 \mapsto 0, x_2 \mapsto 0, x_3 \mapsto 0, x_4 \mapsto 1\}$ ($0 \mathrel{\hat{=}} \mathit{false}, 1 \mathrel{\hat{=}} \mathit{true}$)
  - $\rightsquigarrow$ φ satisfiable

39

Sparsification Lemma

Lemma 6.4 (Sparsification Lemma)

For all $\varepsilon > 0$, there is a constant K so that we can compute for every formula φ in 3-CNF with n clauses over k variables an equivalent formula $\bigvee_{i=1}^{t} \psi_i$ where each ψ_i is in 3-CNF and over the same k variables and has $\leq K \cdot k$ clauses. Moreover, $t \leq 2^{\varepsilon k}$ and the computation takes $O(2^{\varepsilon k} n^c)$ time. ◀

Rough Idea:

Iteratively remove *sunflowers* by retaining only the *heart* or only the *petals*.

Proof in Impagliazzo, Paturi, Zane (2001): *Which Problems Have Strongly Exponential Complexity?*