9 Graph Algorithms

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CS566 (Wintersemester 2025–26)
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Learning Outcomes

Unit 9: ***Graph Algorithms***

1. Know basic terminology from graph theory, including types of graphs.
2. Know adjacency matrix and adjacency list representations and their performance characteristica.
3. Know graph-traversal based algorithm, including efficient implementations.
4. Be able to proof correctness of graph-traversal-based algorithms.
5. Know algorithms for maximum flows in networks.
6. Be able to model new algorithmic problems as graph problems.

Outline

9 Graph Algorithms

9.1 Introduction & Definitions

Clicker Question

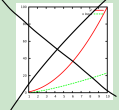
List all matching pairs:

(A) Graph

(B) Graf

(C) Grave

(1)

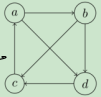


(2)

(3)

(4)

(5)



(6)

à

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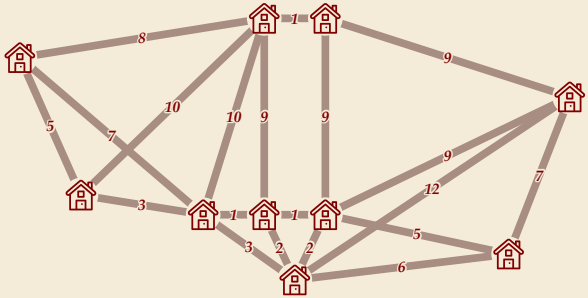
Graphs in real life

- a graph is an abstraction of *entities* with their (pairwise) *relationships*
- abundant examples in real life (often called network there)
  - social networks: e. g. persons and their friendships, . . . *Five/Six? degrees of separation*
  - physical networks: cities and highways, roads networks, power grids etc., the Internet, . . .
  - content networks: world wide web, ontologies, . . .
  - . . .



Many More examples, e. g., in Sedgewick & Wayne's videos:

`https://www.coursera.org/learn/algorithms-part2`

Flavors of Graphs

- Since graphs are used to model so many different entities and relations, they come in several variants

Property	**Yes**	**No**
edges are one-way	*directed* graph (*digraph*)	*undirected* graph
≤ 1 edge between u and v	*simple* graph	*multigraph* / with *parallel* edges
edges can lead from v to v	with *loops*	(loop-free)
edges have weights	*(edge-) weighted* graph	*unweighted* graph

🙃 any combination of the above can make sense . . .

- Synonyms:
  - **vertex** („Knoten“) = node = point = „Ecke”
  - **edge** („Kante“) = arc = line = relation = arrow = „Pfeil“
  - **graph** = network

Graph Theory

- default: unweighted, undirected, loop-free & simple graphs
- *Graph* $G = (V, E)$ with
  - V a finite of *vertices*
  - $E \subseteq [V]^2$ a set of *edges*, which are 2-subsets of V: $[V]^2 = \{e : e \subseteq V \wedge |e| = 2\}$

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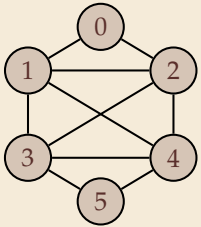
Example

$$\begin{aligned} V &= \{0,1,2,3,4,5\} \\ E &= \big\{\{0,1\},\{1,2\},\{1,4\},\{1,3\},\{0,2\}, \\ &\quad \{2,4\},\{2,3\},\{3,4\},\{3,5\},\{4,5\}\big\}. \end{aligned}$$

Graphical representation



like so . . .

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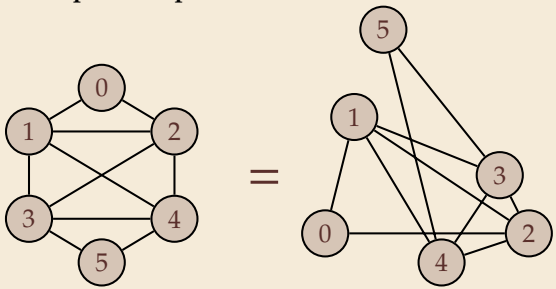

like so . . . . . or so

(same graph)

Digraphs

- default digraph: unweighted, loop-free & simple
- *Digraph (directed graph)* $G = (V, E)$ with
  - V a finite of *vertices*
  - $E \subseteq V^2 \setminus \{(v, v) : v \in V\}$ a set of *(directed) edges*,
    $V^2 = V \times V = \{(x, y) : x \in V \wedge y \in V\}$ 2-tuples / ordered pairs over V

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Example

$V = \{0, 1, 2, 3, 4, 5\}$

$E = \{(0, 2), (1, 0), (1, 4), (2, 1), (2, 4), (3, 1), (3, 2), (4, 3), (4, 5), (5, 3)\}$

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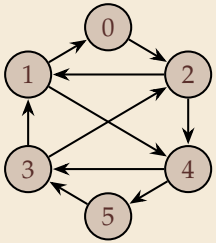

Graph Terminology

Undirected Graphs

- $V(G)$ set of vertices, $E(G)$ set of edges
- write uv (or vu) for edge $\{u, v\}$
- edges *incident* at vertex v: $E(v) = \{e : v \in e\}$
- u and v are *adjacent* iff $\{u, v\} \in E$,
- *neighborhood* $N(v) = \{w \in V : w \text{ adjacent to } v\}$
- *degree* $d(v) = |E(v)|$

Directed Graphs (where different)

- uv for (u, v)
- iff $(u, v) \in E \vee (v, u) \in E$
- in-/out-neighbors $N_{\text{in}}(v)$, $N_{\text{out}}(v)$
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- *walk* („*Weg*“) $w[0..n]$ of length n: sequence of vertices with $\forall i \in [0..n) : w[i]w[i+1] \in E$
- *path* („*Pfad*“) p is a (vertex-) simple walk: no duplicate vertices except possibly its endpoints
- *edge-simple* walk: no edge used twice
- *cycle* c is a closed path, i. e., $c[0] = c[n]$

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- G is *connected*
  iff for all $u \neq v \in V$ there is a path from u to v
- G is *acyclic* iff $\nexists$ cycle (of length $n \geq 1$) in G

- *strongly connected* for digraphs
  (*weakly connected* = connected ignoring directions)

Typical graph-processing problems

- **Path**: Is there a path between s and t?
  Shortest path: What is the shortest path (distance) between s and t?
- **Cycle**: Is there a cycle in the graph?
  Euler tour: Is there a cycle that uses each edge exactly once?
  Hamilton(ian) cycle: Is there a cycle that uses each vertex exactly once.
- **Connectivity**: Is there a way to connect all of the vertices?
  MST: What is the best way to connect all of the vertices?
  Biconnectivity: Is there a vertex whose removal disconnects the graph?
- **Planarity**: Can you draw the graph in the plane with no crossing edges?
- **Graph isomorphism**: Are two graphs the same up to renaming vertices?

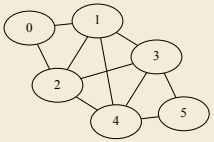
can vary a lot, despite superficial similarity of problems

Challenge: Which of these problems
can be computed in (near) linear time?
in reasonable polynomial time?
are intractable?

Tools to work with graphs

- Convenient GUI to edit & draw graphs: *yEd live*
  `yworks.com/yed-live`
- *graphviz* cmdline utility to draw graphs
  - Simple text format for graphs: `DOT`

```
graph G {
     0 -- 2;     2 -- 4;
     1 -- 0;     2 -- 3;
     1 -- 4;     3 -- 4;
     1 -- 3;     3 -- 5;
     2 -- 1;     4 -- 5;
}
```




```
dot -Tpdf graph.dot -Kfdp > graph.pdf
```

- graphs are typically not built into programming languages, but libraries exist
  - e. g. part of *Google Guava* for Java
  - they usually allow arbitrary objects as vertices
  - aimed at ease of use

9.2 Graph Representations

Graphs in Computer Memory

- We defined graphs in set-theoretic terms. . .
  but computers can't directly deal with sets efficiently

⇝ need to choose a *representation* for graphs.

- which is better depends on the required operations

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Key Operations:

- `isAdjacent(`u`,`v`)`
  Test whether $uv \in E$
- `adj(`v`)`
  Adjacency list of v (iterate through (out-) neighbors of v)
- most others can be computed based on these

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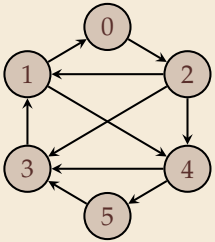
Conventions:

- (di)graph $G = (V, E)$ (omitted if clear from context)
- $n = |V|$, $m = |E|$
- in implementations assume $V = [0..n)$ (if needed, use symbol table to map complex objects to V)

Adjacency Matrix Representation

- adjacency matrix $A \in \{0,1\}^{n \times n}$ of G: matrix with $A[u,v] = [uv \in E]$
  - works for both directed and undirected graphs (undirected $\rightsquigarrow A = A^T$ symmetric)
  - can use a weight $w(uv)$ or multiplicity in $A[u,v]$ instead of $0/1$
  - can represent loops via $A[v,v]$

Example:



$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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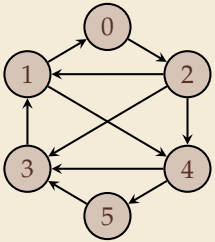

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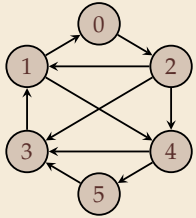
👍 `isAdjacent` in $O(1)$ time

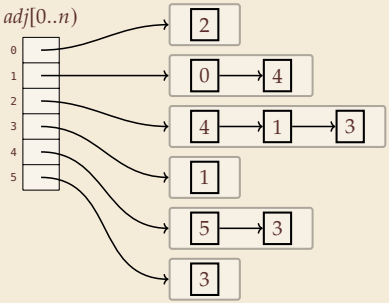
👎 $O(n^2)$ (bits of) space wasteful for sparse graphs

👎 `adj(`v`)` iteration takes $O(n)$ (independent of $d(v)$)

Adjacency List Representation

- Store a linked list of neighbors for each vertex v:
  - $adj[0..n)$ bag of neighbors (as linked list)
  - undirected edge $\{u, v\}$ $\rightsquigarrow$ v in $adj[u]$ and u in $adj[v]$
  - weighted edge uv $\rightsquigarrow$ store pair $(v, w(uv))$ in $adj[u]$
  - multiple edges and loops can be represented





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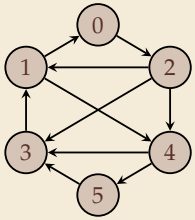
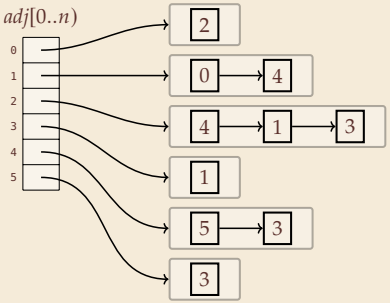




👎 `isAdjacent(`u`,`v`)` takes $\Theta(d(u))$ time (worst case)

👍 `adj(`v`)` iteration $O(1)$ per neighbor

👍 $\Theta(n + m)$ (words of) space for any graph ($\ll \Theta(n^2)$ bits for moderate m)

$\rightsquigarrow$ de-facto standard for graph algorithms

Graph Types and Representations

- Note that adj matrix and lists for undirected graphs effectively are representation of directed graph with directed edges both ways
  - conceptually still important to distinguish!
- multigraphs, loops, edge weights all naturally supported in adj lists
  - good if we allow and use them
  - but requires explicit checks to enforce simple / loopfree / bidirectional!
- we focus on **static graphs**
  dynamically changing graphs much harder to handle

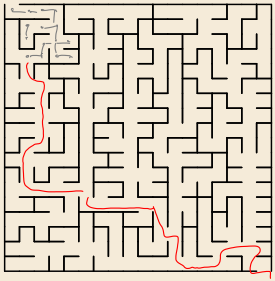
9.3 Graph Traversal

Generic Graph Traversal

- Plethora of graph algorithms can be expressed as a systematic exploration of a graph

visiting all nodes & edges

  - depth-first search, breadth-first search
  - connected components
  - detecting cycles
  - topological sorting
  - Hierholzer's algorithm for Euler walks
  - strong components
  - testing bipartiteness
  - Dijkstra's algorithm
  - Prim's algorithm
  - Lex-BFS for perfect elimination orders of chordal graphs
  - ...



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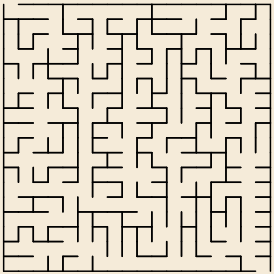

⇝ Formulate generic traversal algorithm

- first in abstract terms to argue about correctness
- then again for concrete instance with efficient data structures

Tricolor Graph Traversal

- maintain vertices in 3 (dynamic) sets
  - **Gray: unseen vertices**
    The traversal has not reached these vertices so far.
  - **Red: active vertices** (a.k.a. frontier („Rand“) of traversal)
    Vertices that have been reached and some unexplored edges remain;
    initially some selected start vertices $\underline{S}$.
  - **Green: done vertices** (a.k.a. visited vertices)
    Visited vertices with all their edges explored.
- maintain edge status: either **unused** or **used**

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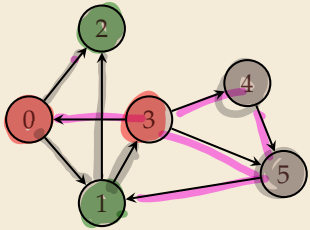
Vertices *“want”* to be ***done***.
To do so, they turn neighbors ***active***.

- maintain edge status: either **unused** or **used**

Tricolor Graph Search:

- Repeat until no more changes:
  - (T1) Pick arbitrary ***active*** vertex v
  - (T2) If no more ***unused*** edges vw, mark v ***done*** (***D step***)
  - (T3) Else pick arbitrary ***unused*** edge vw, mark vw ***used***
  - (T4) If w ***unseen***, mark w ***active*** (***A step***)



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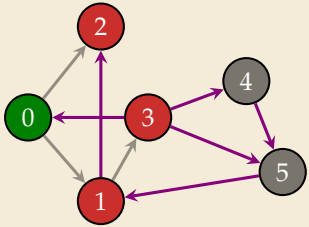

Generic Reachability – Correctness

Theorem 9.1 (Generic Reachability)

When Tricolor Graph Search terminates, the following holds:
$v \in V$ is reachable from S *iff* $v \in$ ***done***.

$\exists$ path $P[0..\ell]$ $\quad P[0] \in S \;\wedge\; P[\ell] = v$

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- **Invariant:** For every ***done*** or ***active*** vertex v, there exists a path from S to v.

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$\rightsquigarrow$ in final state:

- $v \in$ ***done*** $\underset{\text{(inv.}}{\rightsquigarrow}$ $\exists$ path from S to v $\rightsquigarrow$ reachable from S

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$\rightsquigarrow$ in final state:

- $\Leftarrow$: $v \in$ ***done*** $\rightsquigarrow$ $\exists$ path from S to v $\rightsquigarrow$ reachable from S
- $\Rightarrow$: Let v be reachable from S, i. e., $\exists$ path $p[0..\ell]$ from $p[0] \in S$ to $p[\ell] = v$ of length $\ell \leq n$.

  Assume towards a contradiction $v \notin$ ***done***. $\rightsquigarrow$ $v \in$ ***unseen*** (***active*** $= \emptyset$ upon termination).

Generic Reachability – Correctness

Theorem 9.1 (Generic Reachability)

When Tricolor Graph Search terminates, the following holds:
$v \in V$ is reachable from S *iff* $v \in$ ***done***.

Proof:

- We prove the following invariant (next slide)
- **Invariant:** For every ***done*** or ***active*** vertex v, there exists a path from S to v.

$\rightsquigarrow$ in final state:

- $v \in$ ***done*** $\rightsquigarrow$ $\exists$ path from S to v $\rightsquigarrow$ reachable from S
- Let v be reachable from S, i. e., $\exists$ path $p[0..\ell]$ from $p[0] \in S$ to $p[\ell] = v$ of length $\ell \leq n$.

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  Let v be such a vertex with *minimal distance* ℓ from S.

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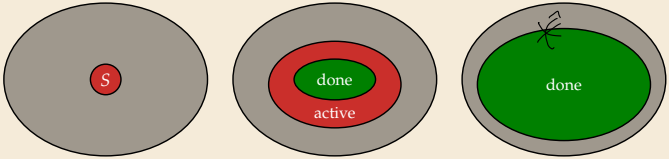
  Assume towards a contradiction $v \notin$ ***done***. $\rightsquigarrow$ $v \in$ ***unseen*** (***active*** $= \emptyset$ upon termination).
  Let v be such a vertex with *minimal distance* ℓ from S. $\rightsquigarrow$ $p[\ell - 1] = w \in$ ***done***.

  But then $wv \in E$, $v \in$ ***unseen***, and yet w was marked ***done*** ↯ (T2).

Generic Reachability – Invariant

- **Invariant:** For every ***done*** or ***active*** vertex v, there exists a path from S to v.



initial state | during traversal | final state

Generic Reachability – Invariant

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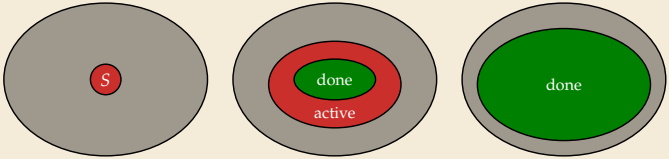


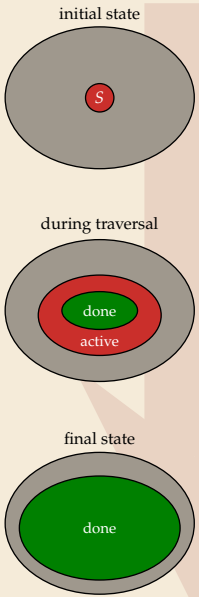
Proof:

By induction over the number of executed steps of tricolor traversal.

- **IB:** (1) no ***done*** vertices yet. (2) ✓ trivial path (w/o edges)
- **IH:** Invariant fulfilled for first k steps.
- **IS:** Step $k+1$ is either *A step* (T3)–(T4) or *D step* (T2)
  - *A step*: new ***active*** vertex w reached via vw with $v \in$ ***active***
    $\exists$ path $P[0..\ell]$ with $P[0] \in S$ and $P[\ell] = v$ by IH $\rightsquigarrow$ path Pw from S to w.
  - *D step*: new ***done*** vertex previously was ***active***, so $\exists$ path from S to v by IH.

Generic Tricolor Graph Traversal – Code

```
1  procedure genericGraphTraversal(G, S):
2      // (di)graph G = (V, E) and start vertices S ⊆ V
3      C[0..n) := unseen // Color array, all cells initialized to unseen
4      for s ∈ S do C[s] := active end for
5      unusedEdges := E
6      while ∃v : C[v] == active
7          v := nextActiveVertex()  // Freedom 1: Which frontier vertex?
8          if ∄vw ∈ unusedEdges // no more edges from v ⇝ done with v
9              C[v] := done
10         else
11             w := nextUnusedEdge(v)  // Freedom 2: Which of its edges?
12             if C[w] == unseen
13                 C[w] := active
14             unusedEdges.remove(vw)
15         end if
16     end while
```



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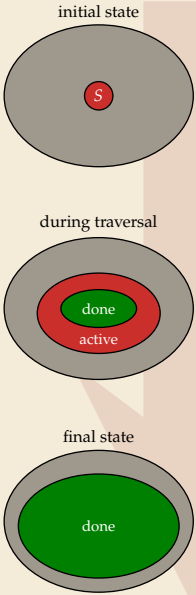

- Any implementation of nextActiveVertex() and nextUnusedEdge(v) suffices for correctness
- Choice depends on (and defines!) specific traversal-based graph algorithms

9.4 Breadth-First Search

Data Structures for Frontier

- We need efficient support for
  - test $\exists v : C[v] =$ ***active***, nextActiveVertex()
  - test $\exists vw \in$ *unusedEdges*, nextUnusedEdge(v)
  - *unusedEdges*.remove(vw)

Data Structures for Frontier

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- Typical solution maintains **bag** "*frontier*" of *pairs* (v, i) where $v \in V$ and i is an **iterator** in `adj`$[v]$
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  - *unusedEdges* represented implicitly: edge used iff previously returned by i
    - $\rightsquigarrow$ *unusedEdges*.remove(vw) doesn't need to do anything
  - Implement $\exists v : C[v] =$ ***active*** via *frontier*.isEmpty()
  - Implement $\exists vw \in$ *unusedEdges* via i.hasNext() assuming $(v, i) \in$ *frontier*
  - Implement nextUnusedEdge(v) via i.next() assuming $(v, i) \in$ *frontier*

$\rightsquigarrow$ all operations apart from nextActiveVertex() in $O(1)$ time

$\rightsquigarrow$ *frontier* requires $O(n)$ extra space

Breadth-First Search

- Maintain *frontier* in a **queue** (FIFO: first in, first out)

Breadth-First Search

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- **Unweighted shortest path distance:**

$$\widetilde{dist_G(s,t)} = \min\left\{\ell \;:\; \exists \text{ path } P[0..\ell] : P[0] = s \wedge P[\ell] = t\right\} \cup \{\infty\}$$

$$dist_G(S,t) = \min_{s \in S} dist_G(s,t)$$

P[0] → P[1] → P[?]

Breadth-First Search

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Theorem 9.2 (BFS Correctness)
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- To preserve paths, we collect extra information during traversal:
  - *parent*$[v]$ stores predecessor on path from S via which v was first reached (made ***active***)
  - *distFromS*$[v]$ stores the length of this path

Breadth-First Search – Code

```
1  procedure bfs(G, S):
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3      C[0..n) := unseen // New array initialized to all unseen
4      frontier := new Queue;
5      parent[0..n) := NOT_VISITED;  distFromS[0..n) := ∞
6      for s ∈ S
7          parent[s] := NONE;  distFromS[s] := 0
8          C[s] := active;  frontier.enqueue((s, G.adj[s].iterator()))
9      end for
10     while ¬frontier.isEmpty()
11         (v, i) := frontier.peek()
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13             C[v] := done;  frontier.dequeue()
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15             w := i.next() // Advance i in adj[v]
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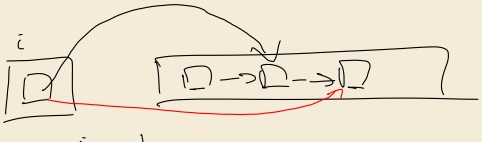
head of adj(s)



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- *parent* stores a *shortest-path tree/forest*
- can retrieve shortest path to v from some vertex $s \in S$ (backwards) by following *parent*$[v]$ iteratively

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- *parent* stores a *shortest-path tree/forest*
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- running time $\Theta(n + m)$
- extra space $\Theta(n)$

Breadth-First Search – Correctness

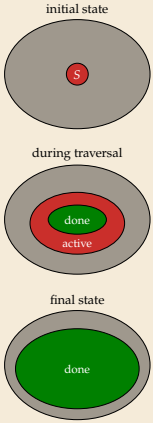
- BFS correctness directly follows from the following invariant.

- **BFS Invariant:**
  1. All ***done*** or ***active*** vertices were reached via a **shortest path** from S (fewest edges)
  2. Vertices enter and leave *frontier* in order of increasing distance from S ($\delta(v)$)

Proof:
By induction over number of steps. Abbreviate $\delta(v) := dist_G(S, v)$

- **IB:** (1) only S ***active***, reached via path of length 0.
  (2) only S in *frontier*, minimal by δ. ✓

- **IH:** Invariant fulfilled for first k steps.



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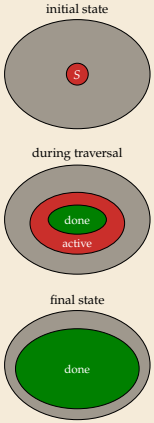

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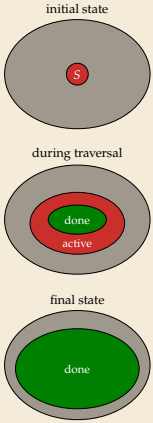
  - *D step:* v moved from ***active*** to ***done*** $\rightsquigarrow$ (1) unchanged. (⊥)
    By IH, v *entered frontier* at correct time, queue keeps order $\rightsquigarrow$ (2) ✓

Breadth-First Search – Correctness



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    By IH, v *entered frontier* at correct time, queue keeps order $\rightsquigarrow$ (2) ✓
  - *A step, (i):* $vw \in$ *unusedEdges* leads to $w \in$ ***active*** $\cup$ ***done***
    no changes, (1) and (2) ✓

Breadth-First Search – Correctness [2]

Proof (cont.):

- *A step, (ii):* $vw \in$ *unusedEdges* leads to $w \in$ ***unseen***
  - $\rightsquigarrow$ w is now marked ***active*** and enqueued in *frontier*.

Breadth-First Search – Correctness [2]

Proof (cont.):

- *A step, (ii):* $vw \in$ *unusedEdges* leads to $w \in$ *unseen*
  $\rightsquigarrow$ w is now marked ***active*** and enqueued in *frontier*.

  By IH, we reached v by shortest path of $\delta(v)$ edges,
  and any node u with $\delta(u) < \delta(v)$ is ***done*** (since dequeued before v).

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- It remains to show that any x with $\delta(x) = \delta(v)$ ($< \delta(w)$) is ***active*** or ***done*** by now;
  then adding w to *frontier* respects the order by δ.

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- It remains to show that any x with $\delta(x) = \delta(v)$ is ***active*** or ***done*** by now;
  then adding w to *frontier* respects the order by δ.

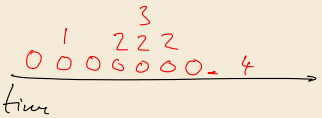
  Any shortest path from S to $x \notin S$ must go via some u with $\delta(u) < \delta(v)$, so u is ***done***.
  - $\rightsquigarrow$ all edges from u, including ux, have been *used*, thus x is ***active*** or ***done***.
  - $\rightsquigarrow$ (2) ✓

9.5 Depth-First Search

Depth-First Search

- Maintain *frontier* in a **stack** (LIFO: last in, first out)
  - only consider $S = \{s\}$
  - usual mode of operation: call `dfs`(v) for all *unseen* v, for $v = 0, \ldots, n-1$

Depth-First Search



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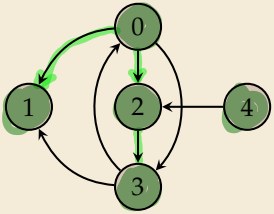
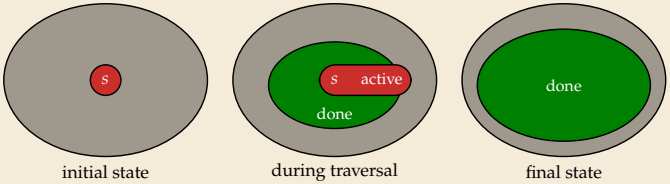
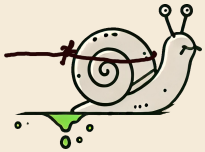
- **DFS Invariant:**
  1. All ***done*** or ***active*** vertices are reached via a path from s
  2. The ***active*** vertices form a **single path** from s







Depth-First Search – Code

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2      C[0..n) := unseen
3      for v := 0, ..., n − 1
4          if C[v] == unseen
5              dfs(G, v)
6
7  procedure dfs(G, s):
8      frontier := new Stack;
9      C[s] := active; frontier.push((s, G.adj[s].iterator()))
10     while ¬frontier.isEmpty()
11         (v, i) := frontier.top()
12         if ¬i.hasNext() // v has no unused edge
13             C[v] := done; frontier.pop(); postorderVisit(v)
14         else
15             w := i.next(); visitEdge(vw)
16             if C[w] == unseen
17                 preorderVisit(w)
18                 C[w] := active; frontier.push((w, G.adj[w].iterator()))
19             end if
20         end if
21     end while
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- define *hooks* to implement further operations
  - preorder: visit v when made ***active*** (start of $T(v)$)
  - postorder: visit v when marked ***done*** (end of $T(v)$)
  - visitEdge: do something for every edge
- if needed, can store DFS forest via *parent* array

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- if needed, can store DFS forest via *parent* array
- running time $\Theta(n + m)$
- extra space $\Theta(n)$

Simple DFS Application: Connected Components

- In an undirected graph, find all *connected components*.
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```
1  procedure connectedComponents(G):
2      // undirected graph G = (V, E) with V = [0..n)
3      C[0..n) := unseen
4      CC[0..n) := NONE
5      id := 0
6      for v := 0, ..., n − 1
7          if C[v] == unseen
8              dfs(G, v)
9              id := id + 1
10     return CC
11
12 procedure preorderVisit(v):
13     CC[v] := id
```

```
1  // same as before
2  procedure dfs(G, s):
3      frontier := new Stack;
4      C[s] := active; frontier.push((s, G.adj[s].iterator()))
5      while ¬frontier.isEmpty()
6          (v, i) := frontier.top()
7          if ¬i.hasNext() // v has no unused edge
8              C[v] := done; frontier.pop()
9              postorderVisit(v)
10         else
11             w := i.next(); visitEdge(vw)
12             if C[w] == unseen
13                 preorderVisit(w)
14                 C[w] := active
15                 frontier.push((w, G.adj[w].iterator()))
16             end if
17         end if
18     end while
```

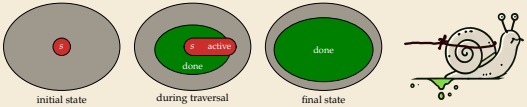
Dijkstra's Algorithm & Prim's Algorithm

- On edge-weighted graphs, we can use tricolor traversal with a *priority queue* as *frontier*
- Dijkstra's Algorithm for shortest paths from s in digraphs with weakly positive edge weights
  - priority of vertex v = length of shortest path known so far from s to v
- Prim's Algorithm for finding a minimum spanning tree
  - priority of vertex v = weight of cheapest edge connecting v to current tree

$\rightsquigarrow$ Detailed discussion in Unit 11

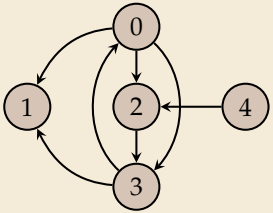
9.6 Advanced Uses of DFS I

Properties of DFS



- **Recall DFS Invariant (2):**
  The *active* vertices form a **single path** from s

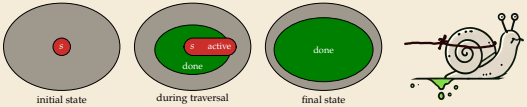
input graph G



DFS forest

stack over time

Properties of DFS



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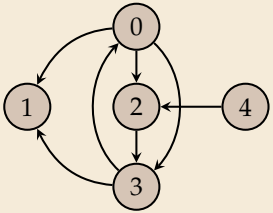

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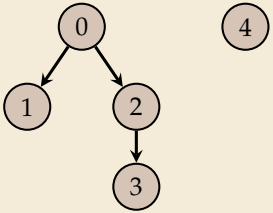

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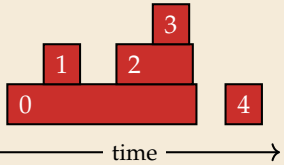

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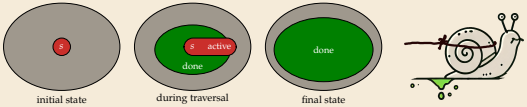

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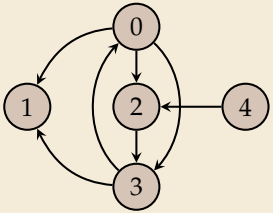

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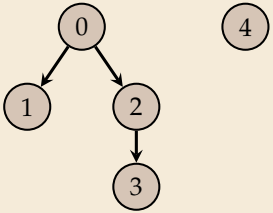

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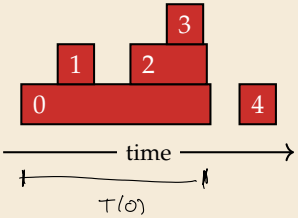

$\rightsquigarrow$ Each vertex v spends *time interval* $T(v)$ as ***active*** vertex

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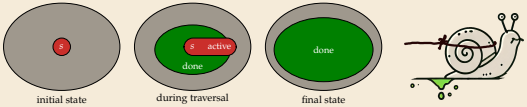

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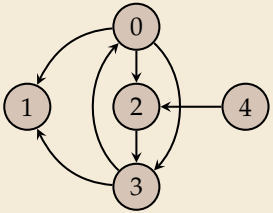

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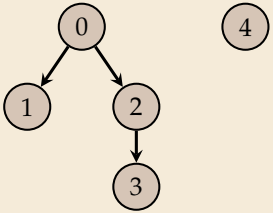

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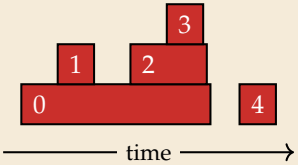

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either $T(v) \cap T(w) = \emptyset$ or $T(v) \subseteq T(w)$ or $T(v) \supseteq T(w)$

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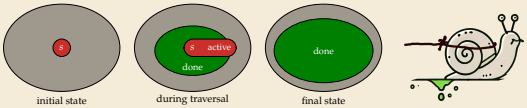

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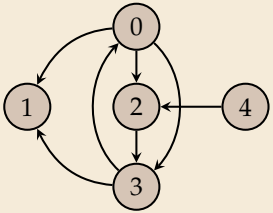

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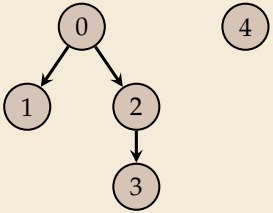

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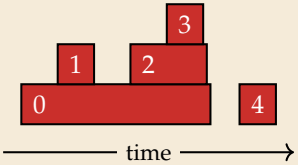

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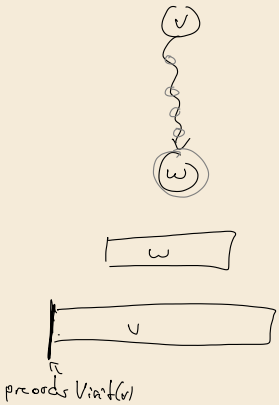
2. **Parenthesis Theorem:** $T(v) \supseteq T(w)$ **iff** v is ancestor of w in DFS tree

'$\Rightarrow$' during $T(v)$, all discovered vertices become descendants of v

'$\Leftarrow$' $T(v)$ covers v's entire subtree, which contains w's subtree

Properties of DFS – Unseen-Path Theorem

- **Unseen-Path Theorem:** In a DFS forest of a (di)graph G, w is a descendant of v **iff** at the time of `preorderVisit`(v), there is a path from v to w using only ***unseen*** vertices.



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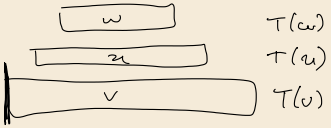

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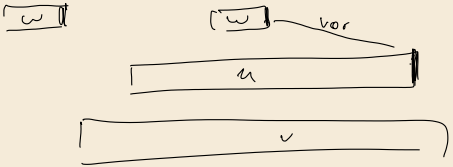

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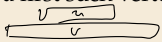

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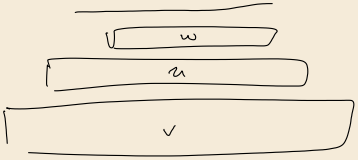
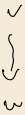
Upon processing u, we will discover edge uw, so whether or not w is already ***done*** at this point, w will be marked ***done*** before u. Hence $\max T(w) \leq \max T(u)$. (3)





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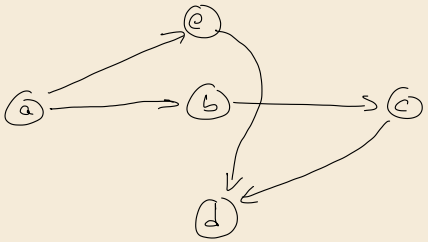
By (1), (2), (3) and laminarity, $T(w) \subset T(u) \subset T(v)$ and w is a descendant of v ↯.

Topological Sorting & Cycle Detection

- **Application:** Given a set of tasks with precedence constraints of the form "a must be done before b", can we find a legal ordering for all tasks?

  ⇝ Model as directed graph!

  - tasks are the vertices V
  - add an edge (a, b) when a must be done before b



a e b c d

a b c e d

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- **Definition:** $R[0..n)$ is a *topological (order) ranking* of **di**graph $G = (V, E)$ if
  $\forall (u, v) \in E : R[u] < R[v]$ think: $R[v]$ = time slot for task v
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- **Topological Sorting**
  - **Given:** simple digraph $G = (V, E)$
  - **Goal:** Compute topological ranking of vertices $R[0..n)$
    or output a directed cycle in G.
- Amazingly, can do all with one pass of DFS!

DFS Postorder & Topological Sort

- **DFS Postorder**: The DFS postorder numbers is a numbering $P[0..n)$ of V such that $P[v] = r$ iff exactly r vertices reached state ***done*** before v in a DFS.

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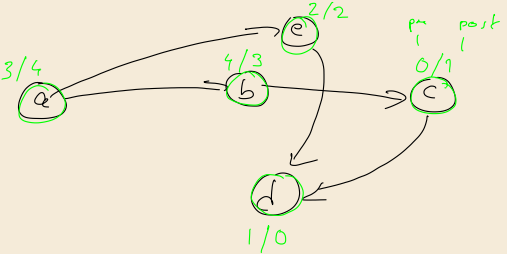
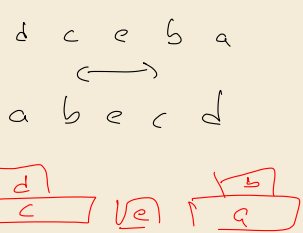
- **Lemma rev postorder:** directed acyclic graph
  Let G be a simple, connected DAG and $R[0..n)$ a *reverse DFS postorder* of G, i. e., $R[v] = n - 1 - P[v]$ for a DFS postorder $P[0..n)$. Then R is a topological ranking of G.

- **Invariant:** If $v \in$ ***done*** and $(v, w) \in E$ then $w \in$ ***done*** and $R[v] < R[w]$.
  - initially true (***done*** $= \emptyset$)
  - upon `postorderVisit`(v), all outgoing edges vw lead to $w \in$ ***done*** (Parenthesis Theorem)

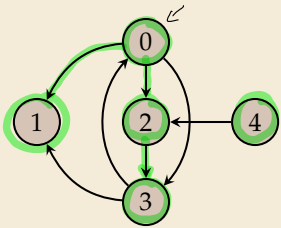
v	a	b	c	d	e
R	0	1	3	4	2





DFS Edge Types

input digraph *G*



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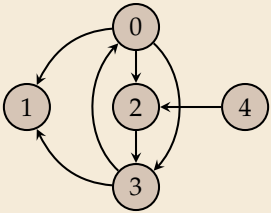

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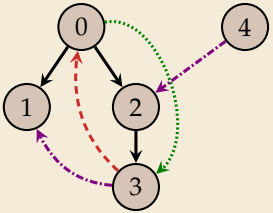
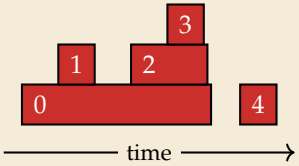
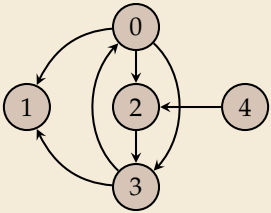


stack over time

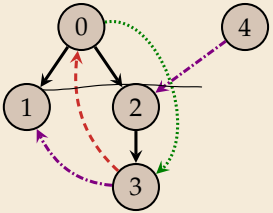


DFS Edge Types

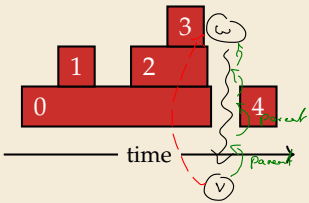
input digraph G



DFS forest



stack over time



- During DFS traversal, an edge vw has one of these 4 types: **example:**
  1. **tree edge:** ⟶ $w \in$ *unseen* $\rightsquigarrow$ vw part of DFS forest. $(0,1), (0,2), (2,3)$
  2. **back edges:** - - -▸ $w \in$ *active*; $\rightsquigarrow$ w points to ancestor of v. $(3,0)$
  3. **forward edges*:** ·····▸ $w \in$ *done* $\wedge$ w is descendant of v in DFS tree. $(0,3)$
  4. **cross edges*:** -·-·▸ $w \in$ *done* $\wedge$ w is not descendant of v. $(3,1), (4,2)$

  *only possible in *directed* graphs

Cycle Detection

If G contains a directed cycle, DFS will find a directed cycle:

- any back edge implies a cycle:
  - DFS visits an edge (v, w) where $w \in$ ***active***, w is already on the stack
  - $\rightsquigarrow$ DFS tree contains path $w \rightsquigarrow v$ and we have edge $v \to w$.

Cycle Detection

If G contains a directed cycle, DFS will find a directed cycle:

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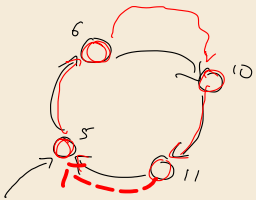
  - DFS visits an edge (v, w) where $w \in$ ***active***, w is already on the stack
  - $\rightsquigarrow$ DFS tree contains path $w \rightsquigarrow v$ and we have edge $v \to w$.

- conversely any cycle $C[0..k]$ once reached must have some back edge or cross edge (tree and forward edges go from smaller to larger preorder index)
  - cannot be a cross edge since cycle is strongly connected
    all cycle vertices must be descendants of first reached cycle vertex
  - $\rightsquigarrow$ cycle contributes a back edge

unseen path flow.



DFS Postorder Implementation

```
1  procedure dfsPostorder(G):
2      C[0..n) := unseen
3      P[0..n) := NONE;  r := 0
4      parent[0..n) := NONE
5      cycle := NONE
6      for v := 0, . . . , n − 1
7          if C[v] == unseen
8              dfs(G, v)
9      return (P, cycle)
10
11 procedure postorderVisit(v):
12     P[v] := r;  r := r + 1
13
14 procedure visitEdge(vw):
15     if C[w] == active
16         if cycle ≠ NONE return
17         while v ≠ w
18             cycle.append(v)
19             v := parent[v]
20         cycle.append(v)
```

```
1  // dfs is as in CC but with parent
2  procedure dfs(G, s):
3      frontier := new Stack;
4      parent[s] := NONE;
5      C[s] := active;  frontier.push((s, G.adj[s].iterator()))
6      while ¬frontier.isEmpty()
7          (v, i) := frontier.top()
8          if ¬i.hasNext() // v has no unused edge
9              C[v] := done;  frontier.pop()
10             postorderVisit(v)
11         else
12             w := i.next() // Advance i in adj[v]
13             visitEdge(vw)
14             if C[w] == unseen
15                 parent[w] := v;
16                 preorderVisit(w)
17                 C[w] := active;  frontier.push((w, G.adj[w].iterator()))
18             end if
19         end if
20     end while
```

Topological Sorting & Cycle Detection – Summary

- Putting everything together we obtain topological sorting
  - can produce either the *ranking* or the *sequence of vertices* in topological order, whatever is more convenient

```
1 procedure topologicalRanking(P):
2     (P[0..n), cycle) := dfsPostorder(G)
3     if cycle ≠ NULL
4         return NOT_A_DAG
5     R[0..n) := NONE
6     for v := 0, ..., n − 1
7         R[v] = n − 1 − P[v]
8     return R
```

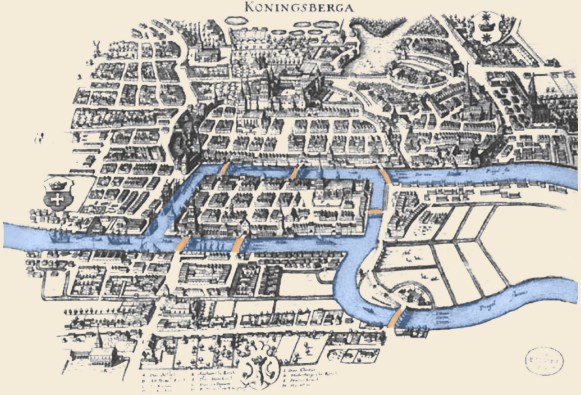
```
1 procedure topologicalSort(P):
2     (P[0..n), cycle) := dfsPostorder(G)
3     if c ≠ NULL
4         return NOT_A_DAG
5     S[0..n) := NONE
6     for v := 0, ..., n − 1
7         S[n − 1 − P[v]] := v
8     return S
```

- $\Theta(n + m)$ time
- $\Theta(n)$ extra space

9.7 Advanced Uses of DFS II

Euler Cycles

Euler Walk: Walk using every edge in $G = (V, E)$ exactly once.



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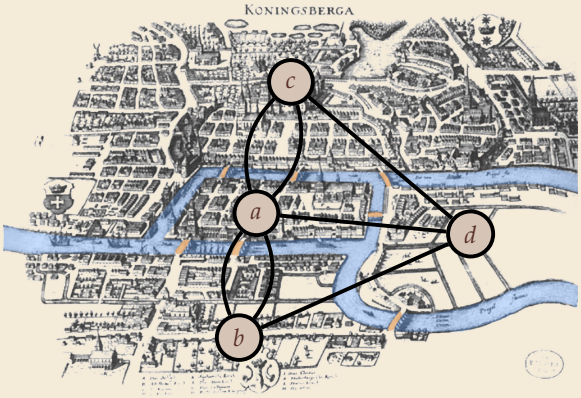

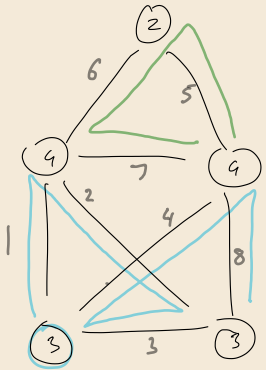
2
6
5
4
7
4
2
4
1
8
3
3
3

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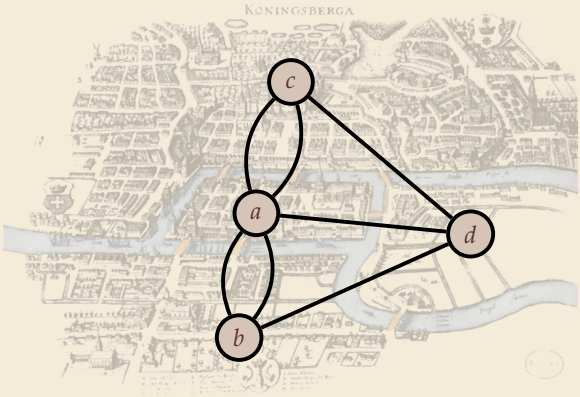


Euler's Theorem:
Euler walk exists iff G connected and 0 or 2 vertices have odd degree.

'$\Rightarrow$' trivial (need to enter and exit intermediate vertices equally often)

'$\Leftarrow$' Following algorithm *constructs* Euler walk under this assumption

Euler Cycles – Hierholzer's Algorithm

- use an ***edge-centric DFS***
  - We mark *edges* (not vertices)
  - ⇝ stack = **edge-simple walk**
  - We remember iterator i globally per v to resume traversal

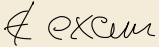
```
1  procedure eulerWalk(G):
2      // Assume G = (V, E) is connected (multi)graph
3      V_odd := {v ∈ V : d(v) odd}
4      if |V_odd| ∉ {0, 2} return NOT_EULERIAN
5      if V_odd = {x, y} then s := x else s := 0
6      euler[0..m) := NONE;  j := m − 1
7      visited[0..n, 0..n) := false // mark edges as visited
8      for v := 0, ..., n − 1
9          // globally remember next unexplored edge
10         nextEdge[v] := G.adj[w].iterator())
11     edgeDFS(s)
12     return euler
```

```
1  procedure edgeDFS(s):
2      frontier := new Stack;
3      frontier.push(s)
4      while ¬frontier.isEmpty()
5          v := frontier.top();  i := nextEdge[v]
6          if ¬i.hasNext() // v has no unused edge
7              frontier.pop()
8              if ¬frontier.isEmpty()
9                  // assign edge leading here largest free index
10                 euler[j] := (frontier.top(), v);  j := j − 1
11             end if
12         else
13             w := i.next()
14             if ¬visited[v, w]
15                 visited[v, w] := true
16                 visited[w, v] := true
17                 frontier.push(w)
18             end if
19         end if
20     end while
```

Euler Cycles – Further Results

- Edge-centric DFS takes $O(n + m)$ time
- also $\Theta(m)$ space in the worst case for the stack
  - can be improved to $\Theta(n)$ with variant of variant of the algorithm

Ismaili Alaoui, Plump, Wild: *Space-Efficient Hierholzer: Eulerian Cycles in $O(m)$ Time and $O(n)$ Space*, SOSA **2026**



Euler Cycles – Further Results

- Edge-centric DFS takes $O(n + m)$ time
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  Ismaili Alaoui, Plump, Wild: *Space-Efficient Hierholzer: Eulerian Cycles in $O(m)$ Time and $O(n)$ Space*, SOSA **2026**

- Same approach can be made to work for *directed* graphs

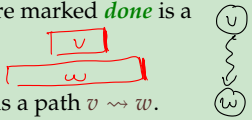
- Euler paths can be used to assemble genomes from fragments
  → *Algorithms of Bioinformatics*

Clicker Question

Mark all correct statements about a `dfsTraversal` (Slide 24) of a **DAG** G:

A Listing vertices in the order they are marked ***done*** is a topological sorting of G.

B Listing vertices in the reverse order they are marked ***done*** is a topological sorting of G.



C If v is marked ***done*** before vertex w, there is a path $v \rightsquigarrow w$.

D If v is marked ***done*** before vertex w, there is a path $w \rightsquigarrow v$.

E If v is marked ***done*** before vertex w, there cannot be a path $v \rightsquigarrow w$.

F If v is marked ***done*** before vertex w, there cannot be a path $w \rightsquigarrow v$.

→ *sli.do/cs566*

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B Listing vertices in the reverse order they are marked *done* is a topological sorting of G. ✓

C ~~If v is marked *done* before vertex w, there is a path $v \rightsquigarrow w$.~~

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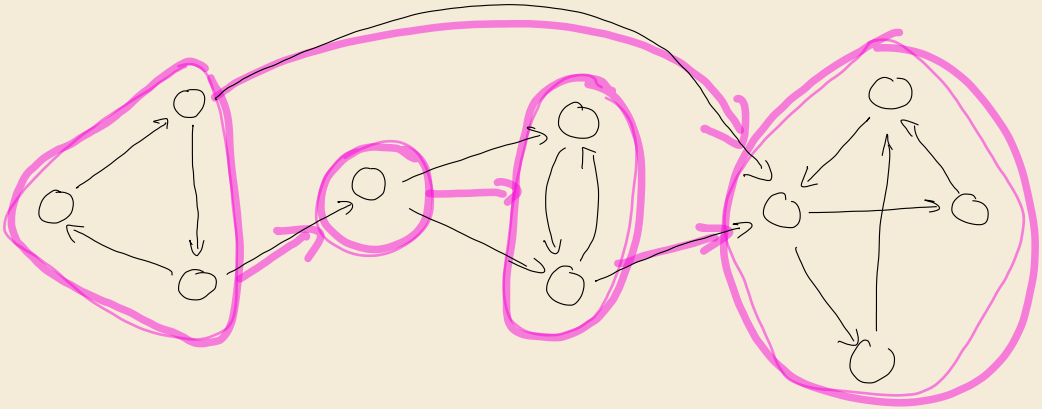
→ *sli.do/cs566*

Strong Components

- **Given:** digraph $G = (V, E)$
- **Goal:** component ids $SCC[0..n)$, s.t. $SCC[v] = SCC[u]$ iff $\exists$ directed path from v to u



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  strongly connected component
- *Component DAG* G^{SCC}*:* contract SCCs intro single vertices
  $V(G^{\text{SCC}}) = \{C_1, \ldots, C_k\}$ with $C_1 \dot\cup \cdots \dot\cup C_k = V$;
  name by smallest vertex s.t. $i \le j$ iff $\min C_i \le \min C_j$
  - can't have cycles (↯ maximality of SCC)

  $\rightsquigarrow$ component DAG has a topological order $R^{\text{SCC}}[1..k]$

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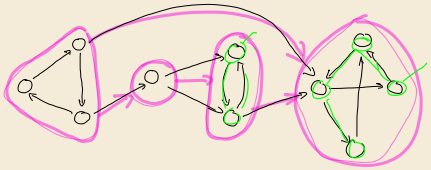
If we call `dfs` on any v in the **last** SCC C, it will discover all vertices in C, and only those!
(any edges between components lead *into* C by topological order)

And we can iterate this backwards through any topological order to get all SCCs!



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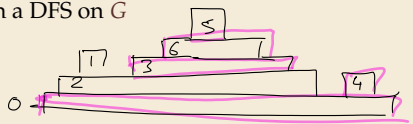
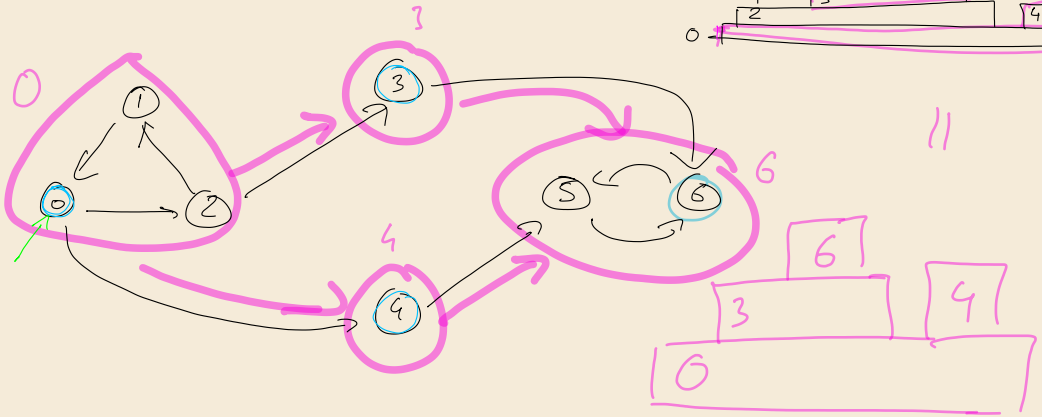
Can we efficiently find the topological order of G^{SCC}?
Without knowing the components to start with??

Amazingly, yes.

Component Graph DFS

- Suppose we run `dfsTraversal` on G.
  - ⇝ We can extend time intervals to SCCs: $T(C_i) := \bigcup_{v \in C_i} T(v)$
  - ⇝ $T(C_i) = T(v_i)$ for $v_i \in C_i$ the first vertex to be explored in a DFS on G (by Unseen Path & Parenthesis Thms)





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$\rightsquigarrow$ DFS on G produces same $T(C_i)$ (up to time scaling) as DFS on G^{SCC}!

$\rightsquigarrow$ reverse DFS postorder on G gives same relative order to $v_1, \ldots, v_k$ as reverse DFS postorder on G^{SCC} gives as relative order to $C_1, \ldots, C_k$

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$\rightsquigarrow$ reverse DFS postorder on G gives same relative order to $v_1, \ldots, v_k$ as reverse DFS postorder on G^{SCC} gives as relative order to $C_1, \ldots, C_k$

We need **reverse** topological order on G^{SCC}, e. g., ***reversed*** *reverse DFS postorder*

- If we had the actual reverse DFS postorder on G^{SCC}, could just reverse again!
- But we only have reverse DFS postorder $S[0..n)$ on G!
- Reversing here would change v_i, i. e., which vertices of an SCC we see first

Kosaraju-Sharir's Algorithm

- **Recall:** Want `reverse`$\big($`topologicalRanking`$(G^{\mathrm{SCC}})\big)$

Kosaraju-Sharir's Algorithm

- **Recall:** Want $\texttt{reverse}(\texttt{topologicalRanking}(G^{\text{SCC}}))$
- **Transpose/Reverse Graph of $G = (V, E)$:** $G^T = (V, E^T)$ where $E^T = \{wv : vw \in E\}$

  Note: A adj matrix of $G \rightsquigarrow A^T$ adj matrix of G^T
- For any DAG, we obtain a reverse topological order from reversing all edges: $\texttt{topologicalSort}(G^T)$

  If we reverse *iteration order* in dfsTraversal, we get $\texttt{reverse}(\texttt{topologicalSort}(G)) = \texttt{topologicalSort}(G^T)$

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  - strong components not affected by edge reversals
- Want: $\texttt{reverse}\big(\texttt{topologicalRanking}(G^{\text{SCC}})\big)$ (any ranking works, need not be reverse DFS postorder)

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$\rightsquigarrow$ Get it from: $\texttt{topologicalRanking}\big((G^{\text{SCC}})^T\big) = \texttt{topologicalRanking}\big((G^T)^{\text{SCC}}\big)$

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$\rightsquigarrow$ Get it from: $\texttt{topologicalRanking}\big((G^{\text{SCC}})^T\big) = \texttt{topologicalRanking}\big((G^T)^{\text{SCC}}\big)$

$\rightsquigarrow$ Get that as induced ranking on $v_1, \ldots, v_k$ from reverse $\texttt{dfsPostorder}(G^T)$

Kosaraju-Sharir's Algorithm – Code

```
1  procedure strongComponents(G):
2      // directed graph G = (V, E) with V = [0..n)
3      G^T = (V, {wv : vw ∈ E})
4      P[0..n) := dfsPostorder(G^T) // postorder numbers
5      for v ∈ V do S[P[v]] := v end for // postorder sequence
6      // Rest like connectedComponents (with permuted vertices)
7      C[0..n) := unseen
8      SCC[0..n) := NONE
9      id := 0
10     for j := n − 1, . . . , 0 // reverse postorder seq
11         v := S[j]
12         if C[v] == unseen
13             dfs(G, v)
14             id := id + 1
15     return SCC
16
17 procedure preorderVisit(v):
18     SCC[v] := id
```

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- correctness follows from our discussion

Kosaraju-Sharir's Algorithm – Code

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8      SCC[0..n) := NONE
9      id := 0
10     for j := n − 1, . . . , 0 // reverse postorder seq
11         v := S[j]
12         if C[v] == unseen
13             dfs(G, v)
14             id := id + 1
15     return SCC
16
17 procedure preorderVisit(v):
18     SCC[v] := id
```

- correctness follows from our discussion
- ordering of SCCs follows reverse topological sort of G^{SCC}
  - some implementations reverse G for 2nd DFS, not 1st
  - $\rightsquigarrow$ output in (forward) topological order
  - but derivation more natural this way?

Kosaraju-Sharir's Algorithm – Code

```
1  procedure strongComponents(G):
2      // directed graph G = (V, E) with V = [0..n)
3      G^T = (V, {wv : vw ∈ E})
4      P[0..n) := dfsPostorder(G^T) // postorder numbers
5      for v ∈ V do S[P[v]] := v end for // postorder sequence
6      // Rest like connectedComponents (with permuted vertices)
7      C[0..n) := unseen
8      SCC[0..n) := NONE
9      id := 0
10     for j := n − 1, . . . , 0 // reverse postorder seq
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  - some implementations reverse G for 2nd DFS, not 1st
  - $\rightsquigarrow$ output in (forward) topological order
  - but derivation more natural this way?
- as all our traversals: $\Theta(n + m)$ time, $\Theta(n)$ extra space

9.8 Network flows

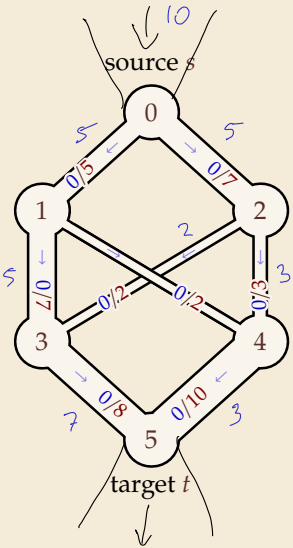
Clicker Question

Prior knowledge from linear optimization; check all apply.

- **A** I've seen LPs in lectures before.
- **B** I could model an application problem as (I)LP.
- **C** I know algorithms for solving LPs.
- **D** I know what weak and strong duality in LPs are.
- **E** I could dualize an LP given to me.
- **F** I know about the complexity of LPs and ILPs.
- **G** LPs for me only mean music on vinyl.

→ *sli.do/cs566*

Networks and Flows – Informal



Informally, imagine a network of water pipes.

- Water can flow through the pipes up to a flow capacity limit (up to $c(e)$ liters per second, say).
- There's infinite water pressing into the source s and infinite drain capacity at the sink / target t
- At all other junctions, inflow = outflow (no leakage)

$\rightsquigarrow$ How much water can flow through the network?

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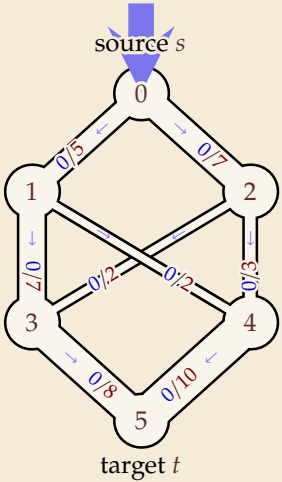

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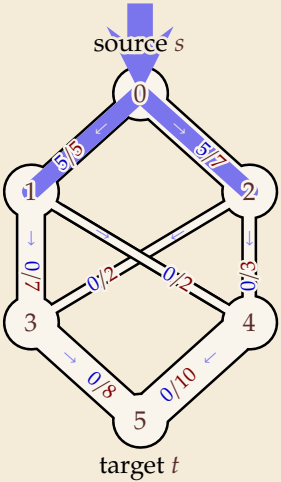

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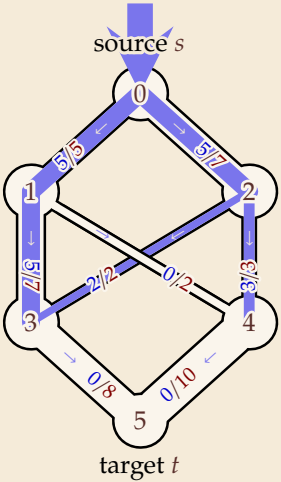

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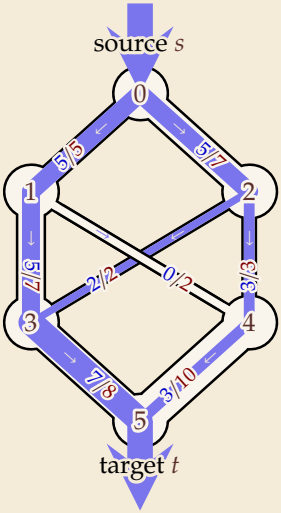

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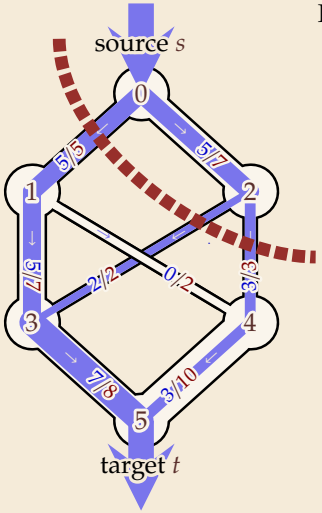

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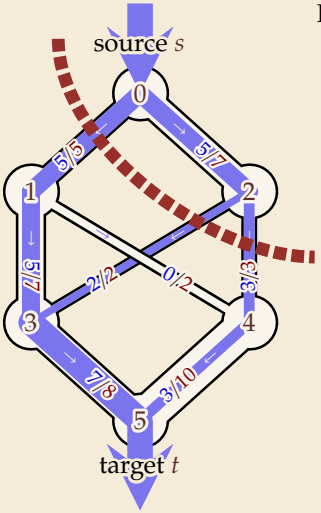
In this example:

- not more than $5+2+3=10$ units of flow out of $\{0,2\}$ possible

⇝ not more than 10 units out of s possible

Networks and Flows – Informal



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In this example:

- not more than $5+2+3=10$ units of flow out of $\{0,2\}$ possible
- ⇝ not more than 10 units out of s possible
- ⇝ shown flow is maximal

Remainder of this unit: general version of above (+ efficient algorithms)

Networks and Flows – Definitions

- *s-t-(flow) network*:
  - **simple**, **directed**, **connected** graph $G = (V, E)$, for notational convenience only: no antiparallel edges ($vw \in E \rightsquigarrow wv \notin E$)
  - *edge capacities* $c : E \to \mathbb{R}_{\neq 0}^{>}$
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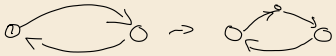

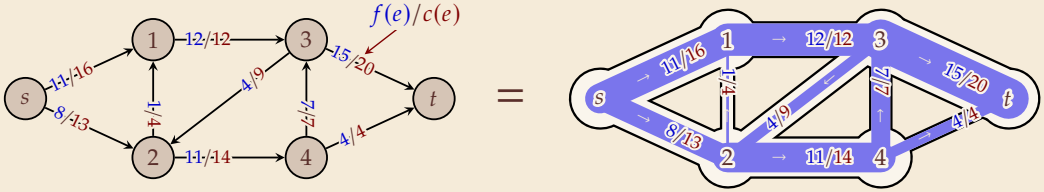

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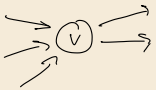

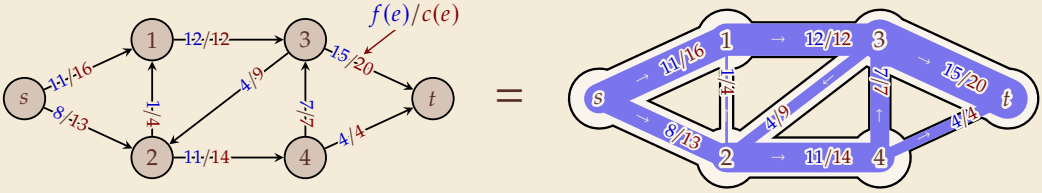

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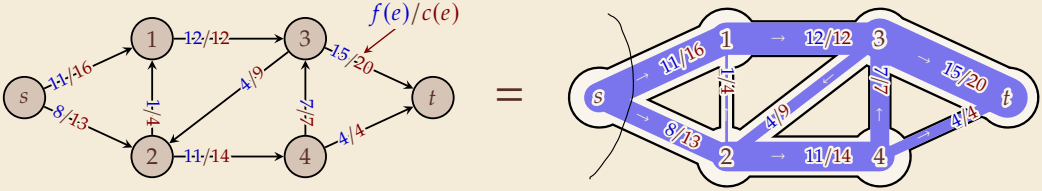
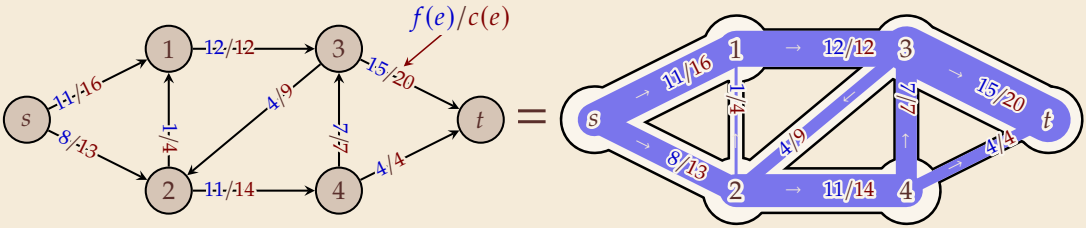
- *value* $|f|$ of flow f: $|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$



Max-Flow Problem



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  - **Given:** s-t-flow network
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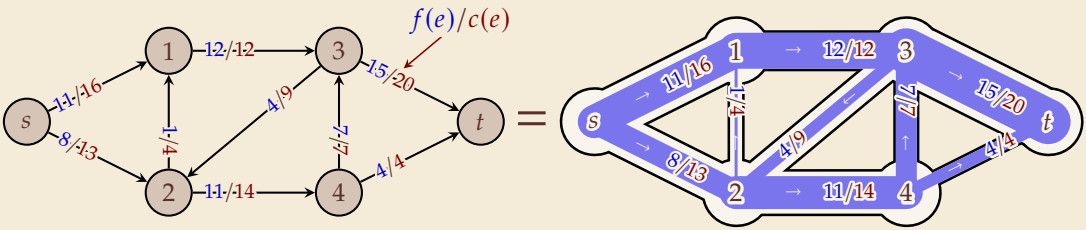

- **Maximum-Flow Problem:**
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- $\mathbb{N}$ vs $\mathbb{R}$
  - We focus on integral capacities here (as we will see) can restrict ourselves to integral flows
  - but: ideally want algorithms that work with arbitrary real numbers, too

Multiple Sources & Sinks, Antiparallel Edges

- Some of the restrictions can be generalized easily.
- We forbid **loops** and **antiparallel** edges.
  - The presented algorithms actually work fine with both!
  - but proofs are cleaner to write without them
  - also: can always remove loops and (anti)parallel edges by adding a new vertex in the middle of the edge

  $\rightsquigarrow$ same maximum $|f|$

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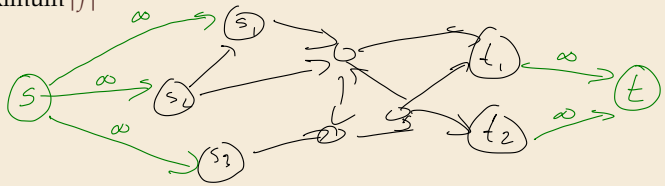
  ⇝ same maximum $|f|$

- We only allow a **single source** and a **single sink**
  - can add a **"supersource"** and **"supersink"** with capacity-∞ edges to all sources resp. sinks

  ⇝ same maximum $|f|$



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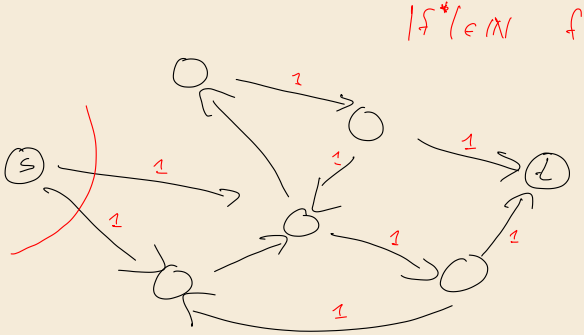

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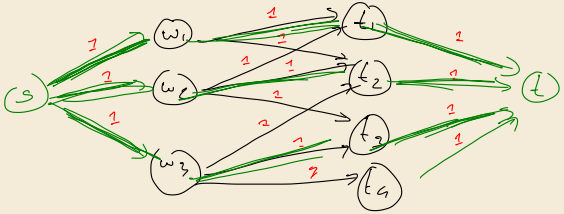
   - **Given:** workers $W = \{w_1, \ldots, w_k\}$ tasks $T = \{t_1, \ldots, t_\ell\}$, *qualified-for* relation $Q \subseteq W \times T$
   - **Goal:** Assignment $a : W \to T \cup \{\perp\}$ of workers to tasks such that
     - workers are qualified: $\forall w \in W : a(w) \neq \perp \implies (w, a(w)) \in Q$
     - $|a(W)|$, the number of tasks assigned, is maximized



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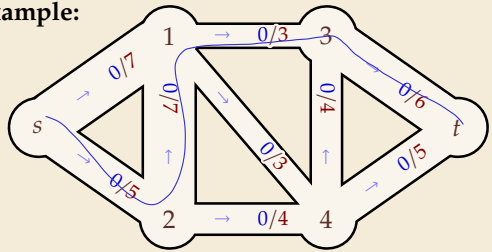
- Both problems can be solved by (in both cases, 1. and 3. are very efficient)
  1. constructing a specific flow network from their input data
  2. computing a maximum flow in that network
  3. "reading off" a solution for the original problem from the max flow

9.9 The Ford-Fulkerson Method

Push Push Push!

- **Simple Idea:** Iteratively find a path from s to t that we can push more flow over.

Example:



1. Push 3 units of flow over $s \to 2 \to 1 \to 3 \to t$

Problem: Cannot undo mistakes.
Here: shouldn't have put so much flow on $(1, 2)$ . . .

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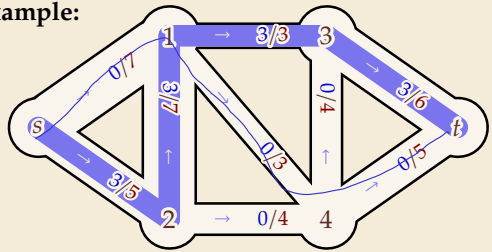

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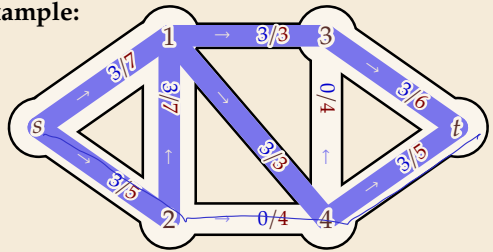

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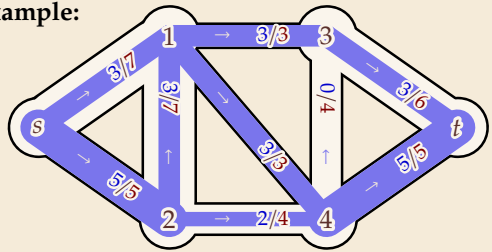

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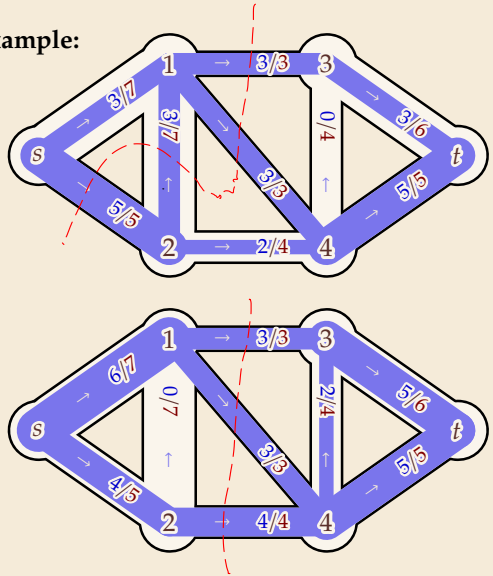

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⚠ **But:** resulting flow is **not** optimal!

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Residual Networks

- Goal: Allow undoing flow (without backtracking)

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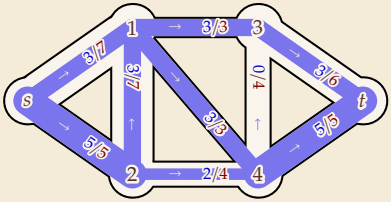
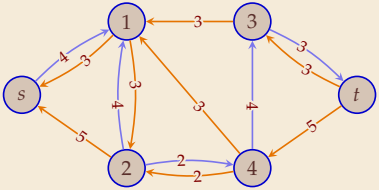
    $E_f = \{vw : c_f(vw) > 0\}$





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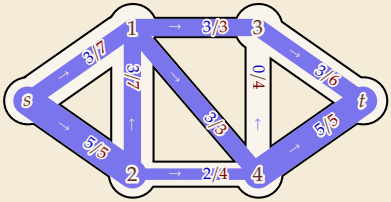

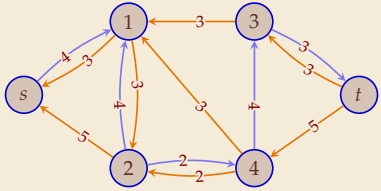

- *residual flow* f': feasible flow in G_f $\qquad (f + f')(vw) = f(vw) + f'(vw) - f'(wv)$
  $\rightsquigarrow$ for any f and residual flow f' in G_f, flow $f + f'$ is a feasible flow in G

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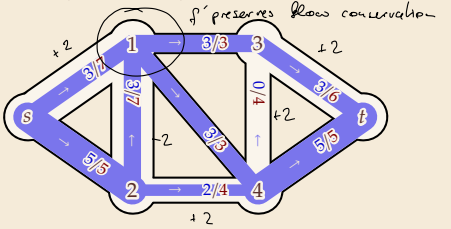

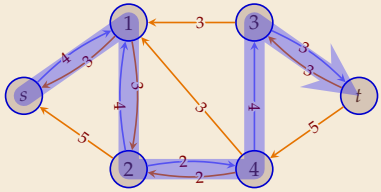

- *residual flow* f': feasible flow in G_f $\qquad (f + f')(vw) = f(vw) + f'(vw) - f'(wv)$
  $\rightsquigarrow$ for any f and residual flow f' in G_f, flow $f + f'$ is a feasible flow in G
- *augmenting path* p: s-t-path G_f $\qquad$ particularly simple f'!

Residual Networks

- Goal: Allow undoing flow (without backtracking)
- *Residual network* G_f: given network $G = (V, E)$ and feasible flow f
  - $G_f = (V, E_f)$ with capacities $c_f(vw) = \begin{cases} c(vw) - f(vw) & vw \in E \quad \text{// add flow} \\ f(wv) & wv \in E \quad \text{// revert flow} \\ 0 & \text{else} \end{cases}$
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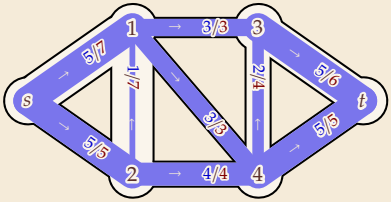

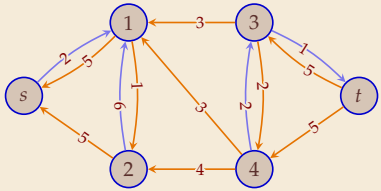

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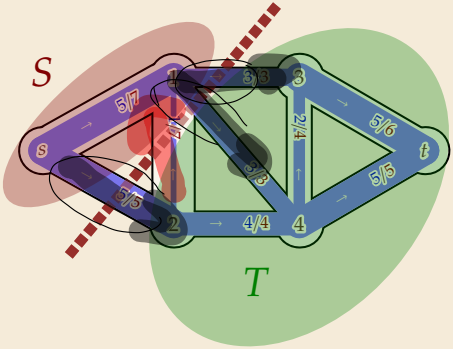
Cuts

- **Goal:** Certificate for maximum flows
- *s-t-cut* (S, T): partition $S \dot\cup T = V$, $s \in S$, $t \in T$
  - *net flow* across cut:
    $$f(S,T) = \sum_{v \in S} \sum_{w \in T} \big(f(vw) - f(wv)\big)$$
  - *capacity* of cut:
    $$c(S,T) = \sum_{v \in S} \sum_{w \in T} f(vw)$$



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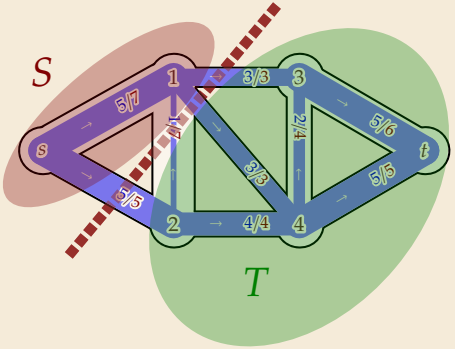

- $f(S,T) = 5 + 3 + 3 - 1 = 10$
- $c(S,T) = 5 + 3 + 3 = 11$

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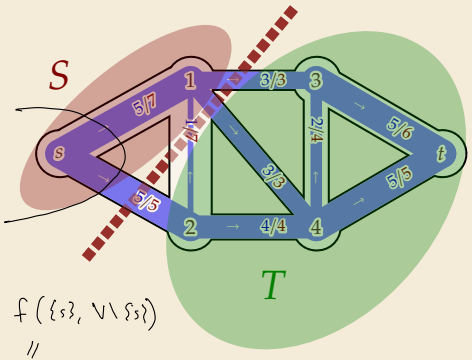


- **Lemma:** For any cut (S,T), we have $f(S,T) = |f|$.
  (flow conservation!)

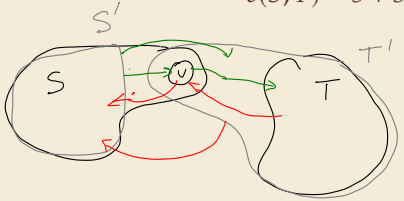
- $f(S,T) = 5+3+3-1 = 10$
- $c(S,T) = 5+3+3 = 11$

Proof: Induction on $|S|$

IA. $|S| = 1$ $\quad s \in S$ ✓

IS $|S| = k+1$

flow conservation at v



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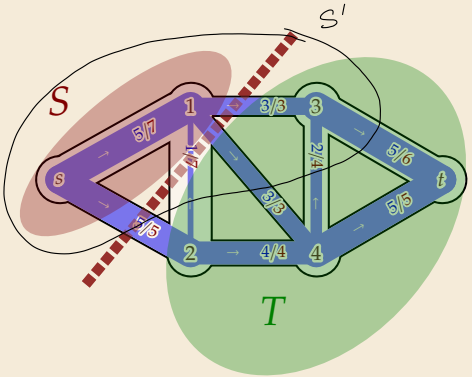


- **Lemma:** For any cut (S,T), we have $f(S,T) = |f|$.
  (flow conservation!)
- **Corollary:** $|f| \le c(S,T)$ for any s-t-cut (S,T)

- $f(S,T) = 5 + 3 + 3 - 1 = 10$
- $c(S,T) = 5 + 3 + 3 = 11$

$f(S',T') = 10$

$c(S',T') = 14$

The Max-Flow Min-Cut Theorem

- **Max-Flow Min-Cut Theorem:**
  Let f be a feasible flow in s-t-network $G = (V, E)$. Then the following conditions are equivalent:

  1. $|f| = c(S,T)$ for some cut (S,T) of G.
  2. f is a maximum flow in G
  3. The residual network G_f has no augmenting path.

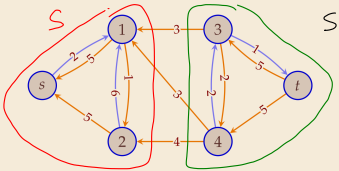
(path from s to t in G_f)

Proof:

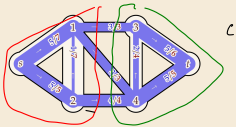
- (1) $\Rightarrow$ (2) $|f| \le c(S,T)$ for any s-t-cut (S,T) $\rightsquigarrow$ any f' has $|f'| \le c(S,T) = |f|$
- (2) $\Rightarrow$ (3) contraposition $\neg(3) \Rightarrow \neg(2)$

  $\Rightarrow$ G_f has path from s to t w/ min. capacity $\varepsilon > 0$ $\rightsquigarrow$ f' in G_f $|f'| = \varepsilon$

  $\Rightarrow$ $f + f'$ is valid flow in G with $|f + f'| = |f| + \varepsilon > |f|$ $\Rightarrow \neg(2)$
- (3) $\Rightarrow$ (1)



$S = \{v : v \text{ reachable from } s\}$



$c(S,T) = 10$

$= |f|$

Generic Ford-Fulkerson Method

```
1  procedure genericFordFulkerson(G = (V, E), s, t, c):
2      // G is a flow network with source s ∈ V, sink t ∈ V and capacities c : E → ℝ≥0
3      for vw ∈ E do f(vw) := 0 end for
4      while ∃ path p from s to t in G_f // Freedom: Which augmenting path?
5          Δ := min{c_f(e) : e ∈ p} // bottleneck capacity
6          for e ∈ p
7              if e ∈ E // forward edge
8                  f(e) := f(e) + Δ
9              else // backward edge
10                 f(e) := f(e) − Δ
11     return f
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- Returned flow is a maximum flow f^* (Max-Flow Min-Cut Theorem)

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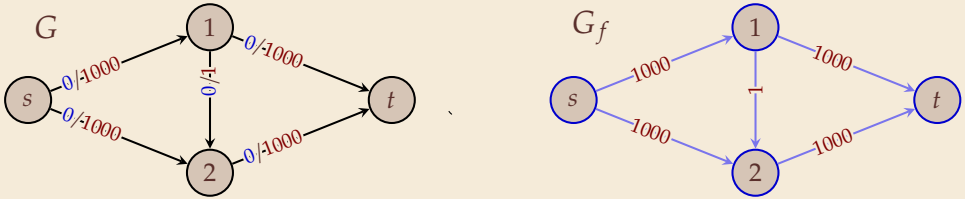
- Returned flow is a maximum flow f^* (Max-Flow Min-Cut Theorem)
- If $c : E \to \mathbb{N}_0$, also $f : E \to \mathbb{N}_0$: For all $v, w \in V$ holds:
  - initially $f(vw) = 0 \in \mathbb{N}_0$
  - $c_f(vw)$ is difference of $c(vw) \in \mathbb{N}_0$ and $f(vw) \in \mathbb{N}_0$
  - Δ equal to some $c_f(v'w') \in \mathbb{N}_{\geq 1}$ (E_f contains only non-zero capacity edges!)
  - $\rightsquigarrow$ new flow $f(vw) \pm \Delta \in \mathbb{N}_0$
- $\rightsquigarrow$ For integral capacities, always terminate after $\leq |f^*|$ iterations

Bad Example

- Unfortunately, we might also take $|f^*|$ iterations!



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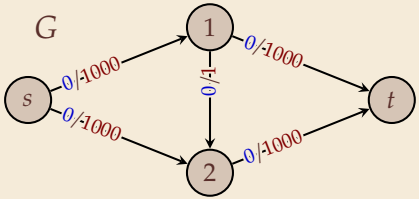

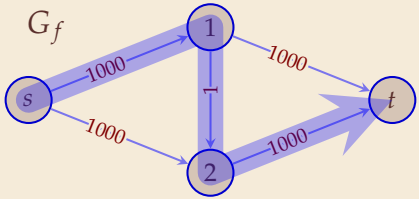

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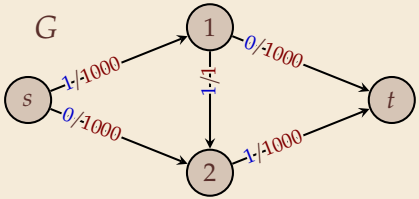

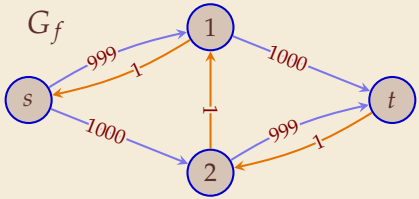

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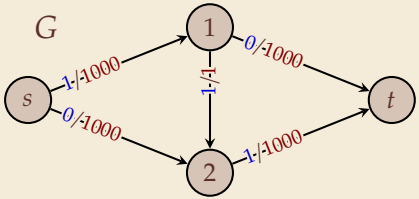

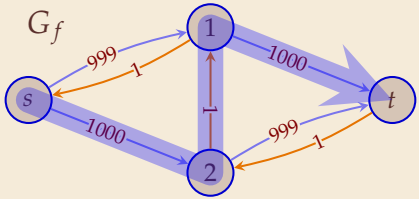

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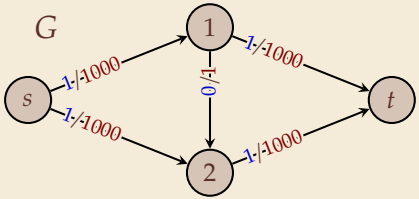

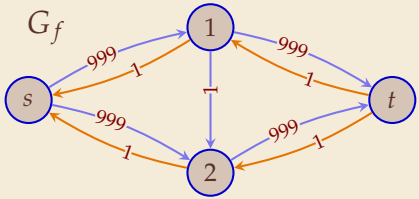

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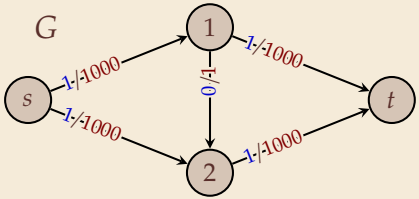

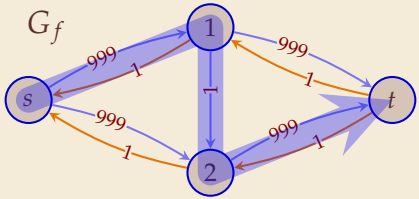

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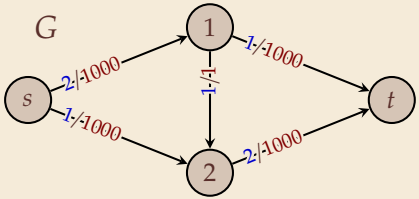

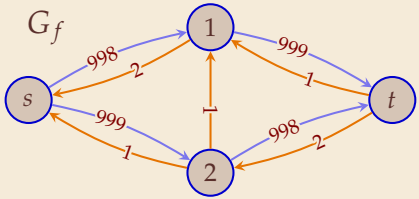

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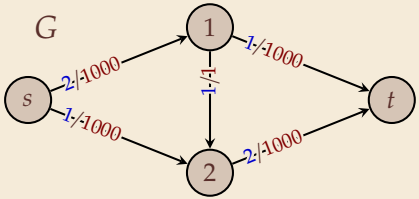

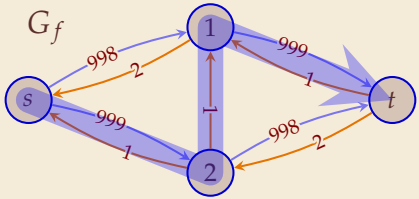

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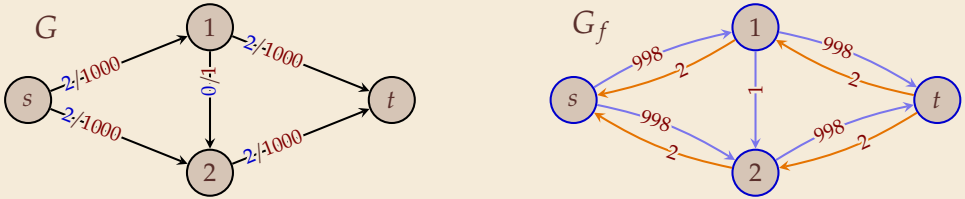

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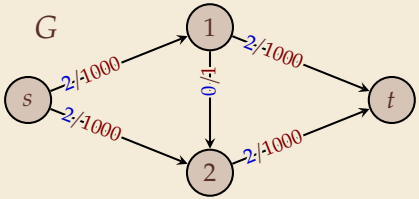

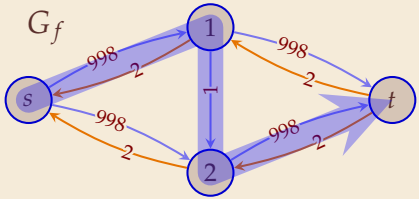

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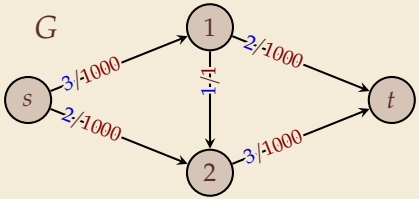

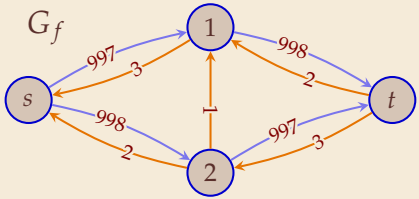

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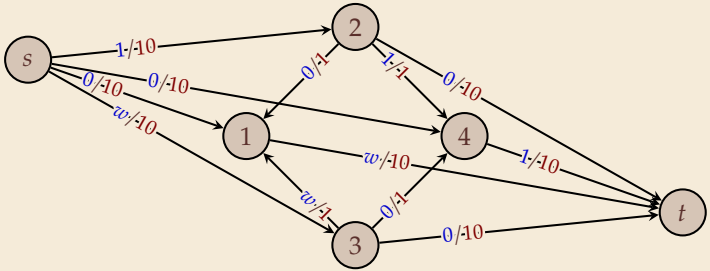
A Very Bad Example

- for irrational flows, might not even terminate
- example network with irrational initial flow
- $w = \varphi - 1 = (\sqrt{5} - 1)/2 \approx 0.618 \quad \rightsquigarrow \quad 1 - w = w^2 \approx 0.382$



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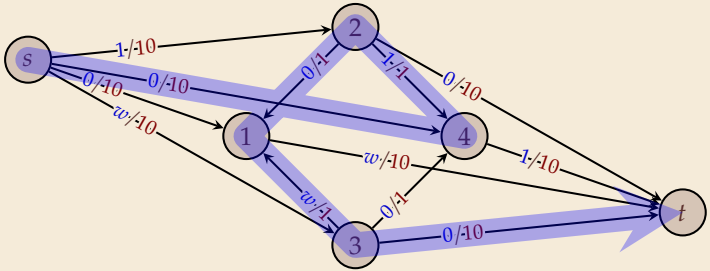

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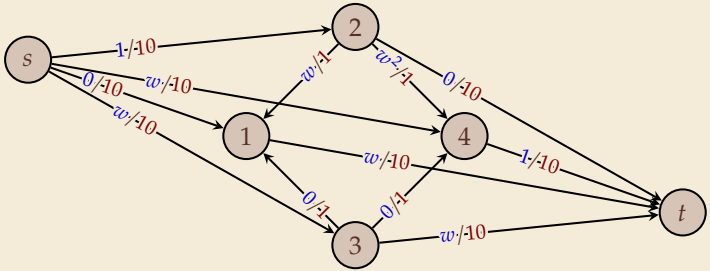

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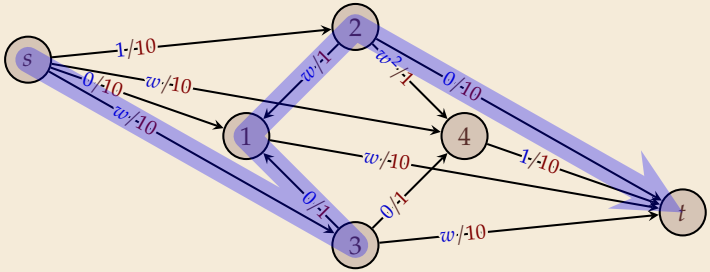

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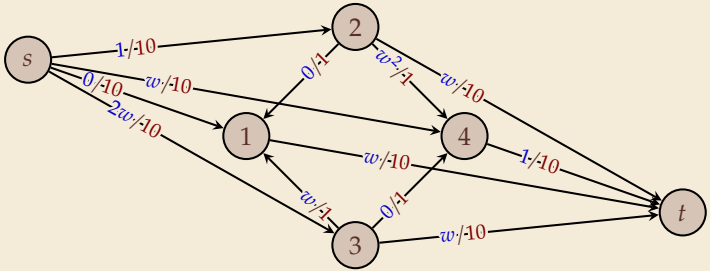

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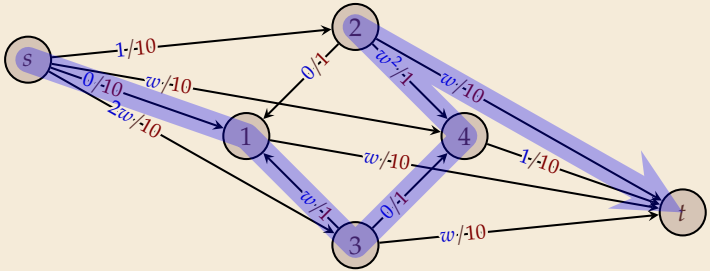

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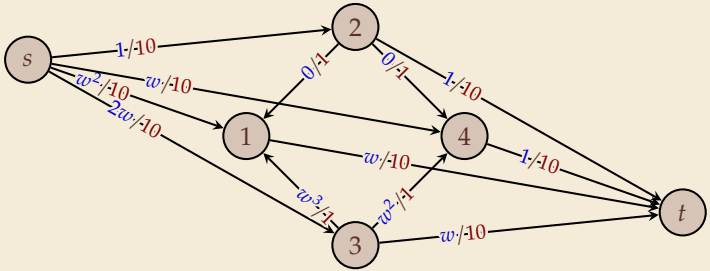

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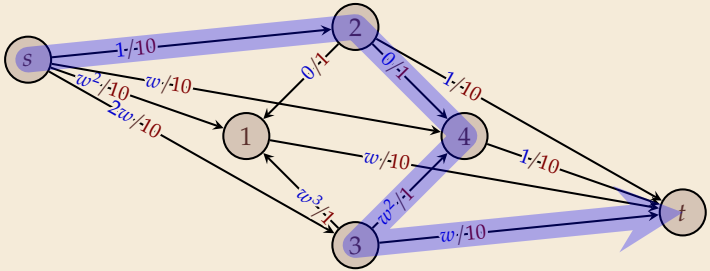

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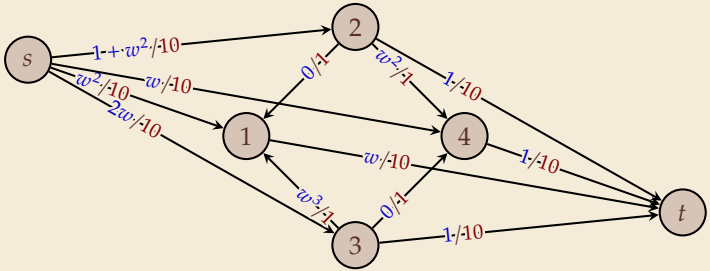

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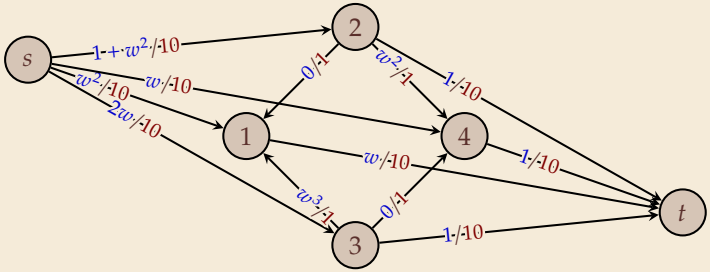

- after 2 paths, situation in 1-2-3-4 restored (rotated), but flows multiplied by w

$\rightsquigarrow$ augmenting paths have capacities w, w, w^2, w^2, w^3, w^3 ...

$\rightsquigarrow$ never terminate, never exceed $|f| \geq 5$

continue 15:31

9.10 The Edmonds-Karp Algorithm

Edmonds-Karp

- It turns out, many ways to choose augmenting paths systematically work fine
- Edmonds & Karp: take a shortest path (in #edges)

Java code Sedgewick & Wayne

```
1  procedure EdmondsKarp(G = (V, E), s, t, c):
2      // G is a flow network with source s ∈ V, sink t ∈ V and capacities c : E → ℝ≥0
3      for vw ∈ E do f(vw) := 0 end for
4      while true
5          bfs(G_f, {s})
6          if distFrom[t] == ∞ return f
7          else p := pathTo(t)
8          Δ := min{c_f(e) : e ∈ p} // bottleneck capacity
9          for e ∈ p
10             if e ∈ E // forward edge
11                 f(e) := f(e) + Δ
12             else // backward edge
13                 f(e) := f(e) − Δ
14     end while
```

new

∓F

Edmonds-Karp – Analysis

- **Theorem:** The Edmonds-Karp algorithm terminates after $O(nm)$ iterations with a maximum flow. The total running time is in $O(nm^2)$.

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- *Proof Plan:*
  - every augmenting path has a *critical* edge vw contributing the bottleneck capacity
  - we will show:
    (1) distances of vertices from s in G_f weakly increase over time
    (2) before vw can be a *critical* edge *again*, v's distance increases by at least 2
  - $\rightsquigarrow$ each edge vw is critical for at most $n/2$ augmenting paths (v's distance $\in [1..n-2]$)
  - $\rightsquigarrow$ $O(nm)$ augmenting paths
  - each iteration runs one BFS, which costs $O(n+m) = O(m)$ times since G is connected.

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- **Notation:**
  - Write $f_0, f_1, \ldots$ for values of f during iterations of while loop
  - $\rightsquigarrow$ G_{f_i} residual network after ith augmentation
  - Write $\delta_i(v)$ for shortest-path distance from s to v in G_{f_i}