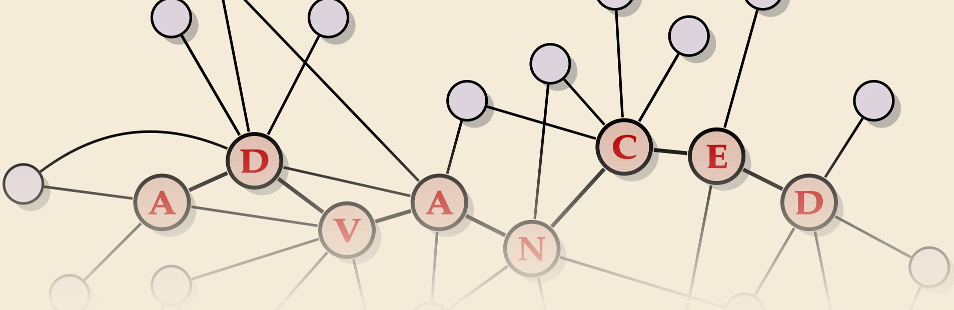

8 Randomized Complexity

18 June 2025

Prof. Dr. Sebastian Wild

CS627 (Summer 2025)
Philipps-Universität Marburg
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Outline

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The Power of Randomness

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$\rightsquigarrow$ back to *decision* problems.

8.1 Randomized Complexity Classes

Randomization for Decision Problems

- Recall: P and NP consider decision problems only

$\rightsquigarrow$ equivalently: languages $L \subseteq \Sigma^\star$

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$\rightsquigarrow$ equivalently: languages $L \subseteq \Sigma^\star$

Can make some simplifications for algorithms:

- Only 3 sensible output values: 0, 1, $\boxed{?}$
- Unless specified otherwise, allow unlimited #random bits,
  i. e., $random_A(x) = time_A(x)$ (Can't read more than one random bit per step)

$\leq$ always

Randomized Complexity Classes

Definition 8.1 (ZPP)

ZPP ***(zero-error probabilistic polytime)*** is the class of all languages L with a polytime **Las Vegas** algorithm A, i. e.,

(a) $\exists c \ : \ Time_A(n) \ = \ O(n^c)$ as $n \to \infty$ (In particular: always terminate!)

(b) $\mathbb{P}\big[A(x) = [x \in L]\big] \ \geq \ \frac{1}{2}$

(c) $A(x) \neq [x \in L]$ implies $A(x) = \boxed{?}$

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Definition 8.2 (BPP)

BPP *(bounded-error probabilistic polytime)* is the class of languages L with a polytime **bounded-error Monte Carlo** algorithm A, i. e.,

(a) $\exists c \;:\; Time_A(n) \;=\; O(n^c)$ as $n \to \infty$

(b) $\exists \varepsilon > 0 \;:\; \mathbb{P}\big[A(x) = [x \in L]\big] \;\geq\; \frac{1}{2} + \varepsilon$

$\forall x \in \Sigma^*$

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Definition 8.3 (PP)

PP *(probabilistic polytime)* is the class of languages L with a polytime **unbounded-error Monte Carlo** algorithm: **(a)** as above **(b)** $\mathbb{P}[A(x) = [x \in L]] \;>\; \frac{1}{2}$. ◀

Error Bounds

Remark 8.4 (Success Probability)

From the point of view of complexity classes, the success probability bounds are flexible:

- BPP only requires success probability $\frac{1}{2} + \varepsilon$, but using *Majority Voting*, we can also obtain any fixed success probability $\delta \in (\frac{1}{2}, 1)$.
- Similarly for ZPP, we can use probability amplification on Las Vegas algorithms

$\rightsquigarrow$ Unless otherwise stated,

$$\boxed{\text{for BPP and ZPP algorithms } A\text{, require } \mathbb{P}\big[A(x) = [x \in L]\big] \geq \tfrac{2}{3}}$$

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But recall: this is *not* true for **unbounded** errors and class PP.
In fact, we have the following result:

Theorem 8.5 (PP can simulate nondeterminism)

NP ∪ co-NP ⊆ PP.

⇝ Useful algorithms must avoid unbounded errors.

PP can simulate nondeterminism [1]

Proof (Theorem 8.5):

PP always allows polytime preprocessing

Given any $L \in NP$, we can use reduction $L \leq_p SAT$ (NP-complete)

$\leadsto$ suffices to show $SAT \in PP$

(TAUT is co-NP-complete $\leadsto$ works similarly for co-NP $\subseteq$ PP)

Given unbounded error MC algo A for SAT (polytime)

Given φ of length n over k variables

$A(\varphi)$: (1) Generate a (uniformly) random assignment $V: \{x_1, \ldots, x_k\} \to \{0,1\}$ (k random bits) $O(k)$

(2) If $V(\varphi) = 1$, output 1 $O(n)$

(3) Otherwise output $\mathcal{B}(p)$ $p = \frac{1}{2} - \frac{1}{2^{k+1}} < \frac{1}{2}$ $O(k)$

PP can simulate nondeterminism [2]

Proof (Theorem 8.5): running time polytime ✓

correctness : $\mathbb{P}[A(\varphi) = [\varphi \text{ sat.}]] \overset{?}{>} \frac{1}{2}$

- $\varphi \in SAT$ $\exists$ sat. assignment for $(x_1, \ldots, x_k)$

$$\mathbb{P}[\text{ step (2) succeeds}] \geq \frac{1}{2^k}$$

independence

$$\mathbb{P}[A(\varphi) = 0] = \mathbb{P}[V(\varphi) = 0] \cdot \mathbb{P}[B(p) = 0]$$

$$\leq \left(1 - \frac{1}{2^k}\right) \cdot \left(\frac{1}{2} + \frac{1}{2^{k+1}}\right) < \frac{1}{2}$$

- $\varphi \notin SAT$ $\mathbb{P}[V(\varphi) = 1] = 0$

$$\mathbb{P}[A(\varphi) = 1] = 1 \cdot \mathbb{P}[B(p) = 1] = p < \frac{1}{2}$$

$\Rightarrow \mathbb{P}[A(\varphi) = [\varphi \text{ sat.}]] > \frac{1}{2}.$

One-Sided Errors

In many cases, errors of MC algorithm are only *one-sided*.

Example: (simplistic) randomized algorithm for SAT:
Guess assignment, output $[\phi \text{ satisfied}]$.
(Note: This is not a MC algorithm, since we cannot give a fixed error bound!)

Observation: No false positives; unsatisfiable ϕ always yield 0.
. . . could this help?

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others: TSE-MC

Definition 8.6 (One-sided error Monte Carlo algorithms)

A randomized algorithm A for language L is a *one-sided-error Monte-Carlo (OSE-MC) algorithm* if we have

(a) $\mathbb{P}[A(x) = 1] \geq \frac{1}{2}$ for all $x \in L$, and

(b) $\mathbb{P}[A(x) = 0] = 1$ for all $x \notin L$.

$\rightsquigarrow$ $\boxed{\text{OSE-MC: } A(x) = 1 \text{ must always be correct; } A(x) = 0 \text{ may be a lie}}$

One-Sided Error Classes

Definition 8.7 (RP, co-RP)

The classes RP and co-RP are the sets of all languages L with a polytime OSE-MC algorithm for L resp. $\overline{L}$. ◀

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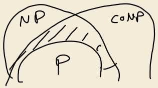
Theorem 8.8 (Complementation feasible → errors avoidable)

$\mathsf{RP} \cap \mathsf{co\text{-}RP} = \mathsf{ZPP}$. ◀

Proof:

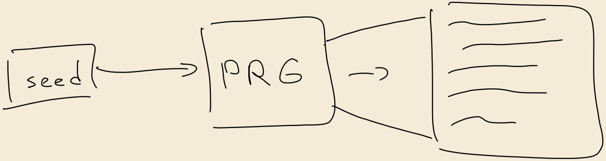
See exercises. ■



Note the similarity to the wide open problem $\mathsf{NP} \cap \mathsf{co\text{-}NP} \overset{?}{=} \mathsf{P}$.
For the latter, the common belief is $\mathsf{NP} \cap \mathsf{co\text{-}NP} \supsetneq \mathsf{P}$, in sharp contrast to the randomized classes.

8.2 Pseudorandom Generators

conventional definition (programming)



pseudorandom numbers = deterministic function of seed

good PRG = output passes statistical tests for randomness

Derandomization

- Suppose we have a BPP algorithm A, i. e., a polytime TSE-MC algorithm

$\rightsquigarrow$ $Random_A(n)$ bounded

$\rightsquigarrow$ There are at most $2^{Random_A(n)}$ different random-bit inputs ρ
and hence at most so many different computations for A on inputs $x \in \Sigma^n$

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- The *derandomization* of A is a deterministic algorithm that simply simulates all these computations one after the other (and outputs the majority).

- In general, the exponential blowup makes this uninteresting.

- **But:** If $Random_A(n) \le c \cdot \underset{}{\overset{= \log_2}{\lg}}(n)$,
the derandomization of A runs in polytime: $n^c \cdot Time_A(n)$

$2^{c \lg n} = (2^{\lg n})^c = n^c$

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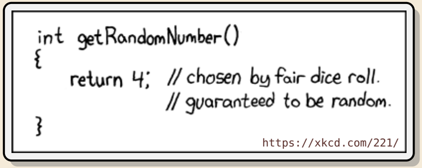

https://xkcd.com/221/

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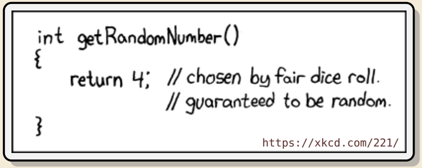

- must somehow keep the random distribution . . .
  in general not clear what "sufficiently random" would mean

$\rightsquigarrow$ Breakthrough idea in TCS: *Pseudorandom Generators*

  - generate an exponential number of bits from a n given truly random bits such that **no efficient** algorithm can distinguish them from truly random

    in a model to be specified

  - **Key (Open!) Question:** *Do they exist?!*

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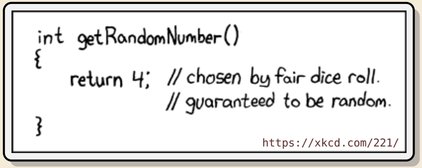

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- **Key (Open!) Question:** *Do they exist?!*
- **Surprising answer:** We have good evidence in favor (!)

8.3 Excursion: Boolean Circuits

Boolean Circuits

For technical reasons (stay tuned . . .), another model of computation more convenient than TM here.

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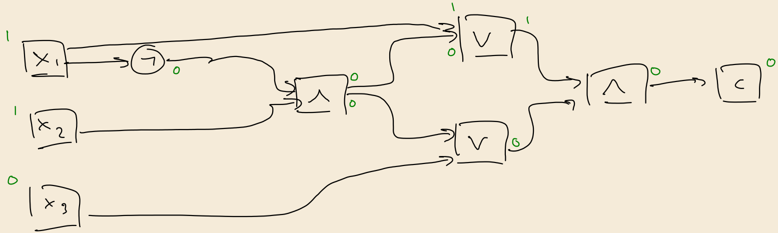
Definition 8.9 (Boolean circuit)

An n-input *Boolean circuit* is a connected DAG $C = (V, E)$

- with n *sources* (labeled $x_1, \ldots, x_n$)
- a single *sink* c (the output)
- any number of *gates* (non-sink vertices) labeled with $\wedge$, $\vee$, or $\neg$.
- All gates have in- and out-degree at most 2 (*fan-in* = *fan-out* = 2). (¬ is always unary)



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Definition 8.10 (Circuit complexity)

The circuit complexity $\mathcal{H}(f)$ of a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is the size of the *smallest* Boolean circuit C that computes f. ◀

Formula vs. Circuit

Parity function: $P_n(x_1, \ldots, x_n) = \bigoplus_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i \bmod 2$ (odd number of 1-bits?)

XOR

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- By associativity, $P_n(x_1, \ldots, x_n) = P_{n-1}(x_1, \ldots, x_{n-1}) \oplus x_n$
- also: $a \oplus b = (a \wedge \neg b) \vee (\neg a \wedge b)$

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$(x_1 \oplus x_2 \oplus x_3) \oplus (x_4 \oplus x_5 \oplus x_6)$

⇝ Can built a circuit for P_n using $5(n-1)$ gates

- Obvious boolean formula: (over basis $\{\wedge, \vee, \neg\}$)
  $P_n(x_1, \ldots, x_n) = (x_n \wedge \neg \boldsymbol{P_{n-1}(x_1, \ldots, x_{n-1})}) \vee (\neg x_n \wedge \boldsymbol{P_{n-1}(x_1, \ldots, x_{n-1})})$

⇝ $5 \cdot 2^{n-1}$ operators

- optimal (assuming $n = 2^k$):
  $$P_n(x_1, \ldots, x_n) = (P_{n/2}(x_1, \ldots, x_{n/2}) \cap \neg P_{n/2}(x_{n/2+1}, \ldots, x_n)) \vee (\neg P_{n/2}(x_1, \ldots, x_{n/2}) \cap P_{n/2}(x_{n/2+1}, \ldots, x_n))$$

⇝ $\Theta(n^2)$ still much more than for circuits!

Circuit Complexity Classes

Poly-size circuits: (somewhat analogous to P, but not quite . . .)

- $\mathsf{P}_{/\mathsf{poly}}$ = all functions computable by ***polynomial-sized*** circuits

$\forall n\ \exists C_n$: C_n computes $f|_{\{0,1\}^n}$
and $|C_n| = O(n^d)$

TM can always simulate circuit for fixed n

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- Can prove: $\mathsf{P} \subseteq \mathsf{P}_{/\mathsf{poly}}$

Theorem 8.11 (TM to circuit)
For $f \in TIME(T(n))$ and input size n, we can compute in polytime a circuit C for f on inputs of size n of size $|C| = O(T(n)^2)$.
(Arora & Barak, Theorem 6.6)

time in TM $\hat{=}$ size of circuit

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  circuits are *non-uniform* model of computation: *different circuit for each* n

$\rightsquigarrow$ has some weird properties in general ($\mathsf{P}_{/\mathsf{poly}}$ contains a version of halting problem . . .)

any $L \subseteq \{1^n : n \in \mathbb{N}\} \in \mathsf{P}_{/\mathsf{poly}}$

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  - ⇝ has some weird properties in general ($\mathsf{P}_{/\mathsf{poly}}$ contains a version of halting problem . . .)
- Probably $\mathsf{NP} \not\subseteq \mathsf{P}_{/\mathsf{poly}}$ (unless polynomial hierarchy collapses)

Circuit Lower Bounds:

- Can show: almost all Boolean functions f have *exponential* $\mathcal{C}(f)$ $\not\in$ NP (counting argument)
- But: *Very* hard to prove circuit lower bounds for *concrete* functions f
  - Showing $\mathcal{H}(f)$ **exponential** for *any* $f \in \mathsf{NP}$ would imply $\mathsf{P} \neq \mathsf{NP}$
  - Proven lower bounds on $\mathcal{H}(f)$ for explicit f are typically **linear** in n

“Monte Carlo” Circuits

We need a somewhat peculiar, weaker form of circuit complexity, where we assume that inputs $\boldsymbol{X} \in \{0,1\}^n$ are chosen *uniformly at random*.

Definition 8.12 (Average-case hardness)

The *ρ-average-case hardness* $\mathcal{H}^{\rho}_{avg}(f)$ of a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is the largest size S, such that every circuit C with $|C| \le S$ we have $\mathbb{P}\left[C(\boldsymbol{X}) = f(\boldsymbol{X})\right] < \rho$.

(Need circuits larger than $\mathcal{H}^{\rho}_{avg}(f)$ for confidence ρ.)

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(Need circuits larger than $\mathcal{H}^{\rho}_{avg}(f)$ for confidence ρ.)

The *average-case hardness* of f then is $\mathcal{H}_{avg}(f) = \max\Big\{S : \mathcal{H}^{\frac{1}{2}+\frac{1}{S}}_{avg} \geq S\Big\}$.
(Allow larger circuits and worse confidence until f probabilistically computable)

Monte Carlo Circuits

We need a somewhat peculiar, weaker form of circuit complexity, where we assume that inputs $\boldsymbol{X} \in \{0,1\}^n$ are chosen *uniformly at random*.

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There exists a function $f \in \mathsf{NP}$ with $\mathcal{H}_{avg}(f) = 2^{\Omega(n)}$.

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- **Deep result** (that we skip): From existence of function with large $\mathcal{H}(f)$, can conclude existence of function with large $\mathcal{H}_{avg}(\overset{g}{\cancel{f}})$.
  (see *Arora & Barak* Chapter 19)
- 3SAT probably has exponential $\mathcal{H}(f)$ ($\approx$ ETH) (and other candidates exist)

Formalization Pseudorandom Generator

Definition 8.14 (Pseudorandom bits)

A r.v. $\boldsymbol{R} \in \{0,1\}^m$ is (S, ε)-*pseudorandom* if for every circuit C with $|C| \le S$

$$\left| \mathbb{P}_{U}\left[C(\boldsymbol{R}) = 1\right] - \mathbb{P}\left[C(\boldsymbol{U}_m\overset{=1}{)}\right] \right| < \varepsilon \qquad \text{where} \qquad \boldsymbol{U}_m \overset{\mathcal{D}}{=} \mathcal{U}(\{0,1\}^m)$$

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Pseudorandom bits are ***indistinguishable*** *from truly random* ***for any small circuit****.*

think: fast-running algorithm

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Definition 8.15 (Pseudorandom generator)

Let $S : \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 1}$.
A function G: $\{0,1\}^\star \to \{0,1\}^\star$ computable in 2^n time ($G \in TIME(2^n)$) is an *$S(\ell)$-pseudorandom generator ($S(\ell)$-PRG)* if

(a) $|G(z)| = S(|z|)$ for every $z \in \{0,1\}^\star$

(b) $\forall \ell \in \mathbb{N}_{\geq 1} \ : \ G(\boldsymbol{U}_\ell)$ is $\left(S(\ell)^3, \frac{1}{10}\right)$-pseudorandom. ◀

Seeding a generator with ℓ truly random bits yields $S(\ell)$ pseudorandom bits.

8.4 Derandomization

Pseudorandom Generator for BPP Derandomization

The *Nisan-Wigderson construction* shows that the existence of any hard-on-average function implies a strong pseudorandom generator.

exponentially many pseudorandom bits(!)

Theorem 8.16 (Strong NW PRG)

Assume Hypothesis 8.13, i. e., $f \in TIME(2^{O(n)})$ exists with $\mathcal{H}_{avg}(f) \geq S$ with $S(n) = 2^{\delta n}$ for a constant $\delta > 0$.
Then there is an $\varepsilon = \varepsilon(\delta)$ such that there is a $2^{\varepsilon \ell}$-pseudorandom generator. ◀

(We will prove this over the course of the next subsection.)

BPP Derandomization

Theorem 8.17 (Hard-on-average function → BPP = P)

Hypothesis 8.13 implies BPP = P.

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Proof:

By Theorem 8.16, Hypothesis 8.13 implies a $S(\ell)$-PRG $G : \{0,1\}^{\ell} \to \{0,1\}^{S(\ell)}$ with $S(\ell) = 2^{\varepsilon \ell}$.

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Let $L \in \mathsf{BPP}$. $\rightsquigarrow$ $\exists$ algorithm $\overset{\text{PTM}}{}$ A with $Time_A(n) \le n^c$ (polytime) and $\mathbb{P}_R[A(x,R) = L(x)] \ge \frac{2}{3}$;

here $R \overset{\mathcal{D}}{=} \mathcal{U}(\{0,1\}^m)$ for $m = Random_A(n) \le Time_A(n) \le n^c$.

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Let $L \in \mathsf{BPP}$. $\rightsquigarrow$ $\exists$ algorithm A with $Time_A(n) \le n^c$ (polytime) and $\mathbb{P}_R[A(x,R) = L(x)] \ge \frac{2}{3}$; here $R \overset{\mathcal{D}}{=} \mathcal{U}(\{0,1\}^m)$ for $m = Random_A(n) \le Time_A(n) \le n^c$.

We now obtain a **deterministic** polytime algorithm B as follows:

1. Replace R by $G(Z)$ for $Z \overset{\mathcal{D}}{=} \mathcal{U}(\{0,1\}^{\ell})$ for $\ell = \ell(n) = \frac{c}{\varepsilon} \lg n$ so that $m \le S(\ell) = 2^{\varepsilon \ell} = n^c$.
2. Instead of this probabilistic TM, simulate $A(x, G(z))$ for **all** possible $z \in \{0,1\}^{\ell}$
3. Output the majority.

The trick here is that number of possible seeds z is $2^{\ell(n)} = n^c$, hence the running time remains polynomial and $B \in \mathsf{P}$!

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It remains to show that B accepts L.
(Intuition: A is too fast to notice a difference of more than $\frac{1}{10}$ between R and $G(Z)$.)

BPP Derandomization [2]

Proof (cont.): $B(x) \neq L(x) \rightsquigarrow$ majority vote wrong

Formally, assume towards a contradiction that there is an infinite sequence of x's with $\mathbb{P}_Z[A(x, G(Z)) = L(x)] < \frac{2}{3} - \frac{1}{10} = 0.5\overline{6} > \frac{1}{2}$.

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Then, we can build a *distinguisher* circuit C for the PRG: C simply computes the function $r \mapsto A(x, r)$, where x is hard-wired into the circuit C.

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Hence, the majority vote in B is correct
(for all but a finite number of inputs, which can be tested in constant time).
$\rightsquigarrow \quad L \in P$. ∎

Consequences

- ⇝ Since the existence of hard-on-average functions is rather likely,
  - it must be assumed that randomization alone does **not** solve NP-hard problems;
  - . . . and it seems that there is some heavy lifting going on in *Nisan-Wigderson*
  - ⇝ Let's see what it does!

8.5 Nisan-Wigderson Pseudorandom Generator

Overview

- In this section, we will describe a conditional construction for pseudorandom generators based on the unproven hard-function hypothesis (Hypothesis 8.13).

  The higher the circuit lower bound $S(n)$ for our hard function f,
  the more pseudorandom bits we can generate from a fixed seed of ℓ truly random bits.

- Key construction is due to *Noam Nisan* and *Avi Wigderson* (2023 Turing Award)
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Theorem 8.18 (PRG from average-case hard function)

Let $S : \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 1}$.
If there exists a function $f \in TIME(2^{O(n)})$ with $\mathcal{H}_{avg}(f)(n) \geq S(n)$ for all n,
then there exists a $S(\delta\ell)^{\delta}$-pseudorandom generator for some constant $\delta > 0$.

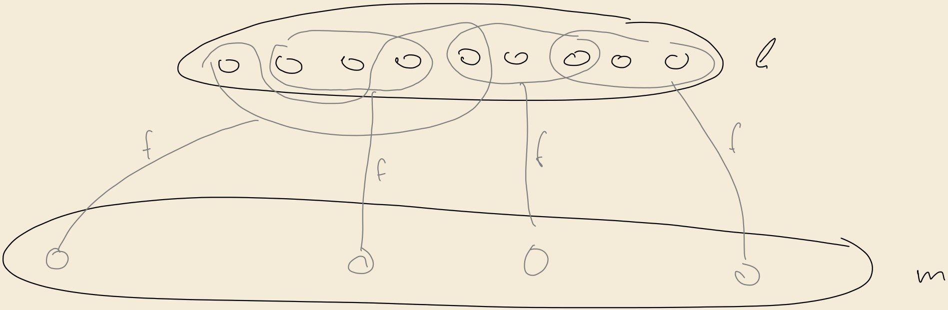
This general result is for a refined construction and works also for weaker assumptions.
We will show the version sufficient for Theorem 8.16; see Arora & Barak Remark 20.8

Nisan-Wigderson Generator

The idea of the ***Nisan-Wigderson (NW) generator*** is to feed many (partially overlapping) subsets $I \in \mathcal{J}$ of ℓ truly random input bits into a (hard) function $f : \{0,1\}^n \rightarrow \{0,1\}$

$$\mathrm{NW}^f_{\mathcal{J}}(Z) \;=\; f(Z_{I_1})\, f(Z_{I_2}) \ldots f(Z_{I_m})$$

where $Z \overset{\mathcal{D}}{=} \mathcal{U}(\{0,1\}^\ell)$ is the random seed and z_I for $I = \{i_1, \ldots, i_n\}$ denotes $(z_{i_1}, \ldots, z_{i_n})$



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A key component is a sufficiently large subset system $\mathcal{I}$ without too much overlap.

Definition 8.19 (Combinatorial Design)

For $\ell > n > d$, a family $\mathcal{I} = \{I_1, \ldots, I_m\}$ of m subsets of $[\ell]$ is an (ℓ, n, d)-*design* if for all j and $k \neq j$,

- we have $|I_j| = n$ and
- $|I_j \cap I_k| \leq d$.

◀

(We will eventually want to use this with $m = 2^{\varepsilon \ell}$.)

Probabilistic Method for Combinatorial Designs

Lemma 8.20 (NW Design)

There is an algorithm A that outputs on input (ℓ, n, d) with $\ell > n > d$ and $\ell > 10n^2/d$ an (ℓ, n, d)-design $\mathcal{I}$ with $\underbrace{|\mathcal{I}| = 2^{d/10}}_{m}$ subsets of $[\ell]$ in time $2^{O(\ell)}$.

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Proof:

A is a simple greedy strategy: We start with $\mathcal{J} = \emptyset$. For $m \in [2^{d/10}]$, iterate over all 2^ℓ subsets of $[\ell]$ and include into $\mathcal{J}$ the first set I with $\max_{J \in \mathcal{J}} |J \cap I| \le d$.

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By Chernoff:

$\mathbb{E}[I] = 2n$

(1) $\mathbb{P}[|I| \geq n] \ \geq \ 0.9$
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$$\bigcup_{J\in\mathcal{J}} \mathbb{P}[|I\cap J| \geq d] \leq \underbrace{|\mathcal{J}|}_{\leq 2^{d/10}} \cdot \frac{1}{2} 2^{-d/10}$$

Since $|\mathcal{J}| \leq 2^{d/10}$ and union bound on (2), $\mathbb{P}[\max_{J\in\mathcal{J}} |J \cap I| \geq d] \leq \frac{1}{2}$.
Hence, with probability at least $0.9 \cdot 0.5 = 0.45$, our random set I has intersection $\leq d$ with all old sets and $\geq n$ elements. Dropping elements until $|I| = n$ does not change that.

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By Chernoff:
(1) $\mathbb{P}[|I| \ge n] \;\ge\; 0.9$
(2) $\mathbb{P}[|I \cap J| \ge d] \;\le\; \frac{1}{2} \cdot 2^{-d/10}$ for any $J \in \mathcal{I}$

Since $|\mathcal{I}| \le 2^{d/10}$ and union bound on (2), $\mathbb{P}[\max_{J \in \mathcal{I}} |J \cap I| \ge d] \le \frac{1}{2}$.
Hence, with probability at least $0.9 \cdot 0.5 = 0.45$, our random set I has intersection $\le d$ with all old sets and $\ge n$ elements. Dropping elements until $|I| = n$ does not change that.

$\rightsquigarrow$ In each step, we have probability ≥ 0.45 to succeed. So picking m random sets succeeds with probability $\ge 0.45^m > 0$, so some choice of sets $\mathcal{I}$ as claimed must exist. ■