1 Machines & Models

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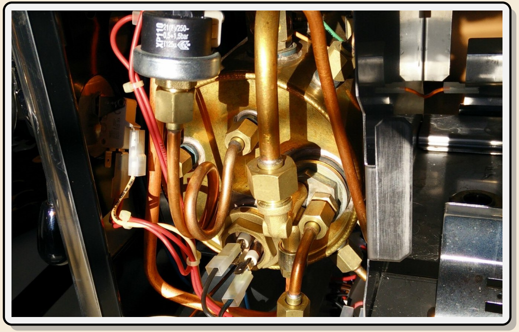
Learning Outcomes

1. Understand the difference between empirical *running time* and algorithm *analysis*.
2. Understand *worst / best / average case* models for input data.
3. Know the *RAM machine* model.
4. Know the definitions of *asymptotic notation* (Big-Oh classes and relatives).
5. Understand the reasons to make *asymptotic approximations*.
6. Be able to *analyze* simple *algorithms*.

Unit 1: ***Machines & Models***



Outline

1 Machines & Models

1.1 Algorithm analysis

1.2 The RAM Model

1.3 Asymptotics & Big-Oh

What is an algorithm?

An algorithm is a sequence of instructions.

think: recipe

e. g. Python script

More precisely:

1. mechanically executable
   ⇝ no "common sense" needed
2. finite description ≠ finite computation!
3. solves a *problem*, i. e., a class of problem instances

$x + y$, not only $17 + 4$

- input-processing-output abstraction

Typical example: *bubblesort*

⇝ not a specific program
but the underlying idea

What is a data structure?

A data structure is

1. a rule for encoding data (in computer memory), plus
2. algorithms to work with it (queries, updates, etc.)

typical example: binary search tree

1.1 Algorithm analysis

Good algorithms

Our goal: Find good (best?) algorithms and data structures for a task.

Good "usually" means

can be complicated in distributed systems

- fast running *time*
- moderate memory *space* usage

Algorithm analysis is a way to

- compare different algorithms,
- predict their performance in an application

Running time experiments

Why not simply run and time it?

- results only apply to
  - single *test* machine
  - tested inputs
  - tested implementation
  - ...
  - $\neq$ ***universal truths***

- instead: consider and analyze algorithms on an abstract machine
  - ⇝ provable statements for model
  - ⇝ testable model hypotheses

survives Pentium 4

⇝ Need precise model of machine (costs), input data and algorithms.

Data Models

Algorithm analysis typically uses one of the following simple data models:

- **worst-case performance:**
  consider the *worst* of all inputs as our cost metric
- **best-case performance:**
  consider the *best* of all inputs as our cost metric
- **average-case performance:**
  consider the average/expectation of a *random* input as our cost metric

Usually, we apply the above for *inputs of same size* n.

$\rightsquigarrow$ performance is only a **function of** n.

1.2 The RAM Model

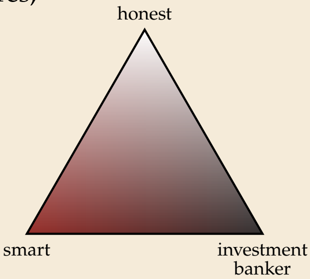
Machine models

The machine model decides

- what algorithms are possible
- how they are described (= programming language)
- what an execution *costs*

Goal: Machine model should be
detailed and powerful enough to reflect actual machines,
abstract enough to unify architectures,
simple enough to analyze.

$\rightsquigarrow$ usually some compromise is needed



Random Access Machines

Random access machine (RAM)

more detail in §2.2 of *Sequential and Parallel Algorithms and Data Structures* by Sanders, Mehlhorn, Dietzfelbinger, Dementiev

- unlimited *memory* $\mathtt{MEM}[0], \mathtt{MEM}[1], \mathtt{MEM}[2], \ldots$
- fixed number of *registers* $R_1, \ldots, R_r$ (say $r = 100$)
- memory cells $\mathtt{MEM}[i]$ and registers R_i store w-bit integers, i. e., numbers in $[0..2^w - 1]$
  w is the word width/size; typically $w \propto \lg n \rightsquigarrow 2^w \approx n$
- Instructions:
  - load & store: $R_i := \mathtt{MEM}[R_j]$ $\quad \mathtt{MEM}[R_j] := R_i$
  - operations on registers: $R_k := R_i + R_j$ (arithmetic is *modulo* 2^w!)
    also $R_i - R_j,\ R_i \cdot R_j,\ R_i \operatorname{div} R_j,\ R_i \bmod R_j$
    C-style operations (bitwise and/or/xor, left/right shift)
  - conditional and unconditional jumps
- cost: number of executed instructions

we will see further models later

$\rightsquigarrow$ The RAM is the standard model for sequential computation.

Pseudocode

Typical simplifications for convenience:

- more abstract *pseudocode* to specify algorithms
  code that humans understand (easily)
- count *dominant operations* (e. g. array accesses) instead of all operations

In both cases: can go to full detail if needed.

1.3 Asymptotics & Big-Oh

Why asymptotics?

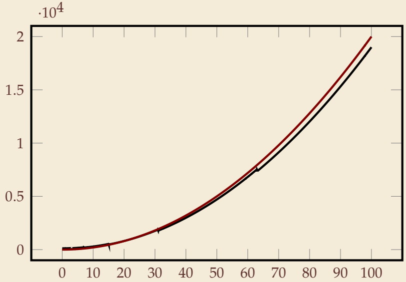
Algorithm analysis focuses on (the limiting behavior for infinitely) **large** inputs.

- abstracts from unnecessary detail
- simplifies analysis
- often necessary for sensible comparison

Asymptotics = approximation around ∞

Example: Consider a function $f(n)$ given by

$2n^2 - 3n\lfloor \log_2(n+1) \rfloor + 7n - 3\lfloor \log_2(n+1) \rfloor + 120 \quad \sim \quad \mathbf{2n^2}$



Asymptotic tools – Formal & definitive definition

- **"Tilde Notation":** $f(n) \sim g(n)$ $\underset{\text{if, and only if}}{\text{iff}}$ $\displaystyle\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

  „f and g are *asymptotically equivalent*"

- **"Big-Oh Notation":** $f(n) \in O\big(g(n)\big)$ iff $\left|\dfrac{f(n)}{g(n)}\right|$ is bounded for $n \geq n_0$

  also write '=' instead

  need supremum since limit might not exist!

  iff $\displaystyle\limsup_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| < \infty$

  Variants:

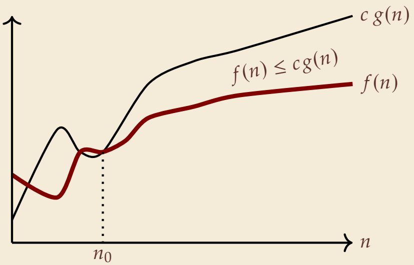
  - $f(n) \in \Omega\big(g(n)\big)$ iff $g(n) \in O\big(f(n)\big)$ "Big-Omega"
  - $f(n) \in \Theta\big(g(n)\big)$ iff $f(n) \in O\big(g(n)\big)$ **and** $f(n) \in \Omega\big(g(n)\big)$ "Big-Theta"

- **"Little-Oh Notation":** $f(n) \in o\big(g(n)\big)$ iff $\displaystyle\lim_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| = 0$

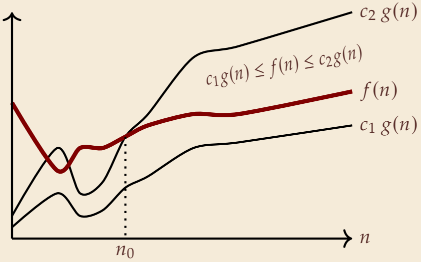
  $f(n) \in \omega\big(g(n)\big)$ if $\lim = \infty$

Asymptotic tools – Intuition

- $f(n) = O(g(n))$: $f(n)$ is **at most** $g(n)$ up to constant factors and for sufficiently large n



- $f(n) = \Theta(g(n))$: $f(n)$ is **equal to** $g(n)$ up to constant factors and for sufficiently large n



Plots can be misleading!

Example

Asymptotics – Example 1

Basic examples:

- $20n^3 + 10n\ln(n) + 5 \;\sim\; 20n^3 \;=\; \Theta(n^3)$
- $3\lg(n^2) + \lg(\lg(n)) \;=\; \Theta(\log n)$
- $10^{100} \;=\; O(1)$

Use *wolfram alpha* to compute/check limits.

Asymptotics – Frequently used facts

- Rules:
  - $c \cdot f(n) \;=\; \Theta(f(n))$ for constant $c \neq 0$
  - $\Theta(f + g) \;=\; \Theta(\max\{f, g\})$ largest summand determines Θ-class
- Frequently used orders of growth:
  - logarithmic $\Theta(\log n)$ Note: $a, b > 0$ constants $\rightsquigarrow$ $\Theta(\log_a(n)) = \Theta(\log_b(n))$
  - linear $\Theta(n)$
  - linearithmic $\Theta(n \log n)$
  - quadratic $\Theta(n^2)$
  - polynomial $O(n^c)$ for constant c
  - exponential $O(c^n)$ for constant c Note: $a > b > 0$ constants $\rightsquigarrow$ $b^n = o(a^n)$

Asymptotics – Example 2

Square-and-multiply algorithm
for computing x^m with $m \in \mathbb{N}$

Inputs:

- m as binary number (array of bits)
- n = #bits in m
- x a floating-point number

```
1 def pow(x, m):
2     # compute binary representation of exponent
3     exponent_bits = bin(m)[2:]
4     result = 1
5     for bit in exponent_bits:
6         result *= result
7         if bit == '1':
8             result *= x
9     return result
```

- Cost: C = # multiplications
- $C = n$ (line 4) + #one-bits binary representation of m (line 5)

$\rightsquigarrow$ $n \le C \le 2n$