5 Parallel Algorithms

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COMP526 (Fall 2022)
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Learning Outcomes

1. Know and apply *parallelization strategies* for embarrassingly parallel problems.
2. Identify *limits of parallel speedups*.
3. Understand and use the *parallel random-access-machine* model in its different variants.
4. Be able to *analyze* and compare simple shared-memory parallel algorithms by determining *parallel time and work*.
5. Understand efficient parallel *prefix sum* algorithms.
6. Be able to devise high-level description of *parallel quicksort and mergesort* methods.

Unit 5: ***Parallel Algorithms***

Outline

5 Parallel Algorithms

5.1 Parallel computation

Clicker Question

Have you ever written a concurrent program (explicit threads, job pools library, or using a framework for distributed computing)?

- **A** Yes
- **B** No
- **C** Concur. . . what?

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Types of parallel computation

£££ can't buy you more time ... but more computers!

$\rightsquigarrow$ Challenge: Algorithms for *parallel* computation.

Types of parallel computation

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There are two main forms of parallelism:

1. **shared-memory parallel computer** $\leftarrow$ *focus of today*
   - p *processing elements* (PEs, processors) working in parallel
   - **single** big memory, **accessible from every PE**
   - communication via shared memory
   - think: a big server, 128 CPU cores, terabyte of main memory
2. **distributed computing**
   - p PEs working in parallel
   - each PE has **private** memory
   - communication by sending **messages** via a network
   - think: a cluster of individual machines

PRAM – Parallel RAM

- extension of the RAM model (recall Unit 1)
- the p PEs are identified by ids $0, \ldots, p-1$
  - like w (the word size), p is a parameter of the model that can grow with n
  - $p = \Theta(n)$ is not unusual   maaany processors!
- the PEs all **independently** run ~~a~~ the same RAM-style program
  (they can use their id there)
- each PE has its own registers, but `MEM` is shared among all PEs
- computation runs in **synchronous** steps:
  in each time step, every PE executes one instruction

PRAM – Conflict management

Problem: What if several PEs simultaneously overwrite a memory cell?

- **EREW**-PRAM (exclusive read, exclusive write)
  any **parallel access** to same memory cell is **forbidden** (crash if happens)
- **CREW**-PRAM (concurrent read, exclusive write)
  parallel **write** access to same memory cell is *forbidden*, but reading is fine
- **CRCW**-PRAM (concurrent read, concurrent write)
  concurrent access is allowed,
  need a rule for write conflicts:
  - common CRCW-PRAM:
    all concurrent writes to same cell must write *same* value
  - arbitrary CRCW-PRAM:
    some unspecified concurrent write wins
  - (more exist . . .)

- no single model is always adequate, but our default is CREW

PRAM – Execution costs

Cost metrics in PRAMs

- **space:** total amount of accessed memory
- **time:** number of steps till all PEs finish assuming sufficiently many PEs!
  sometimes called *depth* or *span*
- **work:** total #instructions executed on **all** PEs

PRAM – Execution costs

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Holy grail of PRAM algorithms:

- minimal time
- work (asymptotically) no worse than running time of best sequential algorithm
  $\rightsquigarrow$ *"work-efficient"* algorithm: work in same Θ-class as best sequential

Clicker Question

Does every computational problem allow a work-efficient algorithm?

A Yes

B No

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Clicker Question

Does every computational problem allow a work-efficient algorithm?

A Yes ✓

B ~~No~~

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The number of processors

Hold on, my computer does not have $\Theta(n)$ processors! Why should I care for span and work!?

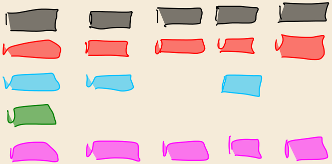
Theorem 5.1 (Brent's Theorem)

If an algorithm has span T and work W (for an arbitrarily large number of processors), it can be run on a PRAM with p PEs in time $O(T + \frac{W}{p})$ (and using $O(W)$ work).

Proof: schedule parallel steps in round-robin fashion on the p PEs.

PRAM algorithm p

time ↓

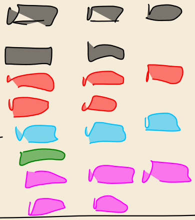


$\# \text{boxes} = W$

simulate on PRAM $p = 3$

PE0 PE1 PE2

$\left\lceil \frac{P}{p} \right\rceil$



$$\sum_{i=1}^{T} \left\lceil \frac{P_i}{p} \right\rceil \le \sum_{i=1}^{T} \frac{P_i}{p} + 1 = \frac{W}{p} + T$$

$\rightsquigarrow$ span and work give guideline for *any* number of processors

5.2 Parallel String Matching

Embarrassingly Parallel

- A problem is called *"embarrassingly parallel"*
  if it can immediately be split into *many, small subtasks*
  that can be solved completely *independently* of each other
- Typical example: sum of two large matrices (all entries independent)

$\rightsquigarrow$ best case for parallel computation (simply assign each processor one subtask)

- Sorting is not embarrassingly parallel
  - no obvious way to define many *small* (= efficiently solvable) subproblems
  - but: some subtasks of our algorithms are (stay tuned . . .)

Clicker Question

Is the string-matching problem "embarrassingly parallel"?

- **A** Yes
- **B** No
- **C** Only for $n \gg m$
- **D** Only for $n \approx m$

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Parallel string matching – Easy?

- We have seen a plethora of string matching methods in Unit 4
- But all efficient methods seem inherently sequential
  Indeed, they became efficient only after building on knowledge from previous steps!

Sounds like the *opposite* of parallel!

$\leadsto$ How well can we parallelize string matching?

Here: string matching = find *all* occurrences of P in T (more natural problem for parallel)
always assume $m \le n$

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$\rightsquigarrow$ How well can we parallelize string matching?

Here: string matching = find *all* occurrences of P in T (more natural problem for parallel)
always assume $m \le n$

Subproblems in string matching:

- string matching = check all guesses $i = 0, \ldots, n - m - 1$
- checking one guess is a subtask!

Parallel string matching – Brute force

- Check all guesses in parallel

---

procedure parallelBruteForce($T[0..n)$, $P[0..m)$)
    for $i := 0, \dots, n - m - 1$ **do in parallel** ← only difference to normal brute force!
        for $j := 0, \dots, m - 1$ **do**
            if $T[i + j] \neq P[j]$ **then break** inner loop
        if $j == m$ **then** report match at i
    end parallel for

- PE k is executing the loop iteration where $i = k$.
  - ⇝ requires that all iterations can be done **independently**!
  - Different PEs work **in lockstep** (synchronized after each instruction)
  - similar to OpenMP `#pragma omp parallel for`
- checking whether *no* match was found by *any* PE more effort ⇝ ... stay tuned

Parallel string matching – Brute force

- Check all guesses in parallel

```
procedure parallelBruteForce(T[0..n), P[0..m))
    for i := 0, ..., n − m − 1 do in parallel
        for j := 0, ..., m − 1 do
            if T[i + j] ≠ P[j] then break inner loop
        if j == m then report match at i
    end parallel for
```

only difference to normal brute force!

- PE k is executing the loop iteration where $i = k$.
  - ⇝ requires that all iterations can be done **independently**!
  - Different PEs work **in lockstep** (synchronized after each instruction)
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⇝ **Time**: $\Theta(m)$ using sequential checks
$\Theta(\log m)$ on CREW-PRAM (⇝ tutorials)
$\Theta(1)$ on CRCW-PRAM (⇝ tutorials)

Work: $\Theta((n-m)m)$ ⇝ not great
. . . much more than best sequential

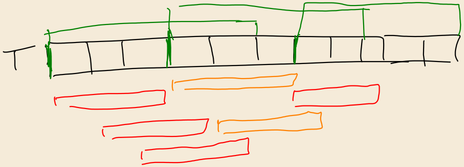
Parallel string matching – Blocking

Divide T into **overlapping** blocks of $2m-1$ characters:
$T[0..2m-1)$, $T[m..3m-1)$, $T[2m..4m-1)$, $T[3m..5m-1)\ldots$

- Find matches inside blocks in parallel, using efficient sequential method

---

procedure blockingStringMatching($T[0..n)$, $P[0..m)$)
  for $b := 0,\ldots,\lceil n/m \rceil$ **do in parallel**
    $result$:= KMP($T[bm \mathrel{..} (b+1)m-1)$, P)
    if $result \neq$ `NO_MATCH` **then** report match at $result$
  end parallel for



$m = 3$

$2m - 1 = 5$

Parallel string matching – Blocking

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procedure blockingStringMatching(T[0..n), P[0..m))
    for b := 0, ..., ⌈n/m⌉ do in parallel
        result := KMP(T[bm .. (b+1)m − 1), P)
        if result ≠ NO_MATCH then report match at result
    end parallel for
```

$\rightsquigarrow$ **Time**:

- loop body has text of length $n' = 2m-1$ and pattern of length m

$\rightsquigarrow$ KPM runtime $\Theta(n'+m) = \Theta(m)$

$\rightsquigarrow$ **Work**: $\Theta(\frac{n}{m} \cdot m) = \Theta(n)$ $\rightsquigarrow$ work efficient!

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Parallel string matching – Discussion

- 👍 very simple methods
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- work-efficient methods with better parallel time possible?
  - $\rightsquigarrow$ must genuinely parallelize the matching process! (and the preprocessing of the pattern)
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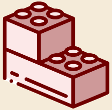
Parallel string matching – Discussion

- very simple methods
- could even run distributed with access to part of T
- parallel speedup only for $m \ll n$

- work-efficient methods with better parallel time possible?
  - $\rightsquigarrow$ must genuinely parallelize the matching process! (and the preprocessing of the pattern)
  - $\rightsquigarrow$ needs new ideas (much more complicated, but possible!)
- **Parallel string matching – State of the art:**
  - $O(\log m)$ time & work-efficient parallel string matching (very complicated)
    - this is optimal for CREW-PRAM
  - on CRCW-PRAM: matching part even in $O(1)$ time (easy)
but preprocessing requires $\Theta(\log \log m)$ time (very complicated)

5.3 Parallel primitives

Building blocks



- Most nontrivial problems need tricks to be parallelized
- Some versatile building blocks are known that help in many problems

$\rightsquigarrow$ We study some of them now, before we apply them to *parallel sorting*

Prefix sums

Prefix-sum problem (also: cumulative sums, running totals)

- Given: array $A[0..n)$ of numbers
- Goal: compute all prefix sums $A[0] + \cdots + A[i]$ for $i = 0, \ldots, n-1$
  may be done "in-place", i. e., by overwriting A

Example:

input:	3	0	0	5	7	0	0	2	0	0	0	4	0	8	0	1
output:	3	3	3	8	15	15	15	17	17	17	17	21	21	29	29	30

Clicker Question

What is the *sequential* running time achievable for prefix sums?

A $O(n^3)$

B $O(n^2)$

C $O(n \log n)$

D $O(n)$

E $O(\sqrt{n})$

F $O(\log n)$

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Clicker Question

What is the *sequential* running time achievable for prefix sums?

- **A** $\cancel{O(n^3)}$
- **B** $\cancel{O(n^2)}$
- **C** $\cancel{O(n \log n)}$
- **D** $O(n)$ ✓
- **E** $\cancel{O(\sqrt{n})}$
- **F** $\cancel{O(\log n)}$

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Prefix sums – Sequential

- sequential solution does $n-1$ additions
- but: cannot parallelize them!
  ↯ data dependencies!

⇝ need a different approach

```
procedure prefixSum(A[0..n])
    for i := 1, ..., n − 1 do
        A[i] := A[i − 1] + A[i]
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Let's try a simpler problem first.

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Excursion: Sum

- Given: array $A[0..n)$ of numbers
- Goal: compute $A[0] + A[1] + \cdots + A[n-1]$
  (solved by prefix sums)

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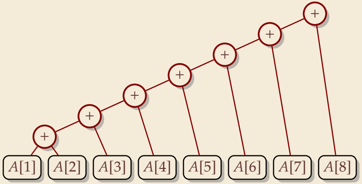

Excursion: Sum

- Given: array $A[0..n)$ of numbers
- Goal: compute $A[0] + A[1] + \cdots + A[n-1]$
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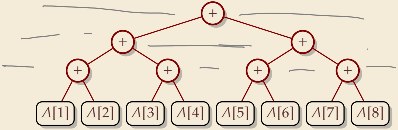
Any algorithm *must* do $n-1$ binary additions

⇝ Height of tree = parallel time!



time

Parallel prefix sums

- Idea: Compute all prefix sums with balanced trees in parallel
  Remember partial results for reuse

input:

3	0	0	5	7	0	0	2	0	0	0	4	0	8	0	1

Parallel prefix sums

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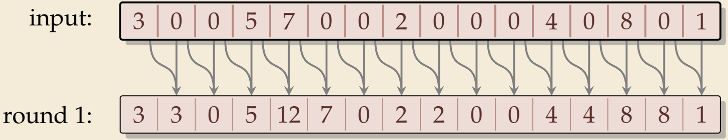

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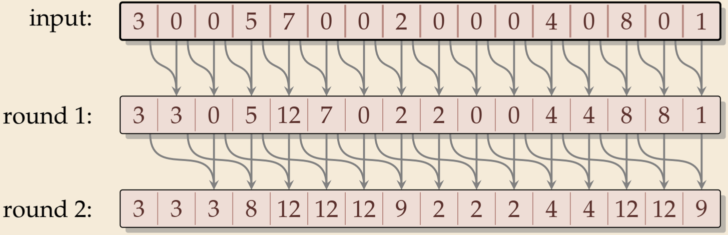

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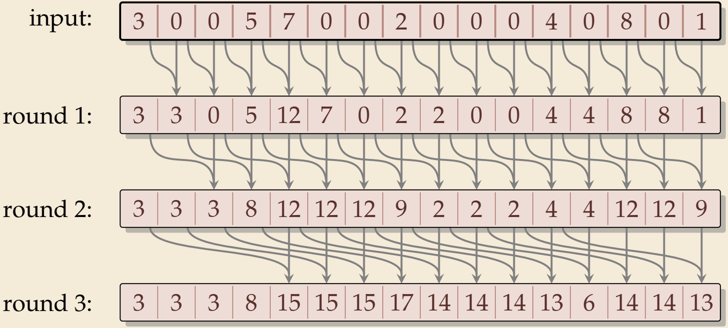

Parallel prefix sums

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input:	3	0	0	5	7	0	0	2	0	0	0	4	0	8	0	1	− 1
round 1:	3	3	0	5	12	7	0	2	2	0	0	4	4	8	8	1	− 2
round 2:	3	3	3	8	12	12	12	9	2	2	2	4	4	12	12	9	− 4
round 3:	3	3	3	8	15	15	15	17	14	14	14	13	6	14	14	13	− 8
round 4:	3	3	3	8	15	15	15	17	17	17	17	21	21	29	29	30	

Parallel prefix sums

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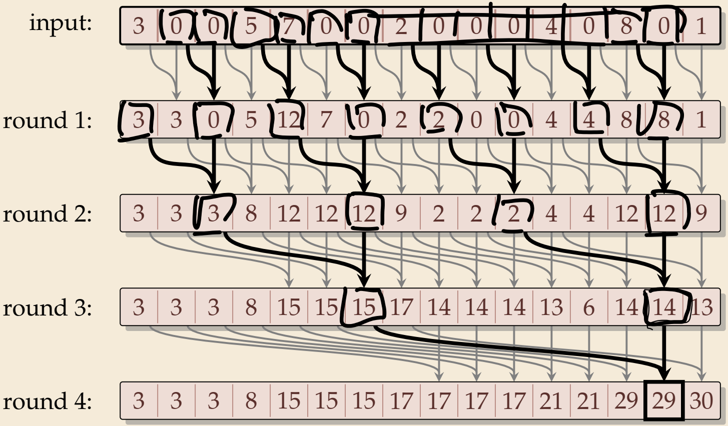

Parallel prefix sums – Code

- can be realized in-place (overwriting A)
- assumption: in each parallel step, all reads precede all writes

procedure parallelPrefixSums($A[0..n)$)
    for $r := 1, \ldots \lceil \lg n \rceil$ **do**
        $\mathit{step} := 2^{r-1}$
        for $i := \mathit{step}, \ldots n-1$ **do in parallel**
            $x := A[i] + A[i - \mathit{step}]$
            $A[i] := x$
        end parallel for
    end for

$O(\log n)$

$O(1)$

$O(1)$

$O(n)$

$O(n \log n)$

Parallel prefix sums – Analysis

- **Time:**
  - all additions of one round run in parallel
  - $\lceil \lg n \rceil$ rounds
  - $\rightsquigarrow$ $\Theta(\log n)$ time   best possible!   (from sum)
- **Work:**
  - $\geq \frac{n}{2}$ additions in all rounds (except maybe last round)
  - $\rightsquigarrow$ $\Theta(n \log n)$ work
  - more than the $\Theta(n)$ sequential algorithm!

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  - ⇝ $\Theta(n \log n)$ work
  - more than the $\Theta(n)$ sequential algorithm!
- Typical trade-off: greater parallelism at the expense of more overall work
- For prefix sums:
  - can actually get $\Theta(n)$ work in *twice* that time!
  - ⇝ algorithm is slightly more complicated
  - instead here: linear work in *thrice* the time using "blocking trick"

Work-efficient parallel prefix sums

recall string matching!

standard trick to improve work: compute small blocks sequentially

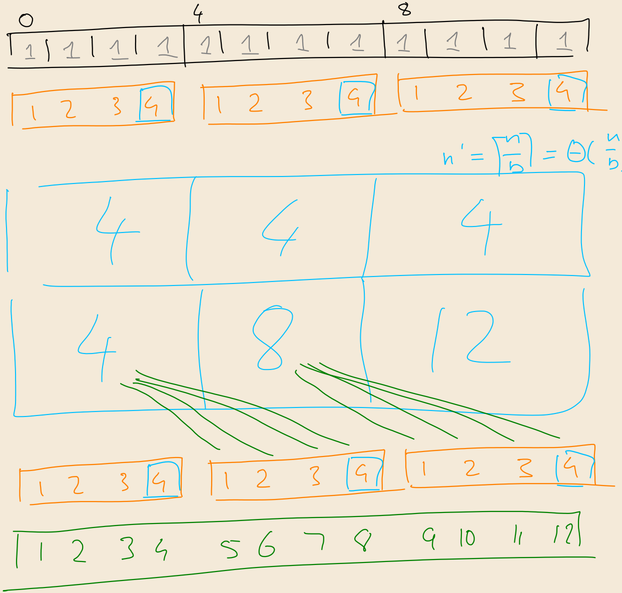
1. Set $b := \lceil \lg n \rceil$
2. For blocks of b consecutive indices, i. e., $A[0..b)$, $A[b..2b)$, ... **do in parallel**:
   - compute local prefix sums with fast **sequential** algorithm
3. Use previous work-inefficient parallel algorithm only on **rightmost elements** of block, i. e., to compute prefix sums of $A[b-1]$, $A[2b-1]$, $A[3b-1]$, ...
4. For blocks $A[0..b)$, $A[b..2b)$, ... do in parallel:
   Add block-prefix sums to local prefix sums

Analysis:

- **Time:**
  - 2. & 4.: $\Theta(b) = \Theta(\log n)$ time
  - 3. $\Theta(\log(n/b)) = \Theta(\log n)$ time
- **Work:**
  - 2. & 4.: $\Theta(b)$ per block $\times$ $\lceil \frac{n}{b} \rceil$ blocks $\rightsquigarrow$ $\Theta(n)$
  - 3. $\Theta\big(\frac{n}{b} \log(\frac{n}{b})\big) = \Theta(n)$

$n = 12$

$b = 4$



$n' = \lceil \frac{n}{b} \rceil = \Theta(\frac{n}{b}) = \Theta(\frac{n}{\log n})$

time	work
$\Theta(b) = \Theta(\log n)$	$\Theta(b)$ per block $\times \frac{n}{b}$ blocks $= \Theta(n)$
$\Theta(\log n') = \Theta(\log n)$	$\Theta(n' \log n') = \Theta(\frac{n}{\cancel{\log n}} \cancel{\log n}) = \Theta(n)$
$\Theta(b)$	$\Theta(\frac{n}{b} \cdot b) = \Theta(n)$

Compacting subsequences

How do prefix sums help with sorting? one more step to go . . .

Goal: *Compact* a subsequence of an array

B:	**1**	0	0	**1**	0	0	0	**1**	**1**	**1**	**1**	0	0	**1**	0	0	0
A:	3	8	9	1	15	13	14	7	2	4	6	17	12	5	10	11	16

S:	3	1	7	2	4	6	5

Compacting subsequences

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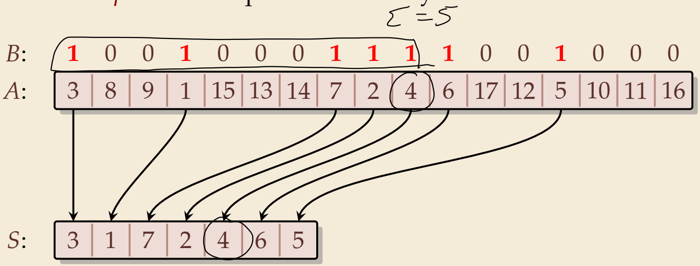

Use prefix sums on bitvector B

⇝ offset of selected cells in S

```
C := B // deep copy of B
parallelPrefixSums(C)
for j := 0, ..., n − 1 do in parallel
    if B[j] == 1 then S[C[j] − 1] := A[j]
end parallel for
```

Clicker Question

What is the parallel time and work achievable for *compacting* a subsequence of an array of size n?

A $O(1)$ time, $O(n)$ work

B $O(\log n)$ time, $O(n)$ work

C $O(\log n)$ time, $O(n \log n)$ work

D $O(\log^2 n)$ time, $O(n^2)$ work

E $O(n)$ time, $O(n)$ work

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