3 Efficient Sorting

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COMP526 (Spring 2022)
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Learning Outcomes

1. Know principles and implementation of *mergesort*.
2. Know principles and implementation of *quicksort*.
3. Know properties and *performance characteristics* of mergesort and quicksort.
4. Know the comparison model and understand the corresponding *lower bound*.
5. Understand *counting sort* and how it circumvents the comparison lower bound.
6. Understand and use the *parallel random-access-machine* model in its different variants.
7. Be able to *analyze* and compare simple shared-memory parallel algorithms by determining *parallel time and work*.
8. Understand efficient parallel *prefix sum* algorithms.
9. Be able to devise high-level description of *parallel quicksort and mergesort* methods.

Unit 3: *Efficient Sorting*

Outline

3 Efficient Sorting

Why study sorting?

- fundamental problem of computer science that is still not solved

  Algorithm with optimal #comparisons in worst case?

- building brick of many more advanced algorithms
  - for preprocessing
  - as subroutine
- playground of manageable complexity
  to practice algorithmic techniques

Here:

- “classic” fast sorting method
- exploit **partially sorted** inputs
- **parallel** sorting

Part I

The Basics

Rules of the game

- **Given:**
  - array $A[0..n) = A[0..n-1]$ of n objects
  - a total order relation $\leq$ among $A[0], \ldots, A[n-1]$
    (a comparison function)
    Python: elements support <= operator (`__lt__()`)
    Java: `Comparable` class (`x.compareTo(y) <= 0`)
- **Goal:** rearrange (i. e., permute) elements within A,
  so that A is *sorted*, i. e., $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

- for now: A stored in main memory (*internal sorting*)
  single processor (*sequential sorting*)

Clicker Question

What is the complexity of sorting? Type you answer, e. g., as "`Theta(sqrt(n))`"

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3.1 Mergesort

Clicker Question

How does mergesort work?

- **A** Split elements around median, then recurse on small / large elements.
- **B** Recurse on left / right half, then combine sorted halves.
- **C** Grow sorted part on left, repeatedly add next element to sorted range.
- **D** Repeatedly choose 2 elements and swap them if they are out of order.
- **E** Don't know.

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Merging sorted lists

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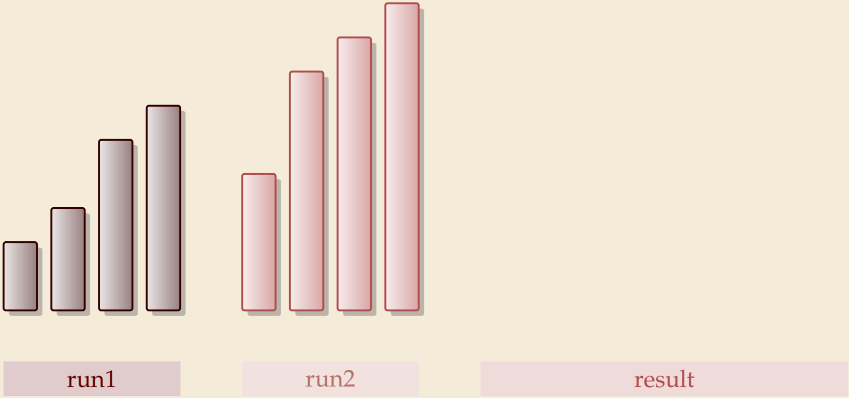

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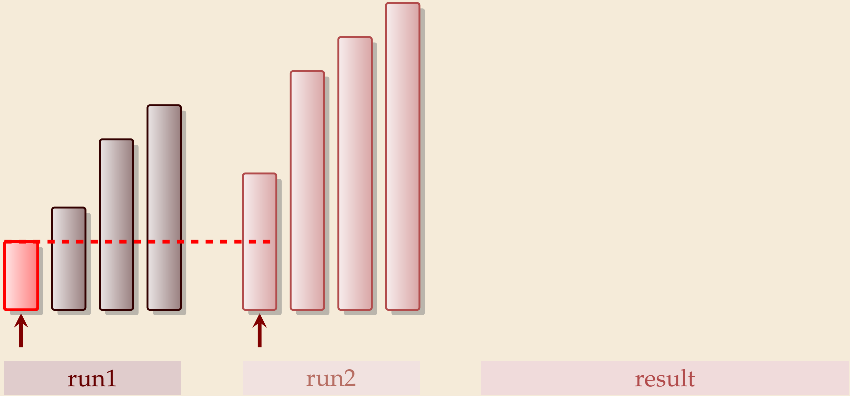

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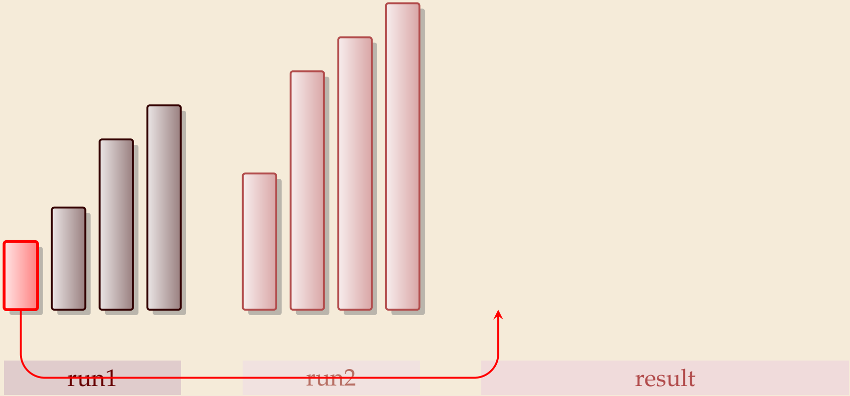

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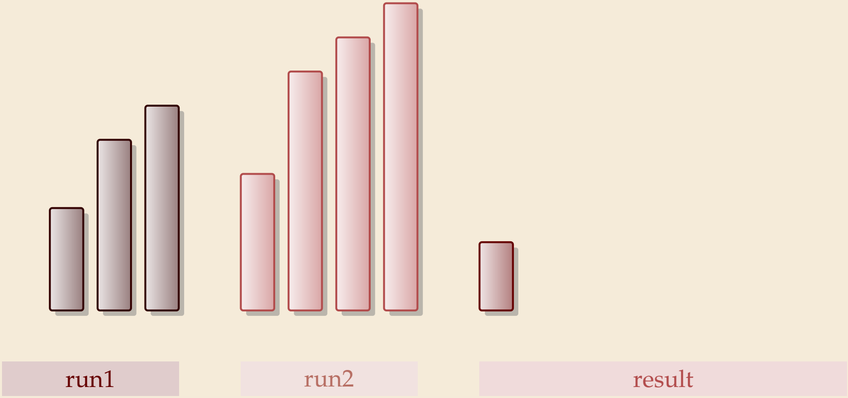

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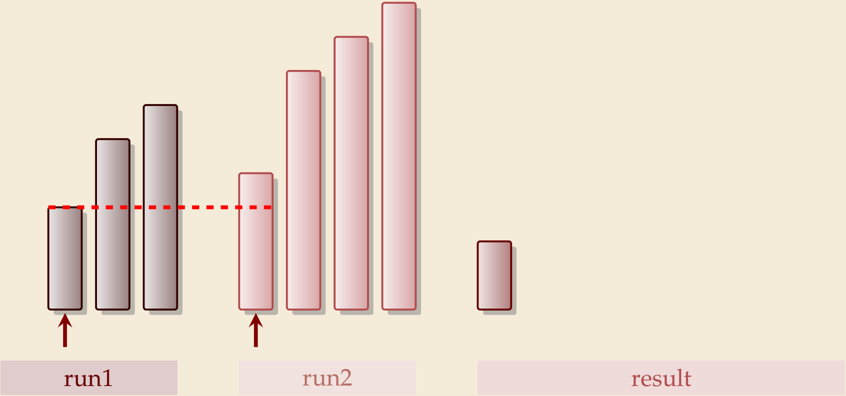

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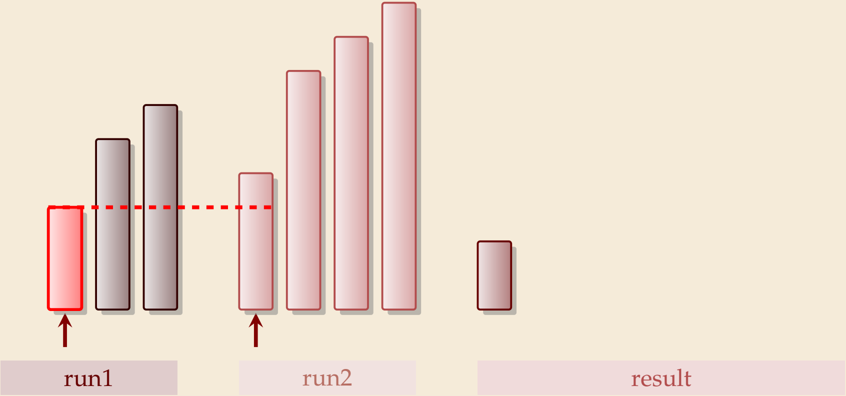

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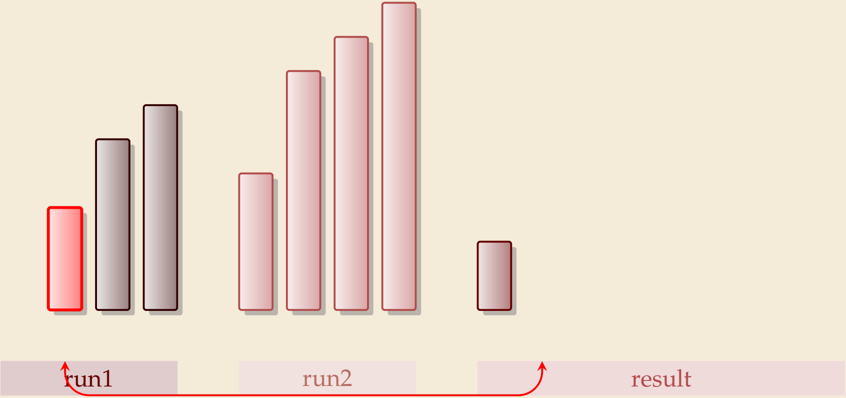

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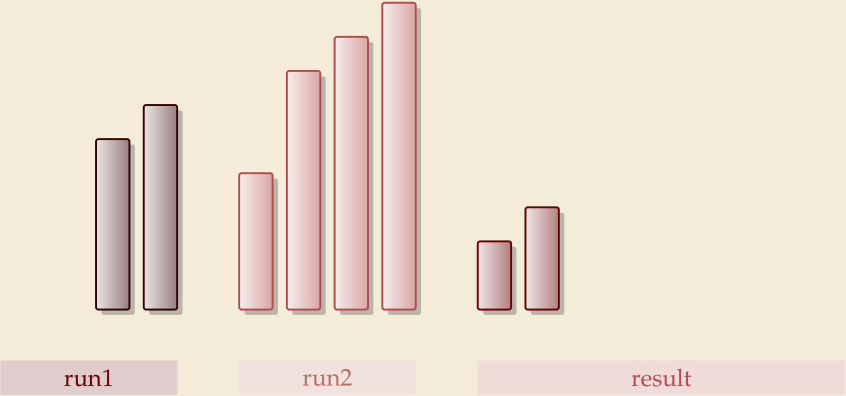

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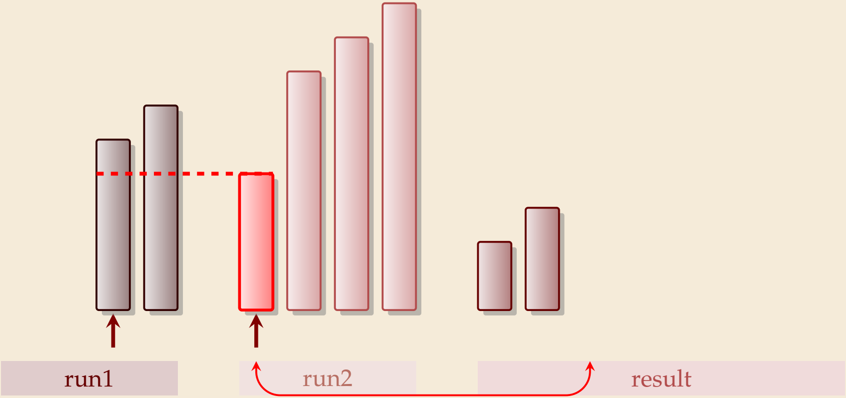

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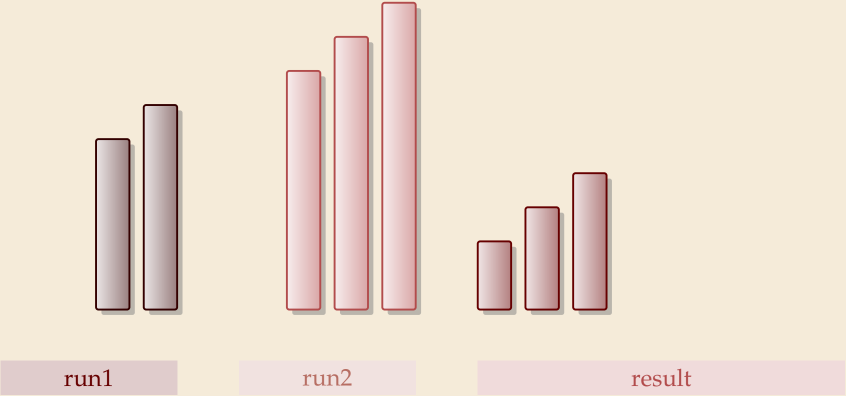

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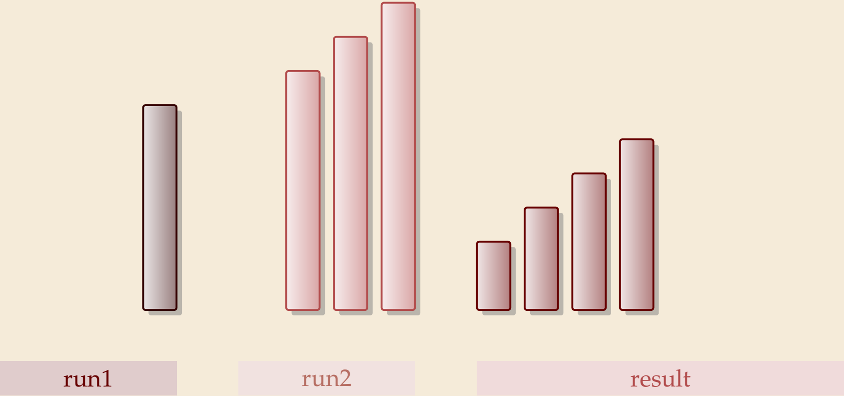

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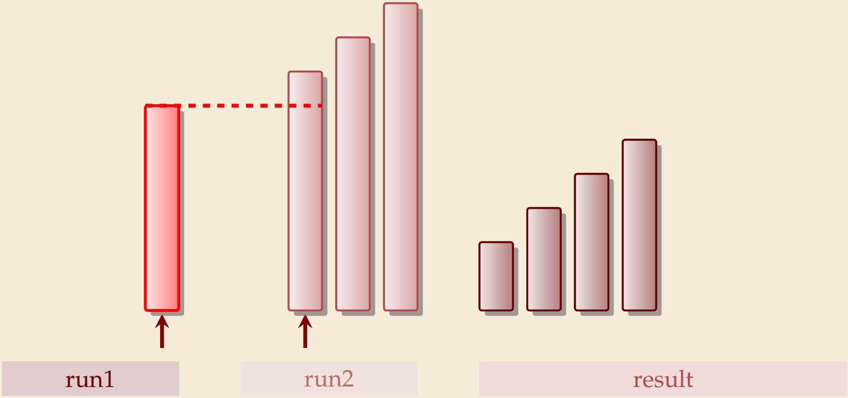

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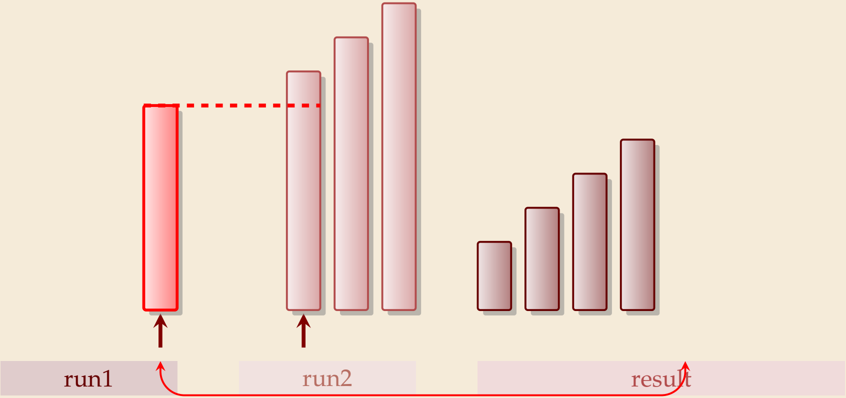

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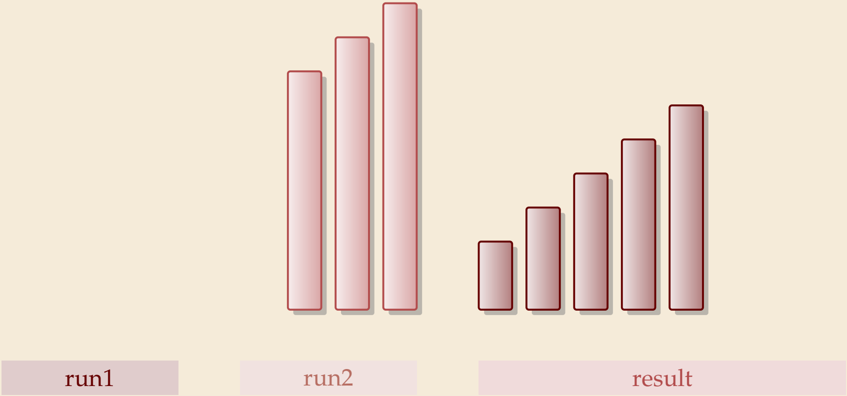

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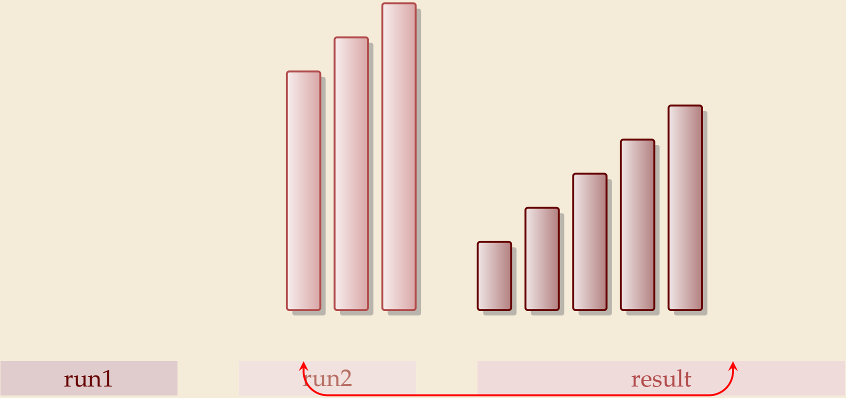

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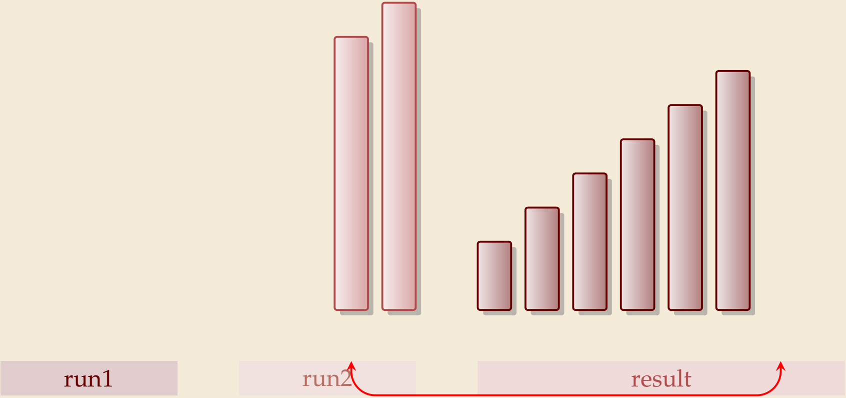

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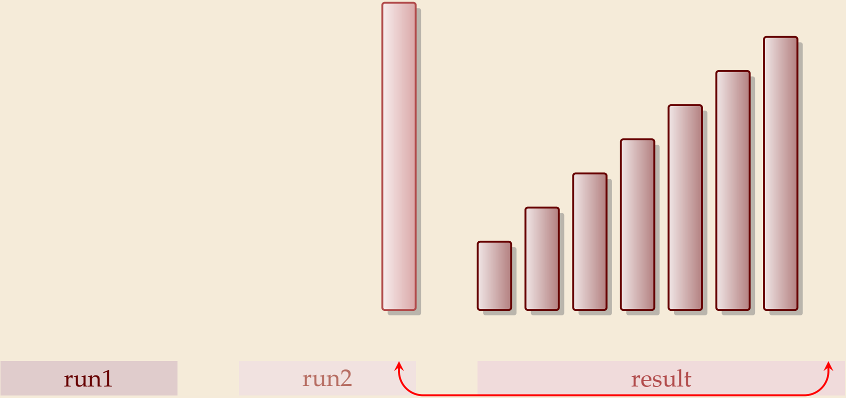

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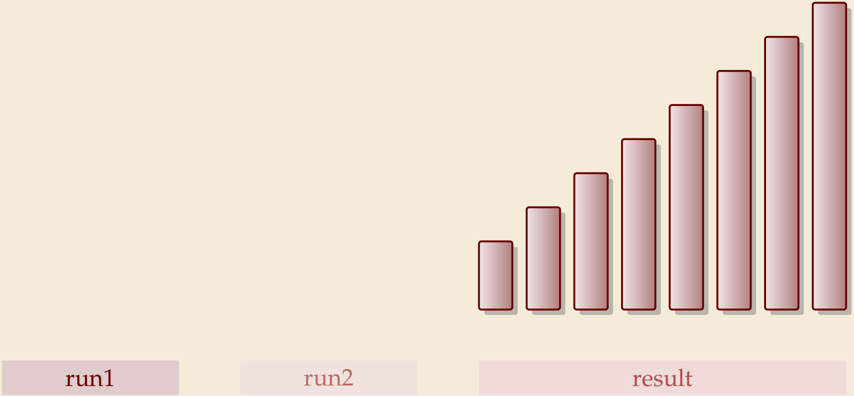

Clicker Question

What is the worst-case running time of mergesort?

A $\Theta(1)$

B $\Theta(\log n)$

C $\Theta(\log \log n)$

D $\Theta(\sqrt{n})$

E $\Theta(n)$

F $\Theta(n \log \log n)$

G $\Theta(n \log n)$

H $\Theta(n \log^2 n)$

I $\Theta(n^{1+\epsilon})$

J $\Theta(n^2)$

K $\Theta(n^3)$

L $\Theta(2^n)$

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- **F** $\cancel{\Theta(n \log\log n)}$
- **G** $\Theta(n \log n)$ $\checkmark$
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Mergesort

```
procedure mergesort(A[l..r))
    n := r − l
    if n ≤ 1 return
    m := l + ⌊n/2⌋
    mergesort(A[l..m))
    mergesort(A[m..r))
    merge(A[l..m), A[m..r), buf)
    copy buf to A[l..r)
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- recursive procedure; *divide & conquer*
- merging needs
  - temporary storage for result of same size as merged runs
  - to read and write each element twice (once for merging, once for copying back)

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Analysis: count *"element visits"* (read and/or write)

$$C(n) = \begin{cases} 0 & n \le 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \ge 2 \end{cases}$$

same for best and worst case!

$n = 2^k$

$k = \lg(n)$

Simplification $\boxed{n = 2^k}$

$$C(2^k) = \begin{cases} 0 & k \le 0 \\ 2 \cdot C(2^{k-1}) + 2 \cdot 2^k & k \ge 1 \end{cases} = \overbrace{2 \cdot 2^k + 2^2 \cdot 2^{k-1} + 2^3 \cdot 2^{k-2} + \cdots + 2^k \cdot 2^1}^{k \text{ summands}} = 2k \cdot 2^k$$

$k+1$

$$C(n) = 2n \lg(n) = \Theta(n \log n)$$

Mergesort – Discussion

- optimal time complexity of $\Theta(n \log n)$ in the worst case
- *stable* sorting method i. e., retains relative order of equal-key items
- memory access is sequential (scans over arrays)
- requires $\Theta(n)$ extra space

there are in-place merging methods,
but they are substantially more complicated
and not (widely) used

3.2 Quicksort

Clicker Question

How does quicksort work?

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Partitioning around a pivot

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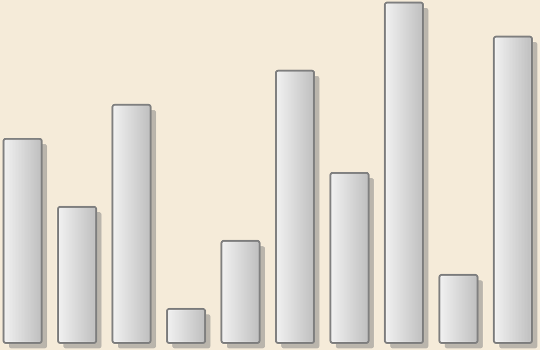

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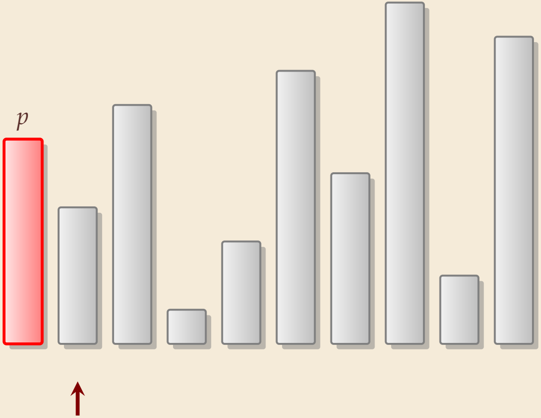

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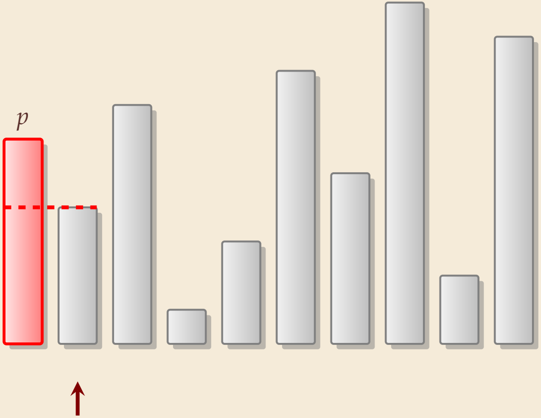

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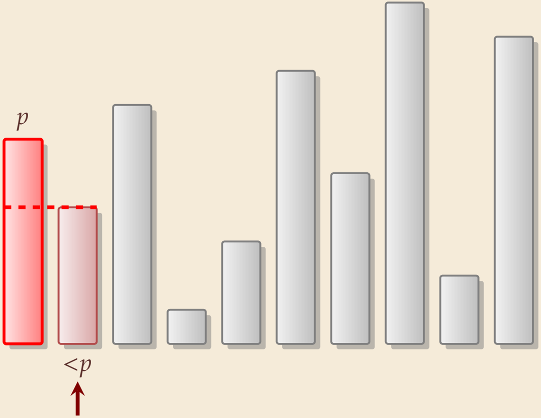

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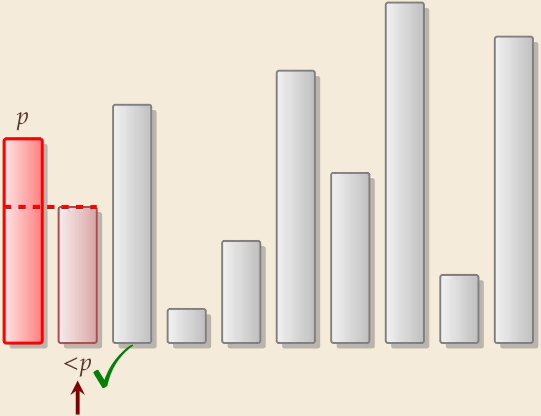

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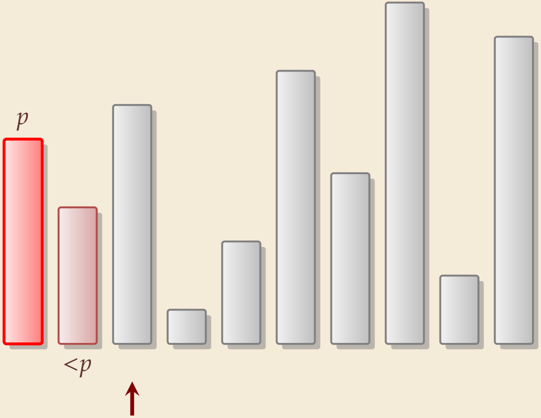

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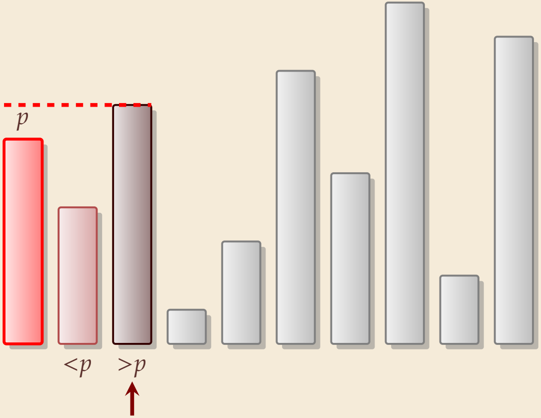

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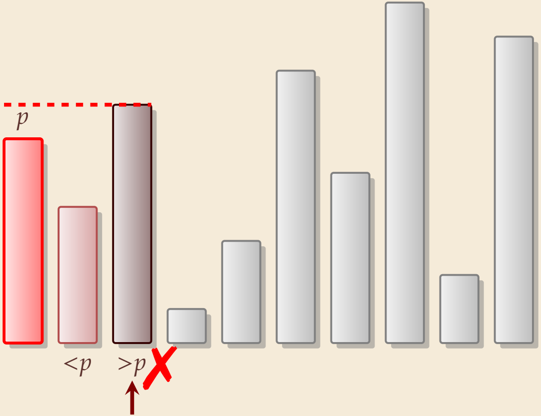

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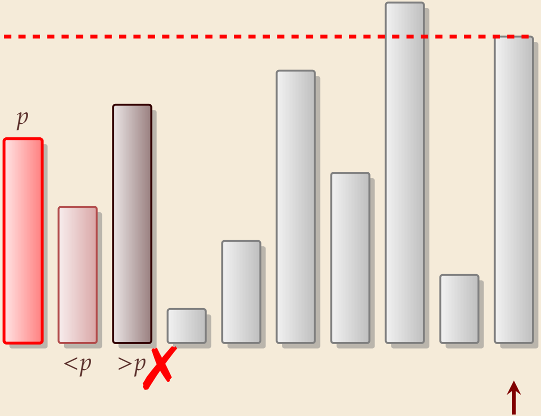

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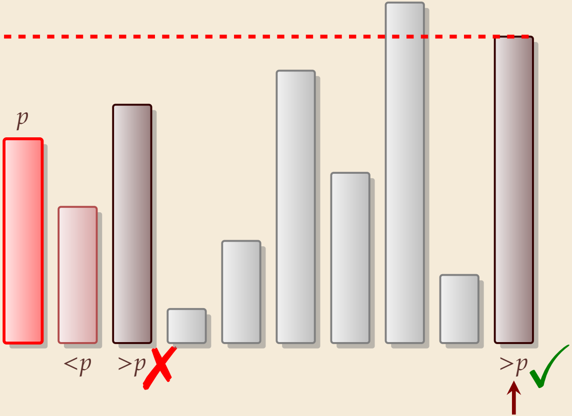

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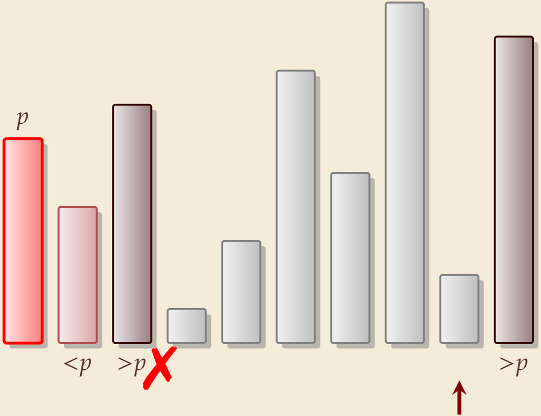

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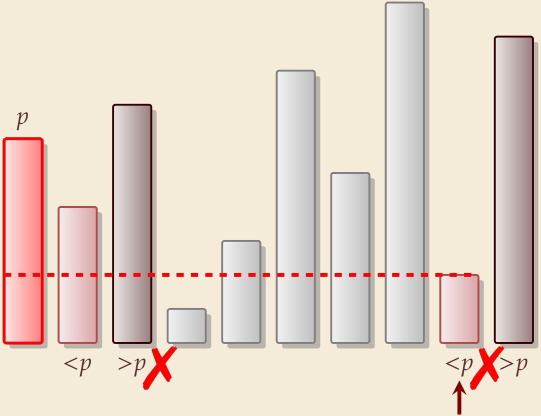

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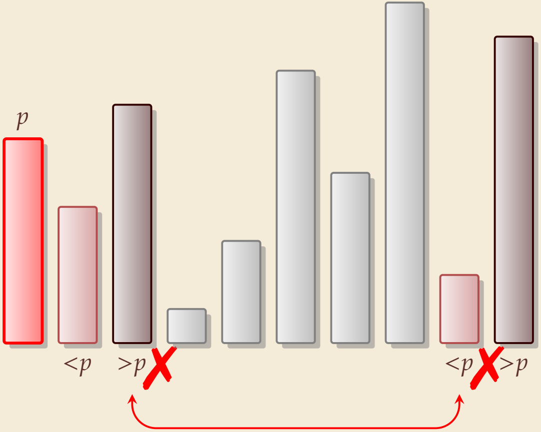

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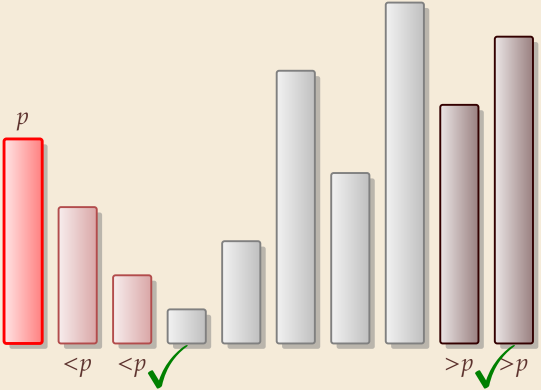

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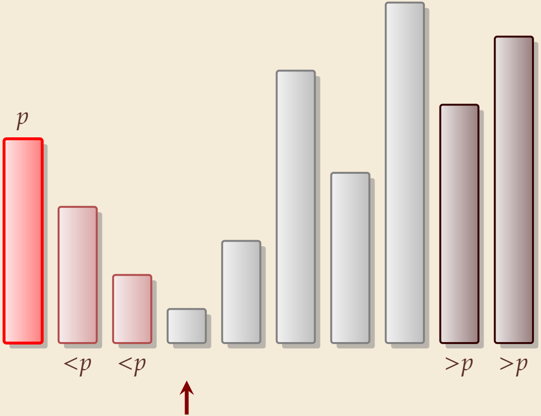

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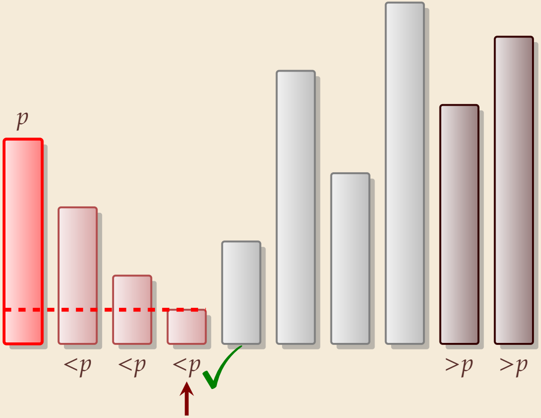

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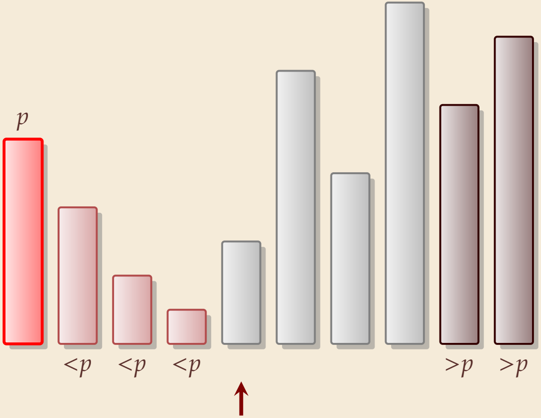

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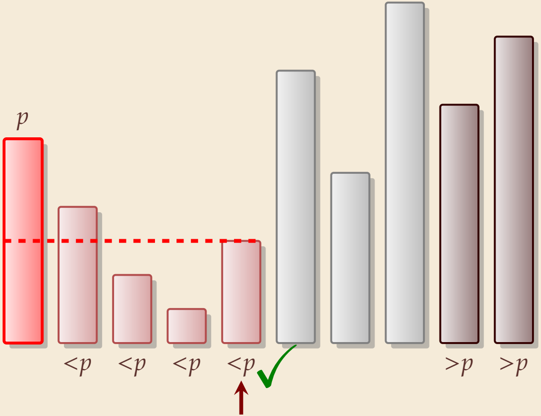

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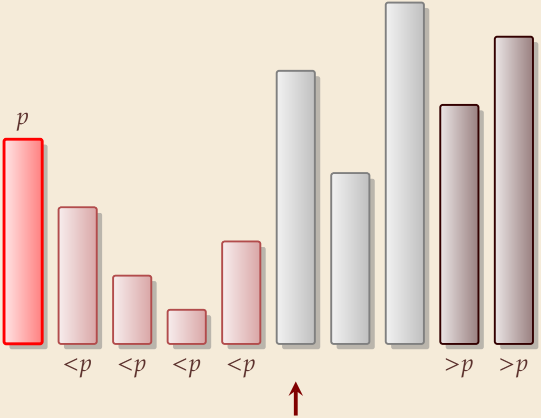

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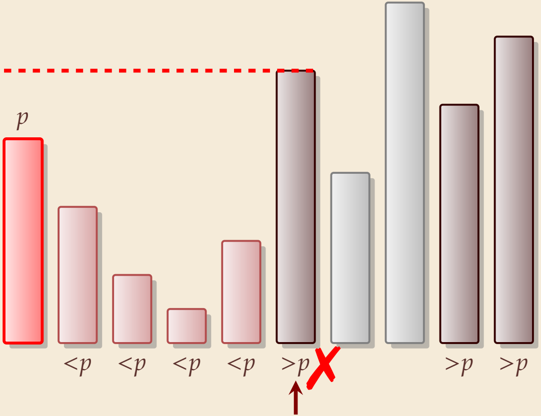

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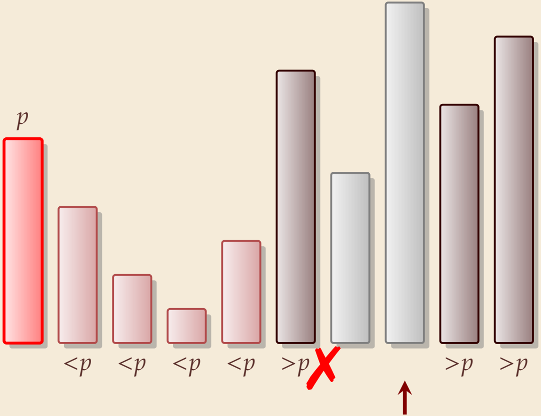

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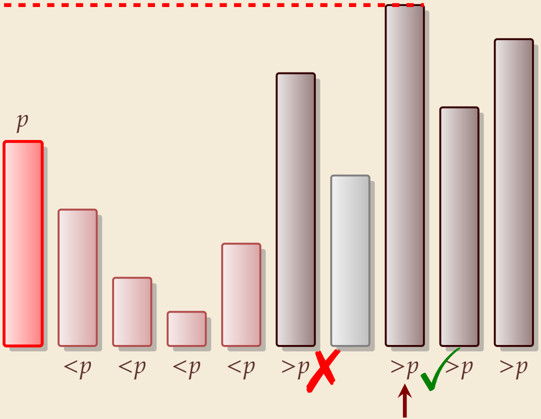

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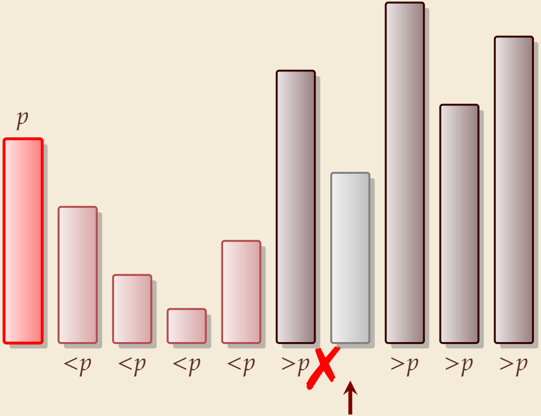

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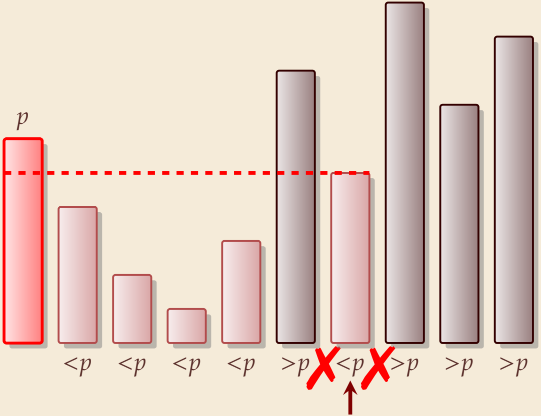

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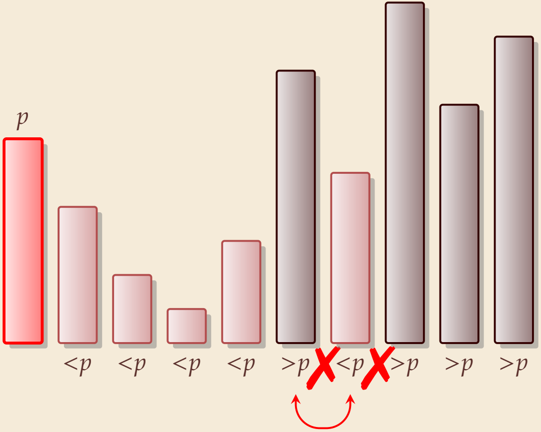

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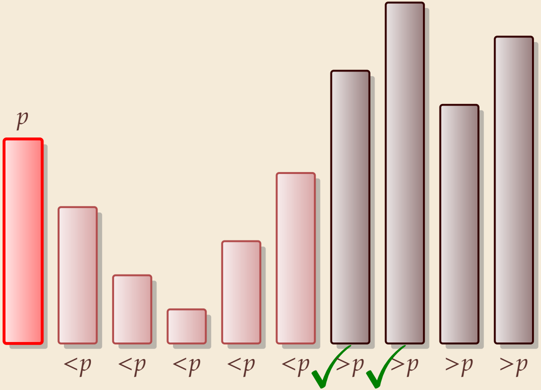

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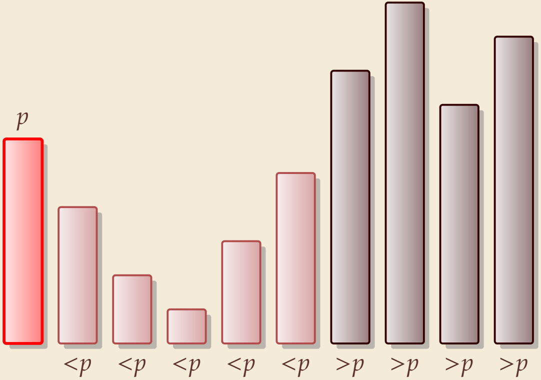

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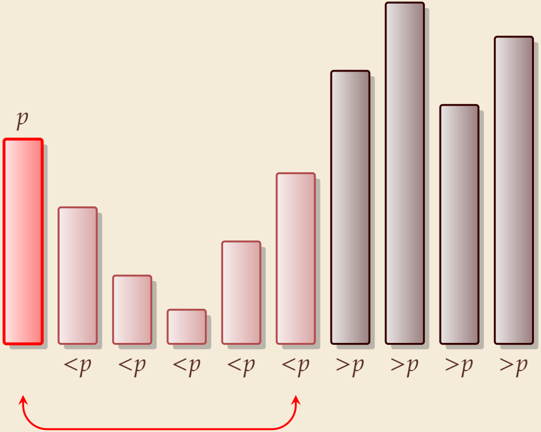

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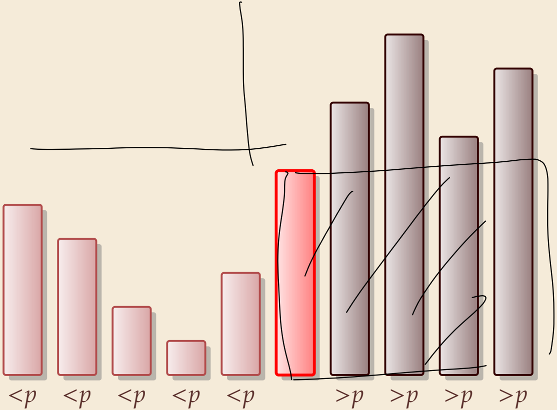

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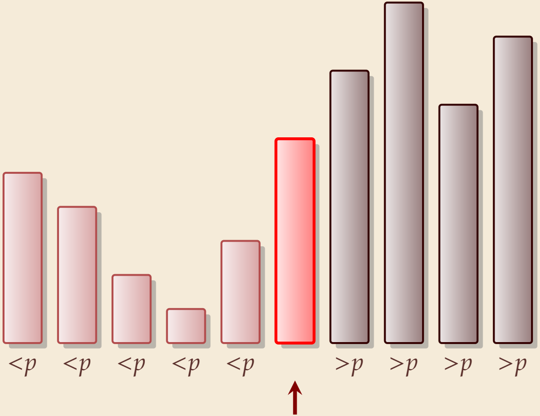


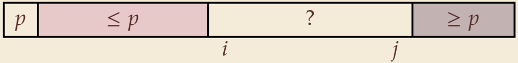
- no extra space needed
- visits each element once
- returns rank/position of pivot

Partitioning – Detailed code

Beware: details easy to get wrong; use this code! (if you ever have to)

```
1  procedure partition(A, b)
2      // input: array A[0..n), position of pivot b ∈ [0..n)
3      swap(A[0], A[b])
4      i := 0,    j := n
5      while true do
6          do i := i + 1 while i < n and A[i] < A[0]
7          do j := j − 1 while j ≥ 1 and A[j] > A[0]
8          if i ≥ j then break    (goto 11)
9          else swap(A[i], A[j])
10     end while
11     swap(A[0], A[j])
12     return j
```

Loop invariant (5–10): A



Quicksort

```
procedure quicksort(A[l..r))
    if r − ℓ ≤ 1 then return
    b := choosePivot(A[l..r))
    j := partition(A[l..r), b)
    quicksort(A[l..j))
    quicksort(A[j + 1..r))
```

- recursive procedure; *divide & conquer*
- choice of pivot can be
  - fixed position $\rightsquigarrow$ dangerous!
  - random
  - more sophisticated, e. g., median of 3

Clicker Question

What is the worst-case running time of quicksort?

- **A** $\Theta(1)$
- **B** $\Theta(\log n)$
- **C** $\Theta(\log \log n)$
- **D** $\Theta(\sqrt{n})$
- **E** $\Theta(n)$
- **F** $\Theta(n \log \log n)$
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- **I** $\Theta(n^{1+\epsilon})$
- **J** $\Theta(n^2)$
- **K** $\Theta(n^3)$
- **L** $\Theta(2^n)$

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Clicker Question

What is the worst-case running time of quicksort?

A $\cancel{\Theta(1)}$

B $\cancel{\Theta(\log n)}$

C $\cancel{\Theta(\log \log n)}$

D $\cancel{\Theta(\sqrt{n})}$

E $\cancel{\Theta(n)}$

F $\cancel{\Theta(n \log \log n)}$

G $\cancel{\Theta(n \log n)}$

H $\cancel{\Theta(n \log^2 n)}$

I $\cancel{\Theta(n^{1+\epsilon})}$

J $\Theta(n^2)$ $\checkmark$

K $\cancel{\Theta(n^3)}$

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Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

Quicksort & Binary Search Trees

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7	4	2	9	1	3	8	5	6

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6

9	8

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6	7	9	8

Quicksort & Binary Search Trees

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7	4	2	9	1	3	8	5	6

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7

9	8

Quicksort & Binary Search Trees

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7	4	2	9	1	3	8	5	6

$\boxed{4}$ 2 1 3 5 6 | 7 | 9 8

2 1 3 | 4 | 5 6

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6

7

9	8

2	1	3

4

5	6

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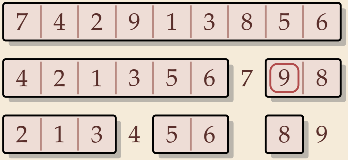
Quicksort



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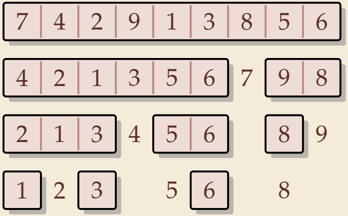

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7	4	2	9	1	3	8	5	6

4	2	1	3	5	6

7

9	8

2	1	3

4

5	6

8

9

1

2

3

5

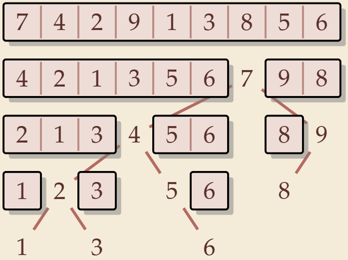
6

8

1 3 6

Quicksort & Binary Search Trees

Quicksort



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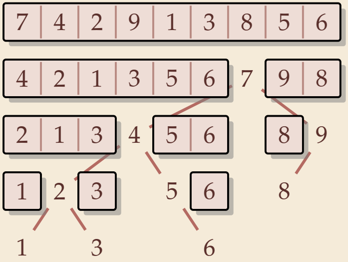
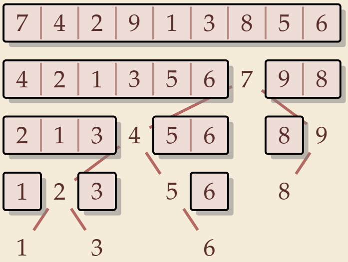


Binary Search Tree (BST)

7 4 2 9 1 3 8 5 6

Quicksort & Binary Search Trees

Quicksort



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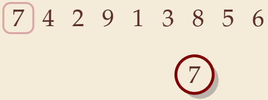

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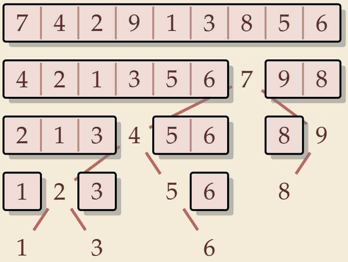

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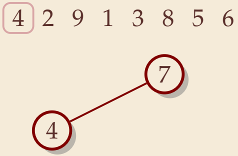

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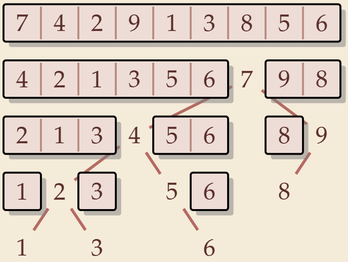

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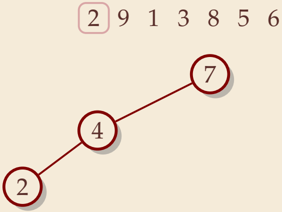

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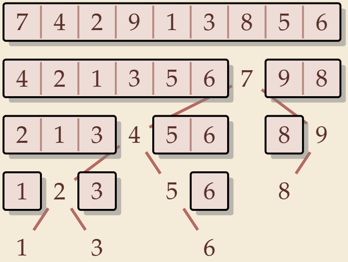

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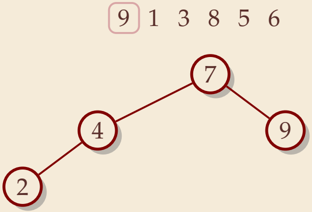

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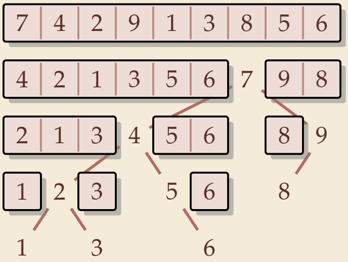

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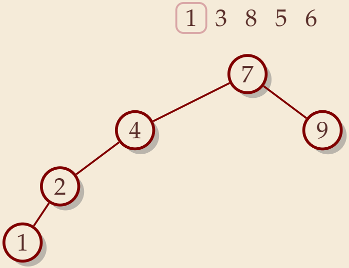

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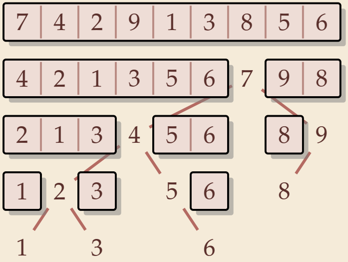

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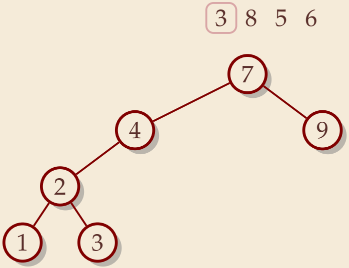

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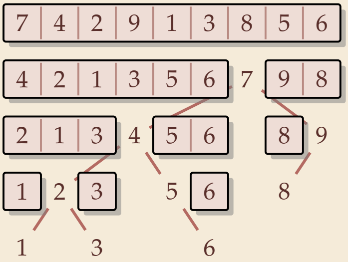

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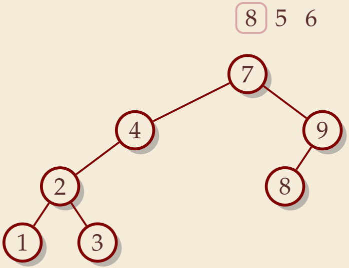

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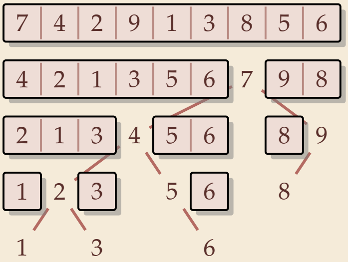
Quicksort



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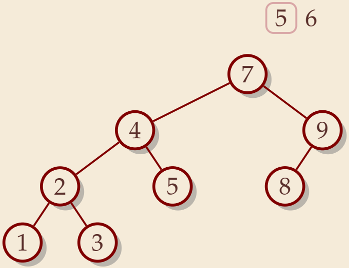

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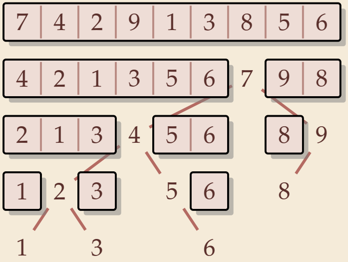

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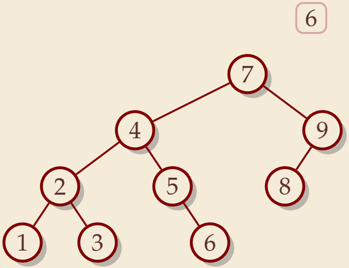

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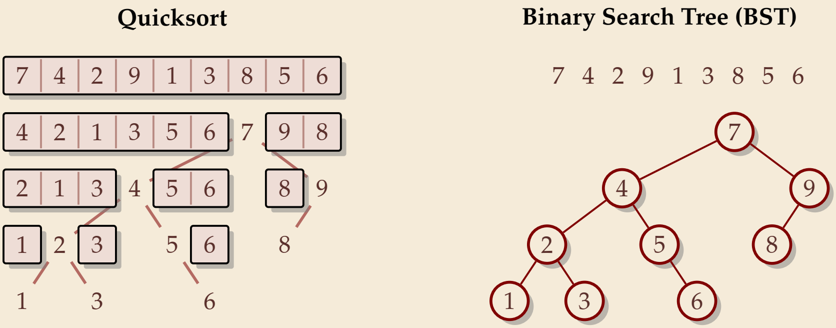
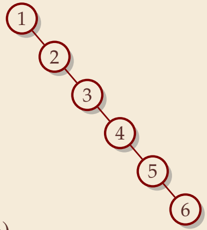


- recursion tree of quicksort $=$ binary search tree from successive insertion
- comparisons in quicksort $=$ comparisons to built BST
- comparisons in quicksort $\approx$ comparisons to search each element in BST

Quicksort – Worst Case

- Problem: BSTs can degenerate
- Cost to search for k is $k - 1$

$\rightsquigarrow$ Total cost $\displaystyle\sum_{k=1}^{n}(k-1) = \frac{n(n-1)}{2} \sim \frac{1}{2}n^2$



$\rightsquigarrow$ quicksort worst-case running time is in $\Theta(n^2)$

terribly slow!

But, we can fix this:

Randomized quicksort:

- choose a *random pivot* in each step

$\rightsquigarrow$ same as randomly *shuffling* input before sorting

Randomized Quicksort – Analysis

- $C(n)$ = element visits (as for mergesort)

$\rightsquigarrow$ quicksort needs $\sim 2\ln(2) \cdot n \lg n \approx 1.39 n \lg n$ *in expectation*

- also: very unlikely to be much worse:
  e. g., one can prove: $\Pr[\text{cost} > 10 n \lg n] = O(n^{-2.5})$
  distribution of costs is "concentrated around mean"
- intuition: have to be *constantly* unlucky with pivot choice

Quicksort – Discussion

- 👍 fastest general-purpose method
- 👍 $\Theta(n \log n)$ average case
- 👍 works *in-place* (no extra space required)
- 👍 memory access is sequential (scans over arrays)
- 👎 $\Theta(n^2)$ worst case (although extremely unlikely)
- 👎 not a *stable* sorting method

Open problem: Simple algorithm that is fast, stable and in-place.

3.3 Comparison-Based Lower Bound

Lower Bounds

- **Lower bound:** mathematical proof that *no algorithm* can do better.
  - very powerful concept: bulletproof *impossibility* result
    $\approx$ *conservation of energy* in physics
  - **(unique?) feature of computer science:**
    for many problems, solutions are known that (asymptotically) **achieve the lower bound**

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  $\rightsquigarrow$ can speak of "*optimal* algorithms"

- To prove a statement about *all algorithms*, we must precisely define what that is!

- already know one option: the word-RAM model

- Here: use a simpler, more restricted model.

The Comparison Model

- In the *comparison model* data can only be accessed in two ways:
  - comparing two elements
  - moving elements around (e. g. copying, swapping)
  - Cost: number of these operations.

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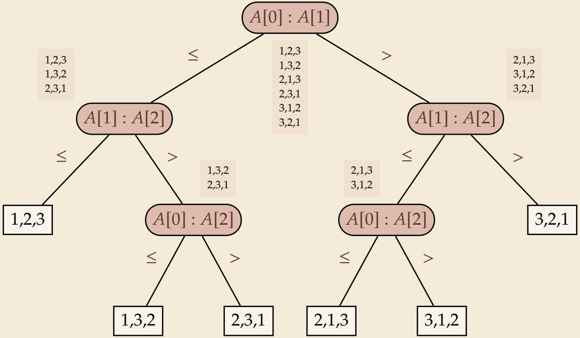
- Mergesort and Quicksort work in the comparison model.

$\rightsquigarrow$ Every comparison-based sorting algorithm corresponds to a ***decision tree***.

  - only model comparisons $\rightsquigarrow$ ignore data movement
  - nodes $=$ comparisons the algorithm does
  - next comparisons can depend on outcomes $\rightsquigarrow$ different subtrees
  - child links $=$ outcomes of comparison
  - leaf $=$ unique initial input permutation compatible with comparison outcomes

Comparison Lower Bound

Example: Comparison tree for a sorting method for $A[0..2]$:



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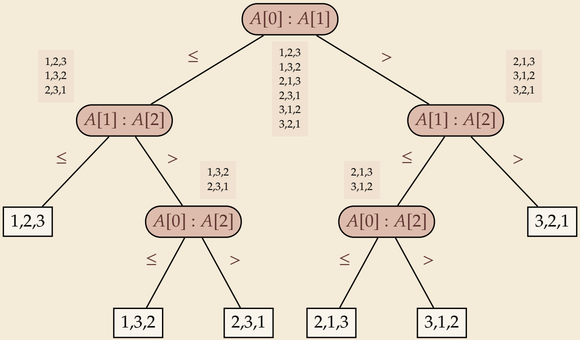

- Execution = follow a path in comparison tree.
- ⇝ height of comparison tree = worst-case # comparisons
- comparison trees are *binary* trees
- ⇝ ℓ leaves ⇝ height $\ge \lceil \lg(\ell) \rceil$
- comparison trees for sorting method must have $\ge n!$ leaves
- ⇝ height $\ge \lg(n!) \sim n \lg n$

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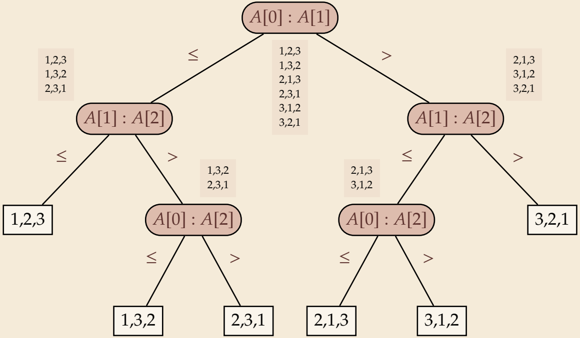

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more precisely: $\lg(n!) = n \lg n - \lg(e)n + O(\log n)$

- Mergesort achieves $\sim n \lg n$ comparisons $\rightsquigarrow$ asymptotically comparison-optimal!
- Open (theory) problem: Sorting algorithm with $n \lg n - \lg(e)n + o(n)$ comparisons?

≈ 1.4427

Clicker Question

Does the comparison-tree from the previous slide correspond to a worst-case optimal sorting method?

A Yes

B No

`sli.do/comp526`

Clicker Question

Does the comparison-tree from the previous slide correspond to a worst-case optimal sorting method?

A Yes ✓

B ~~No~~

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3.4 Integer Sorting

How to beat a lower bound

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- Here: sort n **integers**
  - can do *a lot* with integers: add them up, compute averages, . . . (full power of word-RAM)

  $\rightsquigarrow$ we are **not** working in the comparison model
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  - but: a priori unclear how much arithmetic helps for sorting . . .

Counting sort

- Important parameter: size/range of numbers
  - numbers in range $[0..U) = \{0, \ldots, U-1\}$ typically $U = 2^b$ $\rightsquigarrow$ b-bit binary numbers

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- We can sort n integers in $\boxed{\Theta(n+U)}$ time and $\Theta(U)$ space when $\boxed{b \le w}$:

word size

Counting sort

```
procedure countingSort(A[0..n))
    // A contains integers in range [0..U).
    C[0..U) := new integer array, initialized to 0
    // Count occurrences
    for i := 0, ..., n − 1
        C[A[i]] := C[A[i]] + 1
    i := 0 // Produce sorted list
    for k := 0, ... U − 1
        for j := 1, ... C[k]
            A[i] := k;  i := i + 1
```

- *count* how often each *possible* value occurs
- produce sorted result directly from counts
- circumvents lower bound by using integers as array index / pointer offset

$\rightsquigarrow$ Can sort n integers in range $[0..U)$ with $U = O(n)$ in time and space $\Theta(n)$.

Integer Sorting – State of the art

- $O(n)$ time sorting also possible for numbers in range $U = O(n^c)$ for constant c.
  - *radix sort* with radix 2^w

- **Algorithm theory**
  - suppose $U = 2^w$, but w can be an arbitrary function of n
  - how fast can we sort n such w-bit integers on a w-bit word-RAM?
    - for $w = O(\log n)$: linear time (*radix/counting sort*)
    - for $w = \Omega(\log^{2+\varepsilon} n)$: linear time (*signature sort*)
    - for w in between: can do $O(n\sqrt{\lg\lg n})$ (very complicated algorithm)
      don't know if that is best possible!

$\notin$ exam

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* * *

- for the rest of this unit: back to the comparisons model!