

4

String Matching – What's behind Ctrl+F?

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Outline

4 String Matching

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4.1 Introduction

Ubiquitous strings

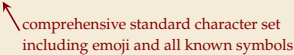
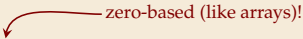
string = sequence of characters

- ▶ universal data type for ... everything!
 - ▶ natural language texts
 - ▶ programs (source code)
 - ▶ websites
 - ▶ XML documents
 - ▶ DNA sequences
 - ▶ bitstrings
 - ▶ ... a computer's memory \rightsquigarrow ultimately any data is a string

\rightsquigarrow many different tasks and algorithms

- ▶ This unit: finding (exact) **occurrences of a pattern** text.
 - ▶ Ctrl+F
 - ▶ grep
 - ▶ computer forensics (e. g. find signature of file on disk)
 - ▶ virus scanner
- ▶ basis for many advanced applications

Notations

- ▶ *alphabet* Σ : finite set of allowed **characters**; $\sigma = |\Sigma|$ “a string over alphabet Σ ”
 - ▶ letters (Latin, Greek, Arabic, Cyrillic, Asian scripts, ...)
 - ▶ “what you can type on a keyboard”, Unicode characters
 - ▶ $\{0, 1\}$; nucleotides $\{A, C, G, T\}$; ...

- ▶ $\Sigma^n = \Sigma \times \cdots \times \Sigma$: strings of **length** $n \in \mathbb{N}_0$ (n -tuples)
- ▶ $\Sigma^* = \bigcup_{n \geq 0} \Sigma^n$: set of **all** (finite) strings over Σ
- ▶ $\Sigma^+ = \bigcup_{n \geq 1} \Sigma^n$: set of **all** (finite) **nonempty** strings over Σ
- ▶ $\varepsilon \in \Sigma^0$: the *empty* string (same for all alphabets)
- ▶ for $S \in \Sigma^n$, write $S[i]$ (other sources: S_i) for *i***th** character ($0 \leq i < n$)

- ▶ for $S, T \in \Sigma^*$, write $ST = S \cdot T$ for **concatenation** of S and T
- ▶ for $S \in \Sigma^n$, write $S[i..j]$ or $S_{i,j}$ for the **substring** $S[i] \cdot S[i+1] \cdots S[j]$ ($0 \leq i \leq j < n$)
 - ▶ $S[0..j]$ is a **prefix** of S ; $S[i..n-1]$ is a **suffix** of S
 - ▶ $S[i..j) = S[i..j-1]$ (endpoint exclusive) $\rightsquigarrow S = S[0..n)$

String matching – Definition

Search for a string (pattern) in a large body of text

► Input:

- $T \in \Sigma^n$: The text (haystack) being searched within
- $P \in \Sigma^m$: The pattern (needle) being searched for; typically $n \gg m$

► Output:

- the *first occurrence (match)* of P in T : $\min\{i \in [0..n - m) : T[i..i + m) = P\}$
- or NO_MATCH if there is no such i (“ P does not occur in T ”)

► Variant: Find **all** occurrences of P in T .

↪ Can do that iteratively (update T to $T[i + 1..n)$ after match at i)

► Example:

- $T = \text{“Where is he?”}$
- $P_1 = \text{“he”} \rightsquigarrow i = 1$
- $P_2 = \text{“who”} \rightsquigarrow \text{NO_MATCH}$

► string matching is implemented in Java in `String.indexOf`

4.2 Brute Force

Abstract idea of algorithms

Pattern matching algorithms consist of *guesses* and *checks*:

- ▶ A **guess** is a position i such that P might start at $T[i]$.
Possible guesses (initially) are $0 \leq i \leq n - m$.
- ▶ A **check** of a guess is a pair (i, j) where we compare $T[i + j]$ to $P[j]$.
- ▶ Note: need all m checks to verify a single **correct** guess i ,
but it may take (many) fewer checks to recognize an **incorrect** guess.
- ▶ Cost measure: #character comparisons = #checks

\rightsquigarrow cost $\leq n \cdot m$ (number of possible checks)

Brute-force method

```
1 procedure bruteForceSM( $T[0..n]$ ,  $P[0..m]$ )  
2   for  $i := 0, \dots, n - m - 1$  do  
3     for  $j := 0, \dots, m - 1$  do  
4       if  $T[i + j] \neq P[j]$  then break inner loop  
5       if  $j == m$  then return  $i$   
6   return NO_MATCH
```

- try all guesses i
- check each guess (left to right); stop early on mismatch
- essentially the implementation in Java!

► **Example:**

$T = \text{abbbababbab}$

$P = \text{abba}$

↪ 15 char cmps
(vs $n \cdot m = 44$)
not too bad!

	a	b	b	b	a	b	a	b	b	a	b
a	a	b	b	a							
		a									
			a								
				a							
					a	b	b				
						a					
							a	b	b	a	

Brute-force method – Discussion



Brute-force method can be good enough

- ▶ typically works well for natural language text
- ▶ also for random strings



but: can be as bad as it gets!

	a	a	a	a	a	a	a	a	a	a
a	a	a	b							
	a	a	a	b						
		a	a	a	b					
			a	a	a	b				
				a	a	a	b			
					a	a	a	b		
						a	a	a	b	
							a	a	a	b

▶ Worst possible input: $P = a^{m-1}b$,
 $T = a^n$

▶ Worst-case performance: $(n - m + 1) \cdot m$

\rightsquigarrow for $m \leq n/2$ that is $\Theta(mn)$

- ▶ Bad input: lots of self-similarity in T ! \rightsquigarrow can we exploit that?
- ▶ brute force does ‘obviously’ stupid repetitive comparisons \rightsquigarrow can we avoid that?

Roadmap

- ▶ **Approach 1** (this week): Use *preprocessing* on the pattern P to eliminate guesses (avoid 'obvious' redundant work)
 - ▶ Deterministic finite automata (DFA)
 - ▶ Knuth-Morris-Pratt algorithm
 - ▶ Boyer-Moore algorithm
 - ▶ Rabin-Karp algorithm
- ▶ **Approach 2** (\rightsquigarrow Unit 6): Do preprocessing on the text T
Can find matches in time *independent of text size(!)*
 - ▶ inverted indices
 - ▶ Suffix trees
 - ▶ Suffix arrays

4.3 String Matching with Finite Automata

Theoretical Computer Science to the rescue!

- ▶ string matching = deciding whether $T \in \Sigma^* \cdot P \cdot \Sigma^*$
- ▶ $\Sigma^* \cdot P \cdot \Sigma^*$ is *regular* formal language
- $\rightsquigarrow \exists$ *deterministic finite automaton* (DFA) to recognize $\Sigma^* \cdot P \cdot \Sigma^*$
- \rightsquigarrow can check for occurrence of P in $|T| = n$ steps!



Job done!



WTF!?

We are not quite done yet.

- ▶ (Problem 0: programmer might not know automata and formal languages ...)
- ▶ Problem 1: existence alone does not give an algorithm!
- ▶ Problem 2: automaton could be very big!

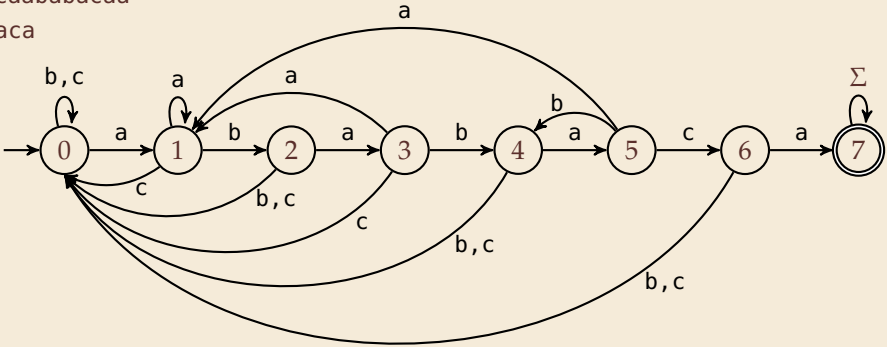
String matching with DFA

- ▶ Assume first, we already have a deterministic automaton
- ▶ How does string matching work?

Example:

$T = \text{aabacaababacaa}$

$P = \text{ababaca}$



text:		a	a	b	a	c	a	a	b	a	b	a	c	a	a
state:	0	1	1	2	3	0	1	1	2	3	4	5	6	7	7

String matching DFA – Intuition

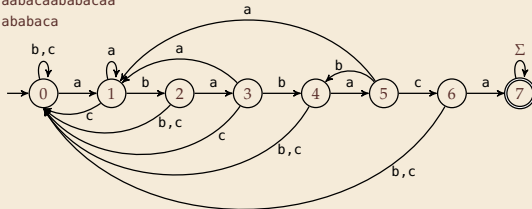
Why does this work?

► Main insight:

State q means:
“we have seen $P[0..q)$ until here
(but not any longer prefix of P)”

$T = \text{aabacaababacaa}$

$P = \text{ababaca}$



text:		a	a	b	a	c	a	a	b	a	b	a	c	a	a
state:	0	1	1	2	3	0	1	1	2	3	4	5	6	7	7

► If the next text character c does not match, we know:

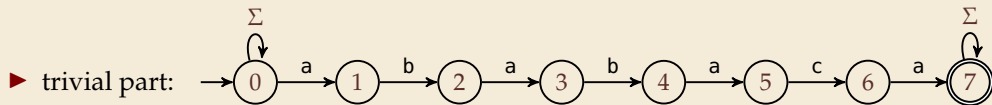
- (i) text seen so far ends with $P[0...q) \cdot c$
- (ii) $P[0...q) \cdot c$ is not a prefix of P
- (iii) without reading c , $P[0..q)$ was the *longest* prefix of P that ends here.

↪ New longest matched prefix will be (weakly) shorter than q

↪ All information about the text needed to determine it is contained in $P[0...q) \cdot c$!

NFA instead of DFA?

It remains to *construct* the DFA.



► that actually is a *nondeterministic finite automaton* (NFA) for $\Sigma^*P\Sigma^*$

~> We *could* use the NFA directly for string matching:

- at any point in time, we are in a *set* of states
- accept when one of them is final state

Example:

text:		a	a	b	a	c	a	a	b	a	b	a	c	a	a
state:	0	0,1	0,1	0,2	0,1,3	0	0,1	0,1	0,2	0,1,3	0,2,4	0,1,3,5	0,6	0,1,7	0,1,7

But maintaining a whole set makes this slow . . .

Computing DFA directly



You have an NFA and want a DFA?
Simply apply the power-set construction
(and maybe DFA minimization)!

The powerset method has exponential state blow up!
I guess I might as well use brute force ...



Ingenious algorithm by Knuth, Morris, and Pratt: construct DFA *inductively*:

Suppose we add character $P[j]$ to automaton A_{j-1} for $P[0..j-1]$

- ▶ add new state and matching transition \rightsquigarrow easy
- ▶ for each $c \neq P[j]$, we need $\delta(j, c)$ (transition from \textcircled{j} when reading c)
- ▶ $\delta(j, c) =$ length of the longest prefix of $P[0..j]c$ that is a suffix of $P[1..j]c$
= state of automaton after reading $P[1..j]c$
 $\leq j \rightsquigarrow$ can use known automaton A_{j-1} for that!

\rightsquigarrow can directly compute A_j from A_{j-1} !



seems to require simulating automata $m \cdot \sigma$ times

State q means:
“we have seen $P[0..q)$ until here
(but not any longer prefix of P)”

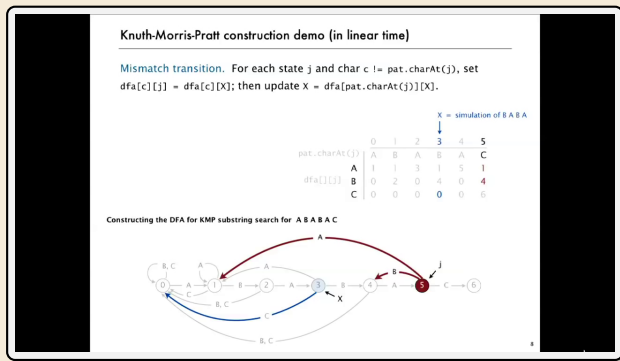
Computing DFA efficiently

► **KMP's second insight:** simulations in one step differ only in last symbol

⇒ simply maintain state x , the state after reading $P[1..j-1]$.

- copy its transitions
- update x by following transitions for $P[j]$

Demo: Algorithms videos of Sedgewick and Wayne



<https://cuvids.io/app/video/194/watch>

String matching with DFA – Discussion


► Time:


- Matching: n table lookups for DFA transitions
- building DFA: $\Theta(m\sigma)$ time (constant time per transition edge).


$\leadsto \Theta(m\sigma + n)$ time for string matching.

► Space:

- $\Theta(m\sigma)$ space for transition matrix.

 **fast matching** time actually: hard to beat!

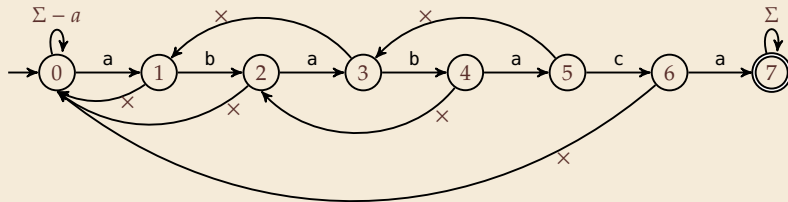
 total time asymptotically optimal for small alphabet (for $\sigma = O(n/m)$)

 substantial **space overhead**, in particular for large alphabets

4.4 The Knuth-Morris-Pratt algorithm

Failure Links

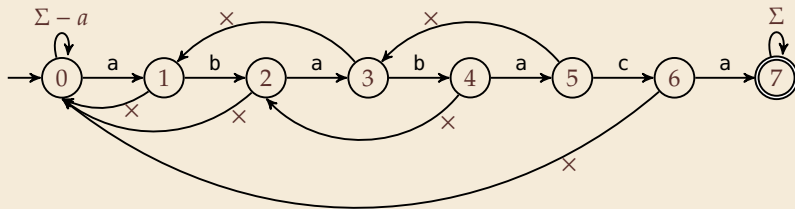
- ▶ Recall: String matching with DFA fast, but needs table of $m \times \sigma$ transitions.
- ▶ in fast DFA construction, we used that all simulations differ only by *last* symbol
- ~> **KMP's third insight:** do this last step of simulation from state x during *matching*!
... but how?
- ▶ **Answer:** Use a new type of transition, the *failure links*
 - ▶ Use this transition (only) if no other one fits.
 - ▶ \times *does not consume a character.* ~> might follow several failure links



~> Computations are deterministic (but automaton is not a real DFA.)

Failure link automaton – Example

Example: $T = \text{abababaaaca}$, $P = \text{ababaca}$



T : a b a b a b a a b a b

P :	a	b	a	b	a	x					
			(a)	(b)	(a)	b	a	x			
								a	b	a	b

to state 3

to state 1

q :	1	2	3	4	5	3,4	5	3,1,0,1	2	3	4
-------	---	---	---	---	---	-----	---	---------	---	---	---

(after reading this character)

The Knuth-Morris-Pratt Algorithm

```
1 procedure KMP( $T[0..n - 1]$ ,  $P[0..m - 1]$ )
2    $fail[0..m] := failureLinks(P)$ 
3    $i := 0$  // current position in  $T$ 
4    $q := 0$  // current state of KMP automaton
5   while  $i < n$  do
6     if  $T[i] == P[q]$  then
7        $i := i + 1$ ;  $q := q + 1$ 
8       if  $q == m$  then
9         return  $i - q$  // occurrence found
10      else // i.e.  $T[i] \neq P[q]$ 
11        if  $q \geq 1$  then
12           $q := fail[q]$  // follow one  $\times$ 
13        else
14           $i := i + 1$ 
15   end while
16   return NO_MATCH
```

► only need single array *fail* for failure links

► (procedure `failureLinks` later)

Analysis: (matching part)

► always have $fail[j] < j$ for $j \geq 1$

↪ in each iteration

► either advance position in text ($i := i + 1$)

► or shift pattern forward (guess $i - j$)

► each can happen at most n times

↪ $\leq 2n$ symbol comparisons!

Computing failure links

► failure links point to error state x (from DFA construction)

↪ run same algorithm, but store $fail[j] := x$ instead of copying all transitions

```
1 procedure failureLinks( $P[0..m-1]$ )
2    $fail[0] := 0$ 
3    $x := 0$ 
4   for  $j := 1, \dots, m-1$  do
5      $fail[j] := x$ 
6     // update failure state using failure links:
7     while  $P[x] \neq P[j]$ 
8       if  $x == 0$  then
9          $x := -1$ ; break
10      else
11         $x := fail[x]$ 
12      end while
13       $x := x + 1$ 
14  end for
```

Analysis:

- m iterations of for loop
- while loop always decrements x
- x is incremented only once per iteration of for loop

↪ $\leq m$ iterations of while loop *in total*

↪ $\leq 2m$ symbol comparisons

Knuth-Morris-Pratt – Discussion

► Time:


- $\leq 2n + 2m = O(n + m)$ character comparisons


- clearly must at least *read* both T and P

~> KMP has optimal worst-case complexity!

► Space:

- $\Theta(m)$ space for failure links

 total time asymptotically optimal (for any alphabet size)

 reasonable extra space

The KMP prefix function

- ▶ It turns out that the failure links are useful beyond KMP
- ▶ a slight variation is more widely used: (for historic reasons)
the (KMP) *prefix function* $F : [1..m - 1] \rightarrow [0..m - 1]$:

$F[j]$ *is the length of the longest prefix of $P[0..j]$
that is a suffix of $P[1..j]$.*

- ▶ Can show: $fail[j] = F[j - 1]$ for $j \geq 1$, and hence

$fail[j] =$ *length of the
longest prefix of $P[0..j]$
that is a suffix of $P[1..j]$.*

← memorize this!

4.5 Beyond Optimal? The Boyer-Moore Algorithm

Motivation


- ▶ KMP is an optimal algorithm, isn't it? What else could we hope for?
- ▶ KMP is “only” optimal in the worst-case (and up to constant factors)
- ▶ how many comparisons do we need for the following instance?
 $T = \text{aaaaaaaaaaaaaaaaa}$, $P = \text{xxxxx}$
 - ▶ there are no matches
 - ▶ we can *certify* the correctness of that output with only 4 comparisons:

T	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
				x											
									x						
														x	
															x

⇒ We did *not* even read most text characters!

Boyer-Moore Algorithm

- ▶ Let's check guesses *from right to left!*
- ▶ If we are lucky, we can eliminate several shifts in one shot!

 must avoid (excessive) redundant checks, e. g., for $T = a^n$, $P = ba^{m-1}$

↪ New rules:

- ▶ **Bad character jumps:** Upon mismatch at $T[i] = c$:
 - ▶ If P does not contain c , shift P entirely past i !
 - ▶ Otherwise, shift P to align the *last occurrence* of c in P with $T[i]$.
- ▶ **Good suffix jumps:**

Upon a mismatch, shift so that the already matched *suffix* of P aligns with a previous occurrence of that suffix (or part of it) in P .

(Details follow; ideas similar to KMP failure links)

↪ two possible shifts (next guesses); use larger jump.

Boyer-Moore Algorithm – Code

```
1 procedure boyerMoore( $T[0..n - 1]$ ,  $P[0..m - 1]$ )
2    $\lambda := \text{computeLastOccurrences}(P)$ 
3    $\gamma := \text{computeGoodSuffixes}(P)$ 
4    $i := 0$  // current guess
5   while  $i \leq n - m$ 
6      $j := m - 1$  // next position in P to check
7     while  $j \geq 0 \wedge P[j] \neq T[i + j]$  do
8        $j := j - 1$ 
9     if  $j == -1$  then
10       return  $i$ 
11     else
12        $i := i + \max\{j - \lambda[T[i + j]], \gamma[j]\}$ 
13   return NO_MATCH
```

- ▶ λ and γ explained below
- ▶ shift forward is larger of two heuristics
- ▶ shift is always positive (see below)

Bad character examples

P = a l d o

T = w h e r e i s w a l d o

			o								
							o				
								a	l	d	o

↪ 6 characters not looked at

P = m o o r e

T = b o y e r m o o r e

				e					
				(r)	e				
					(m)	o	o	r	e

↪ 4 characters not looked at

Last-Occurrence Function

- ▶ Preprocess pattern P and alphabet Σ
- ▶ *last-occurrence function* $\lambda[c]$ defined as
 - ▶ the largest index i such that $P[i] = c$ or
 - ▶ -1 if no such index exists
- ▶ **Example:** $P = \text{moore}$

c	m	o	r	e	all others
$\lambda[c]$	0	2	3	4	-1

P	=	m	o	o	r	e					
T	=	b	o	y	e	r	m	o	o	r	e
					e						
				(r)	e						

$i = 0, j = 4, T[i + j] = r, \lambda[r] = 3$
 \rightsquigarrow shift by $j - \lambda[T[i + j]] = 1$

- ▶ λ easily computed in $O(m + |\Sigma|)$ time.
- ▶ store as array $\lambda[0..|\Sigma| - 1]$.

Good suffix examples

1. $P = \text{sell_shells}$

s	h	e	i	l	a	_	s	e	l	l	s	_	s	h	e	l	l	s
							h	e	l	l	s							
								(e)	(l)	(l)	(s)							

2. $P = \text{odetofood}$

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o
				o	f	o	o	d										
							(o)	(d)										

matched suffix



- **Crucial ingredient:** longest suffix of $P[j+1..m-1]$ that occurs earlier in P .
- 2 cases (as illustrated above)
 1. complete suffix occurs in $P \rightsquigarrow$ characters left of suffix are *not* known to match
 2. part of suffix occurs at beginning of P

Good suffix jumps

- Precompute **good suffix jumps** $\gamma[0..m-1]$:

- For $0 \leq j < m$, $\gamma[j]$ stores shift if search failed at $P[j]$
- At this point, had $T[i+j+1..i+m-1] = P[j+1..m-1]$, but $T[i] \neq P[j]$

$\leadsto \gamma[j]$ is the shift $m-1-\ell$ for the *largest* ℓ such that

- $P[j+1 \dots m-1]$ is a suffix of $P[0 \dots \ell]$ and $P[j] \neq P[\ell-m+j+1]$

							h	e	l	l	s							
							×	(e)	(l)	(l)	(s)							

–OR–

- $P[0 \dots \ell]$ is a suffix of $P[j+1, \dots, m-1]$

					o	f	o	o	d									
								(o)	(d)									

- Computable (similar to KMP failure function) in $\Theta(m)$ time.
- **Note:** You do not need to know how to find the values $\gamma[j]$ for the exam, but you should be able to find the next guess on examples.

Boyer-Moore algorithm – Discussion

👍 Worst-case running time $\in O(n + m + |\Sigma|)$ if P does *not* occur in T .
(follows from not at all obvious analysis!)

👎 As given, worst-case running time $\Theta(nm)$ if we want to report all occurrences

- ▶ To avoid that, have to keep track of implied matches.
(tricky because they can be in the “middle” of P)
- ▶ Note: KMP reports all matches in $O(n + m)$ without modifications!

👍 On typical *English text*, Boyer Moore probes only approx. 25% of the characters in T !
~> Faster than KMP on English text.

👍 requires moderate extra space $\Theta(m + \sigma)$

4.6 The Rabin-Karp Algorithm

Space – The final frontier

- ▶ Knuth-Morris-Pratt has great worst case and real-time guarantees
- ▶ Boyer-Moore has great typical behavior
- ▶ What else to hope for?
- ▶ All require $\Omega(m)$ extra space;
can be substantial for large patterns!
- ▶ Can we avoid that?

Rabin-Karp Fingerprint Algorithm – Idea

Idea: use *hashing* (but without explicit hash tables)

- ▶ Precompute & store only *hash* of pattern
- ▶ Compute hash for each guess
- ▶ If hashes agree, check characterwise

Example: (treat (sub)strings as decimal numbers)

$$P = 59265$$

$$T = 3141592653589793238$$

Hash function: $h(x) = x \bmod 97$

$$\rightsquigarrow h(P) = 95.$$

$$\frac{3 \quad 1 \quad 4 \quad 1 \quad 5}{h(31415) = 84} \quad 9 \quad 2 \quad 6 \quad 5 \quad 3 \quad 5 \quad 8 \quad 9 \quad 7 \quad 9 \quad 3 \quad 2 \quad 3 \quad 8$$

$$h(14159) = 94$$

$$h(41592) = 76$$

$$h(15926) = 18$$

$$h(59262) = 95$$

Rabin-Karp Fingerprint Algorithm – First Attempt

```
1 procedure rabinKarpSimplistic( $T[0..n - 1], P[0..m - 1]$ )
2    $M :=$  suitable prime number
3    $h_P :=$  computeHash( $P[0..m - 1]$ ),  $M$ )
4   for  $i := 0, \dots, n - m$  do
5      $h_T :=$  computeHash( $T[i..i + m - 1]$ ,  $M$ )
6     if  $h_T == h_P$  then
7       if  $T[i..i + m - 1] == P$  //  $m$  comparisons
8         then return  $i$ 
9   return NO_MATCH
```

- ▶ never misses a match since $h(S_1) \neq h(S_2)$ implies $S_1 \neq S_2$ ✓
- ▶ $h(T[k..k+m-1])$ depends on m characters \rightsquigarrow naive computation takes $\Theta(m)$ time
- \rightsquigarrow Running time is $\Theta(mn)$ for search miss ... can we improve this?

Rabin-Karp Fingerprint Algorithm – Fast Rehash

► **Crucial insight:** We can update hashes in constant time.

► Use previous hash to compute next hash

► $O(1)$ time per hash, except first one

for above hash function!



Example:

► Pre-compute: $10000 \bmod 97 = 9$

► Previous hash: $41592 \bmod 97 = 76$

► Next hash: $15926 \bmod 97 = ??$

Observation:

$$\begin{aligned} 15926 \bmod 97 &= (41592 - (4 \cdot 10000)) \cdot 10 + 6 \bmod 97 \\ &= (76 - (4 \cdot 9)) \cdot 10 + 6 \bmod 97 \\ &= 406 \bmod 97 = 18 \end{aligned}$$

Rabin-Karp Fingerprint Algorithm – Code

- ▶ use a convenient radix $R \geq \sigma$ ($R = 10$ in our examples; $R = 2^k$ is faster)
- ▶ Choose modulus M at *random* to be huge prime (randomization against worst-case inputs)
- ▶ all numbers remain $\leq 2R^2 \rightsquigarrow O(1)$ time arithmetic on word-RAM

```
1 procedure rabinKarp( $T[0..n - 1], P[0..m - 1], R$ )
2    $M :=$  suitable prime number
3    $h_P :=$  computeHash( $P[0..m - 1], M$ )
4    $h_T :=$  computeHash( $T[0..m - 1], M$ )
5    $s := R^{m-1} \bmod M$ 
6   for  $i := 0, \dots, n - m$  do
7     if  $h_T == h_P$  then
8       if  $T[i..i + m - 1] = P$ 
9         return  $i$ 
10    if  $i < n - m$  then
11       $h_T := ((h_T - T[i] \cdot s) \cdot R + T[i + m]) \bmod M$ 
12  return NO_MATCH
```

Rabin-Karp – Discussion

- 👍 Expected running time is $O(m + n)$
- 👎 $\Theta(mn)$ worst-case;
but this is very unlikely
- 👍 Extends to 2D patterns and other generalizations
- 👍 Only constant extra space!

String Matching Conclusion

	Brute-Force	DFA	KMP	BM	RK	Suffix trees*
Preproc. time	—	$O(m \Sigma)$	$O(m)$	$O(m + \sigma)$	$O(m)$	$O(n)$
Search time	$O(nm)$	$O(n)$	$O(n)$	$O(n)$ (often better)	$O(n + m)$ (expected)	$O(m)$
Extra space	—	$O(m \Sigma)$	$O(m)$	$O(m + \sigma)$	$O(1)$	$O(n)$

* (see Unit 6)