
# 4 String Matching – What's behind Ctrl+F?

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Sebastian Wild



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University of Liverpool
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# Learning Outcomes

1. Know and use typical notions for *strings* (substring, prefix, suffix, etc.).
2. Understand principles and implementation of the *KMP*, *BM*, and *RK* algorithms.
3. Know the *performance characteristics* of the KMP, BM, and RK algorithms.
4. Be able to solve simple *stringology problems* using the *KMP failure function*.

## Unit 4: *String Matching*

# Outline

## 4.1 Introduction

# Ubiquitous strings

*string* = sequence of characters

- universal data type for . . . everything!
  - natural language texts
  - programs (source code)
  - websites
  - XML documents
  - DNA sequences
  - bitstrings
  - . . . a computer's memory $\rightsquigarrow$ ultimately any data is a string

$\rightsquigarrow$ many different tasks and algorithms

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- This unit: finding (exact) **occurrences of a pattern** text.
  - Ctrl+F
  - grep
  - computer forensics (e. g. find signature of file on disk)
  - virus scanner

- basis for many advanced applications

# Notations

- *alphabet* $\Sigma$: finite set of allowed **characters**; $\sigma = |\Sigma|$ *"a string over alphabet $\Sigma$"*
  - letters (Latin, Greek, Arabic, Cyrillic, Asian scripts, ...)
  - "what you can type on a keyboard", Unicode characters
    comprehensive standard character set including emoji and all known symbols
  - $\{0, 1\}$; nucleotides $\{A, C, G, T\}$; ...

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  - $\{0, 1\}$; nucleotides $\{A, C, G, T\}$; ...
- $\Sigma^n = \Sigma \times \cdots \times \Sigma$: strings of **length** $n \in \mathbb{N}_0$ ($n$-tuples)
- $\Sigma^\star = \bigcup_{n \geq 0} \Sigma^n$: set of **all** (finite) strings over $\Sigma$
- $\Sigma^+ = \bigcup_{n \geq 1} \Sigma^n$: set of **all** (finite) **nonempty** strings over $\Sigma$
- $\varepsilon \in \Sigma^0$: the *empty* string (same for all alphabets)

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- $\varepsilon \in \Sigma^0$: the *empty* string (same for all alphabets) " " Python

  zero-based (like arrays)!
- for $S \in \Sigma^n$, write $S[i]$ (other sources: $S_i$) for $i$**th** character $(0 \le i < n)$
- for $S, T \in \Sigma^\star$, write $ST = S \cdot T$ for **concatenation** of $S$ and $T$ S+T in Python
- for $S \in \Sigma^n$, write $S[i..j]$ or $S_{i,j}$ for the **substring** $S[i] \cdot S[i+1] \cdots S[j]$ $(0 \le i \le j < n)$
  - $S[0..j]$ is a **prefix** of $S$; $S[i..n-1]$ is a **suffix** of $S$
  - $S[i..j) = S[i..j-1]$ (endpoint exclusive) $\rightsquigarrow$ $S = S[0..n)$

# Clicker Question

True or false: $\Sigma^\star = \Sigma^+ \cup \{\varepsilon\}$

**A** True

**B** False

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True or false: $\Sigma^\star = \Sigma^+ \cup \{\varepsilon\}$

**A** True ✓

**B** ~~False~~

→ sli.do/comp526

# String matching – Definition

Search for a string (pattern) in a large body of text

- **Input:**
  - $T \in \Sigma^n$: The *text* (haystack) being searched within
  - $P \in \Sigma^m$: The *pattern* (needle) being searched for; typically $n \gg m$
- **Output:**
  - the *first occurrence (match)* of $P$ in $T$: $\min\{i \in [0..n-m) : T[i..i+m) = P\}$
  - or `NO_MATCH` if there is no such $i$ ("$P$ does not occur in $T$")
- Variant: Find **all** occurrences of $P$ in $T$.
  
  $\rightsquigarrow$ Can do that iteratively (update $T$ to $T[i+1..n)$ after match at $i$)
- **Example:**
  - $T$ = "`Where is he?`"
  - $P_1$ = "`he`" $\rightsquigarrow$ $i = 1$
  - $P_2$ = "`who`" $\rightsquigarrow$ `NO_MATCH`
- string matching is implemented in Java in `String.indexOf`, in Python as `str.find`

# Clicker Question

Let $T$ = `COMP526␣is␣fun.`
What is $T[3..7)$?

→ *sli.do/comp526*

# Clicker Question

Let $T =$ `COMP526␣is␣fun.`
What is $T[3..7)$?

```
012345678901234
COMP526␣is␣fun.
```

→ *sli.do/comp526*

## 4.2 Brute Force

# Abstract idea of algorithms

String matching algorithms typically use *guesses* and *checks*:

- A **guess** is a position $i$ such that $P$ might start at $T[i]$.
  Possible guesses (initially) are $0 \leq i \leq n - m$.

- A **check** of a guess is a comparison of $T[i + j]$ to $P[j]$.

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- A **check** of a guess is a comparison of $T[i + j]$ to $P[j]$.
- Note: need all $m$ checks to verify a single *correct* guess $i$,
  but it may take (many) fewer checks to recognize an *incorrect* guess.

- Cost measure: #character comparisons

$\rightsquigarrow$ #checks $\le n \cdot m$ (number of possible checks)

# Brute-force method

```
procedure bruteForceSM(T[0..n), P[0..m))
    for i := 0, ..., n − m − 1 do
        for j := 0, ..., m − 1 do
            if T[i + j] ≠ P[j] then break inner loop
        if j == m then return i
    return NO_MATCH
```

- try all guesses $i$
- check each guess (left to right); stop early on mismatch
- essentially the implementation in Java!

- **Example:**
  $T$ = `abbbababbab`
  $P$ = `abba`

| i | a | b | b | b | a | b | a | b | b | a | b |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | a | b | b | a | | | | | | | |
| 1 | | a | | | | | | | | | |
| 2 | | | a | | | | | | | | |
| 3 | | | | a | | | | | | | |
| 4 | | | | | a | b | b | | | | |
| 5 | | | | | | a | | | | | |
| 6 | | | | | | | a | b | b | a | |
| 7 | | | | | | | | | | | |

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- **Example:**
  $T$ = `abbbababbab`
  $P$ = `abba`

$\rightsquigarrow$ 15 char cmps
(vs $n \cdot m = 44$)
not too bad!

| a | b | b | b | a | b | a | b | b | a | b |
|---|---|---|---|---|---|---|---|---|---|---|
| a | b | b | a | | | | | | | |
| | a | | | | | | | | | |
| | | a | | | | | | | | |
| | | | a | | | | | | | |
| | | | | a | b | b | | | | |
| | | | | | a | | | | | |
| | | | | | | a | b | b | a | |
| | | | | | | | | | | |

# Brute-force method – Discussion

Brute-force method can be good enough

- typically works well for natural language text
- also for random strings

but: can be as bad as it gets!

| a | a | a | a | a | a | a | a | a | a | a |
|---|---|---|---|---|---|---|---|---|---|---|
| a | a | a | b | | | | | | | |
| | a | a | a | b | | | | | | |
| | | a | a | a | b | | | | | |
| | | | a | a | a | b | | | | |
| | | | | a | a | a | b | | | |
| | | | | | a | a | a | b | | |
| | | | | | | a | a | a | b | |
| | | | | | | | a | a | a | b |

- Worst possible input: $P = a^{m-1}b$, $T = a^n$
- Worst-case performance: $(n - m + 1) \cdot m$

$\rightsquigarrow$ for $m \le n/2$ that is $\Theta(mn)$

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| a | a | a | a | a | a | a | a | a | a | a |
|---|---|---|---|---|---|---|---|---|---|---|
| a | a | a | b | | | | | | | |
| | a | a | a | b | | | | | | |
| | | a | a | a | b | | | | | |
| | | | a | a | a | b | | | | |
| | | | | a | a | a | b | | | |
| | | | | | a | a | a | b | | |
| | | | | | | a | a | a | b | |
| | | | | | | | a | a | a | b |

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- Bad input: lots of self-similarity in $T$! $\rightsquigarrow$ can we exploit that?
- brute force does 'obviously' stupid repetitive comparisons $\rightsquigarrow$ can we avoid that?

# Roadmap

- **Approach 1** (this week): Use *preprocessing* on the **pattern** $P$ to eliminate guesses (avoid 'obvious' redundant work)
  - Deterministic finite automata (**DFA**)
  - **Knuth-Morris-Pratt** algorithm
  - **Boyer-Moore** algorithm
  - **Rabin-Karp** algorithm
- **Approach 2** ($\rightsquigarrow$ Unit 6): Do *preprocessing* on the **text** $T$
  Can find matches in time *independent of text size(!)*
  - **inverted indices**
  - **Suffix trees**
  - **Suffix arrays**

## 4.3 String Matching with Finite Automata

# Clicker Question

Do you know what regular expressions, NFAs and DFAs are, and how to convert between them?

**A** Never heard of this; are these new emoji?

**B** Heard the terms, but don't remember conversion methods.

**C** Had that in my undergrad course (memories fading a bit).

**D** Sure, I could do that blindfolded!

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# Theoretical Computer Science to the rescue!

- string matching $=$ deciding whether $T \in \Sigma^\star \cdot P \cdot \Sigma^\star$
- $\Sigma^\star \cdot P \cdot \Sigma^\star$ is *regular* formal language

$\rightsquigarrow$ $\exists$ *deterministic finite automaton* (DFA) to recognize $\Sigma^\star \cdot P \cdot \Sigma^\star$

$\rightsquigarrow$ can check for occurrence of $P$ in $|T| = n$ steps!

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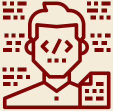

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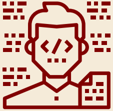


`WTF!?`

We are not quite done yet.

- (Problem 0: programmer might not know automata and formal languages . . . )
- Problem 1: existence alone does not give an algorithm!
- Problem 2: automaton could be very big!

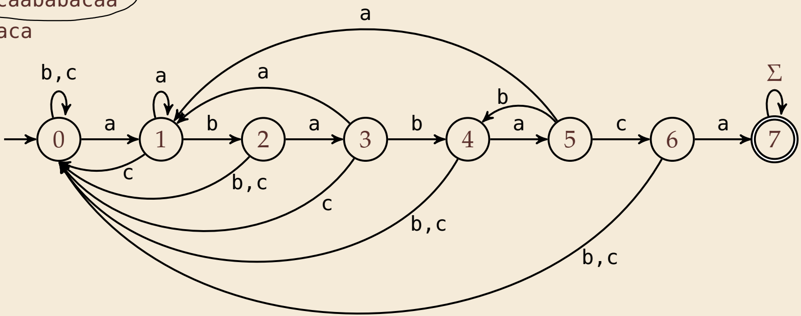
# String matching with DFA

- Assume first, we already have a deterministic automaton
- How does string matching work?

**Example:**

$T$ = aabacaababacaa

$P$ = ababaca




| text: | | a | a | b | a | c | a | a | b | a | b | a | c | a | a |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| state: | 0 | 1 | 1 | 2 | 3 | 0 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |

# String matching with DFA

- Assume first, we already have a deterministic automaton
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find $\delta(2,c) = 0$

$P[0..j)c = abc$

a $P[1..j)c = bc \rightsquigarrow 0$

**Example:**

$T$ = aabacaababacaa

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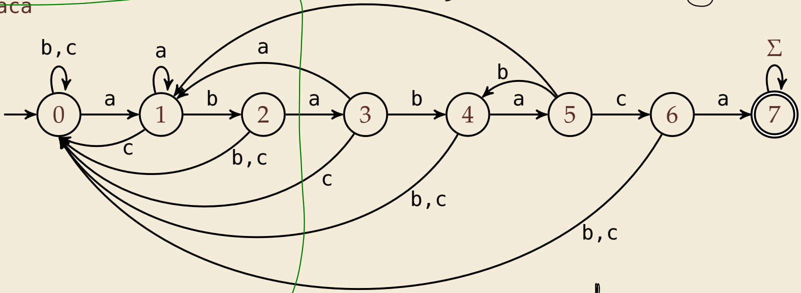
$A_2$




a b a b a c a

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
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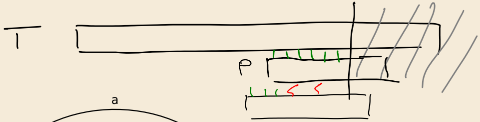
a b a b . .

# String matching DFA – Intuition

Why does this work?

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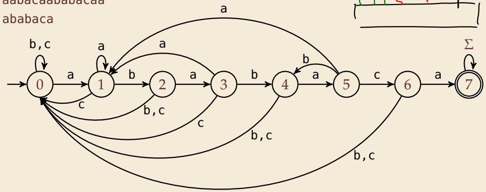


- Main insight:

> State $q$ means:
> *"we have seen $P[0..q)$ until here (but not any longer prefix of $P$)"*

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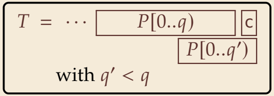
- If the next text character $c$ does not match, we know:
  - (i) text seen so far ends with $P[0\ldots q) \cdot c$
  - (ii) $P[0\ldots q) \cdot c$ is not a prefix of $P$
  - (iii) without reading $c$, $P[0..q)$ was the *longest* prefix of $P$ that ends here.



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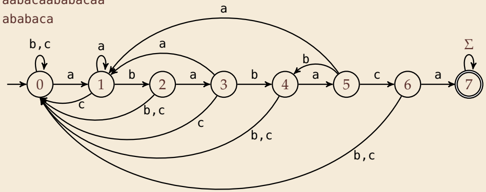


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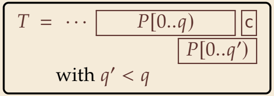


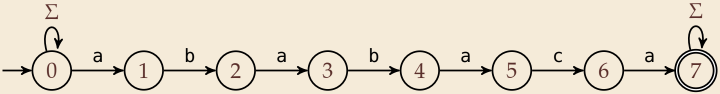

⇝ New longest matched prefix will be (weakly) shorter than $q$

⇝ All information about the text needed to determine it is contained in $P[0...q) \cdot c$!

## 4.4 Constructing String Matching Automata

# NFA instead of DFA?

It remains to *construct* the DFA.

- trivial part: 

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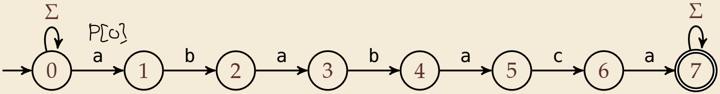
- trivial part: 


- that actually is a ***nondeterministic*** *finite automaton* (NFA) for $\Sigma^\star P\, \Sigma^\star$

⇝ We *could* use the NFA directly for string matching:

- at any point in time, we are in a ***set* of states**
- accept when one of them is final state

**Example:**

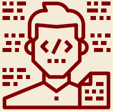
| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| text: | | a | a | b | a | c | a | a | b | a | b | a | c | a | a |
| state: | 0 | 0, 1 | 0, 1 | 0, 2 | 0, 1, 3 | 0 | 0, 1 | 0, 1 | 0, 2 | 0, 1, 3 | 0, 2, 4 | 0, 1, 3, 5 | 0, 6 | 0, 1, 7 | 0, 1, 7 |

But maintaining a whole set makes this slow . . .

# Computing DFA directly

You have an NFA and want a DFA?
Simply apply the power-set construction
(and maybe DFA minimization)!

```
The powerset method has exponential state blow up!
       I guess I might as well use brute force ...
```



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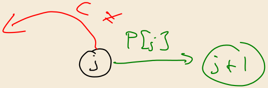
Suppose we add character $P[j]$ to automaton $A_{j-1}$ for $P[0..j)$




- add new state and matching transition $\rightsquigarrow$ easy
- for each $c \neq P[j]$, we need $\delta(j, c)$ (transition from $j$ when reading $c$)

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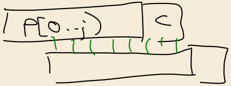



- add new state and matching transition $\rightsquigarrow$ easy
- for each $c \neq P[j]$, we need $\delta(j, c)$ (transition from $j$ when reading $c$)
- $\delta(j,c)$ $=$ length of the longest prefix of $P[0..j)c$ that is a suffix of $P[1..j)c$
  $=$ state of automaton after reading $P[1..j)c$
  $\leq$ $j$ $\rightsquigarrow$ can use known automaton $A_{j-1}$ for that!

State $q$ means:
*"we have seen $P[0..q)$ until here*
*(but not any longer prefix of $P$)"*

$\rightsquigarrow$ can directly compute $A_j$ from $A_{j-1}$!

seems to require simulating automata $m \cdot \sigma$ times

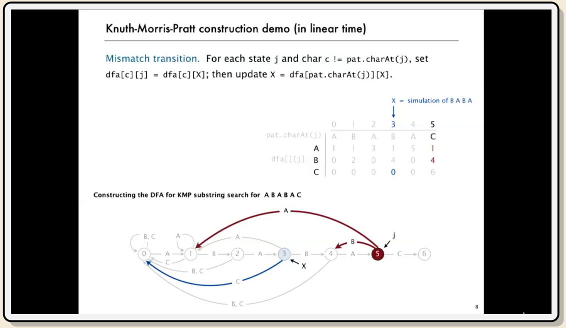
# Computing DFA efficiently

- **KMP's second insight:** simulations in one step differ only in last symbol

⇝ simply maintain state $x$, the state after reading $P[1..j)$.

  - copy its transitions
  - update $x$ by following transitions for $P[j]$

**Demo:** Algorithms videos of Sedgewick and Wayne




https://cuvids.io/app/video/194/watch

# String matching with DFA – Discussion

- **Time:**
  - Matching: $n$ table lookups for DFA transitions
  - building DFA: $\Theta(m\sigma)$ time (constant time per transition edge).
  - $\rightsquigarrow$ $\Theta(m\sigma + n)$ time for string matching.

- **Space:**
  - $\Theta(m\sigma)$ space for transition matrix.

Unicode

$\sigma = 149{,}186$

👍 **fast matching** time  actually: hard to beat!

👍 total time asymptotically optimal for small alphabet  (for $\sigma = O(n/m)$)

👎 substantial **space overhead**, in particular for large alphabets

## 4.5 The Knuth-Morris-Pratt algorithm

# Failure Links

- Recall: String matching with is DFA fast,
  but needs table of $m \times \sigma$ transitions.
- in fast DFA construction, we used that all simulations differ only by *last* symbol

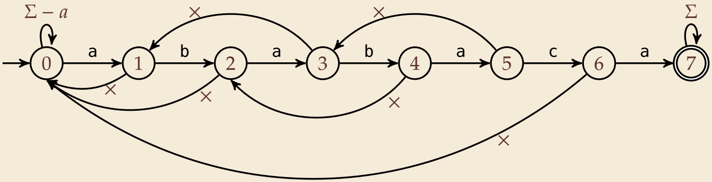
$\rightsquigarrow$ **KMP's third insight:** do this last step of simulation from state $x$ during *matching*!
. . . but how?

# Failure Links

- Recall: String matching with is DFA fast,
  but needs table of $m \times \sigma$ transitions.
- in fast DFA construction, we used that all simulations differ only by *last* symbol

$\rightsquigarrow$ **KMP's third insight:** do this last step of simulation from state $x$ during *matching*!
. . . but how?

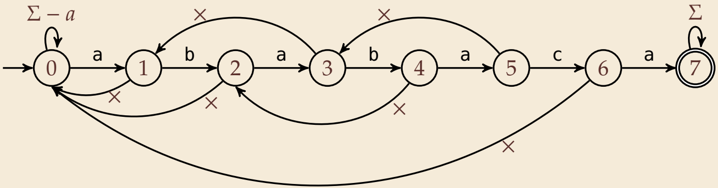
- **Answer:** Use a new type of transition, the *failure links*
  - Use this transition (only) if no other one fits.
  - $\times$ *does not consume a character.* $\rightsquigarrow$ might follow several failure links




$\rightsquigarrow$ Computations are deterministic (but automaton is not a real DFA.)

# Failure link automaton – Example

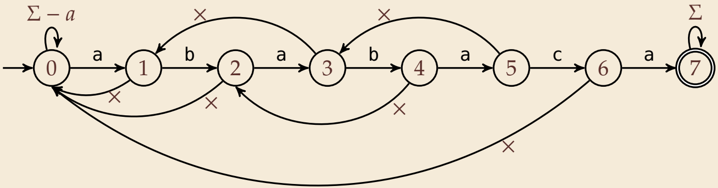
**Example:** $T$ = `abababaaaca`, $P$ = `ababaca`




| $T$ : | a | b | a | b | a | b | a | a | b | a | b |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5,3 | 4 | 5,3,1,0 | 1 | 2 | 3 | 4 |

# Failure link automaton – Example

**Example:** $T$ = `abababaaaca`, $P$ = `ababaca`




| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ : | a | b | a | b | a | b | a | a | b | a | b | |
| $P$ : | a | b | a | b | a | × | | | | | | to state 3 |
| | | | (a) | (b) | (a) | b | a | × | | | | to state 1 |
| | | | | | | | | a | b | a | b | |
| $q$ : | 1 | 2 | 3 | 4 | 5 | 3, 4 | 5 | 3, 1, 0, 1 | 2 | 3 | 4 | |

(after reading this character)

# Clicker Question

What is the worst-case time to process one character in a failure-link automaton for $P[0..m)$?

**A** $\Theta(1)$

**B** $\Theta(\log m)$

**C** $\Theta(m)$

**D** $\Theta(m^2)$

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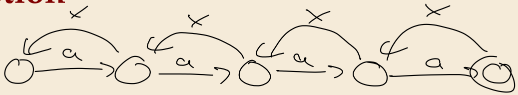
# Clicker Question




What is the worst-case time to process one character in a failure-link automaton for $P[0..m)$?

**A** $\cancel{\Theta(1)}$

**B** $\cancel{\Theta(\log m)}$

**C** $\Theta(m)$ ✓

**D** $\cancel{\Theta(m^2)}$

→ sli.do/comp526

# The Knuth-Morris-Pratt Algorithm

```
procedure KMP(T[0..n − 1], P[0..m − 1])
    fail[0..m] := failureLinks(P)
    i := 0 // current position in T
    q := 0 // current state of KMP automaton
    while i < n do
        if T[i] == P[q] then
            i := i + 1;  q := q + 1
            if q == m then
                return i − q // occurrence found
        else // i.e. T[i] ≠ P[q]
            if q ≥ 1 then
                q := fail[q] // follow one ×
            else
                i := i + 1
    end while
    return NO_MATCH
```

- only need single array *fail* for failure links
- (procedure failureLinks later)