7 Compression

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Outline

7 Compression

7.1 Context

Overview

- Unit 4–6: How to *work* with strings
  - finding substrings
  - finding approximate matches
  - finding repeated parts
  - . . .

- Unit 7–8: How to *store* strings
  - computer memory: must be binary
  - how to compress strings (save space)
  - how to robustly transmit over noisy channels $\rightsquigarrow$ Unit 8

Terminology

- **source text:** string $S \in \Sigma_S^\star$ to be stored / transmitted
  Σ_S is some alphabet
- **coded text:** encoded data $C \in \Sigma_C^\star$ that is actually stored / transmitted
  usually use $\Sigma_C = \{0, 1\}$
- **encoding:** algorithm mapping source texts to coded texts $S \to C$
- **decoding:** algorithm mapping coded texts back to original source text $C \to S$

What is a good encoding scheme?

- Depending on the application, goals can be
  - efficiency of encoding/decoding
  - resilience to errors/noise in transmission
  - security (encryption)
  - integrity (detect modifications made by third parties)
  - size

not here

- Focus in this unit: **size** of coded text $|C|$

  Encoding schemes that (try to) minimize the size of coded texts perform *data compression*.

- We will measure the *compression ratio*: $\dfrac{|C| \cdot \lg |\Sigma_C|}{|S| \cdot \lg |\Sigma_S|} \overset{\Sigma_C=\{0,1\}}{=} \dfrac{|C|}{|S| \cdot \lg |\Sigma_S|}$

  < 1 means successful compression
  $= 1$ means no compression
  > 1 means "compression" made it bigger!? (yes, that happens ...)

Types of Data Compression

- **Logical vs. Physical**
  - **Logical Compression** uses meaning of data
    $\rightsquigarrow$ only applies to a certain domain, e. g., sound recordings
  - **Physical Compression** only knows the (physical) **bits** in the data, not the meaning behind them
- **Lossy vs. Lossless**
  - **lossy compression** can only decode **approximately**; the exact source text S is lost
  - **lossless compression** always decodes S exactly
- For media files, lossy, logical compression is useful (e. g. JPEG, MPEG)
- We will concentrate on *physical, lossless* compression algorithms. These techniques can be used for any application.

What makes data compressible?

- Physical, lossless compression methods mainly exploit two types of redundancies in source texts:

  1. **uneven character frequencies**
     some characters occur more often than others $\rightarrow$ Part I
  2. **repetitive texts**
     different parts in the text are (almost) identical $\rightarrow$ Part II

What makes data compressible?

- Physical, lossless compression methods mainly exploit two types of redundancies in source texts:

  1. **uneven character frequencies**
     some characters occur more often than others $\longrightarrow$ Part I
  2. **repetitive texts**
     different parts in the text are (almost) identical $\longrightarrow$ Part II

There is no such thing as a free lunch!

Not *everything* is compressible ($\longrightarrow$ tutorials)

$\rightsquigarrow$ focus on versatile methods that often work

Part I

Exploiting character frequencies

7.2 Character Encodings

Character encodings

- Simplest form of encoding: Encode each source character individually

$\rightsquigarrow$ encoding function $E : \Sigma_S \to \Sigma_C^\star$

  - typically, $|\Sigma_S| \gg |\Sigma_C|$, so need several bits per character
  - for $c \in \Sigma_S$, we call $E(c)$ the *codeword* of c

- fixed-length code: $|E(c)|$ is the same for all $c \in \Sigma_{\cancel{C}S}$
- variable-length code: not all codewords of same length

Fixed-length codes

- fixed-length codes are the simplest type of character encodings
- Example: **ASCII** (American Standard Code for Information Interchange, 1963)

0000000 NUL	0010000 DLE	0100000	0110000 0	1000000 @	1010000 P	1100000 ‘	1110000 p
0000001 SOH	0010001 DC1	0100001 !	0110001 1	1000001 A	1010001 Q	1100001 a	1110001 q
0000010 STX	0010010 DC2	0100010 "	0110010 2	1000010 B	1010010 R	1100010 b	1110010 r
0000011 ETX	0010011 DC3	0100011 #	0110011 3	1000011 C	1010011 S	1100011 c	1110011 s
0000100 EOT	0010100 DC4	0100100 $	0110100 4	1000100 D	1010100 T	1100100 d	1110100 t
0000101 ENQ	0010101 NAK	0100101 %	0110101 5	1000101 E	1010101 U	1100101 e	1110101 u
0000110 ACK	0010110 SYN	0100110 &	0110110 6	1000110 F	1010110 V	1100110 f	1110110 v
0000111 BEL	0010111 ETB	0100111 ’	0110111 7	1000111 G	1010111 W	1100111 g	1110111 w
0001000 BS	0011000 CAN	0101000 (	0111000 8	1001000 H	1011000 X	1101000 h	1111000 x
0001001 HT	0011001 EM	0101001)	0111001 9	1001001 I	1011001 Y	1101001 i	1111001 y
0001010 LF	0011010 SUB	0101010 *	0111010 :	1001010 J	1011010 Z	1101010 j	1111010 z
0001011 VT	0011011 ESC	0101011 +	0111011 ;	1001011 K	1011011 [	1101011 k	1111011 {
0001100 FF	0011100 FS	0101100 ,	0111100 <	1001100 L	1011100 \	1101100 l	1111100 |
0001101 CR	0011101 GS	0101101 -	0111101 =	1001101 M	1011101]	1101101 m	1111101 }
0001110 SO	0011110 RS	0101110 .	0111110 >	1001110 N	1011110 ^	1101110 n	1111110 ~
0001111 SI	0011111 US	0101111 /	0111111 ?	1001111 O	1011111 _	1101111 o	1111111 DEL

- 7 bit per character
- just enough for English letters and a few symbols (plus control characters)

Fixed-length codes – Discussion

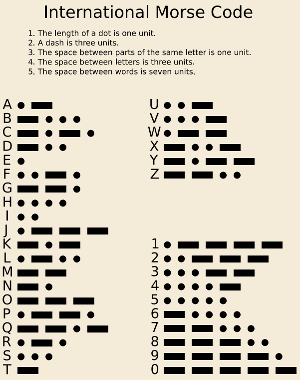
- 👍 Encoding & Decoding as fast as it gets
- 👎 Unless all characters equally likely, it wastes a lot of space
- 👎 inflexible (how to support adding a new character?)

Variable-length codes

- to gain more flexibility, have to allow different lengths for codewords

- actually an old idea: **Morse Code**

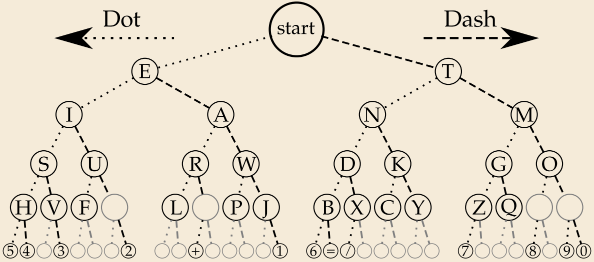
encoding



https://commons.wikimedia.org/wiki/File:International_Morse_Code.svg

trie

decoding



https://commons.wikimedia.org/wiki/File:Morse-code-tree.svg

Variable-length codes – UTF-8

- Modern example: UTF-8 encoding of Unicode:
  default encoding for text-files, XML, HTML since 2009
  - Encodes any Unicode character (137 994 as of May 2019, and counting)
  - uses 1–4 bytes (codeword lengths: 8, 16, 24, or 32 bits)
  - Every ASCII character is encoded in 1 byte with leading bit `0`, followed by the 7 bits for ASCII
  - Non-ASCII charactters start with 1–4 `1`s indicating the total number of bytes, followed by a `0` and 3–5 bits.
    The remaining bytes each start with `10` followed by 6 bits.

Char. number range (hexadecimal)	UTF-8 octet sequence (binary)
`0000 0000-0000 007F`	`0xxxxxxx`
`0000 0080-0000 07FF`	`110xxxxx 10xxxxxx`
`0000 0800-0000 FFFF`	`1110xxxx 10xxxxxx 10xxxxxx`
`0001 0000-0010 FFFF`	`11110xxx 10xxxxxx 10xxxxxx 10xxxxxx`

For English text, most characters use only 8 bit,
but we can include any Unicode character, as well.

Pitfall in variable-length codes

- Suppose we have the following code:

c	a	n	b	s
$E(c)$	0	10	110	100

- Happily encode text S = banana with the coded text C = 1100100100

b a n a n a

Pitfall in variable-length codes

- Suppose we have the following code:

c	a	n	b	s
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- Happily encode text S = banana with the coded text $C = \underbrace{110}_{\texttt{b}}\,\texttt{0}\,\underbrace{10}_{\texttt{n}}\,\texttt{0}\,\underbrace{10}_{\texttt{n}}\,\texttt{0}$

  b a n a n a

↯ C = 1100100100 decodes **both** to banana and to bass: $\underbrace{110}\,\texttt{0}\,\underbrace{100}\,\texttt{100}$

  b a s s

⇝ not a valid code . . . (cannot tolerate ambiguity)

but how should we have known?

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but how should we have known?

$E(\text{n}) = 10$ is a (proper) **prefix** of $E(\text{s}) = 100$ 101

⇝ Leaves decoding wondering whether to stop after reading 10 or continue

⇝ Require a ***prefix-free*** code: $\boxed{\text{No codeword is a prefix of another.}}$

prefix-free $\Longrightarrow$ instantaneously decodable 01001

codeword

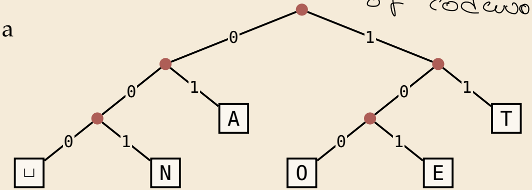
Code tries

- From now on only consider prefix-free codes E: $E(c)$ is not a prefix of $E(c')$ for any $c, c' \in \Sigma_S$.
- **Example:**

c	A	E	N	O	T	␣
$E(c)$	01	101	001	100	11	000

standard trie of codewords

Any prefix-free code corresponds to a ***(code) trie*** (trie of codewords) with characters of Σ_S at **leaves**.



no need for end-of-string symbols $ here
(already prefix-free!)

- Encode `AN␣ANT` 01001000...
- Decode `111000001010111` TO␣

Code tries

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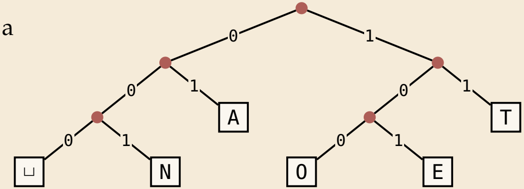

- Encode `AN␣ANT` $\longrightarrow$ `010010000100111`
- Decode `111000001010111` $\longrightarrow$ `TO␣EAT`

Who decodes the decoder?

- Depending on the application, we have to **store/transmit** the **used code**!
- We distinguish:
  - **fixed coding:** code agreed upon in advance, not transmitted (e. g., Morse, UTF-8)
  - **static coding:** code depends on message, but stays same for entire message; it must be transmitted (e. g., Huffman codes → next)
  - **adaptive coding:** code depends on message and changes during encoding; implicitly stored withing the message (e. g., LZW → below)

we have to transmit code

7.3 Huffman Codes

Character frequencies

- **Goal:** Find character encoding that produces short coded text
- Convention here: fix $\Sigma_C = \{0, 1\}$ (binary codes), abbreviate $\Sigma = \Sigma_S$,
- **Observation:** Some letters occur more often than others.

Typical English prose:

e	12.70%	**d**	4.25%	**p**	1.93%
t	9.06%	**l**	4.03%	**b**	1.49%
a	8.17%	**c**	2.78%	**v**	0.98%
o	7.51%	**u**	2.76%	**k**	0.77%
i	6.97%	**m**	2.41%	**j**	0.15%
n	6.75%	**w**	2.36%	**x**	0.15%
s	6.33%	**f**	2.23%	**q**	0.10%
h	6.09%	**g**	2.02%	**z**	0.07%
r	5.99%	**y**	1.97%		

$\leadsto$ Want shorter codes for more frequent characters!

Huffman coding

e. g. frequencies / probabilities

- **Given:** Σ and weights $w : \Sigma \to \mathbb{R}_{\geq 0}$
- **Goal:** prefix-free code E (= code trie) for Σ that minimizes coded text length

  i. e., a code trie minimizing $\displaystyle\sum_{c\in\Sigma} w(c) \cdot |E(c)|$

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- If we use $w(c) = \#\text{occurrences of } c \text{ in } S$,
  this is the character encoding with smallest possible $|C|$

  $\rightsquigarrow$ best possible character-wise encoding

- Quite ambitious! *Is this efficiently possible?*

Huffman's algorithm

- Actually, yes! A greedy/myopic approach succeeds here.

Huffman's algorithm:

1. Find two characters $\mathtt{a}$, $\mathtt{b}$ with lowest weights.
   - We will encode them with the same prefix, plus one distinguishing bit, i. e., $E(\mathtt{a}) = u\mathtt{0}$ and $E(\mathtt{b}) = u\mathtt{1}$ for a bitstring $u \in \{0, 1\}^\star$ (u to be determined)
2. (Conceptually) replace $\mathtt{a}$ and $\mathtt{b}$ by a single character "$\boxed{\mathtt{ab}}$" with $w(\boxed{\mathtt{ab}}) = w(\mathtt{a}) + w(\mathtt{b})$. $\rightarrow$ Σ decreases by 1
3. Recursively apply Huffman's algorithm on the smaller alphabet. This in particular determines $u = E(\boxed{\mathtt{ab}})$.

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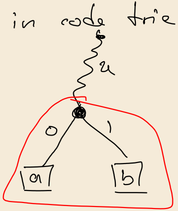


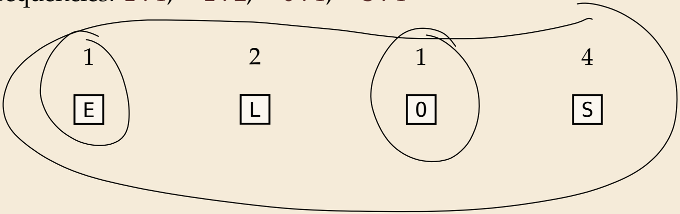
- efficient implementation using a (min-oriented) ***priority queue***
  - start by inserting all characters with their weight as key
  - step 1 uses two `deleteMin` calls
  - step 2 inserts a new character with the sum of old weights as key

Huffman's algorithm – Example

- Example text: $S = \underline{\texttt{LOSSLESS}}$ $\rightsquigarrow$ $\Sigma_S = \{\texttt{E}, \texttt{L}, \texttt{O}, \texttt{S}\}$

- Character frequencies: $\texttt{E} : 1$, $\texttt{L} : 2$, $\texttt{O} : 1$, $\texttt{S} : 4$

priority queue



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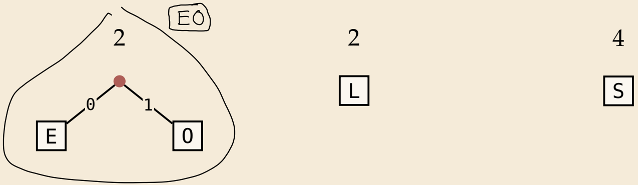

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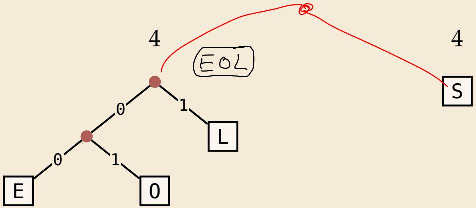

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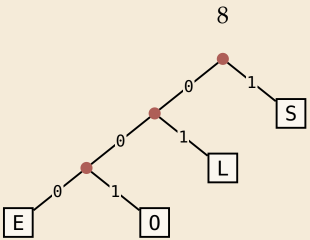

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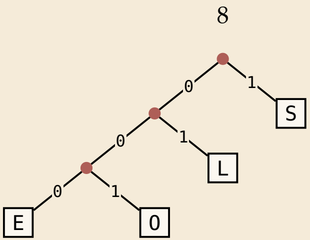

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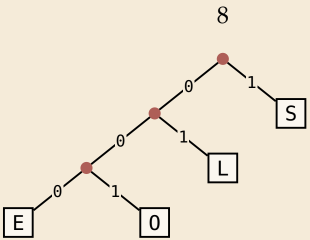

$\rightsquigarrow$ *Huffman tree* (code trie for Huffman code)

$\texttt{LOSSLESS} \to \underbrace{\texttt{01001110100011}}$ compression ratio: $\frac{14}{8 \cdot \log 4} = \frac{14}{16} \approx 88\%$

Huffman tree – tie breaking

- The above procedure is ambiguous:
  - which characters to choose when weights are equal?
  - which subtree goes left, which goes right?

- For COMP 526: always use the following rule:

> 1. To break ties when selecting the two characters, first use the smallest letter according to the alphabetical order, or the tree containing the smallest alphabetical letter.
> 2. When combining two trees of different values, place the lower-valued tree on the left (corresponding to a `0`-bit).
> 3. When combining trees of equal value, place the one containing the smallest letter to the left.

Huffman code – Optimality

Theorem 7.1 (Optimality of Huffman's Algorithm)
Given Σ and $w : \Sigma \to \mathbb{R}_{\geq 0}$, Huffman's Algorithm computes codewords $E : \Sigma \to \{0, 1\}^\star$ with minimal expected codeword length $\ell(E) = \sum_{c \in \Sigma} w(c) \cdot |E(c)|$, among all prefix-free codes for Σ. ◀

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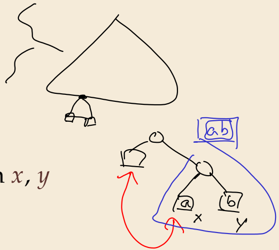
Proof sketch: by induction over $\sigma = |\Sigma|$



- Given any optimal prefix-free code E^* (as its code trie).
- code trie $\rightsquigarrow$ $\exists$ two sibling leaves x, y at largest depth D
- swap characters in leaves to have two lowest-weight characters `a`, `b` in x, y (that can only make ℓ smaller, so still optimal)
- any optimal code for $\Sigma' = \Sigma \setminus \{\mathtt{a}, \mathtt{b}\} \cup \{\boxed{\mathtt{ab}}\}$ yields optimal code for Σ by replacing leaf $\boxed{\mathtt{ab}}$ by internal node with children `a` and `b`.

$\rightsquigarrow$ recursive call yields optimal code for Σ' by inductive hypothesis, so Huffman's algorithm finds optimal code for Σ.