9 Graph Algorithms

9 December 2024

Prof. Dr. Sebastian Wild

CS566 (Wintersemester 2024/25)
Philipps-Universität Marburg

version `2024-12-10 21:11` H

Learning Outcomes

Unit 9: ***Graph Algorithms***

1. Know basic terminology from graph theory, including types of graphs.
2. Know adjacency matrix and adjacency list representations and their performance characteristica.
3. Know graph-traversal based algorithm, including efficient implementations.
4. Be able to proof correctness of graph-traversal-based algorithms.
5. Know algorithms for maximum flows in networks.
6. Be able to model new algorithmic problems as graph problems.

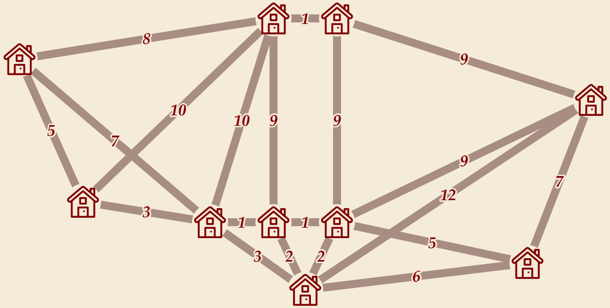
Outline

9 Graph Algorithms

9.1 Introduction & Definitions

Graphs in real life

- a graph is an abstraction of *entities* with their (pairwise) *relationships*
- abundant examples in real life (often called network there)
  - social networks: e. g. persons and their friendships, . . . *Five/Six? degrees of separation*
  - physical networks: cities and highways, roads networks, power grids etc., the Internet, . . .
  - content networks: world wide web, ontologies, . . .
  - . . .



Many More examples, e. g., in Sedgewick & Wayne's videos:

`https://www.coursera.org/learn/algorithms-part2`

Flavors of Graphs

- Since graphs are used to model so many different entities and relations, they come in several variants

Property	**Yes**	**No**
edges are one-way	*directed* graph (*digraph*)	*undirected* graph
≤ 1 edge between u and v	*simple* graph	*multigraph* / with *parallel* edges
edges can lead from v to v	with *loops*	(loop-free)
edges have weights	*(edge-) weighted* graph	*unweighted* graph

🙃 any combination of the above can make sense . . .

- Synonyms:
  - **vertex** („Knoten“) = node = point = „Ecke”
  - **edge** („Kante“) = arc = line = relation = arrow = „Pfeil“
  - **graph** = network

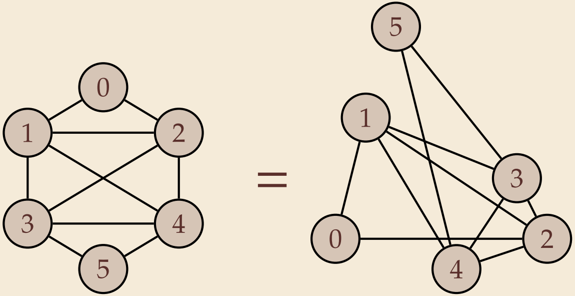
Graph Theory

- default: unweighted, undirected, loop-free & simple graphs
- *Graph* $G = (V, E)$ with
  - V a finite of *vertices*
  - $E \subseteq [V]^2$ a set of *edges*, which are 2-subsets of V: $[V]^2 = \{e : e \subseteq V \wedge |e| = 2\}$

Example

$$
\begin{aligned}
V &= \{0,1,2,3,4,5\} \\
E &= \big\{\{0,1\},\{1,2\},\{1,4\},\{1,3\},\{0,2\}, \\
&\qquad \{2,4\},\{2,3\},\{3,4\},\{3,5\},\{4,5\}\big\}.
\end{aligned}
$$

Graphical representation



like so . . .

. . . or so

(same graph)

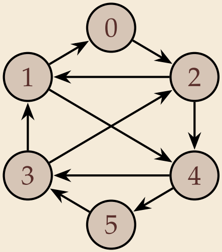
Digraphs

- default digraph: unweighted, loop-free & simple
- *Digraph (directed graph)* $G = (V, E)$ with
  - V a finite of *vertices*
  - $E \subseteq V^2 \setminus \{(v, v) : v \in V\}$ a set of *(directed) edges*,
    $V^2 = V \times V = \{(x, y) : x \in V \wedge y \in V\}$ 2-tuples / ordered pairs over V

Example

$$
\begin{aligned}
V &= \{0, 1, 2, 3, 4, 5\} \\
E &= \{(0, 2), (1, 0), (1, 4), (2, 1), (2, 4), \\
&\quad (3, 1), (3, 2), (4, 3), (4, 5), (5, 3)\}
\end{aligned}
$$

Graphical representation



Graph Terminology

Undirected Graphs

- $V(G)$ set of vertices, $E(G)$ set of edges
- write uv (or vu) for edge $\{u, v\}$
- edges *incident* at vertex v: $E(v)$
- u and v are *adjacent* iff $\{u, v\} \in E$,
- *neighborhood* $N(v) = \{w \in V : w \text{ adjacent to } v\}$
- *degree* $d(v) = |E(v)|$

Directed Graphs (where different)

- uv for (u, v)
- iff $(u, v) \in E \vee (v, u) \in E$
- in-/out-neighbors $N_{\text{in}}(v)$, $N_{\text{out}}(v)$
- in-/out-degree $d_{\text{in}}(v)$, $d_{\text{out}}(v)$

- *walk* w of length n: sequence of vertices $w[0..n]$ with $\forall i \in [0..n) : w[i]w[i+1] \in E$
- *path* p is a (vertex-) simple walk: without duplicate vertices except possibly its endpoints
- *edge-simple* walk: no edge used twice
- *cycle* c is a closed path, i. e., $c[0] = c[n]$

- G is *connected*
  iff for all $u \neq v \in V$ there is a path from u to v
- G is *acyclic* iff $\nexists$ cycle (of length $n \geq 1$) in G

- *strongly connected* for digraphs
  (*weakly connected* = connected ignoring directions)

Typical graph-processing problems

- **Path**: Is there a path between s and t?
  Shortest path: What is the shortest path (distance) between s and t?
- **Cycle**: Is there a cycle in the graph?
  Euler tour: Is there a cycle that uses each edge exactly once?
  Hamilton(ian) cycle: Is there a cycle that uses each vertex exactly once.
- **Connectivity**: Is there a way to connect all of the vertices?
  MST: What is the best way to connect all of the vertices?
  Biconnectivity: Is there a vertex whose removal disconnects the graph?
- **Planarity**: Can you draw the graph in the plane with no crossing edges?
- **Graph isomorphism**: Are two graphs the same up to renaming vertices?

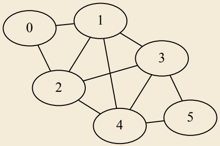
can vary a lot, despite superficial similarity of problems

Challenge: Which of these problems
can be computed in (near) linear time?
in reasonable polynomial time?
are intractable?

Tools to work with graphs

- Convenient GUI to edit & draw graphs: *yEd live*
  `yworks.com/yed-live`
- *graphviz* cmdline utility to draw graphs
  - Simple text format for graphs: `DOT`

```
graph G {
     0 -- 2;     2 -- 4;
     1 -- 0;     2 -- 3;
     1 -- 4;     3 -- 4;
     1 -- 3;     3 -- 5;
     2 -- 1;     4 -- 5;
}
```



`dot -Tpdf graph.dot -Kfdp > graph.pdf`

- graphs are typically not built into programming languages, but libraries exist
  - e. g. part of *Google Guava* for Java
  - they usually allow arbitrary objects as vertices
  - aimed at ease of use

9.2 Graph Representations

Graphs in Computer Memory

- We defined graphs in set-theoretic terms. . .
  but computers can't directly deal with sets efficiently

$\rightsquigarrow$ need to choose a *representation* for graphs.

- which is better depends on the required operations

Key Operations:

- `isAdjacent(`u`,`v`)`
  Test whether $uv \in E$
- `adj(`v`)`
  Adjacency list of v (iterate through (out-) neighbors of v)
- most others can be computed based on these

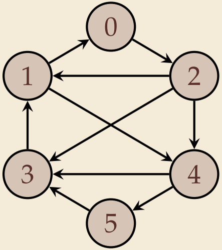
Conventions:

- (di)graph $G = (V, E)$ (omitted if clear from context)
- $n = |V|$, $m = |E|$
- in implementations assume $V = [0..n)$ (if needed, use symbol table to map complex objects to V)

Adjacency Matrix Representation

- adjacency matrix $A \in \{0,1\}^{n \times n}$ of G: matrix with $A[u,v] = [uv \in E]$
  - works for both directed and undirected graphs (undirected $\rightsquigarrow A = A^T$ symmetric)
  - can use a weight $w(uv)$ or multiplicity in $A[u,v]$ instead of $0/1$
  - can represent loops via $A[v,v]$

Example:



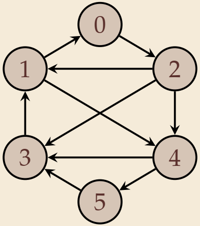
$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

👍 `isAdjacent` in $O(1)$ time

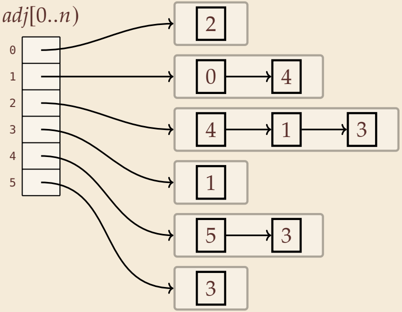
👎 $O(n^2)$ (bits of) space wasteful for sparse graphs

👎 `adj(`v`)` iteration takes $O(n)$ (independent of $d(v)$)

Adjacency List Representation



- Store a linked list of neighbors for each vertex v:
  - $adj[0..n)$ bag of neighbors (as linked list)
  - undirected edge $\{u,v\}$ $\rightsquigarrow$ v in $adj[u]$ and u in $adj[v]$
  - weighted edge uv $\rightsquigarrow$ store pair $(v, w(uv))$ in $adj[u]$
  - multiple edges and loops can be represented



`isAdjacent(`u`,`v`)` takes $\Theta(d(u))$ time (worst case)

`adj(`v`)` iteration $O(1)$ per neighbor

$\Theta(n+m)$ (words of) space for any graph (≪ $\Theta(n^2)$ bits for moderate m)

$\rightsquigarrow$ de-facto standard for graph algorithms

Graph Types and Representations

- Note that adj matrix and lists for undirected graphs effectively are representation of directed graph with directed edges both ways
  - conceptually still important to distinguish!
- multigraphs, loops, edge weights all naturally supported in adj lists
  - good if we allow and use them
  - but requires explicit checks to enforce simple / loopfree / bidirectional!

- we focus on **static graphs**
  dynamically changing graphs much harder to handle

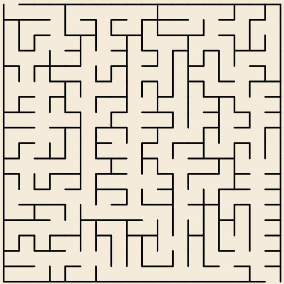
9.3 Graph Traversal

Generic Graph Traversal

- Plethora of graph algorithms can be expressed as a systematic exploration of a graph

  visiting all nodes & edges

  - depth-first search, breadth-first search
  - connected components
  - detecting cycles
  - topological sorting
  - Hierholzer's algorithm for Euler walks
  - strong components
  - testing bipartiteness
  - Dijkstra's algorithm
  - Prim's algorithm
  - Lex-BFS for perfect elimination orders of chordal graphs
  - ...



$\rightsquigarrow$ Formulate generic traversal algorithm

  - first in abstract terms to argue about correctness
  - then again for concrete instance with efficient data structures

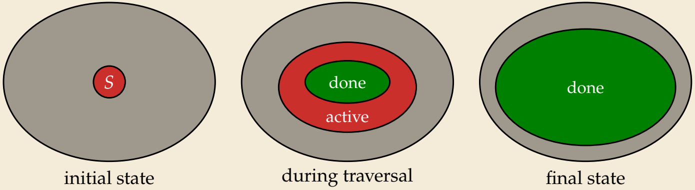
Tricolor Graph Traversal

Tricolor Graph Search:

- maintain vertices in 3 (dynamic) sets
  - **Gray: unseen vertices**
    The traversal has not reached these vertices so far.
  - **Green: done vertices** (a.k.a. visited vertices)
    These vertices have been visited and all their edges have been explored already.
  - **Red: active vertices** (a.k.a. frontier („Rand“) of traversal)
    All others, i.e., vertices that have been reached and some unexplored edges remain; initially some selected start vertices S.

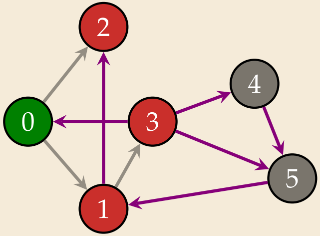
Invariant:
No edges from ***done*** to *unseen* vertices

- (implicitly) maintain status of each edge
  - **not yet used**
  - **used edge**
- Vertices "want" to turn green.



Generic Tricolor Graph Traversal – Code

```
procedure genericGraphTraversal(G, S)
    // (di)graph G = (V, E) and start vertices S ⊆ V
    C[0..n) := unseen // Color array, all cells initialized to unseen
    for s ∈ S do C[s] := active end for
    unusedEdges := E
    while ∃v : C[v] == active
        v := nextActiveVertex() // Freedom 1: Which frontier vertex?
        if ∄vw ∈ unusedEdges // no more edges from v ⇝ done with v
            C[v] := done
        else
            w := nextUnusedEdge(v) // Freedom 2: Which of its edges?
            if C[w] == unseen
                C[w] := active
            end if
            unusedEdges.remove(vw)
        end if
    end while
```

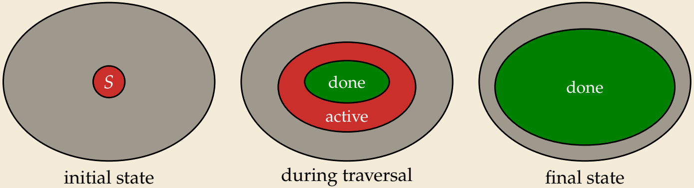


Invariant:
No edges from *done* to *unseen* vertices

- Implementations of nextActiveVertex() and nextUnusedEdge(v) depends on (and defines!) specific traversal-based graph algorithms

Generic Reachability

- Any choices nextActiveVertex() and nextUnusedEdge(v) suffice to find exactly the vertices reachable from S in ***done***
- **Invariant:**
  1. No edges from ***done*** to ***unseen*** vertices
  2. For every ***done*** or ***active*** vertex v, there exists a path from $s \in S$ to v.



$\rightsquigarrow$ in final state:

- $v \in$ ***done*** $\rightsquigarrow$ path from S $\rightsquigarrow$ reachable from S
- $v \in$ ***unseen*** $\rightsquigarrow$ not reachable from ***done*** $\supseteq S$ $\rightsquigarrow$ not reachable from S

Data Structures for Frontier

- We need efficient support for
  - test $\exists v : C[v] = \textbf{\textit{active}}$, nextActiveVertex()
  - test $\exists vw \in \mathit{unusedEdges}$, nextUnusedEdge($v$)
  - $\mathit{unusedEdges}$.remove(vw)
- Typical solution maintains **bag** "*frontier*" of *pairs* (v, i)
  where $v \in V$ and i is an **iterator** in `adj`$[v]$
  - $\mathit{unusedEdges}$ represented implicitly: edge used iff previously returned by i
    $\rightsquigarrow$ don't need $\mathit{unusedEdges}$.remove(vw)
  - Implement $\exists v : C[v] = \textbf{\textit{active}}$ via $\mathit{frontier}$.isEmpty()
  - Implement $\exists vw \in \mathit{unusedEdges}$ via i.hasNext() assuming $(v, i) \in \mathit{frontier}$
  - Implement nextUnusedEdge(v) via i.next() assuming $(v, i) \in \mathit{frontier}$

$\rightsquigarrow$ all operations apart from nextActiveVertex() in $O(1)$ time

$\rightsquigarrow$ $\mathit{frontier}$ requires $O(n)$ extra space

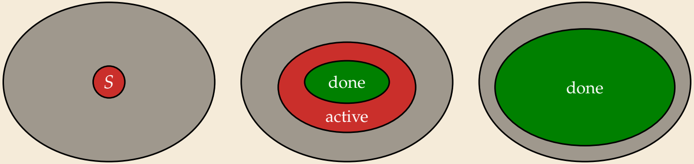
9.4 BFS and DFS

Breadth-First Search

- Maintain *frontier* in a **queue** (FIFO: first in, first out)

- **Invariant:**
  1. No edges from done to unseen vertices
  2. All ***done*** or ***active*** vertices are reached via a **shortest path** from S
  3. Vertices enter and leave *frontier* in order of increasing distance from S

fewest edges



initial state | during traversal | final state

$\rightsquigarrow$ in final state, we reach all reachable vertices via shortest paths

- To preserve that knowledge, we collect extra information during traversal
  - $parent[v]$ stores predecessor on path from S via which v was reached
  - $distFromS[v]$ stores the length of this path

Breadth-First Search – Code

```
procedure bfs(G, S)
    // (di)graph G = (V, E) and start vertices S ⊆ V
    C[0..n) := unseen // New array initialized to all unseen
    frontier := new Queue;
    parent[0..n) := NOT_VISITED;   distFromS[0..n) := ∞
    for s ∈ S
        parent[s] := NONE;   distFromS[s] := 0
        C[s] := active;   frontier.enqueue((s, G.adj[s].iterator()))
    end for
    while ¬frontier.isEmpty()
        (v, i) := frontier.peek()
        if ¬i.hasNext() // v has no unused edge
            C[v] := done;   frontier.dequeue()
        else
            w := i.next() // Advance i in adj[v]
            if C[w] == unseen
                parent[w] := v;   distFromS[w] := distFromS[v] + 1
                C[w] := active;   frontier.enqueue((w, G.adj[w].iterator()))
            end if
        end if
    end while
```

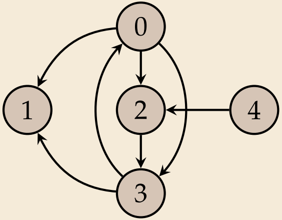
- *parent* stores a *shortest-path tree/forest*
- can retrieve shortest path to v from some vertex $s \in S$ (backwards) by following $parent[v]$ iteratively
- running time $\Theta(n + m)$
- extra space $\Theta(n)$

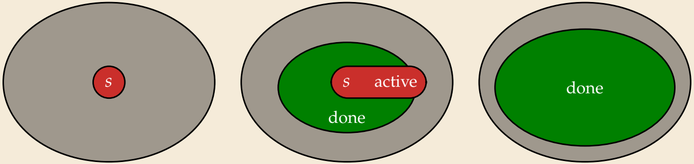
Depth-First Search

- Maintain *frontier* in a **stack** (LIFO: last in, first out)
  - only consider $S = \{s\}$
  - usual mode of operation: call `dfs`(v) for all ***unseen*** v, for $v = 0, \ldots, n-1$

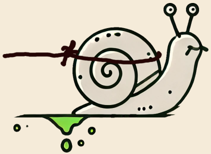
- **Invariant:**

  1. No edges from done to unseen vertices
  2. All ***done*** vertices are reached via a path from s
  3. The ***active*** vertices form a single **path** from s





initial state   during traversal   final state



Depth-First Search – Code

```
procedure dfsTraversal(G)
    C[0..n) := unseen
    for v := 0, ..., n − 1
        if C[v] == unseen
            dfs(G, v)

procedure dfs(G, s)
    frontier := new Stack;
    C[s] := active;  frontier.push((s, G.adj[s].iterator()))
    while ¬frontier.isEmpty()
        (v, i) := frontier.top()
        if ¬i.hasNext() // v has no unused edge
            C[v] := done;  frontier.pop();  postorderVisit(v)
        else
            w := i.next();  visitEdge(vw)
            if C[w] == unseen
                preorderVisit(w)
                C[w] := active;  frontier.push((w, G.adj[w].iterator()))
            end if
        end if
    end while
```

- define *hooks* to implement further operations
  - preorder: visit v when made ***active*** (start of $T(v)$)
  - postorder: visit v when marked ***done*** (end of $T(v)$)
  - visitEdge: do something for every edge
- if needed, can store DFS forest via *parent* array
- running time $\Theta(n + m)$
- extra space $\Theta(n)$

Simple DFS Application: Connected Components

- In an undirected graph, find all *connected components*.
  - **Given:** simple undirected $G = (V, E)$
  - **Goal:** assign component ids $CC[0..n)$, s.t. $CC[v] = CC[u]$ iff $\exists$ path from v to u

```
procedure connectedComponents(G):
    // undirected graph G = (V, E) with V = [0..n)
    C[0..n) := unseen
    CC[0..n) := NONE
    id := 0
    for v := 0, . . . , n − 1
        if C[v] == unseen
            dfs(G, v)
            id := id + 1
    return CC

procedure preorderVisit(v):
    CC[v] := id
```

```
// same as before
procedure dfs(G, s)
    frontier := new Stack;
    C[s] := active; frontier.push((s, G.adj[s].iterator()))
    while ¬frontier.isEmpty()
        (v, i) := frontier.top()
        if ¬i.hasNext() // v has no unused edge
            C[v] := done; frontier.pop()
            postorderVisit(v)
        else
            w := i.next(); visitEdge(vw)
            if C[w] == unseen
                preorderVisit(w)
                C[w] := active
                frontier.push((w, G.adj[w].iterator()))
            end if
        end if
    end while
```

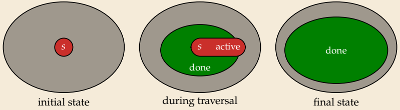
Dijkstra's Algorithm & Prim's Algorithm

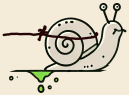
- On edge-weighted graphs, we can use tricolor traversal with a *priority queue* as *frontier*
- Dijkstra's Algorithm for shortest paths from s in digraphs with weakly positive edge weights
  - priority of vertex v = length of shortest path known so far from s to v
- Prim's Algorithm for finding a minimum spanning tree
  - priority of vertex v = weight of cheapest edge connecting v to current tree

⇝ Detailed discussion in Unit 11

9.5 Advanced Uses of DFS

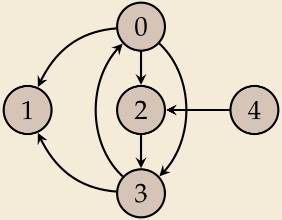
Properties of DFS



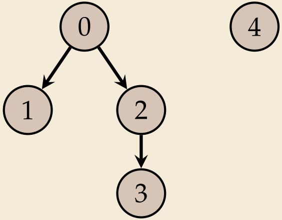


- **Recall DFS Invariant 3:**
  The *active* vertices form a single **path** from s

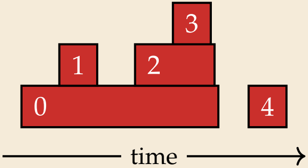
input graph G



DFS forest



stack over time



$\rightsquigarrow$ Each vertex v spends *time interval* $T(v)$ as *active* vertex

1. *frontier* is stack $\rightsquigarrow$ $\{T(v) : v \in V\}$ forms *laminar set family*: ("disjoint or contained")
   either $T(v) \cap T(w) = \emptyset$ or $T(v) \subseteq T(w)$ or $T(v) \supseteq T(w)$

2. **Parenthesis Theorem:** $T(v) \supseteq T(w)$ **iff** v is ancestor of w in DFS tree
   '$\Rightarrow$' during $T(v)$, all discovered vertices become descendants of v
   '$\Leftarrow$' $T(v)$ covers v's entire subtree, which contains w's subtree

Properties of DFS – Unseen-Path Theorem

- **Unseen-Path Theorem:** In a DFS forest of a (di)graph G, w is a descendant of v **iff** at the time of `preorderVisit`(v), there is a path from v to w using only *unseen* vertices.

'$\Rightarrow$' If w is a descendant of v, $T(w) \subseteq T(v)$ by the Parenthesis Theorem.
Hence the path from v to w in the DFS tree consists (at time of `preorderVisit`(v)) of solely *unseen* vertices.

'$\Leftarrow$' Suppose towards a contradiction that there was a w with an *unseen* path $p[0..\ell]$ with $p[0] = v$ and $p[\ell] = w$, but w is not a descendant of v. W.l.o.g. let w be a first such vertex, i. e., $p[0], \ldots, p[\ell - 1] = u$ are descendants of v.
So $T(u) \subset T(v) \quad (*)$.
Upon processing u, we will discover edge uw, so whether or not w is already *done* at this point, w will be marked *done* before u. Hence $\max T(w) \leq \max T(u)$.
With $(*)$, we obtain $\min T(v) \leq \min T(u) \leq \max T(w) \leq \max T(u)$, so by laminarity, $T(w) \subset T(u) \subset T(v)$ and w is a descendant of v ↯.

Topological Sorting & Cycle Detection

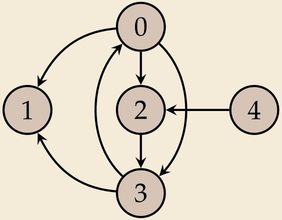
- **Application:** Given a set of tasks with precedence constraints of the form "a must be done before b", can we find a legal ordering for all tasks?

  $\rightsquigarrow$ Model as directed graph!
  - tasks are the vertices V
  - add an edge (a, b) when a must be done before b

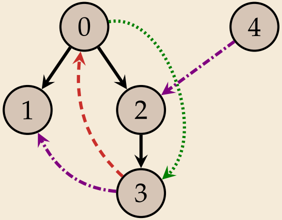
- **Definition:** $R[0..n)$ is a *topological (order) ranking* of digraph $G = (V, E)$ if $\forall (u, v) \in E : R[u] < R[v]$
- **Lemma DAG iff topo:**
  A directed graph G has a topological ranking **iff** it does not contain a directed cycle.
- **Topological Sorting**
  - **Given:** simple digraph $G = (V, E)$
  - **Goal:** Compute topological ranking of vertices $R[0..n)$ or output a directed cycle in G.
- Amazingly, can do all with one pass of DFS!

DFS Edge Types

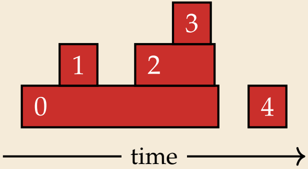
input digraph G



DFS forest



stack over time



- During DFS traversal, an edge vw has one of these 4 types: **example:**

  1. **tree edge:** $\longrightarrow$ $w \in$ ***unseen*** $\rightsquigarrow$ vw part of DFS forest. $(0,1)$, $(0,2)$, $(2,3)$
  2. **back edges:** $\dashrightarrow$ $w \in$ ***active***; $\rightsquigarrow$ w points to ancestor of v. $(3,0)$
  3. **forward edges*:** $\dashrightarrow$ $w \in$ ***done*** $\land$ w is descendant of v in DFS tree. $(0,3)$
  4. **cross edges*:** $\dashrightarrow$ $w \in$ ***done*** $\land$ w is not descendant of v. $(3,0)$

  *only possible in *directed* graphs

Cycle Detection

If G contains a directed cycle, DFS will find a directed cycle:

- any back edge implies a cycle:
  - DFS visits an edge (v, w) where $w \in \textit{active}$, w is already on the stack

  $\rightsquigarrow$ DFS tree contains path $w \rightsquigarrow v$ and we have edge $v \to w$.

- conversely any cycle $C[0..k]$ once reached must have some back edge or cross edge (tree and forward edges go from smaller to larger preorder index)
  - cannot be a cross edge since cycle is strongly connected
    all cycle vertices must be descendants of first reached cycle vertex

  $\rightsquigarrow$ cycle contributes a back edge

DFS Postorder Implementation

```
procedure dfsPostorder(G):
    C[0..n) := unseen
    P[0..n) := NONE;  r := 0
    parent[0..n) := NONE
    cycle := NONE
    for v := 0, ..., n − 1
        if C[v] == unseen
            dfs(G, v)
    return (P, cycle)

procedure postorderVisit(v):
    P[v] := r;  r := r + 1

procedure visitEdge(vw):
    if C[w] == active
        if cycle ≠ NONE return
        while v ≠ w
            cycle.append(v)
            v := parent[v]
        cycle.append(v)
```

```
// dfs is as in CC but with parent
procedure dfs(G, s)
    frontier := new Stack;
    parent[s] := NONE;
    C[s] := active;  frontier.push((s, G.adj[s].iterator()))
    while ¬frontier.isEmpty()
        (v, i) := frontier.top()
        if ¬i.hasNext() // v has no unused edge
            C[v] := done;  frontier.pop()
            postorderVisit(v)
        else
            w := i.next() // Advance i in adj[v]
            visitEdge(vw)
            if C[w] == unseen
                parent[w] := v;
                preorderVisit(w)
                C[w] := active;  frontier.push((w, G.adj[w].iterator()))
            end if
        end if
    end while
```

DFS Postorder & Topological Sort

- **DFS Postorder**: The DFS postorder numbers is a numbering $P[0..n)$ of V such that $P[v] = r$ iff exactly r vertices reached state ***done*** before v in a DFS.

- **Lemma rev postorder:**
  directed acyclic graph
  Let G be a simple, connected DAG and $R[0..n)$ a *reverse DFS postorder* of G, i. e., $R[v] = n - 1 - P[v]$ for a DFS postorder $P[0..n)$. Then R is a topological ranking of G.

- **Invariant:** If $v \in$ ***done*** and $(v, w) \in E$ then $w \in$ ***done*** and $R[v] < R[w]$.
  - initially true (***done*** $= \emptyset$)
  - upon `postorderVisit`(v), all outgoing edges vw lead to $w \in$ ***done*** (Parenthesis Theorem)

Topological Sorting & Cycle Detection – Summary

- Putting everything together we obtain topological sorting
  - can produce either the *ranking* or the *sequence of vertices* in topological order, whatever is more convenient

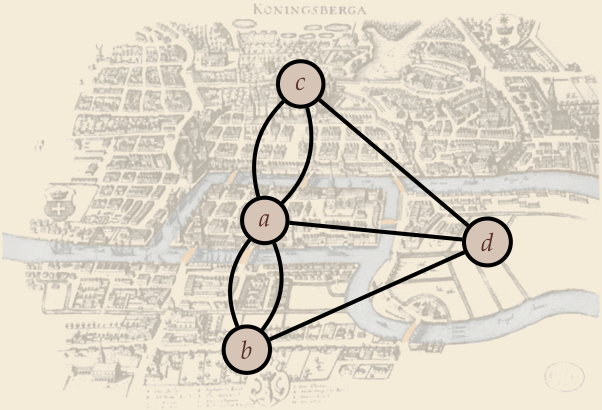
```
procedure topologicalRanking(P):
    (P[0..n), cycle) := dfsPostorder(G)
    if c ≠ NULL
        return NOT_A_DAG
    R[0..n) := NONE
    for v := 0, . . . , n − 1
        R[v] = n − 1 − P[v]
    return P
```

```
procedure topologicalSort(P):
    (P[0..n), cycle) := dfsPostorder(G)
    if c ≠ NULL
        return NOT_A_DAG
    S[0..n) := NONE
    for v := 0, . . . , n − 1
        S[n − 1 − P[v]] := v
    return S
```

- $\Theta(n + m)$ time
- $\Theta(n)$ extra space

Euler Cycles

Euler Walk: Walk using every edge in $G = (V, E)$ exactly once.



Euler's Theorem:

Euler walk exists iff G connected and 0 or 2 vertices have odd degree.

'$\Rightarrow$' trivial (need to enter and exit vertices equally often)

'$\Leftarrow$' Following algorithm constructs Euler walk

Euler Cycles – Hierholzer's Algorithm

- use an *edge-centric DFS*
  - We mark *edges* (not vertices)
  - ⇝ stack = **edge-simple walk**
  - We remember iterator i globally per v to resume traversal

```
procedure eulerWalk(G):
    // Assume G = (V, E) is connected (multi)graph
    V_odd := {v ∈ V : d(v) odd}
    if |V_odd| ∉ {0,2} return NOT_EULERIAN
    if V_odd = {x, y} then s := x else s := 0
    euler[0..m) := NONE;   j := m − 1
    visited[0..n, 0..n) := false // mark edges as visited
    for v := 0, ..., n − 1
        // globally remember next unexplored edge
        nextEdge[v] := G.adj[w].iterator())
    edgeDFS(s)
    return euler
```

```
procedure edgeDFS(s):
    frontier := new Stack;
    frontier.push(s)
    while ¬frontier.isEmpty()
        v := frontier.top()
        if ¬i.hasNext() // v has no unused edge
            frontier.pop()
            if ¬frontier.isEmpty()
                // assign edge to here largest free index
                euler[j] := (frontier.top(), v);   j := j − 1
            end if
        else
            w := i.next()
            if ¬visited[v, w]
                visited[v, w] := true
                visited[w, v] := true
                frontier.push(w)
            end if
        end if
    end while
```

Strong Components

- **Given:** digraph $G = (V, E)$
- **Goal:** component ids $SCC[0..n)$, s.t. $SCC[v] = SCC[u]$ iff $\exists$ directed path from v to u

  strongly connected component
- *Component DAG* G^{SCC}*:* contract SCCs intro single vertices
  $V(G^{SCC}) = \{C_1, \ldots, C_k\}$ with $C_1 \dot\cup \cdots \dot\cup C_k = V$;
  name by smallest vertex s.t. $i \le j$ iff $\min C_i \le \min C_j$
  - can't have cycles (↯ maximality of SCC)

  $\rightsquigarrow$ component DAG has a topological order $R^{SCC}[1..k]$

If we call `dfs` on any v in the **last** SCC C, it will discover all vertices in C, and only those!
(any edges between components lead *into* C by topological order)

And we can iterate this backwards through any topological order to get all SCCs!

Can we efficiently find the topological order of G^{SCC}?
Without knowing the components to start with??

Amazingly, yes.

Component Graph DFS

- Suppose we run `dfsTraversal` on G.
  - ⇝ We can extend time intervals to SCCs: $T(C_i) := \bigcup_{v \in C_i} T(v)$
  - ⇝ $T(C_i) = T(v_i)$ for $v_i \in C_i$ the first vertex to be explored in a DFS on G (by Unseen Path & Parenthesis Thms)

⇝ DFS on G produces same $T(C_i)$ (up to time scaling) as DFS on G^{SCC}!

⇝ reverse DFS postorder on G gives same relative order to $v_1, \ldots, v_k$ as reverse DFS postorder on G^{SCC} gives as relative order to $C_1, \ldots, C_k$

We need **reverse** topological order on G^{SCC}, e. g., ***reversed*** *reverse DFS postorder*

- If we had the actual reverse DFS postorder on G^{SCC}, could just reverse again!
- But we only have reverse DFS postorder $S[0..n)$ on G!
- ϟ Reversing here would change v_i, i. e., which vertices of an SCC we see first

Kosaraju-Sharir's Algorithm

- **Recall:** Want $\texttt{reverse}\big(\texttt{topologicalRanking}(G^{\mathrm{SCC}})\big)$
- **Transpose/Reverse Graph of $G = (V, E)$:** $G^T = (V, E^T)$ where $E^T = \{wv : vw \in E\}$
  Note: A adj matrix of $G \rightsquigarrow A^T$ adj matrix of G^T
- For any DAG, we obtain a reverse topological order from reversing all edges:
  $\texttt{topologicalSort}(G^T)$
- Observation: $\big(G^T\big)^{\mathrm{SCC}} = \big(G^{\mathrm{SCC}}\big)^T$
  - strong components not affected by edge reversals
- Want: $\texttt{reverse}\big(\texttt{topologicalRanking}(G^{\mathrm{SCC}})\big)$ (any ranking works, need not be reverse DFS postorder)

$\rightsquigarrow$ Get it from: $\texttt{topologicalRanking}\big((G^{\mathrm{SCC}})^T\big) = \texttt{topologicalRanking}\big((G^T)^{\mathrm{SCC}}\big)$

$\rightsquigarrow$ Get that as induced ranking on $v_1, \ldots, v_k$ from reverse $\texttt{dfsPostorder}(G^T)$

Kosaraju-Sharir's Algorithm – Code

```
procedure strongComponents(G):
    // directed graph G = (V, E) with V = [0..n)
    G^T = (V, {wv : vw ∈ E})
    P[0..n) := dfsPostorder(G^T) // postorder numbers
    for v ∈ V do S[P[v]] := v end for // postorder sequence
    // Rest mostly like in connectedComponents
    C[0..n) := unseen
    SCC[0..n) := NONE
    id := 0
    for j := n − 1, . . . , 0 // reverse postorder seq
        v := S[j]
        if C[v] == unseen
            dfs(G, v)
            id := id + 1
    return SCC

procedure preorderVisit(v):
    SCC[v] := id
```

- correctness follows from our discussion
- ordering of SCCs follows reverse topological sort of G^{SCC}
  - some implementations reverse G for 2nd DFS, not 1st
  - ⇝ output in (forward) topological order
  - but derivation more natural this way?
- as all our traversals:
  $\Theta(n + m)$ time,
  $\Theta(n)$ extra space

9.6 Network flows

Networks and Flows

Reductions

9.7 The Ford-Fulkerson Method

Residual Networks

Augmenting Paths