12 Dynamic Programming

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CS566 (Wintersemester 2024/25)
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Learning Outcomes

Unit 12: ***Dynamic Programming***

1. Be able to apply the DP paradigm to solve new problems.

Outline

12 Dynamic Programming

12.1 Elements of Dynamic Programming

Introduction

- *Dynamic Programming (DP)* is a powerful algorithm **design pattern** for exact solutions to **optimization** problems

  applicable to many problems

- Some commonalities with Greedy Algorithms,
  but with an element of brute force added in

  DP = "careful brute force" (Erik Demaine)

- often yields polynomial time, but usually not linear time algorithms

- for many problems the *only* way we know to build efficient algorithms

- **Naming fun:** The term "dynamic programming", due to Richard Bellman from around 1953,
  does not refer to computer programming; rather to a program (= plan, schedule) changing with time.
  It seems to have been at least partly marketing babble devoid of technical meaning . . .

Plan of the Unit

1. Abstract steps of DP (briefly)
2. Details on a concrete example (*matrix chain multiplication*)
3. More examples!

The 6 Steps of Dynamic Programming

1. Define **subproblems** (and relate to original problem)
2. **Guess** (part of solution) $\rightsquigarrow$ local brute force
3. Set up **DP recurrence** (for quality of solution)
4. Recursive implementation with **Memoization**
5. Bottom-up **table filling** (topological sort of subproblem dependency graph)
6. **Backtracing** to reconstruct optimal solution

- Steps 1–3 require insight / creativity / intuition;
  Steps 4–6 are mostly automatic / same each time

$\rightsquigarrow$ Correctness proof usually at level of DP recurrence

👍 running time too! worst case time = #subproblems · time to find single best guess

When does DP (not) help?

- *No Silver Bullet*
  DP is the most widely applicable design technique, but can't *always* be applied

1. Vitally important for DP to be correct:

Bellman's Optimality Criterion

For a *correctly guessed* fixed part of the solution,
***any* optimal solution to the corresponding subproblems**
must yield an *optimal solution* to the overall problem (once combined).

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   ***any* optimal solution to the corresponding subproblems**
   must yield an *optimal solution* to the overall problem (once combined).

   at most polynomial in n

2. Also, the total **number of different subproblems** should be *"small"*
   (DP potentially still works correctly otherwise, but won't be *efficient*.)

12.2 DP & Matrix Chain Multiplication

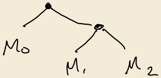
The Matrix-Chain Multiplication Problem

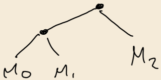
Consider the following exemplary problem

- We have a product $M_0 \cdot M_1 \cdot \cdots \cdot M_{n-1}$ of n matrices to compute
- Since (matrix) multiplication is associative, it can be evaluated in different orders.
- For non-square matrices of different sizes, different order can change costs dramatically
  - Assume elementary matrix multiplication algorithm:
  - $\rightsquigarrow$ Multiplying $a \times b$-matrix with $b \times c$ matrix costs $a \cdot b \cdot c$ integer multiplications
- **Given:** Row and column counts $\overset{r}{c}[0..n)$ and $\overset{c}{r}[0..n)$ with $r[i+1] = c[i]$ for $i \in [0..n-1)$
  (corresponding to matrices $M_0, \ldots, M_{n-1}$ with $M_i \in \mathbb{R}^{r[i] \times c[i]}$)
- **Goal:** parenthesization of the product chain with minimal cost

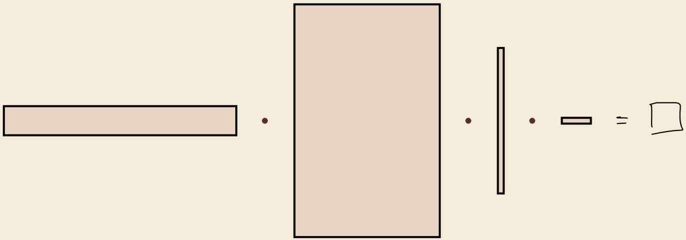
really a binary tree with n leaves!

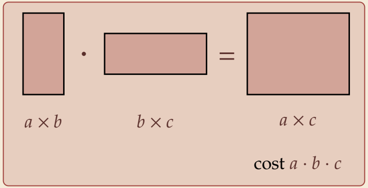
$(M_0 \cdot (M_1 \cdot M_2))$ $((M_0 \cdot M_1) \cdot M_2)$



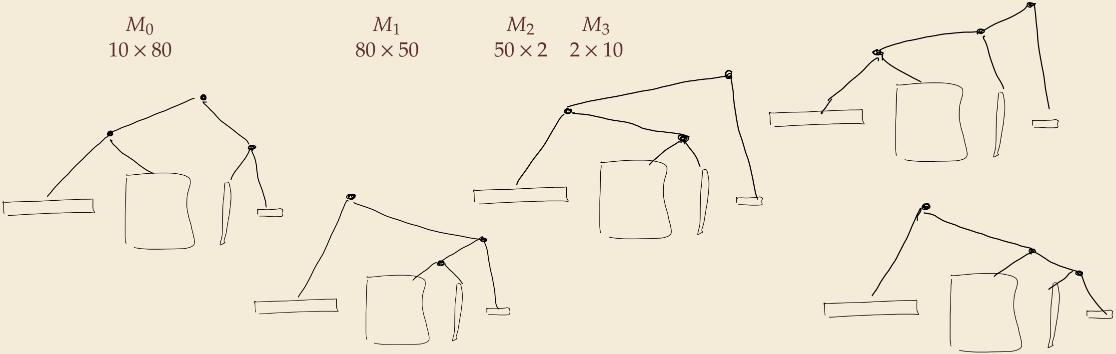


Matrix-Chain Multiplication – Example





M_0	M_1	M_2	M_3
10×80	80×50	50×2	2×10



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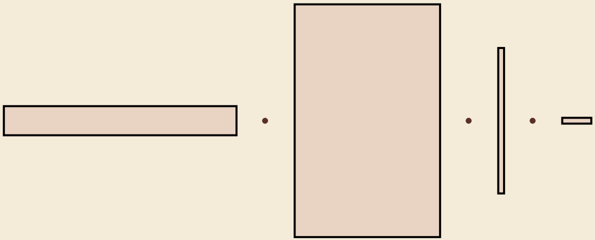
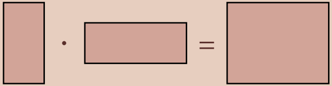




Parenthesization	Cost (integer multiplications)		
$M_0 \cdot (M_1 \cdot (M_2 \cdot M_3))$	$1000 + 40\,000 + 8000$	$=$	$49\,000$
$M_0 \cdot ((M_1 \cdot M_2) \cdot M_3)$	$8000 + 1600 + 8000$	$=$	$17\,600$
$(M_0 \cdot M_1) \cdot (M_2 \cdot M_3)$	$40\,000 + 1000 + 5000$	$=$	$46\,000$
$(M_0 \cdot (M_1 \cdot M_2)) \cdot M_3$	$8000 + 1600 + 200$	$=$	$9\,800$
$((M_0 \cdot M_1) \cdot M_2) \cdot M_3$	$40\,000 + 1000 + 200$	$=$	$41\,200$

first or last operation

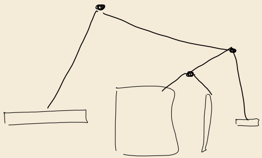
Greedy fails both ways!

Matrix-Chain Multiplication – How about Brute Force?

If Greedy doesn't give optimal parenthesization, maybe just try all?

- parenthesizations for n matrices $=$ binary trees with n leaves (*evalution trees*)
  $=$ binary trees with $n-1$ (internal) nodes

- How many such trees are there?



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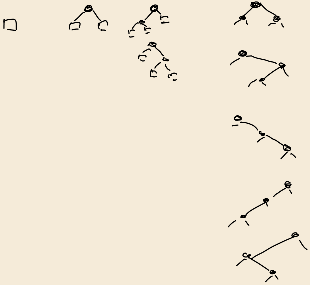
- parenthesizations for n matrices $=$ binary trees with n leaves (*evalution trees*)
  $=$ binary trees with $\underbrace{n-1}_{m}$ (internal) nodes
- How many such trees are there?
  - Let's write $m = n - 1$;
  - $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$



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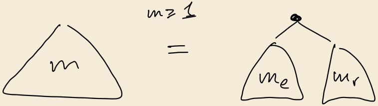
  - $C_m = \sum_{r=1}^{m} C_{r-1} \cdot C_{m-r} \qquad (m \geq 1)$

rank of root



$m_l + m_r = m - 1$

$0 \leq m_l, m_r \leq m - 1$

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    generating functions / combinatorics / guess (OEIS!) & check ...
  - Can show $C_n = \dfrac{1}{n+1}\dbinom{2n}{n} \sim \dfrac{1}{\sqrt{\pi}} \cdot \dfrac{4^n}{n^{3/2}}$

  $\rightsquigarrow$ *exponentially many trees (almost 4^n)* $\qquad C_{20} = 6\,564\,120\,420, \;\; C_{30} = 3\,814\,986\,502\,092\,304$

$\rightsquigarrow$ A brute-force approach is utterly hopeless

$\rightsquigarrow$ *Dynamic programming to the rescue!*

Matrix-Chain Multiplication – Step 1: Subproblems

- Key ingredient for DP: Problem allows for recursive formulation
  Need to decide:
  1. What are the **subproblems** to consider?
  2. How can the **original problem** be expressed as subproblem(s)?

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
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Here:

1. **Subproblems** = Ranges of matrices $[i..j)$ $\quad 0 \le i \le j \le n$
   i. e., optimal parenthesization for each range $M_i, M_{i+1}, \ldots, M_{j-1}$

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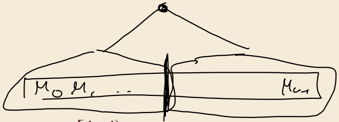

- **Intuition:**
  - Any subtree in binary multiplication tree covers some range $[i..j)$
    (matrix multiplication is not commutative $\rightsquigarrow$ left-right order has to stay)
  - left and right factors of a multiplication don't "see/influence" each other

Matrix-Chain Multiplication – Step 2: Guess

- Usually, any subproblem can be split into smaller subproblems in different ways
- Which way to decompose gives best solution not known *a priori*

$\rightsquigarrow$ Assuming we can correctly *guess* this part; how to solve problem?

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⇝ Assuming we can correctly *guess* this part; how to solve problem?

- Here: **Guess** last multiplication / root of binary tree

⇝ index $k \in [i+1 \mathinner{..} j)$ so that $[i..j)$ computed with **last** multiplication $\underline{(M_i \cdot \dots \cdot M_{k-1})} \cdot \underline{(M_k \cdot \dots \cdot M_{j-1})}$

⇝ optimal parenthesization of $M_i, \dots, M_{k-1}$ and $M_k, \dots, M_{j-1}$ computed recursively (corresponds to subproblems $[i..k)$ and $[k..j)$)

try all k!

Matrix-Chain Multiplication – Step 3: DP Recurrence

- With subproblems and guessed part fixed,
  we try to express total **value/cost of solution** *recursively*

⇝ *We ignore the actual solution and just compute its cost!*

- Often good to prove correctness at level of recurrence

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subproblem [i..j)

- Here: **Recurrence** for $m(i,j)$ = total number of integer multiplications used in best parenthesization of $[i..j)$

⇝ Set up recurrence, including any base cases.

$$m(i,j) = \begin{cases} 0 & \text{if } j-i \le 1 \\ \underbrace{\min}_{\text{best } k \text{ chosen by } \textit{local brute force}} \left\{ \boxed{\overbrace{m(i,k) + m(k,j)}^{\text{recursive cost}} + \overbrace{r[i]\cdot r[k]\cdot c[j-1]}^{\text{cost of last multiplication}}} : \overset{\text{guess}}{k} \in [i+1\,..\,j) \right\} & \text{otherwise} \end{cases}$$

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Matrix-Chain Multiplication – Correctness

Claim: Let $m(i,j)$ for $0 \le i \le j \le n$ be defined by the recurrence

$$m(i,j) = \begin{cases} 0 & \text{if } j - i \le 1 \\ \min\left\{ m(i,k) + m(k,j) + r[i] \cdot r[k] \cdot c[j-1] \;:\; k \in [i+1\,..\,j) \right\} & \text{otherwise} \end{cases}$$

Then $m(i,j)$ = #integer multiplications in best parenthesization of $M_i \cdots M_{j-1}$.

Proof:

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Proof:

- **IB:** When $j - i \le 1$ we have an empty product ($j = i$) or a single matrix ($j = i + 1$)
  In both cases, no multiplications are needed and $m(i,j) = 0$. ✓

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- **IS:** Given $j - i \geq 2$ matrices and an optimal evalution tree T for them.
  - T's root must be a last product of left and right subterms $(M_i \cdots M_{k-1}) \cdot (M_k \cdots M_{j-1})$ for some $i < k < j$, with cost $r[i]r[k]c[j-1]$.

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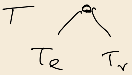

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  - Moreover, left and right subtree T_ℓ and T_r of the root must be optimal evaluation trees for subproblems $[i..k)$ and $[k..j)$; (otherwise can improve T)

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Claim: Let $m(i,j)$ for $0 \le i \le j \le n$ be defined by the recurrence

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Proof: Induction over $j-i$

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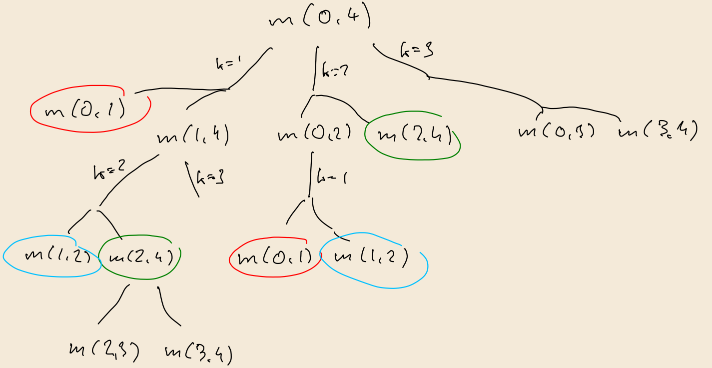
  - Moreover, left and right subtree T_ℓ and T_r of the root must be optimal evaluation trees for subproblems $[i..k)$ and $[k..j)$; (otherwise can improve T)

  $\rightsquigarrow$ By IH, the cost of T_ℓ and T_r are given by $m(i,k)$ and $m(k,j)$

  $\rightsquigarrow$ $m(i,j) =$ cost of T $\square$

$$m(i,j) = \begin{cases} 0 & \text{if } j - i \le 1 \\ \min\bigl\{ m(i,k) + m(k,j) + r[i]\cdot r[k]\cdot c[j-1] \;:\; k \in [i+1\,..\,j) \bigr\} & \text{otherwise} \end{cases}$$



Matrix-Chain Multiplication – Step 4: Memoization

- Write **recursive** function to compute recurrence
- But *memoize* all results! (symbol table: subproblem $\mapsto$ optimal cost)

$\rightsquigarrow$ First action of function: check if subproblem known

  - If so, return cached optimal cost
  - Otherwise, compute optimal cost and remember it!

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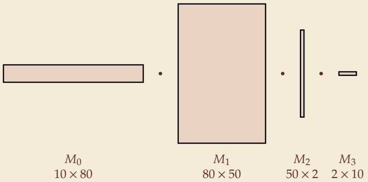
implements recurrence

```
procedure totalMults(r[i..j), c[i..j)):
    if j − i ≤ 1
        return 0
    else
        best := +∞
        for k := i + 1, ..., j − 1
            m_l := cachedTotalMults(r[i..k), c[i..k))
            m_r := cachedTotalMults(r[k..j), c[k..j))
            m := m_l + m_r + r[i] · r[k] · c[j − 1]
            best := min{best, m}
        end for
        return best
```

$$m(i,j) \;=\; \begin{cases} 0 & \text{if } j-i \le 1 \\ \min\Big\{ m(i,k) + m(k,j) + r[i]\cdot r[k]\cdot c[j-1] \;:\; k \in [i+1..j) \Big\} & \text{otherwise} \end{cases}$$

```
procedure cachedTotalMults(r[i..j), c[i..j)):
    // m[0..n)[0..n) initialized to NULL at start
    if m[i][j] == NULL
        m[i][j] := totalMults(r[i..j), c[i..j))
    return m[i, j]
```

Matrix-Chain Multiplication – Example Memoization



$n = 4$

$r[0..n) = [10, 80, 50, 2]$

$c[0..n) = [80, 50, 2, 10]$

$m[i][j]$

$i \backslash j$	0	1	2	3	4
0	0	0			
1	—	0	0		
2	—	—	0	0	
3	—	—	—	0	0
4	—	—	—	—	0

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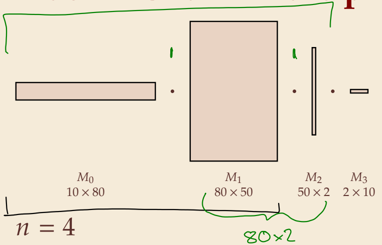


$n = 4$

$r[0..n) = [10, 80, 50, 2]$

$c[0..n) = [80, 50, 2, 10]$

$m(0,2) = 10 \cdot 80 \cdot 50 = 40000$

$m(0,3) = \min\{\overset{1600}{10 \cdot 80 \cdot 2} + \underset{8000}{m(1,3)},\ \overset{40000}{m(0,2)} + 10 \cdot 50 \cdot 2\}$

traceback(0,4) $k=3$

(traceback(0,3)) · (traceback(3,4))

$k=1$ $\quad$ $= M_3$

(traceback(0,1)) (traceback(1,3))

$= M_0$

(1,2) (2,3)

$= M_1$ $\quad$ M_2

$m[i][j]$

$i \backslash j$	0	1	2	3	4
0	0	0	40000	9600	9800
1	—	0	0	8000	9600
2	—	—	0	0	1000
3	—	—	—	0	0
4	—	—	—	—	0

$= ((M_0) \cdot ((M_1) \cdot (M_2))) \cdot (M_3)$

Matrix-Chain Multiplication – Runtime Analyses

```
procedure totalMults(r[i..j), c[i..j)):
    if j − i ≤ 1
        return 0
    else
        best := +∞
        for k := i + 1, . . . , j − 1
            m_l := cachedTotalMults(r[i..k), c[i..k))
            m_r := cachedTotalMults(r[k..j), c[k..j))
            m := m_l + m_r + r[i] · r[k] · c[j − 1]
            best := min{best, m}
        end for
        return best
```

$\rightsquigarrow$ total running time $\overset{O}{\cancel{\Theta}}(n^3)$

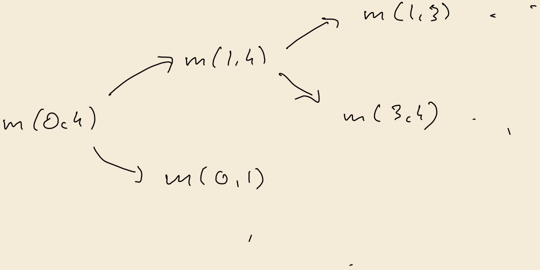
```
procedure cachedTotalMults(r[i..j), c[i..j)):
    // m[0..n)[0..n) initialized to NULL at start
    if m[i][j] == NULL
        m[i][j] := totalMults(r[i..j), c[i..j))
    return m[i, j]
```

- With memoization, compute each subproblem at most once
- nonrecursive cost (totalMults): $O(j - i) = O(n)$
- Number of subproblems $[i..j)$ for $0 \le i \le j \le n$

$$\sum_{0 \le i \le j \le n} 1 \quad = \quad \sum_{i=0}^{n} \sum_{j=i}^{n} 1 \quad = \quad \Theta(n^2)$$

Matrix-Chain Multiplication – Step 5: Table Filling

- Recurrence induces a DAG on subproblems (who calls whom)
  - Memoized recurrence traverses this DAG (DFS!)
  - We can slightly improve performance by systematically computing subproblems following a fixed topological order



1. Subproblems
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$$\underbrace{m(0,4)}_{\ell=4} > \underbrace{m(1,4) > m(0,3)}_{\ell=3} > \underbrace{m(2,4) > m(1,3) > m(0,2)}_{\ell=2} > \dots$$

Matrix-Chain Multiplication – Step 5: Table Filling

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```
procedure totalMultsBottomUp(r[0..n), c[0..n)):
    m[0..n)[0..n) := 0 // initialize to 0          m[i][j] = m(i,j)
    for ℓ = 2, 3, ..., n // iterate over subproblems ...
        for i = 0, 1 ..., n − ℓ // ... in topological order
            j := i + ℓ
            m[i][j] := +∞
            for k := i + 1, ..., j − 1
                q := m[i][k] + m[k][j] + r[i] · r[k] · c[j − 1]
                m[i][j] := min{m[i][j], q}
    return m[0..n)[0..n)
```

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                m[i][j] := min{m[i][j], q}
    return m[0..n)[0..n)
```

- Same Θ-class as memoized recursive function
- In practice usually substantially faster
  - lower overhead
  - predictable memory accesses

Matrix-Chain Multiplication – Step 6: Backtracing

- So far, only determine the **cost** of an optimal solution
  - But we also want the solution itself
- By *retracing* our steps, we can determine/construct one!
- Here: output a parenthesized term recursively

1. Subproblems
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  - But we also want the solution itself
- By *retracing* our steps, we can determine/construct one!
- Here: output a parenthesized term recursively

```
procedure matrixChainMult(r[0..n), c[0..n)):
    m[0..n)[0..n) := totalMultsBottomUp(r[0..n), c[0..n))
    return traceback([0..n))

procedure traceback([i..j)):
    if j − i == 1
        return M_i
    else
        for k := i + 1, ..., j − 1
            q := m[i][k] + m[k][j] + r[i] · r[k] · c[j − 1]
            if m[i][j] == q
                return (traceback([i..k))) · (traceback([k..j)))
        end for
    end if
```

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- follow recurrence a second time

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    end if
```

1. Subproblems
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- follow recurrence a second time
- always have for running time: backtracing $= O(\text{computing } M)$

$\leadsto$ computing optimal cost and computing optimal solution have same complexity

- speedup possible by remembering correct guess k for each subproblem

Summary: The 6 Steps of Dynamic Programming

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

1. Define **subprojects** and how **original problem** is solved
2. What part of solution to **guess**?
3. Set up **DP recurrence** for quality/cost of solution
   - ⇝ Prove **correctness** here: induction over subproblems following recurrence
   - ⇝ Analyze running **time complexity** here: #subproblems · non-recursive time

— (Basically) cookie-cutter approach from here on —

4. Recursive implementation with **Memoization**: mutually recursive functions with cache

or

5. Bottom-up **table filling**: define topological order of subproblem dependency graph
6. **Backtracing** to reconstruct optimal solution: Recursively retrace cost recurrence

12.3 Greedy as Special Case of DP

Dynamic Greedy

- Every Greedy Algorithm can also be seen as a DP algorithm **without guessing**

$\rightsquigarrow$ For new problems, it can help to first follow the DP roadmap and then check if we can select the "correct" guess without local brute force

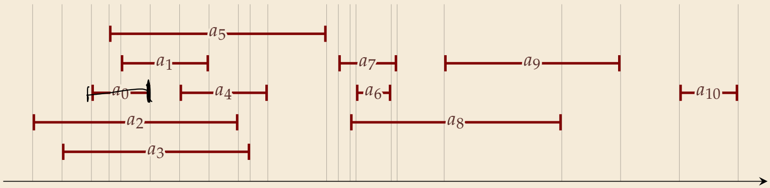
Dynamic Greedy

- Every Greedy Algorithm can also be seen as a DP algorithm **without guessing**
- ⇝ For new problems, it can help to first follow the DP roadmap and then check if we can select the "correct" guess without local brute force
- If so, we then recurse on a single branch of subproblems
- ⇝ Greedy Algorithm doesn't need memoization or bottom-up table filling, but can do direct recursion instead

Recall Unit 11

The Activity selection problem

- **Activity Selection:** scheduling for *single* machine, jobs with *fixed* start and end times
  pick a *subset* of jobs without *conflicts*
  Formally:
  - **Given:** Activities $A = \{a_0, \ldots, a_{n-1}\}$, each with a start time s_i and finish time f_i $(0 \le s_i < f_i < \infty)$
  - **Goal:** Subset $I \subseteq [0..n)$ of tasks such that $i, j \in I \wedge i \ne j \implies [s_i, f_i) \cap [s_j, f_j) = \emptyset$
    and $|I|$ is maximal among all such subsets
  - We further assume that jobs are sorted by finish time, i. e., $f_0 \le f_1 \le \cdots \le f_{n-1}$
    (if not, easy to sort them in $O(n \log n)$ time)



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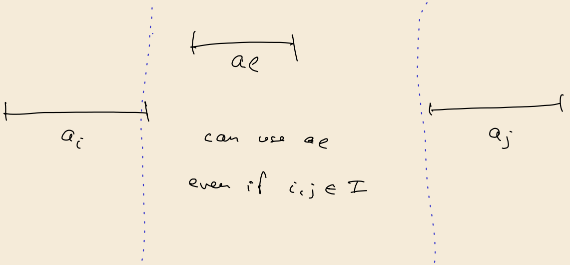
DP Algorithm for Activity Selection

1. **Subproblems:** $A_{i,j} = \{a_\ell \in A : s_\ell \geq f_i \wedge f_\ell \leq s_j\}$
   (after a_i finishes and before a_j begins)

   Original problem: $A_{-1,n}$ with dummy tasks $\cancel{s}f_{-1} = -\infty$, $\cancel{f}s_n = +\infty$



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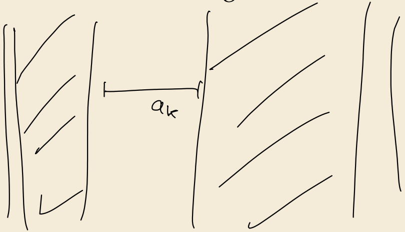
2. Guess: Task $k \in I^*$

3. DP Recurrence: Denote $c(i,j) = |I^*(A_{i,j})|$ = maximum #independent tasks in $A_{i,j}$

$$\rightsquigarrow \quad c(i,j) \quad = \quad \begin{cases} 0\,, & \text{if } A_{i,j} = \emptyset; \\ \max\{c(i,k) + c(k,j) + 1 : a_k \in A_{i,j}\} & \text{otherwise.} \end{cases}$$

4.–6. ***Omitted*** (could be done following the standard scheme)



DP Algorithm for Activity Selection

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4.–6. ***Omitted*** (could be done following the standard scheme)

- Problem-specific insight from Unit 11 $\rightsquigarrow$ Can always use $\underline{k = \min\{k : a_k \in A_{ij}\}}$
  (earliest finish time)

  No guess needed!

12.4 The Bellman-Ford Algorithm

Recall Shortest Paths

- **Single Source Shortest Path Problem (SSSPP)**
  - **Given:** directed, edge-weighted, simple graph $G = (V, E, c)$
    with edge costs $c : E \to \mathbb{R}$, a start vertex $s \in V$
  - **Goal:** a data structure that reports for every $v \in V$:
    $\delta_G(s, v)$: the shortest-path distance from s to v
    $\texttt{spath}(v)$: a shortest path from s to v (if it exists)
- $\delta_G(s, v) = \boxed{\inf\big(\{+\infty\} \cup \{c(w) : w \text{ an } s\text{-}v\text{-walk in } G\}\big)}$
  - Write δ instead of δ_G when graph clear from context

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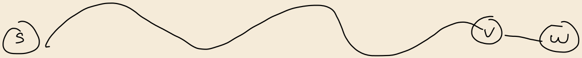
  - Write δ instead of δ_G when graph clear from context
- Here: Assume **negative-weight edges** are present (otherwise Dijkstra suffices)
  - but for now: assume there is **no negative cycle**
  - $\rightsquigarrow$ $\delta(s, v) > -\infty$ and can restrict to shortest **paths** (not walks)

"Pfad" "Weg"

Shortest Paths as DP – Last Edge Decomposition

- Idea: Every nontrivial shortest path has a **last edge**. *We don't know which; so guess!*



Shortest Paths as DP – Last Edge Decomposition

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  - ⇝ Subproblems: for $w \in V$, compute $\delta(s, w)$.
  - ⇝ Recurrence: $\delta(s, w) = \min\{\delta(s, v) + c(vw) : vw \in E\}$ $\delta(s,s) = 0$

Clicker Question

What is the problem with basing a DP algorithm on:
Subproblems: for $w \in V$, compute $\delta(s, w)$.
Recurrence: $\delta(s, w) = \min\{\delta(s, v) + c(vw) : vw \in E\}$

- A Bellman's Optimality Criterion is not satisfied.
- B Does not yield to an efficient algorithm: too many subproblems.
- C Does not yield to an efficient algorithm: non-recursive cost too high.
- D Subproblem dependency graph is cyclic.
- E Subproblem dependency graph is not connected.
- F Does not always compute correct distances.

→ sli.do/cs566

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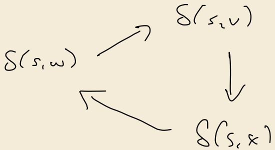
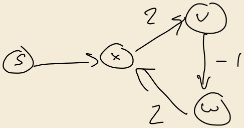
subproblem dependency graph is isomorphic to G^T! ⇝ doesn't work in general
  - ⇝ Yields usable (terminating!) algorithm *iff* G is a DAG.





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- Need to build "layers" in the subproblem dependency graph, so that edges can't go back up.

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- **Original problems:** $\ell = n - 1$ (without negative cycles, paths suffice)

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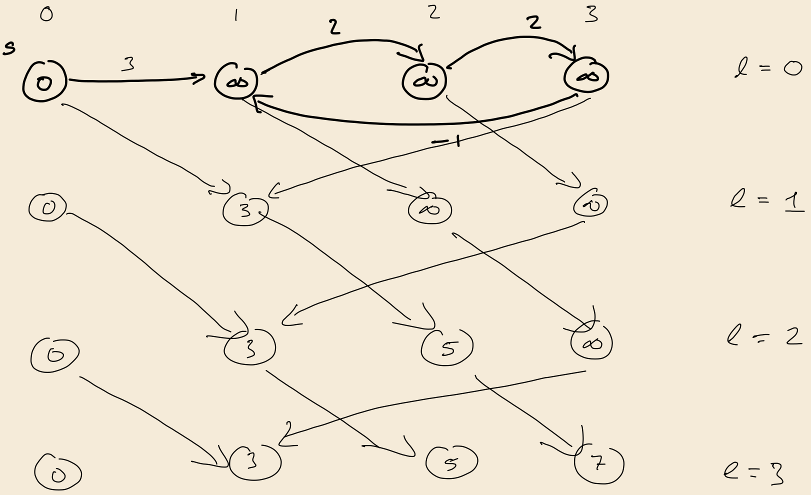
Shortest Paths as DP – Length Layers



Hold On – What about negative cycles?

- The recurrence for $\delta_{\leq \ell}$ seems to work fine with *negative* edges

  But G could contain a **negative-weight cycle** C ...

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Isn't that a contradiction to the non-existence of shortest paths?

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- No. If we restrict the length, shortest walks always exist.
- But: If there is a negative cycle $C[0..k]$ with paths $s \rightsquigarrow C$ and $C \rightsquigarrow w$, then $\delta_{\le\ell}(s,w) > \delta_{\le\ell+k}(s,w) > \delta_{\le\ell+2k}(s,w) > \cdots$ (and $\delta(s,w) = -\infty$)

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$\rightsquigarrow$ We can *detect* if any negative cycle is reachable from s by including more layers $\ell \geq n$ and check if some vertex still improves.

  - *How many further layers do we need / when is it safe to stop?*

Detecting negative cycles

We can detect reachable negative cycles by including just the *single* extra layer $\ell = n$!

Lemma: $\exists w : \delta_{\leq n}(s,w) < \delta_{\leq n-1}(s,w)$ *iff* negative cycle reachable from s

"$\Rightarrow$"
- If some vertex w improves further, i. e., $\delta_{\leq n}(s,w) < \delta_{\leq n-1}(s,w)$
  a walk $W[0..n]$ with $c(W) = \delta_{\leq n}(s,w)$ was the **shortest** way to reach w

$\rightsquigarrow$ W is a non-simple walk, i. e., it contains a cycle

- Let $P[0..k]$ be the path resulting from W by shortcutting all cycles $\rightsquigarrow$ $k \leq n-1$

$\rightsquigarrow$ $c(P) \geq \delta_{\leq n-1}(s,w) > \delta_{\leq n}(s,w) = c(W)$

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$\rightsquigarrow$ W is a non-simple walk, i. e., it contains a cycle

- Let $P[0..k]$ be the path resulting from W by shortcutting all cycles $\rightsquigarrow$ $k \le n-1$

$\rightsquigarrow$ $c(P) \ \ge\ \delta_{\le n-1}(s,w) \ >\ \delta_{\le n}(s,w) \ =\ c(W)$

$\rightsquigarrow$ $\exists$ negative cycle reachable from s

"$\Leftarrow$"

- Conversely, let negative cycle $C[0..k]$ be reachable from s

$\rightsquigarrow$ $c(C) = \sum_{i=0}^{k-1} c(C[i]C[i+1]) < 0$

- Assume towards a contradiction that $\forall w : \delta_{\le n}(s,w) = \delta_{\le n-1}(s,w)$

$\rightsquigarrow$ $\forall vw \in E : \delta_{\le n-1}(s,w) \le \delta_{\le n-1}(s,v) + c(vw)$ (no update in layer $\ell = n$)

- summing this inequality over $C[0..k]$ yields (abbreviating $\delta(w) := \delta_{\le n-1}(s,w)$)

$$\sum_{i=1}^{k}\delta(C[i]) \ \le\ \sum_{i=1}^{k}\Big(\delta(C[i-1]) + c(C[i]C[i+1])\Big) \ =\ \sum_{i=0}^{k-1}\delta(C[i]) + \underbrace{\sum_{i=1}^{k} c(C[i]C[i+1])}_{=\,c(C)\,<\,0}$$

$\rightsquigarrow$ $0 \le c(C) < 0$ ↯

$\square$