7 Compression

20 April 2021

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version `2021-04-21 10:22`

Outline

7 Compression

7.1 Context

Overview

- Unit 4–6: How to *work* with strings
  - finding substrings
  - finding approximate matches
  - finding repeated parts
  - ...
  - assumed character array (random access)!

- Unit 7–8: How to *store/transmit* strings
  - ~~computer memory~~: must be binary
  - how to compress strings (save space)
  - how to robustly transmit over noisy channels ⇝ Unit 8

Clicker Question

What compression methods do you know?

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Terminology

- **source text:** string $S \in \Sigma_S^\star$ to be stored / transmitted
  Σ_S is some alphabet
- **coded text:** encoded data $C \in \Sigma_C^\star$ that is actually stored / transmitted
  usually use $\Sigma_C = \{0, 1\}$
- **encoding:** algorithm mapping source texts to coded texts $S \to C$
- **decoding:** algorithm mapping coded texts back to original source text $S \leftarrow C$

text can be any data

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- **Lossy vs. Lossless** $S \to C \to S' \approx S$
  - **lossy compression** can only decode **approximately**;
    the exact source text S is lost
  - **lossless compression** always decodes S exactly

- For media files, lossy, logical compression is useful (e. g. JPEG, MPEG)

- We will concentrate on *lossless* compression algorithms.
  These techniques can be used for any application.

What is a good encoding scheme?

- Depending on the application, goals can be
  - efficiency of encoding/decoding
  - resilience to errors/noise in transmission
  - security (encryption)
  - integrity (detect modifications made by third parties)
  - size

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- Depending on the application, goals can be
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  - integrity (detect modifications made by third parties)
  - size

- Focus in this unit: **size** of coded text

  Encoding schemes that (try to) minimize the size of coded texts perform *data compression*.

- We will measure the *compression ratio*:

  coded length

  $$\frac{|C| \cdot \lg |\Sigma_C|}{|S| \cdot \lg |\Sigma_S|} \overset{\Sigma_C=\{0,1\}}{=} \frac{|C|}{|S| \cdot \lg |\Sigma_S|}$$

  source length

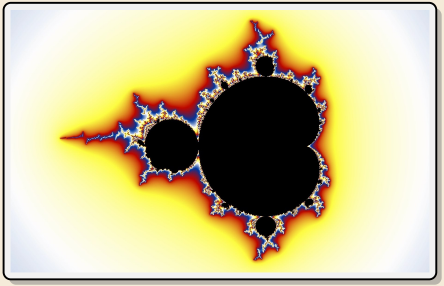
  < 1 means successful compression

  $= 1$ means no compression

  > 1 means "compression" made it bigger!? (yes, that happens ...)

Limits of algorithmic compression

Is this image compressible?

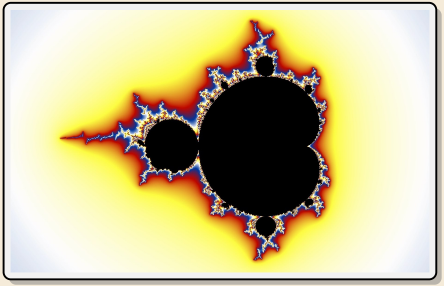


Limits of algorithmic compression

Is this image compressible?

visualization of Mandelbrot set

- Clearly a complex shape!
- Will not compress (too) well using, say, PNG.
- but:
  - completely defined by mathematical formula
  - $\rightsquigarrow$ **can be generated by a very small program!**



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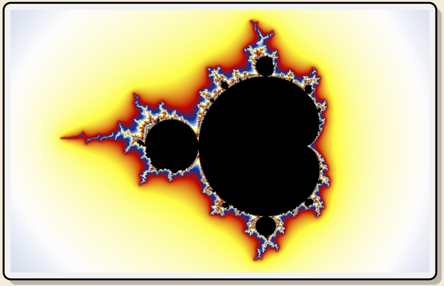

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⇝ ***Kolmogorov complexity***

- C = *any program* that outputs S

  self-extracting archives!
- Kolmogorov complexity = length of smallest such program

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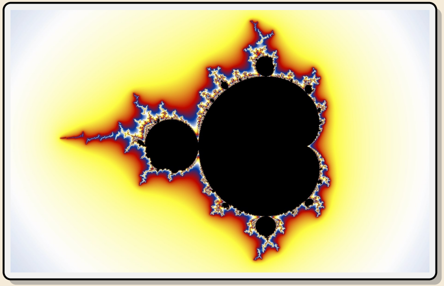

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⇝ ***Kolmogorov complexity***

- C = *any program* that outputs S

  self-extracting archives!
- Kolmogorov complexity = length of smallest such program
- **Problem:** finding smallest such program is *uncomputable*.

⇝ No optimal encoding algorithm is possible!

⇝ must be inventive to get efficient methods

What makes data compressible?

- Lossless compression methods mainly exploit two types of redundancies in source texts:

  1. **uneven character frequencies**
     some characters occur more often than others $\longrightarrow$ Part I
  2. **repetitive texts**
     different parts in the text are (almost) identical $\longrightarrow$ Part II

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     some characters occur more often than others $\longrightarrow$ Part I

  2. **repetitive texts**
     different parts in the text are (almost) identical $\longrightarrow$ Part II

There is no such thing as a free lunch!

Not *everything* is compressible ($\rightarrow$ tutorials)

$\rightsquigarrow$ focus on versatile methods that often work

Part I

Exploiting character frequencies

7.2 Character Encodings

Character encodings

- Simplest form of encoding: Encode each source character individually

$\rightsquigarrow$ encoding function $E : \Sigma_S \to \Sigma_C^\star$

  - typically, $|\Sigma_S| \gg |\Sigma_C|$, so need several bits per character
  - for $c \in \Sigma_S$, we call $E(c)$ the *codeword* of c

- **fixed-length code:** $|E(c)|$ is the same for all $c \in \Sigma_C$
- **variable-length code:** not all codewords of same length

Fixed-length codes

- fixed-length codes are the simplest type of character encodings
- Example: **ASCII** (American Standard Code for Information Interchange, 1963)

```
0000000 NUL   0010000 DLE   0100000       0110000 0   1000000 @   1010000 P   1100000 ‘   1110000 p
0000001 SOH   0010001 DC1   0100001 !     0110001 1   1000001 A   1010001 Q   1100001 a   1110001 q
0000010 STX   0010010 DC2   0100010 "     0110010 2   1000010 B   1010010 R   1100010 b   1110010 r
0000011 ETX   0010011 DC3   0100011 #     0110011 3   1000011 C   1010011 S   1100011 c   1110011 s
0000100 EOT   0010100 DC4   0100100 $     0110100 4   1000100 D   1010100 T   1100100 d   1110100 t
0000101 ENQ   0010101 NAK   0100101 %     0110101 5   1000101 E   1010101 U   1100101 e   1110101 u
0000110 ACK   0010110 SYN   0100110 &     0110110 6   1000110 F   1010110 V   1100110 f   1110110 v
0000111 BEL   0010111 ETB   0100111 ’     0110111 7   1000111 G   1010111 W   1100111 g   1110111 w
0001000 BS    0011000 CAN   0101000 (     0111000 8   1001000 H   1011000 X   1101000 h   1111000 x
0001001 HT    0011001 EM    0101001 )     0111001 9   1001001 I   1011001 Y   1101001 i   1111001 y
0001010 LF    0011010 SUB   0101010 *     0111010 :   1001010 J   1011010 Z   1101010 j   1111010 z
0001011 VT    0011011 ESC   0101011 +     0111011 ;   1001011 K   1011011 [   1101011 k   1111011 {
0001100 FF    0011100 FS    0101100 ,     0111100 <   1001100 L   1011100 \   1101100 l   1111100 |
0001101 CR    0011101 GS    0101101 -     0111101 =   1001101 M   1011101 ]   1101101 m   1111101 }
0001110 SO    0011110 RS    0101110 .     0111110 >   1001110 N   1011110 ^   1101110 n   1111110 ~
0001111 SI    0011111 US    0101111 /     0111111 ?   1001111 O   1011111 _   1101111 o   1111111 DEL
```

- 7 bit per character
- just enough for English letters and a few symbols (plus control characters)

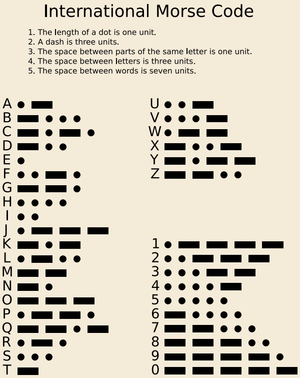
Fixed-length codes – Discussion

- 👍 Encoding & Decoding as fast as it gets & allows random access
- 👎 Unless all characters equally likely, it wastes a lot of space
- 👎 inflexible (how to support adding a new character?)

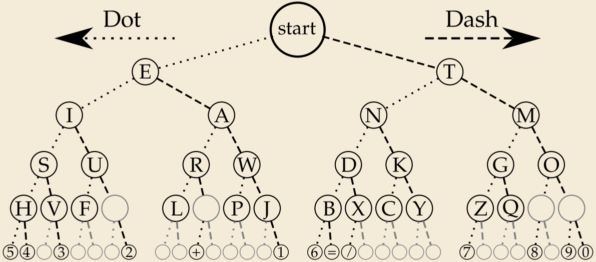
Variable-length codes

- to gain more flexibility, have to allow different lengths for codewords

- actually an old idea: **Morse Code**



https://commons.wikimedia.org/wiki/File:International_Morse_Code.svg



https://commons.wikimedia.org/wiki/File:Morse-code-tree.svg

Clicker Question

How many characters are there in the alphabet of the coded text in Morse Code, i. e., what is $|\Sigma_C|$?

- **A** 1
- **B** 2
- **C** 3
- **D** 4
- **E** 26
- **F** 36
- **G** 256

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Click on "Polls" tab

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- **A** ~~1~~
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- **E** ~~26~~
- **F** ~~36~~
- **G** ~~256~~

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Variable-length codes – UTF-8

- Modern example: UTF-8 encoding of Unicode:
  default encoding for text-files, XML, HTML since 2009
  - Encodes any Unicode character (137 994 as of May 2019, and counting)
  - uses 1–4 bytes (codeword lengths: 8, 16, 24, or 32 bits)
  - Every ASCII character is encoded in 1 byte with leading bit `0`, followed by the 7 bits for ASCII
  - Non-ASCII charactters start with 1–4 `1`s indicating the total number of bytes,
    followed by a `0` and 3–5 bits.
    The remaining bytes each start with `10` followed by 6 bits.

Char. number range (hexadecimal)	UTF-8 octet sequence (binary)
`0000 0000 – 0000 007F`	`0xxxxxxx`
`0000 0080 – 0000 07FF`	`110xxxxx 10xxxxxx`
`0000 0800 – 0000 FFFF`	`1110xxxx 10xxxxxx 10xxxxxx`
`0001 0000 – 0010 FFFF`	`11110xxx 10xxxxxx 10xxxxxx 10xxxxxx`

For English text, most characters use only 8 bit,
but we can include any Unicode character, as well.

↯ random access

Pitfall in variable-length codes

- Suppose we have the following code:

c	a	n	b	s
$E(c)$	0	10	110	100

- Happily encode text $S =$ banana with the coded text $C = \underbrace{110}_{\texttt{b}}\underbrace{0}_{\texttt{a}}\underbrace{10}_{\texttt{n}}\underbrace{0}_{\texttt{a}}\underbrace{10}_{\texttt{n}}\underbrace{0}_{\texttt{a}}$

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ϟ C = 1100100100 decodes **both** to banana and to bass: $\underbrace{\texttt{110}}_{\texttt{b}}\underbrace{\texttt{0}}_{\texttt{a}}\underbrace{\texttt{100}}_{\texttt{s}}\underbrace{\texttt{100}}_{\texttt{s}}$

⇝ not a valid code . . . (cannot tolerate ambiguity)

but how should we have known?

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⇝ not a valid code . . . (cannot tolerate ambiguity)

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$E(\texttt{n}) = \texttt{10}$ is a (proper) **prefix** of $E(\texttt{s}) = \texttt{100}$

⇝ Leaves decoder wondering whether to stop after reading 10 or continue!

⇝ Require a ***prefix-free*** code: No codeword is a prefix of another.

prefix-free $\implies$ instantaneously decodable

Code tries

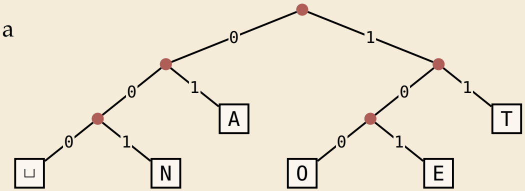
- From now on only consider prefix-free codes E: $E(c)$ is not a prefix of $E(c')$ for any $c, c' \in \Sigma_S$.

- **Example:**

c	A	E	N	O	T	␣
$E(c)$	01	101	001	100	11	000

Any prefix-free code corresponds to a ***(code) trie*** (trie of codewords) with characters of Σ_S at **leaves**.

no need for end-of-string symbols $ here
(already prefix-free!)



- Encode `AN␣ANT` 0100100001
- Decode `111000001010111` T O ␣

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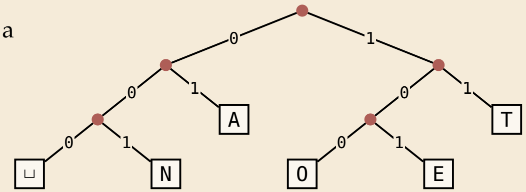

- Encode AN␣ANT $\rightarrow$ 010010000100111
- Decode 111000001010111 $\rightarrow$ TO␣EAT

Who decodes the decoder?

- Depending on the application, we have to **store/transmit** the **used code**!
- We distinguish:
  - **fixed coding:** code agreed upon in advance, not transmitted (e. g., Morse, UTF-8)
  - **static coding:** code depends on message, but stays same for entire message; it must be transmitted (e. g., Huffman codes $\rightarrow$ next)
  - **adaptive coding:** code depends on message and changes during encoding; implicitly stored withing the message (e. g., LZW $\rightarrow$ below)

7.3 Huffman Codes

Character frequencies

- **Goal:** Find character encoding that produces short coded text
- Convention here: fix $\Sigma_C = \{0, 1\}$ (binary codes), abbreviate $\Sigma = \Sigma_S$,
- **Observation:** Some letters occur more often than others.

Typical English prose:

e	12.70%	**d**	4.25%	**p**	1.93%
t	9.06%	**l**	4.03%	**b**	1.49%
a	8.17%	**c**	2.78%	**v**	0.98%
o	7.51%	**u**	2.76%	**k**	0.77%
i	6.97%	**m**	2.41%	**j**	0.15%
n	6.75%	**w**	2.36%	**x**	0.15%
s	6.33%	**f**	2.23%	**q**	0.10%
h	6.09%	**g**	2.02%	**z**	0.07%
r	5.99%	**y**	1.97%		

$\rightsquigarrow$ Want shorter codes for more frequent characters!

Huffman coding

e. g. frequencies / probabilities

- **Given:** Σ and weights $w : \Sigma \to \mathbb{R}_{\geq 0}$
- **Goal:** prefix-free code E (= code trie) for Σ that minimizes coded text length

  i. e., a code trie minimizing $\sum_{c \in \Sigma} w(c) \cdot |E(c)|$

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- If we use $w(c) = \#\text{occurrences of } c \text{ in } S$,
  this is the character encoding with smallest possible $|C|$

  $\rightsquigarrow$ best possible character-wise encoding

- Quite ambitious! *Is this efficiently possible?*

Huffman's algorithm

- Actually, yes! A greedy/myopic approach succeeds here.

Huffman's algorithm:

1. Find two characters `a`, `b` with lowest weights.
   - We will encode them with the same prefix, plus one distinguishing bit, i. e., $E(\texttt{a}) = u\texttt{0}$ and $E(\texttt{b}) = u\texttt{1}$ for a bitstring $u \in \{0,1\}^\star$ (u to be determined)
2. (Conceptually) replace `a` and `b` by a single character "$\boxed{\text{ab}}$" with $w(\boxed{\text{ab}}) = w(\texttt{a}) + w(\texttt{b})$.
3. Recursively apply Huffman's algorithm on the smaller alphabet. This in particular determines $u = E(\boxed{\text{ab}})$.

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ambiguous parts

Huffman's algorithm:

1. Find two characters a, b with lowest weights. which?
   - We will encode them with the same prefix, plus one distinguishing bit, i. e., $E(\texttt{a}) = u\texttt{0}$ and $E(\texttt{b}) = u\texttt{1}$ for a bitstring $u \in \{0,1\}^\star$ (u to be determined)
2. (Conceptually) replace a and b by a single character "$\boxed{\text{ab}}$" with $w(\boxed{\text{ab}}) = w(\texttt{a}) + w(\texttt{b})$. or $\boxed{\text{ba}}$?
3. Recursively apply Huffman's algorithm on the smaller alphabet. This in particular determines $u = E(\boxed{\text{ab}})$.

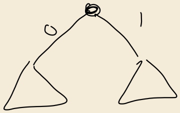
- efficient implementation using a (min-oriented) ***priority queue***
  - start by inserting all characters with their weight as key
  - step 1 uses two `deleteMin` calls
  - step 2 inserts a new character with the sum of old weights as key

Huffman's algorithm – Example

- Example text: S = LOSSLESS $\rightsquigarrow$ $\Sigma_S = \{$E, L, O, S$\}$
- Character frequencies: E : 1, L : 2, O : 1, S : 4

Q

1	2	1	4
$\boxed{\text{E}}$	$\boxed{\text{L}}$	$\boxed{\text{O}}$	$\boxed{\text{S}}$



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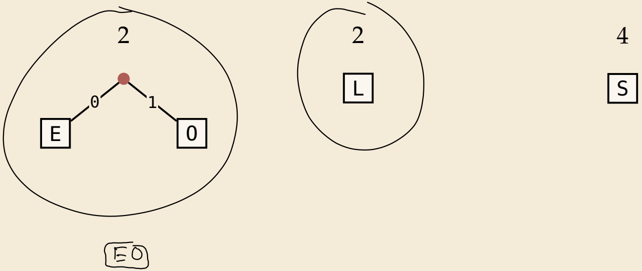

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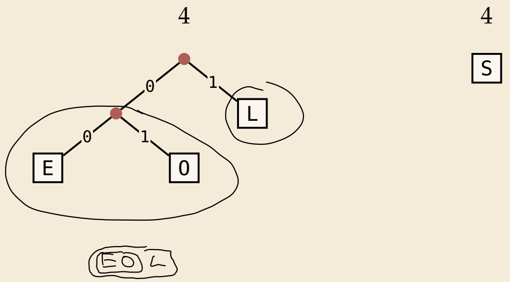

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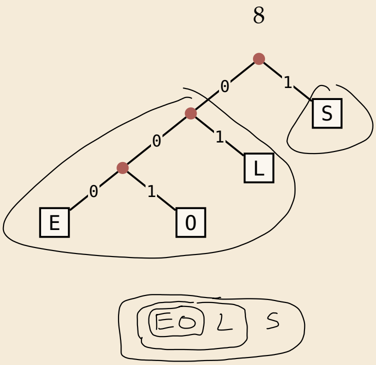

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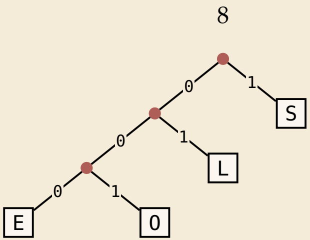

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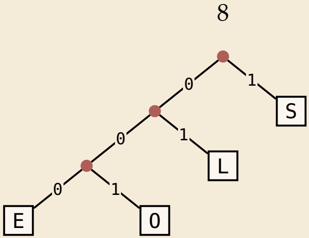
c	E	L	O	S
$E(c)$	000	01	001	1



$\rightsquigarrow$ *Huffman tree* (code trie for Huffman code)

`LOSSLESS` $\rightarrow$ `01001110100011`

compression ratio: $\frac{14}{8 \cdot \log 4} = \frac{14}{16} \approx 88\%$

(but: would also have to store trie)
freqs

Huffman tree – tie breaking

- The above procedure is ambiguous:
  - which characters to choose when weights are equal?
  - which subtree goes left, which goes right?

- For COMP 526: always use the following rule:

> 1. To break ties when selecting the two characters, first use the smallest letter according to the alphabetical order, or the tree containing the smallest alphabetical letter.
> 2. When combining two trees of different values, place the lower-valued tree on the left (corresponding to a `0`-bit).
> 3. When combining trees of equal value, place the one containing the smallest letter to the left.

Huffman code – Optimality

Theorem 7.1 (Optimality of Huffman's Algorithm)

Given Σ and $w : \Sigma \to \mathbb{R}_{\geq 0}$, Huffman's Algorithm computes codewords $E : \Sigma \to \{0,1\}^\star$ with minimal expected codeword length $\underbrace{\ell(E) =}$ $\sum_{c\in\Sigma} w(c) \cdot |E(c)|$ among all prefix-free codes for Σ. ◀

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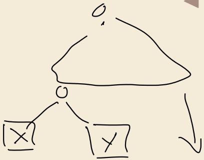

Proof sketch: by induction over $\sigma = |\Sigma|$

- Given any optimal prefix-free code E^* (as its code trie).
- code trie $\rightsquigarrow$ $\exists$ two sibling leaves x, y at largest depth D
- swap characters in leaves to have two lowest-weight characters `a`, `b` in x, y (that can only make ℓ smaller, so still optimal)
- any optimal code for $\Sigma' = \Sigma \setminus \{\texttt{a}, \texttt{b}\} \cup \{\boxed{\texttt{ab}}\}$ yields optimal code for Σ by replacing leaf $\boxed{\texttt{ab}}$ by internal node with children `a` and `b`.

$\rightsquigarrow$ recursive call yields optimal code for Σ' by inductive hypothesis, so Huffman's algorithm finds optimal code for Σ.

Entropy

Definition 7.2 (Entropy)

Given probabilities $p_1, \ldots, p_n$ (for outcomes $1, \ldots, n$ of a random variable), the *entropy* of the distribution is defined as

$$\mathcal{H}(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \lg p_i = \sum_{i=1}^{n} p_i \lg\left(\frac{1}{p_i}\right) = \mathbb{E}\left[\lg \frac{1}{\mathbb{P}}\right]$$

fair die with 6 faces

1 ... 6 with $\frac{1}{6}$

$$\mathcal{H}\left(\frac{1}{6}, \ldots, \frac{1}{6}\right) = \sum_{i=1}^{6} \frac{1}{6} \lg\left(\frac{1}{\frac{1}{6}}\right) = 1 \cdot \lg(6) \approx 2.$$

fair coin heads / tails w/ prob $\frac{1}{2}$

$$\mathcal{H}\left(\frac{1}{2}, \frac{1}{2}\right) = 1$$

Entropy

Definition 7.2 (Entropy)

Given probabilities $p_1, \ldots, p_n$ (for outcomes $1, \ldots, n$ of a random variable), the *entropy* of the distribution is defined as

$$\mathcal{H}(p_1, \ldots, p_n) \;=\; -\sum_{i=1}^{n} p_i \lg p_i \;=\; \sum_{i=1}^{n} p_i \lg\left(\frac{1}{p_i}\right)$$

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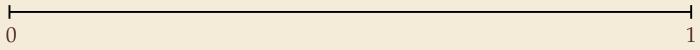

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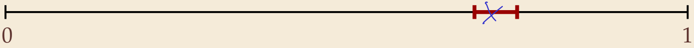

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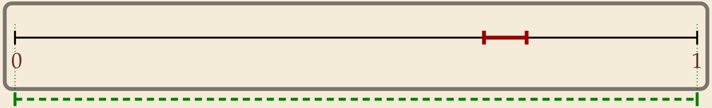

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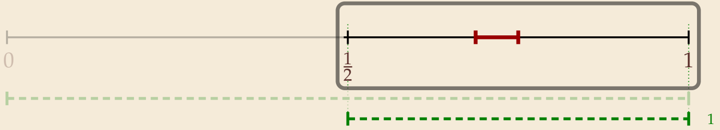

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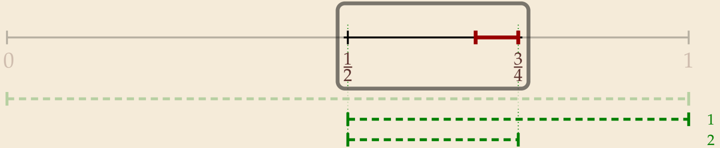

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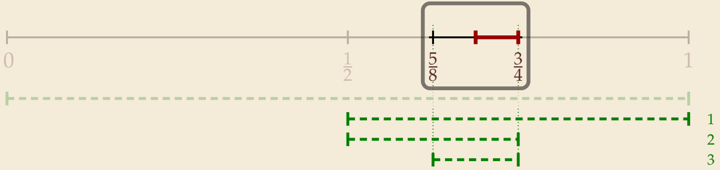

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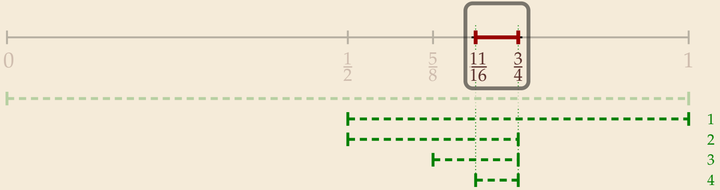

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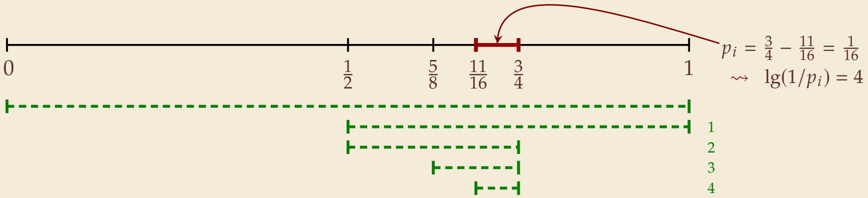

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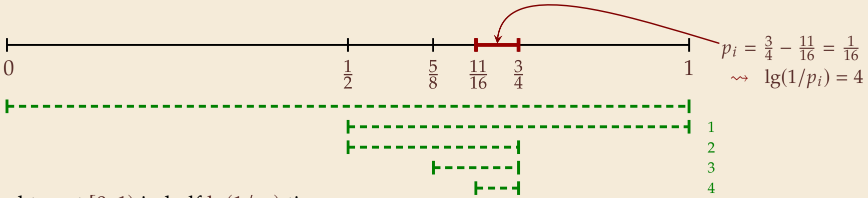

$\rightsquigarrow$ Need to cut $[0,1)$ in half $\lg(1/p_i)$ times

- more precisely: the expected number of bits (Yes/No questions) required to nail down the random value

Entropy and Huffman codes

- would ideally encode value i using $\lg(1/p_i)$ bits

  not always possible; cannot use codeword of 1.5 bits . . .

not for single code; but possible *on average*!

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Theorem 7.3 (Entropy bounds for Huffman codes)

For any $\Sigma = \{a_1, \ldots, a_\sigma\}$ and $w : \Sigma \to \mathbb{R}_{>0}$ and its Huffman code E, we have

$\boxed{\mathcal{H} \;\le\; \ell(E) \;\le\; \mathcal{H} + 1}$ where $\mathcal{H} = \mathcal{H}\left(\frac{w(a_1)}{W}, \ldots, \frac{w(a_\sigma)}{W}\right)$ and $W = w(a_1) + \cdots + w(a_\sigma)$. ◀

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c	a	b	c
$E(c)$	1	00	01
q_c	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Proof sketch:

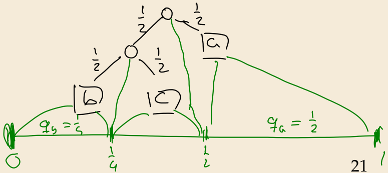
- $\ell(E) \ge \mathcal{H}$

  Any prefix-free code E induces weights $q_i = 2^{-|E(a_i)|}$.

  By *Kraft's Inequality*, we have $q_1 + \cdots + q_\sigma \le 1$.

  Hence we can apply *Gibb's Inequality* to get

$$\mathcal{H} \;=\; \underbrace{\sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{p_i}\right) \;\le\; \sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{\boldsymbol{q_i}}\right)}_{\text{for any } q_i \in [0,1) \quad \sum q_i \le 1} \;=\; \ell(E).$$



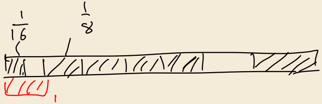
Entropy and Huffman codes [2]

Proof sketch (continued): $\lg\left(\frac{1}{p_i}\right) \mathrel{\hat{=}}$ ideal codeword length

- $\ell(E) \leq \mathcal{H} + 1$ $\to$ round up

  Set $\underline{q_i = 2^{-\lceil \lg(1/p_i) \rceil}}$. We have $\displaystyle\sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{q_i}\right) = \sum_{i=1}^{\sigma} p_i \overbrace{\lceil \lg(1/p_i) \rceil}^{\leq \lg(1/p_i)+1} \leq \mathcal{H} + 1.$ $\quad \sum q_i \leq 1$

  We construct a code E' for Σ with $|E'(a_i)| \leq \lg(1/q_i)$ as follows;
  w.l.o.g. assume $q_1 \leq q_2 \leq \cdots \leq q_\sigma$ $\quad$ $\lceil \lg(1/p_i) \rceil$

  - If $\sigma = 2$, E' uses a single bit each.
    Here, $q_i \leq 1/2$, so $\lg(1/q_i) \geq 1 = |E'(a_i)|$ ✓
  - If $\sigma \geq 3$, we merge a_1 and a_2 to $\boxed{a_1 a_2}$, $\boxed{\text{assign it weight } 2q_2}$ and recurse. $\frac{1}{8}$
    If $q_1 = q_2$, this is like Huffman; otherwise, $\underline{q_1}$ is a unique smallest value and
    $q_2 + q_2 + \cdots + q_\sigma \leq 1$.

    By the inductive hypothesis, we have $\left|E'(\boxed{a_1 a_2})\right| \leq \lg\left(\frac{1}{2q_2}\right) = \lg\left(\frac{1}{q_2}\right) - 1$.

    By construction, $|E'(a_1)| = |E'(a_2)| = \left|E'(\boxed{a_1 a_2})\right| + 1$, so $\underline{|E'(a_1)| \leq \lg(\frac{1}{q_1}) \text{ and } |E'(a_2)| \leq \lg(\frac{1}{q_2})}$.

  By optimality of E, we have $\displaystyle \ell(E) \leq \ell(E') \leq \sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{q_i}\right) \leq \mathcal{H} + 1.$

Clicker Question

When is Huffman coding more efficient than a fixed-length encoding?

A always

B when $\mathcal{H} \approx \lg(\sigma)$

C when $\mathcal{H} < \lg(\sigma)$

D when $\mathcal{H} < \lg(\sigma) - 1$

E when $\mathcal{H} \approx 1$

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Click on “Polls” tab

Clicker Question

When is Huffman coding more efficient than a fixed-length encoding?

- **A** always ✓
- **B** ~~when $\mathcal{H} \approx \lg(\sigma)$~~
- **C** ~~when $\mathcal{H} < \lg(\sigma)$~~
- **D** when $\mathcal{H} < \lg(\sigma) - 1$ ✓
- **E** ~~when $\mathcal{H} \approx 1$~~

$$\ell(E) \le \mathcal{H} + 1 < \lg(\sigma) - 1 + 1$$
$$= \lg(\sigma) = \ell(E^{\text{fixed}})$$

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Encoding with Huffman code

- The overall encoding procedure is as follows:
  - Pass 1: Count character frequencies in S
  - Construct Huffman code E (as above)
  - Store the Huffman code in C (details omitted) → Sedgewick Wayne
  - Pass 2: Encode each character in S using E and append result to C
- Decoding works as follows:
  - Decode the Huffman code E from C. (details omitted)
  - Decode S character by character from C using the code trie.
- Note: Decoding is much simpler/faster!

Huffman coding – Discussion

- running time complexity: $O(\sigma \log \sigma)$ to construct code
  - build PQ + $\sigma \cdot$ (2 deleteMins and 1 insert)
  - can do $\Theta(\sigma)$ time when characters already sorted by weight
  - time for encoding: $O(n + |C|)$
- many variations in use (tie-breaking rules, estimated frequencies, adaptive encoding, ...)

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👍 optimal prefix-free character encoding

👍 very fast decoding

👎 needs 2 passes over source text for encoding

- one-pass variants possible, but more complicated

👎 have to store code alongside with coded text

Part II

Compressing repetitive texts

Beyond Character Encoding

- Many "natural" texts show repetitive redundancy

```
All work and no play makes Jack a dull boy.  All work and no play makes Jack a dull
boy.  All work and no play makes Jack a dull boy.  All work and no play makes Jack
a dull boy.  All work and no play makes Jack a dull boy.  All work and no play makes
Jack a dull boy.  All work and no play makes Jack a dull boy.  All work and no play
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⇝ Have to encode whole ***phrases*** of S by a single codeword

7.4 Run-Length Encoding

Run-Length encoding

- simplest form of repetition: ***runs*** of characters

same character repeated

```
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
0001011001000001111110000000000001111000
0011111111110001111111110000001111111000
0011110110100011100011110000111000000000
0011000000000000000000111000111000000000
0011000000000000000000110011100000000000
0011000000000000000000110011100000000000
0011011000000000000000111001100111110000
0011111111100000000000111001111111111000
0011101111110000000000111000111110011100
0000000001110000000011100001110000001110
0000000001110000000011000001110000001100
0000000000110000000110000000110000001110
0000000000110000001110000000111000001100
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- here: only consider $\Sigma_S = \{0, 1\}$ (work on a binary representation)
  - can be extended for larger alphabets

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001100000000000000000011001110000000000
001100000000000000000011001110000000000
001101100000000000000111001100111110000
001111111110000000000111001111111111000
001110111110000000001110001111100111100
000000000111000000111000001110000001110
000000000111000000110000001110000001100
000000000011000001100000000110000001110
000000000011000001110000011100000001100
000000000111000111000000000110000001110
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$\rightsquigarrow$ **run-length encoding (RLE):**
use runs as phrases: $S = \underbrace{\texttt{00000}}\ \underbrace{\texttt{111}}\ \underbrace{\texttt{0000}}$

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0011011000000000000000111001100111110000
0011111111100000000000111001111111111000
0011101111100000000001110000111110011100
0000000001110000000011100001110000001110
0000000001110000000011000001110000001100
0000000000110000000110000000110000001110
0000000000110000001110000001110000001100
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- Example becomes: $0, 5, 3, 4$

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001110111110000000001110001111100111100
000000000111000001110000011100000001110
000000000111000001100000011100000001100
000000000011000000001100001100000001110
000000000011000000111000011100000001100
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- **Question**: How to encode a run length k in binary? (k can be arbitrarily large!)

Clicker Question

How would you encode a string that can we arbitrarily long?

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Elias codes

- Need a *prefix-free encoding* for $\mathbb{N} = \{1, 2, 3, \ldots, \}$
  - must allow arbitrarily large integers
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- Refinement: ***Elias gamma code***
  - Store the **length** ℓ of the binary representation in **unary**
  - Followed by the binary digits themselves
  - little tricks:
    - always $\ell \geq 1$, so store $\ell - 1$ instead
    - binary representation always starts with `1` $\rightsquigarrow$ don't need terminating `1` in unary

codeword length
for number k
$\leq 2 \lceil \lg k \rceil$

$\rightsquigarrow$ Elias gamma code = $\ell - 1$ zeros, followed by binary representation

Examples: $1 \mapsto$ `1`, $3 \mapsto$ `011`, $5 \mapsto$ `00101`, $30 \mapsto$ `000011110`

Clicker Question

Decode the **first** number in Elias gamma code (at the beginning) of the following bitstream:

$$\underline{000}\underbrace{1101}_{13}11011100110.$$

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Run-length encoding – Examples

- Encoding:
  S = `11111110010000000000000000000011111111111`

  C = `1`

- Decoding:
  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:

  S = `1111111001000000000000000000001111111111`

  $k = 7$ $\qquad$ 00111

  C = `100111`

- Decoding:

  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

  $k = 2$ 010

  C = `100111010`

- Decoding:

  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:

  S = `1111111001000000000000000000011111111111`

  $k = 1$

  C = `1001110101`

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  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:

  S = `1111111001` `00000000000000000000` `11111111111`

  $k = 20$ 10100

  C = `1001110101` `000010100`

- Decoding:

  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:

  $S =$ `111111100100000000000000000000011111111111`

  $k = 11$

  $C =$ `10011101010000101000001011`

- Decoding:

  $C =$ `00001101001001010`

  $S =$

Run-length encoding – Examples

- Encoding:
  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:
  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C =$ `00001101001001010`

  $S =$

Run-length encoding – Examples

- Encoding:
  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  C = `00001101001001010`
  $b = 0$

  S =

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  C = `0000110100100101 0`

  $b = 0$

  $\ell = 3 + 1$

  S =

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  C = `00001101001001010`

  $b = 0$

  $\ell = 3 + 1$

  $k = 13$

  S = `0000000000000`

Run-length encoding – Examples

- Encoding:

  $S = $ `11111110010000000000000000000011111111111`

  $C = $ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  $C = $ `00001101001001010`

  $b = 1$

  $\ell = 2 + 1$

  $k =$

  $S = $ `0000000000000`

Run-length encoding – Examples

- Encoding:

  $S =$ `11111110010000000000000000000001111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  $C =$ `0000110100100 1010`

  $b = 1$

  $\ell = 2 + 1$

  $k = 4$

  $S =$ `00000000000001111`

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000001111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  C = `00001101001001010`

  $b = 0$

  $\ell = 0 + 1$

  $k =$

  S = `00000000000001111`

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000001111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  C = `0000110100100`$\color{red}{\texttt{1}}$`010`

  $b = 0$

  $\ell = 0 + 1$

  $k = 1$

  S = `00000000000001111`$\color{red}{\texttt{0}}$

Run-length encoding – Examples

- Encoding:
  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C =$ `000011010010010` `10`
  $b = 1$
  $\ell = 1 + 1$
  $k =$
  $S =$ `000000000000011110`

Run-length encoding – Examples

- Encoding:
  $S =$ `11111110010000000000000000000001111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C =$ `00001101001001010`
  $b = 1$
  $\ell = 1 + 1$
  $k = 2$
  $S =$ `00000000000001111011`

Run-length encoding – Discussion

- extensions to larger alphabets possible (must store next character then)
- used in some image formats (e. g. TIFF)

Run-length encoding – Discussion

- extensions to larger alphabets possible (must store next character then)
- used in some image formats (e. g. TIFF)

fairly simple and fast

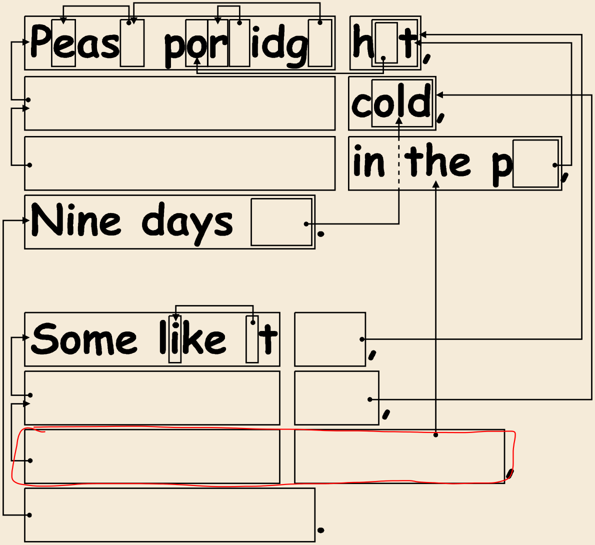
can compress n bits to $\Theta(\log n)$!
for extreme case of constant number of runs

negligible compression for many common types of data

- No compression until run lengths $k \geq 6$
- **expansion** for run length $k = 2$ or 6

7.5 Lempel-Ziv-Welch

Warmup



https://classic.csunplugged.org/text-compression/

https://www.flickr.com/photos/quintanaroo/2742726346

Clicker Question

What is the second-to-last line of the above poem?

`sli.do/comp526`

Click on "Polls" tab

Lempel-Ziv Compression

- Huffman and RLE mostly take advantage of frequent or repeated *single characters*.
- **Observation**: Certain *substrings* are much more frequent than others.
  - in English text: the, be, to, of, and, a, in, that, have, I
  - in HTML: "`<a href`", "`<img src`", "`<br/>`"

Lempel-Ziv Compression

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"
- **Lempel-Ziv** stands for family of *adaptive* compression algorithms.
  - **Idea:** store repeated parts by reference!

  $\rightsquigarrow$ each codeword refers to
    - either a single character in Σ_S,
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- **Lempel-Ziv** stands for family of *adaptive* compression algorithms.
  - **Idea:** store repeated parts by reference!

  ⇝ each codeword refers to
    - either a single character in Σ_S,
    - or a *substring* of S (that both encoder and decoder have already seen).
  - Variants of Lempel-Ziv compression
    - "LZ77" Original version ("sliding window")
      Derivatives: LZSS, LZFG, LZRW, LZP, DEFLATE, ...
      DEFLATE used in (`pk`)`zip`, `gzip`, `PNG`
    - "LZ78" Second (slightly improved) version
      Derivatives: LZW, LZMW, LZAP, LZY, ...
      LZW used in `compress`, `GIF`

Lempel-Ziv-Welch

- here: ***Lempel-Ziv-Welch (LZW)*** (arguably the "cleanest" variant of Lempel-Ziv)
- ***variable-to-fixed*** **encoding**
  - all codewords have k bits (typical: $k = 12$) $\rightsquigarrow$ fixed-length
  - but they represent a variable portion of the source text!