2 Machines & Models

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Learning Outcomes

Unit 2: ***Machines & Models***

1. Understand the difference between empirical *running time* and algorithm *analysis*.
2. Understand *worst/best/average case* models for input data.
3. Know the *RAM machine* model.
4. Know the definitions of *asymptotic notation* (Big-Oh classes and relatives).
5. Understand the reasons to make *asymptotic approximations*.
6. Be able to *analyze* simple *algorithms*.

Outline

2 Machines & Models

What is an algorithm?

An algorithm is a sequence of instructions.

think: recipe

More precisely:

e. g. Python script

1. mechanically executable
   ⇝ no "common sense" needed
2. finite description ≠ finite computation!
3. solves a *problem*, i. e., a class of problem instances

$x + y$, not only $17 + 4$

- input-processing-output abstraction

Typical example: *bubblesort*

⇝ not a specific program
but the underlying idea

What is a data structure?

A data structure is

1. a rule for **encoding data** (in computer memory), plus
2. **algorithms** to work with it (queries, updates, etc.)

typical example: *binary search tree*

2.1 Algorithm analysis

Good algorithms

Our goal: Find good (best?) algorithms and data structures for a task.

Good "usually" means

can be complicated in distributed systems

- fast running *time*
- moderate memory *space* usage

Algorithm analysis is a way to

- compare different algorithms,
- predict their performance in an application

Running time experiments

Why not simply run and time it?

- results only apply to
  - single *test* machine
  - tested inputs
  - tested implementation
  - ...
  - $\neq$ *universal truths*

- instead: consider and analyze algorithms on an abstract machine
  - $\rightsquigarrow$ provable statements for model
  - $\rightsquigarrow$ testable model hypotheses

survives Pentium 4

$\rightsquigarrow$ Need precise model of machine (costs), input data and algorithms.

Data Models

Algorithm analysis typically uses one of the following simple data models:

- **worst-case performance:**
  consider the *worst* of all inputs as our cost metric
- **best-case performance:**
  consider the *best* of all inputs as our cost metric
- **average-case performance:**
  consider the average/expectation of a *random* input as our cost metric

Usually, we apply the above for *inputs of same size* n.

$\rightsquigarrow$ performance is only a **function of** n.

2.2 The RAM Model

Clicker Question

What is the cost of *adding* two d-digit integers?
(For example, for $d = 5$, what is $45\,235 + 91\,342$?)

- A constant time
- B logarithmic in d
- C proportional to d
- D quadratic in d
- E no idea what you are talking about

→ *sli.do/cs566*

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What is the cost of *adding* two d-digit integers?
(For example, for $d = 5$, what is $45\,235 + 91\,342$?)

- A constant time ✓
- B ~~logarithmic in d~~
- C proportional to d ✓
- D ~~quadratic in d~~
- E no idea what you are talking about ✓

64 bit — uniform cost model ⟷ ?

d const. — logarithmic cost model

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Machine models

The machine model decides

- what algorithms are possible
- how they are described (= programming language)
- what an execution *costs*

Goal: Machine models should be
detailed and powerful enough to reflect actual machines,
abstract enough to unify architectures,
simple enough to analyze.

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- how they are described (= programming language)
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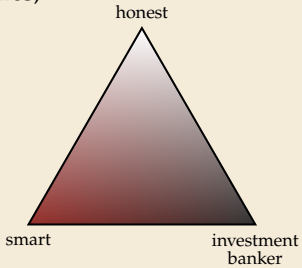
Goal: Machine models should be
detailed and powerful enough to reflect actual machines,
abstract enough to unify architectures,
simple enough to analyze.

⇝ usually some compromise is needed



Random Access Machines

Random access machine (RAM)

more detail in §2.2 of *Sequential and Parallel Algorithms and Data Structures* by Sanders, Mehlhorn, Dietzfelbinger, Dementiev

- unlimited *memory* `MEM[0], MEM[1], MEM[2], ...`
- fixed number of *registers* $R_1, \ldots, R_r$ (say $r = 100$)

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- fixed number of *registers* $R_1, \ldots, R_r$ (say $r = 100$)
- memory cells $\texttt{MEM}[i]$ and registers R_i store w-bit integers, i. e., numbers in $[0..2^w - 1]$
  w is the word width/size; typically $w \propto \lg n$ $\rightsquigarrow$ $2^w \approx n$

$w \in \Theta(\log n)$

proportional to

Random Access Machines

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  w is the word width/size; typically $\boxed{w \propto \lg n}$ $\rightsquigarrow$ $2^w \approx n$
- Instructions:
  - load & store: $R_i := \texttt{MEM}[R_j]$ $\quad \texttt{MEM}[R_j] := R_i$
  - operations on registers: $R_k := R_i + R_j$ (arithmetic is *modulo* 2^w!)
    also $R_i - R_j$, $R_i \cdot R_j$, $R_i \operatorname{div} R_j$, $R_i \bmod R_j$
    C-style operations (bitwise and/or/xor, left/right shift)
  - conditional and unconditional jumps
- cost: number of executed instructions

we will see further models later

$\rightsquigarrow$ The RAM is the standard model for sequential computation.

RAM-Program Example

Example RAM program

```
1  // Assume: R_1 stores number N
2  // Assume: MEM[0..N) contains list of N numbers
3  R_2 := R_1;
4  R_3 := R_1 − 2;
5  R_4 := MEM[R_3];
6  R_5 := R_3 + 1;
7  R_6 := MEM[R_5];
8  if (R_4 ≤ R_6) goto line 12;
9  MEM[R_3] := R_6;
10 MEM[R_5] := R_4;
11 R_3 := R_3 − 1;
12 if (R_3 ≥ 0) goto line 6;
13 R_2 := R_2 − 1;
14 if (R_2 > 0) goto line 5;
15 // Done:
```

Clicker Question

What algorithm does the RAM program on the previous slide implement?

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RAM-Program Example

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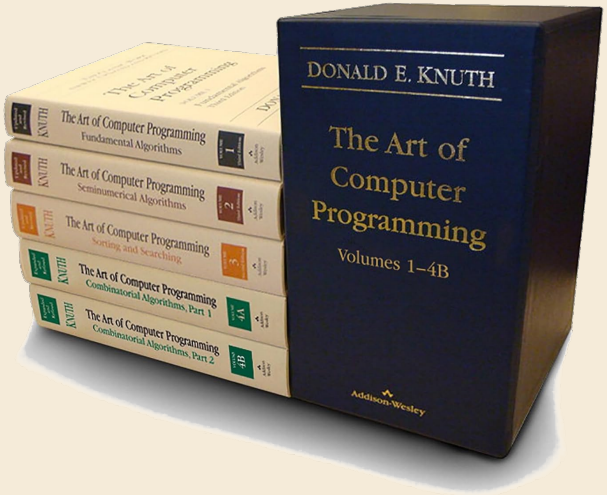
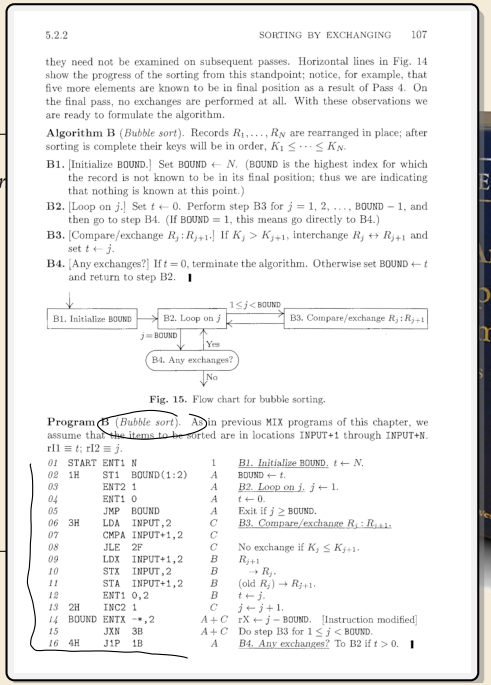


RAM-Program Example

Example RAM program

1 // *Assume:* R_1 *stores number* N
2 // *Assume:* MEM[0..N] *contains list of* N *number*
3 $R_2 := R_1$;
4 $R_3 := R_1 - 2$;
5 $R_4 := \text{MEM}[R_3]$;
6 $R_5 := R_3 + 1$;
7 $R_6 := \text{MEM}[R_5]$;
8 **if** $(R_4 \leq R_6)$ goto line 12;
9 $\text{MEM}[R_3] := R_6$;
10 $\text{MEM}[R_5] := R_4$;
11 $R_3 := R_3 - 1$;
12 **if** $(R_3 \geq 0)$ goto line 6;
13 $R_2 := R_2 - 1$;
14 **if** $(R_2 > 0)$ goto line 5;
15 // *Done:* MEM[0..N] *sorted*



5.2.2 SORTING BY EXCHANGING 107

they need not be examined on subsequent passes. Horizontal lines in Fig. 14 show the progress of the sorting from this standpoint; notice, for example, that five more elements are known to be in final position as a result of Pass 4. On the final pass, no exchanges are performed at all. With these observations we are ready to formulate the algorithm.

Algorithm B (*Bubble sort*). Records $R_1, \ldots, R_N$ are rearranged in place; after sorting is complete their keys will be in order, $K_1 \leq \cdots \leq K_N$.

B1. [Initialize BOUND.] Set BOUND $\leftarrow N$. (BOUND is the highest index for which the record is not known to be in its final position; thus we are indicating that nothing is known at this point.)

B2. [Loop on j.] Set $t \leftarrow 0$. Perform step B3 for $j = 1, 2, \ldots,$ BOUND $- 1$, and then go to step B4. (If BOUND $= 1$, this means go directly to B4.)

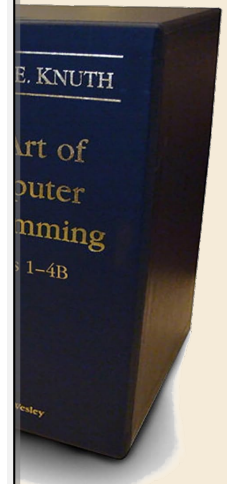
B3. [Compare/exchange $R_j : R_{j+1}$.] If $K_j > K_{j+1}$, interchange $R_j \leftrightarrow R_{j+1}$ and set $t \leftarrow j$.

B4. [Any exchanges?] If $t = 0$, terminate the algorithm. Otherwise set BOUND $\leftarrow t$ and return to step B2. ∎

Fig. 15. Flow chart for bubble sorting.

Program B (*Bubble sort*). As in previous MIX programs of this chapter, we assume that the items to be sorted are in locations INPUT+1 through INPUT+N. rI1 $\equiv t$; rI2 $\equiv j$.

```
01  START ENT1 N              1     B1. Initialize BOUND. t ← N.
02  1H    ST1  BOUND(1:2)     A     BOUND ← t.
03        ENT2 1              A     B2. Loop on j. j ← 1.
04        ENT1 0              A     t ← 0.
05        JMP  BOUND          A     Exit if j ≥ BOUND.
06  3H    LDA  INPUT,2        C     B3. Compare/exchange R_j : R_{j+1}:
07        CMPA INPUT+1,2      C
08        JLE  2F             C     No exchange if K_j ≤ K_{j+1}.
09        LDX  INPUT+1,2      B     R_{j+1}
10        STX  INPUT,2        B       → R_j.
11        STA  INPUT+1,2      B     (old R_j) → R_{j+1}.
12        ENT1 0,2            B     t ← j.
13  2H    INC2 1              C     j ← j + 1.
14  BOUND ENTX -*,2           A+C   rX ← j − BOUND.  [Instruction modified]
15        JXN  3B             A+C   Do step B3 for 1 ≤ j < BOUND.
16  4H    J1P  1B             A     B4. Any exchanges? To B2 if t > 0. ∎
```



Pseudocode

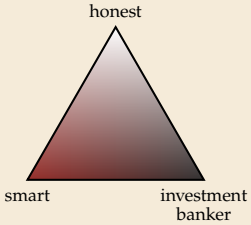
- Programs for the random-access machine are very low level and detailed

≈ assembly/machine language

Typical simplifications when describing and analyzing algorithms:

- more abstract *pseudocode* ← code that humans understand (easily)
  - control flow using **if**, **for**, **while**, etc.
  - variable names instead of fixed registers and memory cells
  - memory management (more below)
- count dominant *elementary operations* (e. g. memory accesses) instead of all RAM instructions

In both cases: We *can* go to full detail where needed/desired.



Pseudocode – Example

RAM-Program

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16 // Done: MEM[0..N) sorted
```

Pseudocode Algorithm

```
1  procedure bubblesort(A[0..N)):
2      for i := N, N − 1, ..., 1
3          for j := N − 2, N − 3, ..., 0
4              if A[j] > A[j + 1]:
5                  Swap A[j] and A[j + 1]
6              end if
7          end for
8      end for
```

- ⇝ much more **readable**
- closer to modern high-level programming languages
- **but:** only allow primitive operations that correspond to $O(1)$ RAM instructions
  - ⇝ analysis

Memory management & Pointers

- A random-access machine is a bit like a bare CPU . . . without any operating system
  - ⇝ cumbersome to use
- All high-level programming languages / operating systems add *memory management*:
  - Instruction to *allocate* a contiguous piece of memory of a given size (like `malloc`).
    - used to allocate a `new array` (of a fixed size) or
    - a `new object/record` (with a known list of instance variables)
    - There's a similar instruction to `free` allocated memory again or an automated garbage collector.
  - ⇝ A *pointer* is a memory address (i. e., the i of $\texttt{MEM}[i]$).
  - Support for procedures (a. k. a. functions, methods) calls including recursive calls
    - (this internally requires maintaining call stack)

We will mostly ignore *how* all this works here.

2.3 Asymptotics & Big-Oh

Clicker Question

What is the correct way to complete the equation?

$8n + \frac{1}{2}n^2 + 1024 = \square$

A $O(1)$

B $O(n)$

C $O(n \log(n))$

D $O(n^2)$

E I don't know $O(\cdot)$

→ sli.do/cs566

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→ sli.do/cs566

Why asymptotics?

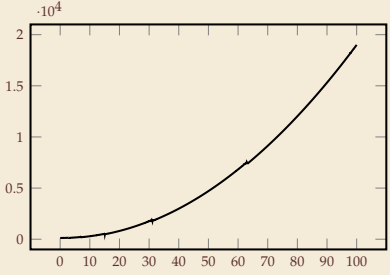
Algorithm analysis focuses on (the limiting behavior for infinitely) **large** inputs.

- abstracts from unnecessary detail
- simplifies analysis
- often necessary for sensible comparison

Asymptotics = approximation around ∞

Example: Consider a function $f(n)$ given by

$$2n^2 - 3n\lfloor \log_2(n+1) \rfloor + 7n - 3\lfloor \log_2(n+1) \rfloor + 120$$



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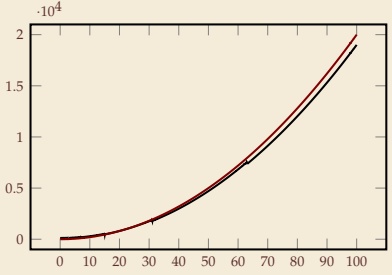

Asymptotic tools – Formal & definitive definition

- **"Tilde Notation":** $f(n) \sim g(n)$ iff (if, and only if) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

  „f and g are *asymptotically equivalent*"

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  (if, and only if; ⟺ iff gdw.)

  „f and g are *asymptotically equivalent*"

- **"Big-Oh Notation":** $f(n) \in O\big(g(n)\big)$ iff $\left|\frac{f(n)}{g(n)}\right|$ is bounded for $n \geq n_0$

  (also write '=' instead)

  iff $\limsup_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| < \infty$

  (need supremum since limit might not exist!)

Variants:

("Big-Omega")

- $f(n) \in \Omega\big(g(n)\big)$ iff $g(n) \in O\big(f(n)\big)$
- $f(n) \in \Theta\big(g(n)\big)$ iff $f(n) \in O\big(g(n)\big)$ **and** $f(n) \in \Omega\big(g(n)\big)$

("Big-Theta")

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  - $f(n) \in \Theta\big(g(n)\big)$ iff $f(n) \in O\big(g(n)\big)$ **and** $f(n) \in \Omega\big(g(n)\big)$

  ("Big-Theta")

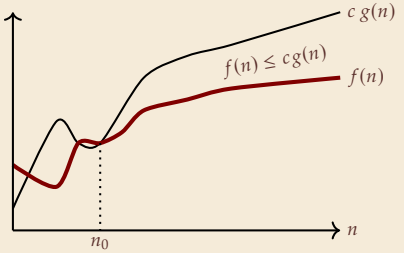
- **"Little-Oh Notation":** $f(n) \in o\big(g(n)\big)$ iff $\lim_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| = 0$

  similarly: $f(n) \in \omega\big(g(n)\big)$ if $\lim = \infty$

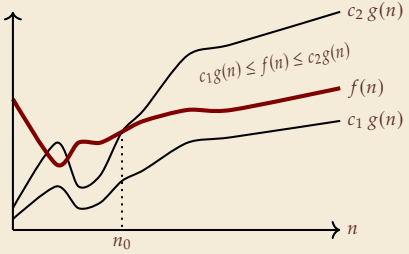
(Benefit of this definition: Works for any $f, g : \mathbb{R} \to \mathbb{R}$ and is easy to generalize to limits other than $n \to \infty$)

Asymptotic tools – Intuition

- $f(n) = O(g(n))$: $f(n)$ is **at most** $g(n)$ up to constant factors and for sufficiently large n



- $f(n) = \Theta(g(n))$: $f(n)$ is **equal to** $g(n)$ up to constant factors and for sufficiently large n



Plots can be misleading!

Example

Clicker Question

Assume $f(n) \in O(g(n))$. What can we say about $g(n)$?

A $g(n) = O(f(n))$

B $g(n) = \Omega(f(n))$

C $g(n) = \Theta(f(n))$

D Nothing (it depends on f and g)

→ sli.do/cs566

Clicker Question

Assume $f(n) \in O(g(n))$. What can we say about $g(n)$?

A ~~$g(n) = O(f(n))$~~

B $g(n) = \Omega(f(n))$ ✓

C ~~$g(n) = \Theta(f(n))$~~

D ~~Nothing (it depends on f and g)~~

→ *sli.do/cs566*

Clicker Question

Assume $f(n) \in O(g(n))$. What can we say about $g(n)$?

A ~~$g(n) = O(f(n))$~~

B $g(n) = \Omega(f(n))$ ✓ (if $f(n) \neq 0$)

C ~~$g(n) = \Theta(f(n))$~~

D Nothing (it depends on f and g) ✓

→ *sli.do/cs566*

Asymptotics – Example 1

Basic examples:

- $20n^3 + 10n\ln(n) + 5 \sim 20n^3 = \Theta(n^3)$
- $3\lg(n^2) + \lg(\lg(n)) = \Theta(\log n)$
- $10^{100} = O(1)$

$$\lim_{n\to\infty} \frac{20n^3 + 10n\ln n + 5}{20n^3} = \lim_{n\to\infty} \frac{20n^3}{20n^3} + \frac{10n\ln n}{20n^3} + \frac{5}{20n^3} \quad (1 + \to 0)$$

$$= 1 + 10 \lim_{n\to\infty} \frac{\ln n}{n^2} + 0$$

$\frac{\ln n}{n^2}$: l'Hôpital

$$= 1$$

$$\lim_{n\to\infty} \frac{(1/n)}{2n} = \lim_{n\to\infty} \frac{1}{2n} = 0$$

$< \infty$

$O(n^3)$

$$\lim_{n\to\infty} \frac{20n^3}{20n^3 + 10n\ln n + 5} = 1 \qquad \Omega(n^3)$$

Use *wolfram alpha* to compute/check limits, but also practice it with pen and paper!

Clicker Question

Is $(\sin(n) + 2)n^2 = \Theta(n^2)$?

A Yes

B No

→ *sli.do/cs566*

Clicker Question

Is $(\sin(n) + 2)n^2 = \Theta(n^2)$?

$\limsup_{n \to \infty} \sin(n) + 2 \leq 3 < \infty$

$\Rightarrow O(n^2)$

A Yes ✓

B ~~No~~

→ sli.do/cs566

Asymptotics – Basic facts

Rules to work with Big-Oh classes:

- $f = \Theta(f)$ (reflexivity)
- $f = \Theta(g) \ \wedge \ g = \Theta(h) \implies f = \Theta(h)$
- $c \cdot f(n) \ = \ \Theta(f(n))$ for constant $c \neq 0$
- $f \sim g \iff f = g \cdot (1 \pm o(1))$
- $\Theta(f) \cdot \Theta(g) = \Theta(f \cdot g)$
- $\Theta(f) + \Theta(g) = \Theta(f + g) \ = \ \Theta(\max\{f, g\})$ largest summand determines Θ-class

Asymptotics – Frequently encountered classes

Frequently used orders of growth:

- constant $\Theta(1)$
- logarithmic $\Theta(\log n)$ Note: $a, b > 0$ constants $\rightsquigarrow$ $\Theta(\log_a(n)) = \Theta(\log_b(n))$
- linear $\Theta(n)$
- linearithmic $\Theta(n \log n)$
- quadratic $\Theta(n^2)$
- cubic $\Theta(n^3)$
- polynomial $O(n^c)$ for some constant c
- exponential $O(c^n)$ for some constant $c > 1$ Note: $a > b > 0$ constants $\rightsquigarrow$ $b^n = o(a^n)$

Asymptotics – Example 2

Square-and-multiply algorithm
for computing x^m with $m \in \mathbb{N}$

Inputs:

- m as binary number (array of bits)
- n = #bits in m
- x a floating-point number

```
1  def pow(x, m):
2      # compute binary representation of exponent
3      exponent_bits = bin(m)[2:]
4      result = 1
5      for bit in exponent_bits:
6          result *= result
7          if bit == '1':
8              result *= x
9      return result
```

- Cost: C = # multiplications
- C = n (line 6) + #one-bits in binary representation of m (line 8)

$\rightsquigarrow$ $n \leq C \leq 2n$

Clicker Question

We showed $n \leq C(n) \leq 2n$; what is the most precise asymptotic approximation for $C(n)$ that we can make?

Write e. g. `O(n^2)` for $O(n^2)$ or `Theta(sqrt(n))` for $\Theta(\sqrt{n})$.

→ *sli.do/cs566*

Asymptotics – Example 2

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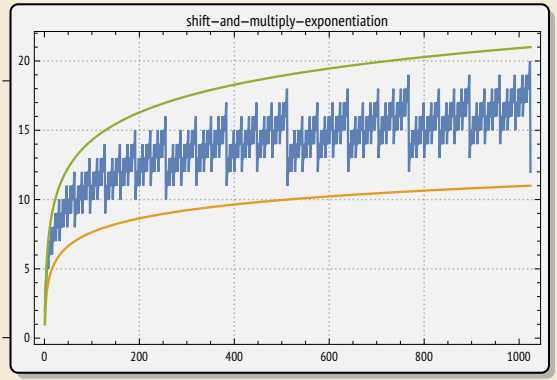

- Cost: C = # multiplications
- $C = n$ (line 6) + #one-bits in binary representation of m (line 8)

$\rightsquigarrow\ n \leq C \leq 2n$

$\rightsquigarrow\ \boxed{C = \Theta(n) = \Theta(\log m)}$

Often, you can pretend Θ is *"like $\sim$ with an unknown constant"*
but in this case, ***no such constant exists!***