3 Efficient Sorting – The Power of Divide & Conquer

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COMP526 (Fall 2022)
University of Liverpool
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Learning Outcomes

1. Know principles and implementation of *mergesort* and *quicksort*.
2. Know properties and *performance characteristics* of mergesort and quicksort.
3. Know the comparison model and understand the corresponding *lower bound*.
4. Understand *counting sort* and how it circumvents the comparison lower bound.
5. Know ways how to exploit *presorted* inputs.

Unit 3: ***Efficient Sorting***

Outline

3 Efficient Sorting

Why study sorting?

- fundamental problem of computer science that is still not solved
  - Algorithm with optimal #comparisons in worst case?
- building brick of many more advanced algorithms
  - for preprocessing
  - as subroutine
- playground of manageable complexity
  to practice algorithmic techniques

Here:

- "classic" fast sorting method
- exploit **partially sorted** inputs
- ~~**parallel sorting**~~ → Unit 5

Part I

The Basics

Rules of the game

- **Given:**
  - array $A[0..n) = A[0..n-1]$ of n objects
  - a total order relation $\leq$ among $A[0], \ldots, A[n-1]$
    (a comparison function)
    Python: elements support `<=` operator (`__le__()`)
    Java: `Comparable` class (`x.compareTo(y) <= 0`)
- **Goal:** rearrange (i. e., permute) elements within A,
  so that A is *sorted*, i. e., $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

- for now: A stored in main memory (*internal sorting*)
  single processor (*sequential sorting*)

Clicker Question

What is the complexity of sorting? Type you answer, e. g., as "`Theta(sqrt(n))`"

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3.1 Mergesort

Clicker Question

How does mergesort work?

- **A** Split elements around median, then recurse on small / large elements.
- **B** Recurse on left / right half, then combine sorted halves.
- **C** Grow sorted part on left, repeatedly add next element to sorted range.
- **D** Repeatedly choose 2 elements and swap them if they are out of order.
- **E** Don't know.

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Clicker Question

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A ~~Split elements around median, then recurse on small / large elements.~~

B Recurse on left / right half, then combine sorted halves. ✓

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D ~~Repeatedly choose 2 elements and swap them if they are out of order.~~

E ~~Don't know.~~

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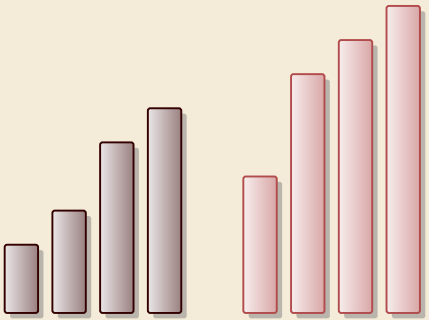
Merging sorted lists

Merging sorted lists



run1 run2 result

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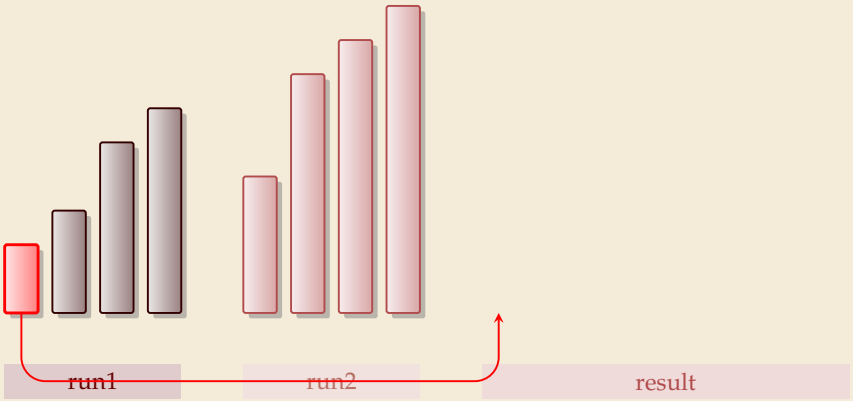

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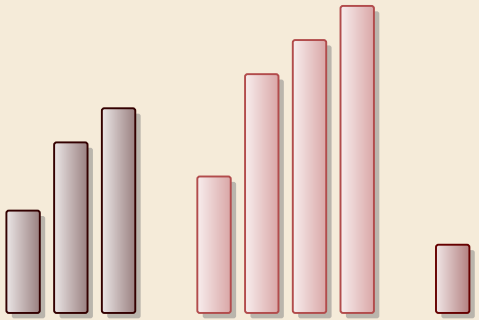

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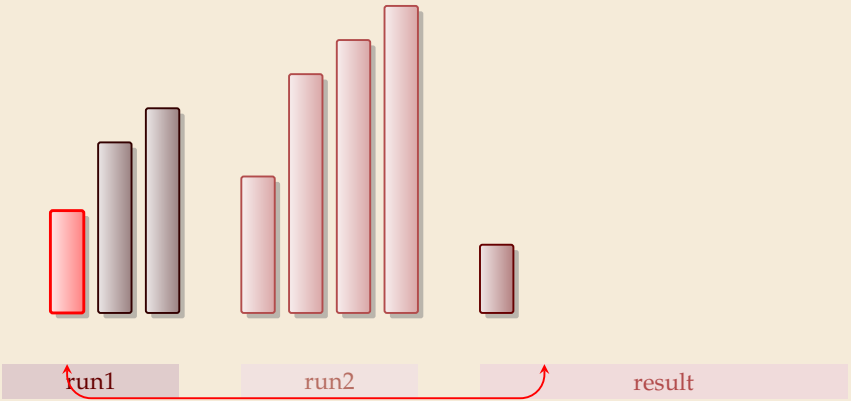

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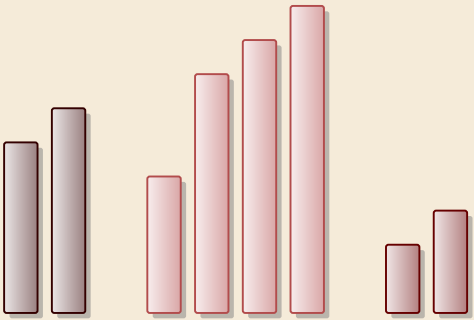

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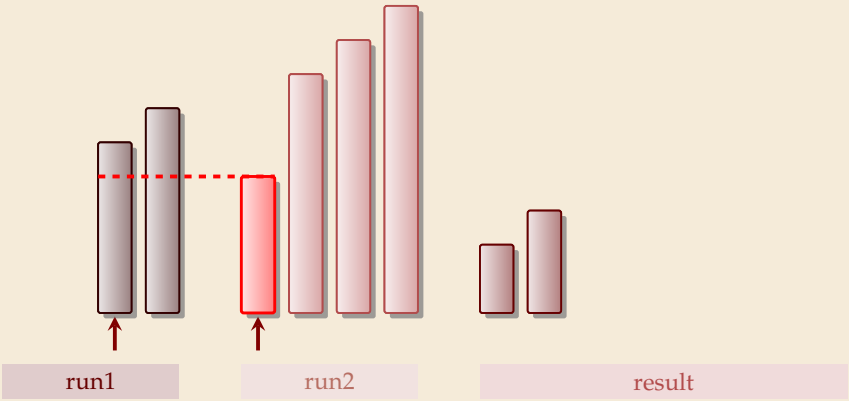

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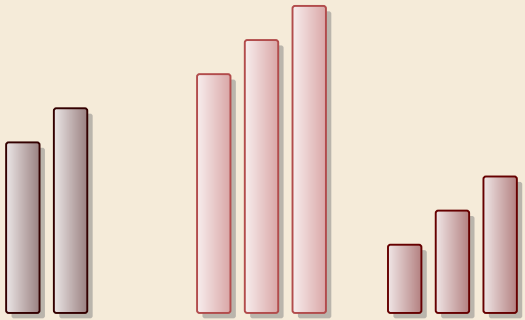

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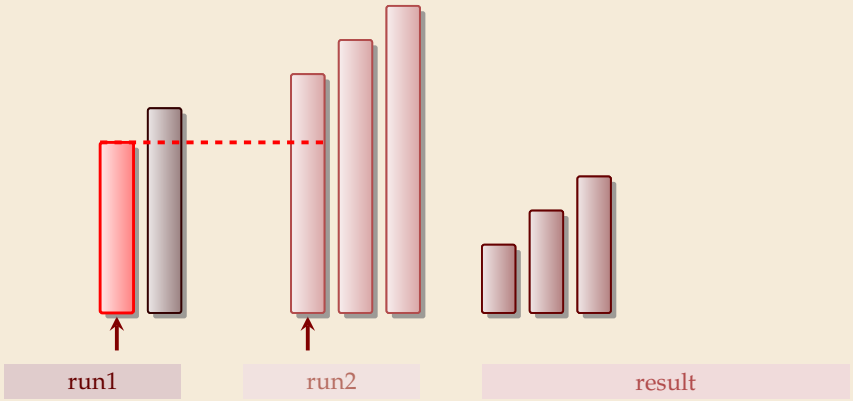

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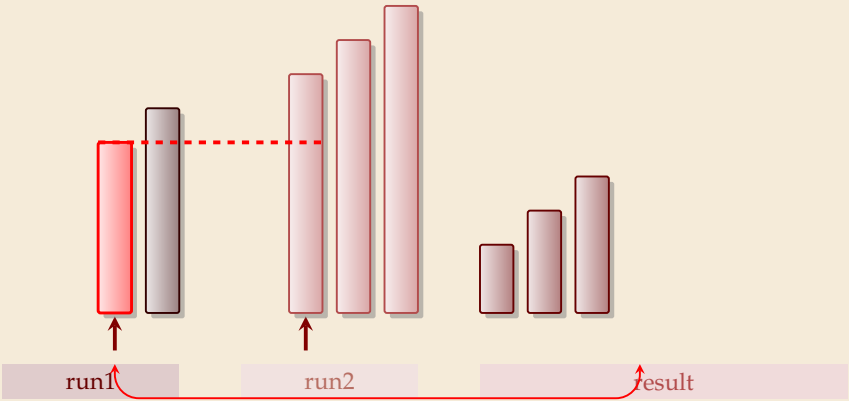

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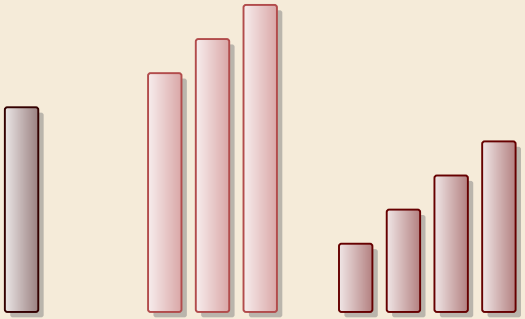

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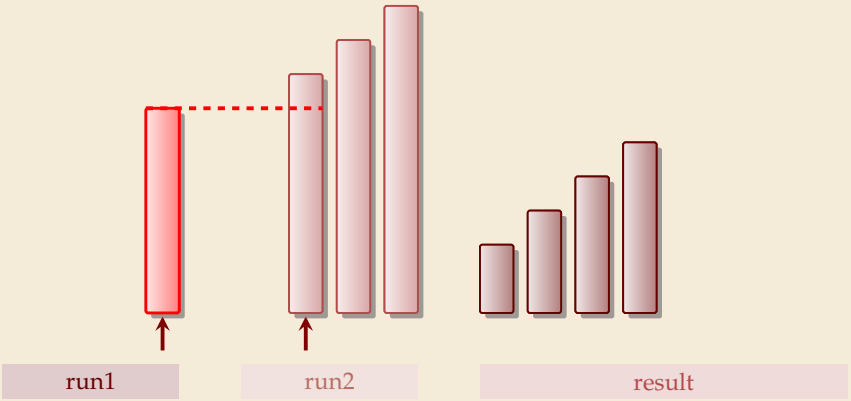

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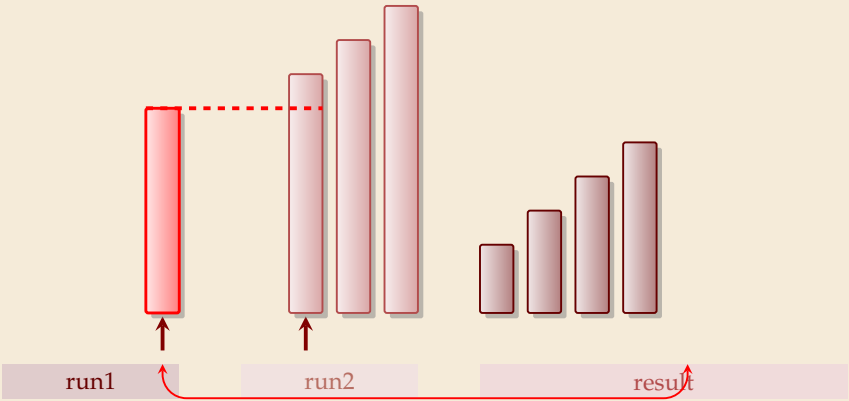

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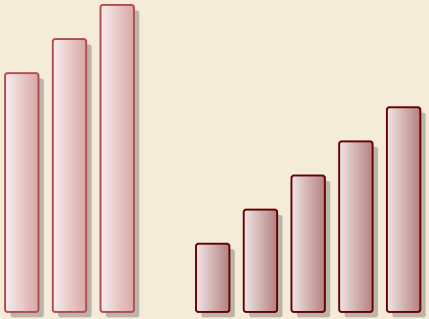

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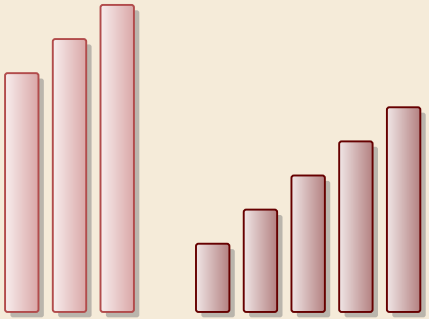

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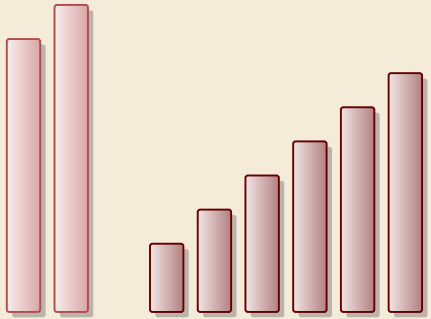

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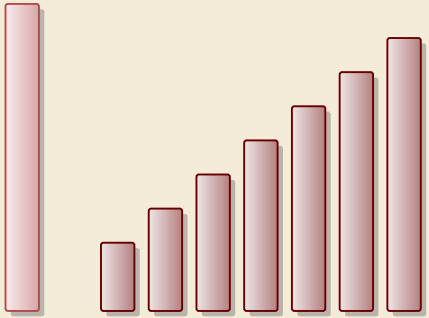

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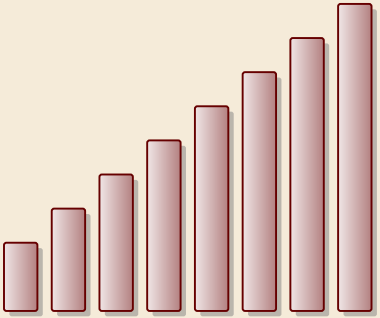

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Clicker Question

What is the worst-case running time of mergesort?

- **A** $\Theta(1)$
- **B** $\Theta(\log n)$
- **C** $\Theta(\log \log n)$
- **D** $\Theta(\sqrt{n})$
- **E** $\Theta(n)$
- **F** $\Theta(n \log \log n)$
- **G** $\Theta(n \log n)$
- **H** $\Theta(n \log^2 n)$
- **I** $\Theta(n^{1+\epsilon})$
- **J** $\Theta(n^2)$
- **K** $\Theta(n^3)$
- **L** $\Theta(2^n)$

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F ~~$\Theta(n \log \log n)$~~

G $\Theta(n \log n)$ ✓

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Mergesort

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1  procedure mergesort(A[l..r))
2      n := r − l
3      if n ≤ 1 return
4      m := l + ⌊n/2⌋
5      mergesort(A[l..m))
6      mergesort(A[m..r))
7      merge(A[l..m), A[m..r), buf)
8      copy buf to A[l..r)
```

proc sort (A[0..n))

mergesort (A[0..n))

- recursive procedure
- merging needs
  - temporary storage *buf* for result (of same size as merged runs)
  - to read and write each element twice (once for merging, once for copying back)

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array access

Analysis: count *"element visits"* (read and/or write)

$$C(n) = \begin{cases} 0 & n \le 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \ge 2 \end{cases}$$

same for best and worst case!

Simplification $n = 2^k$

$$C(2^k) = \begin{cases} 0 & k \le 0 \\ 2 \cdot C(2^{k-1}) + 2 \cdot 2^k & k \ge 1 \end{cases} = 2 \cdot 2^k + 2^2 \cdot 2^{k-1} + 2^3 \cdot 2^{k-2} + \cdots + 2^k \cdot 2^1 = 2k \cdot 2^k$$

$$C(n) = 2n \lg(n) = \Theta(n \log n)$$