6 Text Indexing – Searching whole genomes

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Outline

6 Text Indexing

6.1 Motivation
6.2 Suffix Trees
6.3 Applications
6.4 Longest Common Extensions
6.5 Suffix Arrays
6.6 Linear-Time Suffix Sorting
6.7 The LCP Array

6.1 Motivation

Text indexing

- *Text indexing* (also: *offline text search*):
  - case of string matching: find $P[0..m)$ in $T[0..n)$
  - but with *fixed* text $\rightsquigarrow$ preprocess T (instead of P)
  - $\rightsquigarrow$ expect many queries P, answer them without looking at all of T
  - $\rightsquigarrow$ essentially a data structuring problem: "building an *index* of T"

    Latin: "one who points out"

- application areas
  - web search engines
  - online dictionaries
  - online encyclopedia
  - DNA/RNA data bases
  - ... searching in any collection of text documents (that grows only moderately)

Inverted indices

same as "indexes"

- original indices in books: list of (key) words $\mapsto$ page numbers where they occur
- assumption: searches are only for **whole** (key) **words**

$\rightsquigarrow$ often reasonable for natural language text

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Inverted index:

- collect all words in T
  - can be as simple as splitting T at whitespace
  - actual implementations typically support *stemming* of words
    `goes` $\to$ `go`, `cats` $\to$ `cat`
- store mapping from words to a list of occurrences $\rightsquigarrow$ *how?*

BST? but $O(\log n)$ time

like a dictionary! keys = words
values = list of occurrences

Clicker Question

Do you know what a *trie* is?

- **A** A what? No!
- **B** I have heard the term, but don't quite remember.
- **C** I remember hearing about it in a module.
- **D** Sure.

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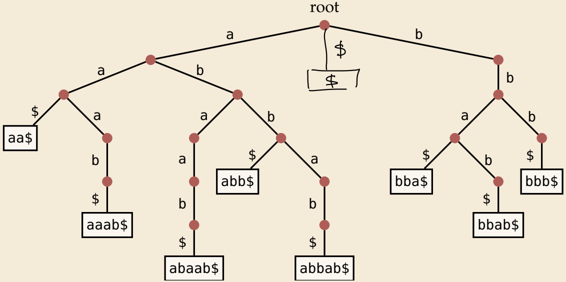
Tries

- efficient dictionary data structure for strings
- name from re**trie**val, but pronounced "try" ≈ tree
- tree based on symbol comparisons

- **Assumption:** stored strings are *prefix-free* (no string is a prefix of another)
  some character $\notin \Sigma$
  - strings of same length ✓
  - strings have "end-of-string" marker \$ ✓

- **Example**: $\Sigma = \{a, b\}$
  {aa\$, aaab\$, abaab\$, abb\$, abbab\$, bba\$, bbab\$, bbb\$}, \$}



Clicker Question

Suppose we have a trie that stores n strings over $\Sigma = \{\texttt{A}, \ldots, \texttt{Z}\}$. Each stored string consists of m characters.

We now search for a query string Q with $|Q| = q$. $(q \le m)$

How many **nodes** in the trie are **visited** during this **query**?

A $\Theta(\log n)$

B $\Theta(\log(nm))$

C $\Theta(m \cdot \log n)$

D $\Theta(m + \log n)$

E $\Theta(m)$

F $\Theta(\log m)$

G $\Theta(q)$

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How many **nodes** does the trie have **in total** *in the worst case*?

A $\Theta(n)$

B $\Theta(n + m)$

C $\Theta(n \cdot m)$

D $\Theta(n \log m)$

E $\Theta(m)$

F $\Theta(m \log n)$

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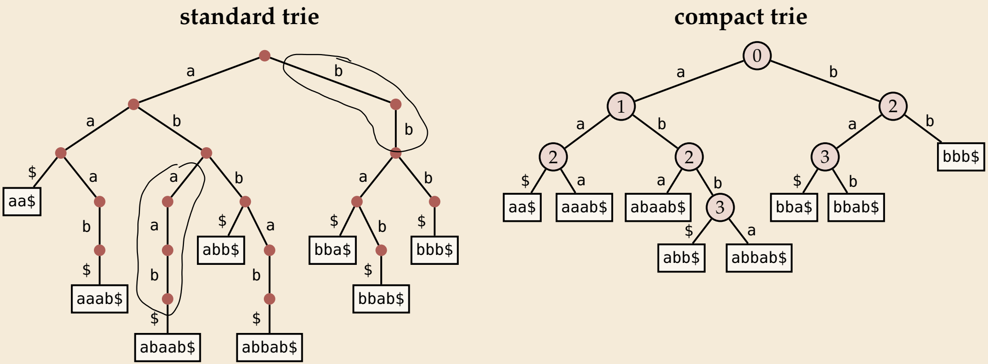
Compact tries

=1 child

- compress paths of unary nodes into single edge
- nodes store index of next character



⇝ searching slightly trickier, but same time complexity as in trie

not $O(n \cdot m)$

- all nodes $\geq$ 2 children ⇝ #nodes $\leq$ #leaves = #strings ⇝ linear space $O(n)$

Tries as inverted index

- simple
- fast lookup

- cannot handle more general queries:
  - search part of a word
  - search phrase (sequence of words)

Tries as inverted index

- 👍 simple
- 👍 fast lookup

- 👎 cannot handle more general queries:
  - search part of a word
  - search phrase (sequence of words)

- 👎 what if the 'text' does not even have words to begin with?!
  - biological sequences

    ```
    ACAAGATGCCATTGTCCCCCGGCCTCCTGCTGCTGCTGCTCTCCGGGGCCACGGCCACCGCTGCCCTGCCCCTGGAGGGTGGCCCCACCGGC
    CGAGACAGCGAGCATATGCAGGAAGCGGCAGGAATAAGGAAAAGCAGCCTCCTGACTTTCCTCGCTTGGTGGTTTGAGTGGACCTCCCAGGC
    CAGTGCCGGGCCCCTCATAGGAGAGGAAGCTCGGGAGGTGGCCAGGCGGCAGGAAGGCGCACCCCCCCAGCAATCCGCGCGCCGGGACAGAA
    TGCCCTGCAGGAACTTCTTCTGGAAGACCTTCTCCTCCTGCAAATAAAACCTCACCCATGAATGCTCACGCAAGTTTAATTACAGACCTGAA
    ```

  - binary streams

    ```
    00000010101001111010111000001111100011111011111001101101000011100010011011110000010001101010
    01101100001101011010000000100000000111010110000010000111101011101100100011001011011101111111
    110001010001011001010000001110101010011000000001101100001100111110000101 0101011101111000011
    10101110010010101010100000111110100110000001111001101010000000100100100000101100011000110111
    ```

⇝ need new ideas

6.2 Suffix Trees

Suffix trees – A 'magic' data structure

Appetizer: Longest common substring problem

- Given: strings $S_1, \ldots, S_k$ **Example:** S_1 = `superiorcalifornialives`, S_2 = `sealiver`
- Goal: find the longest substring that occurs in all k strings

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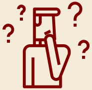

Can we do this in time $O(|S_1| + \cdots + |S_k|)$? How??

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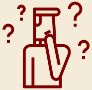

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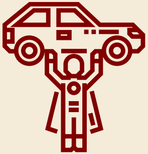
Enter: ***suffix trees***

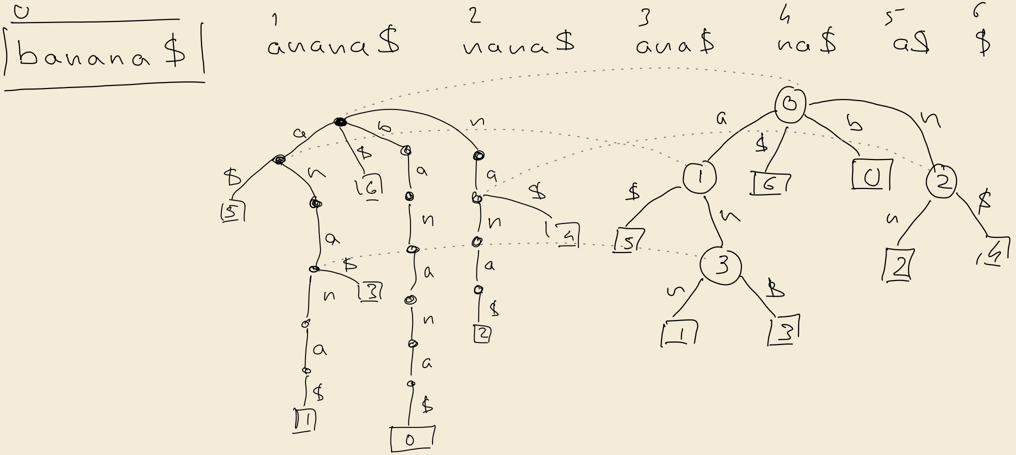


- versatile data structure for index with full-text search
- linear time (for construction) and linear space
- allows efficient solutions for many advanced string problems

"Although the longest common substring problem looks trivial now, given our knowledge of suffix trees, it is very interesting to note that in 1970 Don Knuth conjectured that a linear-time algorithm for this problem would be impossible." *[Gusfield: Algorithms on Strings, Trees, and Sequences (1997)]*

Suffix trees – Definition

- suffix tree $\mathcal{T}$ for text $T = T[0..n)$ $=$ compact trie of all suffixes of $T\$$ (set $T[n] := \$$)



Suffix trees – Definition

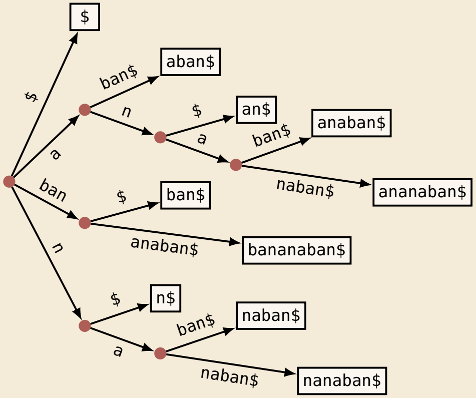
- suffix tree $\mathcal{T}$ for text $T = T[0..n)$ = compact trie of all suffixes of $T\$$ (set $T[n] := \$$)

Example:

T = bananaban$

suffixes: {bananaban$, ananaban$, nanaban$, anaban$, naban$, aban$, ban$, an$, n$, $}

0	1	2	3	4	5	6	7	8	9
b	a	n	a	n	a	b	a	n	$



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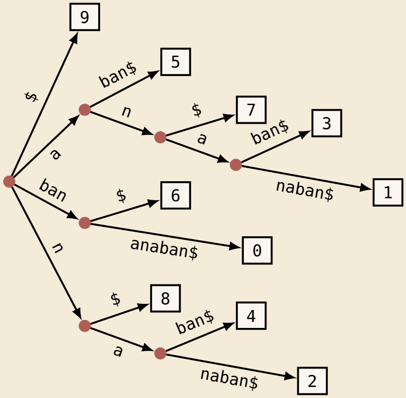

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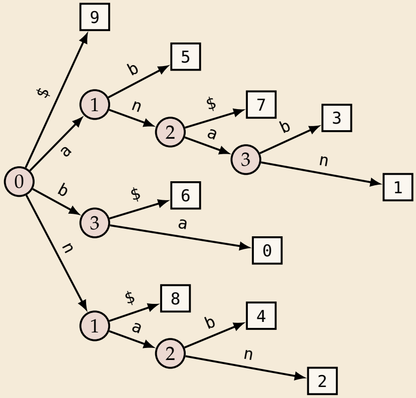

- also: edge labels like in compact trie
- (more readable form on slides to explain algorithms)

Suffix trees – Construction

- $T[0..n)$ has $n + 1$ suffixes (starting at character $i \in [0..n]$)
- We can build the suffix tree by inserting each suffix of T into a compressed trie.
  But that takes time $\Theta(n^2)$. $\rightsquigarrow$ not interesting!

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same order of growth as reading the text!

Amazing result: Can construct the suffix tree of T in $\Theta(n)$ time!

- algorithms are a bit tricky to understand
- but were a theoretical breakthrough
- and they are efficient in practice (and heavily used)!

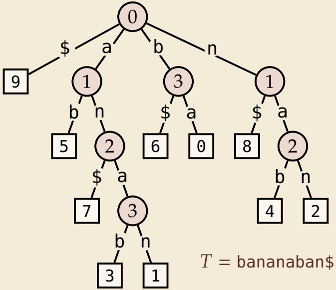
$\rightsquigarrow$ for now, take linear-time construction for granted. What can we do with them?

6.3 Applications

Applications of suffix trees

- In this section, always assume suffix tree $\mathcal{T}$ for T given.

Recall: $\mathcal{T}$ stored like this: but think about this:



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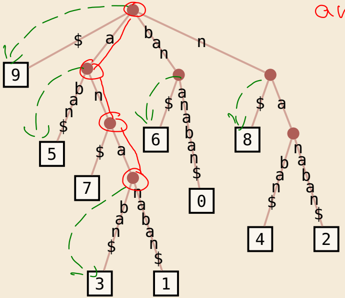
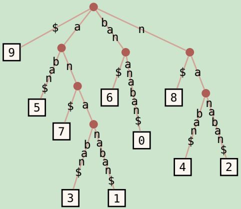


ana

- Moreover: assume internal nodes store pointer to *leftmost leaf in subtree*.
- Notation: $T_i = T[i..n]$ (including $)

Clicker Question



What does T's suffix tree (on the left) tell you about the question whether T contains the pattern $P = \texttt{ana}$?

Check all that apply to this example.

A Nothing.

B P occurs in T.

C P does not occur in T.

D P occurs once in T.

E P occurs twice in T.

F P starts at index 0.

G P starts at index 1.

H P starts at index 2.

I P starts at index 3.

J P starts at index 4.

K P starts at index 7.

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Click on "Polls" tab

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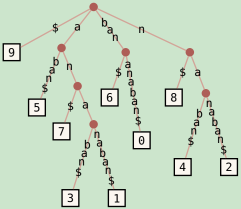

What does T's suffix tree (on the left) tell you about the question whether T contains the pattern P = `ana`?

Check all that apply to this example.

- **A** ~~Nothing.~~
- **B** P occurs in T. ✓
- **C** ~~P does not occur in T.~~
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Application 1: Text Indexing / String Matching

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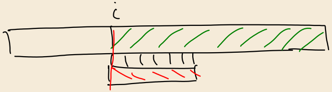
- we have all suffixes in $\mathcal{T}$!



Application 1: Text Indexing / String Matching

- $\boxed{P \text{ occurs in } T \iff P \text{ is a prefix of a suffix of } T}$
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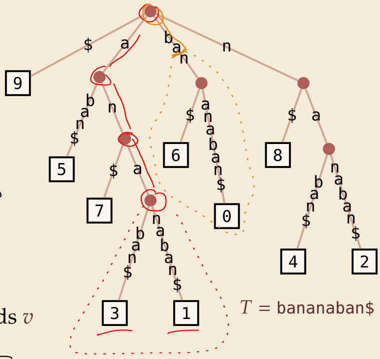
⇝ (try to) follow path with label P, until

1. **we get stuck**
   at internal node (no node with next character of P) nb
   or *inside edge* (mismatch of next characters) baa
   ⇝ P does not occur in T
2. **we run out of pattern** ana ba
   reach end of P at internal node v or inside edge towards v
   ⇝ P occurs at all leaves in subtree of v
3. **we run out of tree**
   reach a leaf ℓ with part of P left ⇝ compare P to ℓ.

   ⚠ This cannot happen when testing edge labels since $\$ \notin \Sigma$,
   but needs check(s) in compact trie implementation!



not possible / relevant
text indexing

- Finding first match (or `NO_MATCH`) takes $O(|P|)$ time!

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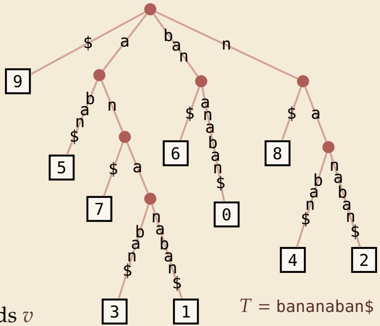


T = `bananaban$`

Examples:

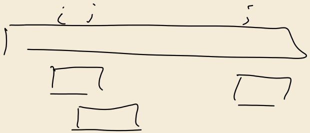
- P = `ann`
- P = `ana`
- P = `briar`

Application 2: Longest repeated substring

- **Goal:** Find longest substring $T[i..i+\ell)$ that occurs also at $j \neq i$: $T[j..j+\ell) = T[i..i+\ell)$.

  e. g. for compression $\rightsquigarrow$ Unit 7

How can we efficiently check *all possible substrings?*



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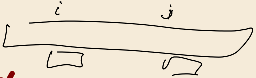

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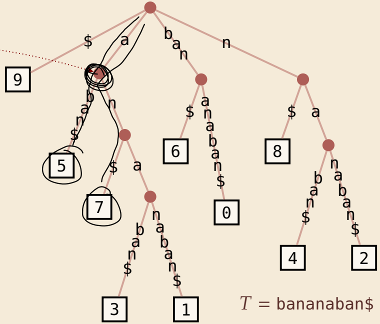
Repeated substrings = shared paths in *suffix tree*

- T_5 = aban$ and T_7 = an$ have *longest common prefix* 'a'

⇝ ∃ internal node with path label 'a'

here single edge, can be longer path



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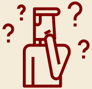

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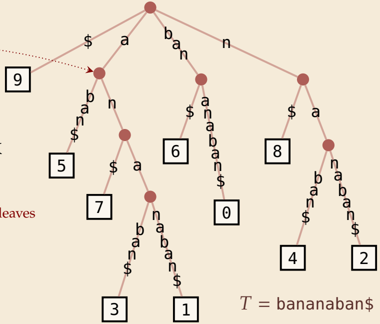
⇝ longest repeated substring = longest common prefix (LCP) of two suffixes

actually: adjacent leaves



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How can we efficiently check *all possible substrings?*

Repeated substrings = shared paths in *suffix tree*

- T_5 = aban\$ and T_7 = an\$ have *longest common prefix* 'a'

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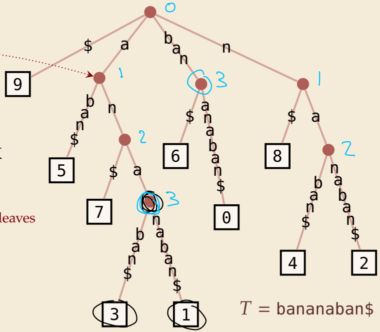

T = bananaban\$

- Algorithm:
  1. Compute *string depth* (=length of path label) of nodes
  2. Find internal nodes with maximal string depth

- Both can be done in depth-first traversal $\rightsquigarrow$ $\Theta(n)$ time

Generalized suffix trees

- longest *repeated* substring (of one string) feels very similar to longest *common* substring of several strings $T^{(1)}, \ldots, T^{(k)}$ with $T^{(j)} \in \Sigma^{n_j}$
- can we solve that in the same way?
- could build the suffix tree for each $T^{(j)}$ ... but doesn't seem to help

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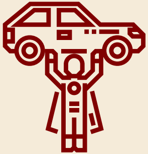
Enter: ***generalized suffix tree***



- Define $\boxed{T := T^{(1)}\$_1 T^{(2)}\$_2 \cdots T^{(k)}\$_k}$ for k new end-of-word symbols
- Construct suffix tree $\mathcal{T}$ for T

$\rightsquigarrow$ $\$_j$-edges always leads to leaves $\rightsquigarrow$ $\exists$ leaf (j, i) for each suffix $T_i^{(j)} = T^{(j)}[i..n_j]$

Clicker Question

What is the longest common substring of the strings `bcabcac`, `aabca` and `bcaa`?

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Application 3: Longest common substring

- With that new idea, we can find longest common superstrings:
  1. Compute generalized suffix tree $\mathcal{T}$.
  2. Store with each node the *subset of strings* that contain its path label:
     - 2.1. Traverse $\mathcal{T}$ bottom-up.
     - 2.2. For a leaf (j, i), the subset is $\{j\}$.
     - 2.3. For an internal node, the subset is the union of its children.
  3. In top-down traversal, compute *string depths* of nodes. (as above)
  4. Report deepest node (by string depth) whose subset is $\{1, \ldots, k\}$.

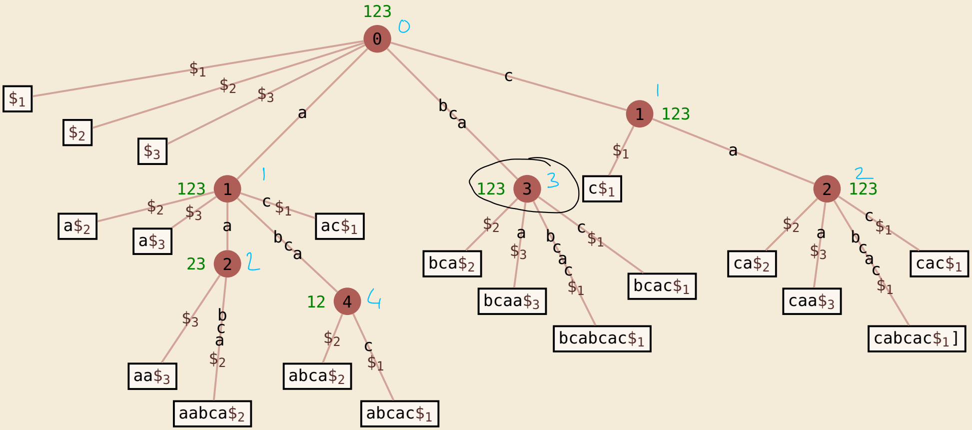
stores set of j so that there is a leaf (j,i) in the subtree

- Each step takes time $\Theta(n)$ for $n = n_1 + \cdots + n_k$ the total length of all texts.

"Although the longest common substring problem looks trivial now, given our knowledge of suffix trees, it is very interesting to note that in 1970 Don Knuth conjectured that a linear-time algorithm for this problem would be impossible." *[Gusfield: Algorithms on Strings, Trees, and Sequences (1997)]*

Longest common substring – Example

$T^{(1)} = \texttt{bcabcac}$, $T^{(2)} = \texttt{aabca}$, $T^{(3)} = \texttt{bcaa}$



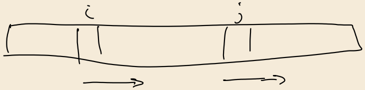
6.4 Longest Common Extensions

Application 4: Longest Common Extensions

- We implicitly used a special case of a more general, versatile idea:

  The ***longest common extension (LCE)*** data structure:

  - **Given:** String $T[0..n)$
  - **Goal:** Answer LCE queries, i. e.,
    given positions i, j in T,
    how far can we read the same text from there?
    formally: $\mathrm{LCE}(i, j) = \max\{\ell : T[i..i+\ell) = T[j..j+\ell)\}$



Application 4: Longest Common Extensions

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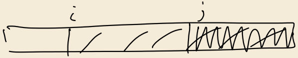
  The ***longest common extension (LCE)*** data structure:

  - **Given:** String $T[0..n)$
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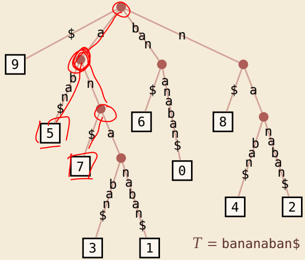
    given positions i, j in T,
    how far can we read the same text from there?
    formally: $\mathrm{LCE}(i,j) = \max\{\ell : T[i..i+\ell) = T[j..j+\ell)\}$

a b a n $
a n $

$\rightsquigarrow$ use suffix tree of T!

longest common prefix of ith and jth suffix

- In $\mathcal{T}$: $\mathrm{LCE}(i,j) = \mathrm{LCP}(T_i, T_j)$ $\rightsquigarrow$ same thing, different name!
  $=$ string depth of *lowest common ancester (LCA)* of leaves $\boxed{i}$ and $\boxed{j}$



T = bananaban$

- in short: $\boxed{\mathrm{LCE}(i,j) = \mathrm{LCP}(T_i, T_j) = \mathrm{stringDepth}(\mathrm{LCA}(\boxed{i}, \boxed{j}))}$

Efficient LCA

How to find lowest common ancestors?

- Could walk up the tree to find LCA $\rightsquigarrow$ $\Theta(n)$ worst case 👎
- Could store all LCAs in big table $\rightsquigarrow$ $\Theta(n^2)$ space and preprocessing 👎

Efficient LCA

How to find lowest common ancestors?

- Could walk up the tree to find LCA $\rightsquigarrow$ $\Theta(n)$ worst case
- Could store all LCAs in big table $\rightsquigarrow$ $\Theta(n^2)$ space and preprocessing

Amazing result: Can compute data structure in $\Theta(n)$ time and space that finds any LCA is **constant(!) time**.

- a bit tricky to understand
- but a theoretical breakthrough
- and useful in practice

and suffix tree construction inside . . .

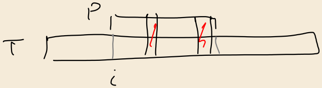
$\rightsquigarrow$ for now, use $O(1)$ LCA as black box.

$\rightsquigarrow$ After linear preprocessing (time & space), we can find LCEs in $O(1)$ time.

Application 5: Approximate matching

***k*-mismatch matching**:

- **Input:** text $T[0..n)$, pattern $P[0..m)$, $k \in [0..m)$ # mismatched characters
- **Output:** "Hamming distance $\leq k$"
  - smallest i so that $T[i..i+m)$ are P differ in at most k characters
  - or `NO_MATCH` if there is no such i



$\rightsquigarrow$ searching with typos

- Assume longest common extensions in $\boxed{T\$_1 P\$_2}$ can be found in $O(1)$
  - $\rightsquigarrow$ generalized suffix tree $\mathcal{T}$ has been built
  - $\rightsquigarrow$ string depths of all internal nodes have been computed
  - $\rightsquigarrow$ constant-time LCA data structure for $\mathcal{T}$ has been built

Clicker Question

What is the Hamming distance between `heart` and `beard`?

2

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Click on "Polls" tab

Clicker Question

Recap: Check all correct statements about suffix tree $\mathcal{T}$ of $T[0..n)$.

- **A** We require T to end with $\texttt{\$}$.
- **B** The size of $\mathcal{T}$ can be $\Omega(n^2)$ in the worst case.
- **C** $\mathcal{T}$ is a standard trie of all suffixes of $T\texttt{\$}$.
- **D** $\mathcal{T}$ is a compact trie of all suffixes of $T\texttt{\$}$.
- **E** The leaves of $\mathcal{T}$ store (a copy of) a suffix of $T\texttt{\$}$.
- **F** Naive construction of $\mathcal{T}$ takes $\Omega(n^2)$ (worst case).
- **G** $\mathcal{T}$ can be computed in $O(n)$ time (worst case).
- **H** $\mathcal{T}$ has n leaves.

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Click on "Polls" tab

Clicker Question

Recap: Check all correct statements about suffix tree $\mathcal{T}$ of $T[0..n)$.

A We require T to end with $\$$. ✓

B ~~The size of $\mathcal{T}$ can be $\Omega(n^2)$ in the worst case.~~

C ~~$\mathcal{T}$ is a standard trie of all suffixes of $T\$$.~~

D $\mathcal{T}$ is a compact trie of all suffixes of $T\$$. ✓

E ~~The leaves of $\mathcal{T}$ store (a copy of) a suffix of $T\$$.~~

F Naive construction of $\mathcal{T}$ takes $\Omega(n^2)$ (worst case). ✓

G $\mathcal{T}$ can be computed in $O(n)$ time (worst case). ✓

H ~~$\mathcal{T}$ has n leaves.~~

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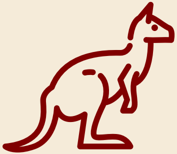
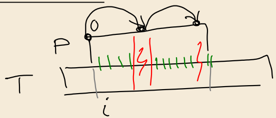
Click on "Polls" tab

Kangaroo Algorithm for approximate matching

easy in $O(n \cdot m)$

```
procedure kMismatch(T[0..n − 1], P[0..m − 1])
    // build LCE data structure
    for i := 0, . . . , n − m − 1 do
        mismatches := 0;  t := i;  p := 0
        while mismatches ≤ k ∧ p < m do
            ℓ := LCE(t, p) // jump over matching part
            t := t + ℓ + 1;  p := p + ℓ + 1
            mismatches := mismatches + 1
        if p == m then
            return i
```





- **Analysis:** $\Theta(n + m)$ preprocessing $+$ $O(n \cdot k)$ matching

$\rightsquigarrow$ very efficient for small k

- State of the art
  - $O\left(n \frac{k^2 \log k}{m}\right)$ possible with complicated algorithms
  - extensions for edit distance $\leq k$ possible

Application 6: Matching with wildcards

- Allow a wildcard character in pattern

  stands for arbitrary (single) character

```
   unit*        P
in␣unit5␣we␣will  T
```

- similar algorithm as for k-mismatch $\rightsquigarrow$ $O(n \cdot k + m)$ when P has k wildcards

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```

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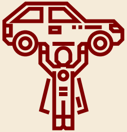
* * *

Many more applications, in particular for problems on biological sequences

20+ described in Gusfield, *Algorithms on strings, trees, and sequences* (1999)

Suffix trees – Discussion

- Suffix trees were a threshold invention



👍 linear time and space

👍 suddenly many questions efficiently solvable in theory

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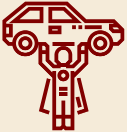

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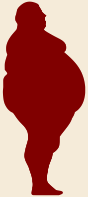
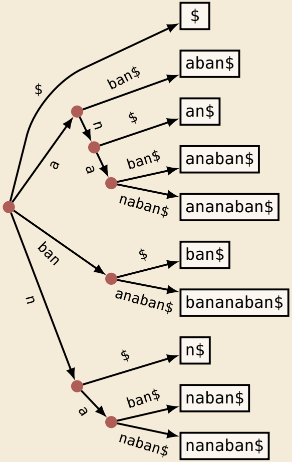


- construction of suffix trees:
  linear time, but significant overhead
- construction methods fairly complicated
- many pointers in tree incur large space overhead

Σ = ASCII $\sigma = 128$

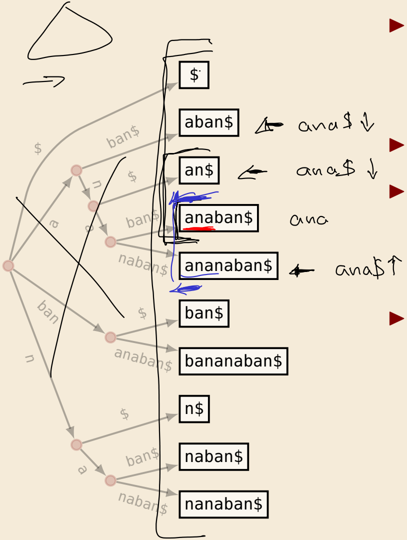
6.5 Suffix Arrays

Putting suffix trees on a diet



- **Observation:** order of leaves in suffix tree
  = suffixes lexicographically *sorted*

Putting suffix trees on a diet



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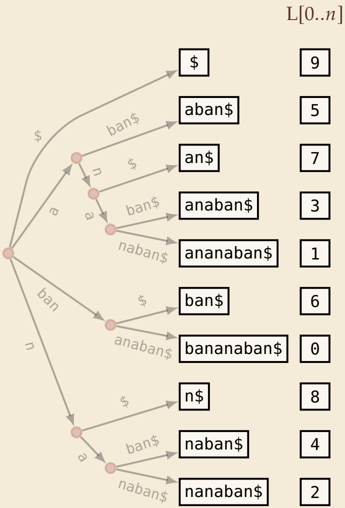
- Idea: only store list of leaves $L[0..n]$
- Enough to do efficient string matching!
  1. Use binary search for pattern P
  2. check if P is prefix of suffix after position found
- **Example:** P = ana $

all occurrences : 2 binary searches

ana#   # > a ∈ Σ

Putting suffix trees on a diet



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- Enough to do efficient string matching!
  1. Use binary search for pattern P
  2. check if P is prefix of suffix after position found
- **Example:** P = ana

⇝ $L[0..n]$ is called ***suffix array***:

$L[r]$ = (start index of) rth suffix in sorted order

- using L, can do string matching with
  $\le (\lg n + 2) \cdot m$ character comparisons

Clicker Question

Check all correct statements about suffix array $L[0..n]$ and suffix tree $\mathcal{T}$ of text $T[0..n)$.

A $L[0..n]$ lists the start indices of leaves of $\mathcal{T}$ in left-to-right order.

B $T\big[L[r]..n\big]$ is the path label in $\mathcal{T}$ to the leaf storing r.

C $T\big[L[r]..n\big]$ is the path label to the rth leaf in $\mathcal{T}$.

D $T_{L[r]}$ is the rth smallest suffix of T (lexicographic order).

E In terms of Θ-classes, $\mathcal{T}$ needs more space than L.

F L (and T) suffice to solve the text indexing problem.

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Click on “Polls” tab

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Suffix arrays – Construction

How to compute $L[0..n]$?

- from suffix tree
  - possible with traversal ...
  - but we are trying to avoid constructing suffix trees!

- sorting the suffixes of T using general purpose sorting method
  - trivial to code!
  - but: comparing two suffixes can take $\Theta(n)$ character comparisons
  - $\Theta(n^2 \log n)$ time in worst case

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  - trivial to code!
  - but: comparing two suffixes can take $\Theta(n)$ character comparisons
  - $\Theta(n^2 \log n)$ time in worst case
- We do better!

Fat-pivot radix quicksort – Example *(corrected version)*

```
she
sells
seashells
by
the
sea
shore
the
shells
she
sells
are
surely
seashells
```

Fat-pivot radix quicksort – Example

she
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seashells
by
the
sea
shore
the
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she
sells
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surely
seashells

Fat-pivot radix quicksort – Example

she	by
sells	are
seashells	she
by	sells
the	seashells
sea	sea
shore	shore
the	shells
shells	she
she	sells
sells	surely
are	seashells
surely	the
seashells	the

Fat-pivot radix quicksort – Example

she	by	are
sells	are	by
seashells	she	
by	sells	
the	seashells	
sea	sea	
shore	shore	
the	shells	
shells	she	
she	sells	
sells	surely	
are	seashells	
surely	the	
seashells	the	

Fat-pivot radix quicksort – Example

she	by	are
sells	are	by
seashells	she	sells
by	sells	seashells
the	seashells	sea
sea	sea	sells
shore	shore	seashells
the	shells	she
shells	she	shore
she	sells	shells
sells	surely	she
are	seashells	surely
surely	the	
seashells	the	

Fat-pivot radix quicksort – Example

she	by	are
sells	are	by
seashells	she	sells
by	sells	seashells
the	seashells	sea
sea	sea	sells
shore	shore	seashells
the	shells	she
shells	she	shore
she	sells	shells
sells	surely	she
are	seashells	surely
surely	the	the
seashells	the	the

Fat-pivot radix quicksort – Example

she	by	are	
sells	are	by	
seashells	she	sells	sells
by	sells	seashells	seashells
the	seashells	sea	sea
sea	sea	sells	sells
shore	shore	seashells	seashells
the	shells	she	
shells	she	shore	
she	sells	shells	
sells	surely	she	
are	seashells	surely	
surely	the	the	
seashells	the	the	

Fat-pivot radix quicksort – Example

she	by	are	
sells	are	by	
seashells	she	sells	sells
by	sells	seashells	seashells
the	seashells	sea	sea
sea	sea	sells	sells
shore	shore	seashells	seashells
the	shells	she	she$
shells	she	shore	shells
she	sells	shells	she$
sells	surely	she	shore
are	seashells	surely	
surely	the	the	
seashells	the	the	

Fat-pivot radix quicksort – Example

she	by	are	
sells	are	by	
seashells	she	sells	sells
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sells	surely	she	shore
are	seashells	surely	
surely	the	the	the
seashells	the	the	the

Fat-pivot radix quicksort – Example

she	by	are		
sells	are	by		
seashells	she	sells	sells	seashells
by	sells	seashells	seashells	sea
the	seashells	sea	sea	seashells
sea	sea	sells	sells	sells
shore	shore	seashells	seashells	sells
the	shells	she	she$	
shells	she	shore	shells	
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Fat-pivot radix quicksort – Example

she	by	are		
sells	are	by		
seashells	she	sells	sells	seashells
by	sells	seashells	seashells	sea
the	seashells	sea	sea	seashells
sea	sea	sells	sells	sells
shore	shore	seashells	seashells	sells
the	shells	she	she$	she
shells	she	shore	shells	she
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sea	sea	sells	sells	sells
shore	shore	seashells	seashells	sells
the	shells	she	she$	she
shells	she	shore	shells	she
she	sells	shells	she$	shells
sells	surely	she	shore	
are	seashells	surely		
surely	the	the	the	the
seashells	the	the	the	the

Fat-pivot radix quicksort – Example

she	by	are			
sells	are	by			
seashells	she	sells	sells	seashells	seashells
by	sells	seashells	seashells	sea	sea$
the	seashells	sea	sea	seashells	seashells
sea	sea	sells	sells	sells	
shore	shore	seashells	seashells	sells	
the	shells	she	she$	she	
shells	she	shore	shells	she	
she	sells	shells	she$	shells	
sells	surely	she	shore		
are	seashells	surely			
surely	the	the	the	the	
seashells	the	the	the	the	

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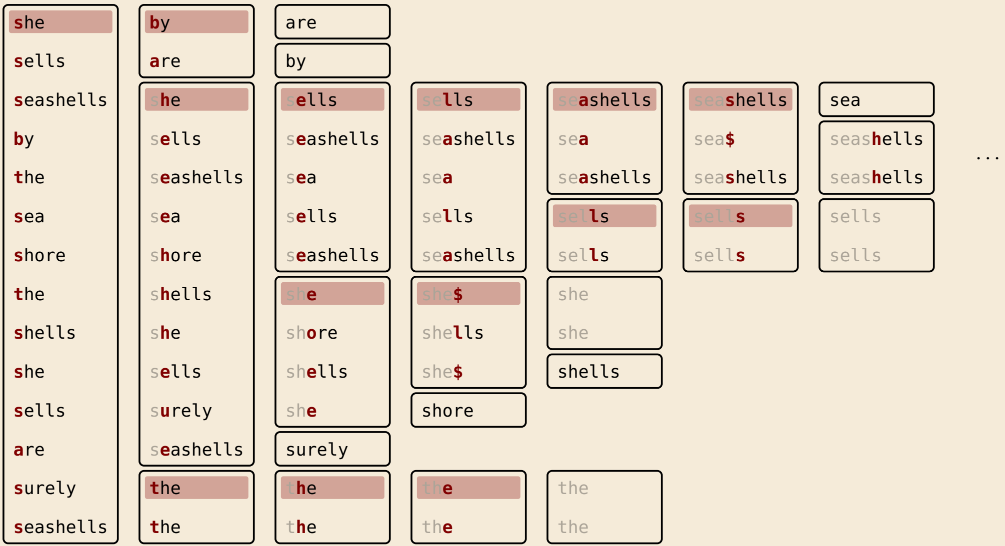

Fat-pivot radix quicksort

details in §5.1 of Sedgewick, Wayne *Algorithms 4th ed.* (2011), Pearson

- **partition** based on d**th character** only (initially $d = 0$)

$\rightsquigarrow$ 3 segments: smaller, equal, or larger than dth symbol of pivot

- recurse on smaller and large with same d, on equal with $d + 1$
  - $\rightsquigarrow$ never compare equal prefixes twice

Fat-pivot radix quicksort

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for random strings

$\rightsquigarrow$ can show: $\sim 2\ln(2) \cdot n \lg n \approx 1.39 n \lg n$ character comparisons on average

👍 simple to code

👍 efficient for sorting many lists of strings

random string

- fat-pivot radix quicksort finds suffix array in $O(n \log n)$ expected time

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choice of pivot
and random strings

- 👍 simple to code
- 👍 efficient for sorting many lists of strings

random string

- fat-pivot radix quicksort finds suffix array in $O(n \log n)$ expected time

*but we can do $O(n)$ time **worst case**!*

6.6 Linear-Time Suffix Sorting

Inverse suffix array: going left & right

- to understand the fastest algorithm, it is helpful to define the *inverse suffix array*:
  - $R[i] = r \iff L[r] = i$ L = *leaf array*
    $\iff$ there are r suffixes that come before T_i in sorted order
    $\iff T_i$ has (0-based) *rank* r $\rightsquigarrow$ call $R[0..n]$ the ***rank array***

text order

i	$R[i]$	T_i
0	6^{th}	bananaban$
1	4^{th}	ananaban$
2	9^{th}	nanaban$
3	3^{th}	anaban$
4	8^{th}	naban$
5	1^{th}	aban$
6	5^{th}	ban$
7	2^{th}	an$
8	7^{th}	n$
9	0^{th}	$

right

$R[0] = 6$

$L[8] = 4$

left

sort suffixes

r	$L[r]$	$T_{L[r]}$
0	9	$
1	5	aban$
2	7	an$
3	3	anaban$
4	1	ananaban$
5	6	ban$
6	0	bananaban$
7	8	n$
8	4	naban$
9	2	nanaban$

sorted

Linear-time suffix sorting

DC3 / Skew algorithm

not a multiple of 3

1. Compute rank array $R_{1,2}$ for suffixes T_i starting at $i \not\equiv 0 \pmod 3$ *recursively.* $\frac{2}{3}n$
2. Induce rank array R_3 for suffixes $T_0, T_3, T_6, T_9, \ldots$ from $R_{1,2}$.
3. Merge $R_{1,2}$ and R_0 using $R_{1,2}$.
   $\rightsquigarrow$ rank array R for entire input

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   $\leq \frac{n}{3}$
3. Merge $R_{1,2}$ and R_0 using $R_{1,2}$.
   $\rightsquigarrow$ rank array R for entire input

- We will show that steps 2. and 3. take $\Theta(n)$ time.

$\rightsquigarrow$ Total complexity is $n + \frac{2}{3}n + \left(\frac{2}{3}\right)^2 n + \left(\frac{2}{3}\right)^3 n + \cdots \leq n \cdot \sum_{i \geq 0} \left(\frac{2}{3}\right)^i = 3n = \Theta(n)$

top level   second level

Linear-time suffix sorting

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$\rightsquigarrow$ Total complexity is $n + \frac{2}{3}n + \left(\frac{2}{3}\right)^2 n + \left(\frac{2}{3}\right)^3 n + \cdots \le n \cdot \sum_{i \ge 0} \left(\frac{2}{3}\right)^i = 3n = \Theta(n)$

- **Note:** L can easily be computed from R in one pass, and vice versa.
  $\rightsquigarrow$ Can use whichever is more convenient.

DC3 / Skew algorithm – Step 2: Inducing ranks

- **Assume:** rank array $R_{1,2}$ known:
  - $R_{1,2}[i] \;=\; \begin{cases} \text{rank of } T_i \text{ among } T_1, T_2, T_4, T_5, T_7, T_8, \ldots & \text{for } i = 1, 2, 4, 5, 7, 8, \ldots \\ \text{undefined} & \text{for } i = 0, 3, 6, 9, \ldots \end{cases}$
- **Task:** sort the suffixes $T_0, T_3, T_6, T_9, \ldots$ in linear time (!)

DC3 / Skew algorithm – Step 2: Inducing ranks

- **Assume:** rank array $R_{1,2}$ known:
  - $R_{1,2}[i] = \begin{cases} \text{rank of } T_i \text{ among } T_1, T_2, T_4, T_5, T_7, T_8, \ldots & \text{for } i = 1,2,4,5,7,8,\ldots \\ \text{undefined} & \text{for } i = 0,3,6,9,\ldots \end{cases}$
- **Task:** sort the suffixes $T_0, T_3, T_6, T_9, \ldots$ in linear time (!)

$T_0 = a \,|\, T_1$

$T_3 = c \,|\, T_4$

- Suppose we want to compare T_0 and T_3.
  - Characterwise comparisons too expensive
  - but: after removing first character, we obtain T_1 and T_4
  - these two can be compared in *constant time* by comparing $R_{1,2}[1]$ and $R_{1,2}[4]$!

$\rightsquigarrow$ T_0 comes before T_3 in lexicographic order
iff pair $(T[0], R_{1,2}[1])$ comes before pair $(T[3], R_{1,2}[4])$ in lexicographic order

DC3 / Skew algorithm – Inducing ranks example

T = `hannahbansbananasman$$$` (append 3 \$ markers)

T_0	`hannahbansbananasman$$$`
T_3	`nahbansbananasman$$$`
T_6	`bansbananasman$$$`
T_9	`sbananasman$$$`
T_{12}	`nanasman$$$`
T_{15}	`asman$$$`
T_{18}	`an$$$`
T_{21}	`$$`

T_1	`annahbansbananasman$$$`	$R_{1,2}[22] = 0$	T_{22}	`$`
T_2	`nnahbansbananasman$$$`	$R_{1,2}[20] = 1$	T_{20}	`$$$`
T_4	`ahbansbananasman$$$`	$R_{1,2}[4] = 2$	T_4	`ahbansbananasman$$$`
T_5	`hbansbananasman$$$`	$R_{1,2}[11] = 3$	T_{11}	`ananasman$$$`
T_7	`ansbananasman$$$`	$R_{1,2}[13] = 4$	T_{13}	`anasman$$$`
T_8	`nsbananasman$$$`	$R_{1,2}[1] = 5$	T_1	`annahbansbananasman$$$`
T_{10}	`bananasman$$$`	$R_{1,2}[7] = 6$	T_7	`ansbananasman$$$`
T_{11}	`ananasman$$$`	$R_{1,2}[10] = 7$	T_{10}	`bananasman$$$`
T_{13}	`anasman$$$`	$R_{1,2}[5] = 8$	T_5	`hbansbananasman$$$`
T_{14}	`nasman$$$`	$R_{1,2}[17] = 9$	T_{17}	`man$$$`
T_{16}	`sman$$$`	$R_{1,2}[19] = 10$	T_{19}	`n$$$`
T_{17}	`man$$$`	$R_{1,2}[14] = 11$	T_{14}	`nasman$$$`
T_{19}	`n$$$`	$R_{1,2}[2] = 12$	T_2	`nnahbansbananasman$$$`
T_{20}	`$$$`	$R_{1,2}[8] = 13$	T_8	`nsbananasman$$$`
T_{22}	`$`	$R_{1,2}[16] = 14$	T_{16}	`sman$$$`

$R_{1,2}$ (known)

DC3 / Skew algorithm – Inducing ranks example

T = `hannahbansbananasman$$$` (append 3 $ markers)

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T_3	`nahbansbananasman$$$`
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T_9	`sbananasman$$$`
T_{12}	`nanasman$$$`
T_{15}	`asman$$$`
T_{18}	`an$$$`
T_{21}	`$$`

`sman$$$` = T_{16}

$R_{1,2}[16] = 14$

T_0	`h05`
T_3	`n02`
T_6	`b06`
T_9	`s07`
T_{12}	`n04`
T_{15}	`a14`
T_{18}	`a10`
T_{21}	`$00`

T_1	`annahbansbananasman$$$`	$R_{1,2}[22] = 0$	T_{22}	`$`
T_2	`nnahbansbananasman$$$`	$R_{1,2}[20] = 1$	T_{20}	`$$$`
T_4	`ahbansbananasman$$$`	$R_{1,2}[\ 4] = 2$	T_4	`ahbansbananasman$$$`
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T_{16}	`sman$$$`	$R_{1,2}[19] = 10$	T_{19}	`n$$$`
T_{17}	`man$$$`	$R_{1,2}[14] = 11$	T_{14}	`nasman$$$`
T_{19}	`n$$$`	$R_{1,2}[\ 2] = 12$	T_2	`nnahbansbananasman$$$`
T_{20}	`$$$`	$R_{1,2}[\ 8] = 13$	T_8	`nsbananasman$$$`
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`sman$$$` = T_{16}

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T_{22}	`$`	$R_{1,2}[16] = 14$	T_{16}	`sman$$$`

$R_{1,2}$ (known)

$R_{1,2}[16] = 14$

T_0	`h05`
T_3	`n02`
T_6	`b06`
T_9	`s07`
T_{12}	`n04`
T_{15}	`a14`
T_{18}	`a10`
T_{21}	`$00`

radix sort

T_{21}	`$00`	$\rightsquigarrow$	$R_0[21] = 0$
T_{18}	`a10`	$\rightsquigarrow$	$R_0[18] = 1$
T_{15}	`a14`	$\rightsquigarrow$	$R_0[15] = 2$
T_6	`b06`	$\rightsquigarrow$	$R_0[\ 6] = 3$
T_0	`h05`	$\rightsquigarrow$	$R_0[\ 0] = 4$
T_3	`n02`	$\rightsquigarrow$	$R_0[\ 3] = 5$
T_{12}	`n04`	$\rightsquigarrow$	$R_0[12] = 6$
T_9	`s07`	$\rightsquigarrow$	$R_0[\ 9] = 7$

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T_{18}	`an$$$`
T_{21}	`$$`

`sman$$$` $= T_{16}$

T_1	`annahbansbananasman$$$`	$R_{1,2}[22] = 0$	T_{22}	`$`
T_2	`nnahbansbananasman$$$`	$R_{1,2}[20] = 1$	T_{20}	`$$$`
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T_0	`h05`
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T_{18}	`a10`
T_{21}	`$00`

radix sort

T_{21}	`$00`	$\rightsquigarrow$	$R_0[21] = 0$
T_{18}	`a10`	$\rightsquigarrow$	$R_0[18] = 1$
T_{15}	`a14`	$\rightsquigarrow$	$R_0[15] = 2$
T_6	`b06`	$\rightsquigarrow$	$R_0[\ 6] = 3$
T_0	`h05`	$\rightsquigarrow$	$R_0[\ 0] = 4$
T_3	`n02`	$\rightsquigarrow$	$R_0[\ 3] = 5$
T_{12}	`n04`	$\rightsquigarrow$	$R_0[12] = 6$
T_9	`s07`	$\rightsquigarrow$	$R_0[\ 9] = 7$

R_0

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`sman$$$` $= T_{16}$

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T_{20}	`$$$`	$R_{1,2}[\ 8] = 13$	T_8	`nsbananasman$$$`
T_{22}	`$`	$R_{1,2}[16] = 14$	T_{16}	`sman$$$`

$R_{1,2}$ (known)

$R_{1,2}[16] = 14$

T_0	`h05`
T_3	`n02`
T_6	`b06`
T_9	`s07`
T_{12}	`n04`
T_{15}	`a14`
T_{18}	`a10`
T_{21}	`$00`

radix sort

T_{21}	`$00`	$\leadsto$	$R_0[21] = 0$
T_{18}	`a10`	$\leadsto$	$R_0[18] = 1$
T_{15}	`a14`	$\leadsto$	$R_0[15] = 2$
T_6	`b06`	$\leadsto$	$R_0[\ 6] = 3$
T_0	`h05`	$\leadsto$	$R_0[\ 0] = 4$
T_3	`n02`	$\leadsto$	$R_0[\ 3] = 5$
T_{12}	`n04`	$\leadsto$	$R_0[12] = 6$
T_9	`s07`	$\leadsto$	$R_0[\ 9] = 7$

R_0

- sorting of pairs doable in $O(n)$ time by 2 iterations of counting sort

$\leadsto$ Obtain R_0 in $O(n)$ time