7 Compression

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COMP526 (Fall 2022)
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Learning Outcomes

1. Understand the necessity for encodings and know *ASCII* and *UTF-8 character encodings*.
2. Understand (qualitatively) the *limits of compressibility*.
3. Know and understand the algorithms (encoding and decoding) for *Huffman codes*, *RLE*, *Elias codes*, *LZW*, *MTF*, and *BWT*, including their *properties* like running time complexity.
4. Select and *adapt* (slightly) a *compression* pipeline for specific type of data.

Unit 7: *Compression*

Outline

7 Compression

7.1 Context

Overview

- Unit 4–6: How to *work* with strings
  - finding substrings
  - finding approximate matches
  - finding repeated parts
  - ...
  - assumed character array (random access)!
- Unit 7–8: How to *store/transmit* strings
  - computer memory: must be binary
  - how to compress strings (save space)
  - how to robustly transmit over noisy channels $\rightsquigarrow$ Unit 8

Clicker Question

What compression methods do you know?

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Terminology

- **source text:** string $S \in \Sigma_S^\star$ to be stored / transmitted
  Σ_S is some alphabet
- **coded text:** encoded data $C \in \Sigma_C^\star$ that is actually stored / transmitted
  usually use $\Sigma_C = \{0, 1\}$
- **encoding:** algorithm mapping source texts to coded texts $S \rightsquigarrow C$
- **decoding:** algorithm mapping coded texts back to original source text $C \rightsquigarrow S$

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- **Lossy vs. Lossless**
  - **lossy compression** can only decode **approximately**;
    the exact source text S is lost
  - **lossless compression** always decodes S exactly

- For media files, lossy, logical compression is useful (e. g. JPEG, MPEG)

- We will concentrate on *lossless* compression algorithms.
  These techniques can be used for any application.

What is a good encoding scheme?

- Depending on the application, goals can be
  - efficiency of encoding/decoding
  - resilience to errors/noise in transmission $\longrightarrow$ Unit 8
  - security (encryption)
  - integrity (detect modifications made by third parties)
  - size

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  - integrity (detect modifications made by third parties)
  - size

- Focus in this unit: **size** of coded text

  Encoding schemes that (try to) minimize the size of coded texts perform *data compression*.

- We will measure the *compression ratio*: $\dfrac{|C| \cdot \lg |\Sigma_C|}{|S| \cdot \lg |\Sigma_S|} \overset{\Sigma_C=\{0,1\}}{=} \dfrac{|C|}{|S| \cdot \lg |\Sigma_S|}$

  < 1 means successful compression

  $= 1$ means no compression

  > 1 means "compression" made it bigger!? (yes, that happens ...)

Clicker Question

Do you know what uncomputable problems (halting problem, Post's correspondence problem, . . .) are?

A Sure, I could explain what it is.

B Heard that in a lecture, but don't quite remember

C No, never heard of it

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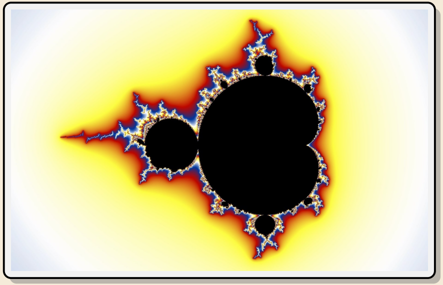
Limits of algorithmic compression

Is this image compressible?



Limits of algorithmic compression

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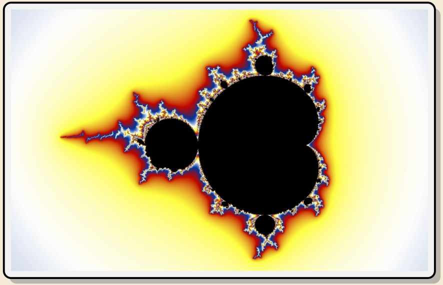
visualization of Mandelbrot set

- Clearly a complex shape!
- Will not compress (too) well using, say, PNG.
- but:
  - completely defined by mathematical formula
  - ⇝ **can be generated by a very small program!**



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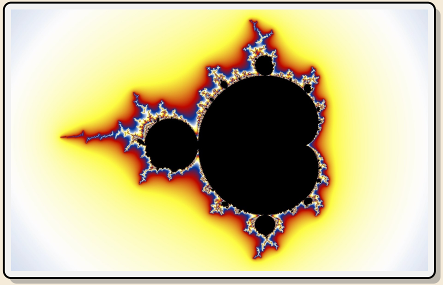

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⇝ ***Kolmogorov complexity***

- C = *any program* that outputs S

  self-extracting archives!
- Kolmogorov complexity = length of smallest such program

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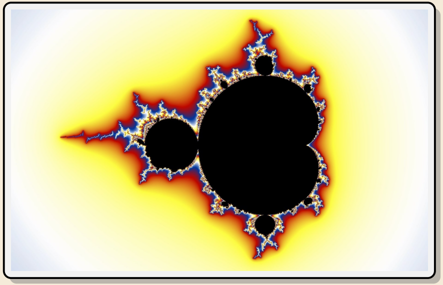

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⇝ ***Kolmogorov complexity***

- C = *any program* that outputs S

  self-extracting archives!

- Kolmogorov complexity = length of smallest such program
- **Problem:** finding smallest such program is *uncomputable*.
- ⇝ No optimal encoding algorithm is possible!
- ⇝ must be inventive to get efficient methods

What makes data compressible?

- Lossless compression methods mainly exploit two types of redundancies in source texts:

  1. **uneven character frequencies**
     some characters occur more often than others $\longrightarrow$ Part I
  2. **repetitive texts**
     different parts in the text are (almost) identical $\longrightarrow$ Part II

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  2. **repetitive texts**
     different parts in the text are (almost) identical $\longrightarrow$ Part II

There is no such thing as a free lunch!

Not *everything* is compressible ($\longrightarrow$ tutorials)
$\rightsquigarrow$ focus on versatile methods that often work

Part I

Exploiting character frequencies

7.2 Character Encodings

Character encodings

- Simplest form of encoding: Encode each source character individually

$\rightsquigarrow$ encoding function $E : \Sigma_S \to \Sigma_C^\star$

  - typically, $|\Sigma_S| \gg |\Sigma_C|$, so need several bits per character
  - for $c \in \Sigma_S$, we call $E(c)$ the *codeword* of c

- **fixed-length code:** $|E(c)|$ is the same for all $c \in \Sigma_C$
- **variable-length code:** not all codewords of same length

Fixed-length codes

- fixed-length codes are the simplest type of character encodings
- Example: **ASCII** (American Standard Code for Information Interchange, 1963)

```
0000000 NUL   0010000 DLE   0100000      0110000 0   1000000 @   1010000 P   1100000 ‘   1110000 p
0000001 SOH   0010001 DC1   0100001 !    0110001 1   1000001 A   1010001 Q   1100001 a   1110001 q
0000010 STX   0010010 DC2   0100010 "    0110010 2   1000010 B   1010010 R   1100010 b   1110010 r
0000011 ETX   0010011 DC3   0100011 #    0110011 3   1000011 C   1010011 S   1100011 c   1110011 s
0000100 EOT   0010100 DC4   0100100 $    0110100 4   1000100 D   1010100 T   1100100 d   1110100 t
0000101 ENQ   0010101 NAK   0100101 %    0110101 5   1000101 E   1010101 U   1100101 e   1110101 u
0000110 ACK   0010110 SYN   0100110 &    0110110 6   1000110 F   1010110 V   1100110 f   1110110 v
0000111 BEL   0010111 ETB   0100111 ’    0110111 7   1000111 G   1010111 W   1100111 g   1110111 w
0001000 BS    0011000 CAN   0101000 (    0111000 8   1001000 H   1011000 X   1101000 h   1111000 x
0001001 HT    0011001 EM    0101001 )    0111001 9   1001001 I   1011001 Y   1101001 i   1111001 y
0001010 LF    0011010 SUB   0101010 *    0111010 :   1001010 J   1011010 Z   1101010 j   1111010 z
0001011 VT    0011011 ESC   0101011 +    0111011 ;   1001011 K   1011011 [   1101011 k   1111011 {
0001100 FF    0011100 FS    0101100 ,    0111100 <   1001100 L   1011100 \   1101100 l   1111100 |
0001101 CR    0011101 GS    0101101 -    0111101 =   1001101 M   1011101 ]   1101101 m   1111101 }
0001110 SO    0011110 RS    0101110 .    0111110 >   1001110 N   1011110 ^   1101110 n   1111110 ~
0001111 SI    0011111 US    0101111 /    0111111 ?   1001111 O   1011111 _   1101111 o   1111111 DEL
```

- 7 bit per character
- just enough for English letters and a few symbols (plus control characters)

Fixed-length codes – Discussion

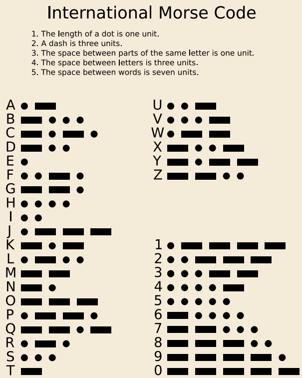
- 👍 Encoding & Decoding as fast as it gets

- 👎 Unless all characters equally likely, it wastes a lot of space
- 👎 inflexible (how to support adding a new character?)

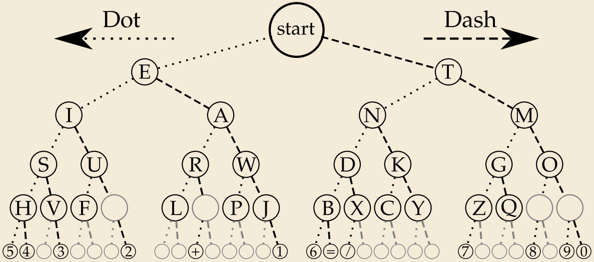
Variable-length codes

- to gain more flexibility, have to allow different lengths for codewords

- actually an old idea: **Morse Code**



https://commons.wikimedia.org/wiki/File:International_Morse_Code.svg



https://commons.wikimedia.org/wiki/File:Morse-code-tree.svg

Clicker Question

How many characters are there in the alphabet of the coded text in Morse Code, i. e., what is $|\Sigma_C|$?

A 1

B 2

C 3

D 4

E 26

F 36

G 256

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- A ~~1~~
- B ~~2~~
- C 3 ✓
- D ~~4~~
- E ~~26~~
- F ~~36~~
- G ~~256~~

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Variable-length codes – UTF-8

- Modern example: UTF-8 encoding of Unicode:
  default encoding for text-files, XML, HTML since 2009
  - Encodes any Unicode character (137 994 as of May 2019, and counting)
  - uses 1–4 bytes (codeword lengths: 8, 16, 24, or 32 bits)
  - Every ASCII character is encoded in 1 byte with leading bit `0`, followed by the 7 bits for ASCII
  - Non-ASCII charactters start with 1–4 `1`s indicating the total number of bytes, followed by a `0` and 3–5 bits.
    The remaining bytes each start with `10` followed by 6 bits.

Char. number range (hexadecimal)	UTF-8 octet sequence (binary)
`0000 0000 – 0000 007F`	`0xxxxxxx`
`0000 0080 – 0000 07FF`	`110xxxxx 10xxxxxx`
`0000 0800 – 0000 FFFF`	`1110xxxx 10xxxxxx 10xxxxxx`
`0001 0000 – 0010 FFFF`	`11110xxx 10xxxxxx 10xxxxxx 10xxxxxx`

For English text, most characters use only 8 bit,
but we can include any Unicode character, as well.

758561

Pitfall in variable-length codes

- Suppose we have the following code:

c	a	n	b	s
$E(c)$	0	10	110	100

- Happily encode text S = `banana` with the coded text C = `1100100100`

b a n a n a

Pitfall in variable-length codes

- Suppose we have the following code:

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ϟ C = 1100100100 decodes **both** to banana and to bass: $\underbrace{\texttt{110}}_{\texttt{b}}\underbrace{\texttt{0}}_{\texttt{a}}\underbrace{\texttt{100}}_{\texttt{s}}\underbrace{\texttt{100}}_{\texttt{s}}$

⇝ not a valid code . . . (cannot tolerate ambiguity)

but how should we have known?

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but how should we have known?

$E(\texttt{n}) = \texttt{10}$ is a (proper) **prefix** of $E(\texttt{s}) = \texttt{100}$

⇝ Leaves decoder wondering whether to stop after reading 10 or continue!

⇝ Require a ***prefix-free*** code: No codeword is a prefix of another.

prefix-free $\Longrightarrow$ instantaneously decodable $\Longrightarrow$ uniquely decodable

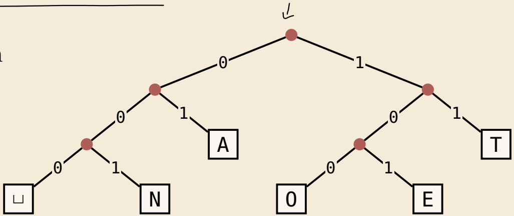
Code tries

- From now on only consider prefix-free codes E: $E(c)$ is not a prefix of $E(c')$ for any $c, c' \in \Sigma_S$.

- **Example:**

c	A	E	N	O	T	␣
$E(c)$	01	101	001	100	11	000

Any prefix-free code corresponds to a ***(code) trie*** (trie of codewords) with characters of Σ_S at **leaves**.



no need for end-of-string symbols $ here (already prefix-free!)

- Encode `AN␣ANT` 01001
- Decode `111000001010111`

T O ␣ E A T

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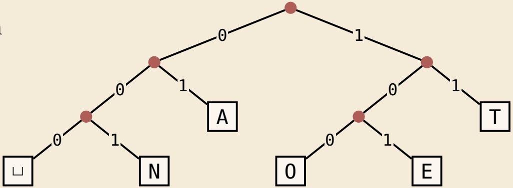

  - Encode `AN␣ANT` $\rightarrow$ `010010000100111`
  - Decode `111000001010111` $\rightarrow$ `TO␣EAT`

Who decodes the decoder?

- Depending on the application, we have to **store/transmit** the **used code**!
- We distinguish:
  - **fixed coding:** code agreed upon in advance, not transmitted (e. g., Morse, UTF-8)
  - **static coding:** code depends on message, but stays same for entire message; it must be transmitted (e. g., Huffman codes $\rightarrow$ next)
  - **adaptive coding:** code depends on message and changes during encoding; implicitly stored withing the message (e. g., LZW $\rightarrow$ below)

7.3 Huffman Codes

Character frequencies

- **Goal:** Find character encoding that produces short coded text
- Convention here: fix $\Sigma_C = \{0,1\}$ (binary codes), abbreviate $\Sigma = \Sigma_S$,
- **Observation:** Some letters occur more often than others.

Typical English prose:

e	12.70%	**d**	4.25%	**p**	1.93%
t	9.06%	**l**	4.03%	**b**	1.49%
a	8.17%	**c**	2.78%	**v**	0.98%
o	7.51%	**u**	2.76%	**k**	0.77%
i	6.97%	**m**	2.41%	**j**	0.15%
n	6.75%	**w**	2.36%	**x**	0.15%
s	6.33%	**f**	2.23%	**q**	0.10%
h	6.09%	**g**	2.02%	**z**	0.07%
r	5.99%	**y**	1.97%		

$\rightsquigarrow$ Want shorter codes for more frequent characters!

Huffman coding

e. g. frequencies / probabilities

- **Given:** Σ and weights $w : \Sigma \to \mathbb{R}_{\geq 0}$
- **Goal:** prefix-free code E (= code trie) for Σ that minimizes coded text length

  i. e., a code trie minimizing $\displaystyle\sum_{c \in \Sigma} w(c) \cdot |E(c)|$

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- If we use $w(c) = \#\text{occurrences of } c \text{ in } S$,
  this is the character encoding with smallest possible $|C|$

  $\rightsquigarrow$ best possible character-wise encoding

- Quite ambitious! *Is this efficiently possible?*

Huffman's algorithm

- Actually, yes! A greedy/myopic approach succeeds here.

Huffman's algorithm:

1. Find two characters $\texttt{a}$, $\texttt{b}$ with lowest weights.
   - We will encode them with the same prefix, plus one distinguishing bit, i. e., $E(\texttt{a}) = u\texttt{0}$ and $E(\texttt{b}) = u\texttt{1}$ for a bitstring $u \in \{0,1\}^\star$ (u to be determined)
2. (Conceptually) replace $\texttt{a}$ and $\texttt{b}$ by a single character "$\boxed{\texttt{ab}}$" with $w(\boxed{\texttt{ab}}) = w(\texttt{a}) + w(\texttt{b})$.
3. Recursively apply Huffman's algorithm on the smaller alphabet. This in particular determines $u = E(\boxed{\texttt{ab}})$.

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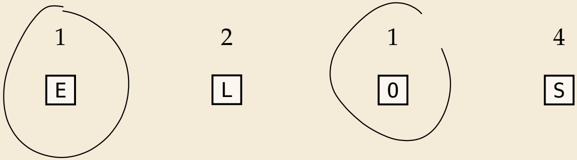
3. Recursively apply Huffman's algorithm on the smaller alphabet. This in particular determines $u = E(\boxed{\mathtt{ab}})$. $\sim \sigma$ iterations

- efficient implementation using a (min-oriented) ***priority queue***
  - start by inserting all characters with their weight as key
  - step 1 uses two `deleteMin` calls
  - step 2 inserts a new character with the sum of old weights as key

Huffman's algorithm – Example

- Example text: $S = \texttt{LOSSLESS}$ $\rightsquigarrow$ $\Sigma_S = \{\texttt{E}, \texttt{L}, \texttt{O}, \texttt{S}\}$
- Character frequencies: $\texttt{E}: 1$, $\texttt{L}: 2$, $\texttt{O}: 1$, $\texttt{S}: 4$



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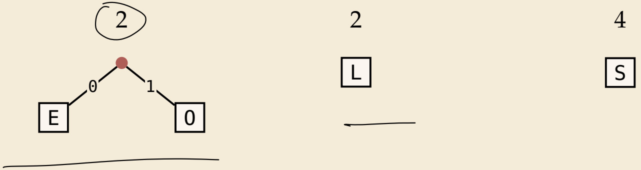

Huffman's algorithm – Example

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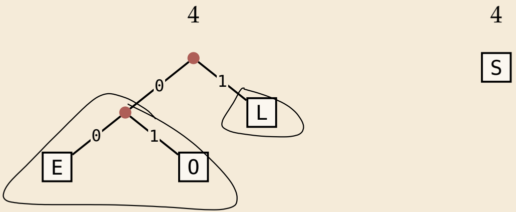

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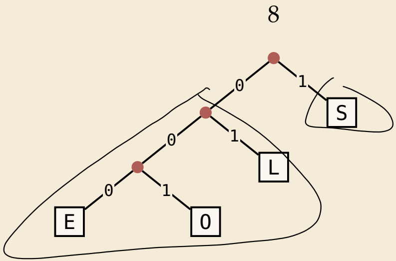

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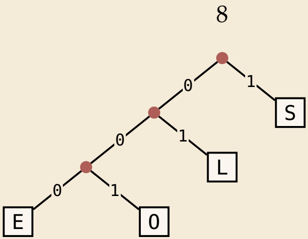

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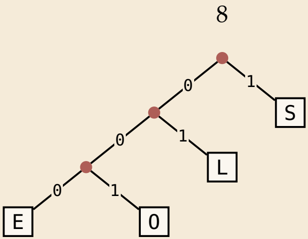

$\rightsquigarrow$ *Huffman tree* (code trie for Huffman code)

`LOSSLESS` $\rightarrow$ `01001110100011` compression ratio: $\frac{14}{8 \cdot \log 4} = \frac{14}{16} \approx 88\%$

Huffman tree – tie breaking

- The above procedure is ambiguous:
  - which characters to choose when weights are equal?
  - which subtree goes left, which goes right?

- For COMP 526: always use the following rule:

1. To break ties when selecting the two characters,
   first use the smallest letter according to the alphabetical order,
   or the tree containing the smallest alphabetical letter.
2. When combining two trees of different values,
   place the lower-valued tree on the left (corresponding to a `0`-bit).
3. When combining trees of equal value,
   place the one containing the smallest letter to the left.

Encoding with Huffman code

- The overall encoding procedure is as follows:
  - Pass 1: Count character frequencies in S
  - Construct Huffman code E (as above)
  - Store the Huffman code in C (details omitted)
  - Pass 2: Encode each character in S using E and append result to C
- Decoding works as follows:
  - Decode the Huffman code E from C. (details omitted)
  - Decode S character by character from C using the code trie.
- Note: Decoding is much simpler/faster!

Huffman code – Optimality

Theorem 7.1 (Optimality of Huffman's Algorithm)
Given Σ and $w : \Sigma \to \mathbb{R}_{\geq 0}$, Huffman's Algorithm computes codewords $E : \Sigma \to \{0,1\}^\star$ with minimal expected codeword length $\ell(E) = \sum_{c \in \Sigma} w(c) \cdot |E(c)|$ among all prefix-free codes for Σ. ◀

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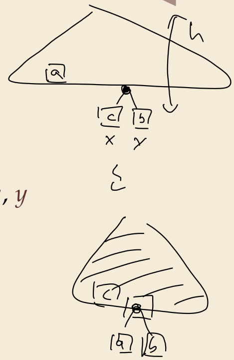
Proof sketch: by induction over $\sigma = |\Sigma| \geq 2$

- Given any optimal prefix-free code E^* (as its code trie).
- code trie $\rightsquigarrow$ $\exists$ two sibling leaves x, y at largest depth D
- swap characters in leaves to have two lowest-weight characters `a`, `b` in x, y (that can only make ℓ smaller, so still optimal)
- any optimal code for $\Sigma' = \Sigma \setminus \{\texttt{a}, \texttt{b}\} \cup \{\boxed{\texttt{ab}}\}$ yields optimal code for Σ by replacing leaf $\boxed{\texttt{ab}}$ by internal node with children `a` and `b`.

$\rightsquigarrow$ recursive call yields optimal code for Σ' by inductive hypothesis, so Huffman's algorithm finds optimal code for Σ.



7.4 Entropy

Entropy

Definition 7.2 (Entropy)

Given probabilities $p_1, \ldots, p_n$ (for outcomes $1, \ldots, n$ of a random variable), the *entropy* of the distribution is defined as

$$\mathcal{H}(p_1, \ldots, p_n) \;=\; -\sum_{i=1}^{n} p_i \lg p_i \;=\; \underbrace{\sum_{i=1}^{n} p_i \lg\left(\frac{1}{p_i}\right)}$$

$n = 6$ standard fair die $p_1 = p_2 = \cdots = p_6 = \frac{1}{6}$

$\mathcal{H}(p_1, \ldots, p_6) = 6 \cdot \frac{1}{6} \cdot \lg(6) = \lg(6)$

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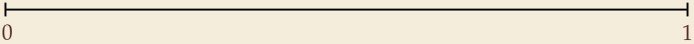
- entropy is a **measure** of **information** content of a distribution
  - *"20 Questions on* $[0,1)$*":* Land inside my interval by halving.



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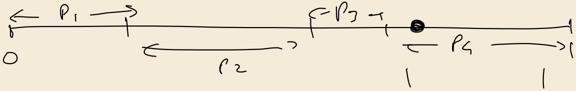

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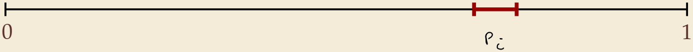

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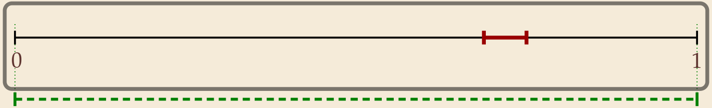

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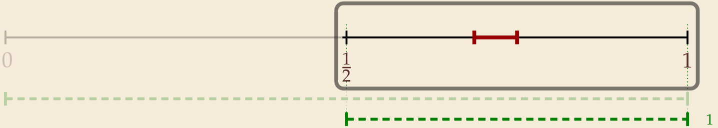

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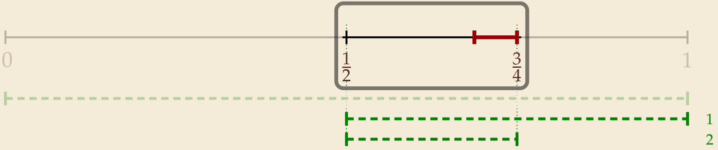

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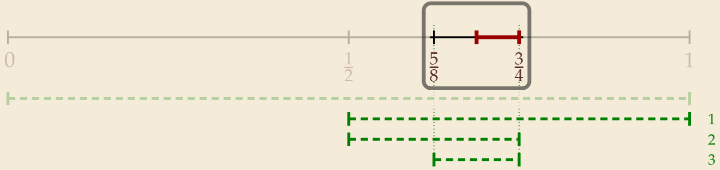

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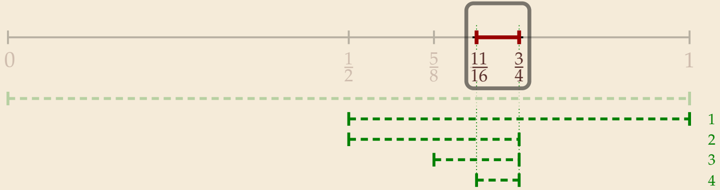

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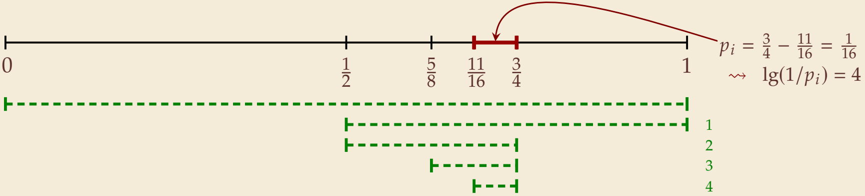

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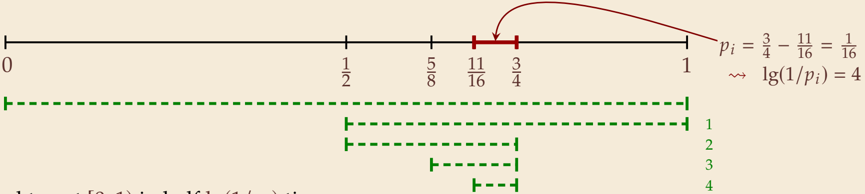

$\rightsquigarrow$ Need to cut $[0,1)$ in half $\lg(1/p_i)$ times

- more precisely: the expected number of bits (Yes/No questions) required to nail down the random value

Entropy and Huffman codes

- would ideally encode value i using $\lg(1/p_i)$ bits

  not always possible; cannot use codeword of 1.5 bits . . .

not as length of single codeword that is;
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Theorem 7.3 (Entropy bounds for Huffman codes)

For any $\Sigma = \{a_1, \ldots, a_\sigma\}$ and $w : \Sigma \to \mathbb{R}_{>0}$ and its Huffman code E, we have

$\boxed{\mathcal{H} \;\le\; \ell(E) \;\le\; \mathcal{H} + 1}$ where $\mathcal{H} = \mathcal{H}\left(\dfrac{w(a_1)}{W}, \ldots, \dfrac{w(a_\sigma)}{W}\right)$ and $W = w(a_1) + \cdots + w(a_\sigma)$. ◀

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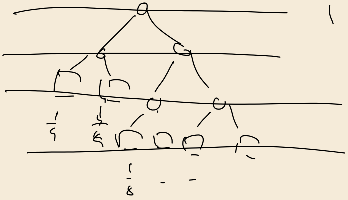
Proof sketch:



- $\ell(E) \ge \mathcal{H}$

  Any prefix-free code E induces weights $q_i = 2^{-|E(a_i)|}$.
  By *Kraft's Inequality*, we have $q_1 + \cdots + q_\sigma \le 1$.
  Hence we can apply *Gibb's Inequality* to get

$$\mathcal{H} \;=\; \sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{p_i}\right) \;\le\; \sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{q_i}\right) \;=\; \ell(E).$$

Entropy and Huffman codes [2]

Proof sketch (continued):

- $\ell(E) \le \mathcal{H} + 1$

  Set $q_i = 2^{-\lceil \lg(1/p_i) \rceil}$. We have $\displaystyle\sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{q_i}\right) = \sum_{i=1}^{\sigma} p_i \lceil \lg(1/p_i) \rceil \le \mathcal{H} + 1.$

We construct a code E' for Σ with $|E'(a_i)| \le \lg(1/q_i)$ as follows;
w.l.o.g. assume $q_1 \le q_2 \le \cdots \le q_\sigma$

not covered
in detail

- If $\sigma = 2$, E' uses a single bit each.
  Here, $q_i \le 1/2$, so $\lceil \lg(1/q_i) \rceil \ge 1 = |E'(a_i)|$ ✓
- If $\sigma \ge 3$, we merge a_1 and a_2 to $\boxed{a_1 a_2}$, assign it weight $2q_2$ and recurse.
  If $q_1 = q_2$, this is like Huffman; otherwise, q_1 is a unique smallest value and
  $q_2 + q_2 + \cdots + q_\sigma \le 1$.

  By the inductive hypothesis, we have $\left|E'(\boxed{a_1 a_2})\right| \le \lg\left(\frac{1}{2q_2}\right) = \lg\left(\frac{1}{q_2}\right) - 1.$

  By construction, $|E'(a_1)| = |E'(a_2)| = \left|E'(\boxed{a_1 a_2})\right| + 1$, so $|E'(a_1)| \le \lg(\frac{1}{q_1})$ and $|E'(a_2)| \le \lg(\frac{1}{q_2})$.

By optimality of E, we have $\displaystyle \ell(E) \le \ell(E') \le \sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{q_i}\right) \le \mathcal{H} + 1.$

Clicker Question

When does Huffman coding yield more efficient compression than a fixed-length character encoding?

A always

B when $\mathcal{H} \approx \lg(\sigma)$

C when $\mathcal{H} < \lg(\sigma)$

D when $\mathcal{H} < \lg(\sigma) - 1$

E when $\mathcal{H} \approx 1$

→ *sli.do/comp526*

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When does Huffman coding yield more efficient compression than a fixed-length character encoding?

- **A** always ✓
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Huffman coding – Discussion

- running time complexity: $O(\sigma \log \sigma)$ to construct code
  - build PQ + $\sigma \cdot$ (2 deleteMins and 1 insert)
  - can do $\Theta(\sigma)$ time when characters already sorted by weight
  - time for encoding text (after Huffman code done): $O(n + |C|)$
- many variations in use (tie-breaking rules, estimated frequencies, adaptive encoding, . . .)

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optimal prefix-free character encoding

very fast decoding

needs 2 passes over source text for encoding

- one-pass variants possible, but more complicated

have to store code alongside with coded text

Part II

Compressing repetitive texts

Beyond Character Encoding

- Many "natural" texts show repetitive redundancy

```
All work and no play makes Jack a dull boy.  All work and no play makes Jack a dull
boy.  All work and no play makes Jack a dull boy.  All work and no play makes Jack
a dull boy.  All work and no play makes Jack a dull boy.  All work and no play makes
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⇝ Have to encode whole ***phrases*** of S by a single codeword

7.5 Run-Length Encoding

Run-Length encoding

- simplest form of repetition: ***runs*** of characters

  same character repeated

```
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
0001011001000000111111000000000011111000
0011111111111000111111110000001111111000
0011110110100011100011110000011100000000
0011000000000000000000111000111000000000
0011000000000000000000011001110000000000
0011000000000000000000011001110000000000
0011011000000000000000111001100111110000
0011111111000000000000111001111111111000
0011101111100000000001110001111100111100
0000000001110000000111000001110000001110
0000000001110000000110000001110000001100
0000000000110000000110000000011000001110
0000000000110000001110000001110000001100
0000000001110001110000000000110000001110
0000000001100001110000000000111000011100
0011011111100011110111010000111111111000
0111111111100011111111111100001111110000
0001011000000010100110010000000100100000
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
```

- here: only consider $\Sigma_S = \{0, 1\}$ (work on a binary representation)
  - can be extended for larger alphabets

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0011000000000000000000011001110000000000
0011000000000000000000011001110000000000
0011011000000000000000111001100111110000
0011111111100000000000111001111111111000
0011101111100000000001110001111100111100
0000000001110000000011100000111000000001110
0000000001110000000011000000111000000001100
0000000000110000000110000000011000000001110
0000000000110000001110000000111000000001100
0000000001110001110000000000110000000001110
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use runs as phrases: $S = \underbrace{\texttt{00000}}\ \underbrace{\texttt{111}}\ \underbrace{\texttt{0000}}$

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0011011000000000000000111001100111110000
0011111111100000000000111001111111111000
0011101111100000000000111000111110011100
0000000001110000000011100001110000001110
0000000001110000000011000001110000001100
0000000000110000001100000000110000001110
0000000000110000011100000011100000001100
0000000001110001110000000000110000001110
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- Note: don't have to store bit for later runs since they must alternate.

- Example becomes: $0, 5, 3, 4$

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001101100000000000000111001100111110000
001111111110000000000111001111111111000
001110111110000000001110001111100111100
000000000111000000001110000011100000001110
000000000111000000001100000111000000001100
000000000011000000011000000001100000001110
000000000011000000111000000011100000001100
000000000111000111000000000011000000001110
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- **Question**: How to encode a run length k in binary? (k can be arbitrarily large!)

Clicker Question

How would you encode a string that can we arbitrarily long?

→ sli.do/comp526

Elias codes

- Need a *prefix-free encoding* for $\mathbb{N} = \{1, 2, 3, \dots,\}$
  - must allow arbitrarily large integers
  - must know when to stop reading

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  $7 \mapsto$ `00000001` $3 \mapsto$ `0001` $0 \mapsto$ `1` $30 \mapsto$ `0000000000000000000000000000001`

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- Refinement: ***Elias gamma code***
  - Store the **length** ℓ of the binary representation in **unary**
  - Followed by the binary digits themselves
  - little tricks:
    - always $\ell \geq 1$, so store $\ell - 1$ instead
    - binary representation always starts with `1` $\rightsquigarrow$ don't need terminating `1` in unary

  $\rightsquigarrow$ Elias gamma code = $\ell - 1$ zeros, followed by binary representation

Examples: $1 \mapsto$ `1`, $\quad 3 \mapsto$ `011`, $\quad 5 \mapsto$ `00101`, $\quad 30 \mapsto$ `000011110`

Clicker Question

Decode the **first** number in Elias gamma code (at the beginning) of the following bitstream:

$$\underline{000}\underbrace{1101}\texttt{11011100110}.$$

⇒ 4 sif $\quad 13 = 1101_2$

→ *sli.do/comp526*

Run-length encoding – Examples

- Encoding:

  S = `1111111001000000000000000000000011111111111`

  C = `1`

- Decoding:

  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:
  $S =$ 1111111001000000000000000000000011111111111
  $k = 7$
  $C =$ 100111

- Decoding:
  $C =$ 00001101001001010

  $S =$

Run-length encoding – Examples

- Encoding:

  $S =$ `1111111``00``10000000000000000000011111111111`

  $k = 2$

  $C =$ `100111``010`

- Decoding:

  $C =$ `00001101001001010`

  $S =$

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

  $k = 1$

  C = `1001110101`

- Decoding:

  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:

  $S =$ `1111111001``00000000000000000000``11111111111`

  $k = 20$

  $C =$ `1001110101000010100`

- Decoding:

  $C =$ `00001101001001010`

  $S =$

Run-length encoding – Examples

- Encoding:

  $S =$ `111111100100000000000000000000` `11111111111`

  $k = 11$

  $C =$ `1001110101000010100` `0001011`

- Decoding:

  $C =$ `00001101001001010`

  $S =$

Run-length encoding – Examples

- Encoding:
  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:
  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  C = `00001101001001010`

  S =

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Run-length encoding – Examples

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  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C =$ `0000`$\color{red}{\texttt{1101}}$`001001010`
  $b = 0$
  $\ell = 3 + 1$
  $k = 13$
  $S =$ $\color{red}{\texttt{0000000000000}}$

Run-length encoding – Examples

- Encoding:
  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C =$ `00001101001001010`
  $b = 1$
  $\ell = 2 + 1$
  $k =$
  $S =$ `0000000000000`

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000001111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  C = `00001101001001010`

  $b = 1$

  $\ell = 2 + 1$

  $k = 4$

  S = `00000000000001111`

Run-length encoding – Examples

- Encoding:

  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  $C =$ `00001101001001010`

  $b = 0$

  $\ell = 0 + 1$

  $k =$

  $S =$ `00000000000001111`

Run-length encoding – Examples

- Encoding:

  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  $C =$ `00001101001001010`

  $b = 0$

  $\ell = 0 + 1$

  $k = 1$

  $S =$ `000000000000011110`

Run-length encoding – Examples

- Encoding:

  S = 11111110010000000000000000000011111111111

  C = 10011101010000101000001011

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  C = 00001101001001010

  $b = 1$

  $\ell = 1 + 1$

  $k =$

  S = 000000000000011110

Run-length encoding – Examples

- Encoding:

  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  $C =$ `00001101001001010`

  $b = 1$

  $\ell = 1 + 1$

  $k = 2$

  $S =$ `00000000000001111011`

Run-length encoding – Discussion

- extensions to larger alphabets possible (must store next character then)
- used in some image formats (e. g. TIFF)

Run-length encoding – Discussion

- extensions to larger alphabets possible (must store next character then)
- used in some image formats (e. g. TIFF)

fairly simple and fast

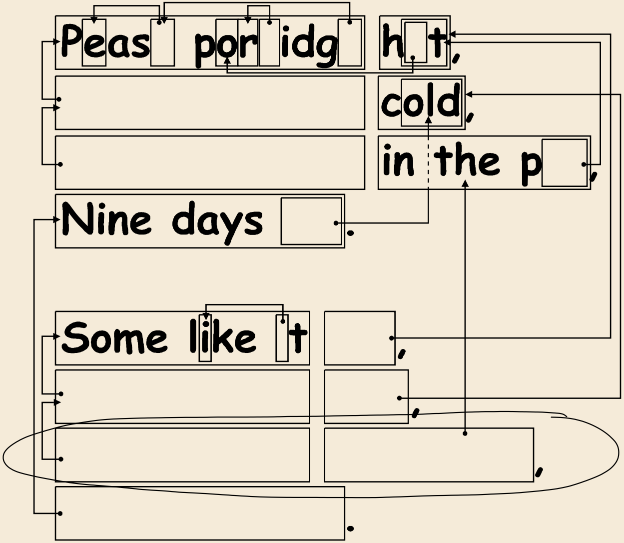
can compress n bits to $\Theta(\log n)$!
for extreme case of constant number of runs

negligible compression for many common types of data

- No compression until run lengths $k \geq 6$
- **expansion** for run length $k = 2$ or 6

7.6 Lempel-Ziv-Welch

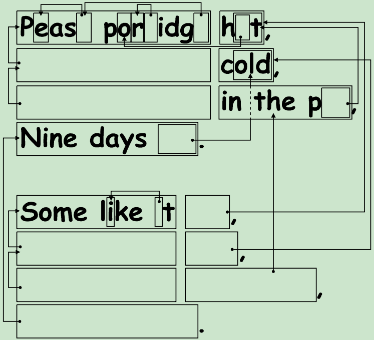
Warmup



https://classic.csunplugged.org/text-compression/

https://www.flickr.com/photos/quintanaroo/2742726346

Clicker Question



What is the **second-to-last line** of the poem to the left?

→ sli.do/comp526

Lempel-Ziv Compression

- Huffman and RLE mostly take advantage of frequent or repeated *single characters*.
- **Observation**: Certain *substrings* are much more frequent than others.
  - in English text: the, be, to, of, and, a, in, that, have, I
  - in HTML: "`<a href`", "`<img src`", "`<br/>`"

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  - in English text: the, be, to, of, and, a, in, that, have, I
  - in HTML: "`<a href`", "`<img src`", "`<br/>`"
- **Lempel-Ziv** stands for family of *adaptive* compression algorithms.
  - **Idea:** store repeated parts by reference!

  $\rightsquigarrow$ each codeword refers to
    - either a single character in Σ_S,
    - or a *substring* of S (that both encoder and decoder have already seen).

Lempel-Ziv Compression

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  - in English text: the, be, to, of, and, a, in, that, have, I
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- **Lempel-Ziv** stands for family of *adaptive* compression algorithms.
  - **Idea:** store repeated parts by reference!

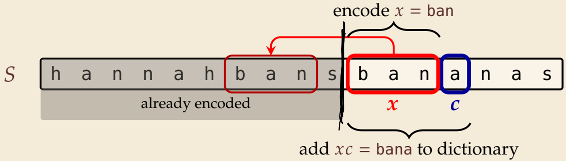
  ⇝ each codeword refers to
    - either a single character in Σ_S,
    - or a *substring* of S (that both encoder and decoder have already seen).
  - Variants of Lempel-Ziv compression
    - "LZ77" Original version ("sliding window")
      Derivatives: LZSS, LZFG, LZRW, LZP, DEFLATE, ...
      DEFLATE used in `(pk)zip`, `gzip`, `PNG`
    - "LZ78" Second (slightly improved) version
      Derivatives: LZW, LZMW, LZAP, LZY, ...
      LZW used in `compress`, `GIF`

Lempel-Ziv-Welch

- here: ***Lempel-Ziv-Welch (LZW)*** (arguably the "cleanest" variant of Lempel-Ziv)
- ***variable-to-fixed* encoding**
  - all codewords have k bits (typical: $k = 12$) $\rightsquigarrow$ fixed-length
  - but they represent a variable portion of the source text!

Lempel-Ziv-Welch

- here: ***Lempel-Ziv-Welch (LZW)*** (arguably the "cleanest" variant of Lempel-Ziv)
- ***variable-to-fixed* encoding**
  - all codewords have k bits (typical: $k = 12$) $\rightsquigarrow$ fixed-length
  - but they represent a variable portion of the source text!
- maintain a **dictionary** D with 2^k entries $\rightsquigarrow$ codewords = indices in dictionary
  - initially, first $|\Sigma_S|$ entries encode single characters (rest is empty)
  - **add** a new entry to D **after each step**:
  - **Encoding:** after encoding a substring x of S,
    add xc to D where c is the character that follows x in S.



$\rightsquigarrow$ new codeword in D

- D actually stores codewords for x and c, not the expanded string

LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

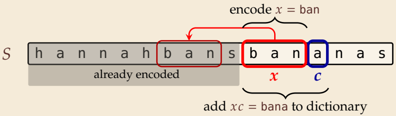
Σ_S = ASCII character set (0–127)

$C =$

$D =$

Code	String
. . .	
32	␣
33	!
. . .	
79	O
. . .	
82	R
. . .	
85	U
. . .	
89	Y
. . .	

Code	String
128	
129	
130	
131	
132	
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `Y0!␣Y0U!␣Y0UR␣Y0Y0!`

Σ_S = ASCII character set (0–127)

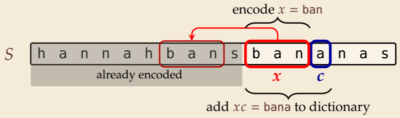
Y

$C =$ 89

$D =$

Code	String
...	
32	␣
33	!
...	
79	0
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	
129	
130	
131	
132	
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

Σ_S = ASCII character set (0–127)

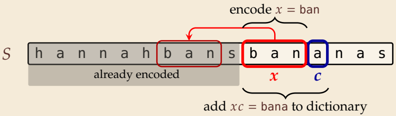
Y

C = 89

D =

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	
130	
131	
132	
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

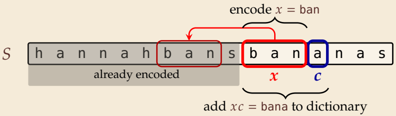
Σ_S = ASCII character set (0–127)

	Y	O
$C =$	89	79

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	
130	
131	
132	
133	
134	
135	
136	
137	
138	
139	



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Input: `YO!␣YOU!␣YOUR␣YOYO!` Σ_S = ASCII character set (0–127)

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$C =$	89	79

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	
131	
132	
133	
134	
135	
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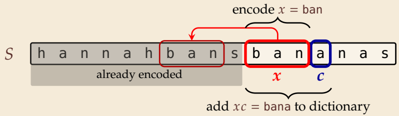
137	
138	
139	



LZW encoding – Example

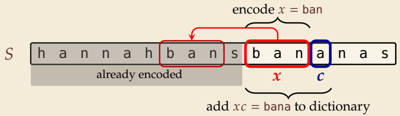
Input: `YO!␣YOU!␣YOUR␣YOYO!` $\qquad \Sigma_S$ = ASCII character set (0–127)

	Y	O	!
$C =$	89	79	33

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	
131	
132	
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

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Σ_S = ASCII character set (0–127)

	Y	O	!
$C =$	89	79	33

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
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135	
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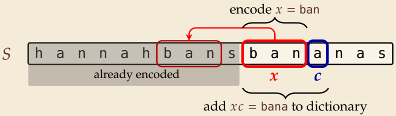
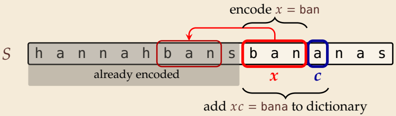
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!` Σ_S = ASCII character set (0–127)

	Y	O	!	␣
C =	89	79	33	32



D =

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	
132	
133	
134	
135	
136	
137	
138	
139	

LZW encoding – Example

Input: Y0!␣Y0U!␣Y0UR␣Y0Y0!

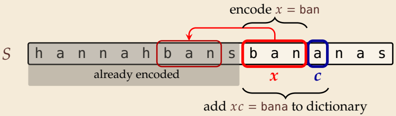
Σ_S = ASCII character set (0–127)

	Y	0	!	␣
$C =$	89	79	33	32

$D =$

Code	String
...	
32	␣
33	!
...	
79	0
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	Y0
129	0!
130	!␣
131	␣Y
132	
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: YO!␣YOU!␣YOUR␣YOYO!

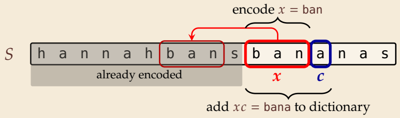
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO
$C =$	89	79	33	32	128

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

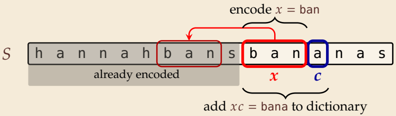
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO
$C =$	89	79	33	32	128

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

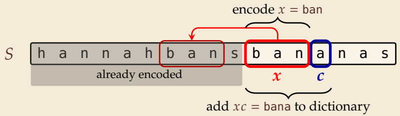
Input: `YO!␣YOU!␣YOUR␣YOYO!` Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U
$C =$	89	79	33	32	128	85

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	
134	
135	
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U
$C =$	89	79	33	32	128	85

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	
135	
136	
137	
138	
139	

encode x = ban

S `h a n n a h b a n s b a n a n a s`

already encoded

x c

add xc = bana to dictionary

LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!` Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣
$C =$	89	79	33	32	128	85	130

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
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134	
135	
136	
137	
138	
139	

encode x = ban

S h a n n a h b a n s b a n a n a s

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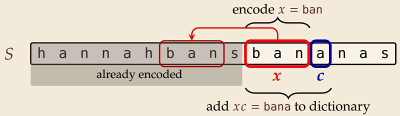
add xc = bana to dictionary

LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣
$C =$	89	79	33	32	128	85	130



$D =$

Code	String
. . .	
32	␣
33	!
. . .	
79	O
. . .	
82	R
. . .	
85	U
. . .	
89	Y
. . .	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	
136	
137	
138	
139	

LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!` Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU
$C =$	89	79	33	32	128	85	130	132

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	
136	
137	
138	
139	

encode x = ban

S h a n n a h b a n s b a n a n a s

already encoded

x c

add xc = bana to dictionary

LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

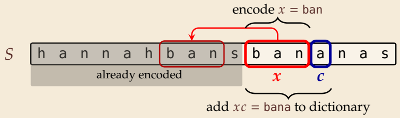
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU
$C =$	89	79	33	32	128	85	130	132

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	
137	
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

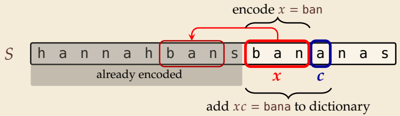
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R
$C =$	89	79	33	32	128	85	130	132	82

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	
137	
138	
139	



LZW encoding – Example

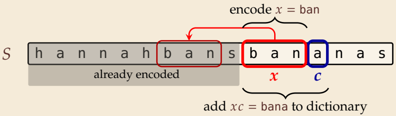
Input: `YO!␣YOU!␣YOUR␣YOYO!` $\qquad \Sigma_S$ = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R
$C =$	89	79	33	32	128	85	130	132	82

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	
138	
139	



LZW encoding – Example

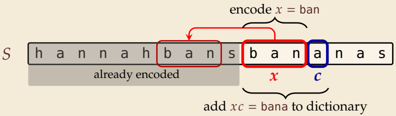
Input: `YO!␣YOU!␣YOUR␣YOYO!` $\quad \Sigma_S$ = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y
$C =$	89	79	33	32	128	85	130	132	82	131

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	
138	
139	



LZW encoding – Example

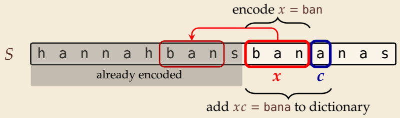
Input: `YO!␣YOU!␣YOUR␣YOYO!` $\quad \Sigma_S$ = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y
$C =$	89	79	33	32	128	85	130	132	82	131

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	
139	



LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!` $\qquad \Sigma_S$ = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y	O
$C =$	89	79	33	32	128	85	130	132	82	131	79

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	
139	

encode x = ban

S h a n n a h b a n s b a n a n a s

already encoded $\qquad x \qquad c$

add xc = bana to dictionary

LZW encoding – Example

Input: `YO!␣YOU!␣YOUR␣YOYO!`

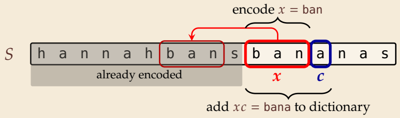
Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y	O
C =	89	79	33	32	128	85	130	132	82	131	79

D =

Code	String
. . .	
32	␣
33	!
. . .	
79	O
. . .	
82	R
. . .	
85	U
. . .	
89	Y
. . .	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	OY
139	



LZW encoding – Example

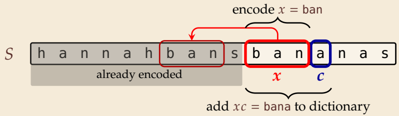
Input: `YO!␣YOU!␣YOUR␣YOYO!` $\qquad \Sigma_S$ = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y	O	YO
$C =$	89	79	33	32	128	85	130	132	82	131	79	128

$D =$

Code	String
. . .	
32	␣
33	!
. . .	
79	O
. . .	
82	R
. . .	
85	U
. . .	
89	Y
. . .	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	OY
139	



LZW encoding – Example

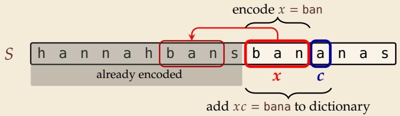
Input: `YO!␣YOU!␣YOUR␣YOYO!` $\quad \Sigma_S$ = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y	O	YO
$C =$	89	79	33	32	128	85	130	132	82	131	79	128

$D =$

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	OY
139	YO!



LZW encoding – Example

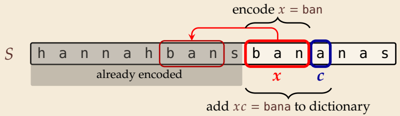
Input: `YO!␣YOU!␣YOUR␣YOYO!` Σ_S = ASCII character set (0–127)

	Y	O	!	␣	YO	U	!␣	YOU	R	␣Y	O	YO	!
C =	89	79	33	32	128	85	130	132	82	131	79	128	33

D =

Code	String
...	
32	␣
33	!
...	
79	O
...	
82	R
...	
85	U
...	
89	Y
...	

Code	String
128	YO
129	O!
130	!␣
131	␣Y
132	YOU
133	U!
134	!␣Y
135	YOUR
136	R␣
137	␣YO
138	OY
139	YO!



LZW encoding – Code

```
procedure LZWencode(S[0..n))
    x := ε // previous phrase, initially empty
    C := ε // output, initially empty
    D := dictionary, initialized with codes for c ∈ Σ_S // stored as trie
    k := |Σ_S| // next free codeword
    for i := 0, . . . , n − 1 do
        c := S[i]
        if D.containsKey(xc) then
            x := xc
        else
            C := C · D.get(x) // append codeword for x
            D.put(xc, k) // add xc to D, assigning next free codeword
            k := k + 1;  x := c
    end for
    C := C · D.get(x)
    return C
```