7 Compression

16 March 2020

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Outline

7 Compression

7.1 Context

Overview

- Unit 4–6: How to *work* with strings
  - finding substrings
  - finding approximate matches
  - finding repeated parts
  - ...

- Unit 7–8: How to *store* strings
  - computer memory: must be binary
  - how to compress strings (save space)
  - how to robustly transmit over noisy channels ⇝ Unit 8

Terminology

- **source text:** string $S \in \Sigma_S^\star$ to be stored / transmitted
  Σ_S is some alphabet
- **coded text:** encoded data $C \in \Sigma_C^\star$ that is actually stored / transmitted
  usually use $\Sigma_C = \{0, 1\}$
- **encoding:** algorithm mapping source texts to coded texts $S \to C$
- **decoding:** algorithm mapping coded texts back to original source text $C \to S$

What is a good encoding scheme?

- Depending on the application, goals can be
  - efficiency of encoding/decoding
  - resilience to errors/noise in transmission
  - security (encryption)
  - integrity (detect modifications made by third parties)
  - size

not here

- Focus in this unit: $\boxed{\textbf{size} \text{ of coded text}}$ $|C|$

  Encoding schemes that (try to) minimize the size of coded texts perform *data compression*.

- We will measure the *compression ratio*: $\dfrac{|C| \cdot \lg|\Sigma_C|}{|S| \cdot \lg|\Sigma_S|} \overset{\Sigma_C=\{0,1\}}{=} \dfrac{|C|}{|S| \cdot \lg|\Sigma_S|}$

  < 1 means successful compression
  $= 1$ means no compression
  > 1 means "compression" made it bigger!? (yes, that happens ...)

Types of Data Compression

- **Logical vs. Physical**
  - **Logical Compression** uses meaning of data
    ⇝ only applies to a certain domain, e. g., sound recordings
  - **Physical Compression** only knows the (physical) **bits** in the data, not the meaning behind them

- **Lossy vs. Lossless**
  - **lossy compression** can only decode **approximately**;
    the exact source text S is lost
  - **lossless compression** always decodes S exactly

- For media files, lossy, logical compression is useful (e. g. JPEG, MPEG)

- We will concentrate on *physical*, *lossless* compression algorithms.
  These techniques can be used for any application.

What makes data compressible?

- Physical, lossless compression methods mainly exploit two types of redundancies in source texts:

  1. **uneven character frequencies**
     some characters occur more often than others $\longrightarrow$ Part I
  2. **repetitive texts**
     different parts in the text are (almost) identical $\longrightarrow$ Part II

What makes data compressible?

- Physical, lossless compression methods mainly exploit two types of redundancies in source texts:

  1. **uneven character frequencies**
     some characters occur more often than others $\longrightarrow$ Part I

  2. **repetitive texts**
     different parts in the text are (almost) identical $\longrightarrow$ Part II

There is no such thing as a free lunch!

Not *everything* is compressible ($\longrightarrow$ tutorials)
$\leadsto$ focus on versatile methods that often work

Part I

Exploiting character frequencies

7.2 Character Encodings

Character encodings

- Simplest form of encoding: Encode each source character individually

$\rightsquigarrow$ encoding function $E : \Sigma_S \to \Sigma_C^\star$

  - typically, $|\Sigma_S| \gg |\Sigma_C|$, so need several bits per character
  - for $c \in \Sigma_S$, we call $E(c)$ the *codeword* of c

- fixed-length code: $|E(c)|$ is the same for all $c \in \Sigma_{\cancel{C}S}$
- variable-length code: not all codewords of same length

Fixed-length codes

- fixed-length codes are the simplest type of character encodings
- Example: **ASCII** (American Standard Code for Information Interchange, 1963)

0000000 NUL	0010000 DLE	0100000	0110000 0	1000000 @	1010000 P	1100000 ‘	1110000 p
0000001 SOH	0010001 DC1	0100001 !	0110001 1	1000001 A	1010001 Q	1100001 a	1110001 q
0000010 STX	0010010 DC2	0100010 "	0110010 2	1000010 B	1010010 R	1100010 b	1110010 r
0000011 ETX	0010011 DC3	0100011 #	0110011 3	1000011 C	1010011 S	1100011 c	1110011 s
0000100 EOT	0010100 DC4	0100100 $	0110100 4	1000100 D	1010100 T	1100100 d	1110100 t
0000101 ENQ	0010101 NAK	0100101 %	0110101 5	1000101 E	1010101 U	1100101 e	1110101 u
0000110 ACK	0010110 SYN	0100110 &	0110110 6	1000110 F	1010110 V	1100110 f	1110110 v
0000111 BEL	0010111 ETB	0100111 ’	0110111 7	1000111 G	1010111 W	1100111 g	1110111 w
0001000 BS	0011000 CAN	0101000 (	0111000 8	1001000 H	1011000 X	1101000 h	1111000 x
0001001 HT	0011001 EM	0101001)	0111001 9	1001001 I	1011001 Y	1101001 i	1111001 y
0001010 LF	0011010 SUB	0101010 *	0111010 :	1001010 J	1011010 Z	1101010 j	1111010 z
0001011 VT	0011011 ESC	0101011 +	0111011 ;	1001011 K	1011011 [	1101011 k	1111011 {
0001100 FF	0011100 FS	0101100 ,	0111100 <	1001100 L	1011100 \	1101100 l	1111100 |
0001101 CR	0011101 GS	0101101 -	0111101 =	1001101 M	1011101]	1101101 m	1111101 }
0001110 SO	0011110 RS	0101110 .	0111110 >	1001110 N	1011110 ^	1101110 n	1111110 ~
0001111 SI	0011111 US	0101111 /	0111111 ?	1001111 O	1011111 _	1101111 o	1111111 DEL

- 7 bit per character
- just enough for English letters and a few symbols (plus control characters)

Fixed-length codes – Discussion

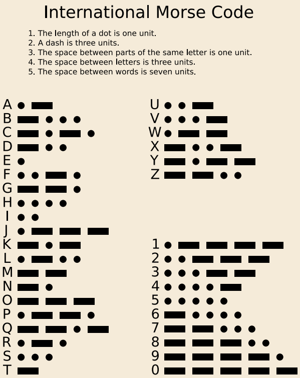
- 👍 Encoding & Decoding as fast as it gets

- 👎 Unless all characters equally likely, it wastes a lot of space
- 👎 inflexible (how to support adding a new character?)

Variable-length codes

- to gain more flexibility, have to allow different lengths for codewords
- actually an old idea: **Morse Code**

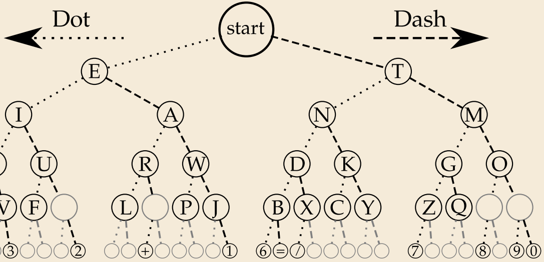
encoding



https://commons.wikimedia.org/wiki/File:International_Morse_Code.svg

trie

decoding



https://commons.wikimedia.org/wiki/File:Morse-code-tree.svg

Variable-length codes – UTF-8

- Modern example: UTF-8 encoding of Unicode:
  default encoding for text-files, XML, HTML since 2009
  - Encodes any Unicode character (137 994 as of May 2019, and counting)
  - uses 1–4 bytes (codeword lengths: 8, 16, 24, or 32 bits)
  - Every ASCII character is encoded in 1 byte with leading bit `0`, followed by the 7 bits for ASCII
  - Non-ASCII charactters start with 1–4 `1`s indicating the total number of bytes, followed by a `0` and 3–5 bits.
    The remaining bytes each start with `10` followed by 6 bits.

Char. number range (hexadecimal)	UTF-8 octet sequence (binary)
`0000 0000-0000 007F`	`0xxxxxxx`
`0000 0080-0000 07FF`	`110xxxxx 10xxxxxx`
`0000 0800-0000 FFFF`	`1110xxxx 10xxxxxx 10xxxxxx`
`0001 0000-0010 FFFF`	`11110xxx 10xxxxxx 10xxxxxx 10xxxxxx`

👍 For English text, most characters use only 8 bit,
but we can include any Unicode character, as well.

Pitfall in variable-length codes

- Suppose we have the following code:

c	a	n	b	s
$E(c)$	0	10	110	100

- Happily encode text S = banana with the coded text C = 1100100100

  b a n a n a

Pitfall in variable-length codes

- Suppose we have the following code:

c	a	n	b	s
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- Happily encode text S = banana with the coded text $C = \underbrace{\texttt{110}}_{\texttt{b}}\underbrace{\texttt{0}}_{\texttt{a}}\underbrace{\texttt{10}}_{\texttt{n}}\underbrace{\texttt{0}}_{\texttt{a}}\underbrace{\texttt{10}}_{\texttt{n}}\underbrace{\texttt{0}}_{\texttt{a}}$

ϟ C = 1100100100 decodes **both** to banana and to bass: $\underbrace{\texttt{110}}_{\texttt{b}}\underbrace{\texttt{0}}_{\texttt{a}}\underbrace{\texttt{100}}_{\texttt{s}}\underbrace{\texttt{100}}_{\texttt{s}}$

⇝ not a valid code . . . (cannot tolerate ambiguity)

but how should we have known?

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⇝ not a valid code . . . (cannot tolerate ambiguity)

but how should we have known?

$E(\texttt{n}) = \texttt{10}$ is a (proper) **prefix** of $E(\texttt{s}) = \texttt{100}$ 101

⇝ Leaves decoding wondering whether to stop after reading 10 or continue

⇝ Require a ***prefix-free*** code: No codeword is a prefix of another.

prefix-free $\implies$ instantaneously decodable

01001

codeword

Code tries

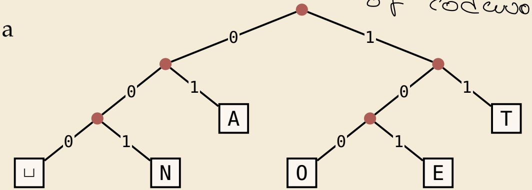
- From now on only consider prefix-free codes E: $E(c)$ is not a prefix of $E(c')$ for any $c, c' \in \Sigma_S$.

- **Example:**

c	A	E	N	O	T	␣
$E(c)$	01	101	001	100	11	000

standard trie of codewords

Any prefix-free code corresponds to a ***(code) trie*** (trie of codewords) with characters of Σ_S at **leaves**.



no need for end-of-string symbols $ here
(already prefix-free!)

- Encode `AN␣ANT` 01001000 ...
- Decode `111000001010111` TO␣

Code tries

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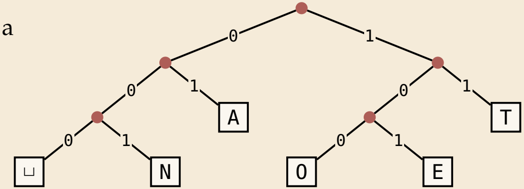

- Encode `AN␣ANT` $\longrightarrow$ `010010000100111`
- Decode `111000001010111` $\longrightarrow$ `TO␣EAT`

Who decodes the decoder?

- Depending on the application, we have to **store/transmit** the **used code**!
- We distinguish:
  - **fixed coding:** code agreed upon in advance, not transmitted (e. g., Morse, UTF-8)
  - **static coding:** code depends on message, but stays same for entire message; it must be transmitted (e. g., Huffman codes → next)
  - **adaptive coding:** code depends on message and changes during encoding; implicitly stored withing the message (e. g., LZW → below)

we have to transmit code

7.3 Huffman Codes

Character frequencies

- **Goal:** Find character encoding that produces short coded text
- Convention here: fix $\Sigma_C = \{0, 1\}$ (binary codes), abbreviate $\Sigma = \Sigma_S$,
- **Observation:** Some letters occur more often than others.

Typical English prose:

e	12.70%	**d**	4.25%	**p**	1.93%
t	9.06%	**l**	4.03%	**b**	1.49%
a	8.17%	**c**	2.78%	**v**	0.98%
o	7.51%	**u**	2.76%	**k**	0.77%
i	6.97%	**m**	2.41%	**j**	0.15%
n	6.75%	**w**	2.36%	**x**	0.15%
s	6.33%	**f**	2.23%	**q**	0.10%
h	6.09%	**g**	2.02%	**z**	0.07%
r	5.99%	**y**	1.97%		

$\rightsquigarrow$ Want shorter codes for more frequent characters!

Huffman coding

e. g. frequencies / probabilities

- **Given:** Σ and weights $w : \Sigma \to \mathbb{R}_{\geq 0}$
- **Goal:** prefix-free code E (= code trie) for Σ that minimizes coded text length

  i. e., a code trie minimizing $\displaystyle\sum_{c \in \Sigma} w(c) \cdot |E(c)|$

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- If we use $w(c) = \#\text{occurrences of } c \text{ in } S$,
  this is the character encoding with smallest possible $|C|$

  $\rightsquigarrow$ best possible character-wise encoding

- Quite ambitious! *Is this efficiently possible?*

Huffman's algorithm

- Actually, yes! A greedy/myopic approach succeeds here.

Huffman's algorithm:

1. Find two characters `a`, `b` with lowest weights.
   - We will encode them with the same prefix, plus one distinguishing bit, i. e., $E(\texttt{a}) = u\texttt{0}$ and $E(\texttt{b}) = u\texttt{1}$ for a bitstring $u \in \{0,1\}^\star$ (u to be determined)
2. (Conceptually) replace `a` and `b` by a single character "$\boxed{\texttt{ab}}$" with $w(\boxed{\texttt{ab}}) = w(\texttt{a}) + w(\texttt{b})$. $\rightarrow$ σ decreases by 1
3. Recursively apply Huffman's algorithm on the smaller alphabet. This in particular determines $u = E(\boxed{\texttt{ab}})$.

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   in code trie
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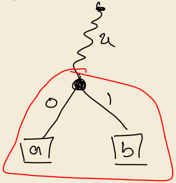
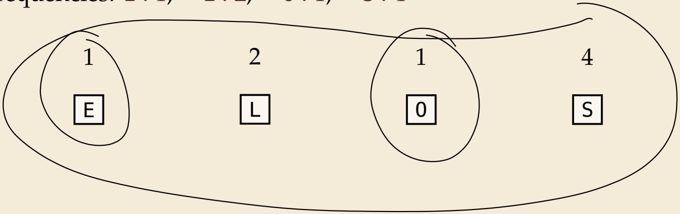


- efficient implementation using a (min-oriented) ***priority queue***
  - start by inserting all characters with their weight as key
  - step 1 uses two `deleteMin` calls
  - step 2 inserts a new character with the sum of old weights as key

Huffman's algorithm – Example

- Example text: S = LOSSLESS $\rightsquigarrow$ $\Sigma_S = \{$E, L, O, S$\}$
- Character frequencies: E : 1, L : 2, O : 1, S : 4

priority queue



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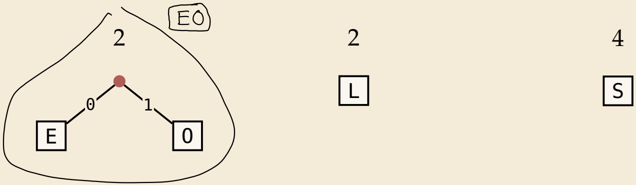

Huffman's algorithm – Example

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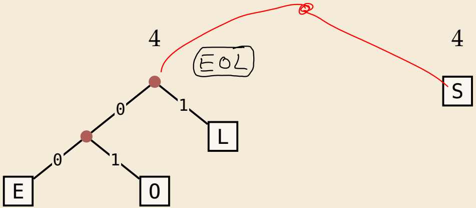
- Character frequencies: $\mathtt{E}: 1$, $\mathtt{L}: 2$, $\mathtt{O}: 1$, $\mathtt{S}: 4$



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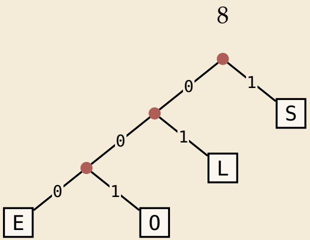
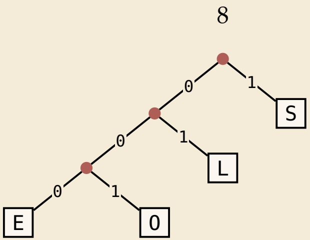


Huffman's algorithm – Example

- Example text: $S = \texttt{LOSSLESS}$ $\leadsto$ $\Sigma_S = \{\texttt{E}, \texttt{L}, \texttt{O}, \texttt{S}\}$
- Character frequencies: $\texttt{E}: 1$, $\texttt{L}: 2$, $\texttt{O}: 1$, $\texttt{S}: 4$



$\leadsto$ *Huffman tree* (code trie for Huffman code)

Huffman's algorithm – Example

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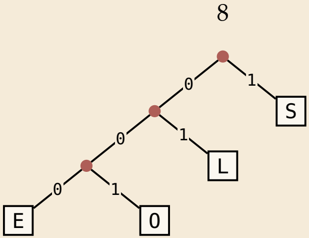

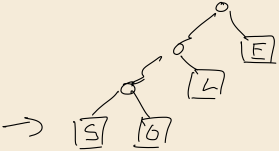

$\rightsquigarrow$ *Huffman tree* (code trie for Huffman code)

`LOSSLESS` $\longrightarrow$ `01001110100011`

compression ratio: $\frac{14}{8 \cdot \log 4} = \frac{14}{16} \approx 88\%$

Huffman tree – tie breaking

- The above procedure is ambiguous:
  - which characters to choose when weights are equal?
  - which subtree goes left, which goes right?

- For COMP 526: always use the following rule:

  1. To break ties when selecting the two characters, first use the smallest letter according to the alphabetical order, or the tree containing the smallest alphabetical letter.
  2. When combining two trees of different values, place the lower-valued tree on the left (corresponding to a `0`-bit).
  3. When combining trees of equal value, place the one containing the smallest letter to the left.

Huffman code – Optimality

Theorem 7.1 (Optimality of Huffman's Algorithm)
Given Σ and $w : \Sigma \to \mathbb{R}_{\geq 0}$, Huffman's Algorithm computes codewords $E : \Sigma \to \{0,1\}^\star$ with minimal expected codeword length $\ell(E) = \sum_{c \in \Sigma} w(c) \cdot |E(c)|$, among all prefix-free codes for Σ. ◀

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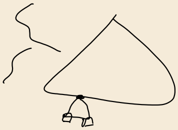
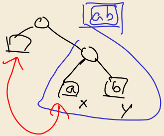


Proof sketch: by induction over $\sigma = |\Sigma|$

- Given any optimal prefix-free code E^* (as its code trie).
- code trie $\rightsquigarrow$ $\exists$ two sibling leaves x, y at largest depth D
- swap characters in leaves to have two lowest-weight characters `a`, `b` in x, y (that can only make ℓ smaller, so still optimal)
- any optimal code for $\Sigma' = \Sigma \setminus \{\texttt{a}, \texttt{b}\} \cup \{\boxed{\texttt{ab}}\}$ yields optimal code for Σ by replacing leaf $\boxed{\texttt{ab}}$ by internal node with children `a` and `b`.



$\rightsquigarrow$ recursive call yields optimal code for Σ' by inductive hypothesis, so Huffman's algorithm finds optimal code for Σ.

Entropy

Definition 7.2 (Entropy)

Given probabilities $p_1, \ldots, p_n$ (for outcomes $1, \ldots, n$ of a random variable), the *entropy* of the distribution is defined as

$$\mathcal{H}(p_1, \ldots, p_n) \;=\; -\sum_{i=1}^{n} p_i \lg p_i \;=\; \sum_{i=1}^{n} p_i \lg\left(\frac{1}{p_i}\right)$$

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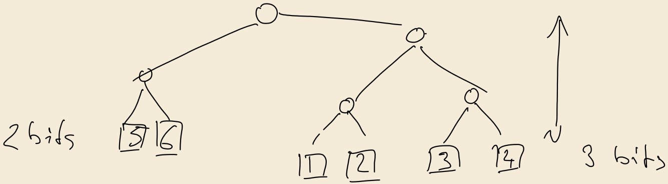
- entropy is a **measure** of **information** content of a distribution
  - more precisely: the expected number of bits (Yes/No questions) required to nail down the random value

$\leadsto$ would ideally encode value i using $\lg(1/p_i)$ bits
that is not always possible; cannot use 1.5 bits ... but:



fair die

$1, 2, \ldots, 6$

$\frac{1}{6} \quad \cdots \quad \frac{1}{6}$

$\mathcal{H}(\frac{1}{6}, \ldots, \frac{1}{6}) = 6 \cdot \frac{1}{6} \cdot \lg(6)$

$= \lg(6) \approx 2.$

$\frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 2 = 2.\overline{6}$

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◀

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Theorem 7.3 (Entropy bounds for Huffman codes)

For any $\Sigma = \{a_1, \ldots, a_\sigma\}$ and $w : \Sigma \to \mathbb{R}_{\geq 0}$ and its Huffman code E, we have

$$\mathcal{H}\left(\frac{w(a_1)}{W}, \ldots, \frac{w(a_\sigma)}{W}\right) \;\leq\; \underline{\ell(E)} \;\leq\; \mathcal{H}\left(\frac{w(a_1)}{W}, \ldots, \frac{w(a_\sigma)}{W}\right) + 1$$

where $W = w(a_1) + \cdots + w(a_\sigma)$. ◀

Clicker Question

When is Huffman coding more efficient than a fixed-length encoding?

$\mathcal{H}$ = entropy

$\sigma = |\Sigma|$

A always

B when $\mathcal{H} \approx \underline{\lg(\sigma)}$

C when $\mathcal{H} < \lg(\sigma)$

D when $\mathcal{H} < \lg(\sigma) - 1$

E when $\mathcal{H} \approx 1$

`pingo.upb.de/622222`

Clicker Question

When is Huffman coding more efficient than a fixed-length encoding?

- **A** always ✓
- **B** ~~when $\mathcal{H} \approx \lg(\sigma)$~~
- **C** ~~when $\mathcal{H} < \lg(\sigma)$~~ — $\ell(E) \leqslant \lg \sigma + 1$
- **D** when $\mathcal{H} < \lg(\sigma) - 1$ ✓ $\underline{\ell(E)} \leqslant \mathcal{H} + 1 < \lceil \lg(\sigma) \rceil$
- **E** ~~when $\mathcal{H} \approx 1$~~

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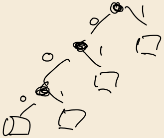
Encoding with Huffman code

- The overall encoding procedure is as follows:
  - Pass 1: Count character frequencies in S
  - Construct Huffman code E (as above)
  - Store the Huffman code in C (details omitted)
  - Pass 2: Encode each character in S using E and append result to C
- Decoding works as follows:
  - Decode the Huffman code E from C. (details omitted)
  - Decode S character by character from C using the code trie.



- Note: Decoding is much simpler/faster!

Huffman coding – Discussion

- running time complexity: $O(\sigma \log \sigma)$ to construct code
  - build PQ + $\sigma \cdot$ (2 deleteMins and 1 insert)
  - can do $\Theta(\sigma)$ time when characters already sorted by weight
  - time for encoding: $O(n + |C|)$
- many variations in use (tie-breaking rules, estimated frequencies, adaptive encoding, ...)

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  - can do $\Theta(\sigma)$ time when characters already sorted by weight
  - time for encoding: $O(n + |C|)$
- many variations in use (tie-breaking rules, estimated frequencies, adaptive encoding, ...)

👍 optimal prefix-free character encoding

👍 very fast decoding

👍 robust encoding local errors only affect 1–2 symbols

flipped bits

👎 needs 2 passes over source text for encoding

- one-pass variants possible, but more complicated

👎 have to store code alongside with coded text → inflation

Part II

Compressing repetitive texts

Beyond Character Encoding

- Many “natural” texts show repetitive redundancy

```
All work and no play makes Jack a dull boy.  All work and no play makes Jack a dull
boy.  All work and no play makes Jack a dull boy.  All work and no play makes Jack
a dull boy.  All work and no play makes Jack a dull boy.  All work and no play makes
Jack a dull boy.  All work and no play makes Jack a dull boy.  All work and no play
makes Jack a dull boy.  All work and no play makes Jack a dull boy.  All work and no
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- character-by-character encoding will **not** capture such repetitions
  - ⇝ Huffman won’t compression this very much

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work and no play makes Jack a dull boy.  All work and no play makes Jack a dull boy.
```

- character-by-character encoding will **not** capture such repetitions
  - ⇝ Huffman won't compression this very much

⇝ Have to encode whole ***phrases*** of S by a single codeword

7.4 Run-Length Encoding

Run-Length encoding

- simplest form of repetition: ***runs*** of characters

  same character repeated

```
000000000000000000000000000000000000000
000000000000000000000000000000000000000
000000000000000000000000000000000000000
000101100100000111111000000000011111000
001111111110001111111110000001111111000
001111011010001110001111000001110000000
001100000000000000000111000111000000000
001100000000000000000011001110000000000
001100000000000000000011001110000000000
001101100000000000000111001100111110000
001111111110000000000111001111111111000
001110111111000000001110001111100111100
000000000111000000011100000111000001110
000000000111000000001100001110000001100
000000000011000000011000001100000001110
000000000011000001110000001110000001100
000000000111000111000000000110000001110
000000000110000111000000000111000011100
001101111111000111101110100001111111000
011111111110001111111111100001111110000
000101100000001010011001000000100100000
000000000000000000000000000000000000000
000000000000000000000000000000000000000
```

- here: only consider $\Sigma_S = \{0, 1\}$ (work on a binary representation)
  - can be extended for larger alphabets

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001100000000000000000011100011100000000
001100000000000000000000110011100000000
001100000000000000000000110011100000000
001101100000000000000111001100111110000
001111111100000000000111001111111111000
001110111110000000000111000111110011100
000000000111000000011100001110000001110
000000000111000000011000001110000001100
000000000011000000110000000110000001110
000000000011000001110000001110000001100
000000000111000111000000000110000001110
000000000110000111000000000111000011100
001101111110001111011101000011111111000
011111111110001111111111000011111110000
000101100000001010011001000000100100000
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⇝ **run-length encoding (RLE):**
use runs as phrases: $S = \underbrace{\texttt{00000}}\ \underbrace{\texttt{111}}\ \underbrace{\texttt{0000}}$

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0000000000000000000000000000000000000000
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0011111111110001111111110000001111111000
0011110110100011100011110000011100000000
0011000000000000000000111000111000000000
0011000000000000000000110011100000000000
0011000000000000000000110011100000000000
0011011000000000000000111001100111110000
0011111111000000000000111001111111111000
0011101111100000000001110001111100111100
0000000001110000000011100001110000001110
0000000001110000000011000001110000001100
0000000000110000000110000000110000001110
0000000000110000011100000001110000001100
0000000001110001110000000000110000001110
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0111111111100011111111111100001111110000
0001011000000001010011001000000100100000
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⇝ We have to store
  - the first bit of S (either `0` or `1`)
  - the length each each run
  - Note: don't have to store bit for later runs since they must alternate.

- Example becomes: $0, 5, 3, 4$

Run-Length encoding

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  same character repeated

```
000000000000000000000000000000000000000000
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000000000000000000000000000000000000000000
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001111011010001110001111000001110000000000
001100000000000000000011100011100000000000
001100000000000000000001100111000000000000
001100000000000000000001100111000000000000
001101100000000000000011100110011111100000
001111111110000000000011100111111111111000
001110111111000000000111000111111001111100
000000000111000000001110000111000000001110
000000000111000000001100000111000000001100
000000000011000000011000000011000000001110
000000000011000000111000000111000000001100
000000000111000111000000000011000000001110
000000000110000111000000000011100000011100
001101111111000111101110100001111111111000
011111111110001111111111110000111111110000
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  - Note: don't have to store bit for later runs since they must alternate.

- Example becomes: $0, 5, 3, 4$

- **Question**: How to encode a run length k in binary? (k can be arbitrarily large!)

Clicker Question

How would you encode a string that can we arbitrarily long?

codius

C string?

$c_1 c_2 c_3 c_4 \ldots$ '\0'

\$

Java arrays store length ...

?

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Elias codes

- Need a prefix-free encoding for $\mathbb{N} = \{1, 2, 3, \ldots, \}$
  - must allow arbitrarily large integers
  - must know when to stop reading

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  $7 \mapsto$ `00000001` $3 \mapsto$ `0001` $0 \mapsto$ `1` $30 \mapsto$ `0000000000000000000000000000001`

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  - Store the **length** ℓ of the binary representation in **unary**
  - Followed by the binary digits themselves

Elias codes

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  Much too long
  - (wasn't the whole point of RLE to get rid of long runs??)
- Refinement: *Elias gamma code*
  - Store the **length** ℓ of the binary representation in **unary**
  - Followed by the binary digits themselves
  - little tricks:
    - always $\ell \geq 1$, so store $\ell - 1$ instead
    - binary representation always starts with `1` $\rightsquigarrow$ don't need terminating `1` in unary

  $\rightsquigarrow$ Elias gamma code = $\ell - 1$ zeros, followed by binary representation

Examples: $1 \mapsto$ `1`, $3 \mapsto$ `011`, $5 \mapsto$ `00101`, $30 \mapsto$ `000011110`

Clicker Question

Decode the **first** number in Elias gamma code (at the beginning) of the following bitstream:

`0001101111011100110.`

3 zeros $\ell = 4$ $1101_2 = 13$

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Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

  C = `1`

- Decoding:

  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:

  $S = $ `11111110010000000000000000000011111111111`

  $k = 7$

  $C = $ `100111`

- Decoding:

  $C = $ `00001101001001010`

  $S = $

Run-length encoding – Examples

- Encoding:
  $S =$ `1111111`00`10000000000000000000011111111111`
  $k = 2$
  $C =$ `100111010`

- Decoding:
  $C =$ `00001101001001010`

  $S =$

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

  $k = 1$

  C = `1001110101`

- Decoding:

  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:

  $S =$ `1111111001000000000000000000000011111111111`

  $k = 20$

  $C =$ `1001110101000010100`

- Decoding:

  $C =$ `00001101001001010`

  $S =$

Run-length encoding – Examples

- Encoding:

  S = `111111100100000000000000000000` `11111111111`

  $k = 11$

  C = `1001110101000010100` `0001011`

- Decoding:

  C = `00001101001001010`

  S =

Run-length encoding – Examples

- Encoding:
  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C =$ `00001101001001010`

  $S =$

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  $S =$ `11111110010000000000000000000011111111111`

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- Decoding:
  $C =$ `00001101001001010`
  $b = 0$

  $S =$

Run-length encoding – Examples

- Encoding:
  $S=$ `11111110010000000000000000000011111111111`

  $C=$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C=$ `0000`$\,$`1101001001010`
  $b=0$
  $\ell=3+1$

  $S=$

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  $C =$ `00001101001001010`

  $b = 0$

  $\ell = 3 + 1$

  $k = 13$

  $S =$ `0000000000000`

Run-length encoding – Examples

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  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C =$ `00001101001001010`
  $b = 1$
  $\ell = 2 + 1$
  $k =$
  $S =$ `0000000000000`

Run-length encoding – Examples

- Encoding:
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  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  C = `00001101001001010`
  $b = 1$
  $\ell = 2 + 1$
  $k = 4$
  S = `00000000000001111`

Run-length encoding – Examples

- Encoding:
  S = `11111110010000000000000000000001111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  C = `00001101001001010`
  $b = 0$
  $\ell = 0 + 1$
  $k =$
  S = `00000000000001111`

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- Encoding:
  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

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  $C =$ `00001101001001010`
  $b = 0$
  $\ell = 0 + 1$
  $k = 1$
  $S =$ `0000000000000011110`

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  $C = \mathtt{00001101001001}\color{red}{\mathtt{0}}\mathtt{10}$

  $b = 1$

  $\ell = 1 + 1$

  $k =$

  S = `000000000000011110`

Run-length encoding – Examples

- Encoding:

  S = `11111110010000000000000000000011111111111`

  C = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  C = `00001101001001010`

  $b = 1$

  $\ell = 1 + 1$

  $k = 2$

  S = `00000000000001111011`

Run-length encoding – Discussion

- extensions to larger alphabets possible (must store next character then)
- used in some image formats (e. g. TIFF)

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- used in some image formats (e. g. TIFF)

👍 fairly simple and fast

👍 can compress n bits to $\Theta(\log n)$!
for extreme case of constant number of runs

👎 negligible compression for many common types of data
- No compression until run lengths $k \geq 6$
- **expansion** when run lengths $k = 2$ or 6