11 Greedy Algorithms

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CS566 (Wintersemester 2024/25)
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Learning Outcomes

Unit 11: ***Greedy Algorithms***

1. Describe informally what greedy algorithms are.
2. Know exemplary problems and their greedy solutions: Change-Making Problem, MSTs, SSSPP, Assignment Problem.
3. Be able to design and proof correctness of greedy algorithms for (simple) algorithmic problems.
4. Be able to explain the matroid properties and its relation to greedy algorithms.

Outline

11 Greedy Algorithms

11.1 Introduction

Myopic Optimization

- In a *"greedy" algorithm*,
  we assemble a solution to an **optimization** problem **step by step**
  always picking the next step to maximize **current** gain,
  and we **never take back** earlier steps.

"Take what you can, give nothing back!"

Myopic Optimization

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  and we **never take back** earlier steps.

"Take what you can, give nothing back!"

- reminiscent of *gradient-descent* algorithms
  but discrete and even more unwilling to undo mistakes

$\rightsquigarrow$ greedy algorithms only yield optimal solutions for certain problems

  - but where they do, their speed is usually unbeatable

  $\rightsquigarrow$ it is understanding where they succeed

- even where they are not optimal, greedy approaches can be efficient heuristics (unknown quality) or approximation algorithms

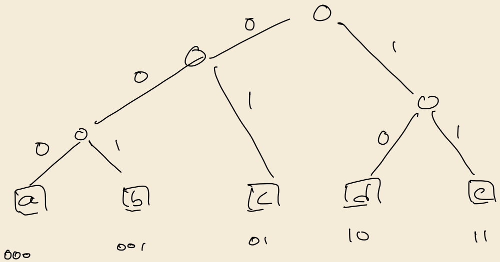
  c-approximation = at most factor c worse than optimum

Plan for the Unit

- We will first see a few examples (known and new) of greedy algorithms to make the vague generic description concrete
  - in particular minimum spanning trees and shortest paths in graphs
- Unlike other algorithm design techniques, greedy algorithms have a formal basis: *matroids* (and *greedoids*)
  - The second part will introduce these and how they can unify correctness proofs

A First Example: Reunion With An Old Friend

- We have seen an example of a Greedy Algorithm in Unit 7: *Huffman Codes!*
- Recall the problem:
  - **Given:** Set of symbols $\Sigma = [0..\sigma)$, weights $w : \Sigma \to \mathbb{R}_{\geq 0}$
  - **Goal:** prefix code E (= code trie) that minimizes $\sum_{c \in \Sigma} w(c) \cdot |E(c)|$



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$\rightsquigarrow$ Since only *code tries* are valid, all solutions consist in repeatedly merging tries (starting from single-leaf tries, until single trie left)

- each merge contributes the subtree's total weight to overall cost (since all leaves in merged tries move one level down / all codewords get one extra bit)
- **Huffman's Algorithm:** Always choose current cheapest merge.

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- each merge contributes the subtree's total weight to overall cost (since all leaves in merged tries move one level down / all codewords get one extra bit)
- **Huffman's Algorithm:** Always choose current cheapest merge. w.r.t
- In the correctness proof, we had to show: There is always an optimal code trie where the two lowest-weight symbols are siblings.

This is typical: To show that Greedy is optimal, we need a structural insight into optimal solutions.

11.2 How Can Greed Succeed?

Greed For Change

The Change-Making Problem (a. k. a. Coin-Exchange Problem)

- **Given:** a set of integer denominations of coins $w_1 < w_2 < \cdots < w_k$ with $w_1 = 1$,
  target value $n \in \mathbb{N}_{\geq 1}$

  (we have sufficient supply of all coins . . .)

- **Goal:** "fewest coins to give change n", i. e.,
  multiplicities $c_1, \ldots, c_k \in \mathbb{N}_{\geq 0}$ with $\sum_{i=1}^{k} c_i \cdot w_i = n$ minimizing $\sum_{i=1}^{k} c_i$

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For Euro coins, denominations are 1¢, 2¢, 5¢, 10¢, 20¢, 50¢, 1€, and 2€.

formally: $\underset{w_1}{1}$, $\underset{w_2}{2}$, $\underset{w_3}{5}$, $\underset{w_4}{10}$, $\underset{w_5}{20}$, $\underset{w_6}{50}$, $\underset{w_7}{100}$, and $\underset{w_8}{200}$.

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$$\text{formally:}\quad \begin{array}{cccccccc} 1, & 2, & 5, & 10, & 20, & 50, & 100, \text{ and} & 200. \\ w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{array}$$

⇝ Simple greedy algorithm: largest coins first

- optimal time ($O(k)$ if coins sorted)
- is $\sum c_i$ minimal?

```
procedure greedyChange(w[1..k], n):
    // Assumes 1 = w[1] < w[2] < ··· < w[k]
    for i := k, k − 1, . . . , 1:
        c[i] := ⌊n / w[i]⌋
        n := n − c[i] · w[i]
    // Now n == 0
    return c[1..k]
```

Clicker Question

Does greedyChange give the optimal answer for the Euro coins change-making problem?

- A Always
- B Sometimes
- C Never

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Optimality of Greedy Euro-Change

- **Theorem:** greedyChange computes an optimal $c[1..8]$ for $w[1..8] = [1, 2, 5, 10, 20, 50, 100, 200]$ for every $n \in N_{\geq 1}$.

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  - The greedy algorithm can be interpreted as picking one coin at a time, each time choosing the largest possible denomination $\hat{w}(n) = \max\{w[i] : w[i] \leq n\}$.
  - We prove by induction over n: Any optimal solution for n must contain $\hat{w}(n)$.
    - $n = 1$: can only use $\hat{w}(n) = 1$ ✓

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    - $n \in [2..5)$: Assume we had a solution without 2¢ $\rightsquigarrow$ must be $n \times$ 1¢ with $n \geq 2$; $\rightsquigarrow$ we can make this strictly better by replacing 1¢ 1¢ by 2¢ ϟ

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      $\rightsquigarrow$ we can make this strictly better by replacing 1¢ 1¢ by 2¢ ↯
    - $n \in [5..10)$: Assume solution without 5¢ summing to $n \geq 5$.
      The solution must fall into one of the following cases:
      (a) $\geq 3 \times$ 2¢ $\rightsquigarrow$ replacing 2¢ 2¢ 2¢ by 5¢ 1¢ strictly better ↯
      (b) $\leq 1 \times$ 2¢ $\rightsquigarrow$ value $n - 2 \geq 3$ without 2¢ ↯ by IH
      (c) $2 \times$ 2¢ and $\geq 1 \times$ 1¢ $\rightsquigarrow$ 2¢ 2¢ 1¢ $\to$ 5¢ strictly better ↯
      (d) $2 \times$ 2¢, no 1¢ $\rightsquigarrow$ only obtain value $\leq 4 < n$ ↯

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    - $n \in [10, 20)$: Any solution without 10¢ contains
      (a) 5¢ 5¢ $\rightsquigarrow$ replace by 10¢; or
      (b) at most one 5¢ $\rightsquigarrow$ at least value 5 without 5¢ ↯ by IH

Optimality of Greedy Euro-Change [2]

- … proof continued
  - $n \in [20..50)$ Without 20¢, we must have
    (a) 10¢ 10¢ $\longrightarrow$ 20¢ ⚡
    (b) at most one 10¢ $\rightsquigarrow$ value $n - 10 \geq 10$ without 10¢ ⚡ by IH

Optimality of Greedy Euro-Change [2]

- ... proof continued
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  - $n \in [50..100)$ Without 50¢, we must have
    (a) $\geq 3 \times$ 20¢ $\rightsquigarrow$ 20¢ 20¢ 20¢ $\rightarrow$ 50¢ 10¢ ϟ
    (b) $\leq 1 \times$ 20¢ $\rightsquigarrow$ value $n - 20 \geq 30$ without 20¢ ϟ by IH
    (c) $2 \times$ 20¢ and $\geq 1 \times$ 10¢ $\rightsquigarrow$ 20¢ 20¢ 10¢ $\rightarrow$ 50¢ ϟ
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  - $n \in [100..200)$: as for $n \in [10, 20)$, *mutatis mutandis*.
  - $n \geq 200$: as for $n \in [20, 50)$. □

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  - $n \in [100..200)$: as for $n \in [10, 20)$, *mutatis mutandis*.
  - $n \geq 200$: as for $n \in [20, 50)$.
- The same arguments work for adding coins $1 \cdot 10^m, 2 \cdot 10^m, 5 \cdot 10^m$ for $m = 3, 4, \ldots$

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  - $n \in [50..100)$ Without 50¢, we must have
    (a) $\geq 3 \times$ 20¢ $\rightsquigarrow$ 20¢ 20¢ 20¢ $\longrightarrow$ 50¢ 10¢ ϟ
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That went smoothly!
And we proved a nice structural statement about how optimal solutions look like as a bonus.

Maybe Greedy always works?

Greed Can Mislead

- *Unfortunately, No.* See $w = (1, 3, 4)$ and $n = 6$.

4 1 1

3 3

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  or $w = (1, 4, 9)$ and $n = 12$

Where/Why does our proof from above fail?

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  Where/Why does our proof from above fail?

- Indeed, Greedy can be **arbitrarily bad** compared to the optimal solution:
  See $w = (1, 999, 1000)$ and $n = 1998$.

$\leadsto$ Need to be careful about the details of a correctness argument for greedy algorithms.

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$\rightsquigarrow$ Need to be careful about the details of a correctness argument for greedy algorithms.

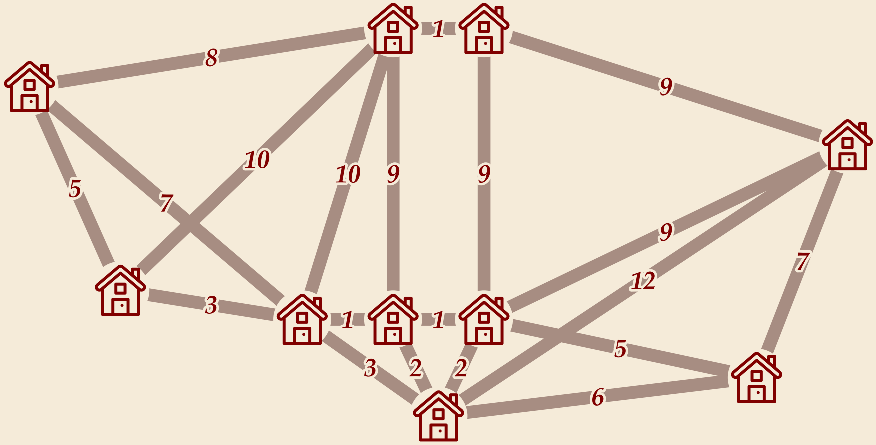
- The Change-Making problem is still only partially understood.
  - Exactly characterizing the denomination sequences that are optimally handled by greedyChange is an **open research problem**.
    - Sufficient criteria for “greed-compatible” denominations found in the literature.
  - The general problem is (weakly) NP-hard
  - Yet, for moderate n, we will see a solution for general denomination sequences later!

11.3 Greed in Graphs I: MSTs

Metaphor: Planning an electricity grid

Given: Houses to be connected to central power grid
Possible connections with building costs given

Goal: Cheapest way to get every house connected



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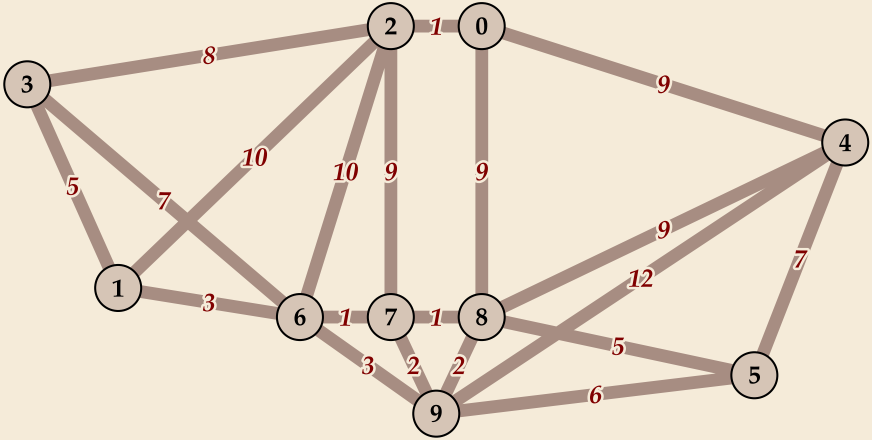

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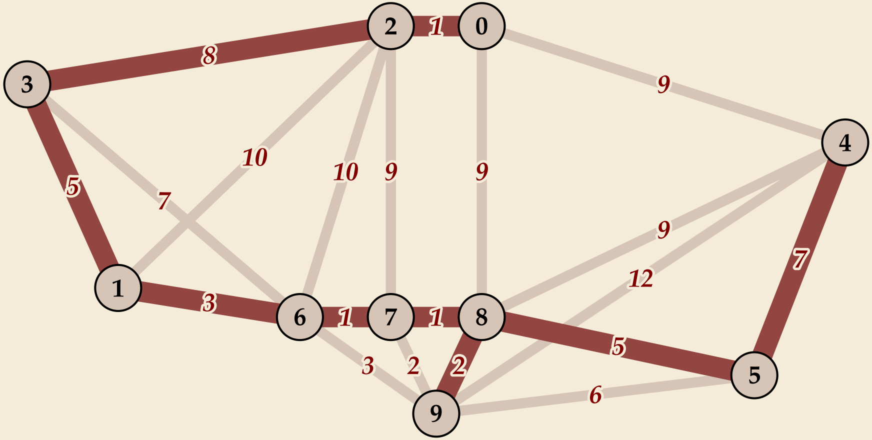
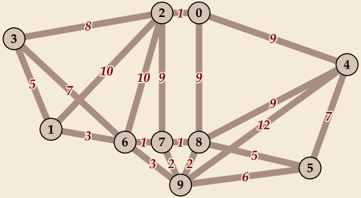


The Minimum Spanning Tree (MST) Problem

Given: **un**directed, edge-**weighted**, simple,
connected graph $G = (V, E, c)$

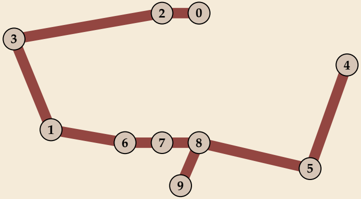
no self loops,
no parallel edges

Formally: Recall assumption $V = [0..n)$ ($\rightsquigarrow$ array indices)
edges $E \subseteq \{\{u, v\} : u, v \in V \wedge u \neq v\}$
edge weights (costs) $c : E \to \mathbb{R}_{\geq 0}$
for all $u, v \in V$ there exists a path $u \rightsquigarrow v$ in (V, E)



Goal: a **spanning tree** (V, T)
with **minimal** total cost $c(T) := \sum_{e \in T} c(e)$

Formally: $T \subseteq E$
(V, T) is connected and acyclic ("spanning tree")
for every spanning tree (V, T') of G we have $c(T') \geq c(T)$.



Further MST Applications

Direct Applictions

- single-linkage hierarchical clustering
- Bottleneck-shortest paths
- Approximation algorithms, e. g.,
  - Christofides's Metric TSP Approximation
  - Steiner-tree problem

As a cheap subroutine

- Routing protocols
- medical image processing
- ...

Interlude: On Varieties of Trees

⚠ *We freely use "tree" to mean different things in different contexts ... mind the confusion.*

- here: "tree" = *undirected, nonplane tree* = an undirected, connected and acyclic graph

in spanning *tree*

no order on edges

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The digraph flavor is a rooted tree: (hence undirected trees sometimes called *unrooted*)

- *rooted (nonplane/unordered) tree* = **digraph** (V, E) with *root* $r \in V$ s.t.
  $\forall v \in V \setminus \{r\} : d_{\text{out}}(v) = 1$ and $d_{\text{out}}(r) = 0$

out-degree = #outgoing edges

We draw trees with the single(!) root on top . . .

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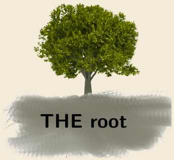

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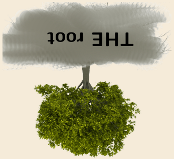

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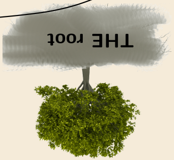

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Other "trees" don't originate from graphs naturally, but rather from recursion / terms:

- *binary tree* = a null pointer or a node with left and right children, each a binary tree
  (formally: the set of binary trees is the smallest fixed point of that construction)

ordered rooted

- *ordinal trees* = a node with a sequence of 0 or more children, each ordinal trees
  = rooted ordered trees (rooted unordered + total order on children)
- plus many more variants out there . . . $\rightsquigarrow$ if in doubt, double check definitions!

Clicker Question

Which algorithm allows to efficiently test whether a given (undirected) graph is connected?

for this problem w/ optimal worst-case Θ-class

- A bubble sort
- B depth-first search
- C breadth-first search
- D generic tricolor search
- E Kosaraju-Sharir's algorithm
- F Dijkstra's algorithm
- G Edmonds-Karp algorithm

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Which algorithm allows to efficiently test whether a given (undirected) graph is connected?

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- B depth-first search ✓
- C breadth-first search ✓
- D (generic tricolor search) ✓
- E Kosaraju-Sharir's algorithm ✓
- F ~~Dijkstra's algorithm~~
- G ~~Edmonds-Karp algorithm~~

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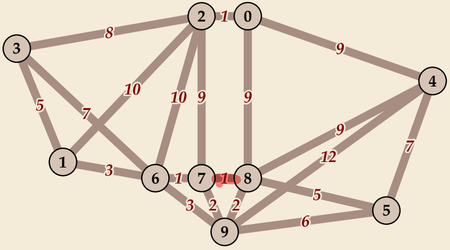
A Naive Approach

How to start finding an MST?

Using the **cheapest** edge shouldn't hurt . . .

---

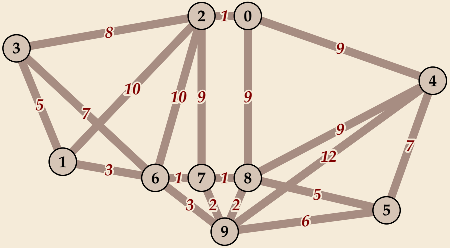
procedure greedyMST(V, E, c):
    // *Assume* (V, E) *is simple & connected,* $c : E \to \mathbb{R}_{\geq 0}$
    $T := \emptyset$
    while (V, T) not connected
        e := cheapest edge that doesn't close a cycle in T
        $T := T \cup \{e\}$
    return T



A Naive Approach Works – Kruskal's Algorithm

How to start finding an MST?

Using the **cheapest** edge shouldn't hurt . . .




```
1  procedure kruskalMST(V, E, c):
2      // Assume (V, E) is simple & connected, c : E → ℝ≥0
3      T := ∅
4      while (V, T) not connected
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6          T := T ∪ {e}
7      return T
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Apart from implementing line 4 and line 5 efficiently, this *is* **Kruskal's Algorithm**!

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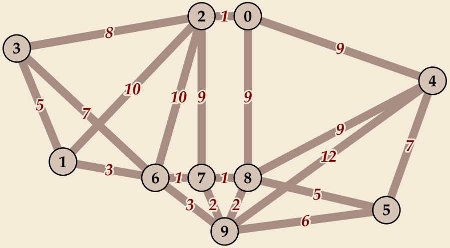

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Apart from implementing line 4 and line 5 efficiently, this *is* **Kruskal's Algorithm**!

As so often with greedy algorithms, the hardest bit is the correctness argument. We have:

Theorem: Kruskal's Algorithm finds a minimum spanning tree.
This immediately follows from proving the following invariant:

Kruskal's Invariant: There is some MST T^* with $T \subseteq T^*$.

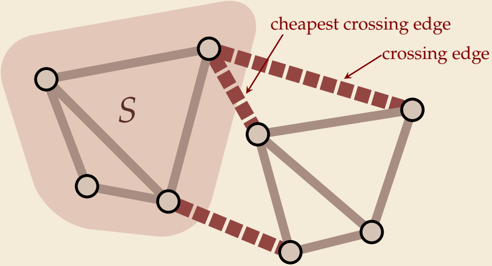
henceforth: identify MST with its edge set

Crossing Edges and the MST-Cut Lemma

To prove the correctness of Kruskal's Algorithm, we need a tool.

Notation:

- **Cut** S:
  non-trivial set of vertices $\emptyset \neq S \subsetneq V$
- **crossing edge** e wrt. cut S:
  $e = \{u, v\}$ with $u \in S, v \in \bar{S} := V \setminus S$

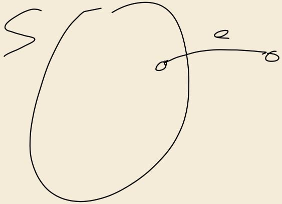


The MST-Cut Lemma:
Let T^* be an MST und $W \subseteq T^*$.
For every cut S, not cutting any edges in W, and
every *cheapest* crossing edge e wrt. S
there is an MST $\hat{T}^*$ that contains $W \cup \{e\}$.

Proof of MST-Cut Lemma

Proof:

- Case 1: $e \in T^*$.
  Then picking $\hat{T}^* = T^*$ proves the claim.



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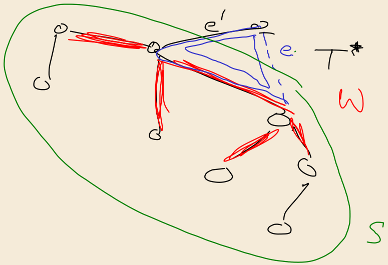
  - Since e crosses cut S, C crosses S
  - $\rightsquigarrow$ There is a second crossing edge $e' \in C$.



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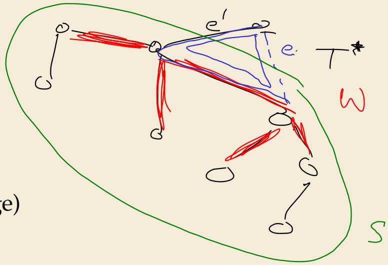
  - Since e' is crossing, $e' \notin W$
  - by assumption, $c(e) \leq c(e')$ (we pick cheapest crossing edge)
  - $\rightsquigarrow$ $\hat{T}^* = T^* \cup \{e\} \setminus \{e'\}$ is a spanning tree, and $W \cup \{e\} \subseteq \hat{T}^*$
  - $c(\hat{T}^*) = c(T^*) + c(e) - c(e') \leq c(T^*)$
  - $\rightsquigarrow$ $\hat{T}^*$ is an **MST**.



Kruskal's Algorithm – Correctness

With these preparations, we can prove

$\boxed{\textbf{Kruskal's Invariant:}\ \text{There is some MST } T^* \text{ with } T \subseteq T^*.}$

Proof: by induction over the loop iterations

- IB: initially $T = \emptyset$ and $\emptyset \subseteq T^*$ for every MST T^*.
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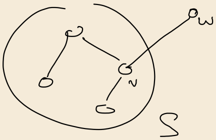

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Since we only terminate when T is spanning, upon termination $T = T^*$ for an MST T^*.

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$W = T$

S

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2. find the next cheapest edge to consider
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∉ exam

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$\rightsquigarrow$ $O(m \log m) = O(m \log n)$ time and $O(m)$ extra space.