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Exercise Sheet 5 for Advanced Algorithms, Summer 2025

Hand In: *Until 2025-06-18 18:00, on ILIAS.*

Problem 1

15 points

a) Prove that it is impossible to perfectly simulate a roll of a fair 6-sided die using random bits in finite worst-case time, i.e., with $t = \mathit{time}_A < \infty$.

b) How do programming libraries deal with this issue?

Problem 2

10 points

Let $P \overset{\mathcal{D}}{=} \mathcal{U}(0,1)$ be a random variable uniformly distributed in $(0,1)$ and let X be a random variable with a Bernoulli $\mathcal{B}(p)$ distribution *conditional* on $P = p$. We also write this as $X \overset{\mathcal{D}}{=} \mathcal{B}(P)$. Compute $\mathbb{E}[X]$.

Problem 3

20 points

Prove that the set $C = \{000, 111\}^\omega$, i.e., the set of infinite bit sequences on with `dieRoll` does *not* terminate is in the σ-algebra $\mathcal{F}$ generated by the cylinder sets $\pi_x = \{xy : y \in \{0,1\}^\omega\} \subseteq \{0,1\}^\omega$ for $x \in \{0,1\}^\star$. Show, by computing along the construction for C, that $\Pr[C] = 0$ in the probability measure induced by $\Pr[\pi_w] = 2^{-|w|}$.

Problem 4

10 + 30 points

a) Prove that every algorithm that randomly shuffles a given list of n items so that afterwards all possible orderings are equally likely must use $\Theta(n \log n)$ random bits.

b) Design a (randomized) algorithm A that generates a random permutation of the numbers $1, \ldots, n$. Each permutation is to have the same probability.

Argue that your algorithm has the desired property and determine $\mathbb{E}\text{-Time}_A(n)$ as well as the *expected* number of random bits to generate a permutation of length n.

Can you find a method with optimal number of random bits (asymptotically and in expectation)?