4 Fixed-Parameter Algorithms

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Outline

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Philosophy of FPT

- **Goal:** Principled theory for studying complexity based on two dimensions:
  input size $n = |x|$ (encoding length) and *some additional parameter k*
  - generalize ideas from $k = MaxInt(x)$
  - investigate influence of k (and n) on running time

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  - generalize ideas from $k = MaxInt(x)$
  - investigate influence of k (and n) on running time

$\leadsto$ Try to find a parameter k such that
(1) the problem can be solved efficiently as long as k is small, and
(2) practical instances have small values of k (even where n gets big).

Motivation: Satisfiability

Consider Satisfiability of CNF formula *the drosophila melanogaster of complexity theory*

- general worst case: NP-complete

$a \to b \equiv \neg a \vee b$

- k = #literals per clause
  - $k \leq 2 \rightsquigarrow$ in P 2SAT $x_i \vee \neg x_j \equiv x_j \to x_i$
  - $k \geq 3$ NP-complete $\equiv \neg x_i \to \neg x_j$

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- k = #literals per clause
  - $k \leq 2$ $\rightsquigarrow$ in P
  - $k \geq 3$ NP-complete
- k = #variables
  - $O(2^k \cdot n)$ time possible (try all assignments)
- k = #clauses?
- k = #literals?
- k = #ones in satisfying assignment
- k = structural property of formula
- for Max-SAT, k = #optimal clauses to satisfy

Parameters

Definition 4.1 (Parameterization)
Let Σ a (finite) alphabet. A *parameterization* (of $\Sigma^\star$) is a mapping $\kappa : \Sigma^\star \longrightarrow \mathbb{N}$ that is polytime computable. ◀

Definition 4.2 (Parameterized problem)
A *parameterized (decision) problem* is a pair (L, κ) of a language $L \subset \Sigma^\star$ and a parameterization κ of $\Sigma^\star$. ◀

Definition 4.3 (Canonical Parameterizations)
We can often specify a parameterized problem conveniently as a language of *pairs* $L \subset \Sigma^\star \times \mathbb{N}$ with

$$(x, k) \in L \ \wedge \ (x, k') \in L \quad \rightarrow \quad k = k'$$

using the *canonical parameterization* $\kappa(x, k) = k$. ◀

Examples

As before: Typically leave encoding implicit.

Definition 4.4 (p-variables-SAT)

Given: formula boolean ϕ (same as before)

Parameter: number of variables

Question: Is there a satisfying assignment $v : [n] \to \{0, 1\}$? ◀

Definition 4.5 (p-Clique)

Given: graph $G = (V, E)$ and $k \in \mathbb{N}$

Parameter: k

Question: $\exists V' \subset V \quad : \quad |V'| \geq k \ \wedge \ \forall u, v \in V' : \{u, v\} \in E$? ◀

Canonical Parameterization

Definition 4.6 (Canonically Parameterized Optimization Problems)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ be an optimization problem.
Then p-U denotes the *(canonically) parameterized (decision) problem* given by the threshold problem $Lang_U$. ◀

Recall: $Lang_U$ is the set of pairs (x, k) of all instances $x \in L_I$ that have solutions that are weakly "better" than k.

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Examples:

- p-Clique
- p-Vertex-Cover
- p-Graph-Coloring
- ...

Naming convention for other parameters:
p-*clause*-CNF-SAT: CNF-SAT with parameter "number of *clauses*"

4.1 Fixed-Parameter Tractability

Exemplary Running Times of Parameterized Problems

- p-*variables*-SAT (consider simplest brute-force methods for problems)
  - k variables, n length of formula
  - $\rightsquigarrow$ $O(2^k \cdot n)$ running time

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  - k threshold (clique size); n vertices, m edges in graph
  - ⇝ $\binom{n}{k}$ candidates to check, each takes time $O(k^2)$ to check
  - ⇝ Total time $O(n^k \cdot k^2)$

$$\binom{n}{k} = \frac{\overbrace{n(n-1)(n-2)\cdots(n-k+1)}^{k}}{k!}$$

$$\sim \frac{n^k}{k!}$$

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- p-VertexCover
  - k threshold (VC size); n vertices, m edges in graph
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  - $\rightsquigarrow$ Total time $O(n^k \cdot m)$

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- p-GraphColoring
  - k threshold (#colors); n vertices, m edges in graph
  - $\rightsquigarrow$ k^n candidates to check, each takes time $O(m)$
  - $\rightsquigarrow$ Total time $O(\underline{k^n} \cdot m)$

$k = 1 \rightsquigarrow$ polynomial

$k = 3$ NP-hard

FPT Running Time

Definition 4.7 (fpt-algorithm)

Let κ be a parameterization for $\Sigma^\star$.
A (deterministic) algorithm A (with input alphabet Σ) is a *fixed-parameter tractable algorithm (fpt-algorithm)* w.r.t. κ if its running time on $x \in \Sigma^\star$ with $\kappa(x) = k$ is at most

only depends on k — polynomial

$$f(k) \cdot p(|x|) = O\big(f(k) \cdot |x|^c\big)$$

where p is a polynomial of degree c and f is an **arbitrary** computable function. ◀

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Definition 4.8 (FPT)

A parameterized problem (L, κ) is *fixed-parameter tractable* if there is an fpt-algorithm that decides it.
The complexity class of all such problems is denoted by FPT. ◀

Intuitively, FPT plays the role of P.

A First FPT Example

Theorem 4.9 (p-variables-SAT is FPT)

p-variables-SAT $\in$ FPT.

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Proof:

Suffices to use brute force satisfiability for *p-variables*-SAT

```
procedure bruteForceSat(φ, 𝒳 = {x_1, ..., x_k})
    if k == 0
        if φ == true return ∅ else UNSATISFIABLE
    for value in {true, false} do
        A := {x_1 ↦ value}
        ψ := φ[x_1/value] // Substitute value for x_1
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$f(k)$

Worst case running time: $O(2^k n)$ for $n = |\varphi|$.

2^k recursive calls;

base case needs time $O(|\phi|)$ to check whether formula evaluates to *true* ∎

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Aren't we all FPT?

Theorem 4.10 (k never decreases → FPT)

Let $g : \mathbb{N} \to \mathbb{N}$ weakly increasing, unbounded and computable, and κ a parameterization with

$$\forall x \in \Sigma^\star : \kappa(x) \geq g(|x|).$$

Then $(L, \kappa) \in \mathsf{FPT}$ for *any* decidable L.

$g(x) = \log \log |x|$
possible

g weakly increasing: $n \leq m \;\to\; g(n) \leq g(m)$
g unbounded: $\forall t\ \exists n\ :\ g(n) \geq t$

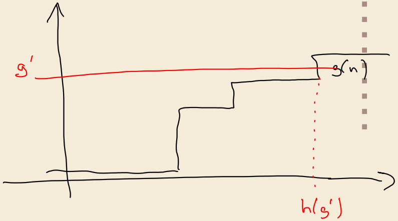
Proof: L decidable $\leadsto$ $\exists$ algorithm to decide L in time $\leq T(|x|)$

w.l.o.g. T weakly increasing

$T(|x|) \geq |x|$

Idea: "hide" $T(|x|)$ in $f(k)$

("g'") $h(g') = \max\{n' \in \mathbb{N} : g(n') \leq g'\} \cup \{1\}$



Aren't we all FPT? – Proof

Proof (cont.):

(1) g weakly incr. & unbounded $\Rightarrow$ h well-defined

(2) h weakly increasing

(3) g computable $\Rightarrow$ h computable

(4) $h(g(n)) \geq n$

time to decide whether $x \in \Sigma^*$ is in L

$n = |x|$

$k = \kappa(x) \geq g(n)$

$$\leq T(n) \underset{\substack{T \text{ incr.} \\ (4)}}{\leq} T(h(g(n))) \underset{T, h \text{ incr.}}{\leq} T(h(k)) =: f(k)$$

∎

Back to “sensible” parameters

$\rightsquigarrow$ always check if parameter is reasonable (can be expected to be small)

- if not, FPT might not even mean in NP!

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- but now, for some positive examples!

4.2 Depth-Bounded Exhaustive Search I

FPT Design Pattern

- The simplest FPT algorithms use exhaustive search
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- The simplest FPT algorithms use exhaustive search
- but with a search tree bounded by $f(k)$

- bruteforceSat was a typical example!
- does this work on other problems?

Depth-Bounded Search for Vertex Cover

Let's try p-VERTEXCOVER. brute force $\binom{n}{k} \cdot \mathrm{poly}(n) = \Theta(n^k \, \mathrm{poly}(n)) \neq$ fpt running time

Key insight: for every edge $\{v, w\}$, any vertex cover must contain v or w

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procedure simpleFptVertexCover($G = (V, E)$, k):
    if $E == \emptyset$ **then return** $\emptyset$
    if $k == 0$ **then return** `NOT_POSSIBLE` // *truncate search*
    Choose $\{v, w\} \in E$ (arbitrarily)
    for u in $\{v, w\}$ **do**:
        $G_u := \big(V \setminus \{u\}, E \setminus \{\{u, x\} \in E\}\big)$ // *Remove u from G*
        $C_u :=$ simpleFptVertexCover(G_u, $k - 1$)
    if $C_v ==$ `NOT_POSSIBLE` **then return** $C_w \cup \{w\}$
    if $C_w ==$ `NOT_POSSIBLE` **then return** $C_v \cup \{v\}$
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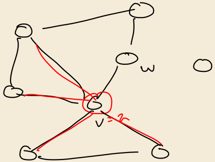

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- Does not need explicit checks of solution candidates!
- runs in time $O(2^k(n + m))$ $\rightsquigarrow$ fpt-algorithm for p-Vertex-Cover $\in$ FPT

Guessing the parameter

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- Running time: $\sum_{k'=0}^{k} O(2^{k'}(n+m)) = O(2^k(n+m))$

⇝ For exponentially growing cost, trying all values up to k costs only constant factor more

4.3 Problem Kernels

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- Trivial example for VertexCover

  $\boxed{\text{Remove vertices of degree } 0 \text{ or } 1.}$ (never needed as part of optimal VC)

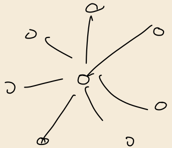
- Here: reduction rules that provably shrink an instance to size $g(k)$

Buss's Reduction Rule for VC

- Given a p-VertexCover instance (G, k)

> **"deg > k" Rule:** If G contains vertex v of degree $\deg(v) > k$, include v in potential solution and remove it from the graph.

- Can apply this simultaneously to degree $> k$ vertices.
- Either rule applies, or all vertices bounded degree(!)



Kernels

Definition 4.11 (Kernelization)

Let (L, κ) be a parameterized problem. A function $K : \Sigma^\star \to \Sigma^\star$ is *kernelization* of L w.r.t. κ if it maps any $x \in L$ to an instance $x' = K(x)$ with $k' = \kappa(x')$ so that

1. (self-reduction) $x \in L \iff x' \in L$
2. (polytime) K is computable in polytime.
3. (kernel-size) $|x'| \le g(k)$ for some computable function g

We call x' the *(problem) kernel* of x and g the *size of the problem kernel*. ◀

Buss's Kernel

Buss's Reduction for Vertex Cover: (repeatedly apply until no more changes)

- $\deg > k$ rule
- Remove degree 0 and 1 vertices

Theorem 4.12 (Buss's Reduction is Kernelization)

Buss' reduction yields a kernelization for p-VERTEX-COVER with kernel size $O(k^2)$. ◀

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Buss' rule for deg > k and 0/1 deg.

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After repeatedly applying Buss's rule as well as the isolated/leaf rule until neither applies further, we have $\forall v \in V : 2 \leq \deg(v) \leq k$.
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In the resulting graph, any VC of size $\le k$ covers $\le k^2$ edges.
If $m > k^2$, we output a trivial No-instance (e. g., a K_{k+1} a complete graph on $k+1$ vertices).
If $m \le k^2$, then the input size is now bounded by $g(k) = 2k^2$. ∎

FPT iff Kernelization

Theorem 4.13 (FPT $\leftrightarrow$ kernel)

A computable, parameterized problem (L, κ) is fixed-parameter tractable if and only if there is a kernelization for L w.r.t. κ.

Proof:

"$\Leftarrow$" kernelization K for (L, κ) given,

L has decider A of running time $T(n)$ (w.l.o.g. weakly increasing)

(1) $x \in \Sigma^*$ to check $x \in L$ $\quad k = \kappa(x) \quad n = |x|$

compute $K(x) = x'$ poly time

$|x'| \leq g(k)$

(2) run A on x'

time $T(|x'|) \underset{\text{incr.}}{\leq} T(g(k)) \Rightarrow f(k)$

$\Rightarrow$ algorithm for (L, κ) w/ time $O(f(k) + \mathrm{poly}(n))$

FPT iff Kernelization [2]

Proof (cont.):

"$\Rightarrow$" Given fpt-algorithm A for (L, x) with time $\leq f(k) \cdot n^c$

(1) Simulate A for $\leq n^{c+1}$ steps (poly time)

(2) • If A terminated

if output Yes : output trivial Yes-instance

if " No - - - No - "

• otherwise $n^{c+1} \leq f(k)\, n^c \Rightarrow n \leq f(k)$

$\Rightarrow$ output original input

Max-SAT Kernel

$k = \#$ clauses to satisfy

Theorem 4.14 (Kernel for Max-SAT)

p-MAX-SAT has a problem kernel of size $O(k^2)$ which can be constructed in linear time. ◀

Proof:

$$(x \vee y \vee \bar{z}) \wedge (x \vee y \vee \bar{z}) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee z)$$

$$\rightsquigarrow \{\{x, y, \bar{z}\}, \{x, y, \bar{z}\}, \{\bar{x}, z\}, \{\bar{y}, z\}\}$$

assumption: each variables occurs at most once per clause

($x \vee \bar{x} \rightsquigarrow$ delete clause)

$m = \#$clauses (n total input size # literals in all clauses

Case 1: $k \leq \lfloor \frac{m}{2} \rfloor$ (output yes)

pick arbitrary assignment A of all variables

under A, ℓ clauses are satisfied $\ell \geq k$ ✓

if $\ell < k \leq \lfloor \frac{m}{2} \rfloor$ $\rightarrow$ then $\bar{A}$ (inverse assignment) satisfies $m - \ell \geq \lfloor \frac{m}{2} \rfloor \geq k$21

Max-SAT Kernel [2]

Proof (cont.):

Case $k > \lfloor \frac{m}{2} \rfloor \Rightarrow k > \frac{m}{2} \Rightarrow \boxed{m < 2k}$

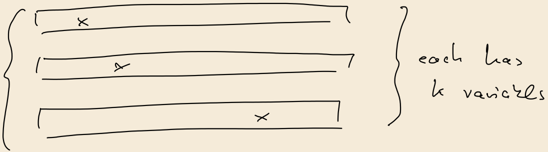
$\Rightarrow$ few clauses but they could be big

consider $F_L = \{ \text{clauses } C : C \text{ has} \geq k \text{ literals} \}$

$F_S = \{ \text{clauses } C : \text{——} < k \text{——} \}$

$|F_L| = \boxed{L \geq k}$

$\Rightarrow$ Yes instance



can pick unique variable per clause to satisfy it

$\boxed{L < k}$ consider $(F_S, k-L)$ if that is a Yes instance

$\Leftrightarrow (F, k)$ is a Yes instance

Max-SAT Kernel [3]

Proof (cont.): "$\Rightarrow$" If A satisfies $k-L$ clauses in F_S

each of the clauses contains a true literal $\leadsto$ $k-L$

$\leadsto$ A only "fixes" $k-L$ variables

$\Rightarrow$ for L long clauses can find L unique variables

that are not fixed in A

"$\Leftarrow$" trivial.

$\Rightarrow$ reduced problem to $(F_S, k-L)$

at most $m-L \leq m$ clauses each has $< k$ literals

$< 2k$

$\Rightarrow$ # literals $= O(k^2)$ (encoding $O(k^2 \log k)$)

Corollary 4.15

p-Max-SAT $\in$ FPT

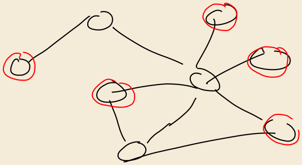
4.4 Depth-Bounded Search II: Planar Independent Set

Deeper results (towards more shallow trees)

- Our previous examples of depth-bounded search were basically brute force
- Here we will see two more examples that exploit the problem structure in more interesting ways

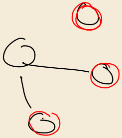
Independent Set on Planar Graphs

We will see
~~Recall~~: general problem p-INDEPENDENT-SET is $\mathcal{W}[1]$-hard.



Definition 4.16 (p-PLANAR-INDEPENDENT-SET)

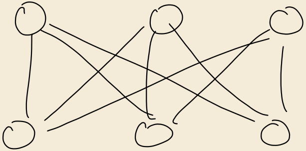
Given: a *planar* graph $G = (V, E)$ and $k \in \mathbb{N}$
Parameter: k
Question: $\exists V' \subset V \;:\; |V'| \geq k \;\wedge\; \forall u, v \in V' : \{u, v\} \notin E$?



planar graph G :

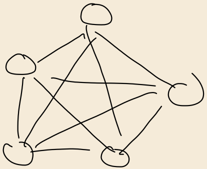
$\exists$ embedding (placement) of vertices in $\mathbb{R}^2$

and a drawing of edges without crossings



$K_{3,3}$ not planar

K_5 not planar



Independent Set on Planar Graphs

Recall: general problem p-Independent-Set is $\mathcal{W}[1]$-hard.

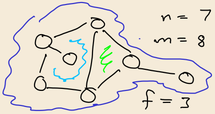
Definition 4.16 (p-Planar-Independent-Set)

Given: a *planar* graph $G = (V, E)$ and $k \in \mathbb{N}$
Parameter: k
Question: $\exists V' \subset V \quad : \quad |V'| \geq k \ \wedge \ \forall u, v \in V' : \{u, v\} \notin E$? ◀

Theorem 4.17 (Depth-Bounded Search for Planar Independent Set)

p-Planar-Independent-Set is in FPT and can be solved in time $O(6^k n)$. ◀

Elementary Knowledge on Planar Graphs



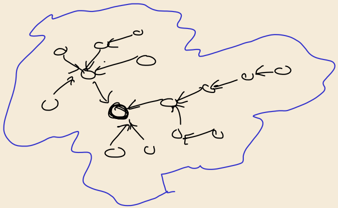
Theorem 4.18 (Euler's formula)

faces

In any finite, connected planar graph G with n nodes, m edges, f holds $n - m + f = 2$.

$7 - 8 + 3 = 2$

Proof idea: Induction on f



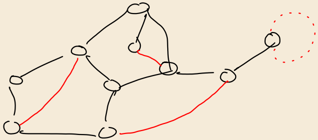
IB $f = 1$ $\Rightarrow$ G is a tree

$\Rightarrow$ $n = m + 1$

faces = regions of $\mathbb{R}^2$

IS "add a new face"

m++ f++



Elementary Knowledge on Planar Graphs

Theorem 4.18 (Euler's formula)

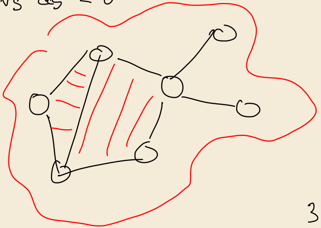
In any finite, connected planar graph G with n nodes, m edges f holds $n - m + f = 2$.

Corollary 4.19

A simple planar graph G on $n \geq 3$ nodes has $m \leq 3n - 6$ edges.
The average degree in G is < 6.

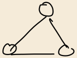
$\sum_v \deg(v) = 2m \leq 6n - 12$

avg deg < 6



is not simple

is not simple



simple $\Rightarrow$ every face is delimited by ≥ 3 edges

$3f$ double counts each edge at most twice

$3f \leq 2m$

$3f = 3(2 - n + m) = 6 - 3n + 3m \quad \Big| \; -2m \; +3n - 6$

$m \leq 3n - 6$

avg deg < 6 $\Rightarrow$ $\boxed{\text{in any planar graph, } \exists v : \deg(v) \leq 5}$

"degeneracy" $d = 5$

always find vertex of degree $\leq d$ in G

and in any induced subgraph

induced subgraph

$G = (V, E)$ $\quad G[V'] = (V', \{\{u,v\} : u,v \in V', \{u,v\} \in E\})$

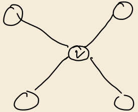
$V' \subseteq V$

Depth-Bounded Search for Planar Independent Set

```
procedure planarIndependentSet(G = (V, E), k):
    if k == 0 then return ∅
    if k > |V| then return NOT_POSSIBLE // truncate search
    Choose v ∈ V with minimal degree; let w_1, ..., w_d be v's neighbors
    // By planarity, we know d ≤ 5.
    for u in {v, w_1, ..., w_d} do
        D := {u} ∪ N(u)
        G_u := (V \ D, E \ {{x, y} ∈ E : x ∈ D}) // Delete u and its neighbors
        I_u := {u} ∪ planarIndependentSet(G_u, k − 1)
    return largest I_u or NOT_POSSIBLE if none exists
```

neighbors of u

$G_u = G[V \setminus D]$



any maximal indep. set

can't add more vertices to this set

(if none of v's neighbors in the set, could add v)

≤ 6 recursive calls

in w.c. recurse until $k=0$

$\Rightarrow 6^k$ recursive calls in total

each take $\Theta(n+m) = \Theta(n)$

$\Rightarrow$ total time $O(6^k \cdot n)$

Summary Planar Independent Set

- Note: IndependentSet is NP-hard on planar graphs even with vertex degrees at most 3
- planarIndependentSet will often be faster than $O(6^k n)$
- works unchanged in $O((d+1)^k n)$ time for any degeneracy-d graph

every (induced) subgraph has vertex of degree at most d

4.5 Depth-Bounded Search III: Closest String

Closest String

Definition 4.20 (p-Closest-String)

Given: S set of m strings $s_1, s_2, \ldots, s_m$ of length L over alphabet Σ and a $k \in \mathbb{N}$.
Parameter: k
Question: Is there a string s for which $d_H(s, s_i) \leq k$ holds for all $i = 1, \ldots, m$?

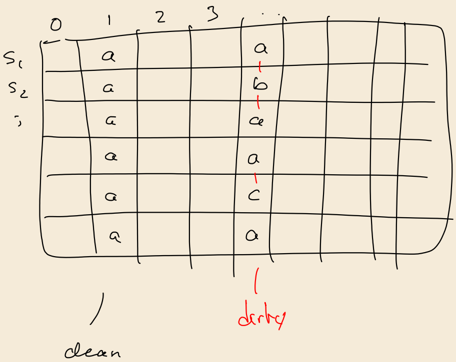
$d_H = 4$

\# mismatched positions

Dirty Columns

Definition 4.21 (Dirty Column)

A column of the $m \times L$ matrix corresponding to m strings of length L is called *dirty* if it contains at least 2 different symbols.



Dirty Columns

Definition 4.21 (Dirty Column)

A column of the $m \times L$ matrix corresponding to m strings of length L is called *dirty* if it contains at least 2 different symbols.

Lemma 4.22 (Many Dirty Columns → No)

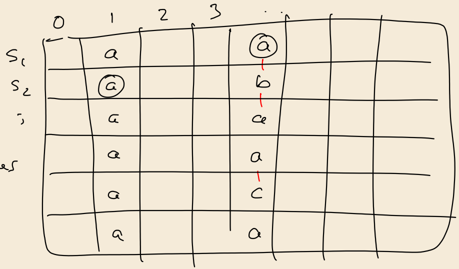
Let an instance to Closest-String with m strings of length L and parameter k be given. If the corresponding $m \times L$ matrix contains more than $m \cdot k$ dirty columns, then no solution for the given instance exists.

If we have $> m \cdot k$ dirty cols

no matter what s,

one s_i must have $\geq k+1$ mismatches

↯



Depth-Bounded Search for Closest String

procedure closestStringFpt(s, d):
    if $d < 0$ **then return** `NOT_POSSIBLE`
    if $d_H(s, s_i) > k + d$ **for an** $i \in \{1, \dots, m\}$ **then**
        return `NOT_POSSIBLE`
    if $d_H(s, s_i) \leq k$ **for all** $i = 1, \dots, m$ **then return** s
    Choose $i \in \{1, \dots, m\}$ arbitrarily with $d_H(s, s_i) > k$
        $P := \{p : s[p] \neq s_i[p]\}$
        Choose arbitrary $P' \subseteq P$ with $|P'| = k + 1$
        for p in P' **do**
            $s' := s$
            $s'[p] := s_i[p]$
            s_{ret} := closestStringFpt($s', d - 1$)
            if $s_{ret} \neq$ `NOT_POSSIBLE` **then return** s_{ret}
    return `NOT_POSSIBLE`

next slide

- initial call closestStringFpt(s_1, k)

search space $(k+1)^k = O(k^k)$

$$\lim_{k\to\infty} \frac{(k+1)^k}{k^k} = \lim \left(\frac{k+1}{k}\right)^k = \lim \left(1 + \frac{1}{k}\right)^k = e = O(1)$$

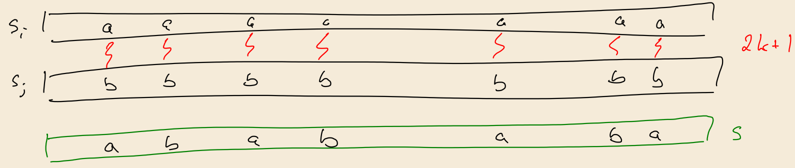
$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$

Too Much Dirt

Lemma 4.23 (Pair Too Different → No)

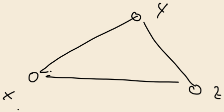
Let $S = \{s_1, s_2, \ldots, s_m\}$ a set of strings and $k \in \mathbb{N}$. If there are $i, j \in \{1, \ldots, m\}$ with $d_H(s_i, s_j) > 2k$, then there is no string s with $\max_{1 \leq i \leq m} d_H(s, s_i) \leq k$. ◀



has distance $\geq k+1$ to either s_i or s_j

d_H is a metric

- $d_H(x,y) \geq 0$
- $d_H(x,x) = 0$
- "Δ-ineq." $\forall x,y,z: d_H(x,z) \leq d_H(x,y) + d_H(y,z)$



Depth-Bounded Search for Closest String

Theorem 4.24 (Search Tree for Closest String)

There is a search tree of size $O(k^k)$ for problem p-CLOSEST-STRING. ◀

Depth-Bounded Search for Closest String

Theorem 4.24 (Search Tree for Closest String)

There is a search tree of size $O(k^k)$ for problem p-CLOSEST-STRING. ◀

Corollary 4.25 (Closest String is FPT)

p-CLOSEST-STRING can be solved in time $O(mL + mk \cdot k^k)$. ◀

maybe can get down to $m \cdot k^k$

- preprocessing ($O(mL)$ time)
  - ignore any clean columns
  - reject if more than mk dirty columns

⇝ effective string length after preprocessing is $L' \leq mk$

- call closestStringFpt(s_1, k)
  - maintain $d_H(s, s_i)$ in an array
    ⇝ checking any distance $d_H(s, s_i)$ takes $O(1)$ time
    - before and after recursive call, update array to reflect $d_H(s', s_i)$
      Single character changed, so update only needs to check single position
    ⇝ Can maintain distances in $O(m)$ time per recursive call
  - P' can be computed in $O(mk)$ time

4.6 Linear Recurrences & Better Vertex Cover

A Better Algorithm for Vertex Cover

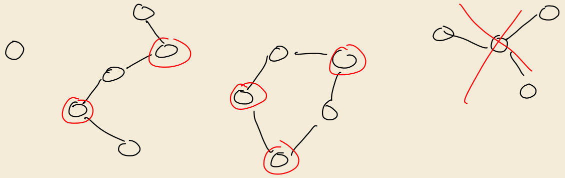
Recall: Branching on endpoints of k edges gives search space of size 2^k for Vertex-Cover.
Can we do better?

A Better Algorithm for Vertex Cover

Recall: Branching on endpoints of k edges gives search space of size 2^k for VERTEX-COVER.
Can we do better?

Idea: Enlarge base case with "easy inputs"

Here: Consider graphs G with $\deg(v) \leq 2$ for all $v \in V(G)$.

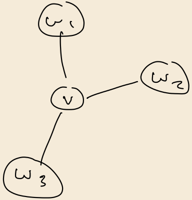


$\Rightarrow$ otherwise $\deg(v) \notin \{0, 1, k+1, k+2 \ldots\}$

$+ \; \exists u: \deg(u) \geq 3$

Depth-Bounded Search for Vertex Cover

procedure betterFptVertexCover($G = (V, E), k$):
    if $E = \emptyset$ **then return** $\emptyset$
    if $k = 0$ **then return** `NOT_POSSIBLE` *// truncate search*
    if all node have degree ≤ 2 **then**
        Find connected components of G
        for each component G_i **do**
            Fill C_i by picking every other node,
            starting with the neighbor of a degree-one node if one exists
        $C := \bigcup C_i$
        if $|C| \leq k$ **then return** C **else return** `NOT_POSSIBLE`
    Choose v with maximal degree, let $w_1, \ldots, w_d$ be its neighbors // $d \geq 3$
    For D in $\{\{v\}, \{w_1, \ldots, w_d\}\}$ **do**:
        $G_D := \big(V \setminus D, E \setminus \{\{x, y\} \in E : x \in D\}\big)$ *// Remove D from G*
        $C_D := D \cup$ betterFptVertexCover($G_u, k - |D|$)
    return smallest C_D or `NOT_POSSIBLE` **if** none exists



recurse on (w_i) $k-1$

$k-3$

Depth-Bounded Search for Vertex Cover

procedure betterFptVertexCover($G = (V, E), k$):
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  Choose v with maximal degree, let $w_1, \ldots, w_d$ be its neighbors // $d \ge 3$
  For D in $\{\{v\}, \{w_1, \ldots, w_d\}\}$ **do**:
    $G_D := \big(V \setminus D, E \setminus \{\{x, y\} \in E : x \in D\}\big)$ *// Remove D from G*
    $C_D := D \cup$ betterFptVertexCover($G_u, k - |D|$)
  return smallest C_D or `NOT_POSSIBLE` **if** none exists

How to analyze running time of betterFptVertexCover?

Analysis of betterFptVertexCover

worst case running time

- never have all degrees ≤ 2
- always need both recursive calls (until base case)
- ignore that graph gets smaller

$$T_0 = \Theta(1)$$
$$T_k = \Theta(\underset{\substack{\shortparallel \\ n}}{|V|} + \underset{\substack{\shortparallel \\ m}}{|E|}) + T_{k-3} + T_{k-1}$$

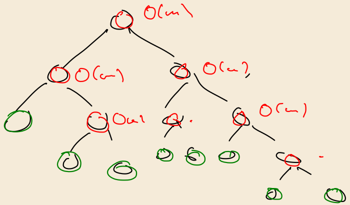
previously (simpleFptVertexCover)

$$T_0 = \Theta(1)$$
$$T_k = 2 \cdot T_{k-1} + \Theta(n+m)$$

Analysis of betterFptVertexCover

worst case running time

- never have all degrees ≤ 2
- always need both recursive calls (until base case)
- ignore that graph gets smaller



$$T_0 = \Theta(1)$$
$$T_k = \Theta(|V| + |E|) + T_{k-3} + T_{k-1}$$

$T_k = \Theta(n+m) \cdot \#\text{non-base cases}$

$= \Theta(n+m)\,(\#\text{base cases} - 1)$

$\approx B_k$

If we only number of base cases B_n, we obtain $T_n = O(B_n n^2)$

$\#$ non-base case calls $\leq$ $\#$ base cases

$$B_0 = 1,\ B_1 = 1,\ B_2 = 1$$
$$B_k = B_{k-3} + B_{k-1} \qquad (k \geq 3)$$

$B_k = \Theta(1.46..^k)$

Solving Linear Recurrences

$B_2 = B_1 = B_0 = 1$

$B_k = B_{k-3} + B_{k-1} \quad (k \geq 3)$

$B(z) = \sum_{k=0}^{\infty} B_k z^k$ ordinary generation function of sequence $(B_k)_{k\geq 0}$

$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ — GF of $1, 1, 1, \ldots$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$[z^k]\, B(z) := B_k$

$\underset{a=0}{=} \frac{B^{(k)}(0)}{k!}$

(not normally convenient to use)

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

approximate $f(x)$ by polynomial for x close to a

$B(z) \overset{a=0}{=} B(0) + z \cdot B'(0) + z^2 \frac{B''(0)}{2} + z^3 \frac{B'''(0)}{3!} + \cdots$

$= \sum_{k=0}^{\infty} \frac{B^{(k)}(0)}{k!} z^k$

$B(z) = \sum_{k\geq 0} B_k \cdot z^k = 1 \cdot z^0 + 1 \cdot z^1 + 1 \cdot z^2 + \sum_{k\geq 3} B_k \cdot z^k$

$B_k = B_{k-3} + B_{k-1}$

$$= 1 + z + z^2 + \sum_{k \geq 3} B_{k-3} z^k + \sum_{k \geq 3} B_{k-1} z^k$$

$$= \text{— '' —} + z^3 \sum_{k \geq 3} B_{k-3} z^{k-3} + z \sum_{k \geq 3} B_{k-1} z^{k-1}$$

$$= \text{— '' —} + z^3 \underbrace{\sum_{k \geq 0} B_k z^k}_{B(z)} + z \left[\underbrace{\sum_{k \geq 2} B_k z^k + (B_1 z^1 + B_0 z^0}_{B(z)} - \underbrace{\overset{1}{\overset{\shortparallel}{B_1}} z^1 - \overset{1}{\overset{\shortparallel}{B_0}} z^0)}_{0} \right]$$

$$B(z) = 1 + z + z^2 + z^3 B(z) + z B(z) - z^2 - z$$

$$= 1 + z^3 B(z) + z B(z)$$

$$B(z) \left(1 - z^3 - z\right) = 1 \qquad B(z) = \frac{1}{1 - z^3 - z}$$

$$B(z) = \frac{1}{1 - z^3 - z} = \frac{A}{z - z_0} + \frac{B}{z - z_1} + \frac{C}{z - z_2}$$ (all roots are distinct)

roots of $\uparrow$ z_0, z_1, z_2

Input interpretation

roots	$1 - z^3 - z = 0$

Results

$z \approx 0.68233$

$z \approx -0.34116 - 1.16154\,i$

$z \approx -0.34116 + 1.16154\,i$

$$1 - z^3 - z = (z - z_0)(z - z_1)(z - z_2)$$

$$\frac{1}{0.6823} \approx 1.4655$$

$$\frac{A}{z - z_0}$$

$$\frac{1}{1 - cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$c = \frac{1}{z_0}$$

$$= \frac{A/z_0}{\frac{z}{z_0} - 1} = \frac{-A/z_0}{1 - \frac{z}{z_0}}$$

$$[z^k]\,\frac{A}{z - z_0} = -\frac{A}{z_0}\,[z^k]\,\frac{1}{1 - \frac{z}{z_0}}$$

$$= -\frac{A}{z_0} \cdot \left(\frac{1}{z_0}\right)^k$$

$$B_k = [z^k]\,B(z) = -\frac{A}{z_0}\left(\frac{1}{z_0}\right)^k - \frac{B}{z_1}\left(\frac{1}{z_1}\right)^k - \frac{C}{z_2}\left(\frac{1}{z_2}\right)^k = \Theta\left(\left|\frac{1}{z_0}\right|^k\right)$$

assuming $|z_0| < |z_1| < |z_2|$

Solving Linear Recurrences – Result

Theorem 4.26 (Linear Recurrences)

Let $d_1, \ldots, d_i \in \mathbb{N}$ and $d = \max d_j$.
The solution to the *homogeneous linear recurrence equation*

$$T_n \quad = \quad T_{n-d_1} + T_{n-d_2} + \cdots + T_{n-d_i}, \qquad (n \geq d)$$

is always given by

$$T_n \quad = \quad \sum_{\ell} \sum_{j=0}^{\mu_\ell - 1} c_{\ell,j}\, z_\ell^n\, n^j$$

where we sum over all roots z_ℓ of multiplicity μ_ℓ of the so-called *characteristic polynomial* $z^d - z^{d-d_1} - z^{d-d_2} \ldots - z^{d-d_i}$.
The d coefficients $c_{\ell,j}$ are determined by the d initial values $T_0, T_1, \ldots, T_{d-1}$. ◀

Corollary 4.27

$T_n = O(z_0^n n^d)$ for z_0 the root of the characteristic polynomial with *largest absolute value*. ◀

Analysis of betterFptVertexCover [2]

$T_0 = \Theta(1)$
$T_k = \Theta(|V| + |E|) + T_{k-3} + T_{k-1}$

If we only number of base cases B_n, we obtain $T_n = O(B_n n^2)$

$B_0 = 1,\ B_1 = 1,\ B_2 = 1$
$B_k = B_{k-3} + B_{k-1} \qquad (k \geq 3)$

Analysis of betterFptVertexCover [2]

$T_0 = \Theta(1)$
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If we only number of base cases B_n, we obtain $T_n = O(B_n n^2)$

$B_0 = 1,\ B_1 = 1,\ B_2 = 1$
$B_k = B_{k-3} + B_{k-1} \qquad (k \geq 3)$

$\rightsquigarrow$ $\vec{d} = (1, 3)$; characteristic polynomial $z^3 - z^2 - 1$
roots at $z_0 \approx 1.4656$ and $z_{1,2} \approx -0.2328 \pm 0.7926i$

Analysis of betterFptVertexCover [2]

$T_0 = \Theta(1)$
$T_k = \Theta(|V| + |E|) + T_{k-3} + T_{k-1}$

If we only number of base cases B_n, we obtain $T_n = O(B_n n^2)$

$B_0 = 1, \;\; B_1 = 1, \;\; B_2 = 1$
$B_k = B_{k-3} + B_{k-1} \qquad (k \ge 3)$

$\rightsquigarrow$ $\vec{d} = (1, 3)$; characteristic polynomial $z^3 - z^2 - 1$
roots at $z_0 \approx 1.4656$ and $z_{1,2} \approx -0.2328 \pm 0.7926i$

Theorem 4.28 (Depth-Bounded Search for Vertex Cover)
p-Vertex-Cover can be solved in time $O(1.4656^k n)$. ◀