3 Fundamental Data Structures

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CS566 (Wintersemester 2024/25)
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Learning Outcomes

Unit 3: ***Fundamental Data Structures***

1. Understand and demonstrate the difference between *abstract data type (ADT)* and its *implementation*
2. Be able to define the ADTs *stack*, *queue*, *priority queue* and *dictionary / symbol table*
3. Understand *array*-based implementations of stack and queue
4. Understand *linked lists* and the corresponding implementations of stack and queue
5. Know *binary heaps* and their performance characteristics
6. Understand *binary search trees* and their performance characteristics
7. Know high-level idea of basic *hashing strategies* and their performance characteristics

Outline

3 Fundamental Data Structures

Clicker Question

What's the running time (on our word-RAM model with word size w) of this Java instruction?

```
Object[] A = new Object[n];
```

A	1	D	$\Theta(w)$	G	$\Theta(n \log n)$
B	$\Theta(1)$	E	$\Theta(n/w)$	H	$\Theta(nw)$
C	$\Theta(\log n)$	F	$\Theta(n)$	I	$\Theta(n^2)$

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What's the running time (on our word-RAM model with word size w) of this Java instruction?

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Object[] A = new Object[n];
```

initialization w/ null

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B	~~$\Theta(1)$~~	E	$\Theta(n/w)$ ✓	H	~~$\Theta(nw)$~~
C	~~$\Theta(\log n)$~~	F	$\Theta(n)$ ✓	I	~~$\Theta(n^2)$~~

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Recap: The Random Access Machine

- Data structures make heavy use of pointers and dynamically allocated memory.

- Recall: Our RAM model supports
  - basic pseudocode ($\approx$ simple Python/Java code)
  - creating arrays of a fixed/known size.
  - creating instances (objects) of a known class.

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Python abstracts this away!

no predefined capacity!

There are *no* *arrays* in Python, only its built-in *lists*.

But: Python *implementations create* lists based on fixed-size arrays (stay tuned!)

Python ≠ RAM: Not every built-in Python instruction runs in $O(1)$ time!

3.1 Stacks & Queues

Abstract Data Types

abstract data type (ADT)

- list of supported operations
- **what** should happen
- **not:** how to do it
- **not:** how to store data

≈ Java `interface`, Python `ABC`s (with comments)

abstract base classes

VS.

data structures

- specify exactly **how** data is represented
- **algorithms** for operations
- has concrete costs (space and running time)

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Why separate?

- Can swap out implementations ⇝ "drop-in replacements"

⇝ **reusable code**!

- (Often) better abstractions
- Prove generic lower bounds (⇝ Unit 4)

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Which of the following are examples of abstract data types?

- A ADT
- B Stack
- C Deque
- D Linked list
- E binary search tree
- F Queue
- G resizable array
- H heap
- I priority queue
- J dictionary/symbol table
- K hash table

→ *sli.do/cs566*

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- A ~~ADT~~
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→ *sli.do/cs566*

Stacks

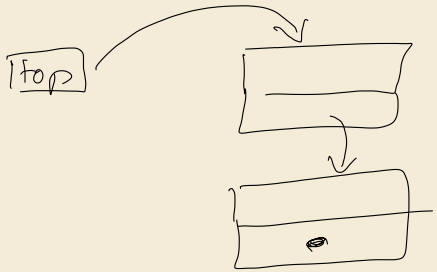
Stack ADT

- `top()`
  Return the topmost item on the stack
  Does not modify the stack.
- `push(`x`)`
  Add x onto the top of the stack.
- `pop()`
  Remove the topmost item from the stack (and return it).
- `isEmpty()`
  Returns `true` iff stack is empty.
- `create()`
  Create and return an new empty stack.

Linked-list implementation for Stack

Invariants:

- maintain pointer *top* to topmost element
- each element points to the element below it (or `null` if bottommost)




```
1  class Node
2      value
3      next
4
5  class Stack
6      top := null
7      procedure top()
8          return top.value
9      procedure push(x)
10         top := new Node(x, top)
11     procedure pop()
12         t := top()
13         top := top.next
14         return t
```

Linked-list implementation for Stack – Discussion

Linked stacks:

- require $\Theta(n)$ space when n elements on stack
- All operations take $O(1)$ time

- require $\Theta(n)$ space when n elements on stack

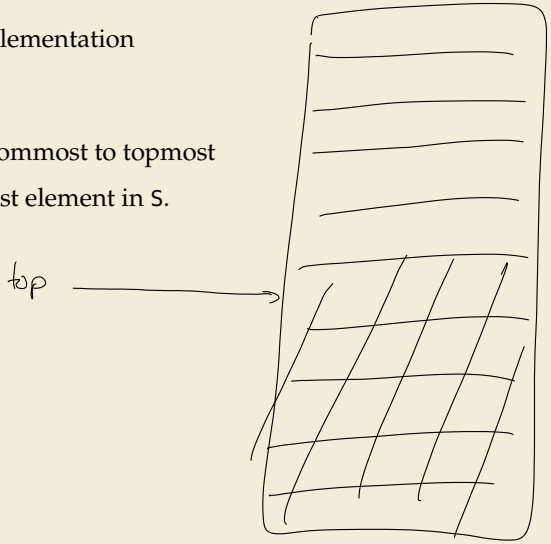
Can we avoid extra space for pointers?

Array-based implementation for Stack

If we want no pointers $\rightsquigarrow$ array-based implementation

Invariants:

- maintain array S of elements, from bottommost to topmost
- maintain index *top* of position of topmost element in `S`.



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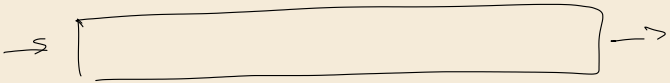
What to do if stack is full upon `push`?

Array stacks:

- require *fixed capacity* C (decided at creation time)!
- require $\Theta(C)$ space for a capacity of C elements
- all operations take $O(1)$ time

Queues



Operations:

- `enqueue(`x`)`
  Add x at the end of the queue.
- `dequeue()`
  Remove item at the front of the queue and return it.

Implementations similar to stacks.

Bags

What do Stack and Queue have in common?

Bags

What do Stack and Queue have in common?

They are special cases of a ***Bag***!

Operations:

- `insert(`x`)`
  Add x to the items in the bag.
- `delAny()`
  Remove any one item from the bag and return it.
  (Not specified which; any choice is fine.)
- roughly similar to Java's `java.util.Collection`
  Python's `collections.abc.Collection`

Sometimes it is useful to state that order is irrelevant $\rightsquigarrow$ Bag
Implementation of Bag usually just a Stack or a Queue

3.2 Resizable Arrays

Digression – Arrays as ADT

Arrays can also be seen as an ADT!

Array operations:

- `create(`n`)` *Java*: `A = new int[`n`];` *Python:* `A = [0] *` n
  Create a new array with n cells, with positions $0, 1, \ldots, n-1$;
  we write $A[0..n) = A[0..n-1]$
- `get(`i`)` *Java/Python*: `A[`i`]`
  Return the content of cell i
- `set(`i`,`x`)` *Java/Python*: `A[`i`] =` x`;`
  Set the content of cell i to x.

⇝ Arrays have *fixed* size (supplied at creation). (≠ `lists` in Python)

Digression – Arrays as ADT

Arrays can also be seen as an ADT! . . . but are commonly seen as specific data structure

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⇝ Arrays have *fixed* size (supplied at creation). (≠ `lists` in Python)

Usually directly implemented by compiler + operating system / virtual machine.

Difference to "real" ADTs: ***Implementation usually fixed***
to "a contiguous chunk of memory".

Doubling trick

Can we have unbounded stacks based on arrays? Yes!

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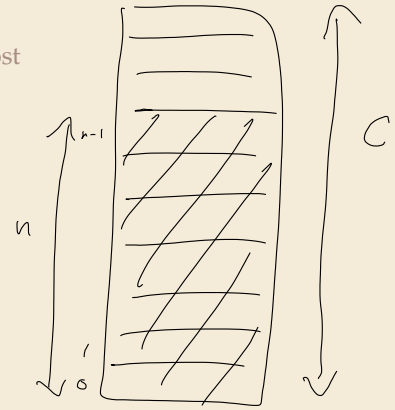
Invariants:

- maintain array S of elements, from bottommost to topmost
- maintain index *top* of position of topmost element in S
- maintain capacity $C = S.length$ so that $\frac{1}{4}C \leq n \leq C$

⇝ can always push more elements!



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for n elements we use $\Theta(n)$ space

How to maintain the last invariant?

- before `push`
  If $n = C$, allocate new array of size $2n$, copy all elements.
- after `pop`
  If $n < \frac{1}{4}C$, allocate new array of size $2n$, copy all elements.

⇝ ***"Resizing Arrays"***

an implementation technique, not an ADT!

Clicker Question

Which of the following statements about resizable array that currently stores n elements is correct?

(always)

- **A** The elements are stored in an array of size $2n$.
- **B** Adding or deleting an element at the end takes constant time.
- **C** A sequence of m insertions or deletions at the end of the array takes time $O(n + m)$.
- **D** Inserting and deleting any element takes $O(1)$ amortized time.

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Amortized Analysis

- Any individual operation `push` / `pop` can be expensive!
  $\Theta(n)$ time to copy all elements to new array.
- **But:** An one expensive operation of cost T means $\Omega(T)$ next operations are cheap!

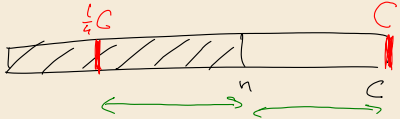
Amortized analysis w/ potential function

c_i = actual cost of operation i

Φ_i = potential after operation i

a_i = amortized cost of operation i

$:= c_i - 4(\Phi_i - \Phi_{i-1})$



Φ_i = distance to dangerous boundary = $\min\{C-n, n-\frac{1}{4}C\}$

Amortized Analysis

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distance to boundary — since $n \le C \le 4n$

Formally: consider "credits/potential" $\Phi = \min\{n - \frac{1}{4}C, C - n\} \in [0, 0.6n]$

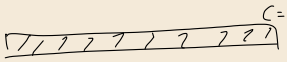
- amortized cost of an operation = actual cost (array accesses) $- 4 \cdot$ change in Φ
  - cheap `push`/`pop`: actual cost 1 array access, consumes ≤ 1 credits $\rightsquigarrow$ amortized cost ≤ 5
  - copying `push`: actual cost $2n+1$ array accesses, creates $\frac{1}{2}n+1$ credits $\rightsquigarrow$ amortized cost ≤ 5
  - copying `pop`: actual cost $2n+1$ array accesses, creates $\frac{1}{2}n-1$ credits $\rightsquigarrow$ amortized cost 5

$\rightsquigarrow$ **sequence** of m operations: total actual cost $\le$ total amortized cost + final credits

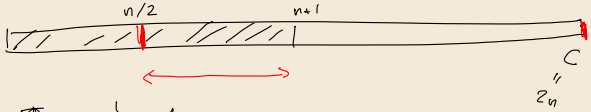
here: $\le \quad 5m \quad + \quad 4 \cdot 0.6n \quad = \Theta(m+n)$

copying push:

before



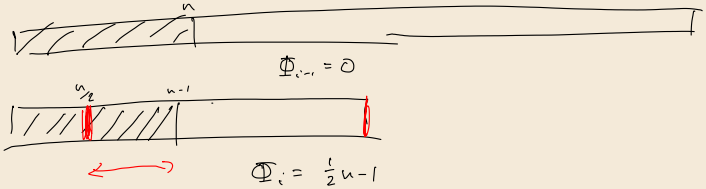
after



$$\Phi_i = \frac{1}{2}n + 1$$

$$a_i = 2n+1 - 4\left(\frac{1}{2}n + 1\right) = 1 - 4 = -3 \leq 5$$

copying pop



$$a_i = 2n+1 - 4\left(\frac{1}{2}n - 1\right) = 5$$

$$S_m \geq \sum_{i=1}^{m} a_i = \sum_{i=1}^{m} c_i - 4\left(\Phi_i - \Phi_{i-1}\right)$$

$$= \sum_{i=1}^{m} c_i - 4 \sum_{i=1}^{m} \left(\Phi_i - \Phi_{i-1}\right)$$

telescoping sum

$$\sum_{i=1}^{m} \Phi_i - \Phi_{i-1}$$

$$= \quad " \quad - 4\left(\Phi_m - \Phi_0\right)$$

$$= (\Phi_4 - \Phi_3) + (\Phi_3 - \Phi_2) + (\Phi_2 - \Phi_1) + (\Phi_1 - \Phi_0)$$

$$\Leftrightarrow \quad \sum_{i=1}^{m} c_i \leq S_m + 4\left(\Phi_m - \Phi_0\right)$$

$$\leq S_m + 4 \cdot 0.6n - 0$$

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Which of the following statements about resizable array that currently stores n elements is correct?

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C A sequence of m insertions or deletions at the end of the array takes time $O(n+m)$. ✓ => "O(1) amortized per operation"

D ~~Inserting and deleting any element takes $O(1)$ amortized time.~~

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Deamortized Resizable Arrays

*What if we need $O(1)$ **worst case** time?*

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- It's possible to *de-amortize* the resizing arrays solution!
- maintain **3 arrays**: S (as before) and S_2 and $S_{1/2}$
  of twice and half the size of S

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- maintain **3 arrays**: S (as before) and S_2 and $S_{1/2}$
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- write operations go to all 3 arrays
- upon resize, "shift" arrays up/down $\rightsquigarrow$ S_2 resp. $S_{1/2}$ become new S
  - allocate new array, but **delay filling it with elements** ← general strategy!
  - every insert or delete copies 2 slots from last resize

$\rightsquigarrow$ by time for next resize, we have caught up and S_2 resp. $S_{1/2}$ ready to use

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Analysis:

- $O(1)$ worst case time for read/write by index, push, and pop! ← assuming memory allocation in $O(1)$ $\rightsquigarrow$ needs to be uninitialized!
- up to 7 array accesses per operation
- up to $7n$ space ← other time-space trade-offs possible

Rabbit Hole: Can we do this more space-efficiently?

- It might appear as if every efficient implementation of a stack needs $\Omega(n)$ extra space on top of space for storing the n elements in the stack.

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- It might appear as if every efficient implementation of a stack needs $\Omega(n)$ extra space on top of space for storing the n elements in the stack.
- But this is not true!
- Can get operations in $O(1)$ worst-case time with $O(\sqrt{n})$ extra space at any time (!)
  - Maintain a collection of small arrays (plus header with pointers to them)
  - Clever choice of block sizes guarantees
    $O(\sqrt{n})$ blocks of $O(\sqrt{n})$ elements throughout
    and fast calculation of address for an index.

    imaginary "superblocks" of sizes 2^k, $k = 0, 1, \ldots, \lg n$
    kth superblock consists of $2^{k/2}$ actual blocks of $2^{k/2}$ elements each.
  - $O(\sqrt{n})$ extra space is best possible

∉ exam

Resizable Arrays in Optimal Time and Space
Andrej Brodnik, Svante Carlsson, Erik D. Demaine, J. Ian Munro & Robert Sedgewick
WADS 1999

3.3 Priority Queues & Binary Heaps

Clicker Question

What is a heap-ordered tree?

A. A tree in which every node has exactly 2 children.

B. A tree where all keys in the left subtree are smaller than the key at the root and all keys in the right subtree are bigger than the key at the root.

C. A tree where all keys in the left subtree and right subtree are smaller than the key at the root.

D. An tree that is stored in the heap-area of the memory.

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Priority Queue ADT

Now: elements in the bag have different *priorities*.

(Max-oriented) Priority Queue (MaxPQ):

- `construct(`A`)`
  Construct from from elements in array A.
- `insert(`x`,`p`)`
  Insert item x with priority p into PQ.
- `max()`
  Return item with largest priority. (Does not modify the PQ.)
- `delMax()`
  Remove the item with largest priority and return it.
- `changeKey(`x`,`p'`)`
  Update x's priority to p'.
  Sometimes restricted to *increasing* priority.
- `isEmpty()`

Fundamental building block in many applications.

Priority Queue ADT – min-oriented version

Now: elements in the bag have different *priorities*.

(~~Max~~-oriented) Priority Queue (~~Max~~PQ):

- `construct(`A`)`
  Construct from from elements in array A.
- `insert(`x`,`p`)`
  Insert item x with priority p into PQ.
- `min`~~`max`~~`()`
  Return item with smallest ~~largest~~ priority. (Does not modify the PQ.)
- `del`~~`Max`~~`()`
  Remove the item with smallest ~~largest~~ priority and return it.
- `changeKey(`x`,`p'`)`
  Update x's priority to p'.
  Sometimes restricted to ~~in~~*de*~~*creasing*~~ priority.
- `isEmpty()`

Fundamental building block in many applications.

PQ implementations

Elementary implementations

- unordered list $\rightsquigarrow$ $\Theta(1)$ `insert`, but $\Theta(n)$ `delMax`
- sorted list $\rightsquigarrow$ $\Theta(1)$ `delMax`, but $\Theta(n)$ `insert`

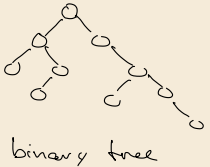
PQ implementations

Elementary implementations

- unordered list $\rightsquigarrow$ $\Theta(1)$ `insert`, but $\Theta(n)$ `delMax`
- sorted list $\rightsquigarrow$ $\Theta(1)$ `delMax`, but $\Theta(n)$ `insert` (both linked & array)

Can we get something between these extremes? Like a "slightly sorted" list?

PQ implementations



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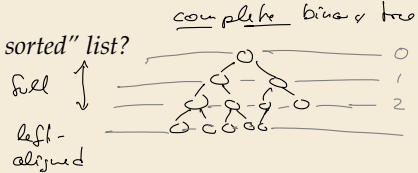
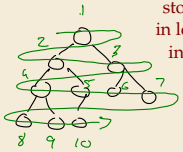


Yes! *Binary heaps.*

Array view

Heap = array A with
$\forall i \in [n] : A[\lfloor i/2 \rfloor] \geq A[i]$



store nodes in level order in $A[1..n]$

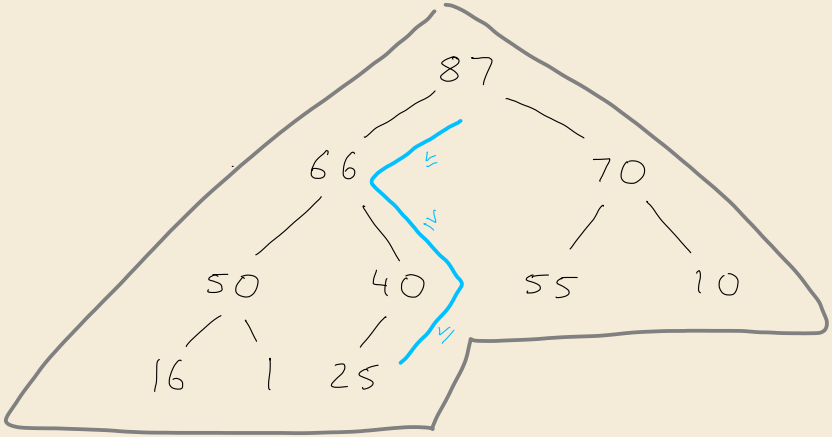
Tree view

Heap = tree that is
(i) a complete binary tree
(ii) heap ordered

all but last level full
last level flush left

parent ≥ children

Binary heap example



Why heap-shaped trees?

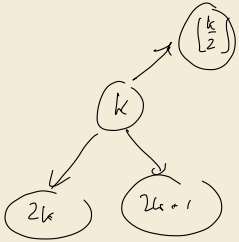
Why complete binary tree shape?

- only one possible tree shape $\rightsquigarrow$ keep it simple!
- complete binary trees have minimal height among all binary trees
- simple formulas for moving from a node to parent or children:

  For a node at index k in A

  - parent at $\lfloor k/2 \rfloor$ (for $k \geq 2$)
  - left child at $2k$
  - right child at $2k + 1$

⚠ Recall: nodes at indices $[1..n]$



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Why complete binary tree shape?

- only one possible tree shape $\rightsquigarrow$ keep it simple!
- complete binary trees have minimal height among all binary trees
- simple formulas for moving from a node to parent or children:
  For a node at index k in A ⚠ Recall: nodes at indices $[1..n]$
  - parent at $\lfloor k/2 \rfloor$ (for $k \geq 2$)
  - left child at $2k$
  - right child at $2k + 1$

Why heap ordered?

- Maximum must be at root! $\rightsquigarrow$ `max()` is trivial!
- But: Sorted only along paths of the tree; leaves lots of leeway for fast inserts

how? . . . stay tuned

Clicker Question

What is a heap-ordered tree?

A ~~A tree in which every node has exactly 2 children.~~

B ~~A tree where all keys in the left subtree are smaller than the key at the root and all keys in the right subtree are bigger than the key at the root.~~

C A tree where all keys in the left subtree and right subtree are smaller than the key at the root. ✓

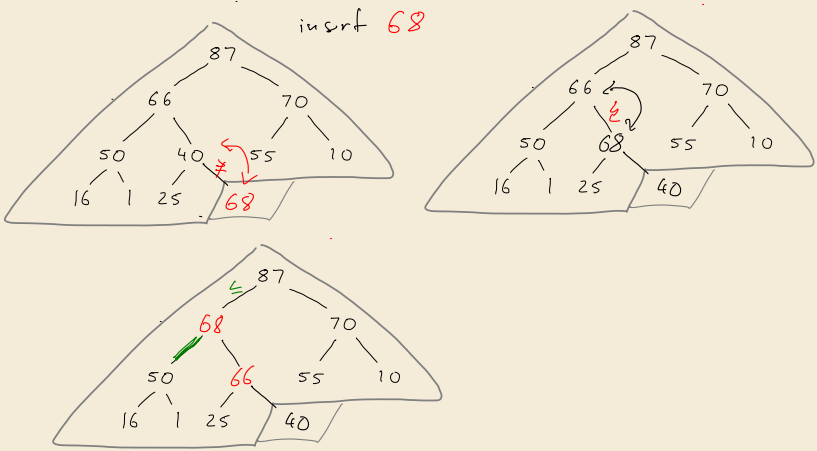
D ~~An tree that is stored in the heap area of the memory.~~

→ *sli.do/cs566*

3.4 Operations on Binary Heaps

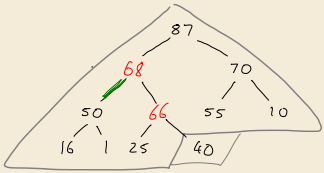
Insert

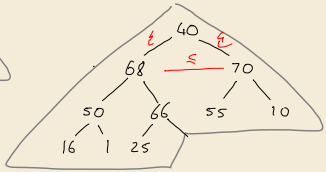
1. Add new element at only possible place: bottom-most level, next free spot.
2. Let element *swim* up to repair heap order.

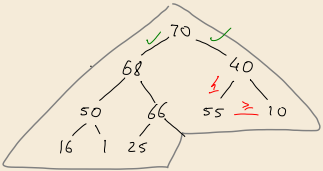


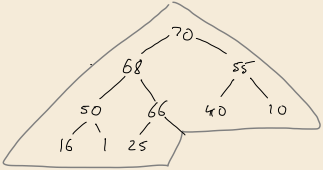
Delete Max

1. Remove max (must be in root).
2. Move last element (bottom-most, rightmost) into root.
3. Let root key *sink* in heap to repair heap order.



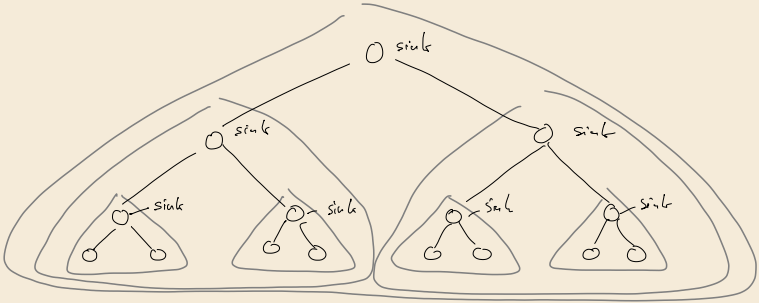






Heap construction

- n times insert $\rightsquigarrow$ $\Theta(n \log n)$ 
- instead:
  1. Start with singleton heaps (one element)
  2. Repeatedly merge two heaps of height k with new element into heap of height $k+1$



Analysis

Height of binary heaps:

- *height* of a tree: # edges on longest root-to-leaf path
- *depth/level* of a node: # edges from root $\rightsquigarrow$ root has depth 0
- How many nodes on first k *full* levels? $\sum_{\ell=0}^{k} 2^\ell = 2^{k+1} - 1$

$\rightsquigarrow$ Height of binary heap: $h = \min k \text{ s.t. } 2^{k+1} - 1 \geq n = \lfloor \lg(n) \rfloor$

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Analysis:

- `insert`: new element "swims" up $\rightsquigarrow$ $\leq h$ steps (h cmps)
- `delMax`: last element "sinks" down $\rightsquigarrow$ $\leq h$ steps ($2h$ cmps)
- `construct` from n elements:

  cost = cost of letting *each node* in heap sink!

$$\leq 1 \cdot h + 2 \cdot (h-1) + 4 \cdot (h-2) + \cdots + 2^\ell \cdot (h-\ell) + \cdots + 2^{h-1} \cdot 1 + 2^h \cdot 0$$

$$= \sum_{\ell=0}^{h} 2^\ell (h-\ell) \;=\; \sum_{i=0}^{h} \frac{2^h}{2^i} i \;=\; 2^h \sum_{i=0}^{h} \frac{i}{2^i} \;\leq\; 2 \cdot 2^h \;\leq\; 4n$$

Binary heap summary

Operation	Running Time
`construct(`$A[1..n]$`)`	$O(n)$
`max()`	$O(1)$
`insert(`x`,`p`)`	$O(\log n)$
`delMax()`	$O(\log n)$
`changeKey(`x`,`p'`)`	$O(\log n)$
`isEmpty()`	$O(1)$
`size()`	$O(1)$