10 Approximation Algorithms

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Outline

10 Approximation Algorithms

10.1 Motivation and Definitions

Recap: Optimization Problems, NPO

Recall general optimization problem $U \in \mathsf{NPO}$:

- each instance x has non-empty set of *feasible solutions* $M(x)$
- objective function $cost$ assigns value $cost(y)$ to all candidate solutions $y \in M(x)$
- can check in polytime
  - whether x is a valid instance
  - whether $y \in M(x)$
  - compute $cost(y) \in \mathbb{Q}$

For each U, consider two variants:

- *optimization problem:* output $y \in M(x)$ s.t. $cost(y) = \overset{\text{min or max}}{goal}_{y' \in M(x)} cost(y')$
- *evaluation problem:* output $goal_{y \in M(x)} cost(y)$

Perfect is the enemy of good

Optimal solutions are great, but if they are too expensive to get,
maybe ***"close-to-optimal"*** suffices?

A "consistent" with problem

A ***heuristic*** is an algorithm A that always computes a feasible solution $A(x) \in M(x)$, but we may not have any guarantees about $cost(A(x))$.

(Sometimes that's all we have . . .)

Our goal: Prove guarantees about worst possible $cost(A(x))$.
Problem: optimal objective function value depends on x,
so how to define *"good enough"*?

Relate $cost(A(x))$ to ***OPT*** $= goal_{y \in M(x)} cost(y)$. $\rightsquigarrow$ *approximation algorithm*

Approximation Algorithms

Definition 10.1 (Approximation Ratio)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ be an optimization problem. For every $x \in L_I$ we denote its *optimal objective value* by $OPT = OPT_U(x) = goal_{y \in M(x)} cost(y)$.

Let further A be an algorithm consistent with U.

The ***approximation ratio $R_A(x)$ of A on x*** is defined as $R_A(x) = \dfrac{cost(A(x))}{OPT_U(x)}$.

Note: For minimization problems, $R_A \geq 1$; for maximization problems $R_A \leq 1$

Definition 10.2 (Approximation Algorithm)

An algorithm A consistent with an optimization problem $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ is called a ***c-approximation (algorithm) for U*** if

- $goal = \min$ and $\forall x \in L_I : R_A(x) \leq c$;
- $goal = \max$ and $\forall x \in L_I : R_A(x) \geq c$.

10.2 Vertex Cover and Matchings

Example: Vertex Cover

Recall the VertexCover optimization problem.
C is a VC iff $\{u, v\} \in E : \{u, v\} \cap C \neq \emptyset$
$goal = \min$
How can we vouch for a VC C to be (close to) optimal?

Definition 10.3 ((Maximal/Maximum/Perfect) Matching)

Given graph $G = (V, E)$, a set $M \subseteq E$ is a *matching* (in G) if (V, M) has max-degree 1.

disjoint pairs of vertices

M is ($\subseteq$-) *maximal* (a.k.a. *saturated*) if no superset of M is a matching.
M is a *maximum matching* is there is no matching of strictly larger cardinality in G.
M is a *perfect matching* if $|M| = |V|/2$. ◀

Note:

- $\subseteq$-maximal matchings easy to find via greedy algorithm.
- Maxim**um** matchings are much more complicated, but also computable in polytime (Edmonds's "Blossom algorithm")

Matching → Vertex Cover

Lemma 10.4 (VC ≥ M)

If M is a matching and C is a vertex cover in G, then $|C| \geq |M|$.

Proof:

Let $\{v, w\} \in M \subseteq E$. $\rightsquigarrow$ C has to contain v or w (or both).

Since all $|M|$ matching edges are disjoint, C must cover them by $\geq |M|$ distinct endpoint. ∎

```
procedure matchingVertexCoverApprox(G = (V, E))
    // greedy maximal matching
    M := ∅
    for e ∈ E // arbitrary order
        if M ∪ {e} is a matching
            M := M ∪ {e}
    return ⋃_{{u,v}∈M} {u, v}
```

Theorem 10.5 (Matching is 2-approx for Vertex Cover)

matchingVertexCoverApprox is a *2-approximation* for VertexCover.

Can we do better?

Maybe do smarter analysis?

A tight example for "VC $\geq$ M": $K_{n,n}$

Assuming the *unique games conjecture*, no polytime $(2 - \varepsilon)$ approx for VC.

Simple matching-based approximation worst-case optimal . . .

10.3 The Drosophila of Approximation: Set Cover

(Weighted) Set Cover

Definition 10.6 (SetCover)

Given: a number n, $\mathcal{S} = \{S_1, \dots, S_k\}$ of k subsets of $U = [n]$,
and a cost function $c : S \to \mathbb{N}$.

Solutions: $\mathcal{C} \subseteq [k]$ with $\bigcup_{i \in \mathcal{C}} S_i = U$

Cost: $\sum_{i \in \mathcal{C}} c(S_i)$

Goal: min

- *cardinality version* a.k.a. UnweightedSetCover has cost $c(S) = |S|$
- UnweightedSetCover generalizes VertexCover:
  For VertexCover instances, the sets S_i are the sets of edges incident at a vertex v
  $\rightsquigarrow$ additional property that each $e \in U$ occurs in **exactly** 2 sets S_i
- general UnweightedSetCover = Vertex Cover on hypergraphs

We will use SetCover to illustrate various techniques for approximation algorithms.

Greedy Algorithm

Arguably simplest approach: **Greedily** pick set with current best *cost-per-new-item* ratio.

procedure greedySetCover$(n, \mathcal{S}, c)$
 $\mathcal{C} := \emptyset; \quad C := \emptyset$
 // For analysis: $i := 1$
 while $C \neq [n]$
  $i^* := \arg\min_{i \in [n]} \dfrac{c(S_i)}{|S_i \setminus C|}$
  $\mathcal{C} := \mathcal{C} \cup \{i^*\}$
  $C := C \cup S_{i^*}$
  // For analysis only:
  // $\alpha_i := \dfrac{c(S_{i^*})}{|S_{i^*} \setminus C|}$
  // for $e \in S_{i^*} \setminus C$ *set* $price(e) := \alpha_i$
  // $i := i + 1$
 return $\mathcal{C}$

Lemma 10.7 (Price Lemma)

Let $e_1, e_2, \ldots, e_n$ the order, in which greedySetCover covers the elements of U.
Then for all $j \in \{1, \ldots, n\}$ we have

$$price(e_j) \le \frac{OPT}{n - j + 1}.$$

Proof:
Consider time when the jth element e_j is covered.
$|\overline{C}| = n - (j-1)$ elements uncovered (for $\overline{C} = U \setminus C$).
Optimal SC $\mathcal{C}^*$ covers $\overline{C}$ with cost $\le OPT$

$$\rightsquigarrow \quad \exists S_{i^*} : \underbrace{\frac{c(S_{i^*})}{|S_{i^*} \setminus C|}}_{\ge price(e_j)} \le \frac{OPT}{|\overline{C}|} \le \frac{OPT}{n - j + 1}.$$

in $\mathcal{C}^*$, but not (yet) in $\mathcal{C}$

Arbitrarily order sets in $\mathcal{C}^*$, assign prices to uncovered elements.
If all prices were $> OPT/|\overline{C}|$, covering $\overline{C}$ would cost $> OPT$. ↯

■

Greedy Set Cover Analysis

Theorem 10.8 (greedySetCover approx)

greedySetCover is an H_n-approximation for WeightedSetCover.

Proof:

$$c(\mathcal{C}) = \sum_{i \in \mathcal{C}} c(S_i) = \sum_{j=1}^{n} price(e_j)$$

$$\underset{[\text{Lemma } 10.7]}{\leq} \sum_{j=1}^{n} \frac{OPT}{n-j+1} = OPT \sum_{i=1}^{n} \frac{1}{n} = H_n \cdot OPT \qquad \blacksquare$$

Greedy Worst Case

$H_n \sim \ln n$ is . . . not amazing. (Guarantee becomes worse with growing input size)

Unfortunately, bound is **tight** for greedySetCover in the worst case
even on WEIGHTEDVERTEXCOVER instances:

- Consider star graph where leaves cost $\frac{1}{n}, \frac{1}{n-1}, \ldots, 1$, and middle vertex costs $1+\varepsilon$.
- greedySetCover picks all leaves $\rightsquigarrow$ H_n
- $OPT = 1+\varepsilon$

More complicated constructions: $\Omega(\log n)$-approx even for (UNWEIGHTED)VERTEXCOVER.

10.4 The Layering Technique for Set Cover

Size-proportional cost functions

Greedy failed on "unfair" costs for sets . . . what if costs are "nicer"?
Larger sets "should" be more costly.

Definition 10.9 (Size-proportional cost function)

A cost function c is called *size proportional* if there is a constant p so that $c(S_i) = p|S_i|$. ◀

Definition 10.10 (Frequency)

The *frequency* f_e of an element $e \in [n]$ is the number of sets in which it occurs:
$f_e = |\{j : e \in S_j\}|$.
The (maximal) *frequency* of a SetCover instance is $f = \max_e f_e$. ◀

Note: (Weighted)VertexCover instance $\rightsquigarrow$ $f = 2$

Size-proportional indeed easier

Lemma 10.11 (size-proportionality → trivial f-approx)

For a size proportional weight function c we have $c(\mathcal{S}) \le f \cdot OPT$.

Proof:

$$c(\mathcal{S}) = \sum_{i=1}^{k} c(S_i) = p \sum_{i=1}^{k} |S_i| = p \sum_{e \in U} f_e \le p \sum_{e \in U} f \overset{\text{size-prop.} \rightsquigarrow OPT \ge p \cdot n}{\le} f \cdot OPT$$

Taking *all* sets gives f-approx, so certainly true for greedySetCover.

But probably not too many problem instances are that simple . . .

Layering Algorithm

Idea: Split cost function into sum of

- size-proportional part c_0 and
- a some residue c_1

```
procedure layeringSetCover(U, S, c)
    p := min{ c(S_j)/|S_j| : j ∈ [k] }
    c_0(S_i) := p · |S_i|   // size-prop. part
    c_1(S_i) := c(S_i) − c_0(S_i)   // ≥ 0
    C_0 := { j ∈ [k] : c_1(S_j) = 0 }
    U_0 := ⋃_{j∈C_0} S_j   // covered by size-prop.
    if U_0 == U
        return C_0
    else
        U_1 := U \ U_0   // rest of universe
        S_1 := { S ∈ {S_1, ..., S_k} | S ∩ U_1 ≠ ∅ }
        C_1 := layeringSetCover(U_1, S_1, c_1)
        return C_0 ∪ C_1
```

Theorem 10.12 (layering f-approx)

layeringSetCover is f-approx. for SetCover.

Proof:

Show by induction over recursive calls that
(a) computes cover (b) of cost $\le f \cdot OPT$.

Basis: $U_0 = U$

All of U covered by size-prop. part/

$\rightsquigarrow$ f-approx by Lemma 10.11

Inductive step:

IH: $\mathcal{C}_1$ covers U_1 at cost $c_1(\mathcal{C}_1) \le f \cdot OPT(U_1, \mathcal{S}_1, c_1)$.

Let $\mathcal{C}^*$ be **optimal** set cover w.r.t. c

Lemma 10.11: $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$ is f-approx w.r.t. $c_{\mathbf{0}}$.

$\rightsquigarrow$ $c_0(\mathcal{C}) \le f \cdot c_0(\mathcal{C}^*)$ $\quad$ (0)

Layering Algorithm [2]

Proof (cont.):

Define $\mathcal{C}_1^* = \{i \in \mathcal{C}^* : S_i \in \mathcal{S}_1\}$

$\mathcal{C}_1^*$ is a set cover for U_1

$$\rightsquigarrow \quad c_1(\mathcal{C}_1) \underset{IH}{\leq} OPT(U_1, \mathcal{S}_1, c_1) \leq f \cdot c_1(\mathcal{C}_1^*) \qquad (1)$$

$$\begin{aligned} c(\mathcal{C}) &= c_0(\mathcal{C}) + c_1(\mathcal{C}) \\ &\underset{\uparrow}{=} c_0(\mathcal{C}) + c_1(\mathcal{C}_{\mathbf{1}}) \end{aligned}$$

$i \in \mathcal{C}_0 \rightsquigarrow c_1 = 0$

$$\begin{aligned} &\underset{(0),(1)}{\leq} f \cdot \big(c_0(\mathcal{C}^*) + c_1(\mathcal{C}_1^*)\big) \\ &\leq f \cdot \big(c_0(\mathcal{C}^*) + c_1(\mathcal{C}^*)\big) \\ &= f \cdot c(\mathcal{C}^*) \end{aligned}$$

∎

Note: For VertexCover, this yields again a 2-approximation.

$\rightsquigarrow$ Same as using maximal matching

But the layering algorithm can handle *arbitrary vertex costs* (WeightedVertexCover)!

10.5 Applications of Set Cover

Shortest Superstrings

Definition 10.13 (ShortestSuperstring)

Given: alphabet Σ, set of strings $W = \{w_1, \ldots, w_n\} \subseteq \Sigma^+$

Feasible Instances: *superstrings* s of S, i. e., s contains w_i as substring for $1 \le i \le n$.

Cost: $|s|$

Goal: min ◀

Remark 10.14

Without-loss-of-generality assumption: no string is a substring of another. ◀

- Motivation: DNA assembly (sequencing from many shorter "reads")
- General problem is NP-complete

Here: Reduce this problem to SetCover!

Shortest Superstring by Set Cover

Construct *all pairwise* superstrings: overlap w_i and w_j by exactly ℓ characters (if possible)

$\sigma_{i,j,\ell} = w_i[0..|w_i|-\ell) \cdot w_j$ valid iff $w_j[0..\ell) = w_i[|w_i|-\ell..|w_i|)$

$M = \Big\{\sigma_{i,j,\ell} : i,j \in [u], \ell \in \big[0..\min\{|w_i|,|w_j|\}\big]\Big\}$

$\rightsquigarrow$ **Set Cover instance:**

- **Universe:** $[n]$ $\quad\rightsquigarrow$ try to *cover* all words in W with superstring ...
- **Subsets:** $S = \{S_\pi : \pi \in W \cup M\}$ $\quad$ ... by combining pairwise superstrings.
  where $S_\pi = \{k \in [n] \ : \ \exists i,j : w_k = \pi[i..j)\}$
- **Cost function:** $c(S_\pi) = |\pi|$

Given set-cover solution $\{S_{\pi_1}, \ldots, S_{\pi_k}\}$

$\rightsquigarrow$ superstring $s = \pi_1 \ldots \pi_k$ $\quad$ (in any order)

Shortest Superstring by Set Cover – Analysis

Lemma 10.15 (Pairwise superstrings yield 2-SC-approx)

Let W be an instance for ShortestSuperstring and (n, S, c) the corresponding SetCover instance. Let further OPT resp. OPT_{SC} be the optimal objective value of W resp. (n, S, c). Then $OPT \leq OPT_{SC} \leq 2 \cdot OPT$. ◀

Corollary 10.16 ($2H_n$ approximation for superstring)

By solving the transformed set cover instance with greedySetCover, we obtain a $2H_n$-approximation for the shortest superstring problem. ◀

Proof (Lemma 10.15):

- "$OPT \leq OPT_{SC}$"
  It suffices to show that $s = \pi_1 \ldots \pi_k$ is a valid superstring.
  By definition, every w_i must be contained in some π_k as a substring.
- "$OPT_{SC} \leq 2 \cdot OPT$"
  $OPT = |s^*|$ for a *shortest* superstring s^* for W.
  Without loss of generality, suppose s^* contains $w_1, \ldots, w_n$ *in this order*.

Shortest Superstring by Set Cover – Analysis [2]

Proof:
Define groups: $i_1 = 1$; $i_j = \min\{i > i_{j-1} :$ first occurrence of w_i does not overlap $w_{i_{j-1}}\}$.

Group j starts with w_{i_j} and ends with $w_{i_{j+1}-1}$
$\rightsquigarrow$ overlap of two strings $\rightsquigarrow$ $\pi_j = \sigma_{i_j, i_{j+1}-1, \ell_j}$

Groups can overlap (so concatenation of σs longer than s^*).

But group j and $j + \mathbf{2}$ cannot overlap!
$\rightsquigarrow$ $|\pi_1 \ldots \pi_k| \le 2|s^*| = 2 \cdot OPT$. ■

(Note: Better approximation algorithms for ShortestSuperstring possible via different techniques.)

10.6 (F)PTAS: Arbitrarily Good Approximations

Approximation Schemes

The problems so far had a barrier to arbitrarily good approximations;
but sometimes we can achieve the latter!

Definition 10.17 ((F)PTAS)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, \mathbf{min})$ an optimization problem.

An algorithm $A = A_\varepsilon(x)$ with input (ε, x) is called
polynomial-time approximation scheme (PTAS) for U,
if for every *constant* $\varepsilon \in \mathbb{Q}_{>0}$, the algorithm A_ε is a $(1+\varepsilon)$-approximation for U with running time polynomial in $|x|$.

If the running time of $A_\varepsilon(x)$ is bounded by a polynomial in $|x|$ **and** $\boldsymbol{\varepsilon^{-1}}$, A is called a
fully polynomial-time approximation scheme (FPTAS) for U. ◀

Note: PTAS could have running time $O(n^c \cdot 2^{2^{1/\varepsilon}})$ or so (akin to fpt running time)

FPTAS much stronger ... *but do they even exist for any* NP*-hard problems?* Yes!

Pseudopolynomial DP Reprise

Recall **0/1-Knapsack**: **Given:** items $1, \ldots, n$ with *weights* $w_1, \ldots, w_n$ and *values* $v_1, \ldots, v_n$;
Feasible solutions: subset of items with total weight $\le b$
Goal: maximize total value

Approximation Idea: Work with *rounded* values (depending on ε)

In Unit 3, we solved Knapsack

- using a DP table $V[n', b']$ = max value from items $1..n'$ and total weight $b' \le b$

$\rightsquigarrow$ $n \cdot b$ entries $\rightsquigarrow$ total time $O(n \cdot b \cdot \log(MaxInt(v)))$

$\rightsquigarrow$ good if *weights* are small, but we want to round *values*

- actually, DP also works with values as index!

$\boxed{\text{Assumption: } w_1, \ldots, w_n,\ v_1, \ldots, v_n \in \mathbb{N}}$

- DP table $W[n', v]$ = min weight from items $1, \ldots, n'$ with value $= v$

$$W[n', v] \;=\; \begin{cases} \min\Big\{ W[n'-1, v],\ W[n'-1, v - v_{n'}] + w_{n'} \Big\} & \text{if } v_{n'} < v \\ W[n'-1, v] & \text{otherwise} \end{cases} \qquad \text{(+ initial values)}$$

$\rightsquigarrow$ $n \cdot nV$ entries for $V = \max v_i$ $\rightsquigarrow$ total time $O(n^2 \cdot V \cdot \log(MaxInt(w)))$

FPTAS for Knapsack

Convenience Assumption: any item fits in the knapsack alone, i. e., $w_i \le b$

procedure knapsackFPTAS(w, v, b, ε)
  $V := \max_{i=1,\dots,n} v_i$
  $K := \varepsilon V / n$
  $\tilde{v} := \left\lfloor \frac{v}{K} \right\rfloor$ // *rounded* v
  return DPKnapsack($w, \tilde{v}, b$)

DPKnapsack is pseudopolynomial DP algorithm with running time $O(n^2 \cdot V \cdot \log(MaxInt(w)))$

Theorem 10.18

approxKnapsack is an FPTAS for 0/1-Knapsack.

Proof:

First consider running time; dominated by DPKnapsack.

$$O(n^2 \tilde{V} \log(MaxInt(w))) \le O(n^2 \tilde{V} |x|) \le O\left(n^2 |x| \frac{V}{K}\right) \le O\left(n^3 |x| \varepsilon^{-1}\right) \le O\left(|x|^4 \varepsilon^{-1}\right)$$

It remains to show that total value of $I = \text{DPKnapsack}(w, \tilde{v}, b)$ is $v(I) \ge (1 - \varepsilon) \cdot OPT$

FPTAS for Knapsack [2]

Proof (cont.):

Let I^* be an optimal solution, $v(I^*) = \sum_{i \in I^*} v_i = OPT$

For each $i \in [n]$, we have by definition $\boxed{v_i - K \; < \; K \cdot \tilde{v}_i \; \leq \; v_i \quad (*)}$.

FPKnapsack returns *optimal* solution for rounded values $\rightsquigarrow$ $\tilde{v}(I) \; \geq \; \tilde{v}(I^*) \quad (o)$

Moreover, $OPT \; \geq \; V$ by our assumption that each item fits into knapsack. $\quad (V)$

We now have

$$v(I) \underset{(*)}{\geq} K \cdot \tilde{v}(I) \underset{(o)}{\geq} K \cdot \tilde{v}(I^*) \underset{(*)}{\geq} v(I^*) - nK = OPT - \varepsilon V \underset{(V)}{\geq} (1 - \varepsilon) \cdot OPT$$

$\blacksquare$

FPTAS asks for much

Theorem 10.19 (FPTAS → FPT and pseudopolynomial)

1. $U \in \mathsf{FPTAS} \quad \Longrightarrow \quad p\text{-}U \in \mathsf{FPT}$
2. $U \in \mathsf{FPTAS}$ and $cost(u, x) < p\big(MaxInt(x)\big)$ for some polynomial p
   $\Longrightarrow \quad \exists$ pseudopolynomial algorithm for U.

◀

10.7 Christofides's Algorithm

Metric TSP – MST Approximation

MetricTravelingSalespersonProblem: TSP where distances obey triangle inequality

Step 1: MST

- Consider edge-weighted complete graph $G = ([n], E, D)$ of cities with pairwise distances $D_{i,j}$.
- Compute a minimum spanning tree T in G.

"Baby-Christofides": Walk around T (Euler tour after doubling all edges)
If this visits a vertex another time, simply skip it
(shortcut edge to next vertex)

Lemma 10.20

Baby-Christofides is a 2-approximation for MetricTSP.

Proof:

$\rightsquigarrow$ Walking around T uses each edge twice: cost $= 2c(T)$.

- Shortcutting does not make the tour longer by the triangle inequality.
- Removing one edge form an optimal TSP tour yields a spanning tree (path)

$\rightsquigarrow$ $OPT \ge c(T)$. ∎

Matchings and Tours

Can we improve upon the specific Euler tour we used?

Doubling edges was costly. For even-degree vertices this is not needed!

Recall: graph has an Euler tour iff all vertices have even degrees.

Lemma 10.21

Let $V' \subseteq V$ with $|V'|$ even and let M be a minimum-cost perfect matching on V' (in the TSP graph). Then $c(M) \leq OPT/2$. ◀

Proof:

Let C^* be the optimal TSP tour and let C' be the your (on V') where we shortcut all vertices not in V'.
By triangle inequality $c(C') \leq c(C^*)$.

Since $|V'|$ is even, C' is the *disjoint union of two perfect matchings* of V' (odd and even steps).

So the cheaper of these two matchings has cost $\leq\ c(C')/2\ \leq\ c(C^*)/2\ =\ OPT/2$.
$\rightsquigarrow$ optimal perfect matching also has $c(M)\ \leq\ OPT/2$. ■

Christofides's 3/2-Approximation

Step 2: Christofides's Algorithm

- T := MST in G.
- V' := vertices with odd degree in T.
- M := minimum-cost perfect matching of V' in G.
- Output Euler cycle C in $([n], E(T) \cup M)$, shortcutting repeated vertices.

Theorem 10.22

Christofides's algorithms is a $\frac{3}{2}$-approximation for METRICTSP.

Proof:

$$c(C) \quad = \quad c(T) + c(M) \quad \le \quad OPT + OPT/2 \quad = \quad \frac{3}{2} \cdot OPT$$ ■

Major open problem: Can $\frac{3}{2}$ be improved?

- Was open since 1976
- (Tiny) improvement published at STOC 2021 ($(\frac{3}{2} - \delta)$-approximation) out of PhD project of Nathan Klein (!)

10.8 Randomized Approximations

Randomized Approximation Guarantees

Definition 10.23 (Randomized δ-approx.)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, \mathbf{max})$ an optimization problem. For $\delta > 1$ a randomized algorithm A is called *randomized δ-approximation algorithm for U*, if

- $\mathbb{P}[A(x) \in M(x)] \;=\; 1,$ (always feasible) and
- $\mathbb{P}[R_A(x) \leq \delta] \;\geq\; \frac{1}{2}$ (typically within δ)

for all $x \in L_I$. ◀

Definition 10.24 (δ-expected approx.)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, \max)$ an optimization problem. For $\delta > 1$ a randomized algorithm A is called *(randomized) δ-expected approximation algorithm for U*, if

- $\mathbb{P}[A(x) \in M(x)] \;=\; 1$ (always feasible) and
- $\dfrac{\mathbb{E}[cost(A(x))]}{OPT_U(x)} \;\leq\; \delta$ (expected within δ)

for all $x \in L_I$. ◀

(Minimization problems similar.)