2 Machines & Models

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Learning Outcomes

Unit 2: ***Machines & Models***

1. Understand the difference between empirical *running time* and algorithm *analysis*.
2. Understand *worst/best/average case* models for input data.
3. Know the *RAM machine* model.
4. Know the definitions of *asymptotic notation* (Big-Oh classes and relatives).
5. Understand the reasons to make *asymptotic approximations*.
6. Be able to *analyze* simple *algorithms*.

Outline

2 Machines & Models

What is an algorithm?

An algorithm is a sequence of instructions.

think: recipe

More precisely:

e. g. Python script

1. mechanically executable
   ⇝ no "common sense" needed
2. finite description ≠ finite computation!
3. solves a *problem*, i. e., a class of problem instances

$x + y$, not only $17 + 4$

- input-processing-output abstraction

Typical example: *bubblesort*

⇝ not a specific program
but the underlying idea

What is a data structure?

A data structure is

1. a rule for **encoding data** (in computer memory), plus
2. **algorithms** to work with it (queries, updates, etc.)

typical example: *binary search tree*

2.1 Algorithm analysis

Good algorithms

Our goal: Find good (best?) algorithms and data structures for a task.

Good "usually" means

- fast running *time* (can be complicated in distributed systems)
- moderate memory *space* usage

Algorithm analysis is a way to

- compare different algorithms,
- predict their performance in an application

Running time experiments

Why not simply run and time it?

- results only apply to
  - single *test* machine
  - tested inputs
  - tested implementation
  - ...
  - $\neq$ *universal truths*

- instead: consider and analyze algorithms on an abstract machine
  - $\rightsquigarrow$ provable statements for model
  - $\rightsquigarrow$ testable model hypotheses

survives Pentium 4

$\rightsquigarrow$ Need precise model of machine (costs), input data and algorithms.

Data Models

Algorithm analysis typically uses one of the following simple data models:

- **worst-case performance:**
  consider the *worst* of all inputs as our cost metric
- **best-case performance:**
  consider the *best* of all inputs as our cost metric
- **average-case performance:**
  consider the average/expectation of a *random* input as our cost metric

Usually, we apply the above for *inputs of same size* n.

$\rightsquigarrow$ performance is only a **function of** n.

2.2 The RAM Model

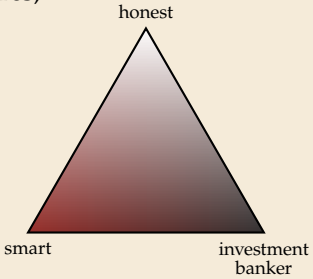
Machine models

The machine model decides

- what algorithms are possible
- how they are described (= programming language)
- what an execution *costs*

Goal: Machine models should be
detailed and powerful enough to reflect actual machines,
abstract enough to unify architectures,
simple enough to analyze.

⇝ usually some compromise is needed



Random Access Machines

Random access machine (RAM)

more detail in §2.2 of *Sequential and Parallel Algorithms and Data Structures* by Sanders, Mehlhorn, Dietzfelbinger, Dementiev

- unlimited *memory* $\texttt{MEM}[0], \texttt{MEM}[1], \texttt{MEM}[2], \ldots$
- fixed number of *registers* $R_1, \ldots, R_r$ (say $r = 100$)
- memory cells $\texttt{MEM}[i]$ and registers R_i store w-bit integers, i. e., numbers in $[0..2^w - 1]$
  w is the word width/size; typically $\boxed{w \propto \lg n}$ $\rightsquigarrow$ $2^w \approx n$
- Instructions:
  - load & store: $R_i := \texttt{MEM}[R_j]$ $\quad \texttt{MEM}[R_j] := R_i$
  - operations on registers: $R_k := R_i + R_j$ (arithmetic is *modulo* 2^w!)
    also $R_i - R_j$, $R_i \cdot R_j$, $R_i \operatorname{div} R_j$, $R_i \bmod R_j$
    C-style operations (bitwise and/or/xor, left/right shift)
  - conditional and unconditional jumps
- cost: number of executed instructions

we will see further models later

$\rightsquigarrow$ The RAM is the standard model for sequential computation.

RAM-Program Example

Example RAM program

```
1  // Assume: R_1 stores number N
2  // Assume: MEM[0..N) contains list of N numbers
3  R_2 := R_1;
4  R_3 := R_1 − 2;
5  R_4 := MEM[R_3];
6  R_5 := R_3 + 1;
7  R_6 := MEM[R_5];
8  if (R_4 ≤ R_6) goto line 11;
9  MEM[R_3] := R_6;
10 MEM[R_5] := R_4;
11 R_3 := R_3 − 1;
12 if (R_3 ≥ 0) goto line 5;
13 R_2 := R_2 − 1;
14 if (R_2 > 0) goto line 4;
15 // Done:
```

Pseudocode

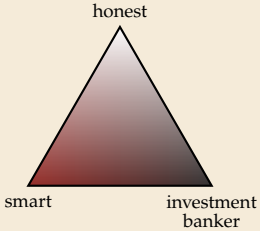
- Programs for the random-access machine are very low level and detailed

≈ assembly/machine language

Typical simplifications when describing and analyzing algorithms:

- more abstract *pseudocode* ← code that humans understand (easily)
  - control flow using **if**, **for**, **while**, etc.
  - variable names instead of fixed registers and memory cells
  - memory management (more below)
- count dominant *elementary operations* (e. g. memory accesses) instead of all RAM instructions

In both cases: We *can* go to full detail where needed/desired.



Pseudocode – Example

RAM-Program

```
1  // Bubblesort
2  // Assume: R_1 stores number N
3  // Assume: MEM[0..N) contains list of N numbers
4  R_2 := R_1;
5  R_3 := R_1 − 2;
6  R_4 := MEM[R_3];
7  R_5 := R_3 + 1;
8  R_6 := MEM[R_5];
9  if (R_4 ≤ R_6) goto line 11;
10 MEM[R_3] := R_6;
11 MEM[R_5] := R_4;
12 R_3 := R_3 − 1;
13 if (R_3 ≥ 0) goto line 5;
14 R_2 := R_2 − 1;
15 if (R_2 > 0) goto line 4;
16 // Done: MEM[0..N) sorted
```

Pseudocode Algorithm

```
1 procedure bubblesort(A[0..N)):
2     for i := N, N − 1, . . . , 1
3         for j := N − 2, N − 3, . . . , 0
4             if A[j] > A[j + 1]:
5                 Swap A[j] and A[j + 1]
6             end if
7         end for
8     end for
```

- ⇝ much more **readable**
- closer to modern high-level programming languages
- **but:** only allow primitive operations that correspond to $O(1)$ RAM instructions
  - ⇝ analysis

Memory management & Pointers

- A random-access machine is a bit like a bare CPU . . . without any operating system
  - ⇝ cumbersome to use
- All high-level programming languages / operating systems add *memory management*:
  - Instruction to *allocate* a contiguous piece of memory of a given size (like `malloc`).
    - used to allocate a `new` array (of a fixed size) or
    - a `new` object/record (with a known list of instance variables)
    - There's a similar instruction to `free` allocated memory again or an automated garbage collector.
  - ⇝ A *pointer* is a memory address (i. e., the i of $\texttt{MEM}[i]$).
  - Support for procedures (a. k. a. functions, methods) calls including recursive calls
    - (this internally requires maintaining call stack)

We will mostly ignore *how* all this works here.

2.3 Asymptotics & Big-Oh

Why asymptotics?

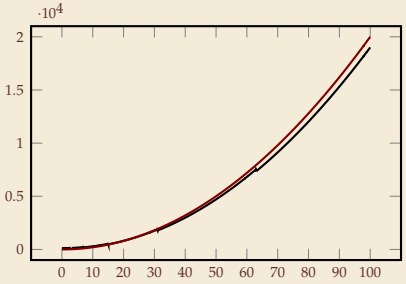
Algorithm analysis focuses on (the limiting behavior for infinitely) **large** inputs.

- abstracts from unnecessary detail
- simplifies analysis
- often necessary for sensible comparison

Asymptotics = approximation around ∞

Example: Consider a function $f(n)$ given by

$$2n^2 - 3n\lfloor \log_2(n+1) \rfloor + 7n - 3\lfloor \log_2(n+1) \rfloor + 120 \quad \sim \quad \mathbf{2n^2}$$



Asymptotic tools – Formal & definitive definition

- **"Tilde Notation":** $f(n) \sim g(n)$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

  (if, and only if)

  „f and g are *asymptotically equivalent*"

- **"Big-Oh Notation":** $f(n) \in O\big(g(n)\big)$ iff $\left|\frac{f(n)}{g(n)}\right|$ is bounded for $n \geq n_0$

  (also write '=' instead)

  iff $\limsup_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| < \infty$

  (need supremum since limit might not exist!)

  Variants:

  ("Big-Omega")

  - $f(n) \in \Omega\big(g(n)\big)$ iff $g(n) \in O\big(f(n)\big)$
  - $f(n) \in \Theta\big(g(n)\big)$ iff $f(n) \in O\big(g(n)\big)$ **and** $f(n) \in \Omega\big(g(n)\big)$

  ("Big-Theta")

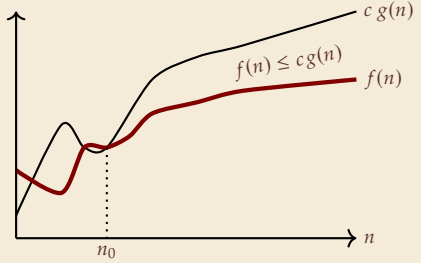
- **"Little-Oh Notation":** $f(n) \in o\big(g(n)\big)$ iff $\lim_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| = 0$

  similarly: $f(n) \in \omega\big(g(n)\big)$ if $\lim = \infty$

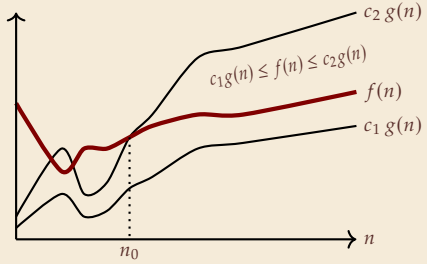
(Benefit of this definition: Works for any $f, g : \mathbb{R} \to \mathbb{R}$ and is easy to generalize to limits other than $n \to \infty$)

Asymptotic tools – Intuition

- $f(n) = O(g(n))$: $f(n)$ is **at most** $g(n)$ up to constant factors and for sufficiently large n



- $f(n) = \Theta(g(n))$: $f(n)$ is **equal to** $g(n)$ up to constant factors and for sufficiently large n



Plots can be misleading!

Example

Asymptotics – Example 1

Basic examples:

- $20n^3 + 10n\ln(n) + 5 \sim 20n^3 = \Theta(n^3)$
- $3\lg(n^2) + \lg(\lg(n)) = \Theta(\log n)$
- $10^{100} = O(1)$

Use *wolframalpha* to compute/check limits, but also practice it with pen and paper!

Asymptotics – Basic facts

Rules to work with Big-Oh classes:

- $f = \Theta(f)$ (reflexivity)
- $f = \Theta(g) \land g = \Theta(h) \implies f = \Theta(h)$
- $c \cdot f(n) = \Theta(f(n))$ for constant $c \neq 0$
- $f \sim g \iff f = g \cdot (1 \pm o(1))$
- $\Theta(f) \cdot \Theta(g) = \Theta(f \cdot g)$
- $\Theta(f) + \Theta(g) = \Theta(f + g) = \Theta(\max\{f, g\})$ largest summand determines Θ-class

Asymptotics – Frequently encountered classes

Frequently used orders of growth:

- constant $\Theta(1)$
- logarithmic $\Theta(\log n)$ Note: $a, b > 0$ constants $\rightsquigarrow$ $\Theta(\log_a(n)) = \Theta(\log_b(n))$
- linear $\Theta(n)$
- linearithmic $\Theta(n \log n)$
- quadratic $\Theta(n^2)$
- cubic $\Theta(n^3)$
- polynomial $O(n^c)$ for some constant c
- exponential $O(c^n)$ for some constant $c > 1$ Note: $a > b > 0$ constants $\rightsquigarrow$ $b^n = o(a^n)$

Asymptotics – Example 2

Square-and-multiply algorithm

for computing x^m with $m \in \mathbb{N}$

Inputs:

- m as binary number (array of bits)
- n = #bits in m
- x a floating-point number

```
1  def pow(x, m):
2      # compute binary representation of exponent
3      exponent_bits = bin(m)[2:]
4      result = 1
5      for bit in exponent_bits:
6          result *= result
7          if bit == '1':
8              result *= x
9      return result
```

- Cost: C = # multiplications
- C = n (line 6) + #one-bits in binary representation of m (line 8)

$\rightsquigarrow$ $n \leq C \leq 2n$

Asymptotics with several variables

- **Example:** Algorithms on graphs with n vertices and m edges.
  - want to say: Algorithm A takes time $\Theta(n+m)$.
  - But what does that even mean formally?!

⚠ Inconsistent and incompatible definitions used in the literature!

- **Here:**
  - (implicitly) always have a single *"main" variable* n: with $n \to \infty$
  - all other variables are *functions* of n: $m = m(n)$
  - must make *conditions* on functions explicit: $m(n) \in \Omega(n)$ and $m(n) \in O(n^2)$.

  $\rightsquigarrow$ Can make statements like

  $$O(n+m) \subseteq O(nm) \qquad (n \to \infty,\ m \in \Omega(1))$$

2.4 Teaser: Maximum subarray problem

Bring on the puzzles!

Time for a concrete example of algorithm design!

- we will illustrate the algorithm design process on a "toy problem"
- clean abstract problem, but nontrivial to solve!

Maximum (sum) subarray problem

- **Given:** $A[0..n)$ with $A[i] \in \mathbb{Z}$ for $0 \le i < n$.
- Abbreviate $s(i,j) := \sum_{k=i}^{j-1} A[k]$
- **Goal:** Compute $s := \max\{s(i,j) : 0 \le i \le j \le n\}$ and a pair (i,j) with $s = s(i,j)$.

will ignore that here; easy to modify algorithms

Modeling decisions:

- input size: # numbers n
- assume all integers (and sums) fit in $O(1)$ words

$\rightsquigarrow$ count # additions as elementary operation

Applications:

- largest gain of a stock $A[i]$ price change on day i
- signal detection in biological sequence analysis
- 2D generalization used in image analysis

Template for Describing an Algorithm

1. **Algorithmic Idea**
   Abstract idea that makes the algorithm work (prose)
   (an expert could fill in the rest from here)

2. **Pseudocode**
   structured description of procedure including edge cases
   should be unambiguous and close to real code

3. **Correctness proof**
   argument why the correct result is computed
   often uses induction and invariants

4. **Algorithm analysis**
   analysis of the efficiency of the algorithm
   usually want Θ-class of worst-case running time
   where interesting, also space usage

Brute force approach

- Let's start with the simplest thinkable solution

Maximal subarray problem

- **Given:** $A[0..n)$ with $A[i] \in \mathbb{Z}$ for $0 \le i < n$.
- Abbreviate $s(i,j) := \sum_{k=i}^{j-1} A[k]$
- **Goal:** Compute $s := \max\{s(i,j) : 0 \le i \le j \le n\}$ and a pair (i,j) with $s = s(i,j)$.

1. Algorithmic Idea

try all contiguous subarrays $A[i..j)$

2. </> Pseudocode

```
1  s = 0
2  for i = 0, ..., n − 1
3      for j = i, ..., n
4          t = 0
5          for k = i, ..., j − 1
6              t = t + A[k]
7          end for
8          if t > s then s := t
9      end for
10 end for
```

3. Correctness proof

direct by definition of s

4. Algorithm analysis

$$
\begin{aligned}
&\#\text{additions} \\
&= \sum_{i=0}^{n-1}\sum_{j=i}^{n}\sum_{k=i}^{j-1} 1 \;=\; \sum_{i=0}^{n-1}\sum_{j=i}^{n}(j-i) \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-i} j \;=\; \sum_{i=0}^{n-1}\frac{(n-i)(n-i+1)}{2} \\
&= \frac{1}{2}\sum_{i=1}^{n} i(i+1) \;=\; \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i \\
&= \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4} \\
&= \frac{n(n+1)(n+2)}{6} \;\sim\; \frac{1}{6}n^3 \;=\; \Theta(n^3)
\end{aligned}
$$

Reusing sums

1. Algorithmic Idea

- brute force algorithm is unnecessarily wasteful!
- can use $s(i,j) = s(i,j-1) + A[j-1]$

2. Pseudocode

```
1 s = 0
2 for i = 0, ..., n − 1
3       t = 0
4       for j = i + 1, ..., n
5             t = t + A[j − 1]
6             if t > s then s := t
7       end for
8 end for
```

Can we possibly do better?

- There are $\binom{n}{2} \sim \frac{1}{2}n^2$ different $s(i,j)$ ...

$\rightsquigarrow$ Can't look at all of them

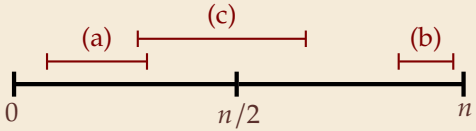
3. Correctness proof: as above

4. Algorithm analysis: $\displaystyle\sum_{i=0}^{n-1}\sum_{j=i+1}^{n} 1 = \frac{n(n+1)}{2} \sim \frac{1}{2}n^2 = \Theta(n^2)$ additions

A subquadratic solution

Algorithmic idea:
Consider $n/2$-mark.
Only 3 options for optimal solution $s(i, j)$:

(a) $0 \le i \le j < \lceil \frac{n}{2} \rceil$ (left)
(b) $\lceil \frac{n}{2} \rceil \le i \le j \le n$ (right)
(c) $i < \lceil \frac{n}{2} \rceil \le j$ (straddle)



- optimal straddle easy to compute!
  - **independently** find best left endpoint i for $s(i, \lceil \frac{n}{2} \rceil)$ and best right endpoint j for $s(\lceil \frac{n}{2} \rceil, j)$
- for (a) and (b), recurse on instance of half the size!

A subquadratic solution – Pseudocode & Correctness

```
1  procedure findMaxSubarraySum(A[ℓ..r]):
2      if r − ℓ ≤ 0
3          return 0
4      if r − ℓ == 1
5          return max{0, A[ℓ]}
6      m := ⌈(ℓ + r)/2⌉
7      s(a) := findMaxSubarraySum(A[ℓ, m))
8      s(b) := findMaxSubarraySum(A[m, r))
9      // Find left endpoint of straddle:
10     sℓ := 0; t := 0
11     for i = m − 1, m − 2, . . . , ℓ
12         t := A[i] + t
13         sℓ := max{sℓ, t}
14     end for
15     // Find right endpoint of straddle:
16     sr := 0; t := 0
17     for j = m + 1, . . . , r
18         t := t + A[j − 1]
19         sr := max{sr, t}
20     end for
21     s(c) := sℓ + sr
22     return max{s(a), s(b), s(c)}
```

- **Correctness proof:**
  - Induction over $n = r - \ell$
    - **basis:** for $n \leq 1$ ✓
    - **hypothesis:** Assume findMaxSubarraySum returns correct result for all arrays of up to $n - 1$ elements
    - **step:** For array of $n \geq 2$ elements, distinguish cases (a), (b), (c)

      (a) and (b) $\rightsquigarrow$ IH ✓

      (c) "from inspection of the code"

A subquadratic solution – Analysis

```
1  procedure findMaxSubarraySum(A[ℓ..r]):
2      if r − ℓ ≤ 0
3          return 0
4      if r − ℓ == 1
5          return max{0, A[ℓ]}
6      m := ⌈(ℓ + r)/2⌉
7      s(a) := findMaxSubarraySum(A[ℓ, m))
8      s(b) := findMaxSubarraySum(A[m, r))
9      // Find left endpoint of straddle:
10     sℓ := 0; t := 0
11     for i = m − 1, m − 2, . . . , ℓ
12         t := A[i] + t
13         sℓ := max{sℓ, t}
14     end for
15     // Find right endpoint of straddle:
16     sr := 0; t := 0
17     for j = m + 1, . . . , r
18         t := t + A[j − 1]
19         sr := max{sr, t}
20     end for
21     s(c) := sℓ + sr
22     return max{s(a), s(b), s(c)}
```

Algorithm analysis:

- Write $n = r - \ell$
- # additions in non-recursive part: $(m - \ell) + (r - m) + 1 = n + 1$
- Write $C(n)$ for total # additions for n elements

$\rightsquigarrow\ C(n) = C(\lceil \frac{n}{2} \rceil) + C(\lfloor \frac{n}{2} \rfloor) + n + 1$

- for $n = 2^k$ for $k \in \mathbb{N}_0$, this simplifies to $C(2^k) = 2C(2^{k-1}) + 2^k + 1$

$\rightsquigarrow\ C(n) \sim n \log_2(n)$

A lower bound

- **Theorem:** Every correct algorithm has a running time of $\Omega(n)$.

An optimal algorithm

Algorithmic idea:

In a clever sweep, we can compute best $s(i, r)$ and best $s(i, j)$ with $i \leq j \leq r$ for all r.

Pseudocode

```
1 procedure findMaxSubarraySum(A[0..n])
2     suffixMax := 0; globalMax := 0
3     for r = 1, ..., n
4         suffixMax := max{suffixMax + A[r − 1], 0}
5         globalMax := max{globalMax, suffixMax}
6     return globalMax
```