1 Proof Techniques

13 October 2025

Prof. Dr. Sebastian Wild

CS566 (Wintersemester 2025–26)
Philipps-Universität Marburg

version `2025-10-13 11:17`

Learning Outcomes

Unit 1: ***Proof Techniques***

1. Know logical *proof strategies* for proving implications, set inclusions, set equalities, and quantified statements.
2. Be able to use *mathematical induction* in simple proofs.
3. Know techniques for *proving termination* and *correctness* of procedures.

Outline

1 Proof Techniques

1.1 Digression: Random Shuffle

Random shuffle

Notation: $[0..n) = \{0, 1, \ldots, n-2, n-1\}$

- **Goal:** Put an array $A[0..n)$ of n numbers into random order.
  More precisely: Any ordering of the elements $A[0], \ldots, A[n-1]$ should be equally likely.
- A natural approach yields the following code

```
procedure myShuffle(A[0..n))
    for i := 0, ..., n − 1
        r := randomInt([0..n)) // A uniformly random number r with 0 ≤ r < n.
        Swap A[i] and A[r] // Swap A[i] to random position.
    end for
```

- Intuitively: *All elements are moved to a random index, so the order is random . . . right?*

Clicker Question

Select all statements that apply to myShuffle (for you).

A I have seen this shuffling algorithm (or a very similar method) before.

B I can understand the pseudocode for myShuffle (so I would be able to do an example by hand).

C It can generate all possible orderings of A (depending on the random numbers).

D myShuffle produces all possible orderings with the same probability.

E Assuming randomInt gives (perfect) uniform random numbers in the given range, myShuffle generates any ordering with equal probability.

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$n = 2$

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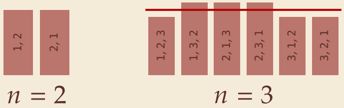

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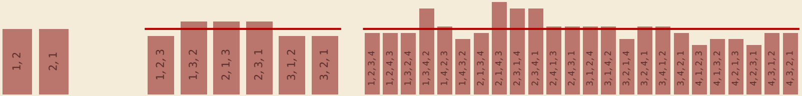

$n = 2$ $\qquad$ $n = 3$ $\qquad$ $n = 4$

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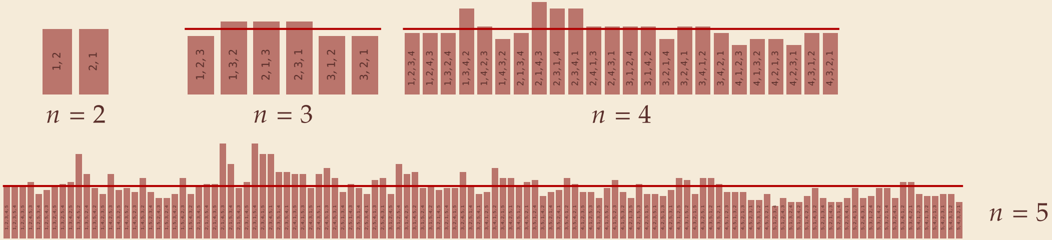

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← **WRONG!**
DO NOT USE

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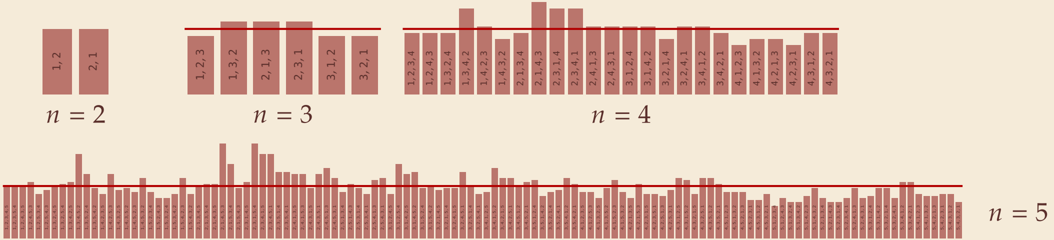

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E ~~Assuming randomInt gives (perfect) uniform random numbers in the given range, myShuffle generates any ordering with equal probability.~~

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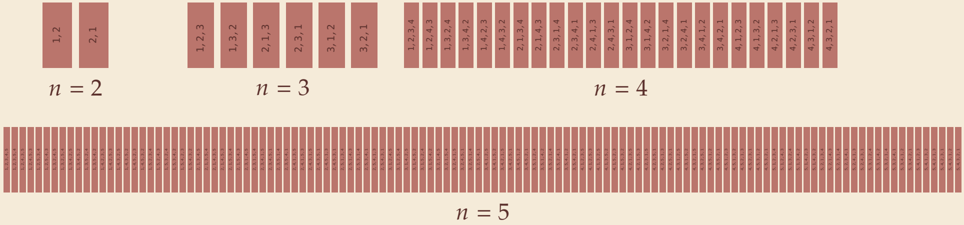
Correct shuffle

- interestingly, a very small change corrects the issue

```
procedure shuffleKnuthFisherYates(A[0..n))
    for i := 0, . . . , n − 1
        r := randomInt([i..n))
        Swap A[i] and A[r]
    end for
```



- looks good . . .
- . . . but how can we convince ourselves that it is correct, *beyond any doubt?*

1.2 Proof Templates

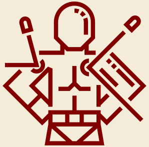
What is a *formal* proof?

A formal proof (in a logical system) is a **sequence of statements** such that each statement

1. is an *axiom* (of the logical system), or
2. follows from previous statements using the *inference rules* (of the logical system).

Among experts: Suffices to *convince a human* that a formal proof *exists*.

But: Use formal logic as guidance against faulty reasoning. $\rightsquigarrow$ bulletproof



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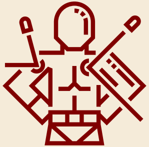

Notation:

- Statements: $A \equiv$ "it rains", $B \equiv$ "the street is wet".
- Negation: $\neg A$ "Not A"
- And/Or: $A \wedge B$ "A and B"; $A \vee B$ "A or B or both"
- Implication: $A \Rightarrow B$ "If A, then B"; $\neg A \vee B$
- Equivalence: $A \Leftrightarrow B$ "A holds true *if and only if ('iff')* B holds true."; $(A \Rightarrow B) \wedge (B \Rightarrow A)$

Clicker Question

$$A \Rightarrow B \equiv \neg A \vee B$$

Is the following statement true?

"If $\underbrace{\textit{the Earth is flat}}_{A}$, then $\underbrace{\textit{ships can fall over its rim}}_{B}$."

A Yes

B No

C Neither

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Clicker Question

Is the following statement true?

"If the Earth is flat, then ships can fall over its rim."

A Yes ✓ **B** ~~No~~ **C** ~~Neither~~

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Implications

$\neg\neg A \equiv A$

To prove $A \Rightarrow B$, we can

$A \Rightarrow B \equiv \neg A \vee B$

$\equiv \neg(\neg B) \vee (\neg A)$

$\equiv \neg B \Rightarrow \neg A$

- directly derive B from A *direct proof*
- prove $(\neg B) \Rightarrow (\neg A)$ *indirect proof, proof by contraposition*
- assume $A \wedge \neg B$ and derive a contradiction *proof by contradiction, reductio ad absurdum*
- distinguish cases, i. e., separately prove $(A \wedge C) \Rightarrow B$ and $(A \wedge \neg C) \Rightarrow B$. *proof by exhaustive case distinction*

Clicker Question

n odd

$\Rightarrow \exists k \in \mathbb{N}_{\geq 0} : n = 2k+1$

Suppose we want to prove:
"If $n^2 \in \mathbb{N}_0$ is an even number, then n is also even."
For that we show that when n is odd, also n^2 is odd.
Which proof template do we follow?

$\Rightarrow n^2 = (2k+1)^2$
$= 4k^2 + 4k + 1$
$= 2\underbrace{(2k^2 + 2k)}_{k' \in \mathbb{N}_{\geq 0}} + 1$

$\Rightarrow n^2$ odd

A direct proof: $A \Rightarrow B$

B indirect proof: $(\neg B) \Rightarrow (\neg A)$ ✓

C proof by contradiction: $A \wedge \neg B \Rightarrow \lightning$

D proof by case distinction: $(A \wedge C) \Rightarrow B$ and $(A \wedge \neg C) \Rightarrow B$

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Clicker Question

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C ~~proof by contradiction:~~ $\cancel{A \wedge \neg B \Rightarrow \lightning}$

D ~~proof by case distinction:~~ $\cancel{(A \wedge C) \Rightarrow B}$ ~~and~~ $\cancel{(A \wedge \neg C) \Rightarrow B}$

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Equivalences

To prove $A \Leftrightarrow B$,
we prove both implications $A \Rightarrow B$ and $B \Rightarrow A$ separately.

(Often, one direction is much easier than the other.)

Set Inclusion and Equality

To prove that a set S contains a set R, i. e., $R \subseteq S$,
we prove the implication $x \in R \Rightarrow x \in S$.

To prove that two sets S and R are equal, $S = R$,
we prove both inclusions, $S \subseteq R$ and $R \subseteq S$ separately.

1.3 Mathematical Induction

Quantified Statements

Notation

- Statements with parameters: $A(x) \equiv$ "x is an even number."
- Existential quantifiers: $\exists x : A(x)$ "There exists some x, so that $A(x)$."
- Universal quantifiers: $\forall x : A(x)$ "For all x it holds that $A(x)$."

  Note: $\forall x : A(x)$ is equivalent to $\neg\exists x : \neg A(x)$

Quantifiers can be nested, e. g., *ε-δ-criterion for limits*:

$$\lim_{x\to\xi} f(x) = a \quad :\Leftrightarrow \quad \underline{\forall\varepsilon > 0\ \exists\delta > 0\ :}\ \big(|x-\xi| < \delta\big) \Rightarrow \big|f(x) - a\big| < \varepsilon.$$

To prove $\exists x : A(x)$, we simply list an example ξ such that $A(\xi)$ is true.

Clicker Question

Have you seen **proofs by *mathematical induction*** before?

- **A** Yes, could do it
- **B** Yes, but only vaguely remember
- **C** I've heard this term before, but ...
- **D** I have not heard "mathematical induction" before

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For-all statements

To prove $\forall x : A(x)$, we can

- derive $A(x)$ for an *"arbitrary but fixed value of x"*, or,
- for $x \in \mathbb{N}_0$, use ***induction***, i. e.,
  - prove $A(0)$, *induction basis*, and
  - prove $\forall n \in \mathbb{N}_0 : A(n) \Rightarrow A(n + 1)$ *inductive step*

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More general variants of induction:

- complete/strong induction
  inductive step shows $\big(A(0) \wedge \cdots \wedge A(n)\big) \Rightarrow A(n+1)$
- structural/transfinite induction
  works on any *well-ordered* set, e. g., binary trees, graphs, Boolean formulas, strings, . . .

no infinite strictly decreasing chains

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