5 Divide & Conquer

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Learning Outcomes

Unit 5: ***Divide & Conquer***

1. Know the steps of the Divide & Conquer paradigm.
2. Be able to solve simple Divide & Conquer recurrences.
3. Be able to design and analyze new algorithms using the Divide & Conquer paradigm.
4. Know the performance characteristics of selection-by-rank algorithms.

Outline

5 Divide & Conquer

Divide and conquer

Divide and conquer *idiom* (Latin: *divide et impera*)
to make a group of people disagree and fight with one another
so that they will not join together against one (Merriam-Webster Dictionary)

$\rightsquigarrow$ in politics & algorithms, many independent, small problems are better than one big one!

Divide-and-conquer algorithms:

1. Break problem into smaller, independent subproblems. (Divide!)
2. Recursively solve all subproblems. (Conquer!)
3. Assemble solution for original problem from solutions for subproblems.

Examples:

- Mergesort
- Quicksort
- Binary search
- (arguably) Tower of Hanoi

5.1 Divide & Conquer Recurrences

Back-of-the-envelope analysis

- before working out the details of a D&C idea,
  it is often useful to get a quick indication of the resulting performance
  - don't want to waste time on something that's not competitive in the end anyways!
- since D&C is naturally recursive, running time often not obvious
  instead: given by a recursive equation
- unfortunately, rigorous analysis often tricky
  - Remember mergesort?

$$C(n) = \begin{cases} 0 & n \le 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \ge 2 \end{cases}$$

$$\rightsquigarrow C(n) = 2n\lfloor \lg(n) \rfloor + 2n - 4 \cdot 2^{\lfloor \lg(n) \rfloor} \text{ 😱}$$
$$= \Theta(n \log n) \text{ 😌}$$

- the following method works for many typical cases to give the right **order of growth**

The Master Method

- Assume a stereotypical D&C algorithm
  - a recursive calls on (for some constant $a \geq 1$)
  - subproblems of size n/b (for some constant $b > 1$)
  - with non-recursive "conquer" effort $f(n)$ (for some function $f : \mathbb{R} \to \mathbb{R}$)
  - base case effort d (some constant $d > 0$)

$\rightsquigarrow$ running time $T(n)$ satisfies

$$\boxed{T(n) \;=\; \begin{cases} a \cdot T\left(\frac{n}{b}\right) \;+\; f(n) & n > 1 \\ d & n \leq 1 \end{cases}}$$

Theorem 5.1 (Master Theorem)

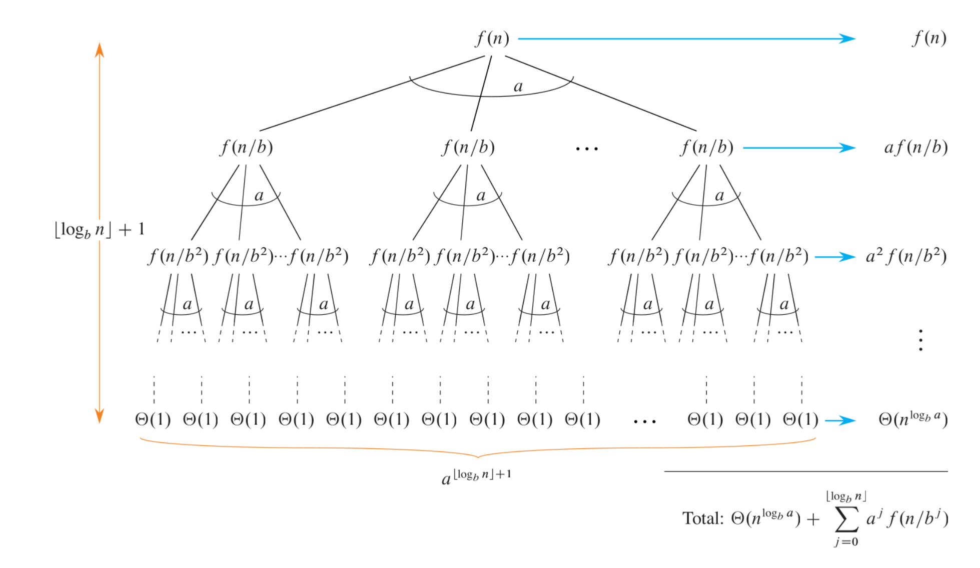
With $c := \log_b(a)$, we have for the above recurrence:

(a) $T(n) = \Theta(n^c)$ if $f(n) = O(n^{c-\varepsilon})$ for constant $\varepsilon > 0$.

(b) $T(n) = \Theta(n^c \log n)$ if $f(n) = \Theta(n^c)$.

(c) $T(n) = \Theta(f(n))$ if $f(n) = \Omega(n^{c+\varepsilon})$ for constant $\varepsilon > 0$ **and** f satisfies the regularity condition $\exists n_0, \alpha < 1 \; \forall n \geq n_0 \;:\; a \cdot f\left(\frac{n}{b}\right) \leq \alpha f(n)$.

Master Theorem – Intuition & Proof Idea



When it's fine to ignore floors and ceilings

The ***polynomial-growth condition***

- $f : \mathbb{R}_{>0} \to \mathbb{R}$ satisfies the *polynomial-growth condition* if

$$\exists n_0 \; \forall C \ge 1 \; \exists D > 1 \quad \forall n \ge n_0 \; \forall c \in [1, C] \;\; : \;\; \frac{1}{D} f(n) \;\le\; f(cn) \;\le\; D f(n)$$

- intuitively: increasing n by up to a factor C (and anywhere in between!) changes the function value by at most a factor $D = D(C)$ (for sufficiently large n)

- examples: $f(n) = \Theta(n^{\alpha} \log^{\beta}(n) \log\log^{\gamma}(n))$ for constants α, β, γ zero allowed
  $\rightsquigarrow$ f satisfies the polynomial-growth condition

Lemma 5.2 (Polynomial-growth master method)

If the toll function $f(n)$ satisfies the polynomial-growth condition, then the Θ-class of the solution of a D&C recurrence remains the same when ignoring floors and ceilings on subproblem sizes.

A Rigorous and Stronger Meta Theorem

Theorem 5.3 (Roura's Discrete Master Theorem)

Let $T(n)$ be recursively defined as

$$T(n) = \begin{cases} b_n & 0 \le n < n_0, \\ f(n) + \sum_{d=1}^{D} a_d \cdot T\left(\frac{n}{b_d} + r_{n,d}\right) & n \ge n_0, \end{cases}$$

where $D \in \mathbb{N}$, $a_d > 0$, $b_d > 1$, for $d = 1, \ldots, D$ are constants, functions $r_{n,d}$ satisfy $|r_{n,d}| = O(1)$ as $n \to \infty$, and function $f(n)$ satisfies $f(n) \sim B \cdot n^{\alpha}(\ln n)^{\gamma}$ for constants $B > 0$, α, γ.
Set $H = 1 - \sum_{d=1}^{D} a_d\,(1/b_d)^{\alpha}$; then we have:

- **(a)** If $H < 0$, then $T(n) = O(n^{\tilde{\alpha}})$, for $\tilde{\alpha}$ the unique value of α that would make $H = 0$.
- **(b)** If $H = 0$ and $\gamma > -1$, then $T(n) \sim f(n)\ln(n)/\tilde{H}$ with constant $\tilde{H} = (\gamma + 1)\sum_{d=1}^{D} a_d\, b_d^{-\alpha} \ln(b_d)$.
- **(c)** If $H = 0$ and $\gamma = -1$, then $T(n) \sim f(n)\ln(n)\ln(\ln(n))/\hat{H}$ with constant $\hat{H} = \sum_{d=1}^{D} a_d\, b_d^{-\alpha} \ln(b_d)$.
- **(d)** If $H = 0$ and $\gamma < -1$, then $T(n) = O(n^{\alpha})$.
- **(e)** If $H > 0$, then $T(n) \sim f(n)/H$.

5.2 Order Statistics

Selection by Rank

- Standard data summary of numerical data: (Data scientists, listen up!)
  - mean, standard deviation
  - min/max (range)
  - histograms

  easy to compute in $\Theta(n)$ time

  - median, quartiles, other quantiles (a.k.a. order statistics)

  
  computable in $\Theta(n)$ time?

General form of problem: **Selection by Rank**

- **Given:** array $A[0..n)$ of numbers and number $k \in [0..n)$.
- **Goal:** find element that would be in position k if A was sorted (kth smallest element).

  but 0-based & counting dups

- $k = \lfloor n/2 \rfloor \rightsquigarrow$ median; $k = \lfloor n/4 \rfloor \rightsquigarrow$ lower quartile
  $k = 0 \rightsquigarrow$ minimum; $k = n - \ell \rightsquigarrow$ ℓth largest

Quickselect

- Key observation: Finding the element of rank k seems hard.
  But computing the rank of a given element is easy!
  count smaller elements

$\rightsquigarrow$ Pick any element $A[b]$ and find its rank j.

- $j = k$? $\rightsquigarrow$ Lucky Duck! Return chosen element and stop
- $j < k$? $\rightsquigarrow$ ... not done yet. But: The $j + 1$ elements smaller than $\le A[b]$ can be excluded!
- $j > k$? $\rightsquigarrow$ similarly exclude the $n - j$ elements $\ge A[b]$

- partition function from Quicksort:
  - returns the rank of pivot
  - separates elements into smaller/larger

$\rightsquigarrow$ can use same building blocks

```
procedure quickselect(A[l..r), k)
    if r − ℓ ≤ 1 then return A[l]
    b := choosePivot(A[l..r))
    j := partition(A[l..r), b)
    if j == k
        return A[j]
    else if j < k
        quickselect(A[j + 1..n), k − j − 1)
    else // j > k
        quickselect(A[0..j), k)
```

Quickselect – Iterative Code

Recursion can be replaced by loop (*tail-recursion elimination*)

```
procedure quickselect(A[l..r), k)
    if r − ℓ ≤ 1 then return A[l]
    b := choosePivot(A[l..r))
    j := partition(A[l..r), b)
    if j == k
        return A[j]
    else if j < k
        quickselect(A[j + 1..n), k − j − 1)
    else // j > k
        quickselect(A[0..j), k)
```

```
procedure quickselectIterative(A[0..n), k)
    l := 0;  r := n
    while r − l > 1
        b := choosePivot(A[l..r))
        j := partition(A[l..r), b)
        if j ≥ k then r := j − 1
        if j ≤ k then l := j + 1
    return A[k]
```

- implementations should usually prefer iterative version
- analysis more intuitive with recursive version

Quickselect – Analysis

```
procedure quickselect(A[l..r), k)
    if r − ℓ ≤ 1 then return A[l]
    b := choosePivot(A[l..r))
    j := partition(A[l..r), b)
    if j == k
        return A[j]
    else if j < k
        quickselect(A[j + 1..n), k − j − 1)
    else // j > k
        quickselect(A[0..j), k)
```

- cost = #cmps
- costs depend on n **and** k
- **worst case:** $k = 0$, but always $j = n - 2$
  - $\rightsquigarrow$ each recursive call makes n one smaller at cost $\Theta(n)$
  - $\rightsquigarrow$ $T(n,k) = \Theta(n^2)$ worst case cost

average case:

- let $T(n,k)$ expected cost when we choose a pivot uniformly from $A[0..n)$

$\rightsquigarrow$ formulate recurrence for $T(n,k)$ similar to BST/Quicksort recurrence

$$T(n,k) \;=\; n \;+\; \frac{1}{n}\sum_{r=0}^{n-1} [r = k]\cdot 0 \;+\; [k < r]\cdot T(r,k) \;+\; [k > r]\cdot T(n-r-1, k-r-1)$$

Quickselect – Average Case Analysis

- $T(n,k) \;=\; n \;+\; \dfrac{1}{n}\displaystyle\sum_{r=0}^{n-1}[r=k]\cdot 0 \;+\; [k<r]\cdot T(r,k) \;+\; [k>r]\cdot T(n-r-1,k-r-1)$
- Set $\hat{T}(n) \;=\; \max_{k\in[0..n)} T(n,k)$

$\rightsquigarrow \hat{T}(n) \;\le\; n \;+\; \dfrac{1}{n}\displaystyle\sum_{r=0}^{n-1}\max\{\hat{T}(r),\hat{T}(n-r-1)\}$

- analyze hypothetical, worse algorithm:
  if $r \notin [\frac14 n, \frac34 n)$, discard pivot and repeat with new one!

$\rightsquigarrow \hat{T}(n) \le \tilde{T}(n)$ defined by $\quad \tilde{T}(n) \;\le\; n \;+\; \frac12\tilde{T}(n) \;+\; \frac12\tilde{T}(\frac34 n)$

$\rightsquigarrow \tilde{T}(n) \;\le\; 2n \;+\; \tilde{T}(\frac34 n)$

- Master Theorem Case 3: $\tilde{T}(n) = \Theta(n)$

Quickselect Discussion

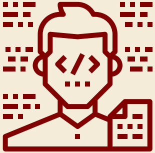
- $\Theta(n^2)$ worst case (like Quicksort)
- expected cost $\Theta(n)$ (best possible)
- no extra space needed
- adaptations possible to find several order statistics at once

- expected cost can be further improved by choosing pivot from a small sorted sample
  - $\rightsquigarrow$ asymptotically optimal randomized cost: $n + \min\{k, n-k\}$ comparisons in expectation achieved asymptotically by the *Floyd-Rivest algorithm*

5.3 Linear-Time Selection

Interlude – A recurring conversation

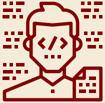
Cast of Characters:



Hi! I'm a *computer science practitioner*.
I love algorithms for the sometimes miraculous applications they enable.
I care for **things** I can implement and **that actually work in practice**.

Hi! I'm a *theoretical computer science researcher*.
I find beauty in elegant and **definitive** answers to questions about complexity.
I care for **eternal truths** and mathematically proven facts;
asymptotically optimal is what counts! (Constant factors are secondary.)

Quickselect Disagreements



For practical purposes, (randomized) Quickselect is perfect.

e. g. used in C++ STL `std::nth_element`

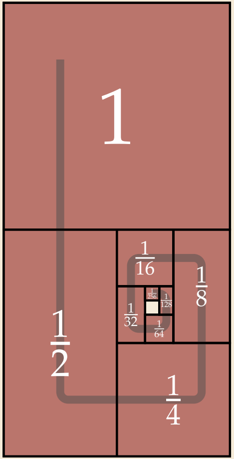
Yeah . . . maybe. But can we select by rank in $O(n)$ deterministic **worst case** time?

Better Pivots

It turns out, we can!

- All we need is better pivots!
  - If pivot was the exact median,
    we would at least halve #elements in each step
  - Then the total cost of all partitioning steps is $\leq 2n = \Theta(n)$.

But: finding medians is (basically) our original problem!



It totally suffices to find an element of rank αn for $\alpha \in (\varepsilon, 1 - \varepsilon)$
to get overall costs $\Theta(n)$!

The Median-of-Medians Algorithm

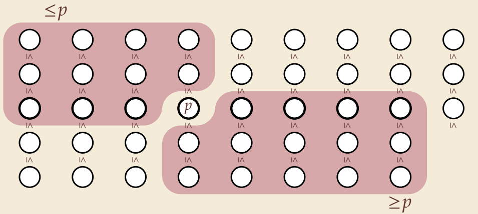
```
procedure choosePivotMoM(A[l..r])
    m := ⌊n/5⌋
    for i := 0, ..., m − 1
        sort(A[5i..5i + 4])
        // collect median of 5
        Swap A[i] and A[5i + 2]
    return quickselectMoM(A[0..m), ⌊(m−1)/2⌋)

procedure quickselectMoM(A[l..r), k)
    if r − ℓ ≤ 1 then return A[l]
    b := choosePivotMoM(A[l..r))
    j := partition(A[l..r), b)
    if j == k
        return A[j]
    else if j < k
        quickselectMoM(A[j + 1..n), k − j − 1)
    else // j > k
        quickselectMoM(A[0..j), k)
```

Analysis:

- Note: 2 mutually recursive procedures
  $\rightsquigarrow$ effectively 2 recursive calls!

1. recursive call inside choosePivotMoM on $m \le \frac{n}{5}$ elements
2. recursive call inside quickselectMoM



$\rightsquigarrow$ partition excludes $\sim 3 \cdot \frac{m}{2} \sim \frac{3}{10} n$ elem.

$$
\begin{aligned}
\rightsquigarrow\ C(n) &\le \Theta(n) + C(\tfrac{1}{5}n) + C(\tfrac{7}{10}n) \\
&\le \Theta(n) + C(\tfrac{1}{5}n + \tfrac{7}{10}n) \\
&= \Theta(n) + C(\tfrac{9}{10}n) \quad \rightsquigarrow \quad C(n) = \Theta(n)
\end{aligned}
$$

ansatz: overall cost linear

5.4 Karatsuba Integer Multiplication

Integer Multiplication

- What's the cost of computing $x \cdot y$ for two integers x and y?

$\rightsquigarrow$ depends on how big the numbers are!

  - If x and y have $O(w)$ bits, multiplication takes $O(1)$ time on word-RAM
  - otherwise, need a dedicated algorithm!

Long multiplication (»Schulmethode«)

- Given $x = \sum_{i=0}^{n-1} x_i 2^i$ and $y = \sum_{i=0}^{n-1} y_i 2^i$, want $z = \sum_{i=0}^{2n-1} z_i 2^i$

```
for i := 0, ..., n − 1
    c := 0
    for j := 0, ..., n − 1
        z_{i+j} := z_{i+j} + c + x_i · y_j
        c := ⌊z_{i+j}/2⌋
        z_{i+j} := z_{i+j} mod 2
    end for
    z_{i+n} := c
end for
```

- $\Theta(n^2)$ bit operations
- could work with base 2^w instead of 2

  $\rightsquigarrow$ $\Theta\big((n/w)^2\big)$ time
- here: count bit operations for simplicity can be generalized

Example:
easier in binary!
("shift and add")

```
1001010101 * 101101
-------------------
    1001010101
     0000000000
      1001010101
       1001010101
        0000000000
         1001010101
-------------------
    110100011110001
```

Divide & Conquer Multiplication

- assume n is power of 2 (fill up with 0-bits otherwise)
- We can write
  - $x = a_1 2^{n/2} + a_2$ and
  - $y = b_1 2^{n/2} + b_2$
  - for a_1, a_2, b_1, b_2 integers with $n/2$ bits

$$\rightsquigarrow\ x \cdot y \;=\; (a_1 2^{n/2} + a_2) \cdot (b_1 2^{n/2} + b_2) \;=\; \boxed{\mathbf{a_1 b_1} 2^n + (\mathbf{a_1 b_2} + \mathbf{a_2 b_1}) 2^{n/2} + \mathbf{a_2 b_2}}$$

  - recursively compute **4** smaller products
  - combine with shifts and additions ($O(n)$ bit operations)

- ... but is this any good?
  - $T(n) \;=\; \mathbf{4} \cdot T(n/2) + \Theta(n)$

  $\rightsquigarrow$ Master Theorem Case 1: $T(n) = \Theta(n^2)$ *... just like the primary school method!?*
    - but Master Theorem gives us a hint: cost is dominated by the leaves

    $\rightsquigarrow$ try to do more work in conquer step!

Karatsuba Multiplication

- how can we do "less divide and more conquer"?

Recall: $x \cdot y \;=\; \boxed{\color{red}{a_1 b_1} 2^n + (\color{red}{a_1 b_2} + \color{red}{a_2 b_1}) 2^{n/2} + \color{red}{a_2 b_2}}$

Let's do some algebra.

$$\begin{aligned} c \;&:=\; (a_1 + a_2) \cdot (b_1 + b_2) \\ &=\; a_1 b_1 + (a_1 b_2 + a_2 b_1) + a_2 b_2 \end{aligned}$$

$\rightsquigarrow$ $(\color{red}{a_1 b_2} + \color{red}{a_2 b_1}) \;=\; c - a_1 b_1 - a_2 b_2$

this can be computed with **3** recursive multiplications

$a_1 + a_2$ and $b_1 + b_2$ still have roughly $n/2$ bits

```
procedure karatsuba(x, y):
    // Assume x and y are n = 2^k bit integers
    a1 := ⌊x/2^(n/2)⌋;  a2 := x mod 2^(n/2) // implemented by shifts
    b1 := ⌊y/2^(n/2)⌋;  b2 := y mod 2^(n/2)
    c1 := karatsuba(a1, b1)
    c2 := karatsuba(a2, b2)
    c  := karatsuba(a1 + a2, b1 + b2) − c1 − c2
    return c1·2^n + c·2^(n/2) + c2 // shifts and additions
```

Analysis:

- nonrecursive cost: only additions and shifts
- all numbers $O(n)$ bits

$\rightsquigarrow$ conquer cost $f(n) = \Theta(n)$

Recurrence:

- $T(n) \;=\; 3T(n/2) + \Theta(n)$
- Master Theorem Case 1

$\rightsquigarrow$ $T(n) = \Theta(n^{\lg 3}) = O(n^{1.585})$

much cheaper (for large n)!

Integer Multiplication

- until 1960, integer multiplication was conjectured to take $\Omega(n^2)$ bit operations

$\rightsquigarrow$ Karatsuba's algorithm was a big breakthrough
  - which he discovered as a student!
- idea can be generalized to breaking numbers into $k \geq 2$ parts (*Toom-Cook algorithm*)
- asymptotically *much* better algorithms are now known!
  - e. g., the *Schönhage-Strassen algorithm* with $O(n \log n \log \log n)$ bit operations (!)
  - these are based on the *Fast Fourier Transform* (FFT) algorithm
    - numbers = polynomials evaluated at base (e. g., $z = 2$)

    $\rightsquigarrow$ multiplication of numbers = convolution of polynomials
    - FFT makes computation of this convolution cheap by computing the polynomial via interpolation
    - Schönhage-Strassen adds careful finite-field algebra to make computations efficient

5.5 Majority

Majority

- **Given:** Array $A[0..n)$ of objects
- **Goal:** Check of there is an object x that occurs at $> \frac{n}{2}$ positions in A
  if so, return x
- Naive solution: check each $A[i]$ whether it is a majority $\rightsquigarrow$ $\Theta(n^2)$ time

Can be solved faster using a simple Divide & Conquer approach:

- If A has a majority, that element must also be a majority of at least one half of A.

$\rightsquigarrow$ Can find majority (if it exists) of left half and right half recursively

$\rightsquigarrow$ Check these ≤ 2 candidates.

- Costs similar to mergesort $\Theta(n \log n)$

Majority – Code

procedure GetMajorityElement(a[1...n])

Input: Array a of objects

Output: Majority element of a

if n = 1: return a[1]

k = $\left\lfloor \frac{n}{2} \right\rfloor$

$elem_{lsub}$ = GetMajorityElement(a[1...k])

$elem_{rsub}$ = GetMajorityElement(a[k+1...n]

if $elem_{lsub}$ = $elem_{rsub}$:

  return $elem_{lsub}$

lcount = GetFrequency(a[1...n],$elem_{lsub}$)

rcount = GetFrequency(a[1...n],$elem_{rsub}$)

if lcount $>$ k+1:

  return $elem_{lsub}$

else if rcount $>$ k+1:

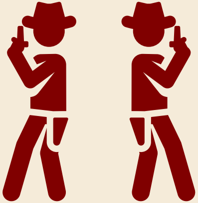
  return $elem_{rsub}$

else return NO-MAJORITY-ELEMENT

Majority – Linear Time

We can actually do much better!

```
def MJRTY(A[0..n))
    c := 0
    for i := 1, ..., n − 1
        if c == 0
            x := A[i]; c := 1
        else
            if A[i] == x then c := c + 1 else c := c − 1
    return x
```



- MJRTY($A[0..n)$) returns *candidate* majority element
- either that candidate is the majority element or none exists(!)

Clearly $\Theta(n)$ time

5.6 Closest Pair of Points in the Plane

Closest Pair of Points in the Plane

- **Given:** Array $P[0..n)$ of points in the plane ($\mathbb{R}^2$)
  each has x and y coordinates: $P[i].x$ and $P[i].y$
- **Goal:** Find pair $P[i], P[j]$ that is closest in (Euclidean) distance

- Naive solution: compute distance of each pair $\rightsquigarrow$ $\Theta(n^2)$ time
  - cost here = # arithmetic operations
  - ignore numerical accuracy

Divide & Conquer

Refined Merge