4 Efficient Sorting

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CS566 (Wintersemester 2025–26)
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Learning Outcomes

Unit 4: ***Efficient Sorting***

1. Know principles and implementation of *mergesort* and *quicksort*.
2. Know properties and *performance characteristics* of mergesort and quicksort.
3. Know the comparison model and understand the corresponding *lower bound*.
4. Understand *counting sort* and how it circumvents the comparison lower bound.
5. Know ways how to exploit *presorted* inputs.

Outline

4 Efficient Sorting

Why study sorting?

- fundamental problem of computer science that is still not solved

Algorithm with optimal #comparisons in worst case?

- building brick of many more advanced algorithms
  - for preprocessing
  - as subroutine
- playground of manageable complexity
  to practice algorithmic techniques

Here:

- "classic" fast sorting method
- exploit **partially sorted** inputs
- **parallel** sorting

later

Part I

The Basics

Rules of the game

- **Given:**
  - array $A[0..n) = A[0..n-1]$ of n objects
  - a total order relation $\leq$ among $A[0], \ldots, A[n-1]$
    (a comparison function)
    Python: elements support <= operator (`__le__()`)
    Java: `Comparable` class (`x.compareTo(y) <= 0`)
- **Goal:** rearrange (i. e., permute) elements within A,
  so that A is *sorted*, i. e., $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

- for now: A stored in main memory (*internal sorting*)
  single processor (*sequential sorting*)

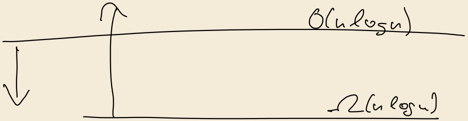
Clicker Question

$\Theta(n \log n)$

What is the complexity of sorting? Type you answer, e. g., as "`Theta(sqrt(n))`"

(a) $O(n \log n)$ algorithm solving the problem

(b) lower bound for problem $\Omega(n \log n)$



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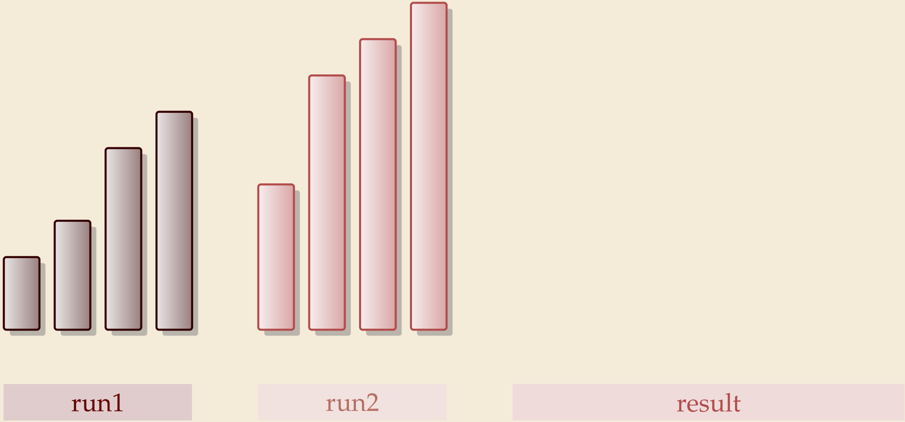
4.1 Mergesort

Merging sorted lists

Merging sorted lists



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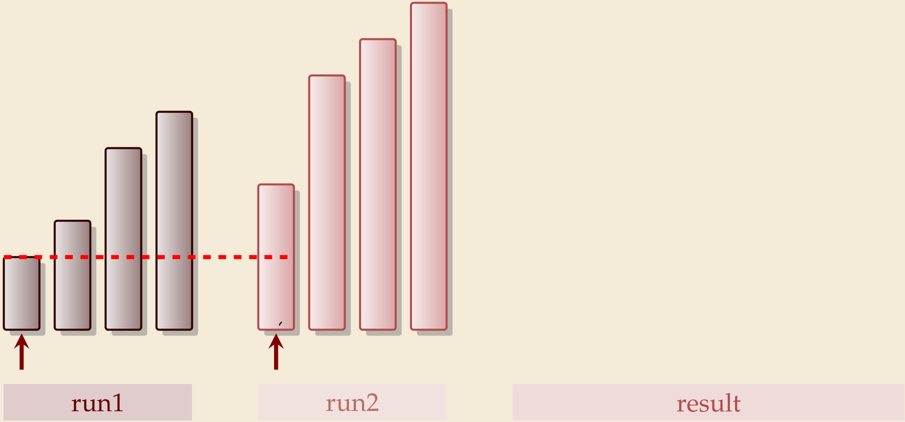

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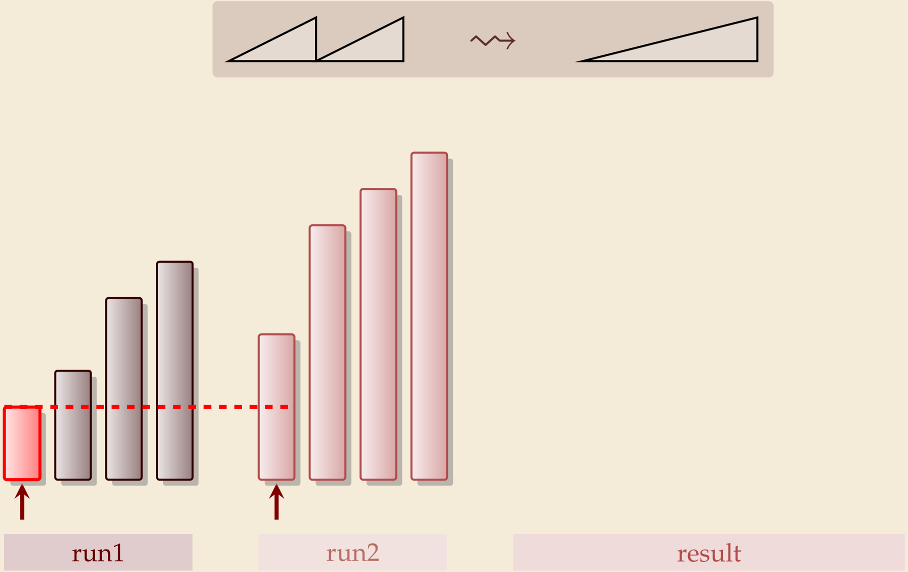

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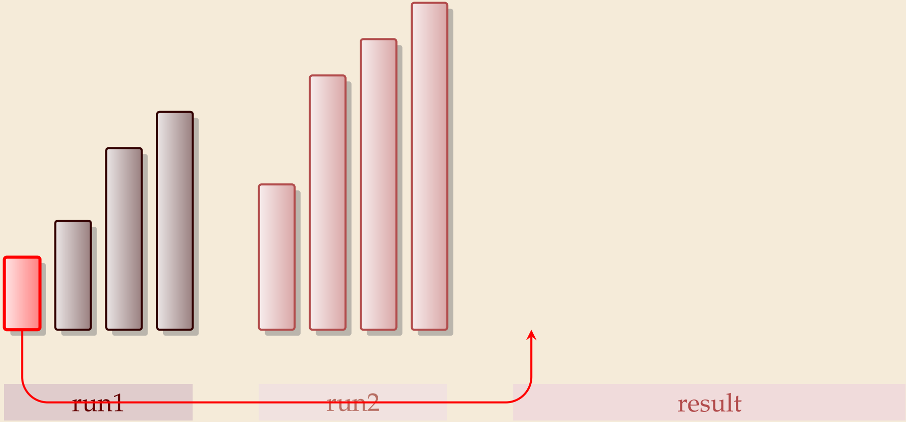

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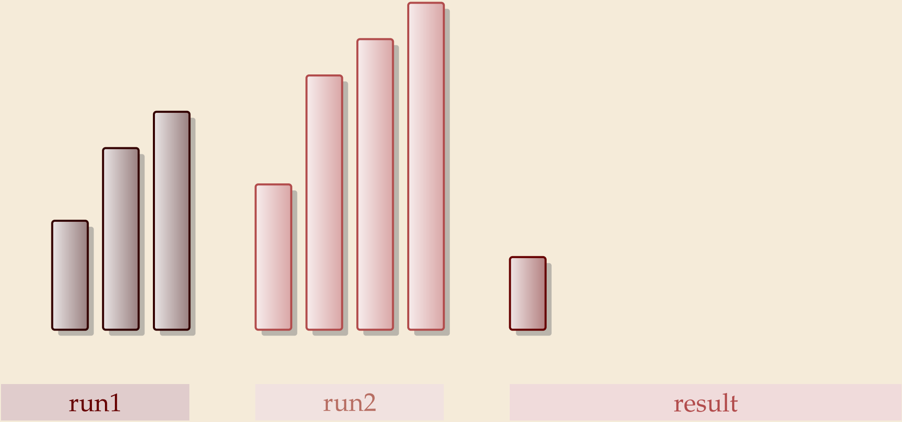

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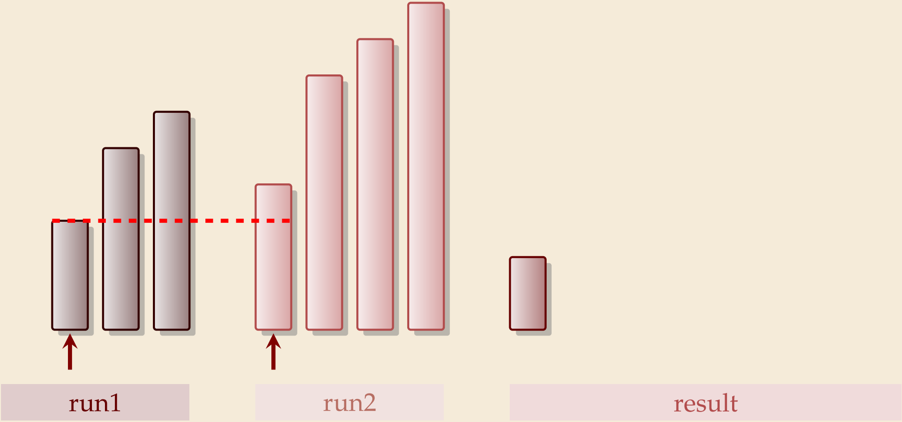

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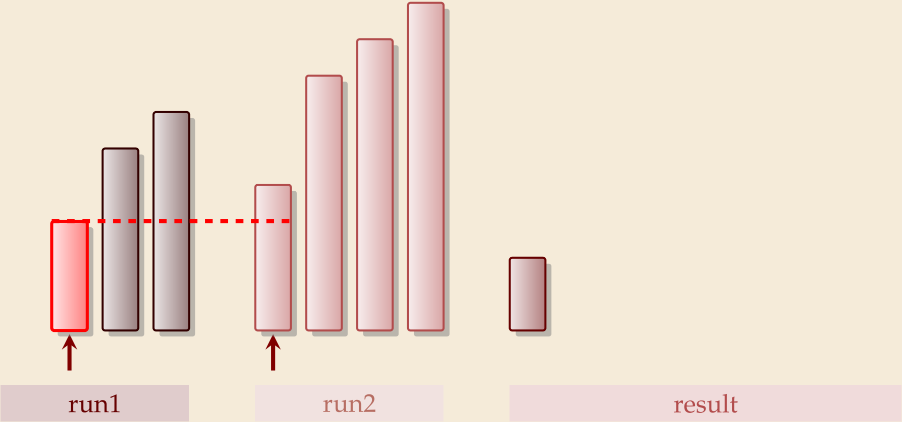

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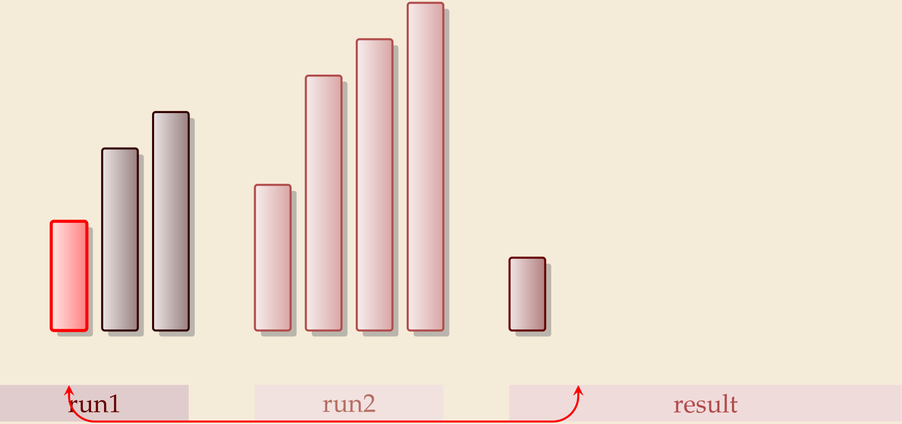

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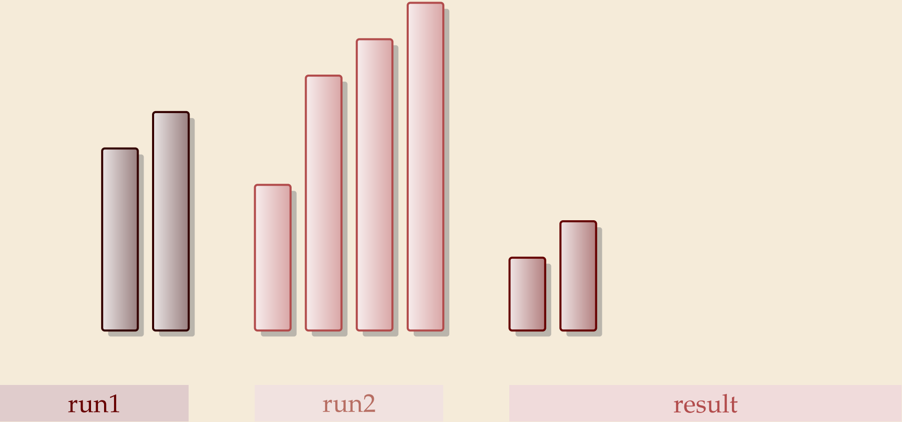

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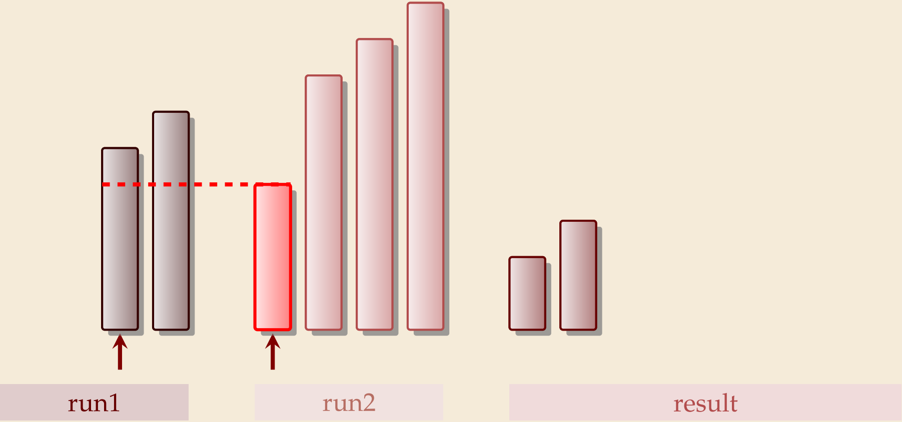

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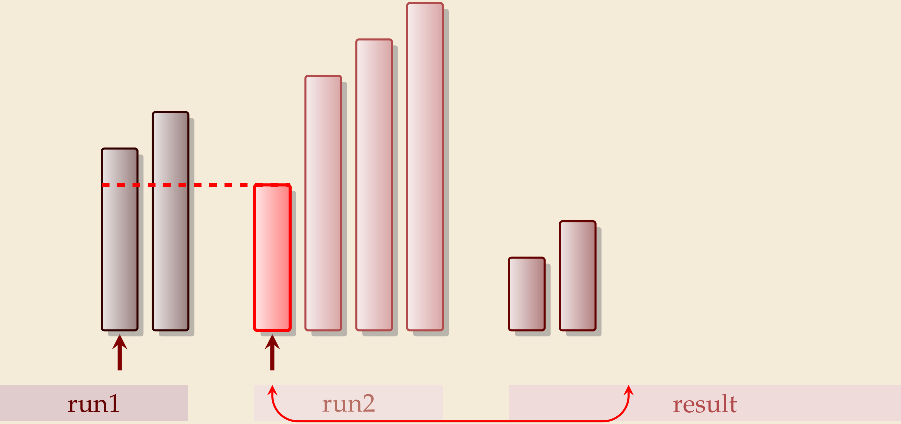

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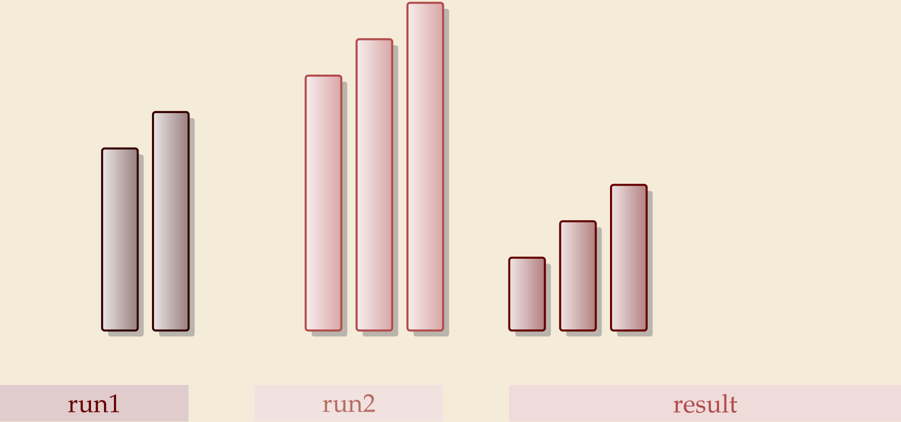

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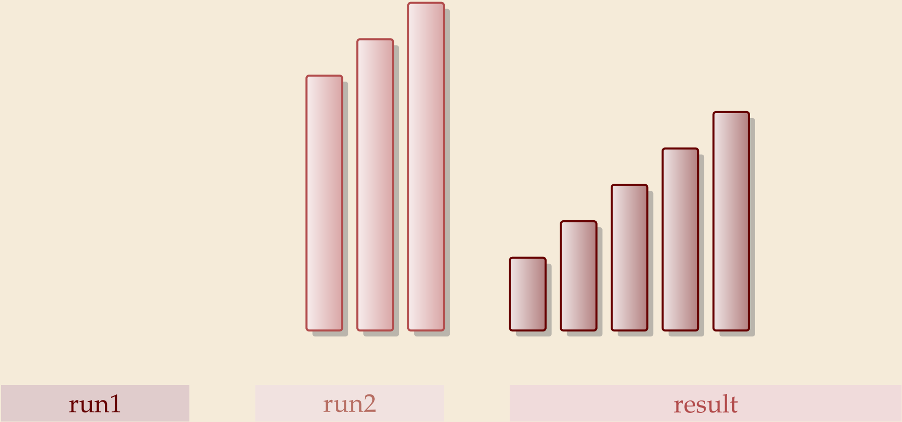

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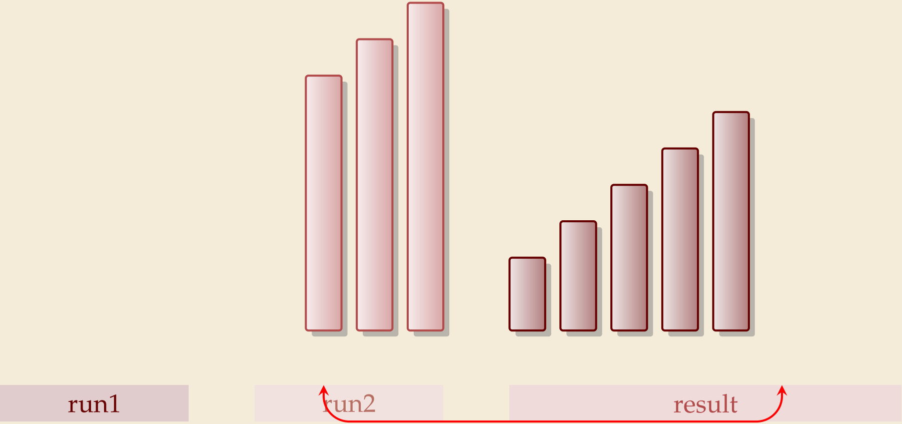

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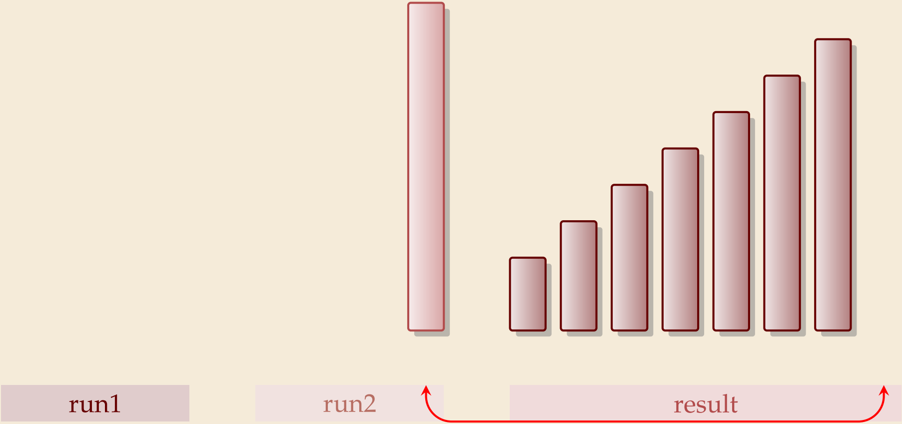

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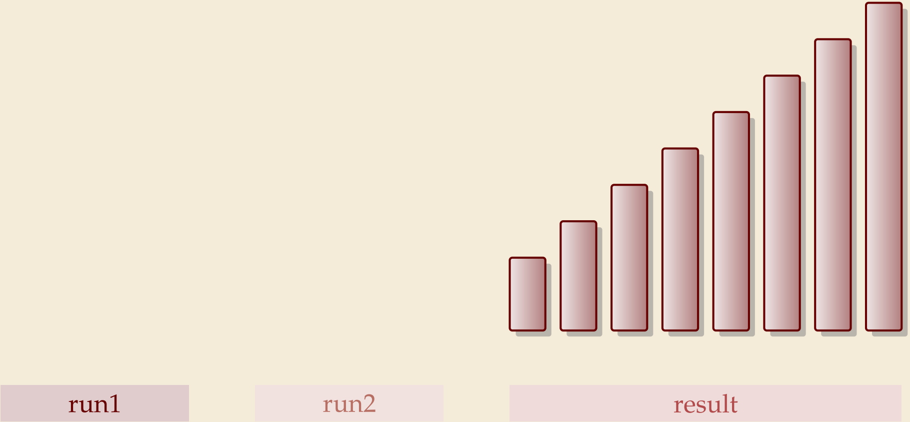

Clicker Question

What is the worst-case running time of mergesort?

A	$\Theta(1)$	**G**	$\Theta(n \log n)$
B	$\Theta(\log n)$	**H**	$\Theta(n \log^2 n)$
C	$\Theta(\log \log n)$	**I**	$\Theta(n^{1+\epsilon})$
D	$\Theta(\sqrt{n})$	**J**	$\Theta(n^2)$
E	$\Theta(n)$	**K**	$\Theta(n^3)$
F	$\Theta(n \log \log n)$	**L**	$\Theta(2^n)$

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What is the worst-case running time of mergesort?

- **A** $\cancel{\Theta(1)}$
- **B** $\cancel{\Theta(\log n)}$
- **C** $\cancel{\Theta(\log \log n)}$
- **D** $\cancel{\Theta(\sqrt{n})}$
- **E** $\cancel{\Theta(n)}$
- **F** $\cancel{\Theta(n \log \log n)}$
- **G** $\Theta(n \log n)$ ✓
- **H** $\cancel{\Theta(n \log^2 n)}$
- **I** $\cancel{\Theta(n^{1+\epsilon})}$
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- **L** $\cancel{\Theta(2^n)}$

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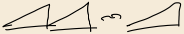
Mergesort

```
procedure mergesort(A[l..r]):
    n := r − l
    if n ≤ 1 return
    m := l + ⌊n/2⌋
    mergesort(A[l..m))
    mergesort(A[m..r))
    merge(A[l..m), A[m..r), buf)
    copy buf to A[l..r)
```



- recursive procedure
- merging needs
  - temporary storage *buf* for result (of same size as merged runs)
  - to read and write each element twice (once for merging, once for copying back)

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- recursive procedure
- merging needs
  - temporary storage *buf* for result (of same size as merged runs)
  - to read and write each element twice (once for merging, once for copying back)

Analysis: count *"element visits"* (read and/or write)

$$C(n) \;=\; \begin{cases} 0 & n \le 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \ge 2 \end{cases}$$

same for best and worst case!

Simplification $\boxed{n = 2^k}$

$$C(2^k) \;=\; \begin{cases} 0 & k \le 0 \\ 2 \cdot C(2^{k-1}) + 2 \cdot 2^k & k \ge 1 \end{cases} \;=\; 2 \cdot 2^k + 2^2 \cdot 2^{k-1} + 2^3 \cdot 2^{k-2} + \cdots + 2^k \cdot 2^1 \;=\; 2k \cdot 2^k$$

$$C(n) \;=\; 2n \lg(n) \;=\; \Theta(n \log n)$$

(arbitrary n: $C(n) \le C(\text{next larger power of 2}) \le 4n\lg(n) + 2n = \Theta(n \log n)$)

Mergesort

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same for best and worst case!

(precisely(!) solvable *without* assumption $n = 2^k$:
$C(n) = 2n\lg(n) + \big(2 - \{\lg(n)\} - 2^{1-\{\lg(n)\}}\big)2n$
with $\{x\} := x - \lfloor x \rfloor$)

Simplification $\boxed{n = 2^k}$

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$$C(n) \;=\; 2n\lg(n) \;=\; \Theta(n \log n)$$

(arbitrary n: $C(n) \le C(\text{next larger power of 2}) \le \underline{4n\lg(n) + 2n} = \Theta(n \log n)$)

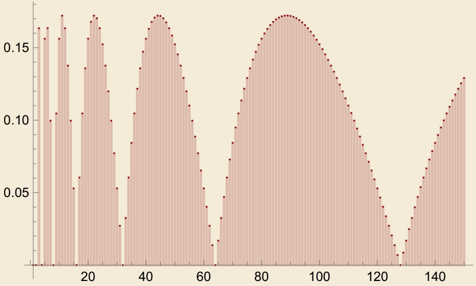
Linear Term of $C(n)$

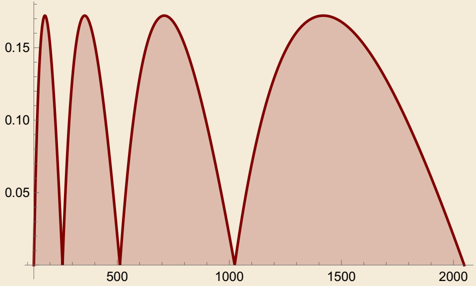
$\notin$ exam

Recall:

$$C(n) \;=\; 2n\lg(n) + \underbrace{\left(2 - \{\lg(n)\} - 2^{1-\{\lg(n)\}}\right)}2n \qquad \text{with } \{x\} := x - \lfloor x \rfloor$$

Plot of $2\left(2 - \{\lg(n)\} - 2^{1-\{\lg(n)\}}\right)$





$\rightsquigarrow$ Can prove: $\boxed{C(n) \;\le\; 2n\lg n + 0.172n}$

Mergesort – Discussion

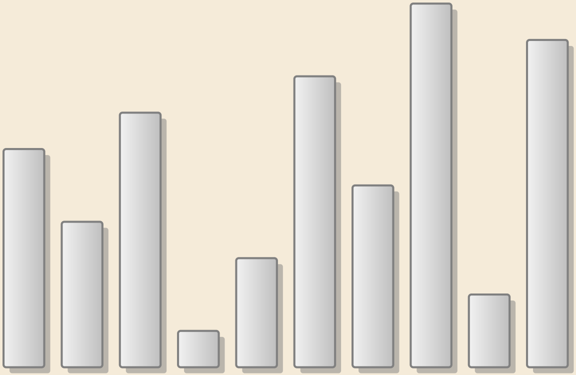
- optimal time complexity of $\Theta(n \log n)$ in the worst case
- *stable* sorting method   i. e., retains relative order of equal-key items
- memory access is sequential (scans over arrays)
- requires $\Theta(n)$ extra space

  there are in-place merging methods,
  but they are substantially more complicated
  and not (widely) used

4.2 Quicksort

Partitioning around a pivot

Partitioning around a pivot



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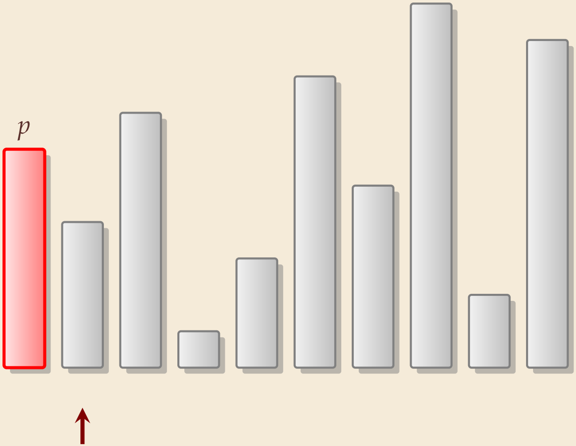

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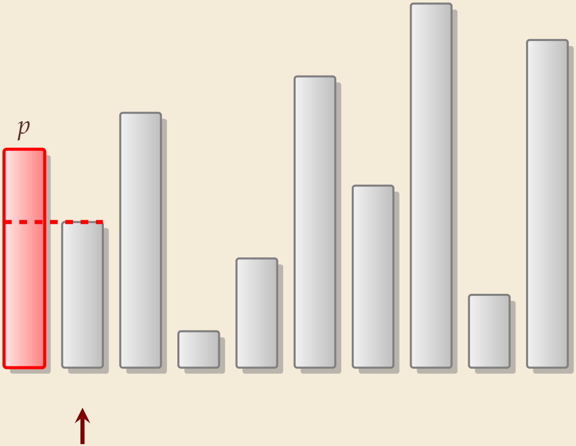

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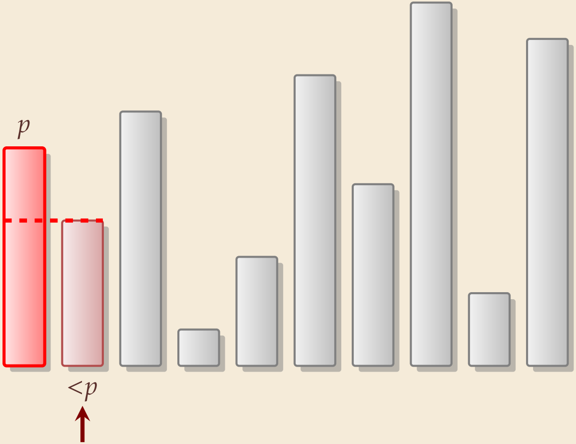

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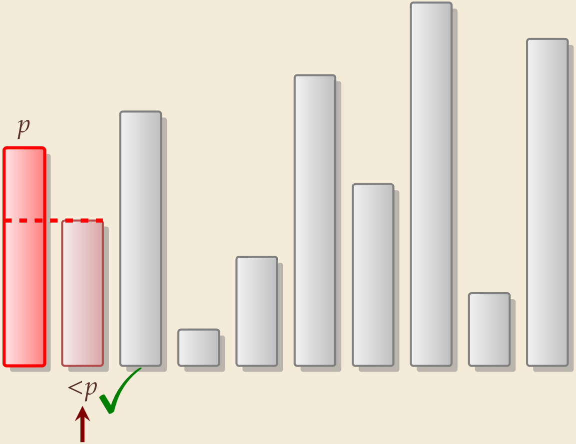

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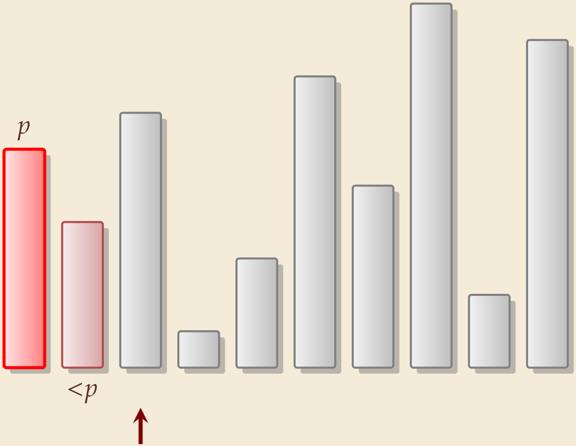

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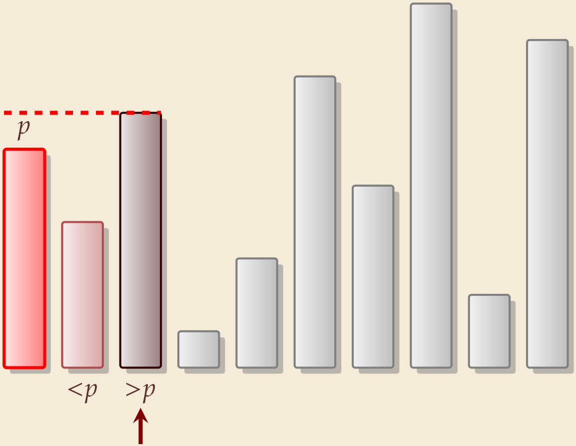

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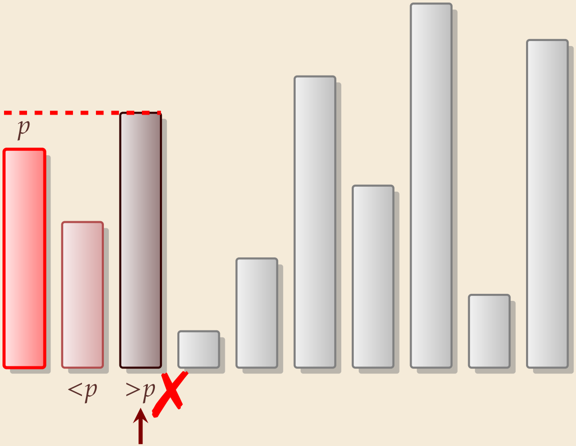

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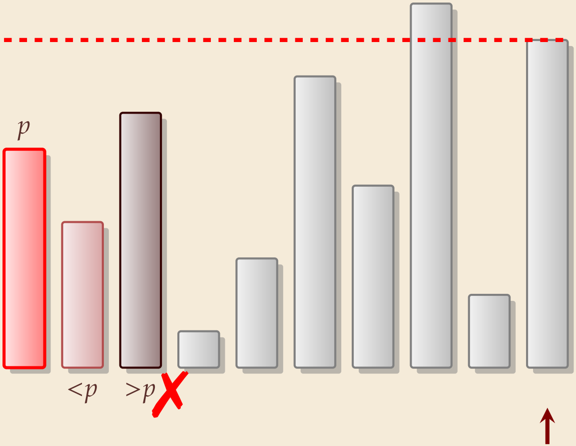

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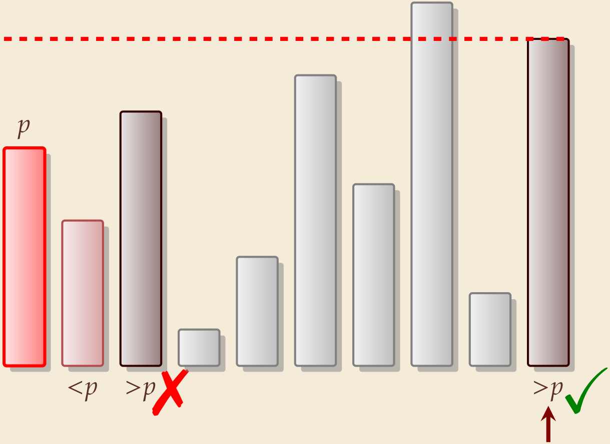

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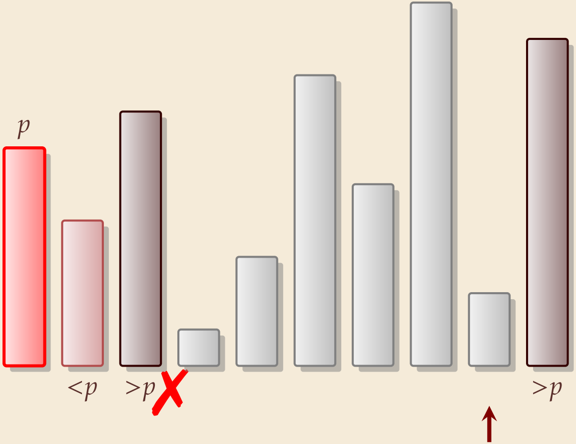

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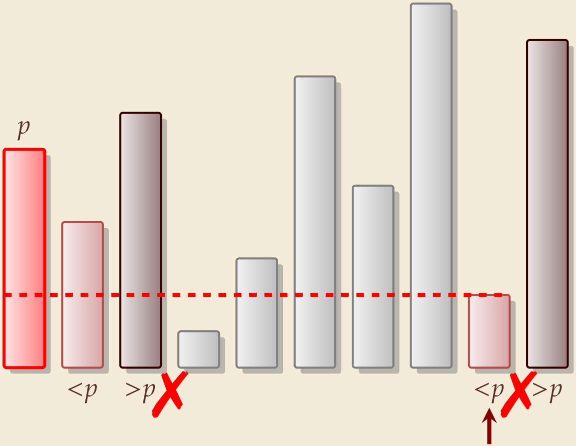

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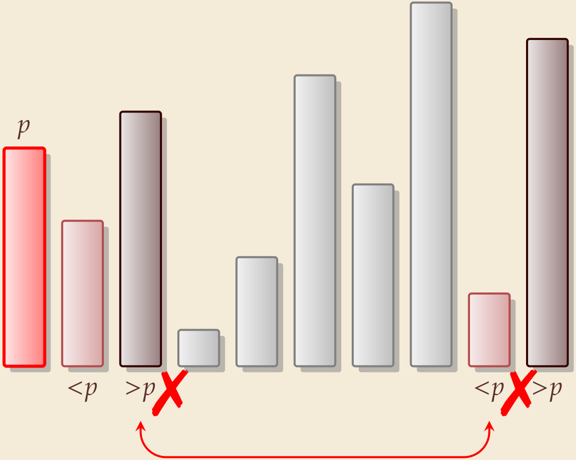

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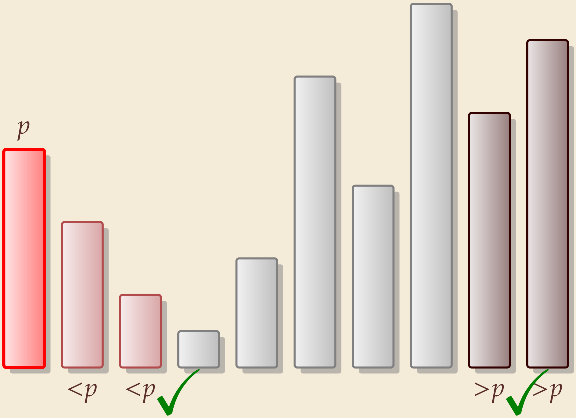

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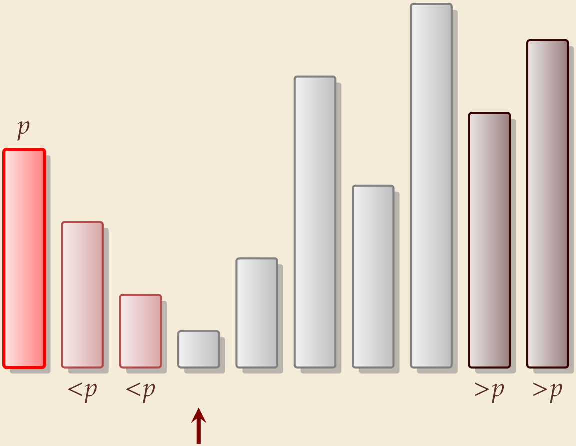

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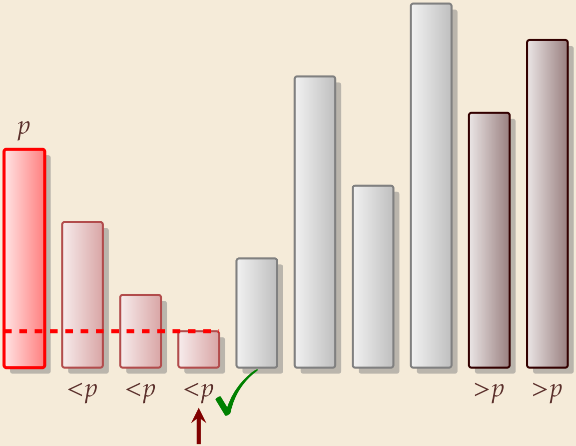

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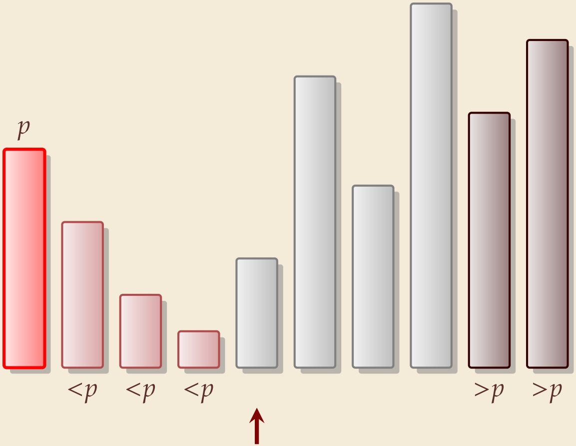

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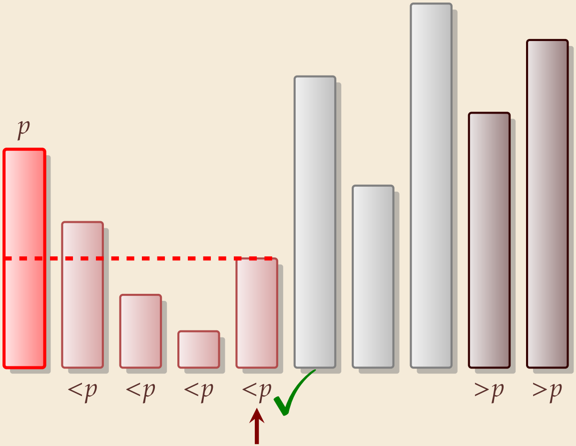

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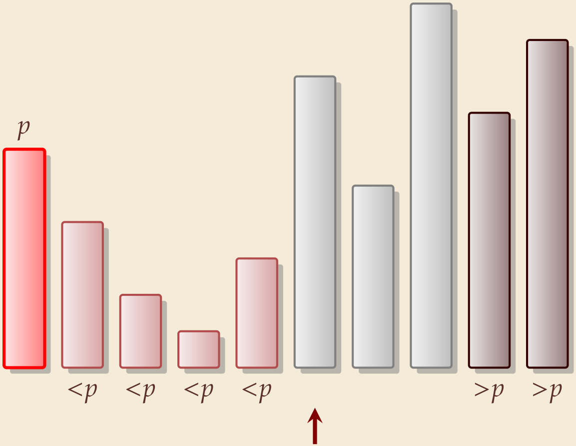

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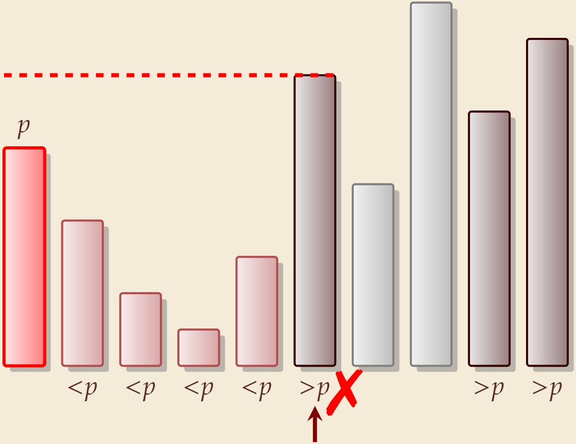

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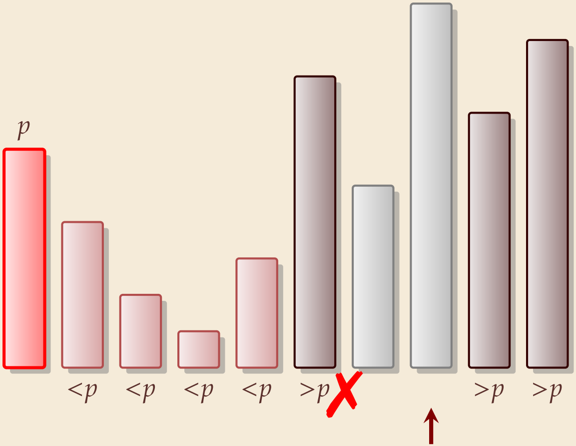

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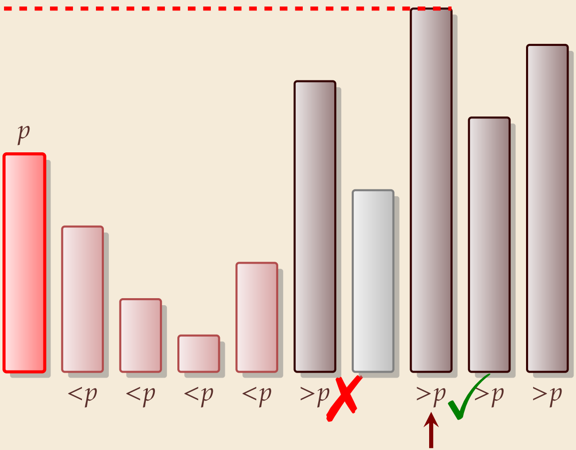

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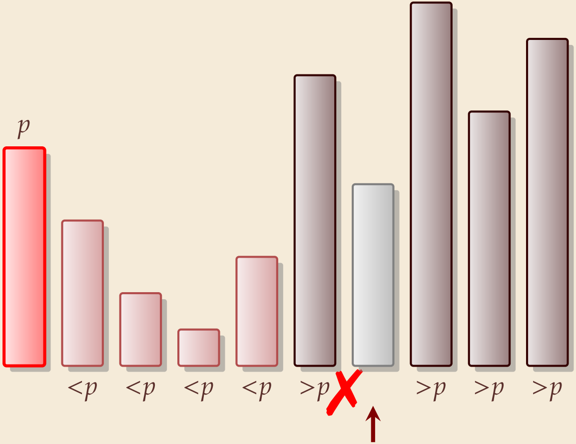

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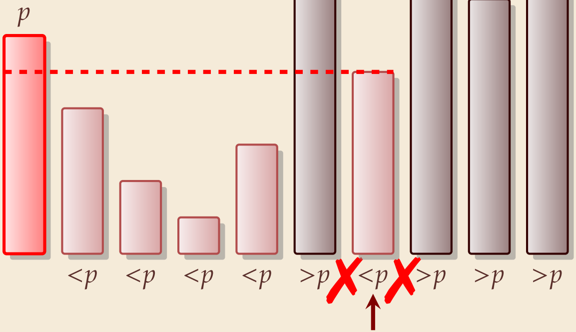

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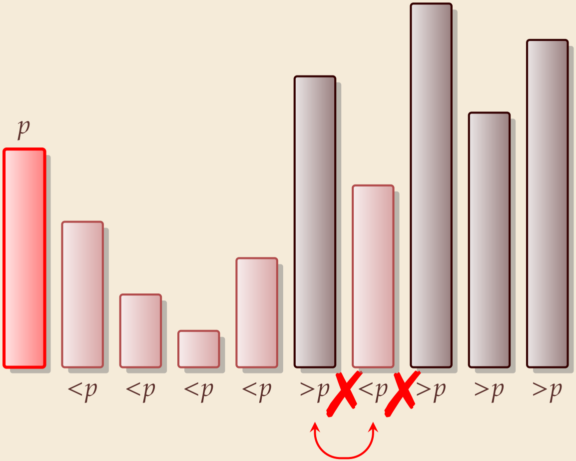

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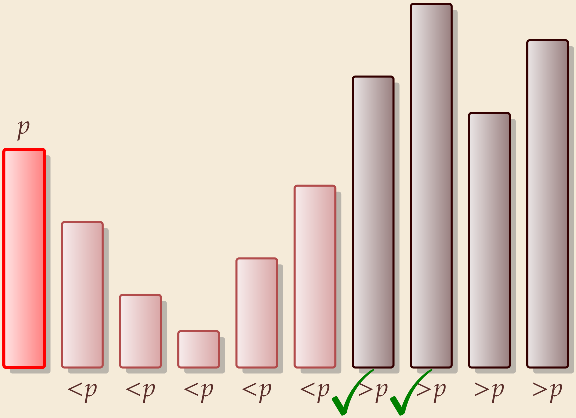

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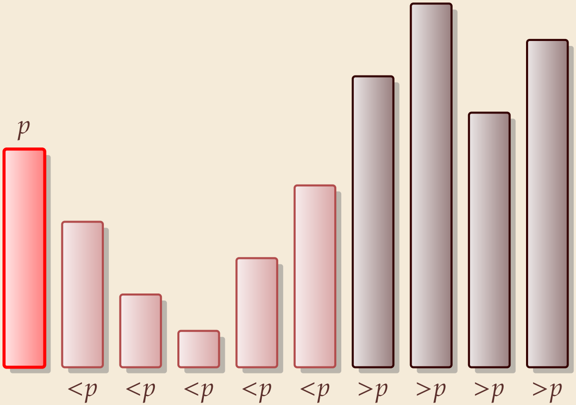

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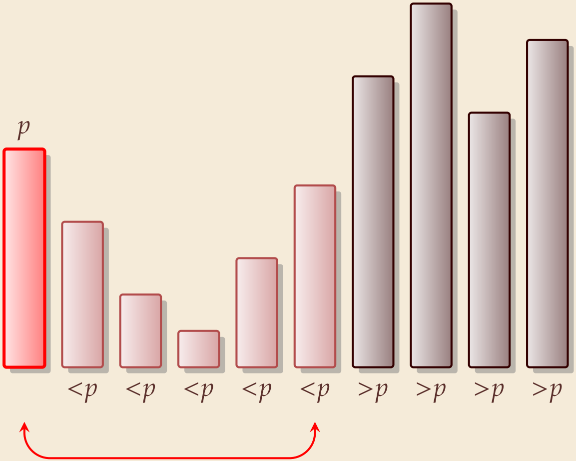

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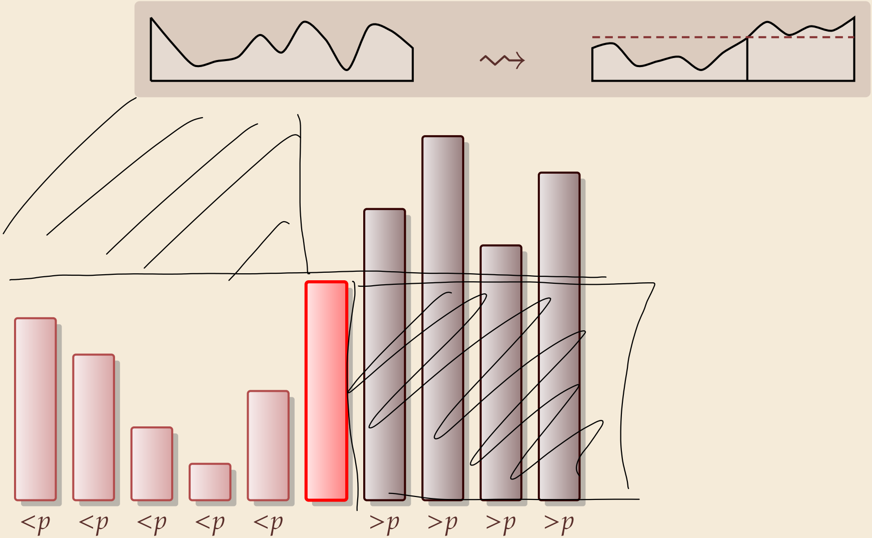

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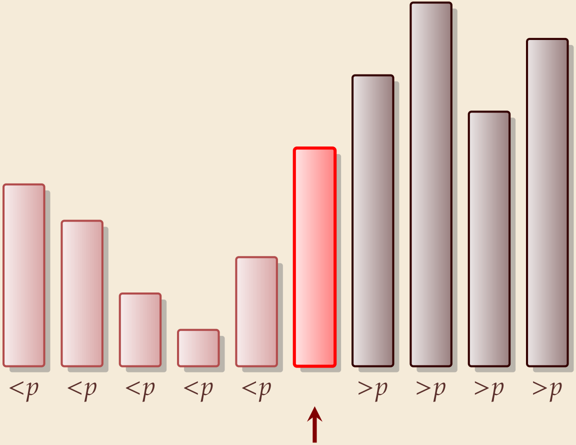

- no extra space needed
- visits each element once
- returns rank/position of pivot

Partitioning – Detailed code

Beware: details easy to get wrong; use this code! (if you ever have to)

```
1  procedure partition(A, b):
2      // input: array A[0..n), position of pivot b ∈ [0..n)
3      swap(A[0], A[b])
4      i := 0,   j := n
5      while true do
6          do i := i + 1 while i < n and A[i] < A[0]
7          do j := j − 1 while j ≥ 1 and A[j] > A[0]
8          if i ≥ j then break     (goto 11)
9          else swap(A[i], A[j])
10     end while
11     swap(A[0], A[j])
12     return j
```

Loop invariant (5–10): A

p	$\leq p$	?	$\geq p$
	i	j	

Quicksort

```
procedure quicksort(A[l..r]):
    if r − ℓ ≤ 1 then return
    b := choosePivot(A[l..r])
    j := partition(A[l..r], b)
    quicksort(A[l..j))
    quicksort(A[j + 1..r))
```

- recursive procedure
- choice of pivot can be
  - fixed position $\rightsquigarrow$ dangerous!
  - random
  - more sophisticated, e. g., median of 3

Clicker Question

What is the worst-case running time of quicksort?

- **A** $\Theta(1)$
- **B** $\Theta(\log n)$
- **C** $\Theta(\log \log n)$
- **D** $\Theta(\sqrt{n})$
- **E** $\Theta(n)$
- **F** $\Theta(n \log \log n)$
- **G** $\Theta(n \log n)$
- **H** $\Theta(n \log^2 n)$
- **I** $\Theta(n^{1+\epsilon})$
- **J** $\Theta(n^2)$
- **K** $\Theta(n^3)$
- **L** $\Theta(2^n)$

→ sli.do/cs566

Clicker Question

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- **A** $\cancel{\Theta(1)}$
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- **F** $\cancel{\Theta(n \log \log n)}$
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- **H** $\cancel{\Theta(n \log^2 n)}$
- **I** $\cancel{\Theta(n^{1+\epsilon})}$
- **J** $\Theta(n^2)$ $\checkmark$
- **K** $\cancel{\Theta(n^3)}$
- **L** $\cancel{\Theta(2^n)}$

→ `sli.do/cs566`

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

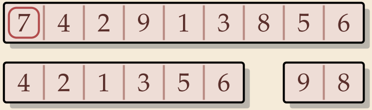
Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

Quicksort & Binary Search Trees

Quicksort



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Quicksort & Binary Search Trees

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7

Quicksort & Binary Search Trees

Quicksort

7 | 4 | 2 | 9 | 1 | 3 | 8 | 5 | 6

4 | 2 | 1 | 3 | 5 | 6   7   9 | 8

2 | 1 | 3   4   5 | 6

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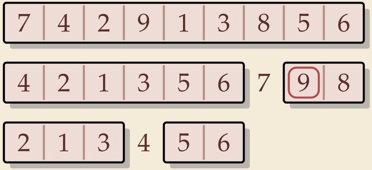

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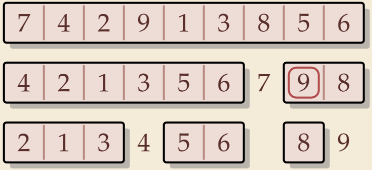

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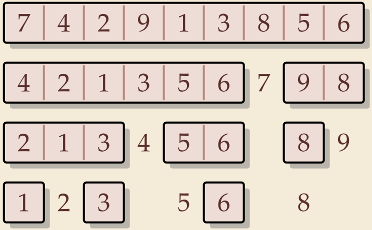

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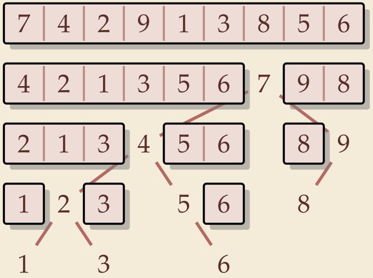

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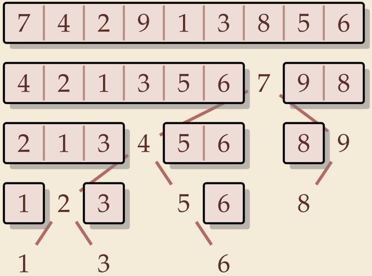
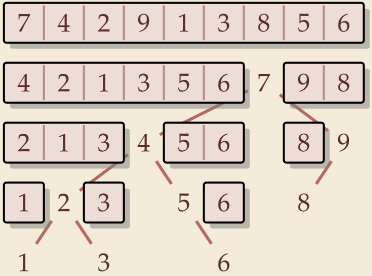


Binary Search Tree (BST)

7 4 2 9 1 3 8 5 6

Quicksort & Binary Search Trees

Quicksort



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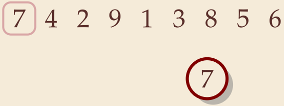

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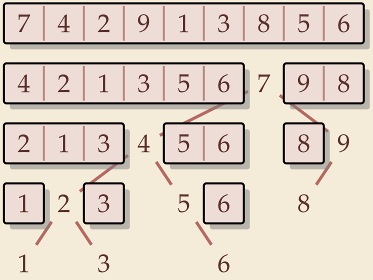
Quicksort



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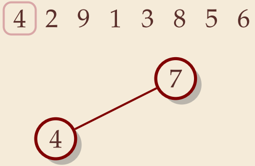

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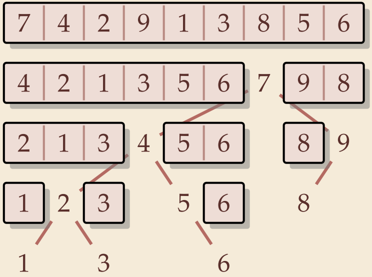

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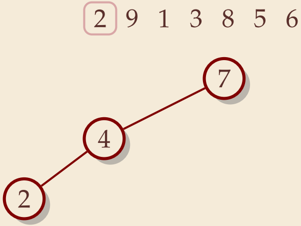

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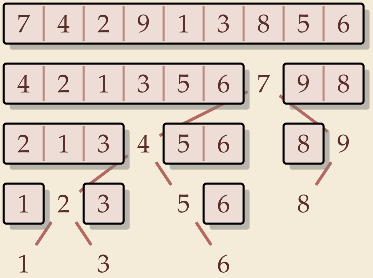

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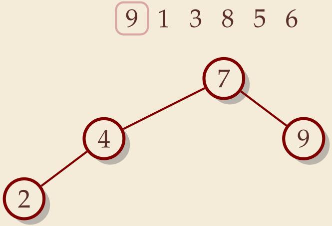

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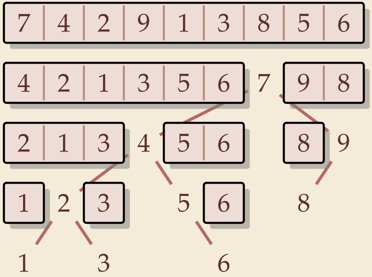

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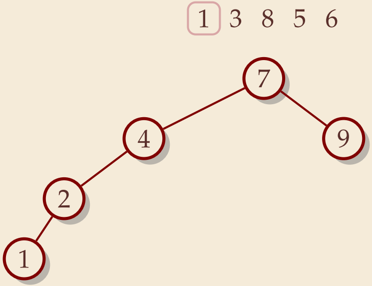

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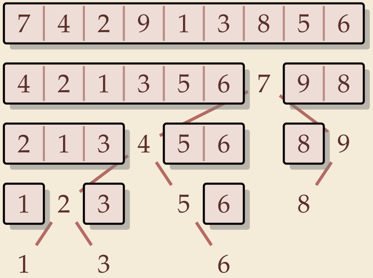

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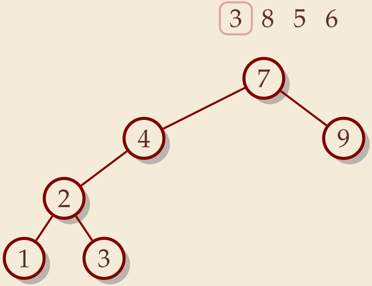

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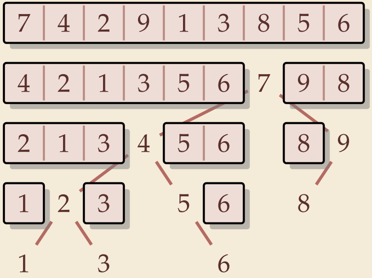

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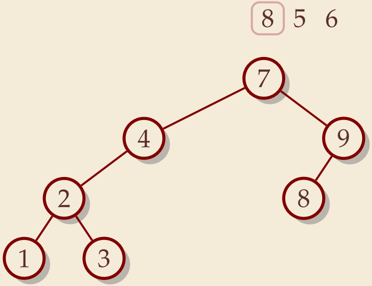

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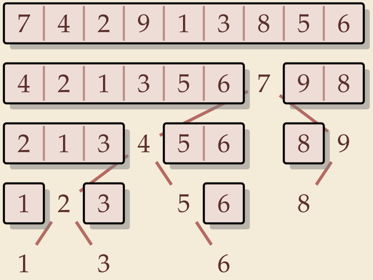

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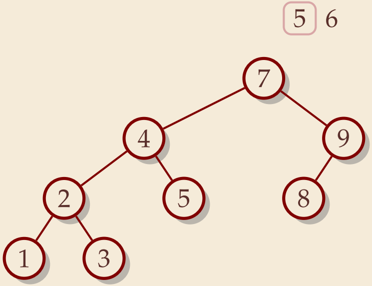

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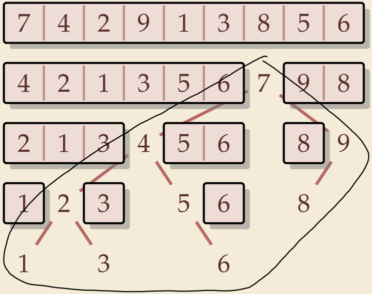

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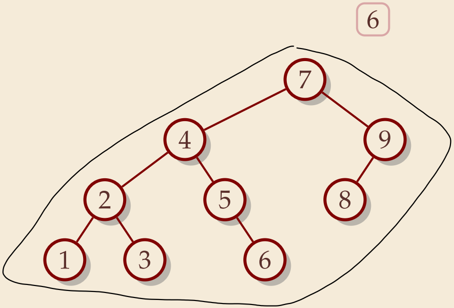

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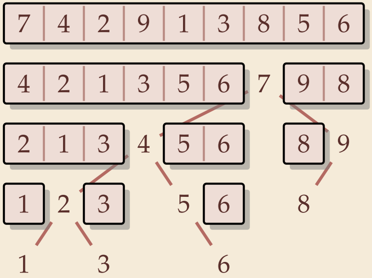
Quicksort



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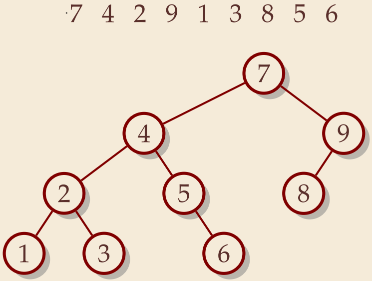


- recursion tree of quicksort $=$ binary search tree from successive insertion
- comparisons in quicksort $=$ comparisons to built BST
- comparisons in quicksort $\approx$ comparisons to search each element in BST

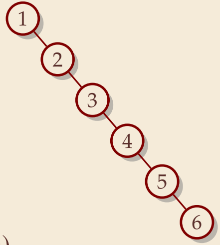
random perm.

$= C_n \sim 2n \ln n$

$\approx 1.39\, n \lg n$

Quicksort – Worst Case

- Problem: BSTs can degenerate
- Cost to search for k is $k - 1$



$\rightsquigarrow$ Total cost $\displaystyle\sum_{k=1}^{n}(k-1) = \frac{n(n-1)}{2} \sim \frac{1}{2}n^2$

$\rightsquigarrow$ quicksort worst-case running time is in $\Theta(n^2)$

terribly slow!

average case $1.39\, n \lg n$

But, we can fix this:

Randomized quicksort:

- choose a *random pivot* in each step

$\rightsquigarrow$ same as randomly *shuffling* input before sorting

Randomized Quicksort – Analysis

- cost measure: element visits (as for mergesort)
- $C(n)$ = #element visits when sorting n randomly permuted elements
  = cost of searching every element in BST built from input

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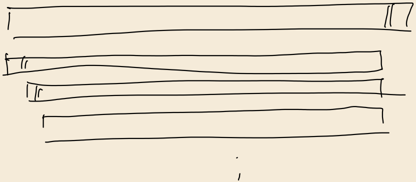
$\rightsquigarrow$ quicksort needs $\sim 2\ln(2) \cdot n \lg n \approx 1.39 n \lg n$ *in expectation*
(see analysis of C_n in Unit 3!)

- also: very unlikely to be much worse:
  e. g., one can prove: $\Pr[\text{cost} > 10n \lg n] \;=\; O(n^{-2.5})$
  distribution of costs is "concentrated around mean"
- intuition: have to be *constantly* unlucky with pivot choice

$\notin$ exam



Quicksort – Discussion

mergesort quicksort

$2 n \lg n$ $1.39 n \lg n$ rand.

- fastest general-purpose method
- $\Theta(n \log n)$ average case #cmps = #element visits
- works *in-place* (no extra space required)
- memory access is sequential (scans over arrays)
- $\Theta(n^2)$ worst case (although extremely unlikely)
- not a *stable* sorting method

Open problem: Simple algorithm that is fast, stable and in-place.

4.3 Comparison-Based Lower Bound

Lower Bounds

- **Lower bound:** mathematical proof that *no algorithm* can do better.
  - very powerful concept: bulletproof *impossibility* result
    $\approx$ *conservation of energy* in physics
  - **(unique?) feature of computer science:**
    for many problems, solutions are known that (asymptotically) **achieve the lower bound**

$\rightsquigarrow$ can speak of "*optimal* algorithms"

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  $\rightsquigarrow$ can speak of "*optimal* algorithms"

- To prove a statement about *all algorithms*, we must precisely define what that is!
- already know one option: the word-RAM model
- Here: use a simpler, more restricted model.

The Comparison Model

- In the *comparison model* data can only be accessed in two ways:
  - comparing two elements
  - moving elements around (e. g. copying, swapping)
  - Cost: number of comparisons.

The Comparison Model

expert note

cell probe model

- In the *comparison model* data can only be accessed in two ways:
  - comparing two elements
  - moving elements around (e. g. copying, swapping)
  - Cost: number of comparisons.

That's good!
Keeps algorithms general!

- This makes very few assumptions on the kind of objects we are sorting.
- Mergesort and Quicksort work in the comparison model.

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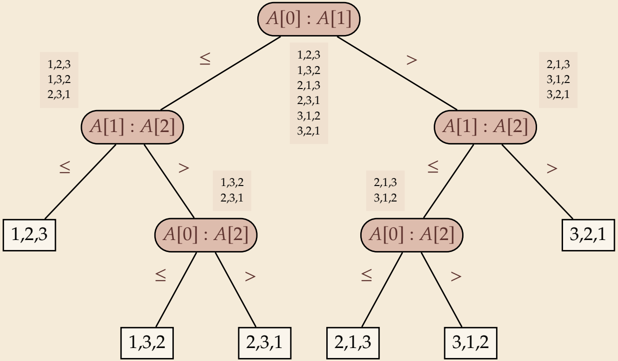
- This makes very few assumptions on the kind of objects we are sorting.
- Mergesort and Quicksort work in the comparison model.

$\rightsquigarrow$ Every comparison-based sorting algorithm corresponds to a ***decision tree***.

  - only model comparisons $\rightsquigarrow$ ignore data movement
  - nodes $=$ comparisons the algorithm does
  - child links $=$ outcomes of comparison
  - leaf $=$ unique initial input permutation compatible with comparison outcomes
  - next comparisons can depend on outcomes $\rightsquigarrow$ child subtrees can look different

Comparison Lower Bound

Example: Comparison tree for a sorting method for $A[0..2]$:



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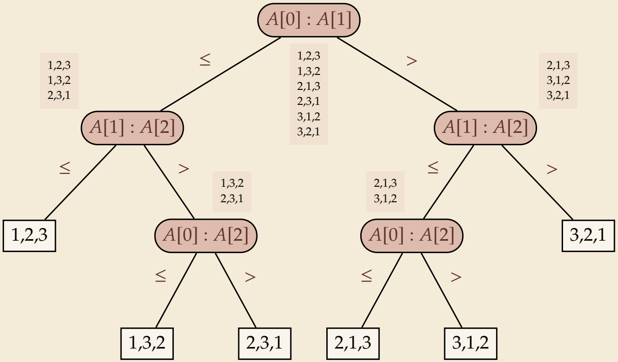

- ▶ Execution $=$ follow a path in comparison tree.
- $\rightsquigarrow$ height of comparison tree $=$ worst-case # comparisons
- ▶ comparison trees are *binary* trees
- $\rightsquigarrow$ ℓ leaves $\rightsquigarrow$ height $\geq \lceil \lg(\ell) \rceil$
- ▶ comparison trees for sorting method must have $\geq n!$ leaves
- $\rightsquigarrow$ $\boxed{\text{height} \geq \lg(n!)}$ $\sim n \lg n$

more precisely: $\lg(n!) = n \lg n - \lg(e)n + O(\log n)$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right).$$

$$\lg(n!) = \lg(\sqrt{2\pi n}) + n \cdot \lg\left(\frac{n}{e}\right) + O\left(\frac{1}{n}\right)$$

$$= n \lg n - n \cdot \lg e \pm O(\log n)$$

Comparison Lower Bound

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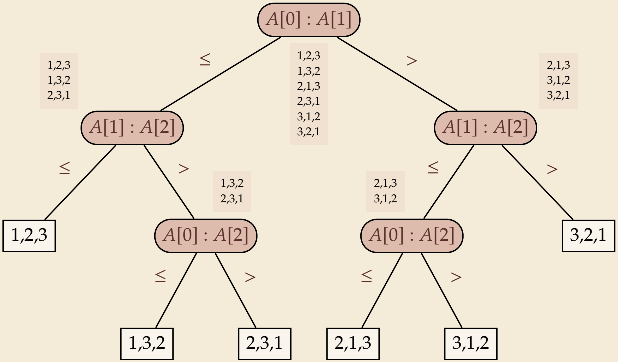

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- comparison trees for sorting method must have $\ge n!$ leaves
- ⇝ height $\ge \lg(n!) \sim n \lg n$

more precisely: $\lg(n!) = n \lg n - \lg(e) n + O(\log n)$

- Mergesort achieves $\sim n \lg n$ comparisons ⇝ asymptotically comparison-optimal!
- Open (theory) problem: Sorting algorithm with $n \lg n - \lg(e) n + o(n)$ comparisons?

≈ 1.4427

Clicker Question

Does the comparison-tree from the previous slide correspond to a worst-case optimal sorting method?

A Yes

B No

→ `sli.do/cs566`

Clicker Question

Does the comparison-tree from the previous slide correspond to a worst-case optimal sorting method?

A Yes ✓

B ~~No~~

→ sli.do/cs566

4.4 Integer Sorting

Clicker Question

Select all **correct formulations** of our **lower bound** from §4.3.

A Any sorting algorithm requires $O(n \log n)$ running time in the worst case.

B Every comparison-based sorting algorithm requires $\Omega(n \log n)$ running time in the worst case for sorting n elements.

C Every comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons in the worst case for sorting n elements.

D Every sorting algorithm requires $\Omega(n \log n)$ comparisons in the worst case for sorting n elements.

E The complexity of sorting n elements in the comparison-model is $\Theta(n \log n)$.

F The complexity of sorting n elements in the comparison-model is $\Omega(n \log n)$.

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D ~~Every sorting algorithm requires $\Omega(n \log n)$ comparisons in the worst case for sorting n elements.~~

E ~~The complexity of sorting n elements in the comparison model is $\Theta(n \log n)$.~~

F The complexity of sorting n elements in the comparison-model is $\Omega(n \log n)$.

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How to beat a lower bound

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- **Not necessarily;** only in the *comparison model!*
  - $\rightsquigarrow$ Lower bounds show where to *change* the model!

- Here: sort n **integers**
  - can do *a lot* with integers: add them up, compute averages, ... (full power of word-RAM)
  - $\rightsquigarrow$ we are **not** working in the comparison model
  - $\rightsquigarrow$ *above lower bound does not apply!*

How to beat a lower bound

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  $\rightsquigarrow$ we are **not** working in the comparison model
  $\rightsquigarrow$ *above lower bound does not apply!*

  - but: a priori unclear how much arithmetic helps for sorting ...

Counting sort

- Important parameter: size/range of numbers
  - numbers in range $[0..U) = \{0, \ldots, U-1\}$ typically $U = 2^b$ $\rightsquigarrow$ b-bit binary numbers

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- Important parameter: size/range of numbers
  - numbers in range $[0..U) = \{0, \ldots, U-1\}$ typically $U = 2^b$ $\rightsquigarrow$ b-bit binary numbers
- We can sort n integers in $\Theta(n+U)$ time and $\Theta(U)$ space when $\boxed{b \le w}$: $\rightsquigarrow$ $U \le 2^w$

word size

Counting sort

procedure countingSort($A[0..n)$):
 // *A contains integers in range $[0..U)$.*
 $C[0..U)$:= new integer array, initialized to 0
 // *Count occurrences*
 for $i := 0, \ldots, n-1$
  $C[A[i]] := C[A[i]] + 1$ indirect addressing
 $i := 0$ // *Produce sorted list*
 for $k := 0, \ldots U-1$
  for $j := 1, \ldots C[k]$
   $A[i] := k$; $i := i+1$

- *count* how often each *possible* value occurs
- produce sorted result directly from counts
- circumvents lower bound by using integers as array index / pointer offset

$\rightsquigarrow$ Can sort n integers in range $[0..U)$ with $\boxed{U = O(n)}$ in time and space $\Theta(n)$.

Larger Universes: Radix Sort

- *MSD Radix Sort:*
  - split numbers into base-R "digits"
  - Use counting sort on <u>m</u>ost <u>s</u>ignificant <u>d</u>igit
    (with variant of counting sort that moves full number)

  ⇝ integers sorted with respect to first digit
  - recurse on sublist for each digit value, using next digit for counting sort

⇝ After $\lfloor \log_R(U) \rfloor + 1$ levels of counting sort, fully sorted!

- For $R \le 2^w$, all counting sort calls on same level cost total of $O(n)$ time
  (requires care to avoid reinitialization cost of array C)

⇝ total time $O(n \log_R(U)) = O\left(n \frac{\log(U)}{\log(R)}\right)$

⇝ $O(n)$ time sorting possible for numbers in range $U = O(n^c)$ for constant c.

Integer Sorting – State of the art

Algorithm theory

- integer sorting on the w-bit word-RAM
- suppose $U = 2^w$, but w can be an arbitrary function of n
- how fast can we sort n such w-bit integers on a w-bit word-RAM?
  - for $w = O(\log n)$: linear time (*radix/counting sort*) // standard assumption
  - for $w = \Omega(\log^{2+\varepsilon} n)$: linear time (*signature sort*)
  - for w in between: can do $O(n\sqrt{\lg\lg n})$ (very complicated algorithm)
    don't know if that is best possible!

$\notin$ exam

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  - for w in between: can do $O(n\sqrt{\lg \lg n})$ (very complicated algorithm)
    don't know if that is best possible!

* * *

... for the rest of this unit: back to the comparisons model!

Clicker Question

Which statements are correct? Select all that apply.

My computer has 64-bit words, so an `int` has 64 bits. Hence I can sort any `int[]` of length n ...

- **A** in constant time.
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- **C** in $O(n)$ time.
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→ sli.do/cs566

Clicker Question

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Part II

Exploiting presortedness

4.5 Adaptive Sorting

Adaptive sorting

- Comparison lower bound also holds for the *average case* $\rightsquigarrow$ $\lfloor \lg(n!) \rfloor$ cmps necessary
- Mergesort and Quicksort from above use $\sim n \lg n$ cmps even in best case

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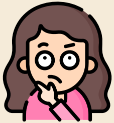

Can we do better if the input is already "almost sorted"?

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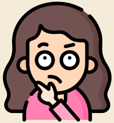

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Scenarios where this may arise naturally:

- Append new data as it arrives, regularly sort entire list (e. g., log files, database tables)
- Compute summary statistics of time series of measurements that change slowly over time (e. g., weather data)
- Merging locally sorted data from different servers (e. g., map-reduce frameworks)

$\rightsquigarrow$ Ideally, algorithms should *adapt* to input: *the more sorted the input, the faster the algorithm*

. . . but how to do that!?

Warmup: check for sorted inputs

- Any method could first check if input already completely in order!
  - 👍 Best case becomes $\Theta(n)$ with $n-1$ comparisons!
  - 👎 Usually $n-1$ extra comparisons and pass over data "wasted"
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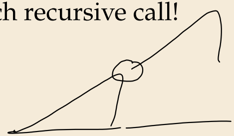


For Mergesort, can instead check before merge with a **single** comparison

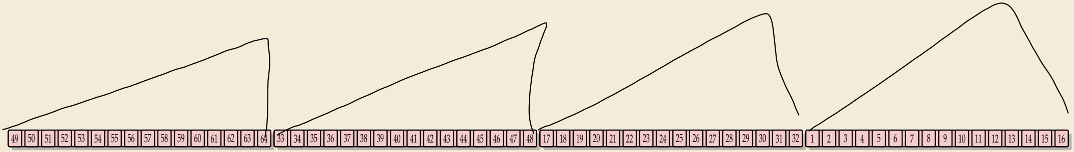
- If last element of first run $\le$ first element of second run, skip merge

How effective is this idea?

```
procedure mergesortCheck(A[l..r)):
    n := r − l
    if n ≤ 1 return
    m := l + ⌊n/2⌋
    mergesortCheck(A[l..m))
    mergesortCheck(A[m..r))
    if A[m − 1] > A[m]
        merge(A[l..m), A[m..r), buf)
        copy buf to A[l..r)
```

Mergesort with sorted check – Analysis

- Simplified cost measure: *merge cost* = size of output of merges
  ≈ number of comparisons
  ≈ number of memory transfers / cache misses
- Example input: $n = 64$ numbers in sorted *runs* of 16 numbers each:



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49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

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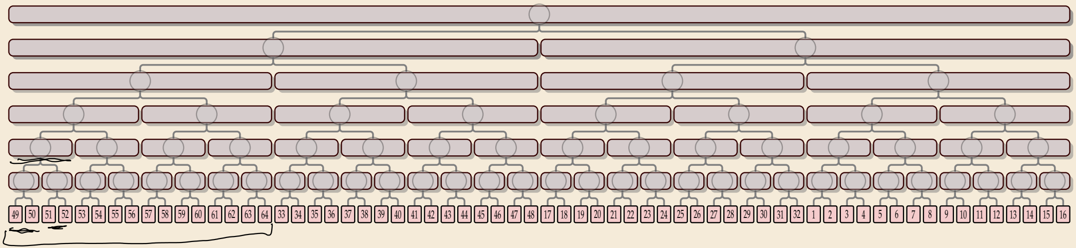

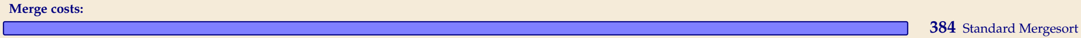

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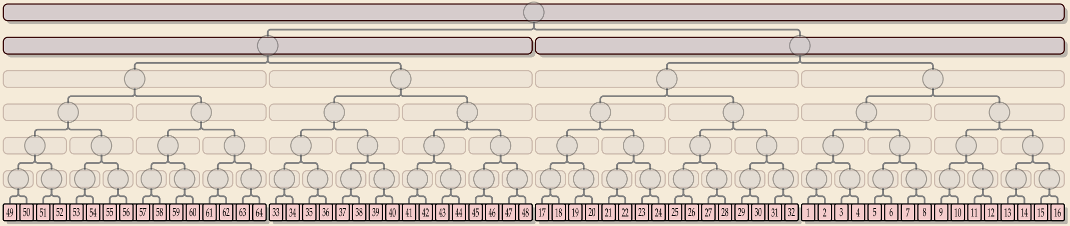

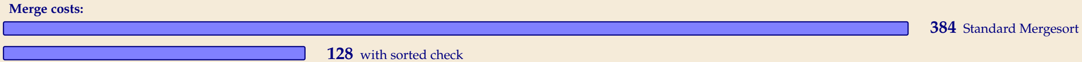

Sorted check can help a lot!