9 Random Tricks

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Outline

9 Random Tricks

Uses of Randomness

- Since it is likely that $\mathsf{BPP} = \mathsf{P}$, we focus on the more fine-grained benefits of randomization:
  - simpler algorithms (with same performance)
  - improving performance (but not jumping from exponential to polytime)
  - improved robustness
- Here: Collection of examples illustrating different techniques
  - fingerprinting / hashing
  - exploiting abundance of witnesses
  - random sampling

9.1 Hashing – Balls Into Bins

Fingerprinting / Hashing

- Often have elements from huge universe $U = [0..u)$ of possible values, but only deal with few actual items $\underbrace{x_1, \ldots, x_n}_{\in \mathcal{U}}$ at one time.
  Think: $n \ll u$
- Fingerprinting can help to be more efficient in this case
  - fingerprints from $[0..m)$
  - $m \ll u$
  - *Hash Function* $h : U \to [0..m)$

h will have collisions

$(x, y \in \mathcal{U} : h(x) = h(y))$

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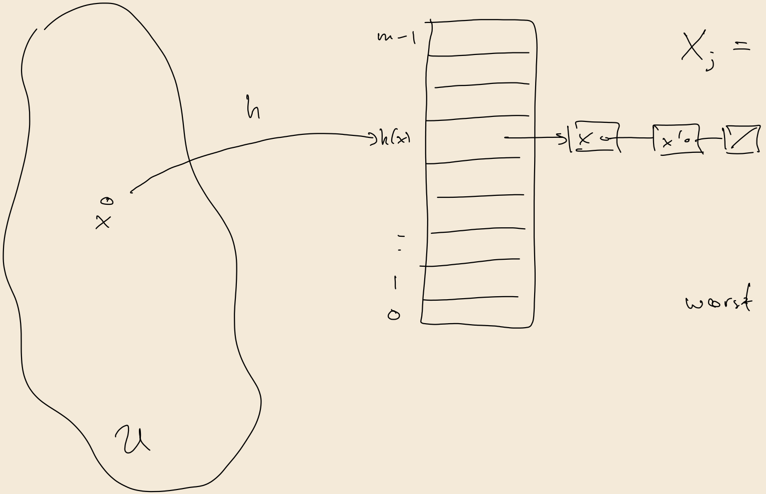
- Classic Example: hash tables and Bloom filters

Hash Tables

indirect chaining



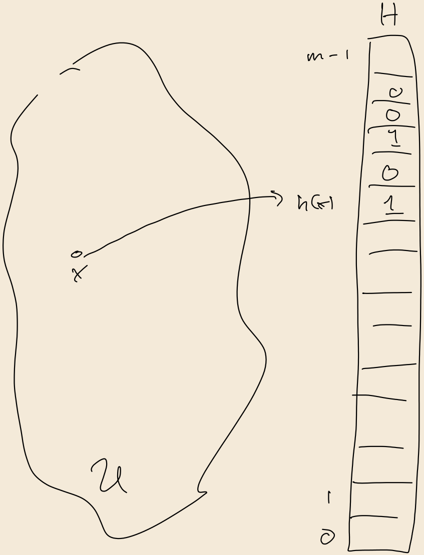
Performance:

How big are buckets?

X_j = # keys x with $h(x) = j$ in our HT

worst case $X_1 = n$

Bloom Filters



insert(x) : $H[h(x)] := 1$

query(x) : $H[h(x)]$

└ output 1 (Yes) can be a <u>false positive</u>!

output 0 (No) correct

(reduce false positive rate using independent h_1, h_2, h_3, h_4 ...)

application: segmented database
"cheap first checks"

Uniform – Universal – Perfect

Randomness is essential for hashing to make any sense! Three very different uses

1. *uniform hashing assumption*: (optimistic, often roughly right in practice!)
   How good is hashing if input is "as nicely random" as possible?

Uniform hashing assumption:

All m^n possible hash seq. $h(x_1), h(x_2), h(x_3), \ldots, h(x_n)$ are equally likely.

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   $\rightsquigarrow$ *universal hashing*: pick h at random from class H of suitable functions

   universal class of hash functions

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3. For given keys, can construct collision-free hash function
   $\rightsquigarrow$ *perfect hashing*

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Uniform Hashing Assumption:
When n elements $x_1, \ldots, x_n$ are inserted, for their *hash sequence* $h(x_1), \ldots, h(x_n)$, all m^n possible values are **equally likely**.

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$\rightsquigarrow$ might as well forget data!

Balls into bins model (a.k.a. balanced allocations)

- throw n balls into m bins

  ⚠ Literature usually swaps n and m!
- each ball picks bin *i.i.d. uniformly* at random

  B_i = bin of ith ball $\overset{\mathcal{D}}{=} \mathcal{U}([0..m))$
- classic abstract model to study randomized algorithms
  - For hashing, effectively the best imaginable case tends to be a bit optimistic!
  - but: data in applications often not far from this

A Paradox?

- X_j: Number of balls in bin j:

$\rightsquigarrow$ $X_1 \stackrel{\mathcal{D}}{=} \cdots \stackrel{\mathcal{D}}{=} X_m \stackrel{\mathcal{D}}{=} \mathrm{Bin}(n, \frac{1}{m})$

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actually, just shows $X_i = n/m \pm n^{0.501}$

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  Compute counter-probability: $\mathbb{P}[\max X_j \leq 1]$

$$\underbrace{1}_{\text{ball 1}} \cdot \underbrace{\left(1 - \frac{1}{m}\right)}_{\text{ball 2}} \cdot \left(1 - \frac{2}{m}\right) \cdots \left(1 - \frac{n-1}{m}\right)$$

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  Taylor series $e^x = \underbrace{1 + x} \pm O(x^2)$ as $x \to 0$

$$1 \cdot \left(1 - \frac{1}{m}\right) \cdot \left(1 - \frac{2}{m}\right) \cdots \left(1 - \frac{n-1}{m}\right) \quad = \quad e^{-\frac{1}{m}} \cdot e^{-\frac{2}{m}} \cdots e^{-\frac{n-1}{m}} \cdot \left(1 \pm O\left(\left(\frac{n}{m}\right)^2\right)\right)$$

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- $\mathbb{P}[\max X_j \leq 1] = \frac{1}{2}$ for $n \approx \sqrt{2m \ln(2)}$, so for $m = 365$ days, need $n \approx 22.49$ people

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⇝ *Can't expect to see **all** bins close to **expected** occupancy.*

Fullest Bin

$\hat{X} = \max_{j \in [m]} X_j$

Theorem 9.1

If we throw n balls into n bins, then w.h.p., the *fullest bin* has $O\left(\frac{\log n}{\log\log n}\right)$ balls.

Proof:

$$\mathbb{P}[\max X_j \geq M] \underset{\text{union bound}}{\leq} m \cdot \mathbb{P}[X_1 \geq M]$$

$$\mathbb{P}[X_1 \geq M] = \mathbb{P}\Big[\bigcup_{\substack{I \subseteq [n] \\ |I| = M}} \text{balls } i \in I \text{ land in bin } 1\Big]$$

$$\leq \binom{n}{M} \mathbb{P}(|I| \text{ balls land in bin } 1]$$

$$\leq \binom{n}{M}\left(\frac{1}{m}\right)^M$$

$\boxed{n = m}$

$$= \binom{n}{M}\left(\frac{1}{n}\right)^M = \frac{n!}{M!\,(n-M)!\,n^M}$$

$$\frac{n^M}{n^M} \leq 1$$

Fullest Bin [2]

Proof (cont.):

$$\leq \frac{1}{M!} \qquad \text{Stirling} \quad M! \geq \left(\frac{M}{e}\right)^M \sqrt{2\pi M}$$

$$\leq \left(\frac{e}{M}\right)^M \cdot \frac{1}{\sqrt{2\pi M}} \quad \leq 1$$

$$\leq \left(\frac{e}{M}\right)^M \qquad \text{need this} \leq \frac{1}{n}$$

$$\left(\frac{e}{M}\right)^M \approx \frac{1}{n} \qquad M = c\,\frac{\ln n}{\ln\ln n}$$

$$\exp\left(\ln\left(\left(\frac{e}{M}\right)^M\right)\right) = \exp\left(M\left(\cancel{\ln e}^{\,1} - \ln M\right)\right) = \exp\left(-\ln n - \ln\ln n - \ln\ln n\right) \approx \frac{1}{n}$$

to show w.h.p. $\mathbb{P}[\hat{X} \geq M] = O(n^{-d})$

$$n \cdot \left(\frac{e}{M}\right)^M = n\left(\underbrace{\frac{e}{c}}_{<1} \cdot \frac{\ln\ln n}{\ln n}\right)^{c\frac{\ln n}{\ln\ln n}} \qquad c > e$$

$$= \exp\left(\ln n + c \cdot \frac{\ln n}{\ln\ln n}\,\ln\left(\frac{\ln\ln n}{\ln n}\right)\right)$$

$$= \exp\left(\ln n + \ln n \, \frac{c \ln\ln\ln n}{\ln\ln n} - c \ln n \cdot \frac{\cancel{\ln\ln n}}{\cancel{\ln\ln n}} \right)$$

$$= \exp\left((1-c)\ln n \overset{-\ln n + 2\ln n}{\vee} + \ln n \, \frac{c \ln\ln\ln n}{\ln\ln n} \right)$$

$$= n^{2-c} \cdot \underbrace{\exp\left(\ln n \left(\underbrace{\frac{c \ln\ln\ln n}{\ln\ln n}}_{o(1) \,\le\, 1 \text{ for large } n} - 1 \right)\right)}_{\le 1 \text{ for large } n}$$

$$\le n^{2-c} = O(n^{-d}) \quad \text{for } c > d+2$$

Fullest Bin – Consequences

- Closer analysis shows for $n = \alpha m$, constant α ("load factor"),

  $$\max X_j = \frac{\ln n}{\ln(\ln(n)/\alpha)} \cdot \bigl(1 + o(1)\bigr) \quad \text{w.h.p.}$$

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What can we learn from this?

1. Under *uniform hashing assumption*, even **worst case** of chaining hashing cost beats BST.
2. . . . but not by much.
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- Cool trick: *Power of 2 choices*
  Assume ***two*** candidate bins per ball (hash functions), take less loaded bin

$\rightsquigarrow$ $\max X_j = \ln\ln n / \ln 2 \pm O(1)$ (!) analysis more technical; details in *Mitzenmacher & Upfal*

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  - $\mathbb{E}[S] = \sum_{i=1}^{m} \mathbb{E}[S_i] = m \sum_{i=1}^{m} \frac{1}{i} = mH_m = m \ln m \pm O(m)$
- Can similarly show $\mathrm{Var}[S] = \Theta(m^2)$
  (since S_i are independent, stdev is linear + using $\mathrm{Var}[S_i] = \dfrac{1-p_i}{p_i^2}$)

$\rightsquigarrow$ $\sigma[S] = \Theta(m) = o(\mathbb{E}[S])$, so S converges in probability to $\mathbb{E}[S]$ (Chebyshev)

9.2 Universal Hashing

Randomized Hashing

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  - $\rightsquigarrow$ Assumes both that input is not adversarial **and** that hash functions work well.

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⇝ Pick h at random, but from a smaller class $\mathcal{H}$ of "convenient" functions

Universal Hashing

What's a convenient class?

Definition 9.2 (Universal Family)

Let $\mathcal{H}$ be a set of hash functions from U to $[m]$ and $|U| \geq m$.
Assume $h \in \mathcal{H}$ is chosen uniformly at random.

(a) Then $\mathcal{H}$ is called a *universal* if

$$\forall x_1, x_2 \in U \quad : \quad x_1 \neq x_2 \quad \Longrightarrow \quad \mathbb{P}_h\big[h(x_1) = h(x_2)\big] \;\leq\; \frac{1}{m}.$$

(b) $\mathcal{H}$ is called *strongly universal* or *pairwise independent* if

$$\forall x_1, x_2 \in U, y_1, y_2 \in R \quad : \quad x_1 \neq x_2 \quad \Longrightarrow \quad \mathbb{P}_h\big[h(x_1) = y_1 \wedge h(x_2) = y_2\big] \;\leq\; \frac{1}{m^2}.$$

Universal Hashing

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◀

- strong universal implies universal
- In the following, always assume (uniformly) **random** $h \in \mathcal{H}$.
- by contrast, $x_1, \ldots, x_n$ may be chosen adversarially (but all distinct) from $[u]$

Examples of universal families

$$h_{ab}(x) \;=\; \Big(a \cdot x + b \bmod p\Big) \bmod m \qquad p \text{ prime}, p \geq m$$

$$h_a(x) \;=\; \Big(a \cdot x \bmod 2^k\Big) \operatorname{div} 2^{k-\ell} \qquad u = 2^k, m = 2^\ell$$

- $\mathcal{H}_1 = \{h_{ab} : a \in [1..p), b \in [0..p)\}$ is universal
- $\mathcal{H}_0 = \{h_{ab} : a \in [0..p), b \in [0..p)\}$ is strongly universal
- $\mathcal{H}_2 = \{h_a : a \in [1..2^k), a \text{ odd}\}$ is universal

How good is universal hashing?

balls into bins

$\hat{X} \approx \frac{\ln n}{\ln \ln n}$

Theorem 9.3

Assign $x_1, \ldots, x_n \in [u]$ to bins $h(x_i) \in [m]$ using hash function h, uniformly chosen from a universal family of hash functions $\mathcal{H}$.

Let X_j be the load of bin $j \in [m]$.

$n = m$ $\sqrt{2n}$

Then $\mathbb{P}\left[\underbrace{\max X_j}_{\hat{X}} \geq \sqrt{2} \cdot \frac{n}{\sqrt{m}}\right] \leq \frac{1}{2}.$

$\boxed{X_j \overset{\mathcal{D}}{\neq} \mathrm{Bin}(n,p)}$

Proof:

C_{ij} = 'x_i and x_j collide' $= [h(x_i) = h(x_j)]$

$\underset{\text{univ.}}{\Rightarrow}$ $\mathbb{P}[C_{ij}] \leq \frac{1}{m}$

$C = \sum_{1 \leq i < j \leq n} C_{ij}$ $\qquad \mathbb{E}[C] = \sum \mathbb{E}[C_{ij}] \leq \binom{n}{2} \cdot \frac{1}{m} < \frac{n^2}{2m}$

$\hat{X}$ itself implies $\binom{\hat{X}}{2}$ collisions

$\Rightarrow \quad C \geq \binom{\hat{X}}{2} = \frac{\hat{X}(\hat{X}-1)}{2} \geq \frac{(\hat{X}-1)^2}{2}$

How good is universal hashing [2]

Proof:

$$\mathbb{P}\left[\hat{X} \geq n\cdot\sqrt{\frac{2}{m}}\right] \leq \mathbb{P}\left[C \geq \frac{n^2}{m}\right] = \mathbb{P}\left[C \geq 2\cdot\mathbb{E}[C]\right] \underset{\text{Markov}}{\leq} \frac{1}{2}$$

$+1$

then $\hat{X}^2 \geq n\sqrt{\frac{2}{m}} + 1$ implies

$\frac{(\hat{X}-1)^2}{2} \geq \frac{n^2}{m}$ which implies

$C \geq \frac{n^2}{m}$

$\square$.,

So, how good is universal hashing?

- For $n = m$, fullest bin $\leq \sqrt{2n}$
- Much worse than $\Theta(\log n / \log \log n)$!

So, how good is universal hashing?

- For $n = m$, fullest bin $\leq \sqrt{2n}$
- Much worse than $\Theta(\log n / \log\log n)$!
- Note that we only proved an upper bound, however
  - bound is tight in the worst case
    (if all we know is pairwise independence of hash values)
    $\rightsquigarrow$ exercises
  - for practical choices like $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ better bounds are proven
    (close to $O(n^{1/3})$ instead of $O(n^{1/2})$)
    but still far worse than uniform hashing

9.3 Perfect Hashing

Perfect Hashing: Random Sampling

A hash function $h : [u] \to [m]$ is called

- *perfect* for a set $\mathcal{X} = \{x_1, \ldots, x_n\} \subset [u]$ if $i \neq j$ implies $h(x_i) \neq h(x_j)$
- *minimal* for set $\mathcal{X} = \{x_1, \ldots, x_n\} \subset [u]$ if $m = n$

Perfect Hashing

- only possible for $n \leq m$
- stringent requirement $\rightsquigarrow$ here focus on static $\mathcal{X}$
  - carefully chosen variants with partial rebuilding allow insertion and deletion in $O(1)$ amortized expected time

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Perfect Hashing

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  - carefully chosen variants with partial rebuilding allow insertion and deletion in $O(1)$ amortized expected time
- further requirements
  1. Hash function must be fast to evaluate (ideally $O(1)$ time)
  2. Hash function must be small to store (ideally $O(n)$ space)
  3. should be fast to compute given $\mathcal{X}$ (ideally $O(n)$ time)
  4. Have small m (ideally $m = \Theta(n)$)

Perfect Hashing: Simple, but space inefficient

Start simple: pick $h \in \mathcal{H}$ from universal class

what can we guarantee? n fixed

$\Rightarrow$ increase m until likely that h perfect

birthday paradox! $n \approx \sqrt{m}$

So let's try $m \geq n^2$

$\hat{X} \geq 2$ implies $C \geq 1$

which implies $C \geq \frac{n^2}{m}$

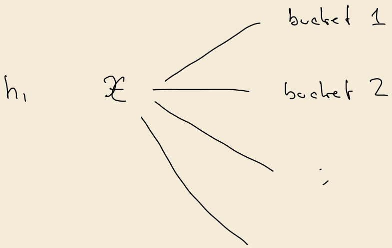
$$\mathbb{P}[\hat{X} \geq 2] \leq \mathbb{P}[C \geq \frac{n^2}{m}] = \mathbb{P}[C \geq 2\,\mathbb{E}(C)] \leq \frac{1}{2}$$

as for universal hashing above

$\Rightarrow \mathbb{P}\{\text{no collisions}\} \geq \frac{1}{2}$

good prob. to get a perfect hash function! $\lightning$ HT has n^2 space

Perfect Hashing: Two-tier solution



$O(n)$ tier 1 buckets

for each tier 1 bucket i

choose $h_2^{(i)}$ as above

using $m_i = X_i^2$ X_i = # elements in bucket i

↳ after expected linear time

over hash function is perfect

$x \mapsto h_2^{(h_1(x))}(x)$

To show: overall space (for all secondary hash tables) small