3 Efficient Sorting

18 February 2020

Sebastian Wild

version 2020-02-18 18:28

Outline

3 Efficient Sorting

Why study sorting?

- fundamental problem of computer science that is still not solved

  Algorithm with optimal #comparisons in worst case?

- building brick of many more advanced algorithms
  - for preprocessing
  - as subroutine
- playground of manageable complexity
  to practice algorithmic techniques

Here:

- "classic" fast sorting method
- **parallel** sorting

Part I

The Basics

Rules of the game

- **Given:**
  - array $A[0..n-1]$ of n objects
  - a total order relation $\leq$ among $A[0], \ldots, A[n-1]$
    (a comparison function)
- **Goal:** rearrange (=permute) elements within A,
  so that A is *sorted*, i. e., $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

- for now: A stored in main memory (*internal sorting*)
  single processor (*sequential sorting*)

3.1 Mergesort

Clicker Question

How does mergesort work?

- **A** Split elements around median, then recurse on small / large elements.
- **B** Recurse on left / right half, then combine sorted halves.
- **C** Grow sorted part on left, repeatedly add next element to sorted range.
- **D** Repeatedly choose 2 elements and swap them if they are out of order.
- **E** Don't know.

pingo.upb.de/622222

Clicker Question

How does mergesort work?

A ~~Split elements around median, then recurse on small / large elements.~~

B Recurse on left / right half, then combine sorted halves. ✓

C ~~Grow sorted part on left, repeatedly add next element to sorted range.~~

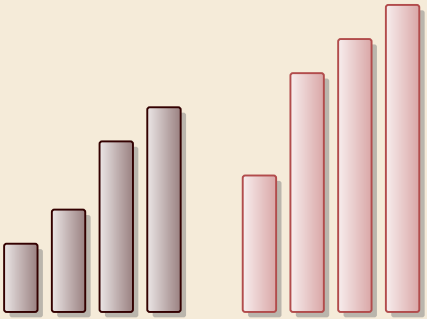
D ~~Repeatedly choose 2 elements and swap them if they are out of order.~~

E ~~Don't know.~~

pingo.upb.de/622222

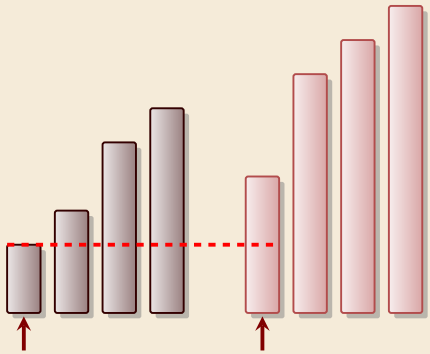
Merging sorted lists

Merging sorted lists

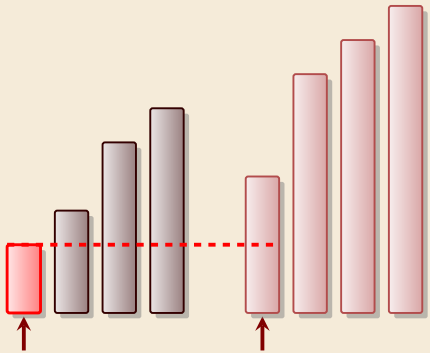


run1 run2 result

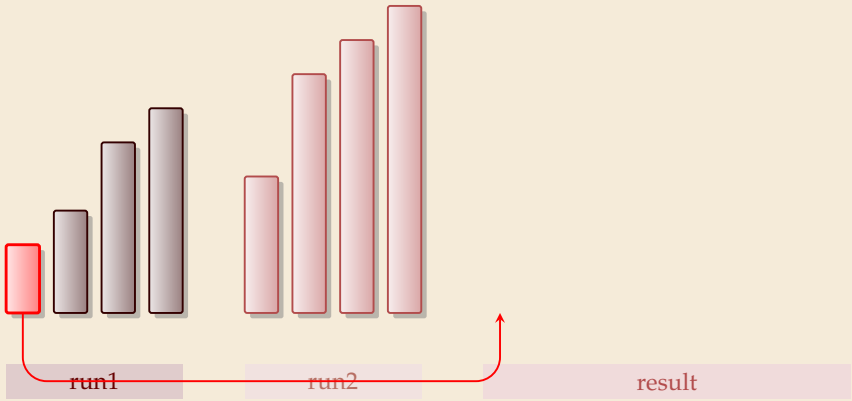
Merging sorted lists



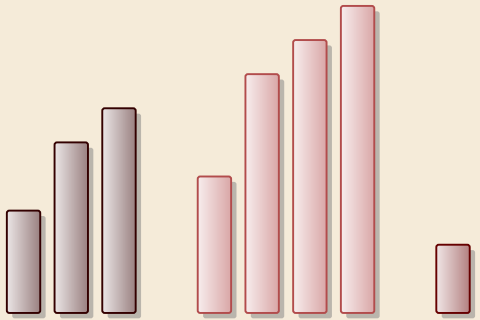
Merging sorted lists



Merging sorted lists

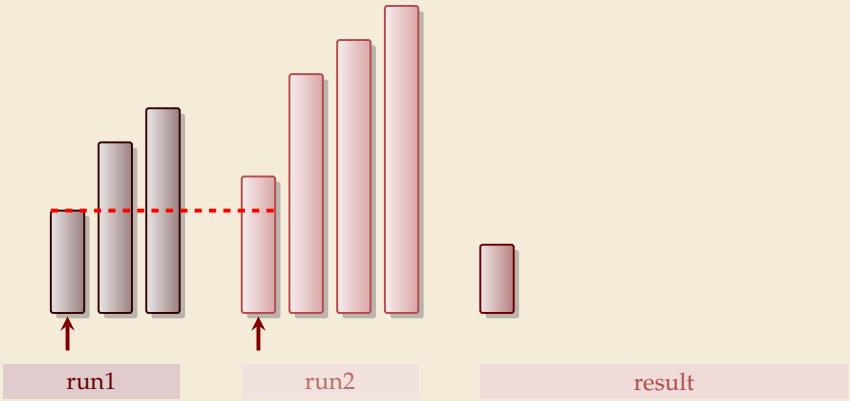


Merging sorted lists

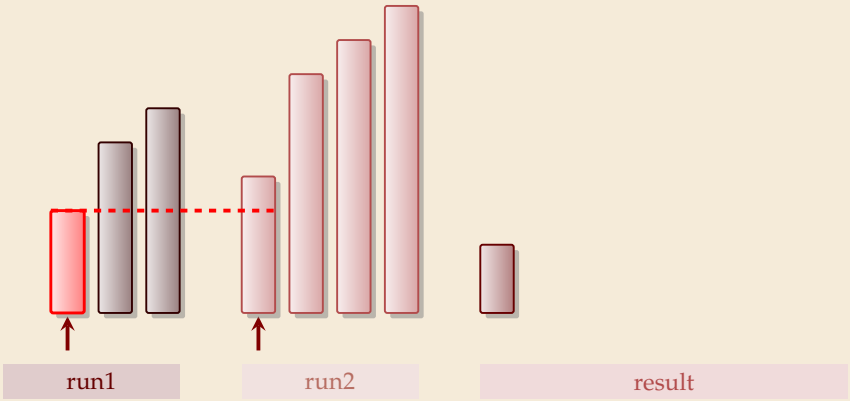


run1 run2 result

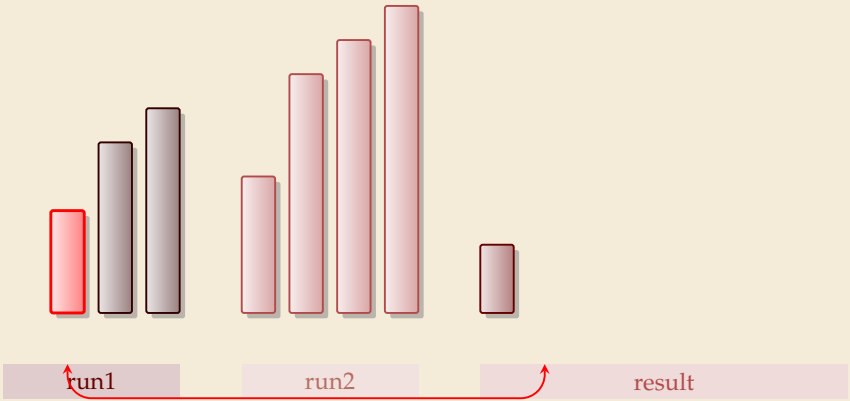
Merging sorted lists



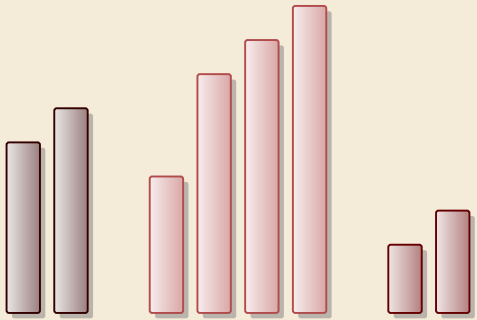
Merging sorted lists



Merging sorted lists

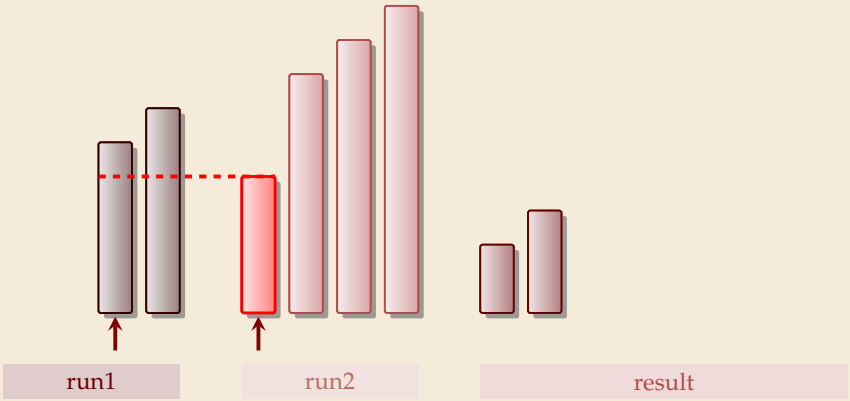


Merging sorted lists



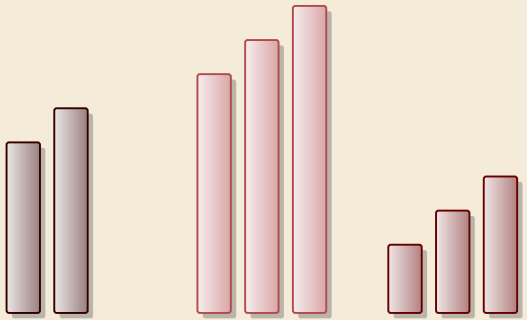
run1 run2 result

Merging sorted lists



Merging sorted lists

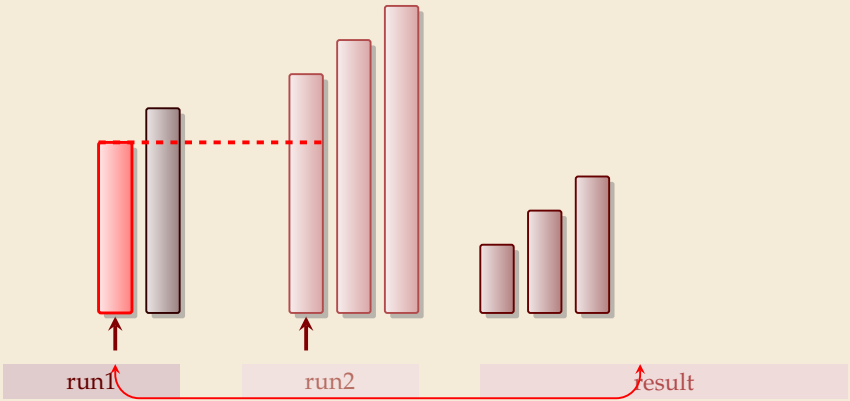
Merging sorted lists



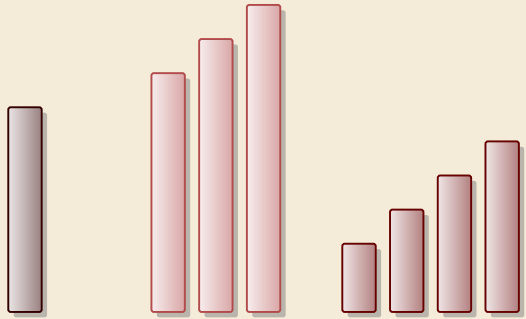
run1 run2 result

Merging sorted lists

Merging sorted lists

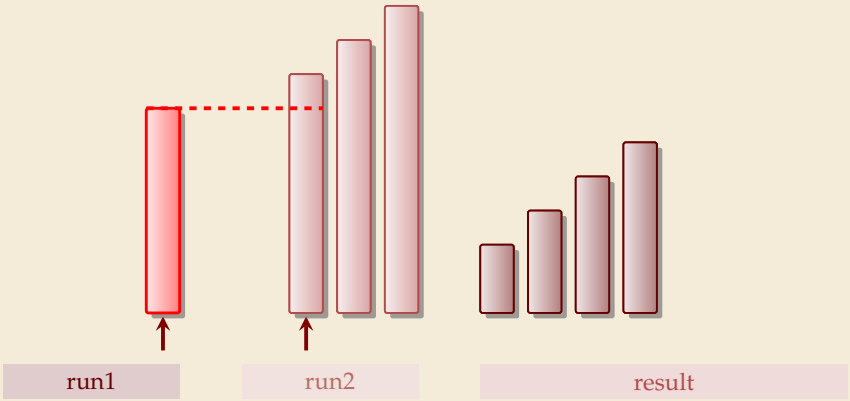


Merging sorted lists

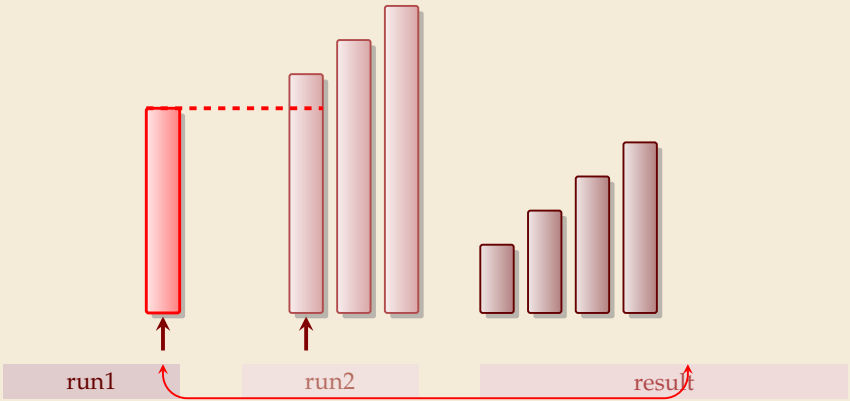


run1 run2 result

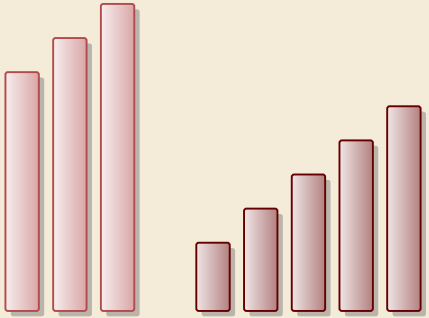
Merging sorted lists



Merging sorted lists

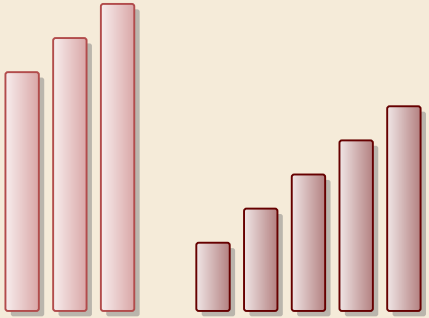


Merging sorted lists



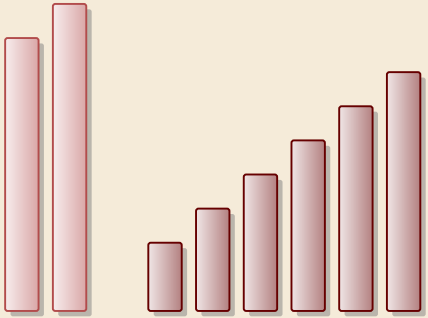
run1 run2 result

Merging sorted lists



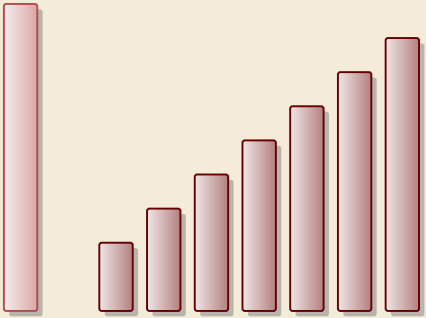
run1 run2 result

Merging sorted lists



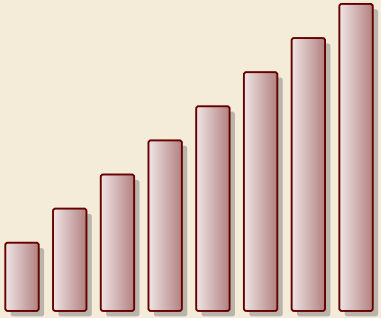
run1 run2 result

Merging sorted lists



run1 run2 result

Merging sorted lists



run1 run2 result

Mergesort

```
1 procedure mergesort(A[l..r])
2     n := r − l + 1
3     if n ≥ 1 return
4     m := l + ⌊n/2⌋
5     mergesort(A[l..m − 1])
6     mergesort(A[m..r])
7     merge(A[l..m − 1], A[m..r], buf)
8     copy buf to A[l..r]
```

- recursive procedure; *divide & conquer*
- merging needs
  - temporary storage for result of same size as merged runs
  - to read and write each element twice (once for merging, once for copying back)

Mergesort

```
1 procedure mergesort(A[l..r])
2     n := r − l + 1
3     if n ≥ 1 return
4     m := l + ⌊n/2⌋
5     mergesort(A[l..m − 1])
6     mergesort(A[m..r])
7     merge(A[l..m − 1], A[m..r], buf)
8     copy buf to A[l..r]
```

- recursive procedure; *divide & conquer*
- merging needs
  - temporary storage for result of same size as merged runs
  - to read and write each element twice (once for merging, once for copying back) $2n$

Analysis: count "element visits" (read and/or write)

$$C(n) = \begin{cases} 0 & n \le 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \ge 2 \end{cases}$$

same for best and worst case!

Simplification $n = 2^k$

$$C(2^k) = \begin{cases} 0 & k \le 0 \\ 2 \cdot C(2^{k-1}) + 2 \cdot 2^k & k \ge 1 \end{cases} = 2 \cdot 2^k + 2^2 \cdot 2^{k-1} + 2^3 \cdot 2^{k-2} + \cdots + 2^k \cdot 2^1 = \underbrace{2k}_{2n} \cdot \underbrace{2^k}_{n}$$

$$C(n) = 2n \lg(n) = \Theta(n \log n)$$

Mergesort – Discussion

- optimal time complexity of $\Theta(n \log n)$ in the worst case
- *stable* sorting method — i. e., retains relative order of equal-key items

  2 3 1 2 2 5

  → 1 2 2 2 3 5 sorted

  but not stably sorted
- memory access is sequential (scans over arrays)
- requires $\Theta(n)$ extra space

  there are in-place merging methods,
  but they are substantially more complicated
  and not (widely) used

3.2 Quicksort

Clicker Question

How does quicksort work?

- **A** split elements around median, then recurse on small / large elements.
- **B** recurse on left / right half, then combine sorted halves.
- **C** grow sorted part on left, repeatedly add next element to sorted range.
- **D** repeatedly choose 2 elements and swap them if they are out of order.
- **E** Don't know.

pingo.upb.de/622222

Clicker Question

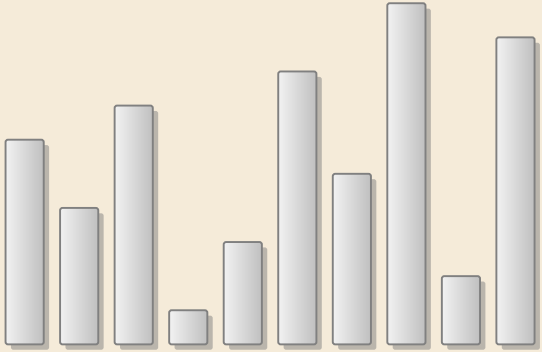
How does quicksort work?

- **A** split elements around median, then recurse on small / large elements. ✓
- **B** ~~recurse on left / right half, then combine sorted halves.~~
- **C** ~~grow sorted part on left, repeatedly add next element to sorted range.~~
- **D** ~~repeatedly choose 2 elements and swap them if they are out of order.~~
- **E** ~~Don't know.~~

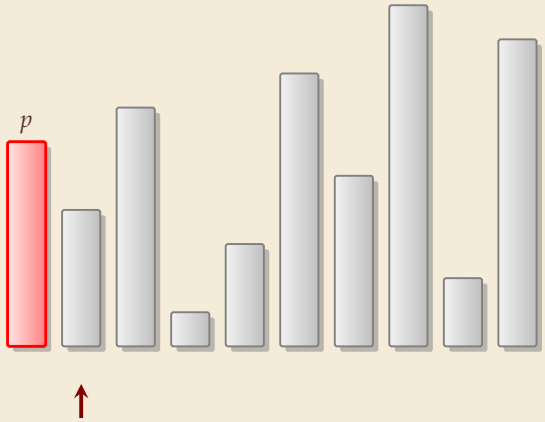
pingo.upb.de/622222

Partitioning around a pivot

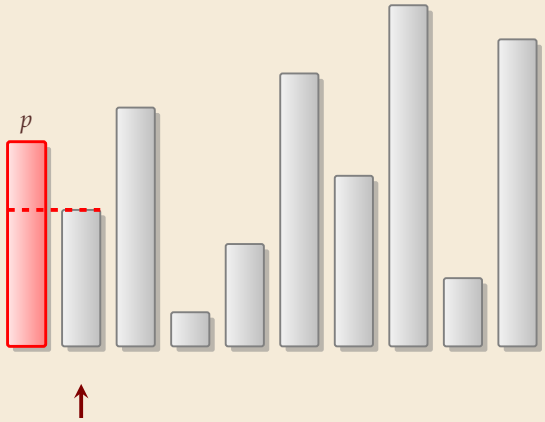
Partitioning around a pivot



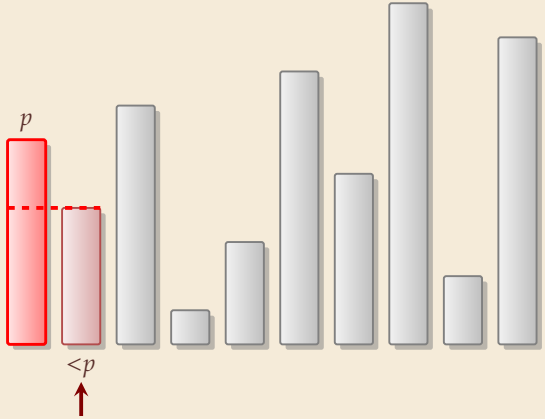
Partitioning around a pivot



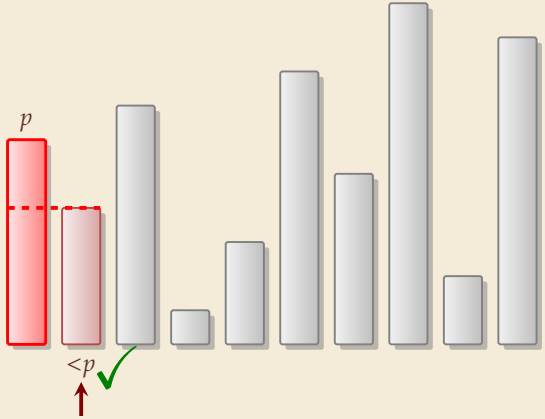
Partitioning around a pivot



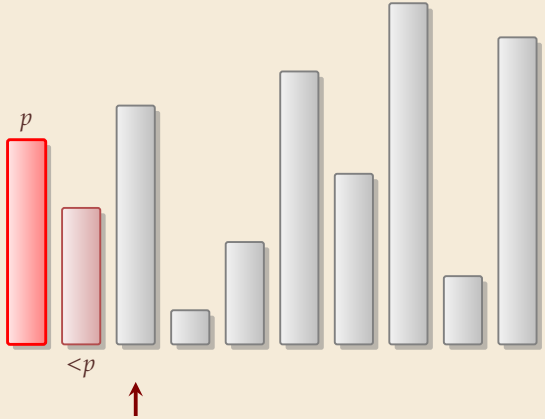
Partitioning around a pivot



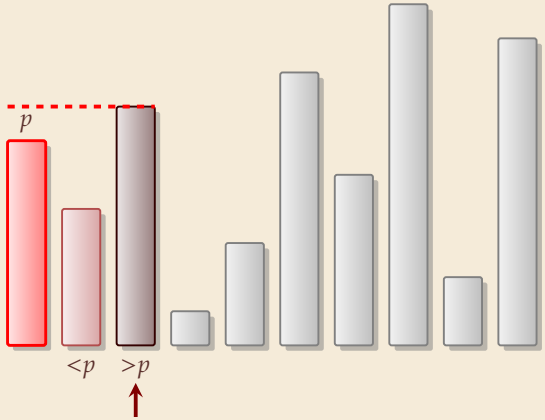
Partitioning around a pivot



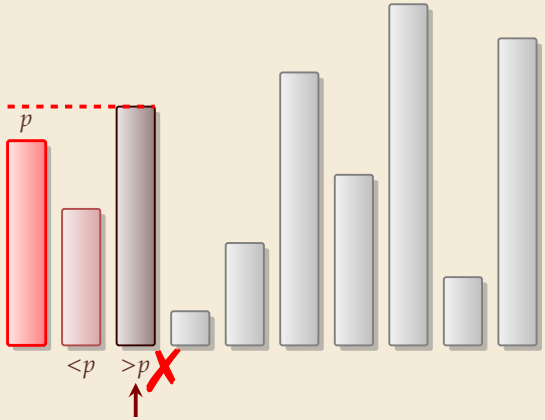
Partitioning around a pivot



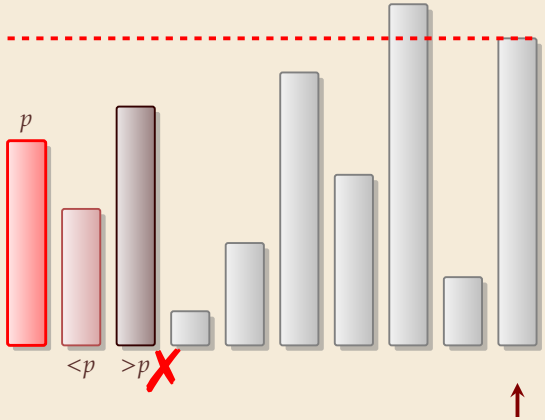
Partitioning around a pivot



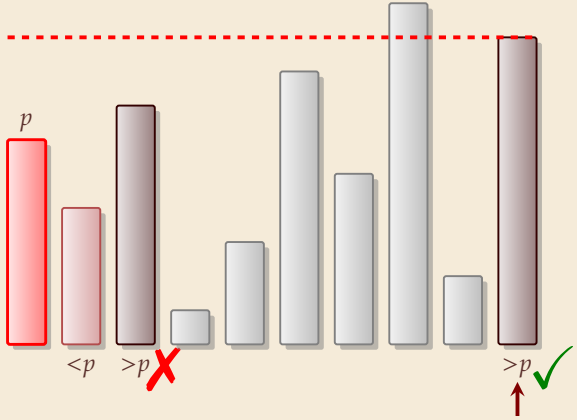
Partitioning around a pivot



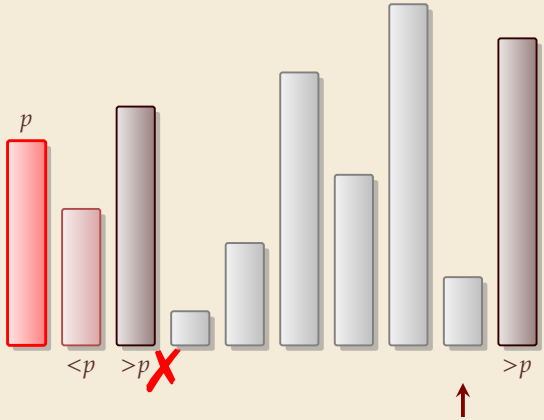
Partitioning around a pivot



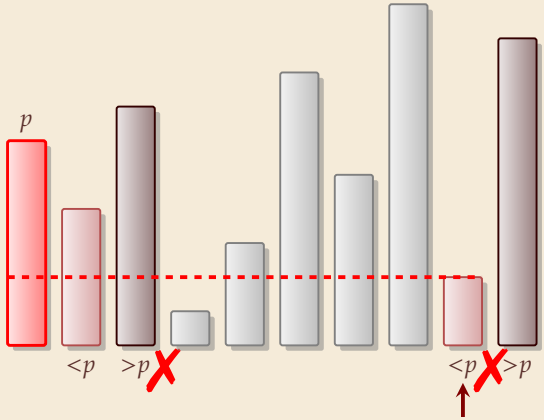
Partitioning around a pivot



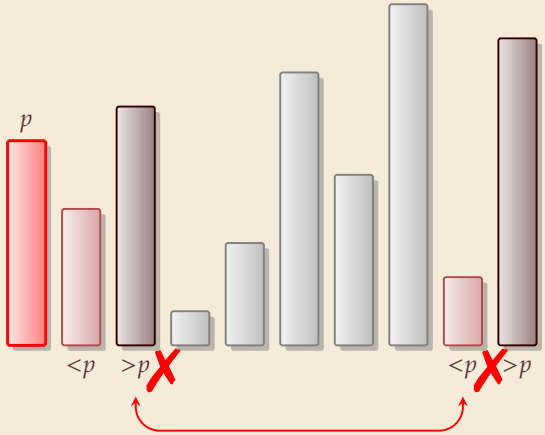
Partitioning around a pivot



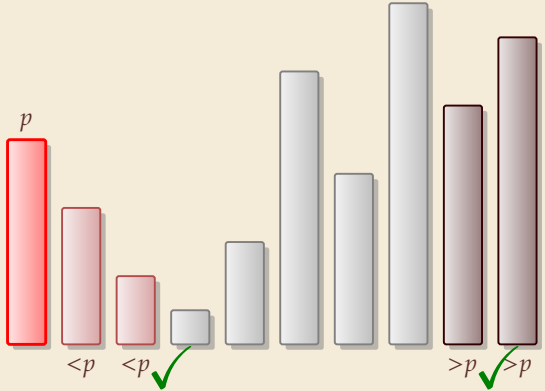
Partitioning around a pivot



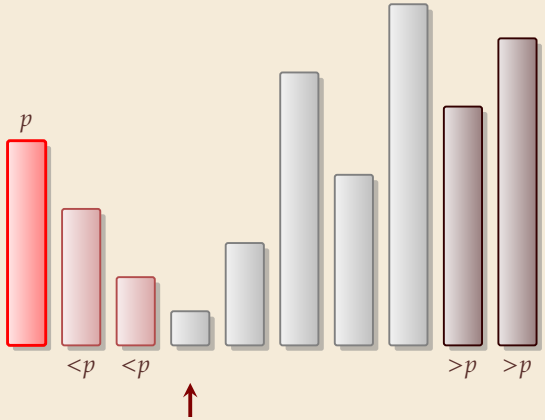
Partitioning around a pivot



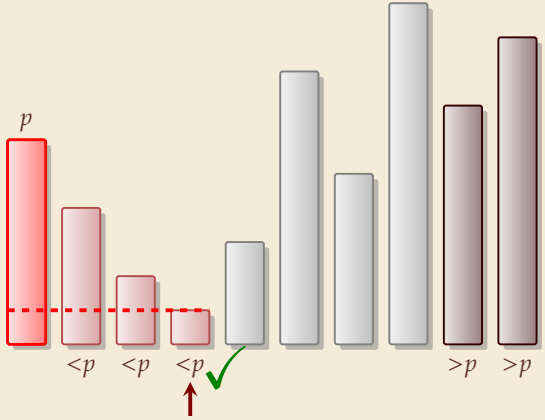
Partitioning around a pivot



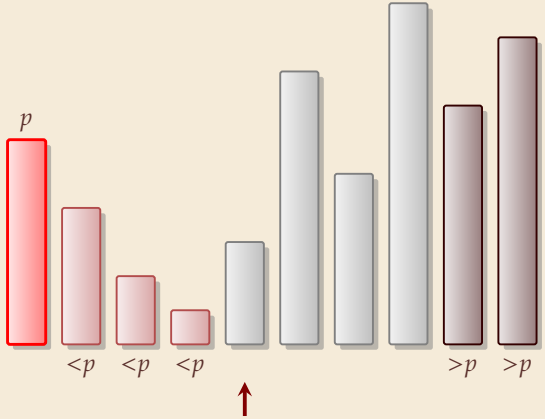
Partitioning around a pivot



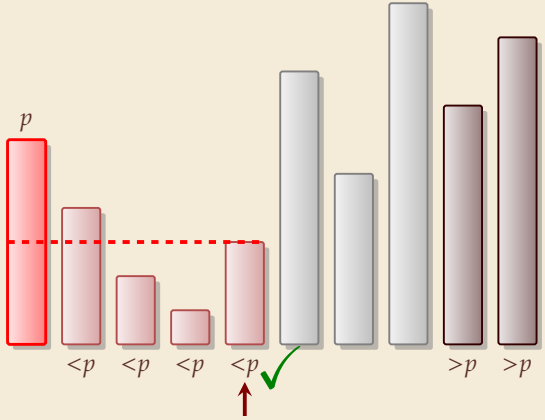
Partitioning around a pivot



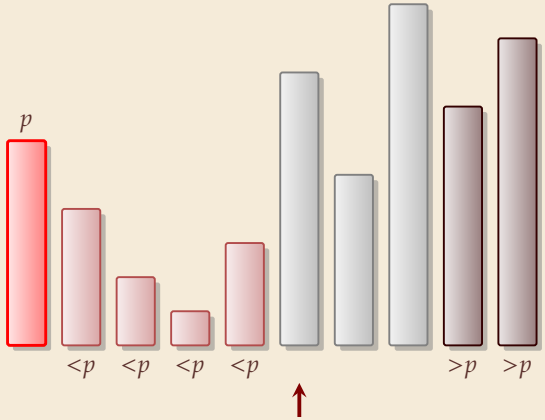
Partitioning around a pivot



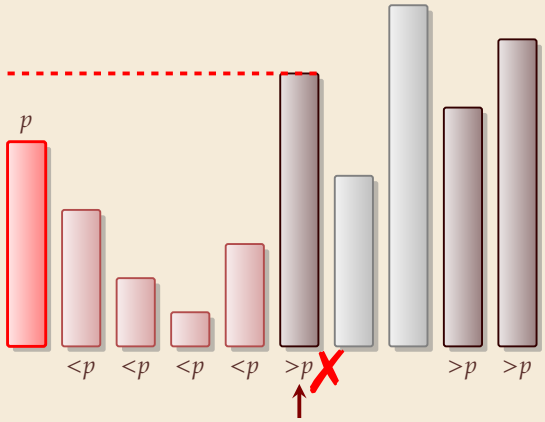
Partitioning around a pivot



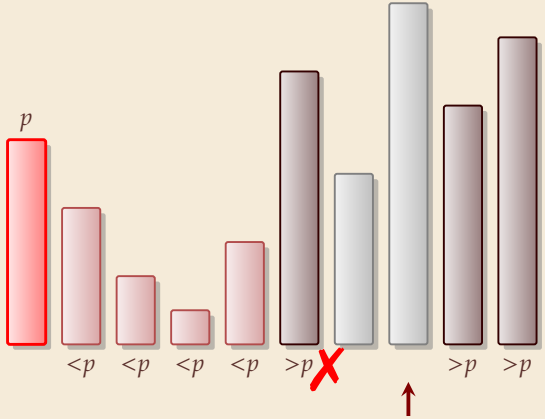
Partitioning around a pivot



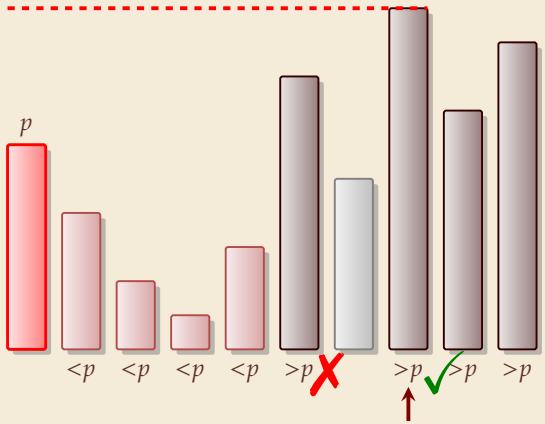
Partitioning around a pivot



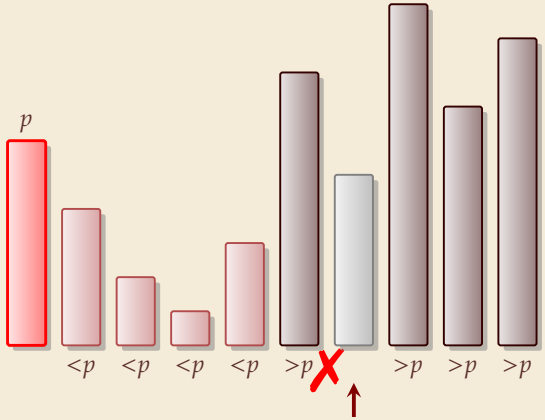
Partitioning around a pivot



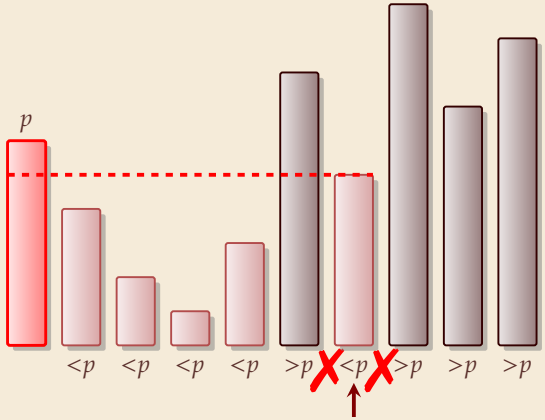
Partitioning around a pivot



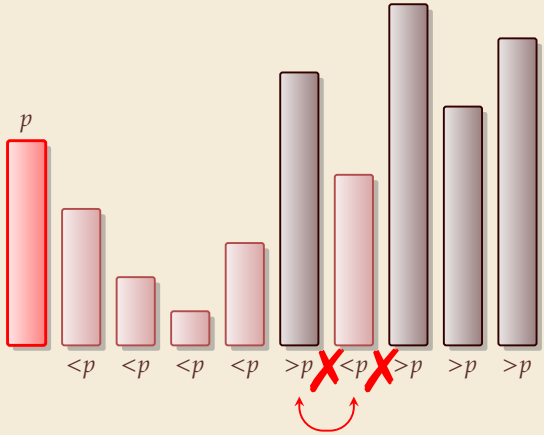
Partitioning around a pivot



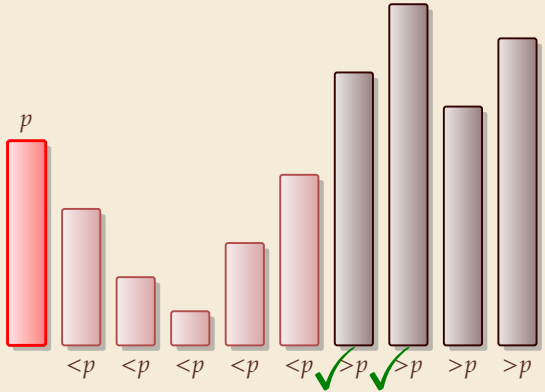
Partitioning around a pivot



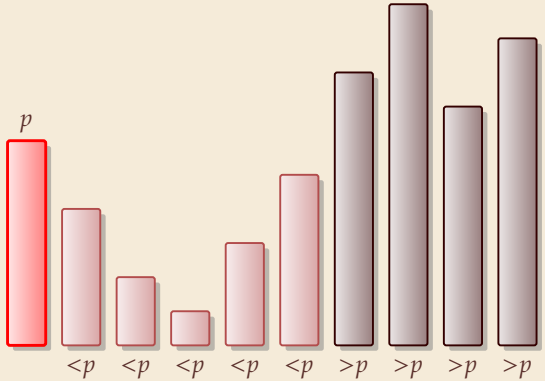
Partitioning around a pivot



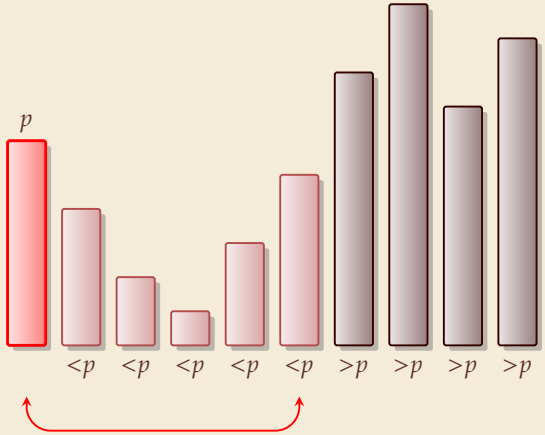
Partitioning around a pivot



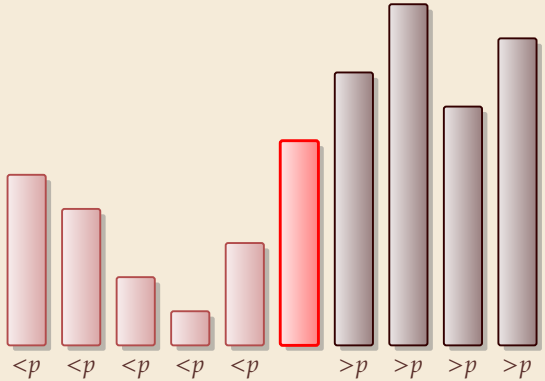
Partitioning around a pivot



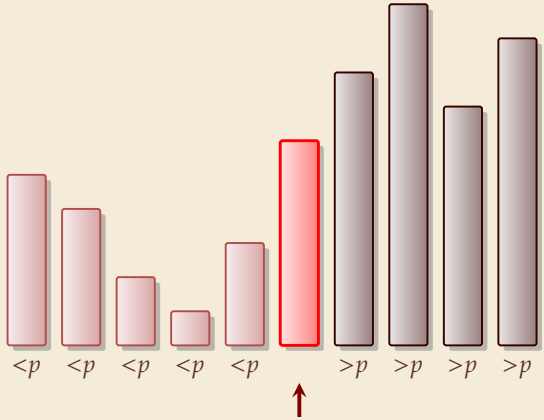
Partitioning around a pivot

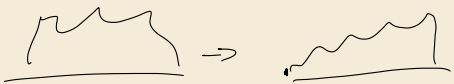


Partitioning around a pivot



Partitioning around a pivot





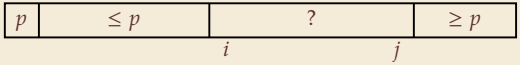
- no extra space needed
- visits each element once
- returns rank/position of pivot

Partitioning – Detailed code

Beware: details easy to get wrong; use this code!

```
1  procedure partition(A, b)
2      // input: array A[0..n − 1], position of pivot b ∈ [0..n − 1]
3      swap(A[0], A[b])
4      i := 0,    j := n
5      while true do
6          do i := i + 1 while i < n and A[i] < A[0]
7          do j := j − 1 while j ≥ 1 and A[j] > A[0]
8          if i ≥ j then break     (goto 8)
9          else swap(A[i], A[j])
10     end while
11     swap(A[0], A[j])
12     return j
```

Loop invariant (5–10): A



Quicksort

```
1 procedure quicksort(A[l..r])
2     if l ≥ r then return
3     b := choosePivot(A[l..r])
4     j := partition(A[l..r], b)
5     quicksort(A[l..j − 1])
6     quicksort(A[j + 1..r])
```

- recursive procedure; *divide & conquer*
- choice of pivot can be
  - fixed position ⇝ dangerous!
  - random
  - more sophisticated, e. g., median of 3

Quicksort & Binary Search Trees

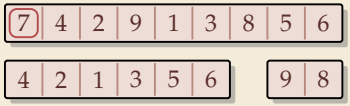
Quicksort

Quicksort & Binary Search Trees

Quicksort

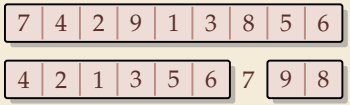
Quicksort & Binary Search Trees

Quicksort



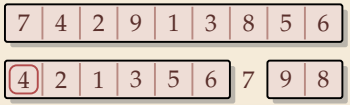
Quicksort & Binary Search Trees

Quicksort



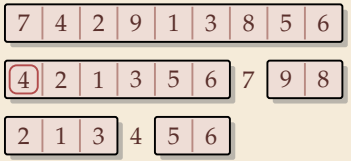
Quicksort & Binary Search Trees

Quicksort



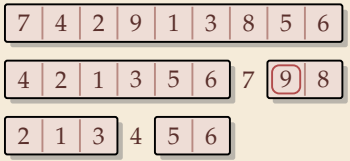
Quicksort & Binary Search Trees

Quicksort



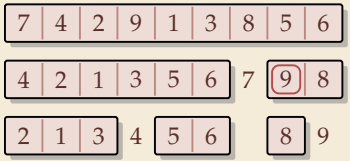
Quicksort & Binary Search Trees

Quicksort



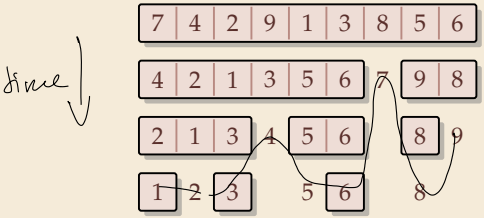
Quicksort & Binary Search Trees

Quicksort



Quicksort & Binary Search Trees

Quicksort

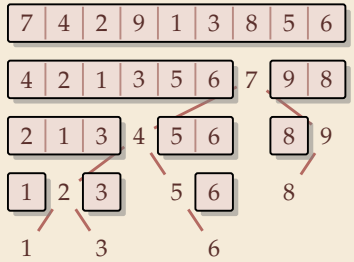


Quicksort & Binary Search Trees

Quicksort

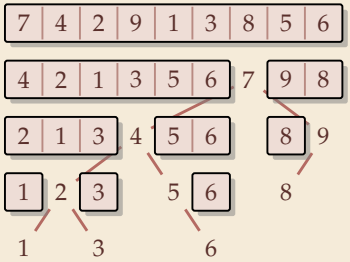
Quicksort & Binary Search Trees

Quicksort



Quicksort & Binary Search Trees

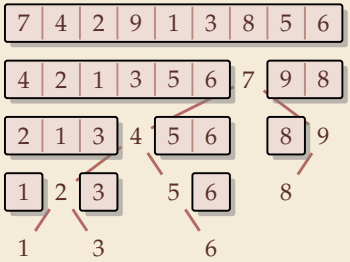
Quicksort



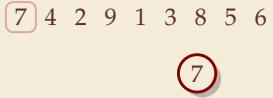
Binary Search Tree (BST)

Quicksort & Binary Search Trees

Quicksort

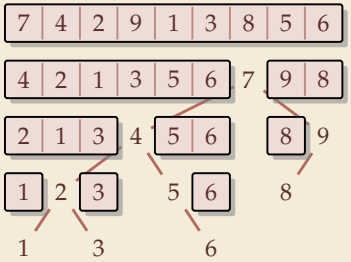


Binary Search Tree (BST)

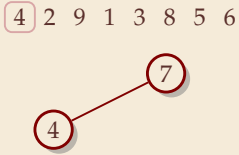


Quicksort & Binary Search Trees

Quicksort

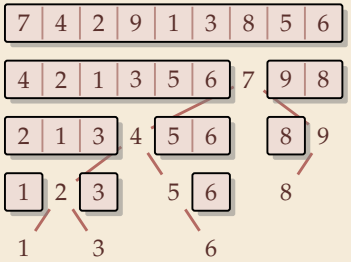


Binary Search Tree (BST)

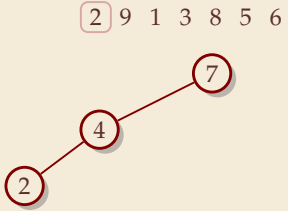


Quicksort & Binary Search Trees

Quicksort

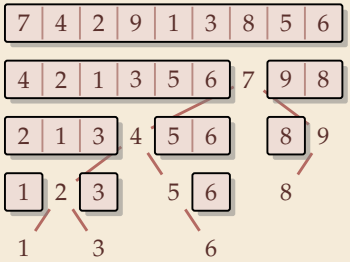


Binary Search Tree (BST)

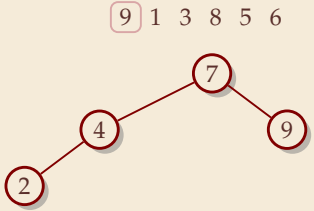


Quicksort & Binary Search Trees

Quicksort

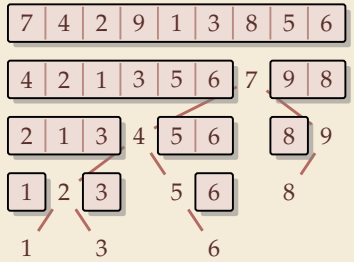


Binary Search Tree (BST)

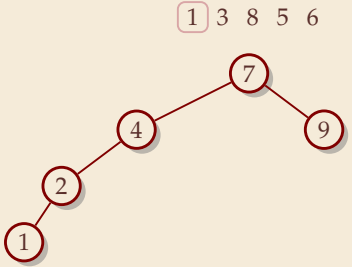


Quicksort & Binary Search Trees

Quicksort

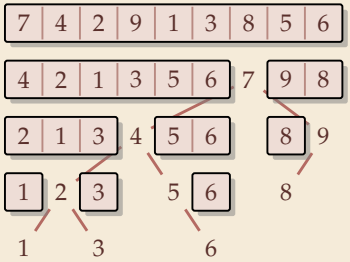


Binary Search Tree (BST)

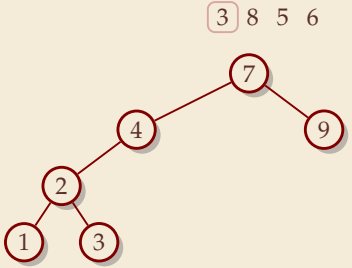


Quicksort & Binary Search Trees

Quicksort

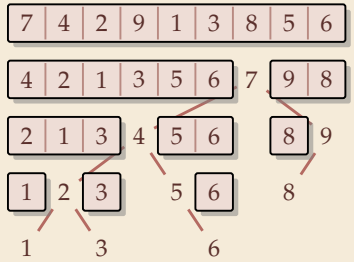


Binary Search Tree (BST)

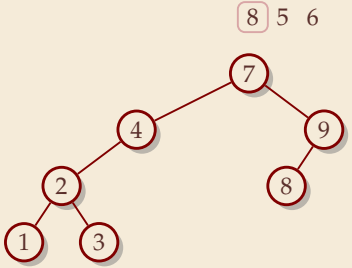


Quicksort & Binary Search Trees

Quicksort

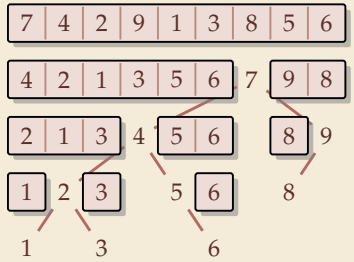


Binary Search Tree (BST)

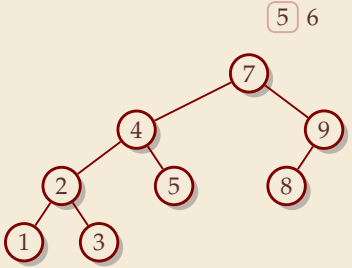


Quicksort & Binary Search Trees

Quicksort

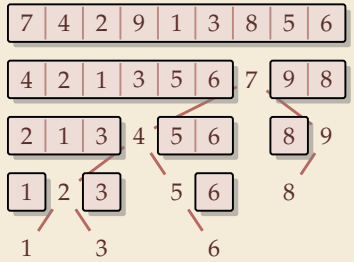


Binary Search Tree (BST)

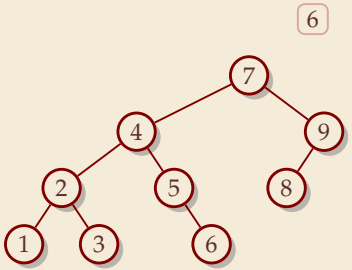


Quicksort & Binary Search Trees

Quicksort

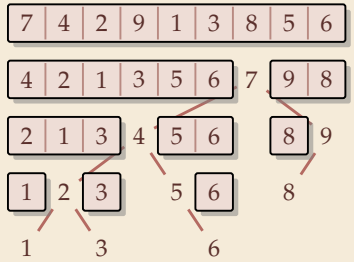


Binary Search Tree (BST)

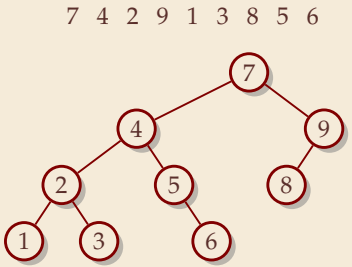


Quicksort & Binary Search Trees

Quicksort



Binary Search Tree (BST)

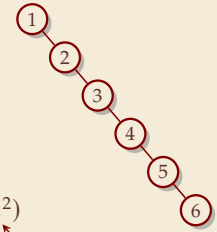


- recursion tree of quicksort = binary search tree from successive insertion
- comparisons in quicksort = comparisons to built BST
- comparisons in quicksort $\approx$ comparisons to search each element in BST

Quicksort – Worst Case

- Problem: BSTs can degenerate
- Cost to search for k is $k - 1$

$\rightsquigarrow$ Total cost $\sum_{k=1}^{n}(k-1) = \frac{n(n-1)}{2} \sim \frac{1}{2}n^2$



$\rightsquigarrow$ quicksort worst-case running time is in $\Theta(n^2)$

terribly slow!

But, we can fix this:

Randomized quicksort:

- choose a *random pivot* in each step

$\rightsquigarrow$ same as randomly *shuffling* input before sorting

Randomized Quicksort – Analysis

- $C(n)$ = element visits (as for mergesort)

$\rightsquigarrow$ quicksort needs $\sim 2\ln(2) \cdot n \lg n \approx \underline{1.39n} \lg n$ *in expectation*

- also: very unlikely to be much worse:
  e. g., one can prove: $\Pr[\text{cost} > 10n \lg n] = O(n^{-2.5})$
  distribution of costs is "concentrated around mean"
- intuition: have to be constantly unlucky with pivot choice $\int$

Quicksort – Discussion

- fastest general-purpose method
- $\Theta(n \log n)$ average case
- works *in-place* (no extra space required)
- memory access is sequential (scans over arrays)
- $\Theta(n^2)$ worst case (although extremely unlikely)
- not a *stable* sorting method

Open problem: Simple algorithm that is fast, stable and in-place.

3.3 Comparison-Based Lower Bound

Lower Bounds

- **Lower bound:** mathematical proof that no algorithm can do better.
  - very powerful concept: bulletproof *impossibility* result
    - $\approx$ *conservation of energy* in physics
  - **(unique?) feature of computer science:**
    for many problems, solutions are known that (asymptotically) *achieve the lower bound*
  - $\rightsquigarrow$ can speak of "*optimal* algorithms"

Lower Bounds

- **Lower bound:** mathematical proof that no algorithm can do better.
  - very powerful concept: bulletproof *impossibility* result
    - ≈ *conservation of energy* in physics
  - **(unique?) feature of computer science:**
    for many problems, solutions are known that (asymptotically) *achieve the lower bound*
  - ⇝ can speak of "*optimal* algorithms"

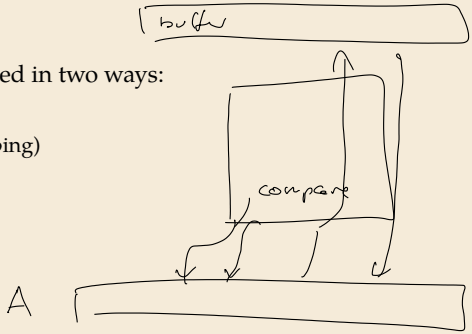
- To prove a statement about *all algorithms*, we must precisely define what that is!

- already know one option: the word-RAM model

- Here: use a simpler, more restricted model.

The Comparison Model

- In the *comparison model* data can only be accessed in two ways:
  - comparing two elements
  - moving elements around (e. g. copying, swapping)
  - Cost: number of these operations.



The Comparison Model

- In the *comparison model* data can only be accessed in two ways:
  - comparing two elements
  - moving elements around (e. g. copying, swapping)
  - Cost: number of these operations.

That's good!
Keeps algorithms general!

- This makes very few assumptions on the kind of objects we are sorting.
- Mergesort and Quicksort work in the comparison model.

The Comparison Model

- In the *comparison model* data can only be accessed in two ways:
  - comparing two elements
  - moving elements around (e. g. copying, swapping)
  - Cost: number of these operations.

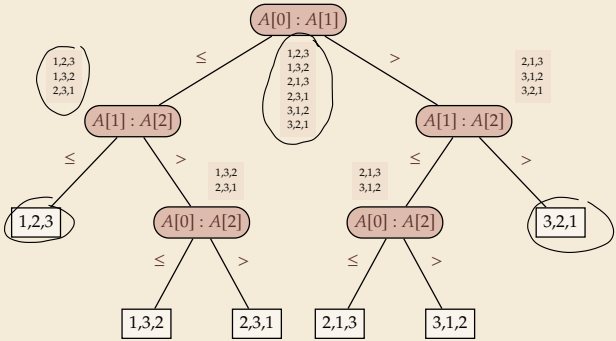
That's good!
Keeps algorithms general!

- This makes very few assumptions on the kind of objects we are sorting.
- Mergesort and Quicksort work in the comparison model.

⇝ Every comparison-based sorting algorithm corresponds to a ***decision tree***.
- only model comparisons ⇝ ignore data movement
- nodes = comparisons the algorithm does
- next comparisons can depend on outcomes ⇝ different subtrees
- child links = outcomes of comparison
- leaf = unique initial input permutation compatible with comparison outcomes

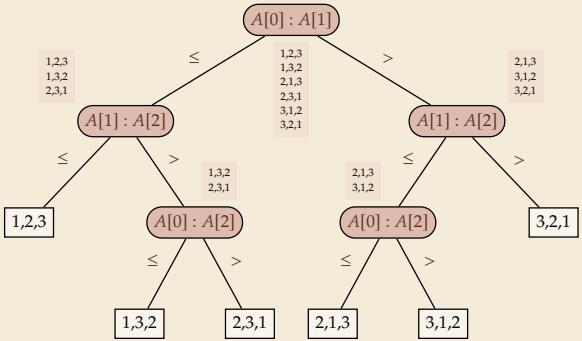
Comparison Lower Bound

Example: Comparison tree for a sorting method for $A[0..2]$:



Comparison Lower Bound

Example: Comparison tree for a sorting method for $A[0..2]$:



- Execution = follow a path in comparison tree.

  ⇝ height of comparison tree = worst-case # comparisons

- comparison trees are *binary* trees

  ⇝ ℓ leaves ⇝ height $\geq \lceil \lg(\ell) \rceil$

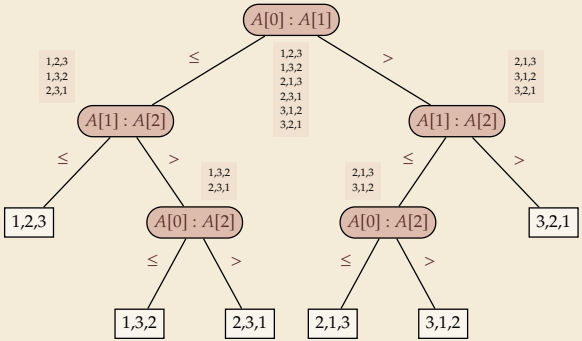
- comparison trees for sorting method must have $\geq n!$ leaves

  ⇝ height $\geq \lg(n!) \sim n \lg n$

more precisely: $\lg(n!) = n \lg n - \lg(e)n + O(\log n)$

Comparison Lower Bound

Example: Comparison tree for a sorting method for $A[0..2]$:



- Execution = follow a path in comparison tree.

  ⇝ height of comparison tree = worst-case # comparisons

- comparison trees are *binary* trees

  ⇝ ℓ leaves ⇝ height $\geq \lceil \lg(\ell) \rceil$

- comparison trees for sorting method must have $\geq n!$ leaves

  ⇝ height $\geq \lg(n!) \sim n \lg n$

more precisely: $\lg(n!) = n \lg n - \lg(e)n + O(\log n)$

- Mergesort achieves $\sim n \lg n$ comparisons ⇝ asymptotically comparison-optimal!
- Open (theory) problem: Can we sort with $n \lg n - \lg(e)n + o(n)$ comparisons?

≈ 1.4427