3 Efficient Sorting – The Power of Divide & Conquer

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COMP526 (Fall 2023)
University of Liverpool
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Learning Outcomes

1. Know principles and implementation of *mergesort* and *quicksort*.
2. Know properties and *performance characteristics* of mergesort and quicksort.
3. Know the comparison model and understand the corresponding *lower bound*.
4. Understand *counting sort* and how it circumvents the comparison lower bound.
5. Know ways how to exploit *presorted* inputs.

Unit 3: *Efficient Sorting*

Outline

3 Efficient Sorting

Why study sorting?

- fundamental problem of computer science that is still not solved

  Algorithm with optimal #comparisons in worst case?
- building brick of many more advanced algorithms
  - for preprocessing
  - as subroutine
- playground of manageable complexity
  to practice algorithmic techniques

Here:

- "classic" fast sorting method
- exploit **partially sorted** inputs
- **parallel** sorting

Part I

The Basics

Rules of the game

- **Given:**
  - array $A[0..n) = A[0..n-1]$ of n objects
  - a total order relation $\leq$ among $A[0], \ldots, A[n-1]$
    (a comparison function)
    Python: elements support <= operator (`__le__()`)
    Java: `Comparable` class (`x.compareTo(y) <= 0`)
- **Goal:** rearrange (i. e., permute) elements within A,
  so that A is *sorted*, i. e., $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

- for now: A stored in main memory (*internal sorting*)
  single processor (*sequential sorting*)

Clicker Question

What is the complexity of sorting? Type you answer, e. g., as "`Theta(sqrt(n))`"

$\Theta(n \log n)$ ← $O(n \log n)$ & $\Omega(n \log n)$

algorithm of that cost

lower bound impossible to do better

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3.1 Mergesort

Clicker Question

How does mergesort work?

A Split elements around median, then recurse on small / large elements.

B Recurse on left / right half, then combine sorted halves.

C Grow sorted part on left, repeatedly add next element to sorted range.

D Repeatedly choose 2 elements and swap them if they are out of order.

E Don't know.

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Merging sorted lists

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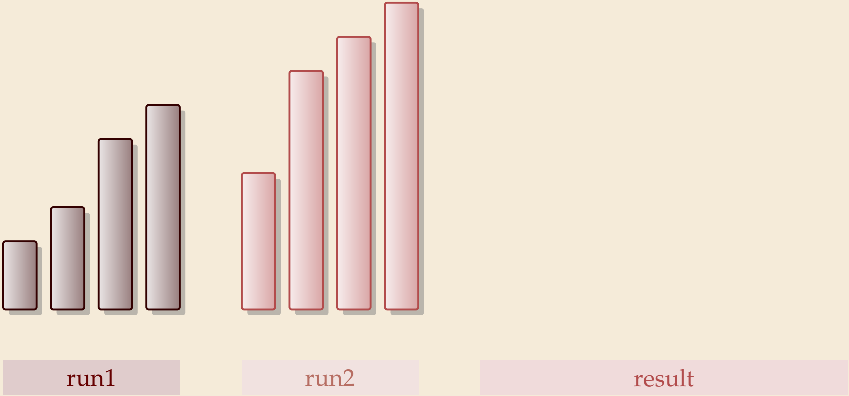

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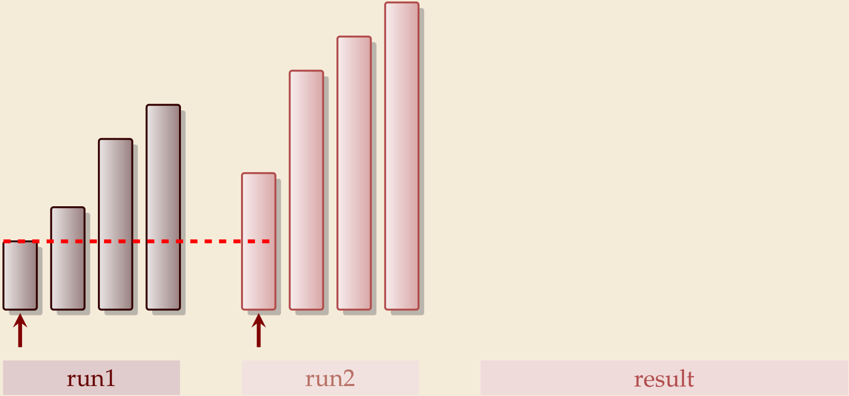

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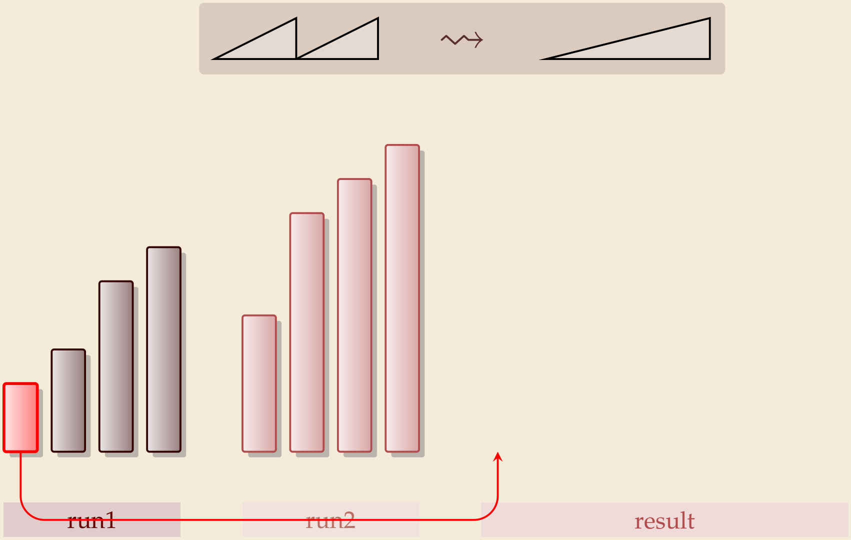

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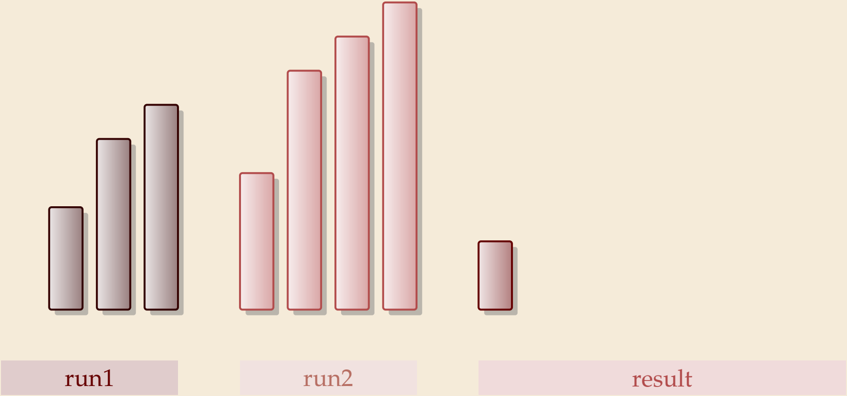

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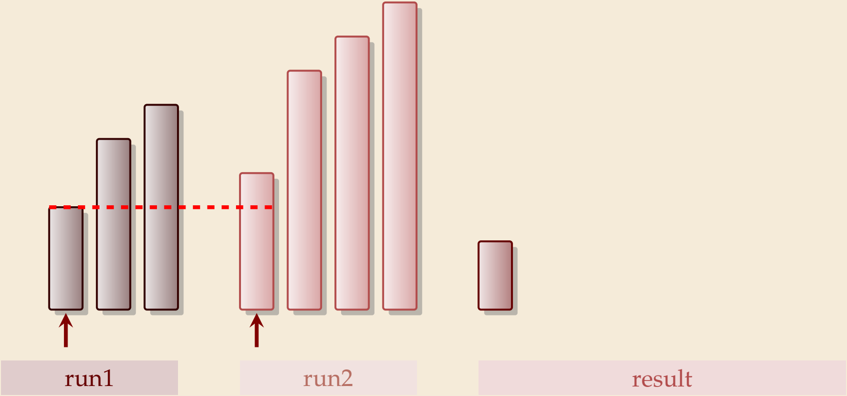

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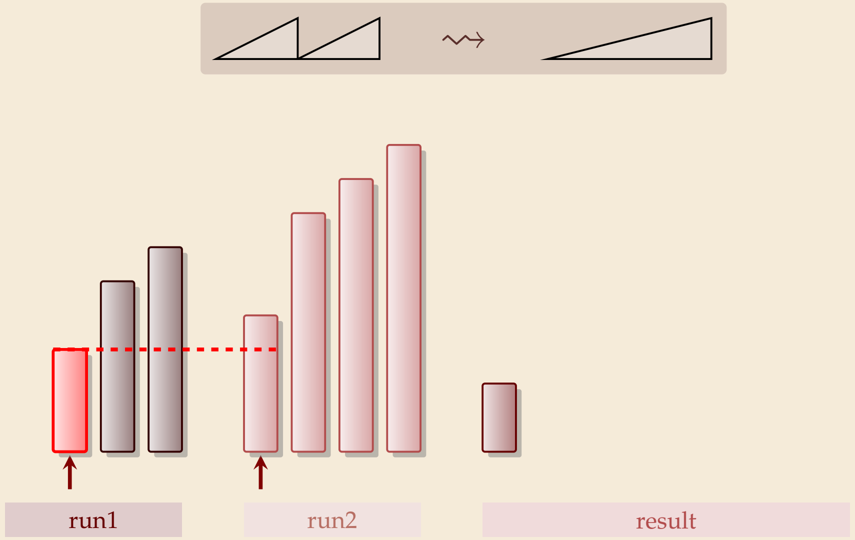

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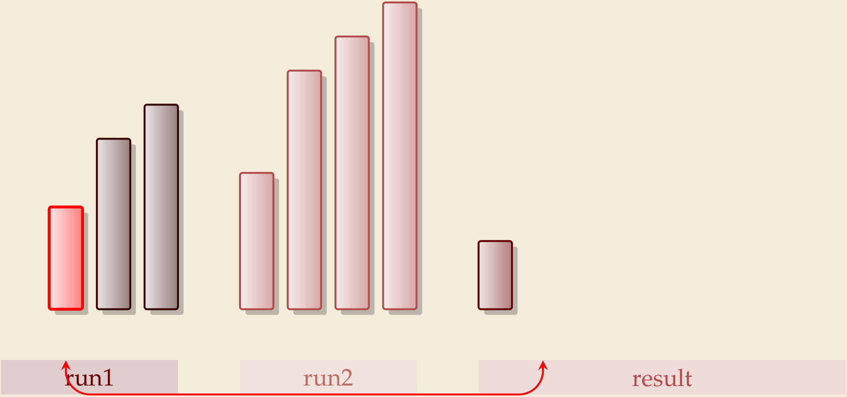

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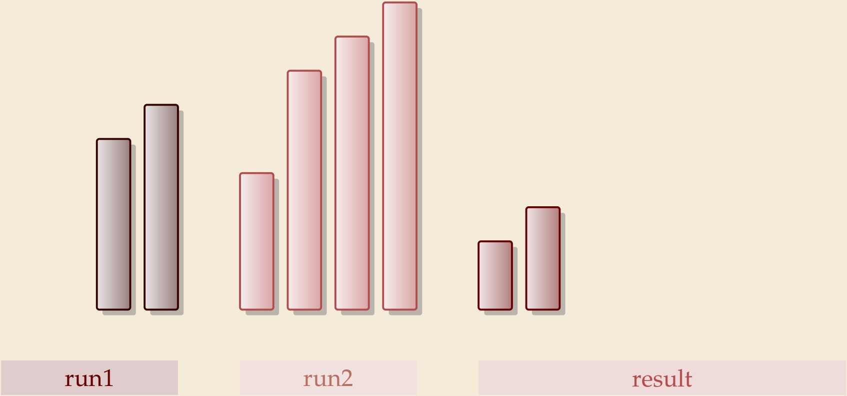

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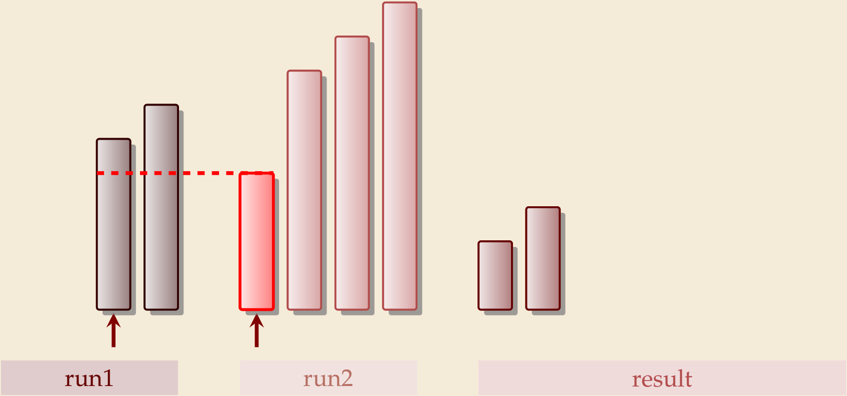

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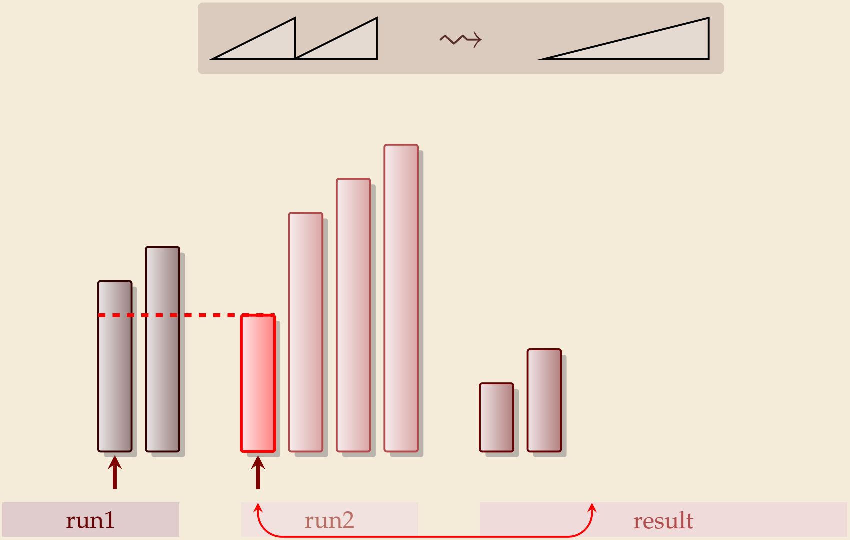

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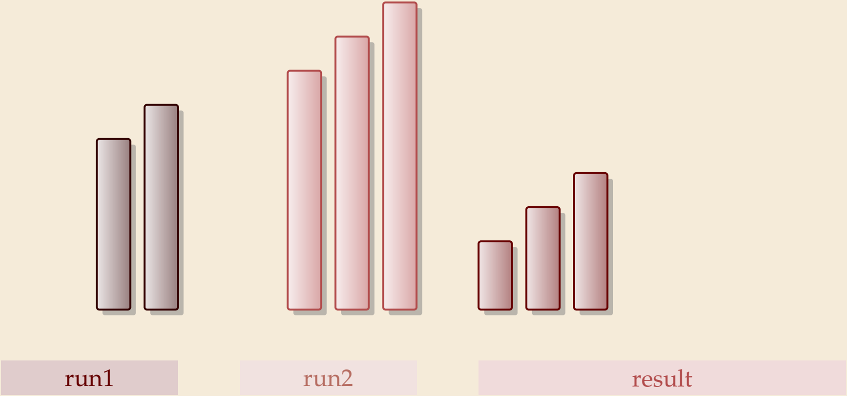

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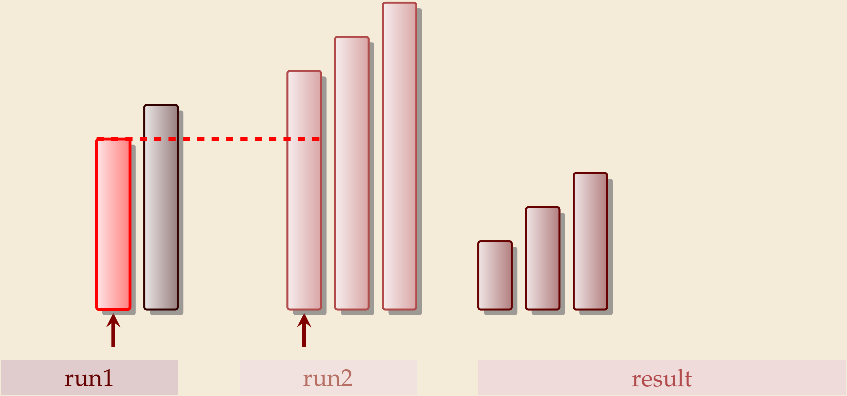

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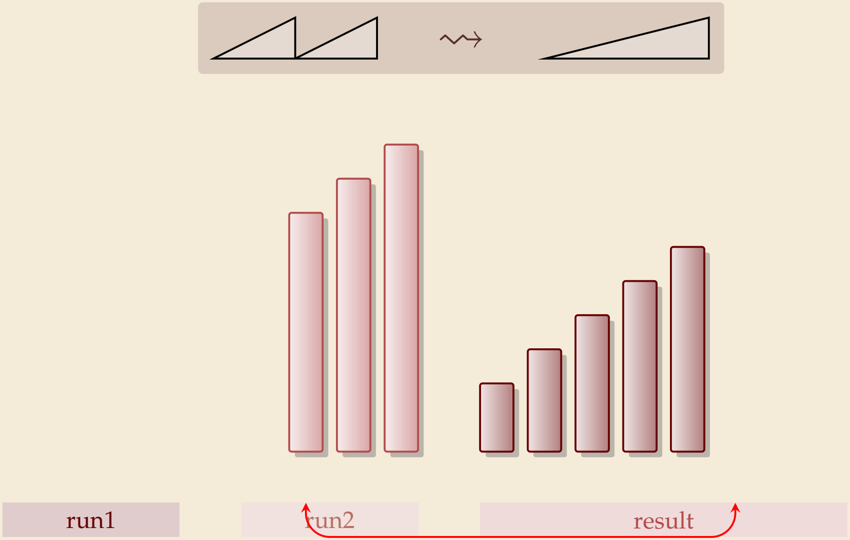

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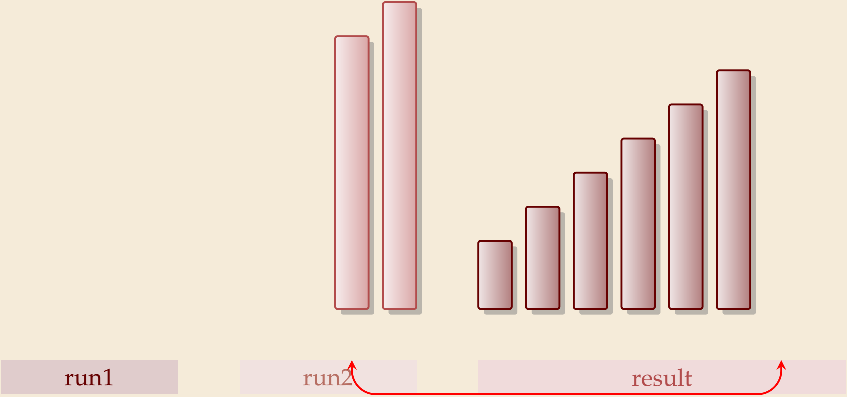

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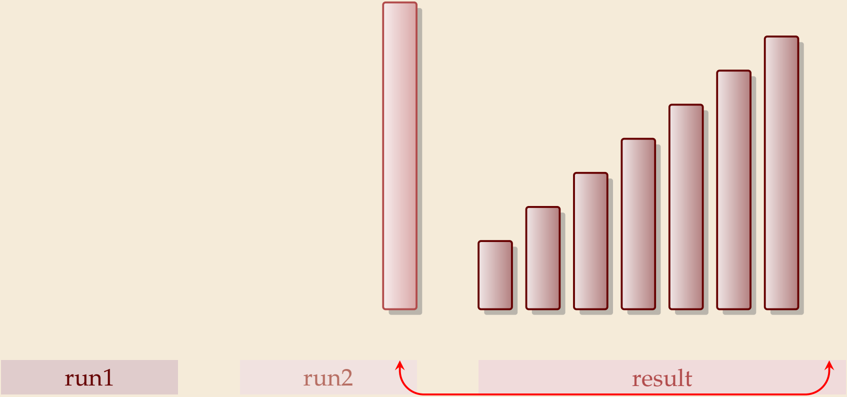

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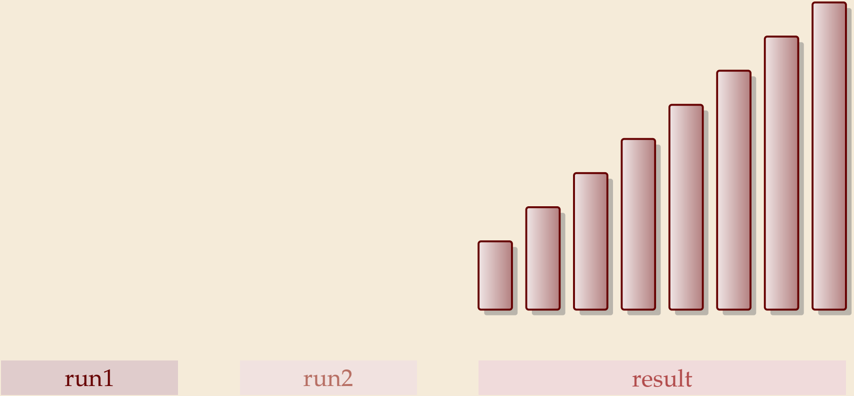

Clicker Question

What is the worst-case running time of mergesort?

- **A** $\Theta(1)$
- **B** $\Theta(\log n)$
- **C** $\Theta(\log \log n)$
- **D** $\Theta(\sqrt{n})$
- **E** $\Theta(n)$
- **F** $\Theta(n \log \log n)$
- **G** $\Theta(n \log n)$
- **H** $\Theta(n \log^2 n)$
- **I** $\Theta(n^{1+\epsilon})$
- **J** $\Theta(n^2)$
- **K** $\Theta(n^3)$
- **L** $\Theta(2^n)$

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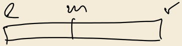
Mergesort

mergesort$(A[0..n))$

procedure mergesort($A[l..r)$)
    $n := r - l$
    if $n \leq 1$ **return**
    $m := l + \left\lfloor \frac{n}{2} \right\rfloor$
    mergesort($A[l..m)$)
    mergesort($A[m..r)$)
    merge($A[l..m)$, $A[m..r)$, *buf*)
    copy *buf* to $A[l..r)$



- recursive procedure
- merging needs
  - temporary storage *buf* for result (of same size as merged runs)
  - to read and write each element twice (once for merging, once for copying back)

Mergesort

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procedure mergesort(A[l..r])
    n := r − l
    if n ≤ 1 return
    m := l + ⌊n/2⌋
    mergesort(A[l..m])
    mergesort(A[m..r])
    merge(A[l..m], A[m..r], buf)
    copy buf to A[l..r]
```

- recursive procedure
- merging needs
  - temporary storage *buf* for result (of same size as merged runs)
  - to read and write each element twice (once for merging, once for copying back)

Analysis: count *"element visits"* (read and/or write)

cmps between $\frac{n}{2}$ and n

$$C(n) = \begin{cases} 0 & n \le 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \ge 2 \end{cases}$$

same for best and worst case!

Simplification $\boxed{n = 2^k}$

$$C(2^k) = \begin{cases} 0 & k \le 0 \\ 2 \cdot C(2^{k-1}) + 2 \cdot 2^k & k \ge 1 \end{cases} = 2 \cdot 2^k + 2^2 \cdot 2^{k-1} + 2^3 \cdot 2^{k-2} + \cdots + 2^k \cdot 2^1 = 2k \cdot 2^k$$

$$C(n) = 2n\lg(n) = \Theta(n \log n)$$

(arbitrary n: $C(n) \le C(\text{next larger power of 2}) \le 4n\lg(n) + 2n = \Theta(n \log n)$)

Mergesort

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$$C(n) \;=\; \begin{cases} 0 & n \le 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \ge 2 \end{cases}$$

same for best and worst case!

(precisely(!) solvable *without* assumption $n = 2^k$: $C(n) = 2n\lg(n) + \left(2 - \{\lg(n)\} - 2^{1-\{\lg(n)\}}\right)2n$ with $\{x\} := x - \lfloor x \rfloor$)

Simplification $\boxed{n = 2^k}$

$$C(2^k) \;=\; \begin{cases} 0 & k \le 0 \\ 2 \cdot C(2^{k-1}) + 2 \cdot 2^k & k \ge 1 \end{cases} \;=\; 2 \cdot 2^k + 2^2 \cdot 2^{k-1} + 2^3 \cdot 2^{k-2} + \cdots + 2^k \cdot 2^1 \;=\; 2k \cdot 2^k$$

$$C(n) \;=\; 2n\lg(n) \;=\; \Theta(n \log n)$$

(arbitrary n: $C(n) \le C(\text{next larger power of 2}) \le 4n\lg(n) + 2n = \Theta(n \log n)$)

Mergesort – Discussion

- optimal time complexity of $\Theta(n \log n)$ in the worst case
- *stable* sorting method i. e., retains relative order of equal-key items
- memory access is sequential (scans over arrays)
- requires $\Theta(n)$ extra space

there are in-place merging methods,
but they are substantially more complicated
and not (widely) used

3.2 Quicksort

Clicker Question

How does quicksort work?

A split elements around median, then recurse on small / large elements.

B recurse on left / right half, then combine sorted halves.

C grow sorted part on left, repeatedly add next element to sorted range.

D repeatedly choose 2 elements and swap them if they are out of order.

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Partitioning around a pivot

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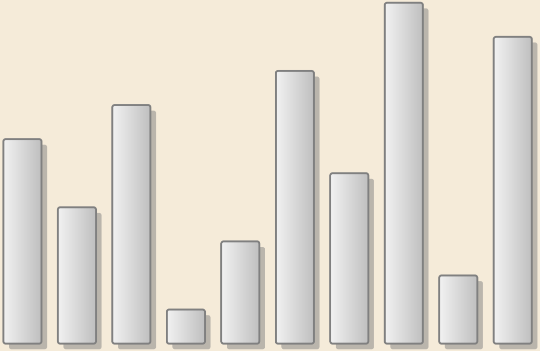

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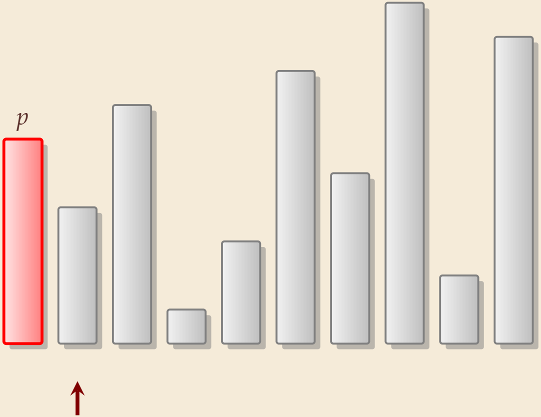

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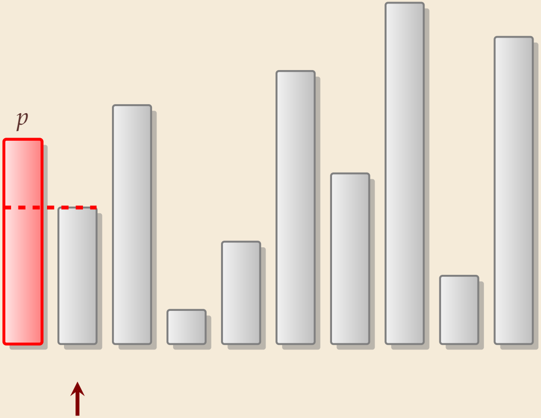

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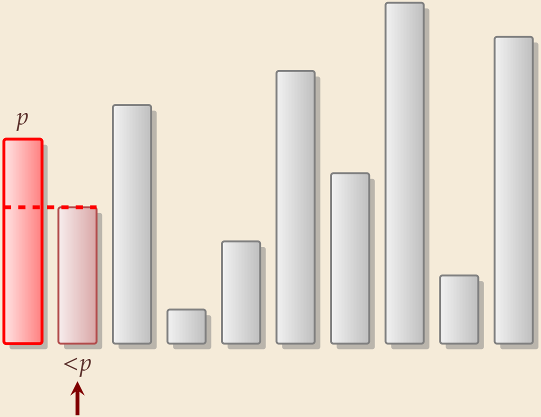

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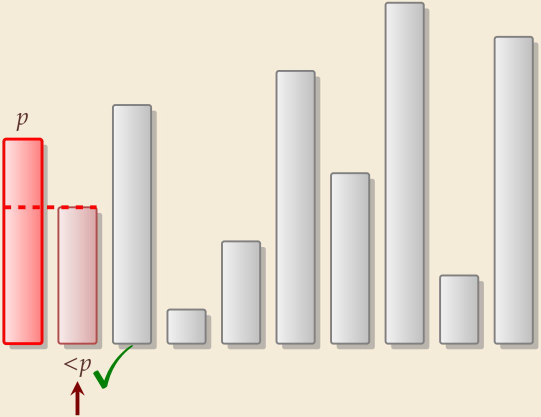

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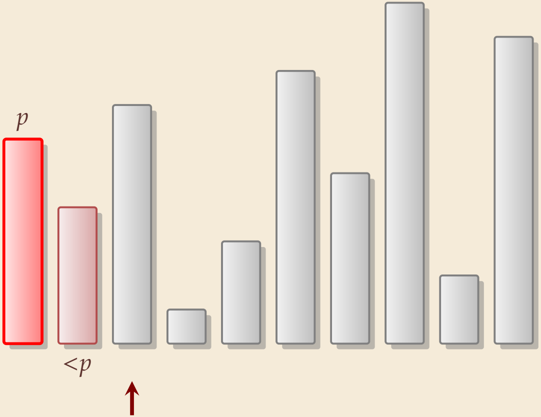

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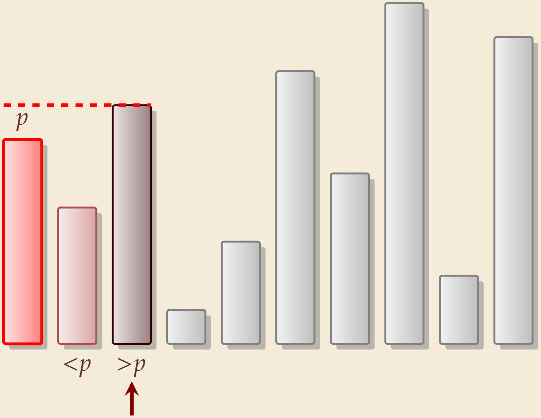

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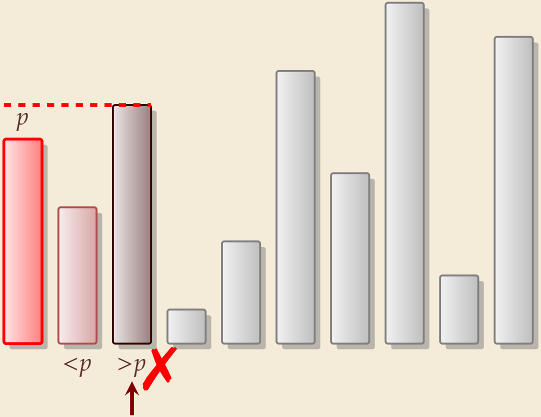

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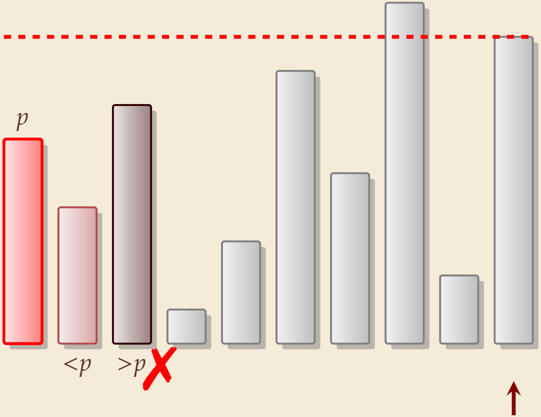

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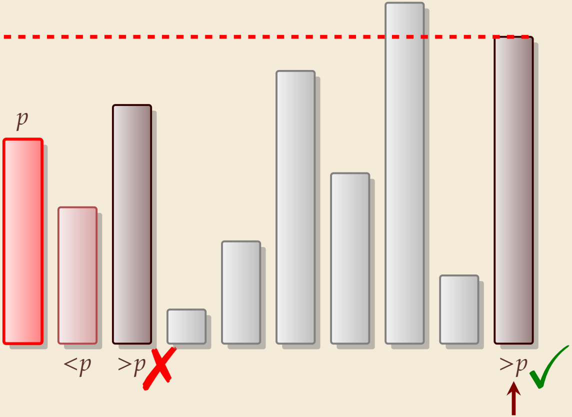

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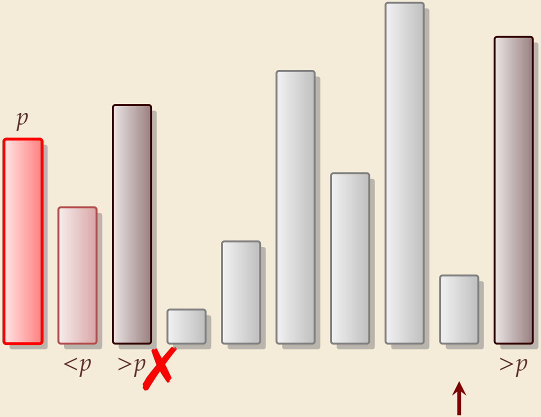

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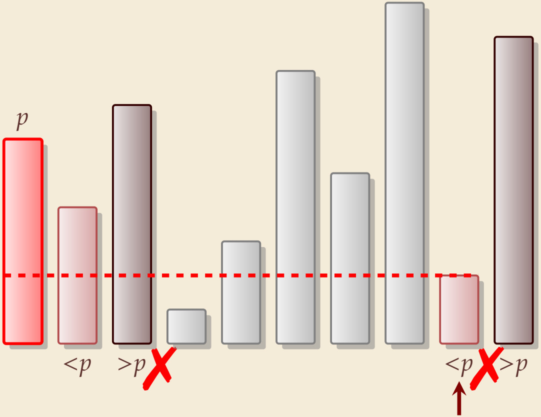

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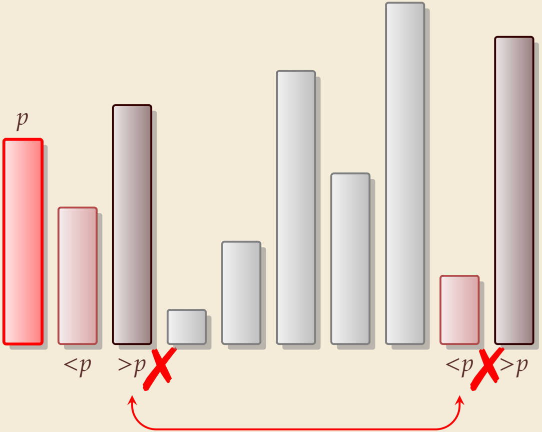

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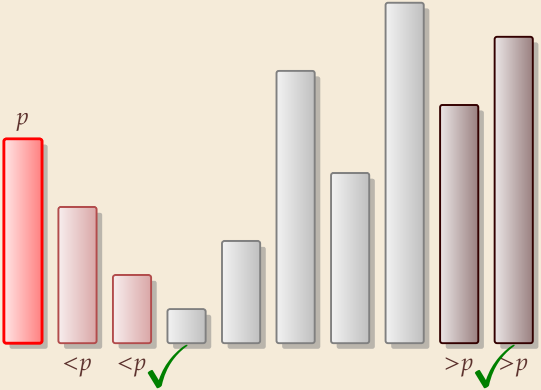

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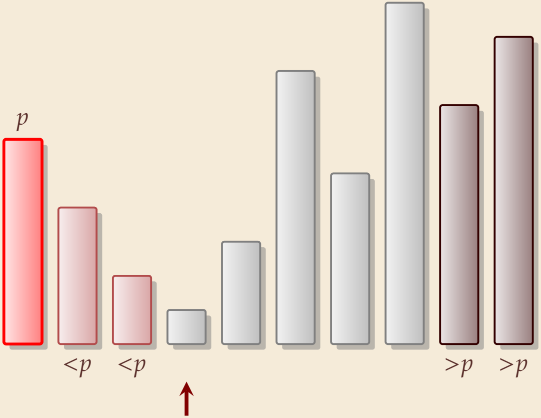

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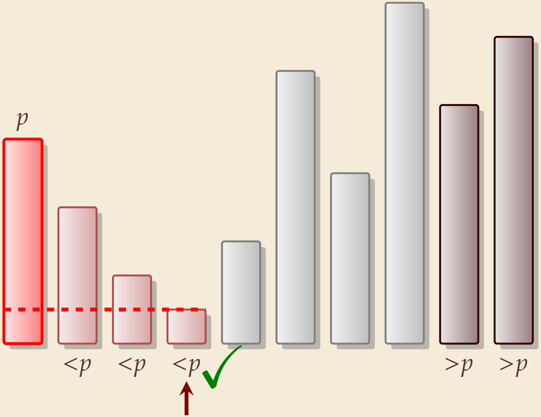

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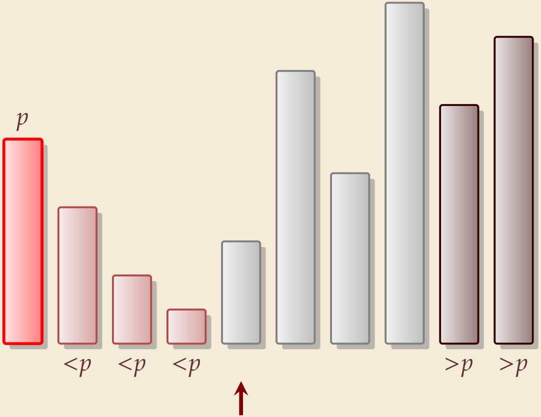

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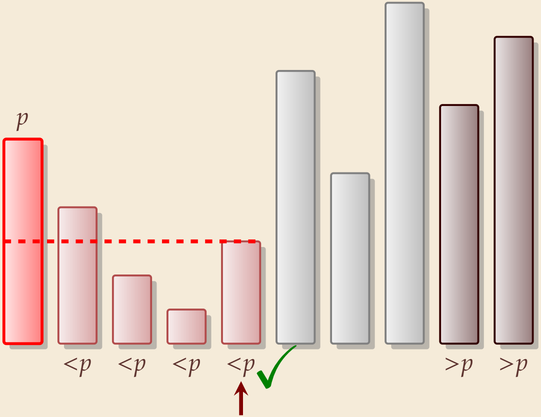

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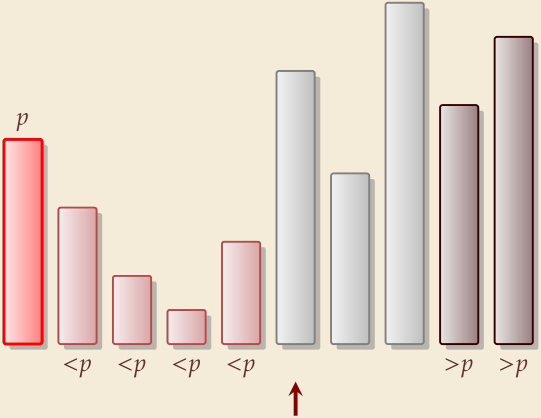

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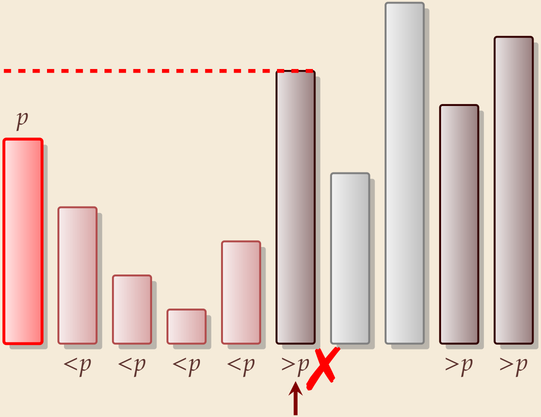

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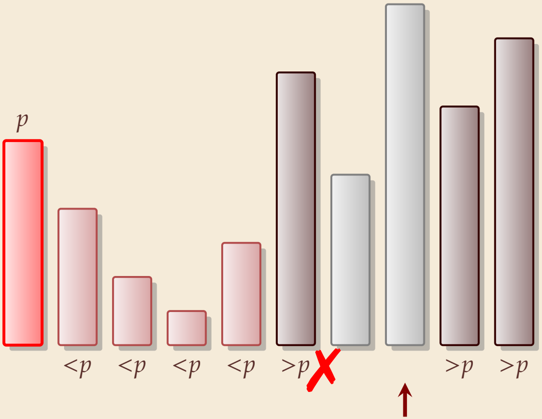

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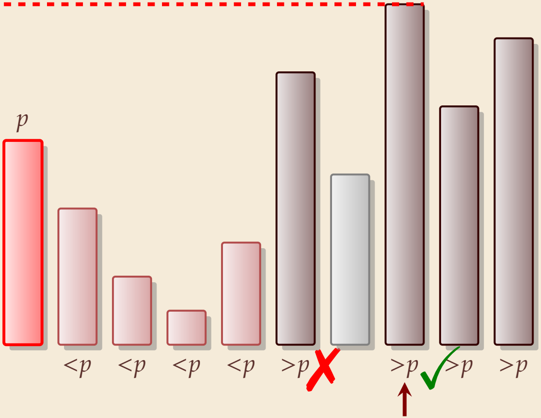

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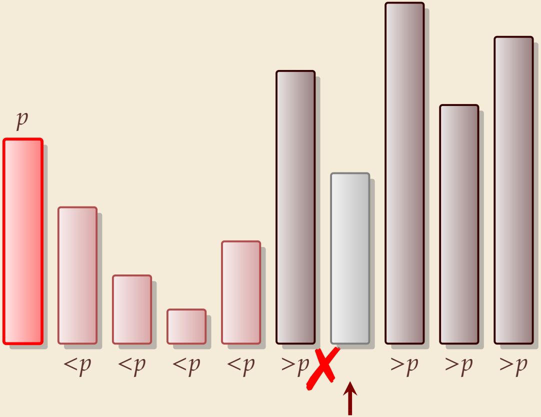

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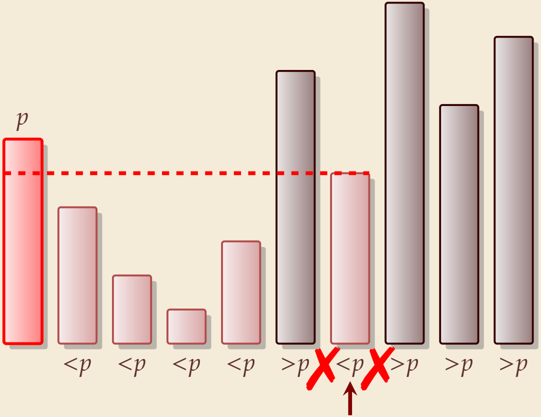

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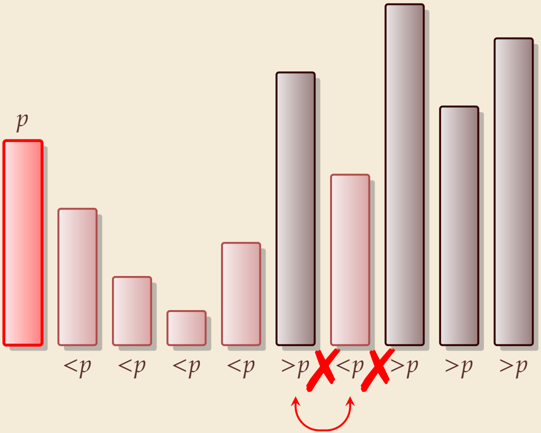

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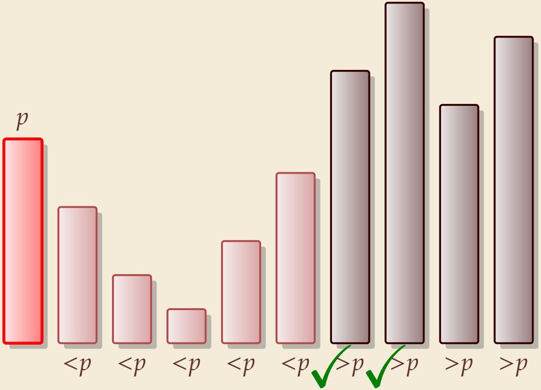

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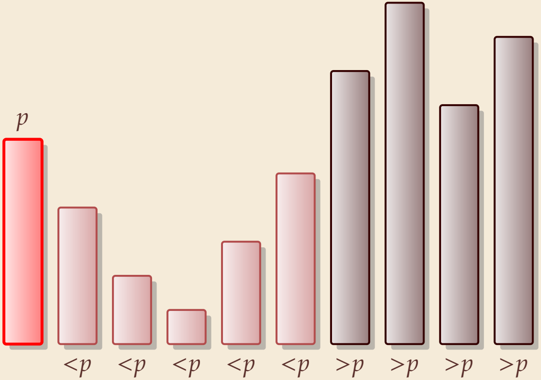

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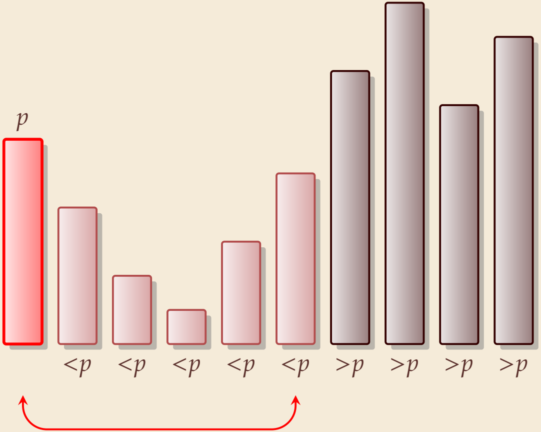

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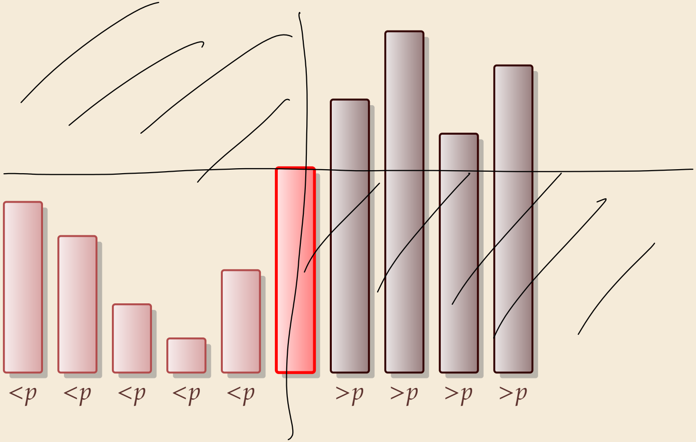

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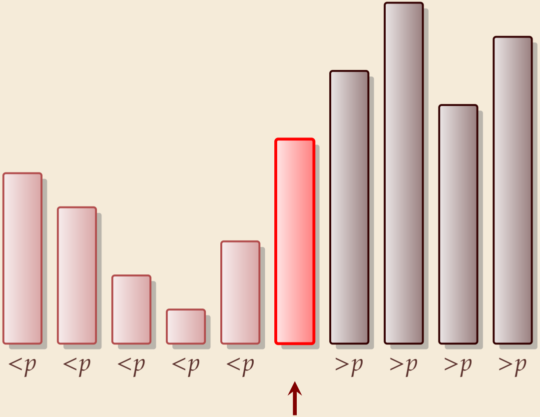


- no extra space needed
- visits each element once
- returns rank/position of pivot

Partitioning – Detailed code

Beware: details easy to get wrong; use this code! (if you ever have to)

```
1  procedure partition(A, b)
2      // input: array A[0..n), position of pivot b ∈ [0..n)
3      swap(A[0], A[b])
4      i := 0,   j := n
5      while true do
6          do i := i + 1 while i < n and A[i] < A[0]
7          do j := j − 1 while j ≥ 1 and A[j] > A[0]
8          if i ≥ j then break     (goto 11)
9          else swap(A[i], A[j])
10     end while
11     swap(A[0], A[j])
12     return j
```

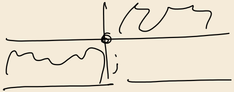
Loop invariant (5–10): A

p	$\le p$	?	$\ge p$
		i	j

Quicksort

initial call
quicksort(A[0..n)]

```
procedure quicksort(A[l..r])
    if r − ℓ ≤ 1 then return
    b := choosePivot(A[l..r])
    j := partition(A[l..r], b)
    quicksort(A[l..j])
    quicksort(A[j + 1..r])
```



- recursive procedure
- choice of pivot can be
  - fixed position $\rightsquigarrow$ dangerous!
  - random
  - more sophisticated, e. g., median of 3

Clicker Question

What is the worst-case running time of quicksort?

- **A** $\Theta(1)$
- **B** $\Theta(\log n)$
- **C** $\Theta(\log \log n)$
- **D** $\Theta(\sqrt{n})$
- **E** $\Theta(n)$
- **F** $\Theta(n \log \log n)$
- **G** $\Theta(n \log n)$
- **H** $\Theta(n \log^2 n)$
- **I** $\Theta(n^{1+\epsilon})$
- **J** $\Theta(n^2)$
- **K** $\Theta(n^3)$
- **L** $\Theta(2^n)$

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Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6

9	8

Quicksort & Binary Search Trees

Quicksort

7	4	2	9	1	3	8	5	6

4	2	1	3	5	6	7	9	8

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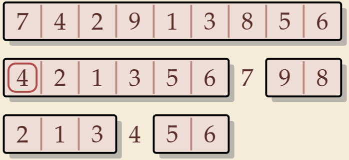

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7	4	2	9	1	3	8	5	6

4	2	1	3	5	6

7

9	8

2	1	3

4

5	6

Quicksort & Binary Search Trees

Quicksort

| 7 | 4 | 2 | 9 | 1 | 3 | 8 | 5 | 6 |

| 4 | 2 | 1 | 3 | 5 | 6 | 7 | 9 | 8 |

| 2 | 1 | 3 | 4 | 5 | 6 | | 8 | 9

Quicksort & Binary Search Trees

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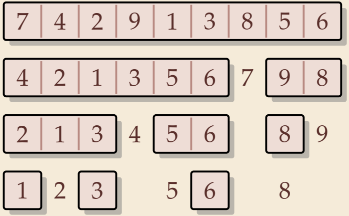

Quicksort & Binary Search Trees

Quicksort

7 4 2 9 1 3 8 5 6

4 2 1 3 5 6 7 9 8

2 1 3 4 5 6 8 9

1 2 3 5 6 8

1 3 6

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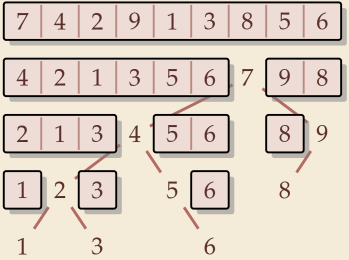
Quicksort



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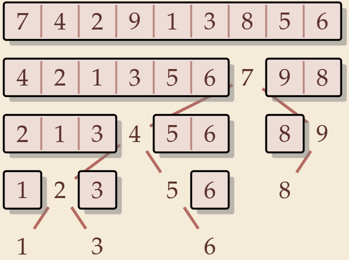
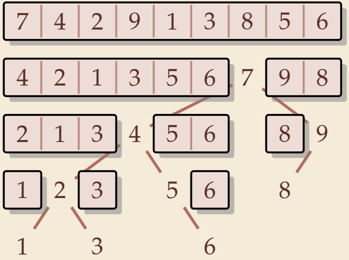


Binary Search Tree (BST)

7 4 2 9 1 3 8 5 6

Quicksort & Binary Search Trees

Quicksort



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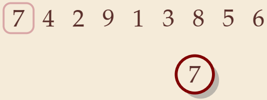

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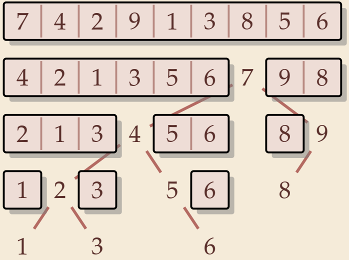
Quicksort



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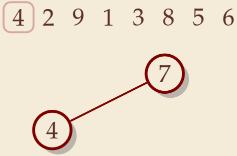

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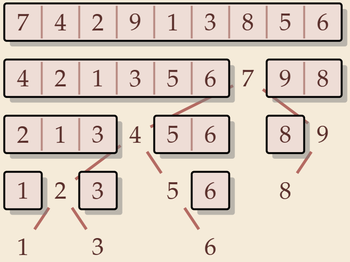
Quicksort



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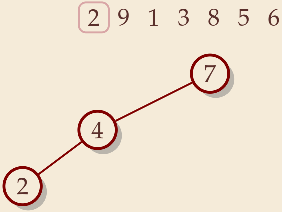

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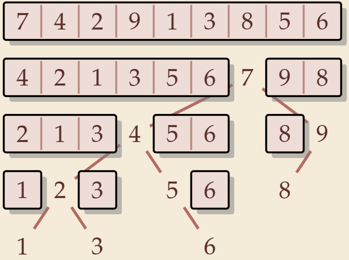

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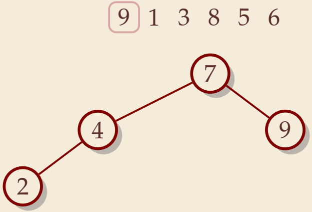

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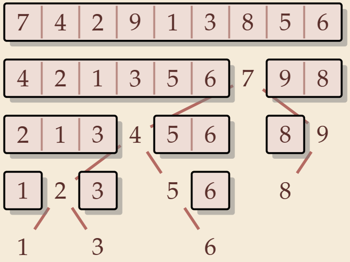

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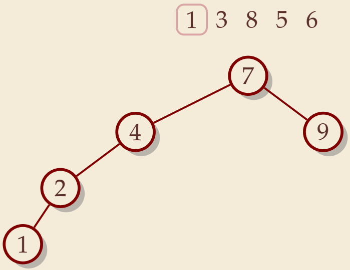

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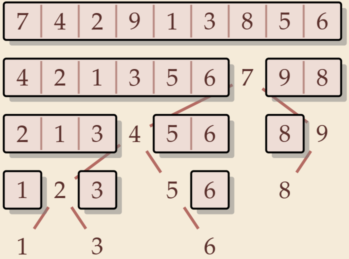

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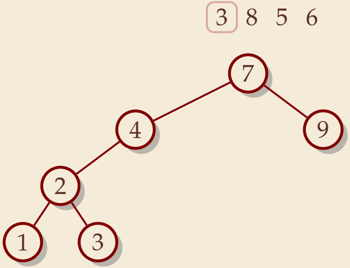

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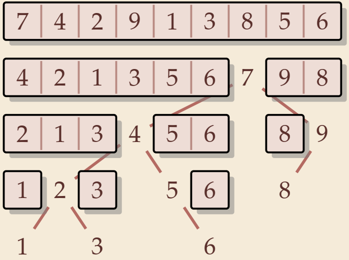

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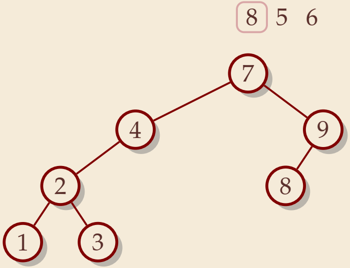

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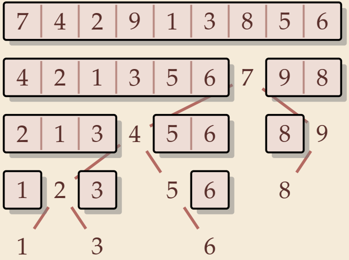

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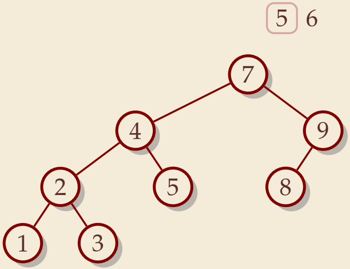

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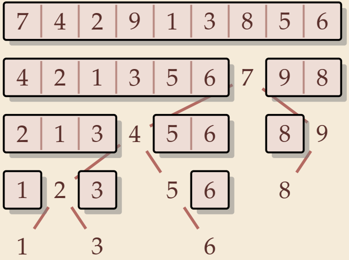

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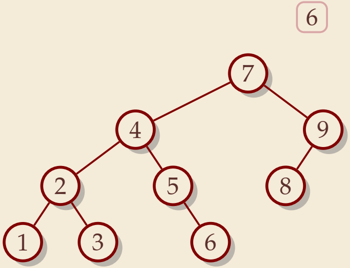

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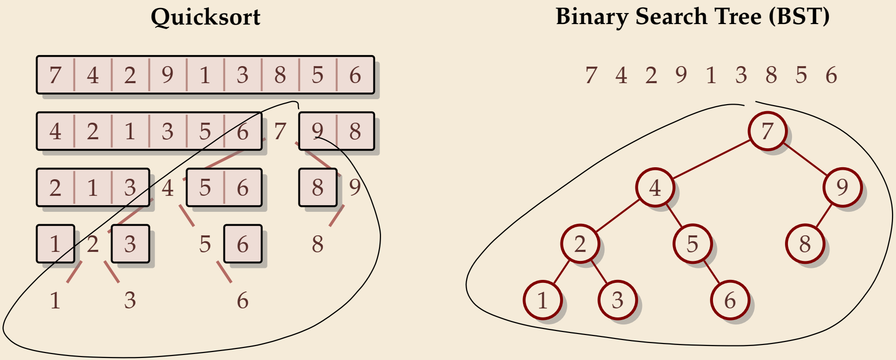
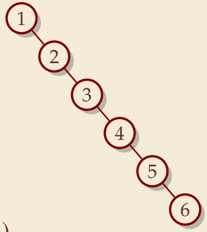


- recursion tree of quicksort $=$ binary search tree from successive insertion
- comparisons in quicksort $=$ comparisons to built BST
- comparisons in quicksort $\approx$ comparisons to search each element in BST

Quicksort – Worst Case

- Problem: BSTs can degenerate
- Cost to search for k is $k - 1$



⇝ Total cost $\displaystyle\sum_{k=1}^{n}(k-1) = \frac{n(n-1)}{2} \sim \frac{1}{2}n^2$

⇝ quicksort worst-case running time is in $\Theta(n^2)$

terribly slow!

But, we can fix this:

Randomized quicksort:

- choose a *random pivot* in each step

⇝ same as randomly *shuffling* input before sorting

Randomized Quicksort – Analysis

- $C(n)$ = element visits (as for mergesort)

$\leadsto$ quicksort needs $\sim 2\ln(2) \cdot n \lg n \approx 1.39 n \lg n$ *in expectation*

- also: very unlikely to be much worse:
  e. g., one can prove: $\Pr[\text{cost} > 10 n \lg n] = O(n^{-2.5})$
  distribution of costs is "concentrated around mean"
- intuition: have to be *constantly* unlucky with pivot choice

Quicksort – Discussion

- 👍 fastest general-purpose method
- 👍 $\Theta(n \log n)$ average case
- 👍 works *in-place* (no extra space required)
- 👍 memory access is sequential (scans over arrays)
- 👎 $\Theta(n^2)$ worst case (although extremely unlikely)
- 👎 not a *stable* sorting method

Open problem: Simple algorithm that is fast, stable and in-place.

3.3 Comparison-Based Lower Bound

Lower Bounds

- **Lower bound:** mathematical proof that *no algorithm* can do better.
  - very powerful concept: bulletproof *impossibility* result
    $\approx$ *conservation of energy* in physics
  - **(unique?) feature of computer science:**
    for many problems, solutions are known that (asymptotically) **achieve the lower bound**

  $\rightsquigarrow$ can speak of "*optimal* algorithms"

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- To prove a statement about *all algorithms*, we must precisely define what that is!
- already know one option: the word-RAM model
- Here: use a simpler, more restricted model.

The Comparison Model

- In the *comparison model* data can only be accessed in two ways:
  - comparing two elements
  - moving elements around (e. g. copying, swapping)
  - Cost: number of these operations.

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That's good!
Keeps algorithms general!

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  - comparing two elements "$\leq$"
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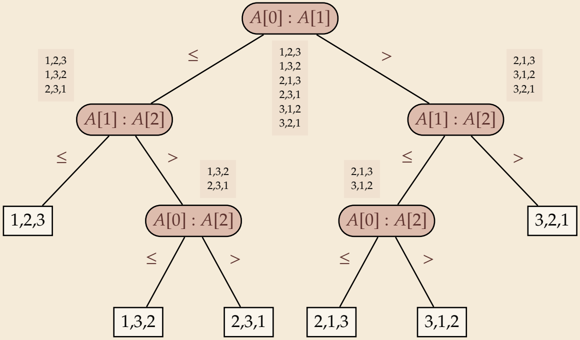
- This makes very few assumptions on the kind of objects we are sorting.
- Mergesort and Quicksort work in the comparison model.

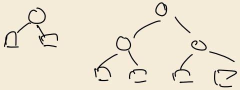
$\boxed{\rightsquigarrow}$ Every comparison-based sorting algorithm corresponds to a ***decision tree***.
  - only model comparisons $\rightsquigarrow$ ignore data movement
  - nodes $=$ comparisons the algorithm does
  - next comparisons can depend on outcomes $\rightsquigarrow$ different subtrees
  - child links $=$ outcomes of comparison
  - leaf $=$ unique initial input permutation compatible with comparison outcomes

Comparison Lower Bound

Example: Comparison tree for a sorting method for $A[0..2]$:



Comparison Lower Bound



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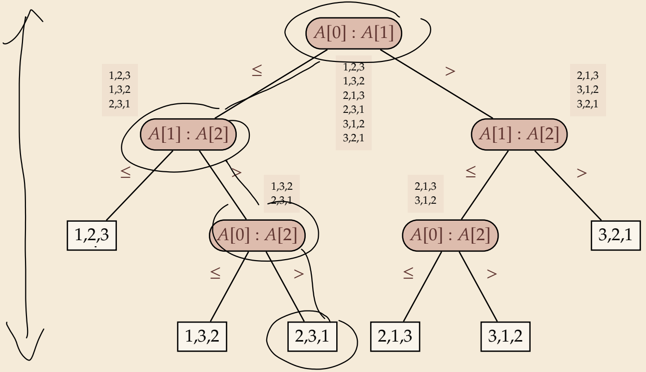

- Execution $=$ follow a path in comparison tree.

$\rightsquigarrow$ height of comparison tree $=$ worst-case # comparisons

- comparison trees are *binary* trees

$\rightsquigarrow$ ℓ leaves $\rightsquigarrow$ height $\geq \lceil \lg(\ell) \rceil$

- comparison trees for sorting method must have $\geq n!$ leaves

$\rightsquigarrow$ height $\geq \lg(n!) \sim n \lg n$

more precisely: $\lg(n!) = n \lg n - \lg(e) n + O(\log n)$

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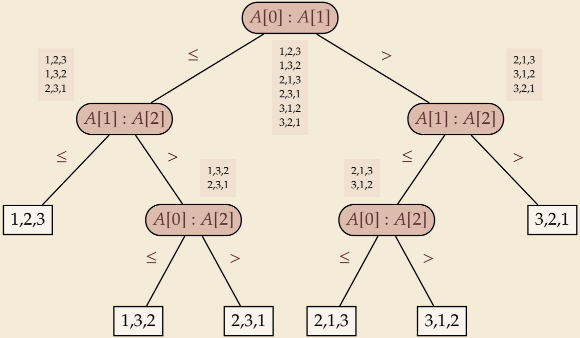

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more precisely: $\lg(n!) = n \lg n - \lg(e)n + O(\log n)$

- Mergesort achieves $\sim n \lg n$ comparisons $\rightsquigarrow$ asymptotically comparison-optimal!
- Open (theory) problem: Sorting algorithm with $n \lg n - \lg(e)n + o(n)$ comparisons?

≈ 1.4427

Clicker Question

Does the comparison-tree from the previous slide correspond to a worst-case optimal sorting method?

A Yes

B No

→ sli.do/comp526

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Does the comparison-tree from the previous slide correspond to a worst-case optimal sorting method?

A Yes ✓

B ~~No~~

→ sli.do/comp526