5 Divide & Conquer

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CS566 (Wintersemester 2025–26)
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Learning Outcomes

Unit 5: ***Divide & Conquer***

1. Know the steps of the Divide & Conquer paradigm.
2. Be able to solve simple Divide & Conquer recurrences.
3. Be able to design and analyze new algorithms using the Divide & Conquer paradigm.
4. Know the performance characteristics of selection-by-rank algorithms.
5. Know the divide and conquer approaches for integer multiplication, matrix multiplication, finding majority elements, and the closest-pair-of-points problem.

Outline

5 Divide & Conquer

Divide and conquer

Divide and conquer *idiom* (Latin: *divide et impera*)
to make a group of people disagree and fight with one another
so that they will not join together against one (Merriam-Webster Dictionary)

$\rightsquigarrow$ in politics & algorithms, many independent, small problems are better than one big one!

Divide-and-conquer algorithms:

1. Break problem into smaller, independent subproblems. (Divide!)
2. Recursively solve all subproblems. (Conquer!)
3. Assemble solution for original problem from solutions for subproblems.

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Examples:

- Mergesort
- Quicksort
- Binary search
- (arguably) Tower of Hanoi

Clicker Question

Have you seen the *Master Method* before?

- **A** Sure, could apply it blindfolded
- **B** Vaguely remember
- **C** Never heard of it

→ sli.do/cs566

5.1 Divide & Conquer Recurrences

Back-of-the-envelope analysis

- before working out the details of a D&C idea,
  it is often useful to get a quick indication of the resulting performance
  - don't want to waste time on something that's not competitive in the end anyways!
- since D&C is naturally recursive, running time often not obvious
  instead: given by a recursive equation

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  instead: given by a recursive equation
- unfortunately, rigorous analysis often tricky
  - Remember mergesort?

$$C(n) = \begin{cases} 0 & n \le 1 \\ C(\lfloor n/2 \rfloor) + C(\lceil n/2 \rceil) + 2n & n \ge 2 \end{cases}$$

$$\rightsquigarrow \quad C(n) = 2n\lfloor \lg(n) \rfloor + 2n - 4 \cdot 2^{\lfloor \lg(n) \rfloor} \quad 😱$$

$$= \Theta(n \log n) \quad 😌$$

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$$= \Theta(n \log n) \text{ 😌}$$

- the following method works for many typical cases to give the right **order of growth**

The Master Method

Mergesort

- Assume a stereotypical D&C algorithm
  - a recursive calls on (for some constant $a > 0$)
  - subproblems of size n/b (for some constant $b > 1$)
  - with non-recursive “conquer” effort $f(n)$ (for some function $f : \mathbb{R} \to \mathbb{R}$)
  - base case effort d (some constant $d > 0$)

$a = 2$

$b = 2$

$f(n) = 2 \cdot n$

$n = 2 \quad d = 2$

$(n = 1 \leadsto d = 0)$

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$\rightsquigarrow$ running time $T(n)$ satisfies

$$\boxed{T(n) = \begin{cases} a \cdot T\left(\frac{n}{b}\right) + f(n) & n > 1 \\ d & n \leq 1 \end{cases}}$$

n_0 also possible

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Theorem 5.1 (Master Theorem)
With $c := \log_b(a)$, we have for the above recurrence:

(a) $T(n) = \Theta(n^c)$ if $f(n) = O(n^{c-\varepsilon})$ for constant $\varepsilon > 0$.

(b) $T(n) = \Theta(n^c \log n)$ if $f(n) = \Theta(n^c)$.

(c) $T(n) = \Theta(f(n))$ if $f(n) = \Omega(n^{c+\varepsilon})$ for constant $\varepsilon > 0$ **and** f satisfies the regularity condition $\exists n_0, \alpha < 1 \; \forall n \ge n_0 : a \cdot f\left(\frac{n}{b}\right) \le \alpha f(n)$.

Example: Mergesort $\quad a = b = 2 \quad f(n) = 2n$

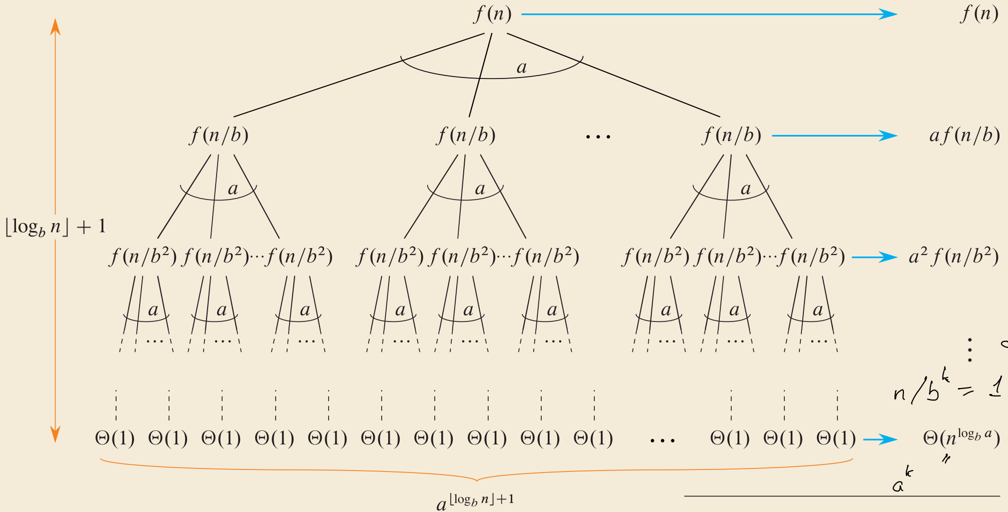
$$c = \log_2(2) = 1$$

$$f(n) = \Theta(n^1) \rightsquigarrow \text{case (b)}$$

$$\Rightarrow \text{cost } \Theta(n \log n)$$

MT

Master Theorem – Intuition & Proof Idea



depth i

$n/b^k = 1$

$n = b^k$

a^k

$k = \log_b(n)$

$$\text{Total: } \Theta(n^{\log_b a}) + \sum_{j=0}^{\lfloor \log_b n \rfloor} a^j f(n/b^j)$$

Figure 4.3 of Cormen et al. *Introduction to Algorithms* 4th ed.

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

$$= a\left(a\,T\left(\frac{n}{b^2}\right) + f\left(\frac{n}{b}\right)\right) + f(n)$$

$$\vdots$$

$$= a^k \cdot T(1) + \sum_{j=0}^{k} a^j f\left(\frac{n}{b^j}\right)$$

$k = \log_b(n)$

$$= a^{\log_b(n)} \cdot d + \sum_{j=0}^{\log_b(n)} a^j f\left(\frac{n}{b^j}\right)$$

$$= n^{\log_b(a)} \cdot d + \sum_{j=0}^{\log_b(n)} a^j f\left(\frac{n}{b^j}\right)$$

$$a^{\log_b(n)} = e^{\ln(a) \cdot \ln(n) / \ln(b)}$$

$$= n^{\ln(a) / \ln(b)} = n^{\log_b(a)}$$

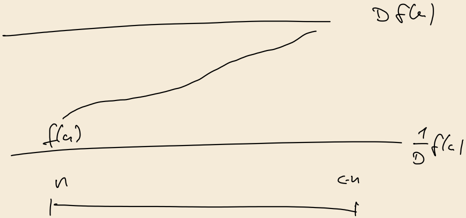
proof not in exam

When it’s fine to ignore floors and ceilings

The ***polynomial-growth condition***

- $f : \mathbb{R}_{>0} \to \mathbb{R}$ satisfies the *polynomial-growth condition* if

$$\exists n_0 \; \forall C \geq 1 \; \exists D > 1 \quad \forall n \geq n_0 \; \forall c \in [1, C] \;\; : \;\; \tfrac{1}{D} f(n) \;\leq\; f(cn) \;\leq\; D f(n)$$



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- intuitively: increasing n by up to a factor C (and anywhere in between!)
  changes the function value by at most a factor $D = D(C)$
  (for sufficiently large n)

- examples: $f(n) = \Theta(n^{\alpha} \log^{\beta}(n) \log\log^{\gamma}(n))$ for constants α, β, γ zero allowed
  $\rightsquigarrow$ f satisfies the polynomial-growth condition

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  $\rightsquigarrow$ f satisfies the polynomial-growth condition

Lemma 5.2 (Polynomial-growth master method)

If the toll function $f(n)$ satisfies the polynomial-growth condition, then the Θ-class of the solution of a D&C recurrence remains the same when ignoring floors and ceilings on subproblem sizes.

A Rigorous and Stronger Meta Theorem

£ exam

Theorem 5.3 (Roura's Discrete Master Theorem)

Let $T(n)$ be recursively defined as

$$T(n) = \begin{cases} b_n & 0 \leq n < n_0, \\ f(n) + \displaystyle\sum_{d=1}^{D} a_d \cdot T\left(\frac{n}{b_d} + r_{n,d}\right) & n \geq n_0, \end{cases}$$

where $D \in \mathbb{N}$, $a_d > 0$, $b_d > 1$, for $d = 1, \ldots, D$ are constants, functions $r_{n,d}$ satisfy $|r_{n,d}| = O(1)$ as $n \to \infty$, and function $f(n)$ satisfies $f(n) \sim B \cdot n^{\alpha} (\ln n)^{\gamma}$ for constants $B > 0$, α, γ.
Set $H = 1 - \sum_{d=1}^{D} a_d \, (1/b_d)^{\alpha}$; then we have:

(a) If $H < 0$, then $T(n) = O(n^{\tilde{\alpha}})$, for $\tilde{\alpha}$ the unique value of α that would make $H = 0$.

(b) If $H = 0$ and $\gamma > -1$, then $T(n) \sim f(n) \ln(n) / \tilde{H}$ with constant $\tilde{H} = (\gamma + 1) \sum_{d=1}^{D} a_d \, b_d^{-\alpha} \ln(b_d)$.

(c) If $H = 0$ and $\gamma = -1$, then $T(n) \sim f(n) \ln(n) \ln(\ln(n)) / \hat{H}$ with constant $\hat{H} = \sum_{d=1}^{D} a_d \, b_d^{-\alpha} \ln(b_d)$.

(d) If $H = 0$ and $\gamma < -1$, then $T(n) = O(n^{\alpha})$.

(e) If $H > 0$, then $T(n) \sim f(n)/H$.

5.2 Order Statistics

Selection by Rank

- Standard data summary of numerical data: (Data scientists, listen up!)
  - mean, standard deviation
  - min/max (range)
  - histograms

  easy to compute in $\Theta(n)$ time
  - median, quartiles, other quantiles (a.k.a. order statistics)

computable in $\Theta(n)$ time?

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General form of problem: **Selection by Rank**

- **Given:** array $A[0..n)$ of numbers and number $k \in [0..n)$.

but 0-based & counting dups

- **Goal:** find element that would be in position k if A was sorted (kth smallest element).

- $k = \lfloor n/2 \rfloor \rightsquigarrow$ median; $k = \lfloor n/4 \rfloor \rightsquigarrow$ lower quartile
  $k = 0 \rightsquigarrow$ minimum; $k = n - \ell \rightsquigarrow$ ℓth largest

Quickselect

- Key observation: Finding the element of rank k seems hard.
  But computing the rank of a given element is easy!
  count smaller elements

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$\rightsquigarrow$ Pick any element $A[b]$ and find its rank j.

- $j = k$? $\rightsquigarrow$ Lucky Duck! Return chosen element and stop
- $j < k$? $\rightsquigarrow$ ... not done yet. But: The $j + 1$ elements smaller than $\leq A[b]$ can be excluded!
- $j > k$? $\rightsquigarrow$ similarly exclude the $n - j$ elements $\geq A[b]$

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- partition function from Quicksort:
  - returns the rank of pivot
  - separates elements into smaller/larger

$\rightsquigarrow$ can use same building blocks

```
procedure quickselect(A[l..r), k):
    if r − l ≤ 1 then return A[l]   // l ≤ k < r
    b := choosePivot(A[l..r))
    j := partition(A[l..r), b)
    if j == k
        return A[j]
    else if j < k
        quickselect(A[j + 1..r), k)
    else // j > k
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```

Quickselect – Iterative Code

Recursion can be replaced by loop (*tail-recursion elimination*)

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procedure quickselect(A[l..r], k):
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```

```
procedure quickselectIterative(A[0..n), k):
    l := 0;   r := n
    while r − l > 1
        b := choosePivot(A[l..r))
        j := partition(A[l..r), b)
        if j ≥ k then r := j − 1
        if j ≤ k then l := j + 1
    return A[k]
```

- implementations should usually prefer iterative version
- analysis more intuitive with recursive version

Quickselect – Analysis

```
procedure quickselect(A[l..r), k):
    if r − l ≤ 1 then return A[l]
    b := choosePivot(A[l..r))
    j := partition(A[l..r), b)      r-l ± 1 cmps
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- cost = #cmps
- costs depend on n **and** k

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- **worst case:** $k = 0$, but always $j = n - 2$
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  - $\rightsquigarrow$ $T(n, k) = \Theta(n^2)$ worst case cost

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average case:

- let $T(n, k)$ expected cost when we choose a pivot uniformly from $A[0..n)$

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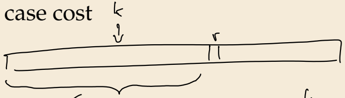

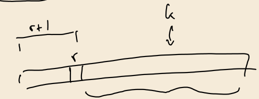

average case:

- let $T(n,k)$ expected cost when we choose a pivot uniformly from $A[0..n)$

$\rightsquigarrow$ formulate recurrence for $T(n,k)$ similar to BST/Quicksort recurrence

$$T(n,k) = \underbrace{n}_{\text{partition}} + \frac{1}{n}\sum_{r=0}^{n-1}\Big([r = k]\cdot 0 + [k < r]\cdot T(r,k) + [k > r]\cdot T(n-r-1, k-r-1)\Big)$$

$[r=k] = \begin{cases} 1 & r=k \\ 0 & \text{else} \end{cases}$ Iverson Bracket

Quickselect – Average Case Analysis

- $T(n,k) \;=\; n \;+\; \dfrac{1}{n}\displaystyle\sum_{r=0}^{n-1} [r=k]\cdot 0 \;+\; \underbrace{[k<r]\cdot \underbrace{T(r,k)}_{\leq\, \hat{T}(r)} \;+\; [k>r]\cdot \underbrace{T(n-r-1,k-r-1)}_{\leq\, \hat{T}(n-r-1)}}_{\leq\, \max\{\hat{T}(r),\, \hat{T}(n-r-1)\}}$
- Set $\hat{T}(n) \;=\; \max_{k\in[0..n)} T(n,k)$

Quickselect – Average Case Analysis

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- Set $\hat{T}(n) \;=\; \max_{k\in[0..n)} T(n,k)$ $\qquad \forall k \;\; T(n,k) \le X \qquad \Rightarrow \;\; \hat{T}(n) \le X$

$\rightsquigarrow \hat{T}(n) \;\le\; n \;+\; \dfrac{1}{n}\displaystyle\sum_{r=0}^{n-1}\max\bigl\{\hat{T}(r),\hat{T}(n-r-1)\bigr\}$

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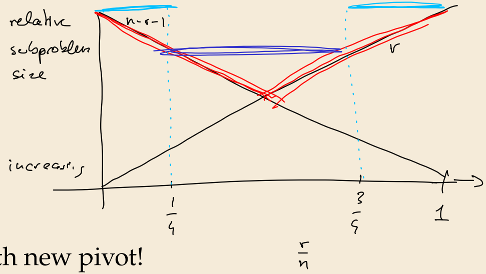
$\hat{T}$ monoton increasing



- analyze hypothetical, worse algorithm:
  if $r \notin [\frac{1}{4}n, \frac{3}{4}n)$, discard partition and repeat with new pivot!

$\rightsquigarrow\; \hat{T}(n) \;\le\; \tilde{T}(n)$ defined by $\quad \tilde{T}(n) \;\le\; n + \frac{1}{2}\tilde{T}(n) + \frac{1}{2}\tilde{T}(\frac{3}{4}n)$

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$\rightsquigarrow \tilde{T}(n) \;\le\; 2n \;+\; \tilde{T}(\frac{3}{4}n)$ $\leftarrow$ MT!

$a = 1$ $\quad f(n) = 2n$

$b = \frac{4}{3}$

$c = \log_b(a) = 0$

$f(n)$ vs. n^0

$f(n) = \Omega(n^{0+\varepsilon})$

Quickselect – Average Case Analysis

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$\rightsquigarrow \tilde{T}(n) \;\le\; 2n \;+\; \tilde{T}(\frac{3}{4}n)$

- Master Theorem Case 3: $\tilde{T}(n) = \Theta(n)$

Quickselect Discussion

- $\Theta(n^2)$ worst case (like Quicksort)
- expected cost $\Theta(n)$ (best possible)
- no extra space needed
- adaptations possible to find several order statistics at once

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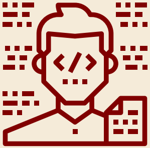
- expected cost can be further improved by choosing pivot from a small sorted sample
  - $\rightsquigarrow$ asymptotically optimal randomized cost: $n + \min\{k, n-k\}$ comparisons in expectation achieved asymptotically by the *Floyd-Rivest algorithm*

$\notin$ exam

5.3 Linear-Time Selection

Interlude – A recurring conversation

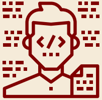
Cast of Characters:



Hi! I'm a *computer science practitioner*.
I love algorithms for the sometimes miraculous **applications** they enable.
I care for things I can **implement** and **that actually work in practice**.

Hi! I'm a *theoretical computer science researcher*.
I find beauty in elegant and **definitive** answers to questions about complexity.
I care for **eternal truths** and mathematically proven facts;
asymptotically optimal is what counts! (Constant factors are secondary.)

Quickselect Disagreements

 For practical purposes, (randomized) Quickselect is perfect.

e. g. used in C++ STL `std::nth_element`

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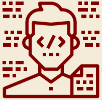


For practical purposes, (randomized) Quickselect is perfect.

e. g. used in C++ STL `std::nth_element`

Yeah ... maybe. But can we select by rank in $O(n)$ deterministic **worst case** time?

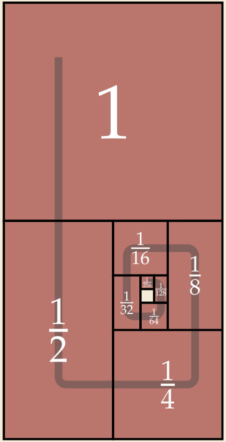
Better Pivots

It turns out, we can!

- All we need is better pivots!
  - If pivot was the exact median,
    we would at least halve #elements in each step
  - Then the total cost of all partitioning steps is $\leq 2n = \Theta(n)$.

$$\sum_{i=0}^{\infty} z^i = \frac{1}{1-z} \qquad |z| < 1$$

$$z = \frac{1}{2}$$



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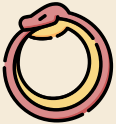
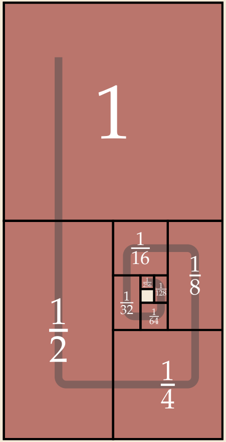


But: finding medians is (basically) our original problem!



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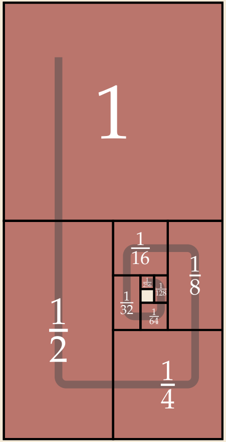

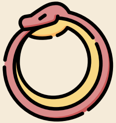

But: finding medians is (basically) our original problem!

It totally suffices to find an element of rank αn for $\alpha \in (\varepsilon, 1 - \varepsilon)$
to get overall costs $\Theta(n)$!

The Median-of-Medians Algorithm

```
procedure choosePivotMoM(A[l..r]):
    m := ⌊n/5⌋
    for i := 0, ..., m − 1
        sort(A[5i..5i + 4])
        // collect median of 5
        Swap A[i] and A[5i + 2]
    return quickselectMoM(A[0..m), ⌊(m−1)/2⌋)

procedure quickselectMoM(A[l..r), k):
    if r − l ≤ 1 then return A[l]
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Analysis:

- Note: 2 mutually recursive procedures
  $\rightsquigarrow$ effectively 2 recursive calls!

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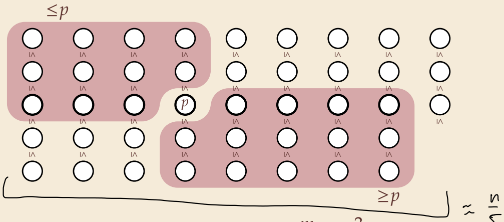
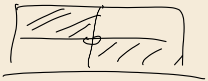


$\rightsquigarrow$ partition excludes $\sim 3 \cdot \frac{m}{2} \sim \frac{3}{10}n$ elem.



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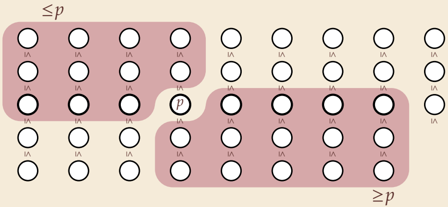

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$$\rightsquigarrow \; C(n) \;\le\; \Theta(n) + C(\tfrac{1}{5}n) + C(\tfrac{7}{10}n)$$

partition + work per group

choose pivot

The Median-of-Medians Algorithm

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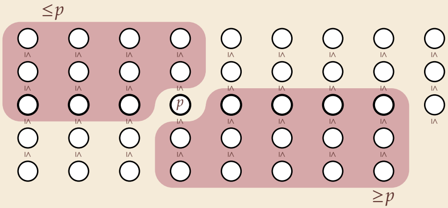

$\rightsquigarrow$ partition excludes $\sim 3 \cdot \frac{m}{2} \sim \frac{3}{10}n$ elem.

$$
\begin{aligned}
\rightsquigarrow\ C(n) &\le \Theta(n) + C(\tfrac{1}{5}n) + C(\tfrac{7}{10}n) \\
&\le \Theta(n) + C(\tfrac{1}{5}n + \tfrac{7}{10}n) \\
&= \Theta(n) + C(\tfrac{9}{10}n) \quad \rightsquigarrow \quad C(n) = \Theta(n)
\end{aligned}
$$

ansatz: overall cost linear

5.4 Fast Multiplication

Clicker Question

How many **bit operations** does it take to multiply two n-bit integers?

- **A** $O(1)$
- **B** $O(\log\log n)$
- **C** $O(\log n)$
- **D** $O(\log^2 n)$
- **E** $O(\sqrt{n})$
- **F** $O(n)$
- **G** $O(n \log n)$
- **H** $O(n \log n \log\log n)$
- **I** $O(n^2)$
- **J** $O(n^2 \log n)$
- **K** $O(n^3)$
- **L** $O(2^n)$

→ sli.do/cs566

Integer Multiplication

- What's the cost of computing $x \cdot y$ for two integers x and y?

$\rightsquigarrow$ depends on how big the numbers are!

  - If x and y have $O(w)$ bits, multiplication takes $O(1)$ time on word-RAM
  - otherwise, need a dedicated algorithm!

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  - otherwise, need a dedicated algorithm!

Long multiplication (»Schulmethode«)

- Given $x = \sum_{i=0}^{n-1} x_i 2^i$ and $y = \sum_{i=0}^{n-1} y_i 2^i$, want $z = \sum_{i=0}^{2n-1} z_i 2^i$

```
for i := 0,...,n-1
    c := 0
    for j := 0,...,n-1
        z_{i+j} := z_{i+j} + c + x_i · y_j
        c := ⌊z_{i+j}/2⌋
        z_{i+j} := z_{i+j} mod 2
    end for
    z_{i+n} := c
end for
```

- $\Theta(n^2)$ bit operations
- could work with base 2^w instead of 2

  $\rightsquigarrow$ $\Theta\big((n/w)^2\big)$ time
- here: count bit operations for simplicity can be generalized

Example:
easier in binary!
("shift and add")

```
1001010101 * 101101
-------------------
   1001010101
    0000000000
     1001010101
      1001010101
       0000000000
        1001010101
-------------------
   110100011110001
```

Divide & Conquer Multiplication

- assume n is power of 2 (fill up with 0-bits otherwise)
- We can write
  - $x = a_1 2^{n/2} + a_2$ and
  - $y = b_1 2^{n/2} + b_2$
  - for a_1, a_2, b_1, b_2 integers with $n/2$ bits

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$$\rightsquigarrow \quad x \cdot y \; = \; (a_1 2^{n/2} + a_2) \cdot (b_1 2^{n/2} + b_2) \; = \; \boxed{\boldsymbol{a_1 b_1} 2^n + (\boldsymbol{a_1 b_2} + \boldsymbol{a_2 b_1}) 2^{n/2} + \boldsymbol{a_2 b_2}}$$

  - recursively compute **4** smaller products
  - combine with shifts and additions ($O(n)$ bit operations)

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  - recursively compute **4** smaller products
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- ...but is this any good?
  - $T(n) \;=\; \mathbf{4} \cdot T(n/2) + \Theta(n)$

  $\rightsquigarrow$ Master Theorem Case 1: $T(n) = \Theta(n^2)$ *...just like the primary school method!?*

    - but Master Theorem gives us a hint: cost is dominated by the leaves

    $\rightsquigarrow$ try to do more work in conquer step!

Karatsuba Multiplication

- how can we do "less divide and more conquer"?

  Recall: $x \cdot y = \boxed{a_1 b_1 2^n + (a_1 b_2 + a_2 b_1) 2^{n/2} + a_2 b_2}$

Karatsuba Multiplication

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Let’s do some algebra.

$$\begin{aligned} c &:= (a_1 + a_2) \cdot (b_1 + b_2) \\ &= a_1 b_1 + (a_1 b_2 + a_2 b_1) + a_2 b_2 \end{aligned}$$

$\rightsquigarrow$ $(\mathbf{a_1 b_2} + \mathbf{a_2 b_1}) \;=\; c - a_1 b_1 - a_2 b_2$

this can be computed with **3** recursive multiplications

$a_1 + a_2$ and $b_1 + b_2$ still have roughly $n/2$ bits

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```
procedure karatsuba(x, y):        condition on n ≤ w
    // Assume x and y are n = 2^k bit integers
    a1 := ⌊x/2^(n/2)⌋;  a2 := x mod 2^(n/2)  // implemented by shifts
    b1 := ⌊y/2^(n/2)⌋;  b2 := y mod 2^(n/2)
    c1 := karatsuba(a1, b1)
    c2 := karatsuba(a2, b2)
    c  := karatsuba(a1 + a2, b1 + b2) − c1 − c2
    return c1 2^n + c 2^(n/2) + c2  // shifts and additions
```

Karatsuba Multiplication

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  Recall: $x \cdot y \;=\; \boxed{a_1 b_1 2^n + (a_1 b_2 + a_2 b_1) 2^{n/2} + a_2 b_2}$

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Analysis:

- nonrecursive cost: only additions and shifts
- all numbers $O(n)$ bits

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Recurrence: $a = 3$, $b = 2$, $c = \log_2(3)$

- $T(n) \;=\; 3T(n/2) + \Theta(n)$
- Master Theorem Case 1

$\rightsquigarrow$ $T(n) = \Theta(n^{\lg 3}) = O(n^{1.585})$

much cheaper (for large n)!

Integer Multiplication

- until 1960, integer multiplication was conjectured to take $\Omega(n^2)$ bit operations

$\rightsquigarrow$ Karatsuba's algorithm was a big breakthrough
  - which he discovered as a student!

- idea can be generalized to breaking numbers into $k \geq 2$ parts (*Toom-Cook algorithm*)

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- asymptotically *much* better algorithms are now known!
  - e. g., the *Schönhage-Strassen algorithm* with $O(n \log n \log\log n)$ bit operations (!)
  - these are based on the *Fast Fourier Transform* (FFT) algorithm
    - numbers = polynomials evaluated at base (e. g., $z = 2$)

    $\leadsto$ multiplication of numbers = convolution of polynomials

    - FFT makes computation of this convolution cheap by computing the polynomial via interpolation
    - Schönhage-Strassen adds careful finite-field algebra to make computations efficient

$\notin$ exam

Clicker Question

What's the product $A \cdot B$ of the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}?$$

A $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

B $\begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix}$

C 9

D $\begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix}$

E $\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{9} & \frac{2}{9} \end{pmatrix}$

→ sli.do/cs566

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A $\cancel{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

B $\cancel{\begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix}}$

C $\cancel{9}$

D $\begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix}$ $\checkmark$

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→ sli.do/cs566

Matrix Multiplication

- The same trick can also be used for faster matrix multiplication
- Recall: For $A, B \in \mathbb{R}^{n \times n}$ we define $C = A \cdot B$ via $c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$

  entry of A in row i and column k

$\rightsquigarrow$ Naive cost: n^2 sums with n terms each $\rightsquigarrow$ $\Theta(n^3)$ arithmetic operations

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- Can use D&C as follows (assuming n is a power of 2 again)
  - Decompose (cut in half hor. & vert.) $A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$, $B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$, $C = \begin{pmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{pmatrix}$

  $\rightsquigarrow$ We get C as

$$\begin{aligned} C_{1,1} &= A_{1,1} \cdot B_{1,1} + A_{1,2} \cdot B_{2,1} \\ C_{1,2} &= A_{1,1} \cdot B_{1,2} + A_{1,2} \cdot B_{2,2} \\ C_{2,1} &= A_{2,1} \cdot B_{1,1} + A_{2,2} \cdot B_{2,1} \\ C_{2,2} &= A_{2,1} \cdot B_{1,2} + A_{2,2} \cdot B_{2,2} \end{aligned}$$

(note "$\cdot$" and "$+$" operate on matrices here)

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  - Decompose (cut in half hor. & vert.) $A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$, $B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$, $C = \begin{pmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{pmatrix}$

  $\rightsquigarrow$ We get C as

$$\begin{aligned} C_{1,1} &= A_{1,1} \cdot B_{1,1} + A_{1,2} \cdot B_{2,1} \\ C_{1,2} &= A_{1,1} \cdot B_{1,2} + A_{1,2} \cdot B_{2,2} \\ C_{2,1} &= A_{2,1} \cdot B_{1,1} + A_{2,2} \cdot B_{2,1} \\ C_{2,2} &= A_{2,1} \cdot B_{1,2} + A_{2,2} \cdot B_{2,2} \end{aligned}$$

(note "·" and "+" operate on matrices here)

  - 8 recursive matrix multiplications on two $\frac{n}{2} \times \frac{n}{2}$ matrices + $\Theta(n^2)$ summations
    4 matrix sums with $(\frac{n}{2})^2$ entries each
  - # operations $T(n) = 8T(n/2) + \Theta(n^2)$

Matrix Multiplication

- The same trick can also be used for faster matrix multiplication
- Recall: For $A, B \in \mathbb{R}^{n\times n}$ we define $C = A \cdot B$ via $c_{i,j} = \sum_{k=1}^{n} \underbrace{a_{i,k}}_{\text{entry of } A \text{ in row } i \text{ and column } k} b_{k,j}$

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  $\rightsquigarrow$ Master Theorem Case 1: $T(n) = \Theta(n^3)$ 😞 (but: still useful for better memory locality!)

Strassen Algorithm for Matrix Multiplication

- Observation (again): Can do more conquer for less divide!
- We recursively compute the following **7** products:

$$
\begin{aligned}
M_1 &:= (A_{1,2} - A_{2,2}) \cdot (B_{2,1} + B_{2,2})\\
M_2 &:= (A_{1,1} + A_{2,2}) \cdot (B_{1,1} + B_{2,2})\\
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M_5 &:= A_{1,1} \cdot (B_{1,2} - B_{2,2})\\
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\end{aligned}
$$

$\rightsquigarrow$ We then obtain the 4 parts of C as

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\begin{aligned}
C_{1,1} &= M_1 + M_2 - M_4 + M_6\\
C_{1,2} &= M_4 + M_5\\
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(Proof: left as exercise 😉)

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$\rightsquigarrow$ $\Theta(n^2)$ operations for conquer

$\rightsquigarrow$ total # arithmetic operations
$T(n) = 7\,T(n/2) + \Theta(n^2)$

$\rightsquigarrow$ Master Theorem Case 1:
$T(n) = \Theta(n^{\lg 7}) = O(n^{2.808})$

Open Problems

Multiplication is extremely fundamental, but its ***computational complexity*** *is an* ***open problem*** *and subject of active research!*

Integer multiplication:

- **conjectured** to require $\Omega(n \log n)$ bit operations (no proof known!)
- Harvey & van der Hoeven **2021**: $O(n \log n)$ algorithm possible!

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Matrix multiplication (MM):

- more relevant than it might seem since complexity identical to
  - computing inverse matrices, determinants
  - Gaussian elimination ($\rightsquigarrow$ solving systems of linear equations)
  - recognition of context free languages

$\rightsquigarrow$ best exponent even has standard notation:
smallest $\omega \in [2, 3)$ so that MM takes $O(n^{\omega})$ operations

- Big open question: Is $\omega > 2$?
- best known bound: $\omega \leq 2.371339$ (from 2024!)

Timeline of matrix multiplication exponent

Year	Bound on omega	Authors
1969	2.8074	Strassen[1]
1978	2.796	Pan[10]
1979	2.780	Bini, Capovani [it], Romani[11]
1981	2.522	Schönhage[12]
1981	2.517	Romani[13]
1981	2.496	Coppersmith, Winograd[14]
1986	2.479	Strassen[15]
1990	2.3755	Coppersmith, Winograd[16]
2010	2.3737	Stothers[17]
2012	2.3729	Williams[18][19]
2014	2.3728639	Le Gall[20]
2020	2.3728596	Alman, Williams[21][22]
2022	2.371866	Duan, Wu, Zhou[23]
2024	2.371552	Williams, Xu, Xu, and Zhou[2]
2024	2.371339	Alman, Duan, Williams, Xu, Xu, and Zhou[24]

Clicker Question

How many **bit operations** does it take to multiply two n-bit integers?

- **A** $O(1)$
- **B** $O(\log\log n)$
- **C** $O(\log n)$
- **D** $O(\log^2 n)$
- **E** $O(\sqrt{n})$
- **F** $O(n)$
- **G** $O(n \log n)$
- **H** $O(n \log n \log\log n)$
- **I** $O(n^2)$
- **J** $O(n^2 \log n)$
- **K** $O(n^3)$
- **L** $O(2^n)$

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- **F** $\cancel{O(n)}$
- **G** $O(n \log n)$ ✓
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