

2

Machines & Models

14 October 2025

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Learning Outcomes

Unit 2: *Machines & Models*

1. Understand the difference between empirical *running time* and algorithm *analysis*.
2. Understand *worst / best / average case* models for input data.
3. Know the *RAM machine* model.
4. Know the definitions of *asymptotic notation* (Big-Oh classes and relatives).
5. Understand the reasons to make *asymptotic approximations*.
6. Be able to *analyze* simple *algorithms*.

Outline

2 Machines & Models

- 2.1 Algorithm analysis
- 2.2 The RAM Model
- 2.3 Asymptotics & Big-Oh
- 2.4 Asymptotic Pitfalls
- 2.5 Teaser: Maximum subarray problem

What is an algorithm?

An algorithm is a sequence of instructions.

think: recipe

More precisely:

e. g. Python script

1. mechanically executable
~~ no “common sense” needed
2. finite description ≠ finite computation!
3. solves a *problem*, i. e., a class of problem instances
 $x + y$, not only $17 + 4$

- ▶ input-processing-output abstraction



Typical example: *bubblesort*

~~ not a specific program
but the underlying idea

What is a data structure?

A data structure is

1. a rule for **encoding data**
(in computer memory), plus
2. **algorithms** to work with it
(queries, updates, etc.)

typical example: *binary search tree*



2.1 Algorithm analysis

Good algorithms

Our goal: Find good (best?) algorithms and data structures for a task.

Good “usually” means

can be complicated in distributed systems

- ▶ fast running *time*
- ▶ moderate memory *space* usage

Algorithm analysis is a way to

- ▶ compare different algorithms,
- ▶ predict their performance in an application

Running time experiments

Why not simply run and time it?

- ▶ results only apply to
 - ▶ single *test* machine
 - ▶ tested inputs
 - ▶ tested implementation
 - ▶ ...
- ≠ *universal truths*
- ▶ instead: consider and analyze algorithms on an abstract machine
 - ~~ provable statements for model
 - ~~ testable model hypotheses
 - ~~ Need precise model of machine (costs), input data and algorithms.



survives Pentium 4

Data Models

Algorithm analysis typically uses one of the following simple data models:

- ▶ **worst-case performance:**

consider the *worst* of all inputs as our cost metric

- ▶ **best-case performance:**

consider the *best* of all inputs as our cost metric

- ▶ **average-case performance:**

consider the average/expectation of a *random* input as our cost metric

Usually, we apply the above for *inputs of same size n* .

↝ performance is only a **function of n** .

2.2 The RAM Model

Clicker Question



What is the cost of *adding* two d -digit integers?
(For example, for $d = 5$, what is $45\,235 + 91\,342$?)

- A** constant time
- B** logarithmic in d
- C** proportional to d
- D** quadratic in d
- E** no idea what you are talking about



→ *sli.do/cs566*

Clicker Question



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→ *sli.do/cs566*

Machine models

The machine model decides

- ▶ what algorithms are possible
- ▶ how they are described (= programming language)
- ▶ what an execution *costs*

Goal: Machine models should be

detailed and powerful enough to reflect actual machines,
abstract enough to unify architectures,
simple enough to analyze.

Machine models

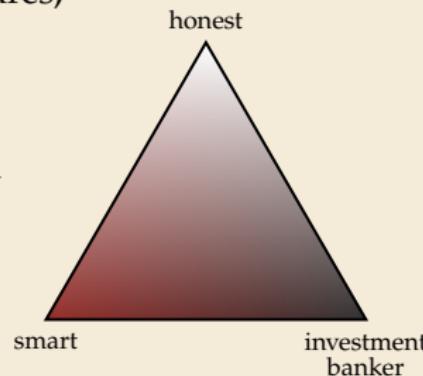
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detailed and powerful enough to reflect actual machines,
abstract enough to unify architectures,
simple enough to analyze.

↝ usually some compromise is needed



Random Access Machines

Random access machine (RAM)

more detail in §2.2 of *Sequential and Parallel Algorithms and Data Structures*
by Sanders, Mehlhorn, Dietzfelbinger, Dementiev

- ▶ unlimited *memory* $\text{MEM}[0], \text{MEM}[1], \text{MEM}[2], \dots$
- ▶ fixed number of *registers* R_1, \dots, R_r (say $r = 100$)

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- ▶ fixed number of *registers* R_1, \dots, R_r (say $r = 100$)
- ▶ memory cells $\text{MEM}[i]$ and registers R_i store w -bit integers, i. e., numbers in $[0..2^w - 1]$
 w is the word width/size; typically $w \propto \lg n \rightsquigarrow 2^w \approx n$

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 w is the word width/size; typically $w \propto \lg n \rightsquigarrow 2^w \approx n$
 - ▶ Instructions:
 - ▶ load & store: $R_i := \text{MEM}[R_j] \quad \text{MEM}[R_j] := R_i$
 - ▶ operations on registers: $R_k := R_i + R_j$ (arithmetic is *modulo 2^w !*)
also $R_i - R_j, R_i \cdot R_j, R_i \text{ div } R_j, R_i \text{ mod } R_j$
C-style operations (bitwise and/or/xor, left/right shift)
 - ▶ conditional and unconditional jumps
 - ▶ cost: number of executed instructions
- rightsquigarrow The RAM is the standard model for sequential computation.
we will see further models later

RAM-Program Example

Example RAM program

```
1 // Assume: R1 stores number N
2 // Assume: MEM[0..N) contains list of N numbers
3 R2 := R1;
4 R3 := R1 - 2;
5 R4 := MEM[R3];
6 R5 := R3 + 1;
7 R6 := MEM[R5];
8 if (R4 ≤ R6) goto line 11;
9 MEM[R3] := R6;
10 MEM[R5] := R4;
11 R3 := R3 - 1;
12 if (R3 ≥ 0) goto line 5;
13 R2 := R2 - 1;
14 if (R2 > 0) goto line 4;
15 //Done:
```

Clicker Question



What algorithm does the RAM program on the previous slide implement?



→ *sli.do/cs566*

RAM-Program Example

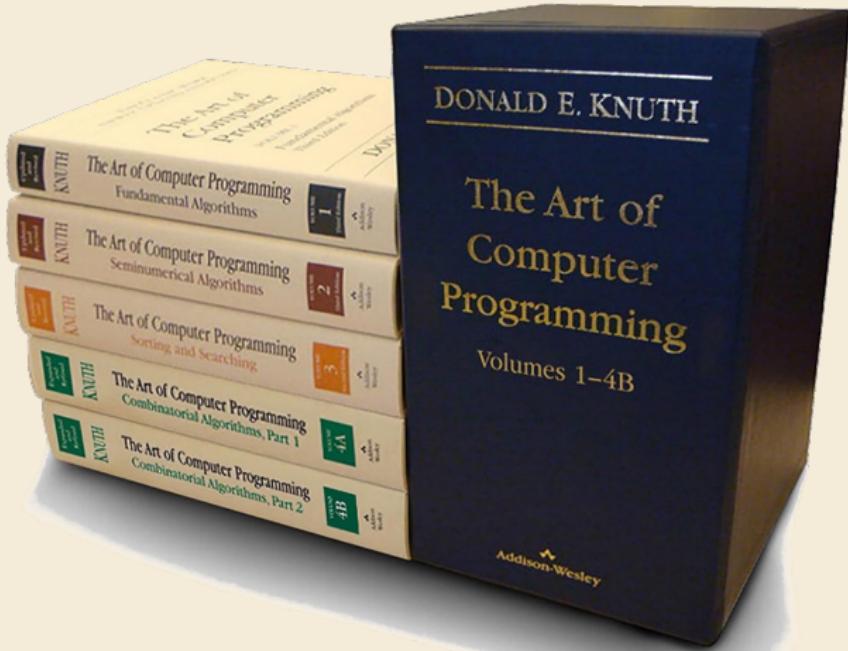
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```

they need not be examined on subsequent passes. Horizontal lines in Fig. 14 show the progress of the sorting from this standpoint; notice, for example, that five more elements are known to be in final position as a result of Pass 4. On the final pass, no exchanges are performed at all. With these observations we are ready to formulate the algorithm.

Algorithm B (*Bubble sort*). Records R_1, \dots, R_N are rearranged in place; after sorting is complete their keys will be in order, $K_1 \leq \dots \leq K_N$.

- B1. [Initialize BOUND.] Set $\text{BOUND} \leftarrow N$. (BOUND is the highest index for which the record is not known to be in its final position; thus we are indicating that nothing is known at this point.)
- B2. [Loop on j .] Set $t \leftarrow 0$. Perform step B3 for $j = 1, 2, \dots, \text{BOUND} - 1$, and then go to step B4. (If $\text{BOUND} = 1$, this means go directly to B4.)
- B3. [Compare/exchange $R_j : R_{j+1}$.] If $K_j > K_{j+1}$, interchange $R_j \leftrightarrow R_{j+1}$ and set $t \leftarrow j$.
- B4. [Any exchanges?] If $t = 0$, terminate the algorithm. Otherwise set $\text{BOUND} \leftarrow t$ and return to step B2. ■

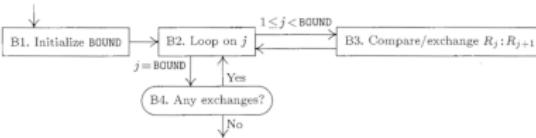
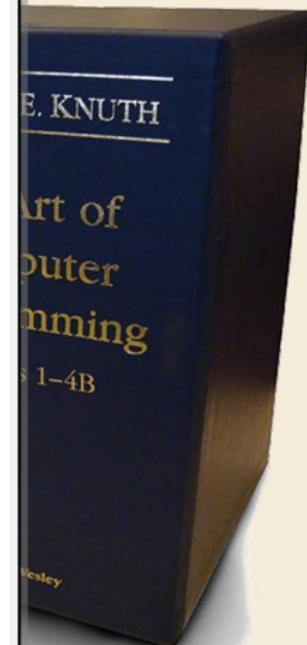


Fig. 15. Flow chart for bubble sorting.

Program B (*Bubble sort*). As in previous MIX programs of this chapter, we assume that the items to be sorted are in locations INPUT+1 through INPUT+N. $r11 \equiv t$; $r12 \equiv j$.

01 START ENT1 N	1 B1. Initialize BOUND, $t \leftarrow N$.
02 1H ST1 BOUND(1:2)	A BOUND $\leftarrow t$.
03 ENT2 1	A B2. Loop on j , $j \leftarrow 1$.
04 ENT1 0	A $t \leftarrow 0$.
05 JMP BOUND	Exit if $j \geq \text{BOUND}$.
06 3H LDA INPUT,2	C B3. Compare/exchange $R_j : R_{j+1}$.
07 CMPA INPUT+1,2	C
08 JLE 2F	C No exchange if $K_j \leq K_{j+1}$.
09 LDX INPUT+1,2	B R_{j+1}
10 STX INPUT,2	B $\rightarrow R_j$.
11 STA INPUT+1,2	B (old R_j) $\rightarrow R_{j+1}$.
12 ENT1 0,2	B $t \leftarrow j$.
13 2H INC2 1	C $j \leftarrow j + 1$.
14 BOUND ENTX -*,2	A + C $rX \leftarrow j - \text{BOUND}$. [Instruction modified]
15 JXN 3B	A + C Do step B3 for $1 \leq j < \text{BOUND}$.
16 4H J1P 1B	A B4. Any exchanges? To B2 if $t > 0$. ■



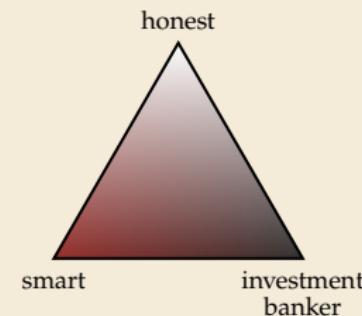
Pseudocode

- ▶ Programs for the random-access machine are very low level and detailed
 - ≈ assembly/machine language

Typical simplifications when describing and analyzing algorithms:

- ▶ more abstract *pseudocode* code that humans understand (easily)
 - ▶ control flow using **if**, **for**, **while**, etc.
 - ▶ variable names instead of fixed registers and memory cells
 - ▶ memory management (more below)
- ▶ count dominant *elementary operations* (e.g. memory accesses) instead of all RAM instructions

In both cases: We *can* go to full detail where needed/desired.



Pseudocode – Example

RAM-Program

```
1 // Bubblesort
2 // Assume:  $R_1$  stores number  $N$ 
3 // Assume:  $\text{MEM}[0..N]$  contains list of  $N$  numbers
4  $R_2 := R_1;$ 
5  $R_3 := R_1 - 2;$ 
6  $R_4 := \text{MEM}[R_3];$ 
7  $R_5 := R_3 + 1;$ 
8  $R_6 := \text{MEM}[R_5];$ 
9 if ( $R_4 \leq R_6$ ) goto line 12;
10  $\text{MEM}[R_3] := R_6;$ 
11  $\text{MEM}[R_5] := R_4;$ 
12  $R_3 := R_3 - 1;$ 
13 if ( $R_3 \geq 0$ ) goto line 6;
14  $R_2 := R_2 - 1;$ 
15 if ( $R_2 > 0$ ) goto line 5;
16 // Done:  $\text{MEM}[0..N]$  sorted
```

Pseudocode Algorithm

```
1 procedure bubblesort( $A[0..N]$ ):
2     for  $i := N, N - 1, \dots, 1$ 
3         for  $j := N - 2, N - 3, \dots, 0$ 
4             if  $A[j] > A[j + 1]$ :
5                 Swap  $A[j]$  and  $A[j + 1]$ 
6             end if
7         end for
8     end for
```

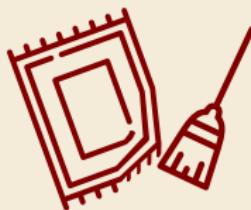
↔ much more **readable**

- ▶ closer to modern high-level programming languages
- ▶ **but:** only allow primitive operations that correspond to $O(1)$ RAM instructions

↔ analysis

Memory management & Pointers

- ▶ A random-access machine is a bit like a bare CPU . . . without any operating system
 - ~~> cumbersome to use
- ▶ All high-level programming languages / operating systems add *memory management*:
 - ▶ Instruction to *allocate* a contiguous piece of memory of a given size (like `malloc`).
 - ▶ used to allocate a new array (of a fixed size) or
 - ▶ a new object/record (with a known list of instance variables)
 - ▶ There's a similar instruction to *free* allocated memory again or an automated garbage collector.
 - ~~> A *pointer* is a memory address (i. e., the *i* of `MEM[i]`).
 - ▶ Support for procedures (a. k. a. functions, methods) calls including recursive calls
 - ▶ (this internally requires maintaining call stack)



We will mostly ignore *how* all this works here.

2.3 Asymptotics & Big-Oh

Clicker Question



What is the correct way to complete the equation?

$$8n + \frac{1}{2}n^2 + 1024 = \boxed{}$$

- A** $O(1)$
- B** $O(n)$
- C** $O(n \log(n))$
- D** $O(n^2)$
- E** I don't know $O(\cdot)$



→ *sli.do/cs566*

Clicker Question



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→ *sli.do/cs566*

Why asymptotics?

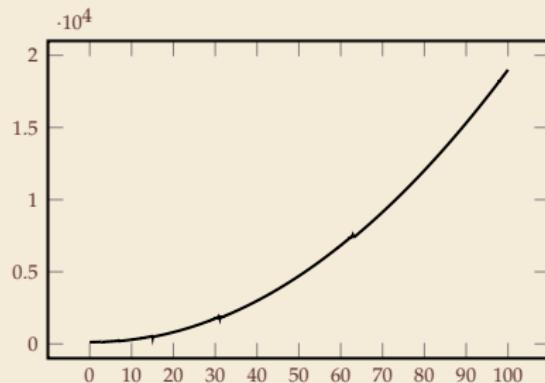
Algorithm analysis focuses on (the limiting behavior for infinitely) **large** inputs.

- ▶ abstracts from unnecessary detail
- ▶ simplifies analysis
- ▶ often necessary for sensible comparison

Asymptotics = approximation around ∞

Example: Consider a function $f(n)$ given by

$$2n^2 - 3n\lfloor \log_2(n+1) \rfloor + 7n - 3\lfloor \log_2(n+1) \rfloor + 120$$



Why asymptotics?

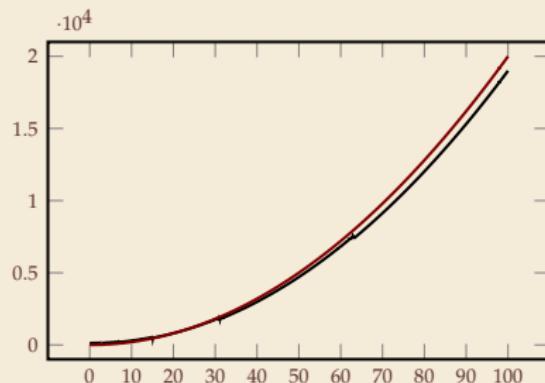
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Example: Consider a function $f(n)$ given by

$$2n^2 - 3n\lfloor \log_2(n+1) \rfloor + 7n - 3\lfloor \log_2(n+1) \rfloor + 120 \sim 2n^2$$



Asymptotic tools – Formal & definitive definition

► “Tilde Notation”: $f(n) \sim g(n)$ iff $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$
„ f and g are *asymptotically equivalent*“

Asymptotic tools – Formal & definitive definition

if, and only if

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„ f and g are *asymptotically equivalent*“

also write ‘=’ instead

► “Big-Oh Notation”: $f(n) \in O(g(n))$ iff $\left| \frac{f(n)}{g(n)} \right|$ is bounded for $n \geq n_0$
need supremum since limit might not exist!

iff $\limsup_{n \rightarrow \infty} \left| \frac{f(n)}{g(n)} \right| < \infty$

Variants:

- “Big-Omega”
- $f(n) \in \Omega(g(n))$ iff $g(n) \in O(f(n))$
- “Big-Theta”
- $f(n) \in \Theta(g(n))$ iff $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$

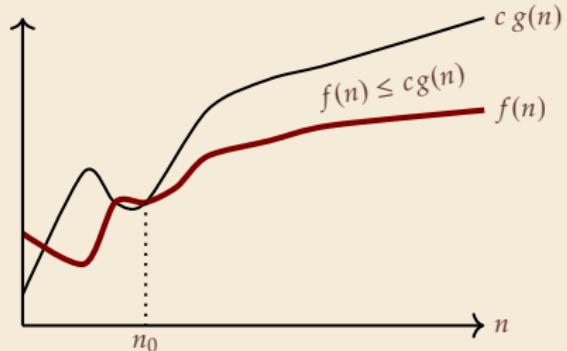
Asymptotic tools – Formal & definitive definition

- if, and only if
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„ f and g are *asymptotically equivalent*“
- “Big-Oh Notation”: $f(n) \in O(g(n))$ iff $\left| \frac{f(n)}{g(n)} \right|$ is bounded for $n \geq n_0$
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- Variants:
- $f(n) \in \Omega(g(n))$ iff $g(n) \in O(f(n))$
“Big-Omega”
 - $f(n) \in \Theta(g(n))$ iff $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$
“Big-Theta”
- “Little-Oh Notation”: $f(n) \in o(g(n))$ iff $\lim_{n \rightarrow \infty} \left| \frac{f(n)}{g(n)} \right| = 0$
similarly: $f(n) \in \omega(g(n))$ if $\lim = \infty$

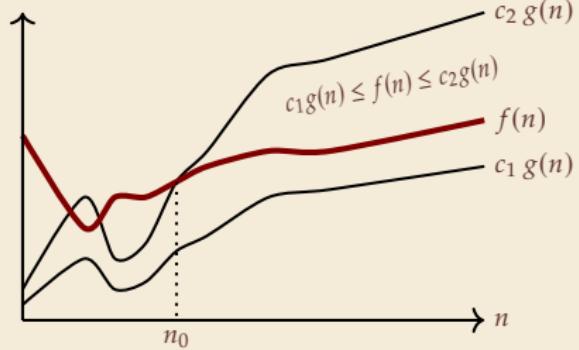
(Benefit of this definition: Works for any $f, g : \mathbb{R} \rightarrow \mathbb{R}$ and is easy to generalize to limits other than $\underline{n \rightarrow \infty}$)

Asymptotic tools – Intuition

- $f(n) = O(g(n))$: $f(n)$ is **at most** $g(n)$ up to constant factors and for sufficiently large n



- $f(n) = \Theta(g(n))$: $f(n)$ is **equal to** $g(n)$ up to constant factors and for sufficiently large n



Plots can be misleading!

Example ↗

Clicker Question

Assume $f(n) \in O(g(n))$. What can we say about $g(n)$?



- A** $g(n) = O(f(n))$
- B** $g(n) = \Omega(f(n))$
- C** $g(n) = \Theta(f(n))$
- D** Nothing (it depends on f and g)



→ *sli.do/cs566*

Clicker Question

Assume $f(n) \in O(g(n))$. What can we say about $g(n)$?



- A** ~~$g(n) = O(f(n))$~~
- B** $g(n) = \Omega(f(n))$ ✓
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Clicker Question

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- A** ~~$g(n) = O(f(n))$~~
- B** $g(n) = \Omega(f(n))$ ✓ (if $f(n) \neq 0$)
- C** ~~$g(n) = \Theta(f(n))$~~
- D** Nothing (it depends on f and g) ✓



→ *sli.do/cs566*

Asymptotics – Example 1

Basic examples:

- $20n^3 + 10n \ln(n) + 5 \sim 20n^3 = O(n^3)$
 $= \Omega(n^3)$
- $3\lg(n^2) + \lg(\lg(n)) = \Theta(\log n)$
- $10^{100} = O(1)$

$$\lim_{n \rightarrow \infty} \left| \frac{20n^3 + 10n \ln(n) + 5}{20n^3} \right| =$$
$$\lim_{n \rightarrow \infty} \frac{20n^3}{20n^3} + \lim_{n \rightarrow \infty} \underbrace{\frac{10n \ln(n)}{20n^3}}_{\leq \frac{1}{2n}} + \lim_{n \rightarrow \infty} \frac{5}{20n^3} = 0$$
$$= 1 < \infty$$

$$\lim_{n \rightarrow \infty} \frac{20n^3}{20n^3 + 10n \ln(n) + 5} = \lim_{n \rightarrow \infty} \frac{20n^3}{20n^3} \left(1 + \underbrace{\frac{10n \ln(n)}{20n^3}}_{\rightarrow 0} + \underbrace{\frac{5}{20n^3}}_{\rightarrow 0} \right)$$
$$= 1$$

Use *wolfram alpha* to compute/check limits, but also practice it with pen and paper!

Clicker Question



Is $(\sin(n) + 2)n^2 = \Theta(n^2)$?

A Yes

B No



→ *sli.do/cs566*

Clicker Question



Is $(\sin(n) + 2)n^2 = \Theta(n^2)$?

A Yes ✓

$$\lim_{n \rightarrow \infty} \left| \frac{(\sin(n) + 2)n^2}{n^2} \right| \text{ ex. nicht!}$$

B No ✗

aber:

$$1 \leq \left| \frac{(\sin(n) + 2)n^2}{n^2} \right| \leq 3$$
$$\Rightarrow (\sin(n) + 2)n^2 = O(n^2)$$



→ sli.do/cs566

Asymptotics – Basic facts

Rules to work with Big-Oh classes:

- ▶ $f = \Theta(f)$ (reflexivity)
- ▶ $f = \Theta(g) \wedge g = \Theta(h) \implies f = \Theta(h)$
- ▶ $c \cdot f(n) = \Theta(f(n))$ for constant $c \neq 0$
- ▶ $f \sim g \iff f = g \cdot (1 \pm o(1))$
- ▶ $\Theta(f) \cdot \Theta(g) = \Theta(f \cdot g)$
- ▶ $\Theta(f) + \Theta(g) = \Theta(f + g) = \Theta(\max\{f, g\})$ largest summand determines Θ -class

Asymptotics – Frequently encountered classes

Frequently used orders of growth:

- ▶ constant $\Theta(1)$
- ▶ logarithmic $\Theta(\log n)$ Note: $a, b > 0$ constants $\rightsquigarrow \Theta(\log_a(n)) = \Theta(\log_b(n))$
- ▶ linear $\Theta(n)$
- ▶ linearithmic $\Theta(n \log n)$
- ▶ quadratic $\Theta(n^2)$
- ▶ cubic $\Theta(n^3)$
- ▶ polynomial $O(n^c)$ for some constant c
- ▶ exponential $O(c^n)$ for some constant $c > 1$ Note: $a > b > 0$ constants $\rightsquigarrow b^n = o(a^n)$

2.4 Asymptotic Pitfalls

Asymptotics – Example 2

Square-and-multiply algorithm

for computing x^m with $m \in \mathbb{N}$

Inputs:

- ▶ m as binary number (array of bits)
- ▶ $n = \#$ bits in m
- ▶ x a floating-point number

```
1 def pow(x, m):
2     # compute binary representation of exponent
3     exponent_bits = bin(m)[2:]
4     result = 1
5     for bit in exponent_bits:
6         result *= result      // n executions
7         if bit == '1':
8             result *= x      // ?    # 1s
9     return result
```

- ▶ Cost: $C = \#$ multiplications
- ▶ $C = n$ (line 6) + $\#\text{one-bits in binary representation of } m$ (line 8)
 $\rightsquigarrow n \leq C \leq 2n$

Clicker Question



We showed $n \leq C(n) \leq 2n$; what is the most precise asymptotic approximation for $C(n)$ that we can make?

Write e.g. $O(n^2)$ for $O(n^2)$ or $\Theta(\sqrt{n})$ for $\Theta(\sqrt{n})$.



→ sli.do/cs566

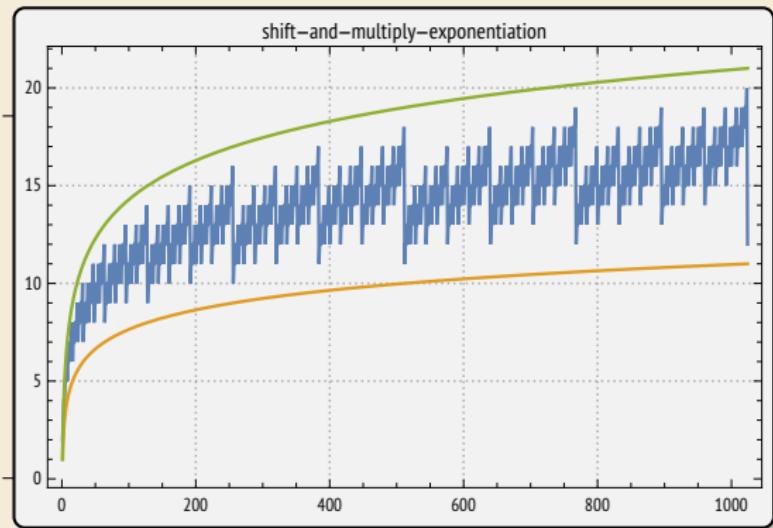
Asymptotics – Example 2

Square-and-multiply algorithm

for computing x^m with $m \in \mathbb{N}$

Inputs:

- ▶ m as binary number (array of bits)
- ▶ $n = \#$ bits in m
- ▶ x a floating-point number



- ▶ Cost: $C = \#$ multiplications
- ▶ $C = n$ (line 6) + #one-bits in binary representation of m (line 8)

$$\rightsquigarrow n \leq C \leq 2n$$

$$\rightsquigarrow C = \Theta(n) = \Theta(\log m)$$



Often, you can pretend Θ is “*like \sim with an unknown constant*”
but in this case, *no such constant exists!*

Asymptotics with several variables

- ▶ Example: Algorithms on graphs with n vertices and m edges.
 - ▶ want to say: Algorithm A takes time $\Theta(n + m)$.
 - ▶ But what does that even mean, formally?!

Schlechte Alternative

$$\exists c \exists n_0 \exists m_0 \forall n \geq n_0 \forall m \geq m_0 f(n, m) \leq c \cdot g(n, m)$$

Asymptotics with several variables

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- ▶ want to say: Algorithm A takes time $\Theta(n + m)$.
- ▶ But what does that even mean, formally?!

⚠ Inconsistent and incompatible definitions used in the literature!

- ▶ Here:

- ▶ (implicitly) always have a single “*main*” variable n : with $n \rightarrow \infty$
- ▶ all other variables are *functions* of n : $m = m(n)$
- ▶ must make *conditions* on functions explicit: $m(n) \in \Omega(n)$ and $m(n) \in O(n^2)$.

~~ Can make statements like

$$O(n + m) \subseteq O(nm) \quad (n \rightarrow \infty, m \in \Omega(1))$$

$$\frac{n + m(u)}{nm(u)} = \frac{\cancel{n}}{\cancel{nm(u)}} + \frac{\cancel{n} \cancel{m(u)}}{\cancel{nm(u)}}$$

2.5 Teaser: The Maximum subarray problem

Bring on the puzzles!

Time for a concrete example of algorithm design!

- ▶ we will illustrate the algorithm design process on a “toy problem”
- ▶ clean abstract problem, but nontrivial to solve!

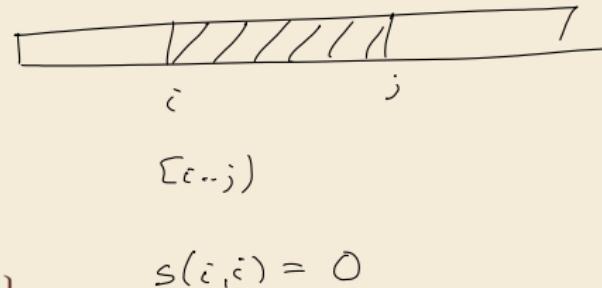
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Maximum (sum) subarray problem

- ▶ Given: $A[0..n]$ with $A[i] \in \mathbb{Z}$ for $0 \leq i < n$.
- ▶ Abbreviate $s(i, j) := \sum_{k=i}^{j-1} A[k]$
- ▶ Goal: Compute $\textcircled{s} := \max\{s(i, j) : 0 \leq i \leq j \leq n\}$
(and a pair (i, j) with $s = s(i, j)$.)
will ignore that here; easy to modify algorithms



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Applications:

- ▶ largest gain of a stock $A[i]$ price change on day i
- ▶ signal detection in biological sequence analysis
- ▶ 2D generalization used in image analysis

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will ignore that here; easy to modify algorithms

Modeling decisions:

- ▶ input size: # numbers n
- ▶ assume all integers (and sums) fit in $O(1)$ words
- ~~ count # additions as elementary operation

Applications:

- ▶ largest gain of a stock $A[i]$ price change on day i
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Clicker Question



What do you think is the Θ -class of the running time of the fastest algorithm for the maximal sum subarray problem?



→ *sli.do/cs566*

Template for Describing an Algorithm

1.💡 Algorithmic Idea

Abstract idea that makes the algorithm work (prose)
(an expert could fill in the rest from here)

2.UserCode Pseudocode

structured description of procedure including edge cases
should be unambiguous and close to real code

3.⌚ Correctness proof

argument why the correct result is computed
often uses induction and invariants

4.💻 Algorithm analysis

analysis of the efficiency of the algorithm
usually want Θ -class of worst-case running time
where interesting, also space usage

Brute force approach

- Let's start with the simplest thinkable solution

1. ♀ Algorithmic Idea

try all contiguous subarrays $A[i..j]$

Maximal subarray problem

- **Given:** $A[0..n]$ with $A[i] \in \mathbb{Z}$ for $0 \leq i < n$.

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3. ◎ Correctness proof

direct by definition of s

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additions

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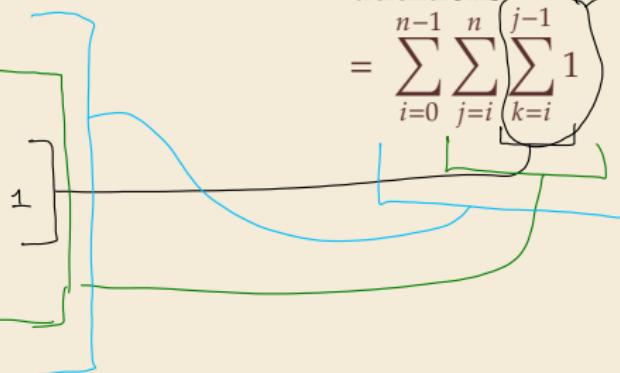
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4. 📈 Algorithm analysis

$$\begin{aligned} \text{\# additions} &= j - l - c + 1 = j - c \\ &= \sum_{i=0}^{n-1} \sum_{j=i}^n \sum_{k=i}^{j-1} 1 \end{aligned}$$

3. ◎ Correctness proof

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$$= \sum_{i=0}^{n-1} \sum_{j=i}^n \sum_{k=i}^{j-1} 1 = \sum_{i=0}^{n-1} \sum_{j=i}^n (j - i)$$

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4. ▲ Algorithm analysis

$$\begin{aligned} & \text{j} = i \rightsquigarrow j - i = 0 \\ & \text{j} = n \rightsquigarrow j - i = n - i \\ \# \text{additions} &= \sum_{i=0}^{n-1} \sum_{j=i}^n \sum_{k=i}^{j-1} 1 = \sum_{i=0}^{n-1} \sum_{j=i}^n (j - i) \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} j = \frac{(n-i)(n-i+1)}{2} \\ \sum_{j=1}^m j &= 1 + 2 + 3 + 4 + \dots + (m-1) + m \\ + \sum_{j=1}^m j &= m + (m-1) + (m-2) + (m-3) + \dots + 2 + 1 \\ &= (m+1) + (m+1) + (m+1) + (m+1) + \dots + (m+1) + (m+1) \\ &= m(m+1) \end{aligned}$$

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$$\begin{aligned} &= \sum_{i=0}^{n-1} \sum_{j=i}^n \sum_{k=i}^{j-1} 1 = \sum_{i=0}^{n-1} \sum_{j=i}^n (j - i) \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} j = \sum_{i=0}^{n-1} \frac{(n - i)(n - i + 1)}{2} \end{aligned}$$

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$$\begin{aligned} &= \sum_{i=0}^{n-1} \sum_{j=i}^n \sum_{k=i}^{j-1} 1 = \sum_{i=0}^{n-1} \sum_{j=i}^n (j-i) \quad \begin{matrix} i=0 \rightsquigarrow n-i=n \\ i=n-1 \rightsquigarrow n-i=1 \end{matrix} \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-i} j = \sum_{i=0}^{n-1} \frac{(n-i)(n-i+1)}{2} \\ &= \frac{1}{2} \sum_{i=1}^n i(i+1) = \frac{1}{2} \sum_{i=1}^n i^2 + \frac{1}{2} \sum_{i=1}^n i \end{aligned}$$

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Reusing sums

1.💡 Algorithmic Idea

- ▶ brute force algorithm is unnecessarily wasteful!
- ▶ can use $s(i, j) = s(i, j - 1) + A[j - 1]$

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2. </> Pseudocode

```
1  s = 0
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3      t = 0
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8  end for
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3. ◎ Correctness proof: as above

4. ── Algorithm analysis: $\sum_{i=0}^{n-1} \sum_{j=i+1}^n 1 = \frac{n(n+1)}{2} \sim \frac{1}{2}n^2 = \Theta(n^2)$ additions

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2.UserCode

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Can we possibly do better?

- ▶ There are $\binom{n}{2} \sim \frac{1}{2}n^2$ different $s(i, j) \dots$
- ~~~ Can't look at all of them

3.◎ Correctness proof: as above

4. **Algorithm analysis:** $\sum_{i=0}^{n-1} \sum_{j=i+1}^n 1 = \frac{n(n+1)}{2} \sim \frac{1}{2}n^2 = \Theta(n^2)$ additions