# 7 Text Compression

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Prof. Dr. Sebastian Wild

CS566 (Wintersemester 2024/25)
Philipps-Universität Marburg

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# Learning Outcomes

**Unit 7:** ***Text Compression***

1. Understand the necessity for encodings and know *ASCII* and *UTF-8 character encodings*.
2. Understand (qualitatively) the *limits of compressibility*.
3. Know and understand the algorithms (encoding and decoding) for *Huffman codes*, *RLE*, *Elias codes*, *LZW*, *MTF*, and *BWT*, including their *properties* like running time complexity.
4. Select and *adapt* (slightly) a *compression* pipeline for a specific type of data.

# Outline

## 7 Text Compression

## 7.1 Context

# Overview

- Unit 6 & 13: How to *work* with strings
  - finding substrings
  - finding approximate matches $\rightsquigarrow$ Unit ~~8~~ 13
  - finding repeated parts $\rightsquigarrow$ Unit ~~8~~ 13
  - ...
  - assumed character array (random access)!
- Unit 7 & 8: How to *store/transmit* strings
  - computer memory: must be binary
  - how to compress strings (save space)
  - how to robustly transmit over noisy channels $\rightsquigarrow$ Unit 8

# Clicker Question

What compression methods do you know?

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# Terminology

- **source text:** string $S \in \Sigma_S^\star$ to be stored / transmitted
  $\Sigma_S$ is some alphabet
- **coded text:** encoded data $C \in \Sigma_C^\star$ that is actually stored / transmitted
  usually use $\Sigma_C = \{0, 1\}$
- **encoding:** algorithm mapping source texts to coded texts $S \rightsquigarrow C$
- **decoding:** algorithm mapping coded texts back to original source text $C \rightsquigarrow S$

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- **encoding:** algorithm mapping source texts to coded texts
- **decoding:** algorithm mapping coded texts back to original source text
- **Lossy vs. Lossless**
  - **lossy compression** can only decode **approximately**;
    the exact source text $S$ is lost
  - **lossless compression** always decodes $S$ exactly
- For media files, lossy, logical compression is useful (e. g. JPEG, MPEG)
- We will concentrate on *lossless* compression algorithms.
  These techniques can be used for any application.

# What is a good encoding scheme?

- Depending on the application, goals can be
  - efficiency of encoding/decoding
  - resilience to errors/noise in transmission
  - security (encryption)
  - integrity (detect modifications made by third parties)
  - size

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- Focus in this unit: **size** of coded text

  Encoding schemes that (try to) minimize the size of coded texts perform *data compression*.

- We will measure the *compression ratio*: $\dfrac{|C| \cdot \lg |\Sigma_C|}{|S| \cdot \lg |\Sigma_S|} \overset{\Sigma_C = \{0,1\}}{=} \dfrac{|C|}{|S| \cdot \lg |\Sigma_S|}$

  $< 1$ means successful compression

  $= 1$ means no compression

  $> 1$ means "compression" made it bigger!? (yes, that happens . . . )

# Clicker Question

Do you know what uncomputable/undecidable problems (halting problem, Post's correspondence problem, . . . ) are?

- A Sure, I could explain what it is.
- B Heard that in a lecture, but don't quite remember
- C No, never heard of it

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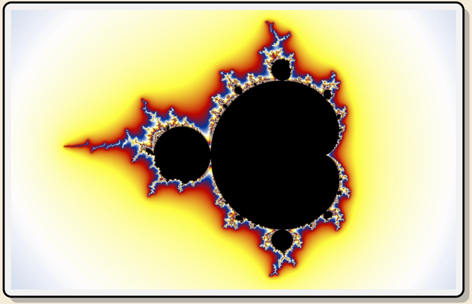
# Limits of algorithmic compression

*Is this image compressible?*

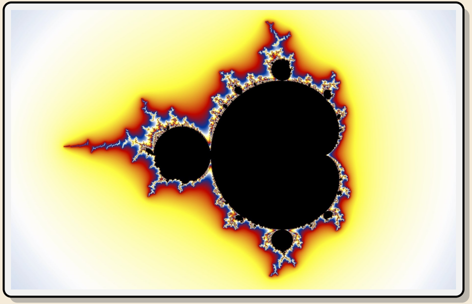


# Limits of algorithmic compression

*Is this image compressible?*

visualization of Mandelbrot set

- Clearly a complex shape!
- Will not compress (too) well using, say, PNG.
- but:
  - completely defined by mathematical formula
  - $\rightsquigarrow$ **can be generated by a very small program!**



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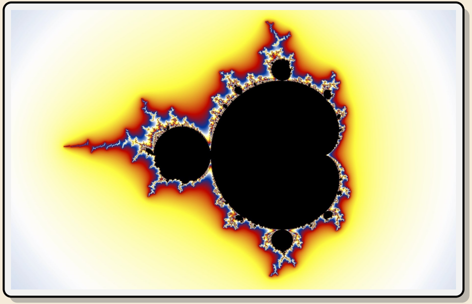

⇝ ***Kolmogorov complexity***

- $C$ = *any program* that outputs $S$

  self-extracting archives!

  needs fixed machine model, but compilers transfer results

- Kolmogorov complexity = length of smallest such program

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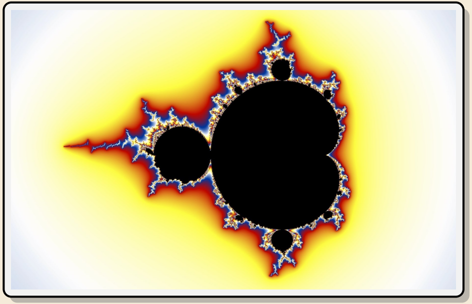

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⇝ ***Kolmogorov complexity***

- $C$ = *any program* that outputs $S$

  self-extracting archives!

  needs fixed machine model, but compilers transfer results
- Kolmogorov complexity = length of smallest such program
- **Problem:** finding smallest such program is *uncomputable*.
- ⇝ No optimal encoding algorithm is possible!
- ⇝ must be inventive to get efficient methods

# What makes data compressible?

- Lossless compression methods mainly exploit two types of redundancies in source texts:

  1. **uneven character frequencies**
     some characters occur more often than others $\rightarrow$ Part I
  2. **repetitive texts**
     different parts in the text are (almost) identical $\rightarrow$ Part II

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     different parts in the text are (almost) identical $\longrightarrow$ Part II

*There is no such thing as a free lunch!*

Not *everything* is compressible ($\longrightarrow$ tutorials)

$\rightsquigarrow$ focus on versatile methods that often work

# Part I

## *Exploiting character frequencies*

## 7.2 Character Encodings

# Character encodings

- Simplest form of encoding: Encode each source character individually

$\rightsquigarrow$ encoding function $E : \Sigma_S \to \Sigma_C^\star$

  - typically, $|\Sigma_S| \gg |\Sigma_C|$, so need several bits per character
  - for $c \in \Sigma_S$, we call $E(c)$ the *codeword* of $c$

- **fixed-length code:** $|E(c)|$ is the same for all $c \in \Sigma_{\cancel{C}S}$

- **variable-length code:** not all codewords of same length

# Fixed-length codes

- fixed-length codes are the simplest type of character encodings
- Example: **ASCII** (American Standard Code for Information Interchange, 1963)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0000000 NUL | 0010000 DLE | 0100000 | 0110000 0 | 1000000 @ | 1010000 P | 1100000 ‘ | 1110000 p |
| 0000001 SOH | 0010001 DC1 | 0100001 ! | 0110001 1 | 1000001 A | 1010001 Q | 1100001 a | 1110001 q |
| 0000010 STX | 0010010 DC2 | 0100010 " | 0110010 2 | 1000010 B | 1010010 R | 1100010 b | 1110010 r |
| 0000011 ETX | 0010011 DC3 | 0100011 # | 0110011 3 | 1000011 C | 1010011 S | 1100011 c | 1110011 s |
| 0000100 EOT | 0010100 DC4 | 0100100 $ | 0110100 4 | 1000100 D | 1010100 T | 1100100 d | 1110100 t |
| 0000101 ENQ | 0010101 NAK | 0100101 % | 0110101 5 | 1000101 E | 1010101 U | 1100101 e | 1110101 u |
| 0000110 ACK | 0010110 SYN | 0100110 & | 0110110 6 | 1000110 F | 1010110 V | 1100110 f | 1110110 v |
| 0000111 BEL | 0010111 ETB | 0100111 ’ | 0110111 7 | 1000111 G | 1010111 W | 1100111 g | 1110111 w |
| 0001000 BS | 0011000 CAN | 0101000 ( | 0111000 8 | 1001000 H | 1011000 X | 1101000 h | 1111000 x |
| 0001001 HT | 0011001 EM | 0101001 ) | 0111001 9 | 1001001 I | 1011001 Y | 1101001 i | 1111001 y |
| 0001010 LF | 0011010 SUB | 0101010 * | 0111010 : | 1001010 J | 1011010 Z | 1101010 j | 1111010 z |
| 0001011 VT | 0011011 ESC | 0101011 + | 0111011 ; | 1001011 K | 1011011 [ | 1101011 k | 1111011 { |
| 0001100 FF | 0011100 FS | 0101100 , | 0111100 < | 1001100 L | 1011100 \ | 1101100 l | 1111100 \| |
| 0001101 CR | 0011101 GS | 0101101 - | 0111101 = | 1001101 M | 1011101 ] | 1101101 m | 1111101 } |
| 0001110 SO | 0011110 RS | 0101110 . | 0111110 > | 1001110 N | 1011110 ^ | 1101110 n | 1111110 ~ |
| 0001111 SI | 0011111 US | 0101111 / | 0111111 ? | 1001111 O | 1011111 _ | 1101111 o | 1111111 DEL |

- 7 bit per character
- just enough for English letters and a few symbols (plus control characters)

# Fixed-length codes – Discussion

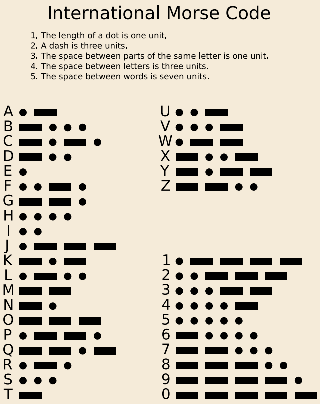
- 👍 Encoding & Decoding as fast as it gets

- 👎 Unless all characters equally likely, it wastes a lot of space
- 👎 inflexible (how to support adding a new character?)

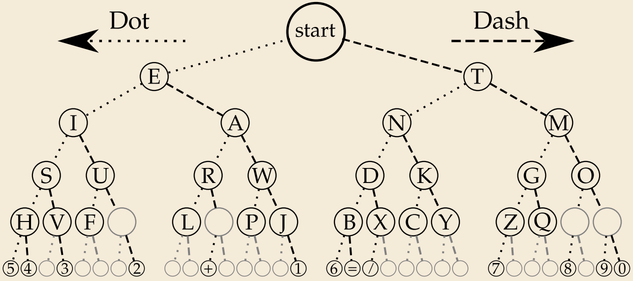
# Variable-length codes

- to gain more flexibility, have to allow different lengths for codewords

- actually an old idea: **Morse Code**




https://commons.wikimedia.org/wiki/File:International_Morse_Code.svg




https://commons.wikimedia.org/wiki/File:Morse-code-tree.svg

# Clicker Question

How many characters are there in the alphabet of the coded text in Morse Code, i. e., what is $|\Sigma_C|$?

- A 1
- B 2
- C 3
- D 4
- E 26
- F 36
- G 256

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- F ~~36~~
- G ~~256~~

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# Variable-length codes – UTF-8

- Modern example: UTF-8 encoding of Unicode:
  default encoding for text-files, XML, HTML since 2009
  - Encodes any Unicode character (154 998 as of Nov 2024, and counting)
  - uses 1 – 4 bytes (codeword lengths: 8, 16, 24, or 32 bits)
  - Every ASCII character is encoded in 1 byte with leading bit `0`, followed by the 7 bits for ASCII
  - Non-ASCII characters start with 1 – 4 `1`s indicating the total number of bytes, followed by a `0` and 3–5 bits.
    The remaining bytes each start with `10` followed by 6 bits.

| Char. number range (hexadecimal) | UTF-8 octet sequence (binary) |
|---|---|
| `0000 0000 – 0000 007F` | `0xxxxxxx` |
| `0000 0080 – 0000 07FF` | `110xxxxx 10xxxxxx` |
| `0000 0800 – 0000 FFFF` | `1110xxxx 10xxxxxx 10xxxxxx` |
| `0001 0000 – 0010 FFFF` | `11110xxx 10xxxxxx 10xxxxxx 10xxxxxx` |

👍 For English text, most characters use only 8 bit,
but we can include any Unicode character, as well. 🤓

# Pitfall in variable-length codes

- Suppose we have the following code:

| $c$ | a | n | b | s |
|---|---|---|---|---|
| $E(c)$ | 0 | 10 | 110 | 100 |

- Happily encode text $S$ = banana with the coded text $C = \underbrace{110}_{\texttt{b}}\underbrace{0}_{\texttt{a}}\underbrace{10}_{\texttt{n}}\underbrace{0}_{\texttt{a}}\underbrace{10}_{\texttt{n}}\underbrace{0}_{\texttt{a}}$

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ϟ $C$ = 1100100100 decodes **both** to banana and to bass: $\underbrace{\texttt{110}}_{\texttt{b}}\underbrace{\texttt{0}}_{\texttt{a}}\underbrace{\texttt{100}}_{\texttt{s}}\underbrace{\texttt{100}}_{\texttt{s}}$

⇝ not a valid code . . . (cannot tolerate ambiguity)

but how should we have known?

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$E(\texttt{n}) = \texttt{10}$ is a (proper) **prefix** of $E(\texttt{s}) = \texttt{100}$

⇝ Leaves decoder wondering whether to stop after reading 10 or continue!

⇝ Usually require a ***prefix-free*** code: No codeword is a prefix of another.

prefix-free $\implies$ instantaneously decodable $\implies$ uniquely decodable

# Code tries

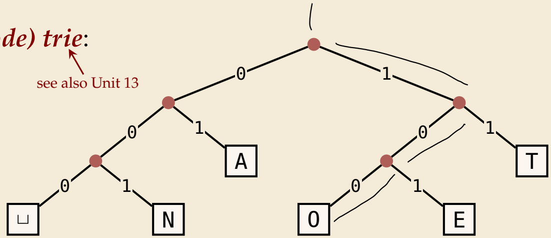
- From now on only consider prefix-free codes $E$:
  $E(c)$ is not a proper prefix of $E(c')$ for any $c, c' \in \Sigma_S$.

- **Example:**

| $c$ | A | E | N | O | T | ␣ |
|---|---|---|---|---|---|---|
| $E(c)$ | 01 | 101 | 001 | 100 | 11 | 000 |

Any prefix-free code corresponds to a ***(code) trie***:

see also Unit 13




  - binary tree
  - one **leaf** for each characters of $\Sigma_S$
  - path from root to leave = codeword
    left child = `0`; right child = `1`

- Example for using the code trie:
  - Encode `AN␣ANT`
  - Decode `$\boxed{11}\boxed{100}$0001010111`

T O

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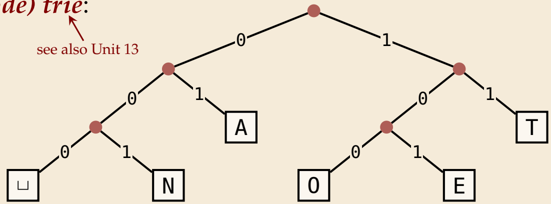
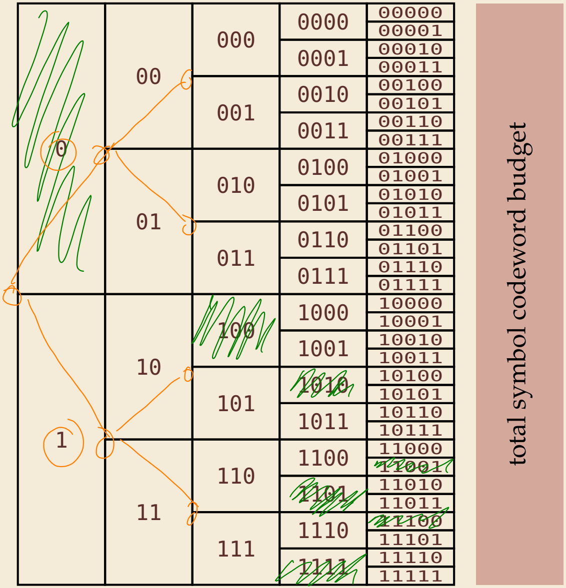



- Example for using the code trie:
  - Encode AN␣ANT $\longrightarrow$ 010010000100111
  - Decode 111000001010111 $\longrightarrow$ TO␣EAT

# The Codeword Supermarket



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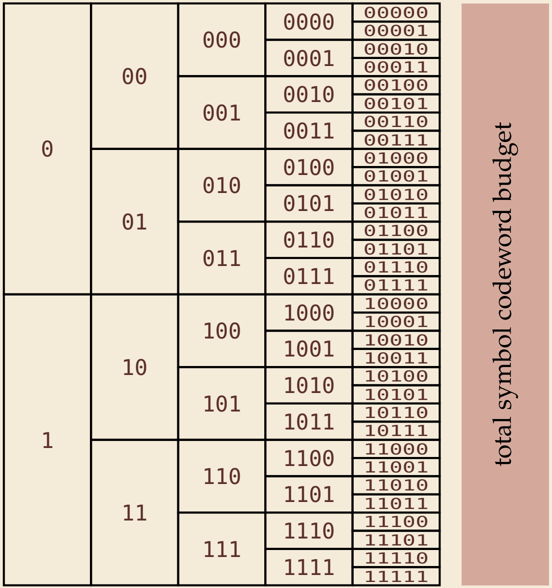


- Can "spend" at most budget of 1 across all codewords
  - Codeword with $\ell$ bits costs $2^{-\ell}$
- *Kraft-McMillan inequality:* any uniquely decodable code with codeword lengths $\ell_1, \ldots, \ell_\sigma$ satisfies

$$\sum_{i=1}^{\sigma} 2^{-\ell_i} \le 1$$

and for any such lengths there is a prefix-free code

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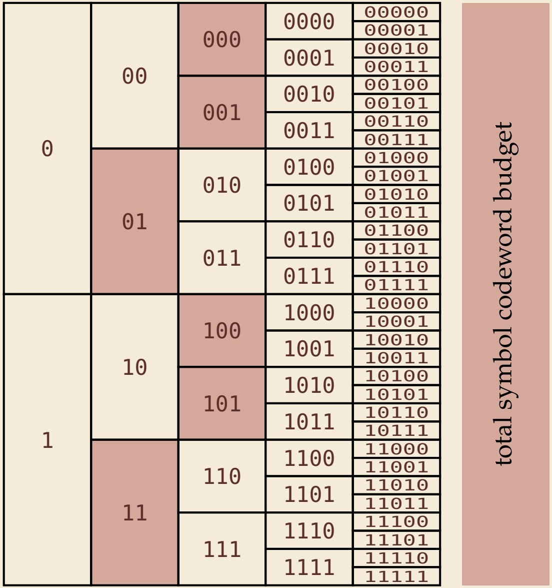


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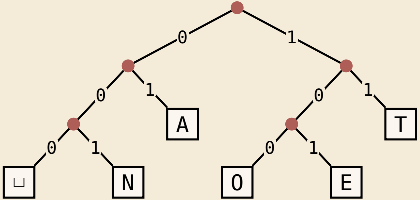

# Who decodes the decoder?

- Depending on the application, we have to **store/transmit** the **used code**!
- We distinguish:
  - **fixed coding:** code agreed upon in advance, not transmitted (e. g., Morse, UTF-8)
  - **static coding:** code depends on message, but stays same for entire message; it must be transmitted (e. g., Huffman codes $\rightarrow$ next)
  - **adaptive coding:** code depends on message and changes during encoding; implicitly stored withing the message (e. g., LZW $\rightarrow$ below)

## 7.3 Huffman Codes

# Character frequencies

- **Goal:** Find character encoding that produces short coded text
- Convention here: fix $\Sigma_C = \{0, 1\}$ (binary codes), abbreviate $\Sigma = \Sigma_S$,
- **Observation:** Some letters occur more often than others.

**Typical English prose:**

| | | | | | |
|---|---|---|---|---|---|
| **e** | 12.70% | **d** | 4.25% | **p** | 1.93% |
| **t** | 9.06% | **l** | 4.03% | **b** | 1.49% |
| **a** | 8.17% | **c** | 2.78% | **v** | 0.98% |
| **o** | 7.51% | **u** | 2.76% | **k** | 0.77% |
| **i** | 6.97% | **m** | 2.41% | **j** | 0.15% |
| **n** | 6.75% | **w** | 2.36% | **x** | 0.15% |
| **s** | 6.33% | **f** | 2.23% | **q** | 0.10% |
| **h** | 6.09% | **g** | 2.02% | **z** | 0.07% |
| **r** | 5.99% | **y** | 1.97% | | |

$\leadsto$ Want shorter codes for more frequent characters!

# Huffman coding

e. g. frequencies / probabilities

- **Given:** $\Sigma$ and weights $w : \Sigma \to \mathbb{R}_{\geq 0}$
- **Goal:** prefix-free code $E$ (= code trie) for $\Sigma$ that minimizes coded text length

  i. e., a code trie minimizing $\underline{\displaystyle\sum_{c\in\Sigma} w(c)\cdot|E(c)|}$

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  i. e., a code trie minimizing $\sum_{c \in \Sigma} w(c) \cdot |E(c)|$

- Let's abbreviate $|S|_c$ = #occurrences of $c$ in $S$
- If we use $w(c) = |S|_c$,
  this is the character encoding with smallest possible $|C|$

  $\rightsquigarrow$ **best possible *character-wise* encoding**

- Quite ambitious! *Is this efficiently possible?*

# Huffman's algorithm

- Actually, yes! A greedy/myopic approach succeeds here.

**Huffman's algorithm:**

1. Find two characters $\mathtt{a}$, $\mathtt{b}$ with lowest weights.
   - We will encode them with the same prefix, plus one distinguishing bit, i. e., $E(\mathtt{a}) = u\mathtt{0}$ and $E(\mathtt{b}) = u\mathtt{1}$ for a bitstring $u \in \{0,1\}^\star$ ($u$ to be determined)
2. (Conceptually) replace $\mathtt{a}$ and $\mathtt{b}$ by a single character "$\boxed{\mathtt{ab}}$" with $w(\boxed{\mathtt{ab}}) = w(\mathtt{a}) + w(\mathtt{b})$.
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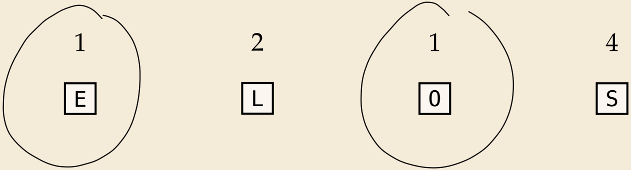
- efficient implementation using a (min-oriented) ***priority queue***
  - start by inserting all characters with their weight as key
  - step 1 uses two `deleteMin` calls
  - step 2 inserts a new character with the sum of old weights as key

# Huffman's algorithm – Example

- Example text: $S$ = LOSSLESS $\rightsquigarrow$ $\Sigma_S = \{$E, L, O, S$\}$

- Character frequencies: E : 1, L : 2, O : 1, S : 4



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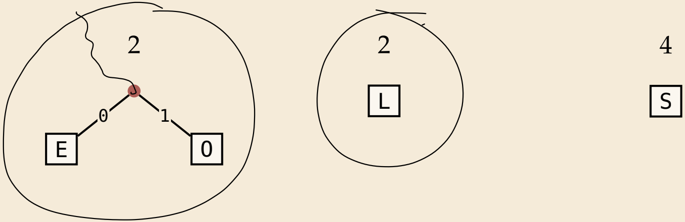

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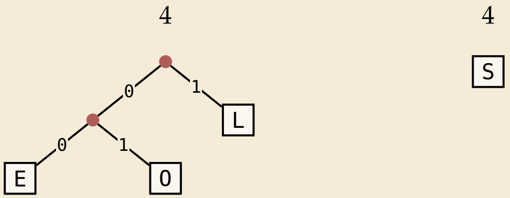

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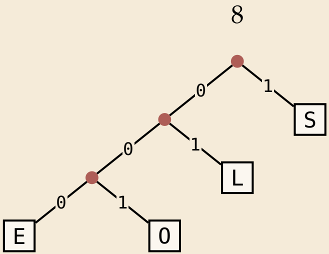

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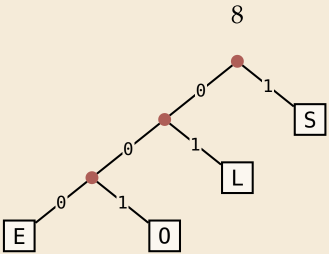
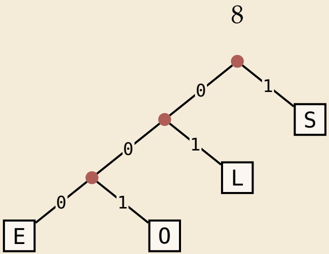



⇝ *Huffman tree* (code trie for Huffman code)

# Huffman's algorithm – Example

- Example text: $S = \texttt{LOSSLESS}$ $\rightsquigarrow$ $\Sigma_S = \{\texttt{E}, \texttt{L}, \texttt{O}, \texttt{S}\}$
- Character frequencies: $\texttt{E} : 1$, $\texttt{L} : 2$, $\texttt{O} : 1$, $\texttt{S} : 4$




$\rightsquigarrow$ *Huffman tree* (code trie for Huffman code)

$\texttt{LOSSLESS} \rightarrow \texttt{01001110100011}$ compression ratio: $\frac{14}{8 \cdot \log 4} = \frac{14}{16} \approx 88\%$

# Huffman tree – tie breaking

- The above procedure is ambiguous:
  - which characters to choose when weights are equal?
  - which subtree goes left, which goes right?

- For CS 566: always use the following rule:

> ***1.*** To break ties when selecting the two characters,
> first use the smallest letter according to the alphabetical order,
> or the tree containing the smallest alphabetical letter.
>
> ***2.*** When combining two trees of different values,
> place the lower-valued tree on the left (corresponding to a `0`-bit).
>
> ***3.*** When combining trees of equal value,
> place the one containing the smallest letter to the left.

$\rightsquigarrow$ practice in tutorials

# Encoding with Huffman code

- The overall encoding procedure is as follows:
  - **Pass 1:** Count character frequencies in $S$
  - Construct Huffman code $E$ (as above)
  - Store the Huffman code in $C$ (details omitted)
  - **Pass 2:** Encode each character in $S$ using $E$ and append result to $C$

- Decoding works as follows:
  - Decode the Huffman code $E$ from $C$. (details omitted)
  - Decode $S$ character by character from $C$ using the code trie.

- Note: Decoding is much simpler/faster!

# Huffman code – Optimality

### Theorem 7.1 (Optimality of Huffman's Algorithm)

Given $\Sigma$ and $w : \Sigma \to \mathbb{R}_{\geq 0}$, Huffman's Algorithm computes codewords $E : \Sigma \to \{0,1\}^\star$ with minimal expected codeword length $\ell(E) = \sum_{c\in\Sigma} w(c) \cdot |E(c)|$ among all prefix-free codes for $\Sigma$. ◀

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IB: $\sigma = 2$ ✓

*Proof sketch:* by induction over $\sigma = |\Sigma|$

IH: $\forall \sigma' < \sigma$ Huffman's algorithm computes optimal PF code

- Given any optimal prefix-free code $E^*$ (as its code trie).
- code trie $\rightsquigarrow$ $\exists$ two sibling leaves $x, y$ at largest depth $D$

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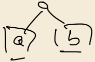


- swap characters in leaves to have two lowest-weight characters $\texttt{a}$, $\texttt{b}$ in $x$, $y$ (that can only make $\ell$ smaller, so still optimal)
- any optimal code for $\Sigma' = \Sigma \setminus \{\texttt{a}, \texttt{b}\} \cup \{\boxed{\texttt{ab}}\}$ yields optimal code for $\Sigma$ by replacing leaf $\boxed{\texttt{ab}}$ by internal node with children $\texttt{a}$ and $\texttt{b}$.

$\rightsquigarrow$ recursive call yields optimal code for $\Sigma'$ by inductive hypothesis, so Huffman's algorithm finds optimal code for $\Sigma$.

## 7.4 Entropy

# Entropy

## Definition 7.2 (Entropy)

Given probabilities $p_1, \ldots, p_n$ (for outcomes $1, \ldots, n$ of a random variable), the *entropy* of the distribution is defined as

$$\mathcal{H}(p_1, \ldots, p_n) \;=\; -\sum_{i=1}^{n} p_i \lg p_i \;=\; \underline{\sum_{i=1}^{n} p_i \lg\left(\frac{1}{p_i}\right)}$$

$$0 \leq \mathcal{H}(p_1, \ldots, p_n) \leq \lg n = \mathcal{H}\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$$

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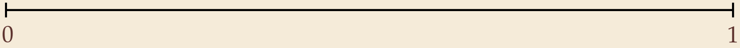
- entropy is a **measure** of **information** content of a distribution
  - *"20 Questions on $[0, 1)$":* Land inside my interval by halving.



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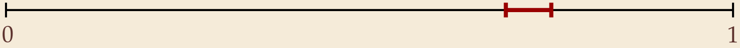

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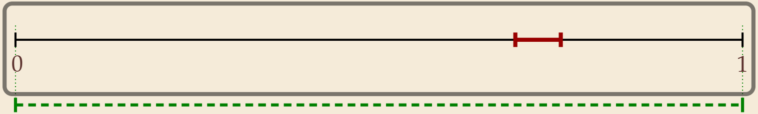

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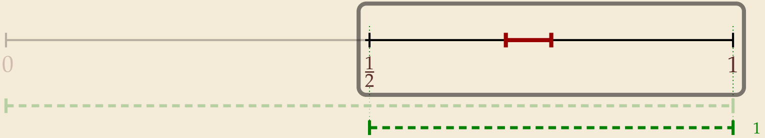

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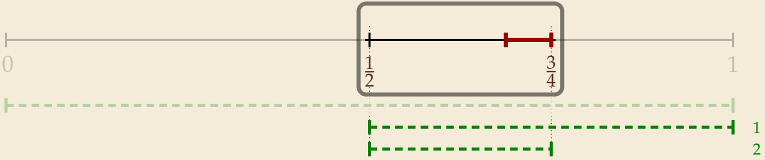

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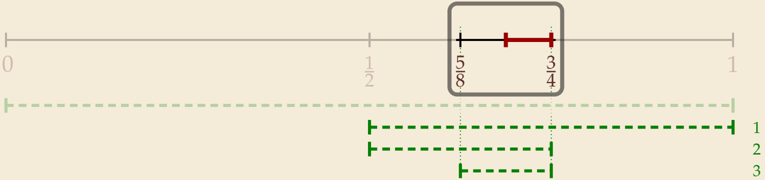

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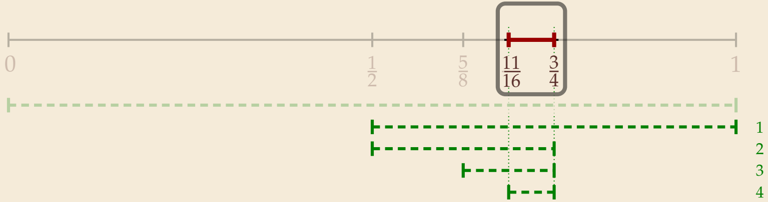

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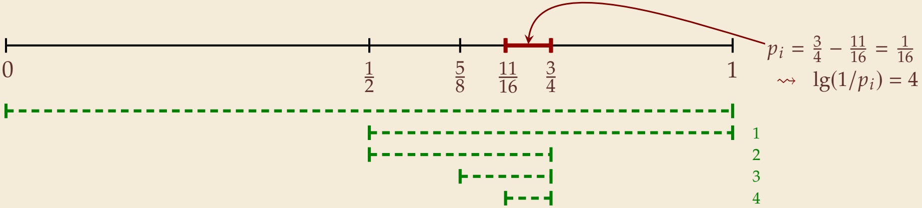

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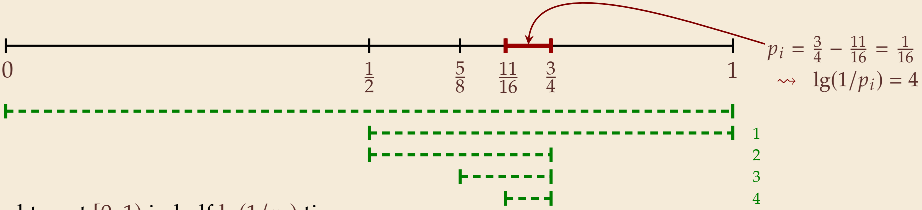



$\rightsquigarrow$ Need to cut $[0,1)$ in half $\lg(1/p_i)$ times

- more precisely: the expected number of bits (Yes/No questions) required to nail down the random value

# Entropy and Huffman codes

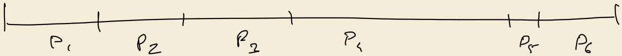
- would ideally encode value $i$ using $\lg(1/p_i)$ bits

  not always possible; cannot use codeword of 1.5 bits . . .

not as length of single codeword that is; but can be possible *on average*!

positive example:

all $p_i = 2^{-k_i}$



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## Theorem 7.3 (Entropy bounds for Huffman codes)

For any probabilities $p_1, \ldots, p_\sigma$ for $\Sigma = \{a_1, \ldots, a_\sigma\}$, the Huffman code $E$ for $\Sigma$ with weights $p(a_i) = p_i$ satisfies $\boxed{\mathcal{H} \le \ell(E) \le \mathcal{H} + 1}$ where $\mathcal{H} = \mathcal{H}(p_1, \ldots, p_\sigma)$. ◀

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*Proof sketch:*

- $\ell(E) \ge \mathcal{H}$

  Any prefix-free code $E$ induces weights $q_i = 2^{-|E(a_i)|}$.

  By *Kraft's Inequality*, we have $q_1 + \cdots + q_\sigma \le 1$.

in general $p_i \neq q_i$

$\sigma = 2$ $\quad p_1 = \varepsilon$

$p_2 = 1 - \varepsilon$

$q_1 = q_2 = \frac{1}{2}$

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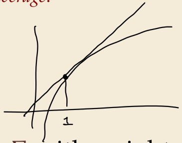


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$\boxed{\ln(x) \le x - 1 \quad x > 0}$

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  Hence we can apply *Gibb's Inequality* to get

$$\mathcal{H} = \underbrace{\sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{p_i}\right) \le \sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{q_i}\right)}_{\sum p_i = 1, \quad \sum q_i \le 1} = \ell(E).$$

$$0 \overset{!}{\ge} \sum p_i \ln\left(\frac{1}{p_i}\right) - \sum p_i \ln\left(\frac{1}{q_i}\right)$$

$$= \sum p_i \ln\left(\frac{q_i}{p_i}\right)$$

$$\le \sum p_i \left(q_i/p_i - 1\right)$$

$$= \underbrace{\sum q_i}_{\le 1} - \underbrace{\sum p_i}_{1} \le 0$$

# Entropy and Huffman codes [2]

*Proof sketch (continued):*

- $\ell(E) \leq \mathcal{H} + 1$

  Set $q_i = 2^{-\lceil \lg(1/p_i) \rceil}$. We have $\displaystyle\sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{q_i}\right) = \sum_{i=1}^{\sigma} p_i \lceil \lg(1/p_i) \rceil \leq \mathcal{H} + 1.$

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  - If $\sigma = 2$, $E'$ uses a single bit each.
    Here, $q_i \le 1/2$, so $\lg(1/q_i) \ge 1 = |E'(a_i)|$ ✓

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  - If $\sigma \ge 3$, we merge $a_1$ and $a_2$ to $\boxed{a_1 a_2}$, assign it weight $\underline{2q_2}$ and recurse.
    If $q_1 = q_2$, this is like Huffman; otherwise, $\underline{q_1}$ is a unique smallest value and
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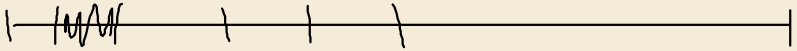

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    By construction, $|E'(a_1)| = |E'(a_2)| = \left|E'(\boxed{a_1 a_2})\right| + 1$, so $|E'(a_1)| \leq \lg(\frac{1}{q_1})$ and $|E'(a_2)| \leq \lg(\frac{1}{q_2})$.

  By optimality of $E$, we have $\displaystyle\ell(E) \leq \ell(E') \leq \sum_{i=1}^{\sigma} p_i \lg\left(\frac{1}{q_i}\right) \leq \mathcal{H} + 1.$

# Clicker Question

When does Huffman coding yield more efficient compression than a fixed-length character encoding?

- A always
- B when $\mathcal{H} \approx \lg(\sigma)$
- C when $\mathcal{H} < \lg(\sigma)$
- D when $\mathcal{H} < \lg(\sigma) - 1$
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→ sli.do/cs566

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use relative frequencies: $p_i = \dfrac{|S|_{a_i}}{|S|} = \dfrac{\#\text{occurences of } a_i \text{ in string } S}{\text{length of } S}$

- Recall: For $S[0..n)$ over $\Sigma = \{a_1, \ldots, a_\sigma\}$,
  length of Huffman-coded text is

$$|C| \;=\; \sum_{i=1}^{\sigma} |S|_{a_i} \cdot |E(a_i)| \;=\; n \sum_{i=1}^{\sigma} \overbrace{\frac{|S|_{a_i}}{n}}^{=p_i} \cdot |E(a_i)| \;=\; \underline{n\ell(E)}$$

$\rightsquigarrow$ Theorem 7.3 tells us rather precisely how well Huffman compresses:
$\mathcal{H}_0(S) \cdot n \;\le\; |C| \;\le\; (\mathcal{H}_0(S) + 1)n$

zero-th order empirical entropy

- $\mathcal{H}_0(S) \;=\; \mathcal{H}\left(\dfrac{|S|_{a_1}}{n}, \ldots, \dfrac{|S|_{a_\sigma}}{n}\right) \;=\; \displaystyle\sum_{i=1}^{\sigma} \frac{n}{|S|_{a_i}} \log_2\left(\frac{|S|_{a_i}}{n}\right)$ is called the *empirical entropy* of $S$

# Huffman coding – Discussion

- running time complexity: $O(\sigma \log \sigma)$ to construct code
  - build PQ + $\sigma \cdot$ (2 deleteMins and 1 insert)
  - can do $\Theta(\sigma)$ time when characters already sorted by weight
  - time for encoding text (after Huffman code done): $O(n + |C|)$
- many variations in use (tie-breaking rules, estimated frequencies, adaptive encoding, . . . )

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  - time for encoding text (after Huffman code done): $O(n + |C|)$
- many variations in use (tie-breaking rules, estimated frequencies, adaptive encoding, . . . )

👍 optimal prefix-free character encoding

👍 very fast decoding

👎 needs 2 passes over source text for encoding

- one-pass variants possible, but more complicated

👎 have to store code alongside with coded text

# Part II

## *Compressing repetitive texts*

# Beyond Character Encoding

- Many "natural" texts show repetitive redundancy

```
All work and no play makes Jack a dull boy.  All work and no play makes Jack a dull
boy.  All work and no play makes Jack a dull boy.  All work and no play makes Jack
a dull boy.  All work and no play makes Jack a dull boy.  All work and no play makes
Jack a dull boy.  All work and no play makes Jack a dull boy.  All work and no play
makes Jack a dull boy.  All work and no play makes Jack a dull boy.  All work and no
play makes Jack a dull boy.  All work and no play makes Jack a dull boy.  All work and
no play makes Jack a dull boy.  All work and no play makes Jack a dull boy.  All work
and no play makes Jack a dull boy.  All work and no play makes Jack a dull boy.  All
work and no play makes Jack a dull boy.  All work and no play makes Jack a dull boy.
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  - ⇝ Huffman won't compression this very much

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work and no play makes Jack a dull boy.  All work and no play makes Jack a dull boy.
```

- character-by-character encoding will **not** capture such repetitions
  - ⇝ Huffman won't compression this very much

⇝ Have to encode whole *phrases* of $S$ by a single codeword

## 7.5 Run-Length Encoding

# Run-Length encoding

- simplest form of repetition: ***runs*** of characters

  same character repeated

```
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
0001011001000000000111110000000000111111000
0011111111110000001111111000000001111111000
0011110110100000011100000000000011100000000
0011000000000000111000000000000111000000000
0011000000000001110000000000001110000000000
0011000000000001110000000000001110000000000
0011011000000011001111100000110011111000
0011111111100001111111111000011111111111000
0011101111110001111100111100011111001111100
0000000001110011100000011100111000000111 0
0000000001110011100000011000111000000110 0
0000000000110001100000011100011000000111 0
0000000000110011100000011000111000000110 0
0000000001110001100000011100011000000111 0
0000000001100001110000111000011100001110 0
0011011111110000011111111000000111111111000
0111111111100000001111111000000000111111 0000
0001011000000000000100100000000000100100000
0000000000000000000000000000000000000000
0000000000000000000000000000000000000000
```

- here: only consider $\Sigma_S = \{0, 1\}$ (work on a binary representation)
  - can be extended for larger alphabets

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00000000000000000000000000000000000000000
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00010110010000000001111100000000011111000
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00111101101000001110000000000011100000000
00110000000000011100000000000111000000000
00110000000000111000000000001110000000000
00110000000000111000000000001110000000000
00110110000000110011111000001100111110000
00111111110000111111111100001111111111000
00111011111000111110011110001111100111100
00000000011100111000000111001110000001110
00000000011100111000000110001110000001100
00000000001100011000000111000110000001110
00000000001100111000000110001110000001100
00000000011100011000000111000110000001110
00000000011000011100001110000111000011100
00110111111000001111111100000011111111000
01111111110000000111111000000001111110000
00010110000000000010010000000000100100000
00000000000000000000000000000000000000000
00000000000000000000000000000000000000000
```

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$\rightsquigarrow$ **run-length encoding (RLE):**
use runs as phrases: $S = \underbrace{\texttt{00000}}\ \underbrace{\texttt{111}}\ \underbrace{\texttt{0000}}$

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00110000000000111000000000001110000000000
00110110000000110011111000001100111110000
00111111110000111111111100001111111111000
00111011111000111110011111000111110011100
00000000011100111000000111001110000001110
00000000011100111000000110001110000001100
00000000001100011000000111000110000001110
00000000001100111000000110001110000001100
00000000011100011000000111000110000001110
00000000011000011100001110000111000011100
00110111111000001111111100000011111111000
01111111110000000111111000000001111110000
00010110000000000001001000000000100100000
00000000000000000000000000000000000000000
00000000000000000000000000000000000000000
```

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use runs as phrases: $S = \underbrace{\texttt{00000}}\ \underbrace{\texttt{111}}\ \underbrace{\texttt{0000}}$

$\rightsquigarrow$ We have to store

- the first bit of $S$ (either `0` or `1`)
- the length of each subsequent run
- Note: don't have to store bit for later runs since they must alternate.

- Example becomes: $0, 5, 3, 4$

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00000000000000000000000000000000000000000
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00111111111000000111111100000001111111000
00111101101000001110000000000011100000000
00110000000000011100000000000111000000000
00110000000000111000000000001110000000000
00110000000000111000000000001110000000000
00110110000000110011111000001100111110000
00111111110000111111111100001111111111000
00111011111000111110011110001111100111100
00000000011100111000000111001110000001110
00000000011100111000000110001110000001100
00000000001100011000000111000110000001110
00000000001100111000000110001110000001100
00000000011100011000000111000110000001110
00000000011000011100001110000111000011100
00110111111000001111111100000011111111000
01111111100000000111111000000001111110000
00010110000000000010010000000000100100000
00000000000000000000000000000000000000000
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⇝ **run-length encoding (RLE)**:
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  - the first bit of $S$ (either `0` or `1`)
  - the length of each subsequent run
  - Note: don't have to store bit for later runs since they must alternate.

- Example becomes: $0, 5, 3, 4$

- **Question**: How to encode a run length $k$ in binary? ($k$ can be arbitrarily large!)

# Clicker Question

How would you encode a string that can we arbitrarily long?

→ sli.do/cs566

# Elias codes

- Need a *prefix-free encoding* for $\mathbb{N} = \{1, 2, 3, \ldots, \}$
  - must allow arbitrarily large integers
  - must know when to stop reading

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  $7 \mapsto$ `00000001` $3 \mapsto$ `0001` $0 \mapsto$ `1` $30 \mapsto$ `0000000000000000000000000000001`

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- Refinement: ***Elias gamma code***
  - Store the **length** $\ell$ of the binary representation in **unary**
  - Followed by the binary digits themselves

# Elias codes

- Need a *prefix-free encoding* for $\cancel{\mathbb{N}}$ $\mathbb{N}_{\geq 1}$ $= \{1, 2, 3, \ldots, \}$
  - must allow arbitrarily large integers
  - must know when to stop reading

- But that's simple!   Just use ***unary* encoding**!
  $7 \mapsto$ `00000001`   $3 \mapsto$ `0001`   $0 \mapsto$ `1`   $30 \mapsto$ `0000000000000000000000000000001`

  Much too long
  - (wasn't the whole point of RLE to get rid of long runs??)

- Refinement: ***Elias gamma code***
  - Store the **length** $\ell$ of the binary representation in **unary**
  - Followed by the binary digits themselves
  - little tricks:
    - always have $\ell \geq 1$, so store $\ell - 1$ instead
    - binary representation always starts with `1` $\rightsquigarrow$ don't need terminating `1` in unary

$\rightsquigarrow$ Elias gamma code = $\ell - 1$ zeros, followed by binary representation

**Examples:** $1 \mapsto$ `1`,   $3 \mapsto$ `011`,   $5 \mapsto$ `00101`,   $30 \mapsto$ `000011110`

# Clicker Question

Decode the **first** number in Elias gamma code (at the beginning) of the following bitstream:

$$\texttt{000}\underbrace{\texttt{1101}}_{13}\texttt{11011100110}.$$

→ *sli.do/cs566*

# Run-length encoding – Examples

- Encoding:
  $S$ = `11111110010000000000000000000011111111111`

  $C$ = `1`

- Decoding:
  $C$ = `00001101001001010`

  $S$ =

# Run-length encoding – Examples

- Encoding:
  $S$ = `11111110010000000000000000000011111111111`
  $k = 7$
  $C$ = `100111`

- Decoding:
  $C$ = `00001101001001010`

  $S$ =

# Run-length encoding – Examples

- Encoding:

  $S$ = `11111110010000000000000000000011111111111`

  $k = 2$

  $C$ = `100111010`

- Decoding:

  $C$ = `00001101001001010`

  $S$ =

# Run-length encoding – Examples

- Encoding:
  $S$ = `11111110010000000000000000000011111111111`
  $k = 1$
  $C$ = `1001110101`

- Decoding:
  $C$ = `00001101001001010`

  $S$ =

# Run-length encoding – Examples

- Encoding:
  $S = $ `1111111001``00000000000000000000``11111111111`
  $k = 20$
  $C = $ `1001110101``000010100`

- Decoding:
  $C = $ `00001101001001010`

  $S = $

# Run-length encoding – Examples

- Encoding:
  $S =$ `111111100100000000000000000000011111111111`
  $k = 11$
  $C =$ `10011101010000101000001011`

- Decoding:
  $C =$ `00001101001001010`

  $S =$

# Run-length encoding – Examples

- Encoding:
  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C =$ `00001101001001010`

  $S =$

# Run-length encoding – Examples

- Encoding:
  $S$ = `11111110010000000000000000000011111111111`

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# Run-length encoding – Examples

- Encoding:
  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C =$ $\cancel{\texttt{0}}$`0001101001001010`
  $b = 0$

  $S =$

# Run-length encoding – Examples

- Encoding:

  $S$ = `11111110010000000000000000000011111111111`

  $C$ = `10011101010000101000001011`

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  $b = 0$

  $\ell = 3 + 1$

  $S =$

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- Encoding:

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  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  $C$ = `0000110100100 1010`

  $b = 0$

  $\ell = 3 + 1$

  $k = 13$

  $S$ = `0000000000000`

# Run-length encoding – Examples

- Encoding:

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- Decoding:

  $C$ = `00001101001001010`

  $b = 1$

  $\ell = 2 + 1$

  $k =$

  $S$ = `0000000000000`

# Run-length encoding – Examples

- Encoding:

  $S =$ `11111110010000000000000000000011111111111`

  $C =$ `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:

  $C =$ `00001101001001010`

  $b = 1$

  $\ell = 2 + 1$

  $k = 4$

  $S =$ `00000000000001111`

# Run-length encoding – Examples

- Encoding:
  $S$ = `11111110010000000000000000000011111111111`

  $C$ = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C$ = `00001101001001010`
  $b = 0$
  $\ell = 0 + 1$
  $k =$
  $S$ = `00000000000001111`

# Run-length encoding – Examples

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  $S$ = `11111110010000000000000000000011111111111`

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  Compression ratio: $26/41 \approx 63\%$

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  $b = 0$

  $\ell = 0 + 1$

  $k = 1$

  $S$ = `000000000000011110`

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- Encoding:

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- Encoding:
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  $C$ = `10011101010000101000001011`

  Compression ratio: $26/41 \approx 63\%$

- Decoding:
  $C$ = `00001101001001010`

  $b = 1$

  $\ell = 1 + 1$

  $k = 2$

  $S$ = `0000000000000111101`

# Run-length encoding – Discussion

- extensions to larger alphabets possible (must store next character then)
- used in some image formats (e. g. TIFF)

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- extensions to larger alphabets possible (must store next character then)
- used in some image formats (e. g. TIFF)

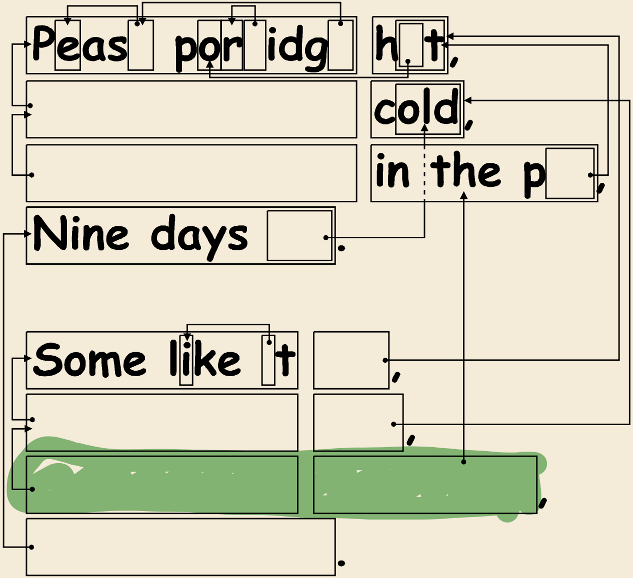
👍 fairly simple and fast

👍 can compress $n$ bits to $\Theta(\log n)$!
for extreme case of constant number of runs

👎 negligible compression for many common types of data
- No compression until run lengths $k \geq 6$
- **expansion** for run length $k = 2$ or $6$

## 7.6 Lempel-Ziv-Welch

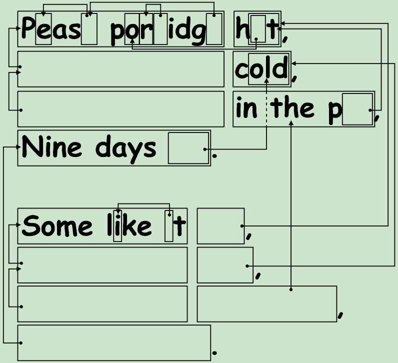
# Warmup




https://classic.csunplugged.org/text-compression/

https://www.flickr.com/photos/quintanaroo/2742726346

# Clicker Question




What is the **second-to-last line** of the poem to the left?

→ sli.do/cs566

# Lempel-Ziv Compression

- Huffman and RLE mostly take advantage of frequent or repeated *single characters*.
- **Observation**: Certain *substrings* are much more frequent than others.
  - in English text: the, be, to, of, and, a, in, that, have, I
  - in HTML: "`<a href`", "`<img src`", "`<br/>`"

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  - in HTML: "`<a href`", "`<img src`", "`<br/>`"
- **Lempel-Ziv** stands for family of *adaptive* compression algorithms.
  - **Idea:** store repeated parts by reference!

  ⇝ each codeword refers to
    - either a single character in $\Sigma_S$,
    - or a *substring* of $S$ (that both encoder and decoder have seen before).

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  - in English text: the, be, to, of, and, a, in, that, have, I
  - in HTML: "`<a href`", "`<img src`", "`<br/>`"
- **Lempel-Ziv** stands for family of *adaptive* compression algorithms.
  - **Idea:** store repeated parts by reference!
  - ⇝ each codeword refers to
    - either a single character in $\Sigma_S$,
    - or a *substring* of $S$ (that both encoder and decoder have seen before).
  - Variants of Lempel-Ziv compression
    - **"LZ77"** Original version (sliding window, overlapping phrases)
      Derivatives: LZSS, LZFG, LZRW, LZP, DEFLATE, ...
      DEFLATE used in (`pk`)`zip`, `gzip`, `PNG`
    - **"LZ78"** Second version (whole-phrase references)
      Derivatives: LZW, LZMW, LZAP, LZY, ...
      LZW used in `compress`, `GIF`

# Lempel-Ziv-Welch

- here: ***Lempel-Ziv-Welch (LZW)*** (arguably the "cleanest" variant of Lempel-Ziv)
- ***variable-to-fixed* encoding**
  - all codewords have $k$ bits (typical: $k = 12$) $\rightsquigarrow$ fixed-length
  - but they represent a variable portion of the source text!

# Lempel-Ziv-Welch

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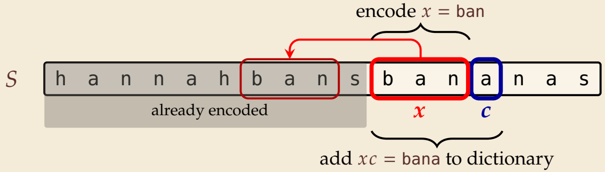
- *variable-to-fixed* **encoding**
  - all codewords have $k$ bits (typical: $k = 12$) $\rightsquigarrow$ fixed-length
  - but they represent a variable portion of the source text!
- maintain a **dictionary** $D$ with $2^k$ entries $\rightsquigarrow$ codewords $=$ indices in dictionary
  - initially, first $|\Sigma_S|$ entries encode single characters (rest is empty)
  - **add** a new entry to $D$ **after each step**:
  - **Encoding:** after encoding a substring $x$ of $S$,
    add $xc$ to $D$ where $c$ is the character that follows $x$ in $S$.




$\rightsquigarrow$ new codeword in $D$

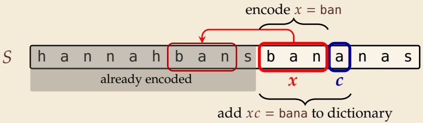
  - $D$ actually stores codewords for $x$ and $c$, not the expanded string

# LZW encoding – Example

**Input:** `YO!␣YOU!␣YOUR␣YOYO!`

$\Sigma_S$ = ASCII character set (0–127)

$C$ =




$D$ =

| Code | String |
|---|---|
| . . . | |
| 32 | ␣ |
| 33 | ! |
| . . . | |
| 79 | O |
| . . . | |
| 82 | R |
| . . . | |
| 85 | U |
| . . . | |
| 89 | Y |
| . . . | |

| Code | String |
|---|---|
| 128 | |
| 129 | |
| 130 | |
| 131 | |
| 132 | |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |

# LZW encoding – Example

**Input:** Y O ! ␣ Y O U ! ␣ Y O U R ␣ Y O Y O !

$\Sigma_S$ = ASCII character set (0–127)

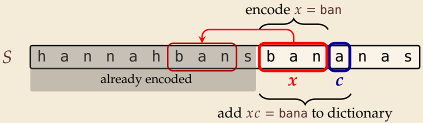
Y

$C$ = 89

$D$ =

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | |
| 129 | |
| 130 | |
| 131 | |
| 132 | |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input:** `YO!␣YOU!␣YOUR␣YOYO!`

$\Sigma_S$ = ASCII character set (0–127)

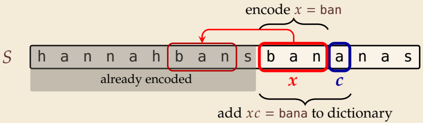
Y

$C = 89$

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | |
| 130 | |
| 131 | |
| 132 | |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input:** `YO!␣YOU!␣YOUR␣YOYO!`

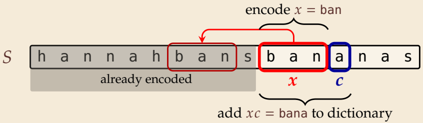
$\Sigma_S$ = ASCII character set (0–127)

| | Y | O |
|---|---|---|
| $C =$ | 89 | 79 |

$D =$

| Code | String |
|---|---|
| . . . | |
| 32 | ␣ |
| 33 | ! |
| . . . | |
| 79 | O |
| . . . | |
| 82 | R |
| . . . | |
| 85 | U |
| . . . | |
| 89 | Y |
| . . . | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | |
| 130 | |
| 131 | |
| 132 | |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input:** `YO!␣YOU!␣YOUR␣YOYO!` $\qquad \Sigma_S$ = ASCII character set (0–127)

| | Y | O |
|---|---|---|
| $C =$ | 89 | 79 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | |
| 131 | |
| 132 | |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |

encode $x$ = ban

$S$ `h a n n a h b a n s b a n a n a s`

already encoded $\qquad x \qquad c$

add $xc$ = bana to dictionary

# LZW encoding – Example

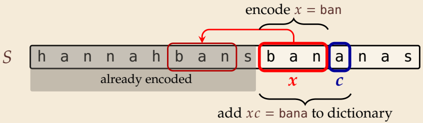
**Input:** `YO!␣YOU!␣YOUR␣YOYO!` $\qquad\qquad \Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! |
|---|---|---|---|
| $C =$ | 89 | 79 | 33 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | |
| 131 | |
| 132 | |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input:** `YO!␣YOU!␣YOUR␣YOYO!`

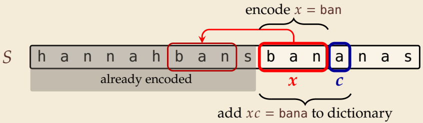
$\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! |
|---|---|---|---|
| $C =$ | 89 | 79 | 33 |

$D =$

| Code | String |
|---|---|
| . . . | |
| 32 | ␣ |
| 33 | ! |
| . . . | |
| 79 | O |
| . . . | |
| 82 | R |
| . . . | |
| 85 | U |
| . . . | |
| 89 | Y |
| . . . | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | |
| 132 | |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

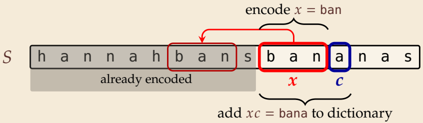
**Input:** `YO!␣YOU!␣YOUR␣YOYO!` $\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ |
|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | |
| 132 | |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input:** `YO!␣YOU!␣YOUR␣YOYO!`

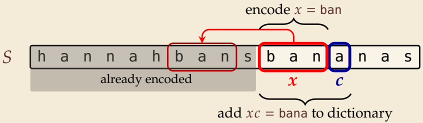
$\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ |
|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input:** `YO!␣YOU!␣YOUR␣YOYO!`

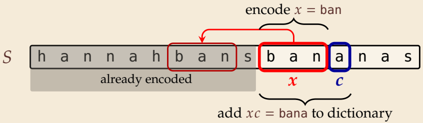
$\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO |
|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input:** `YO!␣YOU!␣YOUR␣YOYO!`

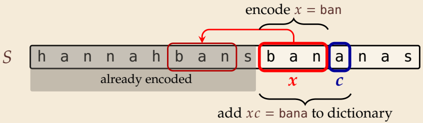
$\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO |
|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input:** `YO!␣YOU!␣YOUR␣YOYO!`

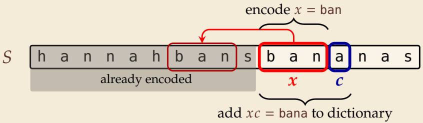
$\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U |
|---|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 | 85 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

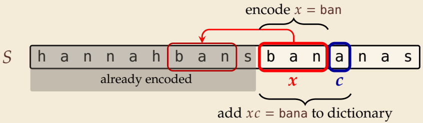
**Input:** `YO!␣YOU!␣YOUR␣YOYO!` $\qquad \Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U |
|---|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 | 85 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input:** `YO!␣YOU!␣YOUR␣YOYO!`

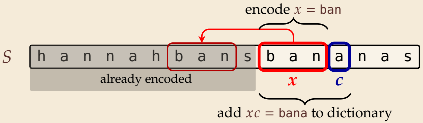
$\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U | !␣ |
|---|---|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 | 85 | 130 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

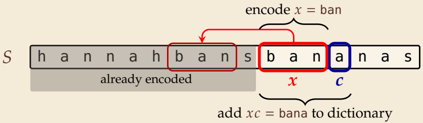
**Input**: `YO!␣YOU!␣YOUR␣YOYO!` $\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U | !␣ |
|---|---|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 | 85 | 130 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input**: `YO!␣YOU!␣YOUR␣YOYO!`

$\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U | !␣ | YOU |
|---|---|---|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 | 85 | 130 | 132 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | |
| 136 | |
| 137 | |
| 138 | |
| 139 | |

encode $x$ = ban

$S$ h a n n a h b a n s b a n a n a s

already encoded $x$ $c$

add $xc$ = bana to dictionary

# LZW encoding – Example

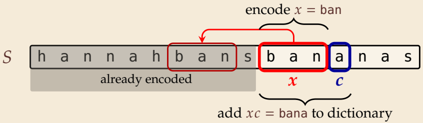
**Input:** `YO!␣YOU!␣YOUR␣YOYO!` $\qquad \Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U | !␣ | YOU |
|---|---|---|---|---|---|---|---|---|
| $C$ = | 89 | 79 | 33 | 32 | 128 | 85 | 130 | 132 |

$D$ =

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | YOUR |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

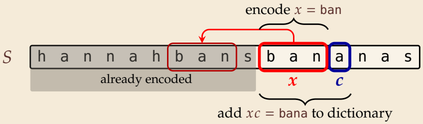
**Input**: `YO!␣YOU!␣YOUR␣YOYO!` $\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U | !␣ | YOU | R |
|---|---|---|---|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 | 85 | 130 | 132 | 82 |

$D =$

| Code | String |
|---|---|
| . . . | |
| 32 | ␣ |
| 33 | ! |
| . . . | |
| 79 | O |
| . . . | |
| 82 | R |
| . . . | |
| 85 | U |
| . . . | |
| 89 | Y |
| . . . | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | YOUR |
| 136 | |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input**: `YO!␣YOU!␣YOUR␣YOYO!`

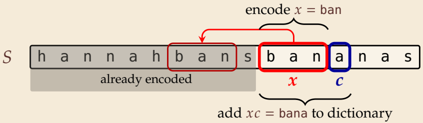
$\Sigma_S$ = ASCII character set (0–127)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Y | O | ! | ␣ | YO | U | !␣ | YOU | R |
| $C$ = | 89 | 79 | 33 | 32 | 128 | 85 | 130 | 132 | 82 |

$D$ =

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | YOUR |
| 136 | R␣ |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

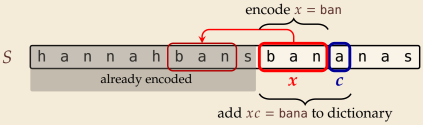
**Input:** `YO!␣YOU!␣YOUR␣YOYO!` $\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U | !␣ | YOU | R | ␣Y |
|---|---|---|---|---|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 | 85 | 130 | 132 | 82 | 131 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | YOUR |
| 136 | R␣ |
| 137 | |
| 138 | |
| 139 | |



# LZW encoding – Example

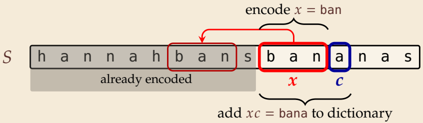
**Input**: `YO!␣YOU!␣YOUR␣YOYO!` $\qquad \Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U | !␣ | YOU | R | ␣Y |
|---|---|---|---|---|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 | 85 | 130 | 132 | 82 | 131 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | YOUR |
| 136 | R␣ |
| 137 | ␣YO |
| 138 | |
| 139 | |



# LZW encoding – Example

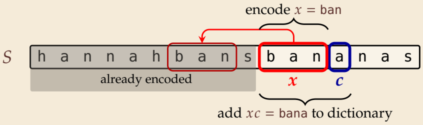
**Input:** `YO!␣YOU!␣YOUR␣YOYO!` $\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U | !␣ | YOU | R | ␣Y | O |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C$ = | 89 | 79 | 33 | 32 | 128 | 85 | 130 | 132 | 82 | 131 | 79 |

$D$ =

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | YOUR |
| 136 | R␣ |
| 137 | ␣YO |
| 138 | |
| 139 | |



# LZW encoding – Example

**Input**: `YO!␣YOU!␣YOUR␣YOYO!`

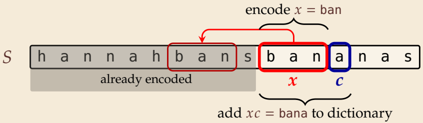
$\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U | !␣ | YOU | R | ␣Y | O |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 | 85 | 130 | 132 | 82 | 131 | 79 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | YOUR |
| 136 | R␣ |
| 137 | ␣YO |
| 138 | OY |
| 139 | |



# LZW encoding – Example

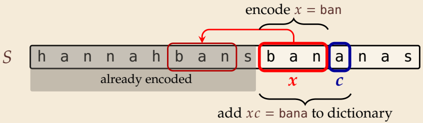
**Input:** `YO!␣YOU!␣YOUR␣YOYO!` $\qquad \Sigma_S$ = ASCII character set (0–127)

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Y | O | ! | ␣ | YO | U | !␣ | YOU | R | ␣Y | O | YO |
| $C =$ | 89 | 79 | 33 | 32 | 128 | 85 | 130 | 132 | 82 | 131 | 79 | 128 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | YOUR |
| 136 | R␣ |
| 137 | ␣YO |
| 138 | OY |
| 139 | |



# LZW encoding – Example

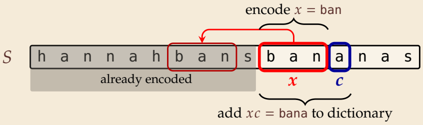
**Input**: `YO!␣YOU!␣YOUR␣YOYO!` $\Sigma_S$ = ASCII character set (0–127)

| | Y | O | ! | ␣ | YO | U | !␣ | YOU | R | ␣Y | O | YO |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C =$ | 89 | 79 | 33 | 32 | 128 | 85 | 130 | 132 | 82 | 131 | 79 | 128 |

$D =$

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | YOUR |
| 136 | R␣ |
| 137 | ␣YO |
| 138 | OY |
| 139 | YO! |

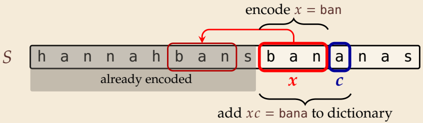


# LZW encoding – Example

**Input**: `YO!␣YOU!␣YOUR␣YOYO!`

$\Sigma_S$ = ASCII character set (0–127)

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Y | O | ! | ␣ | YO | U | !␣ | YOU | R | ␣Y | O | YO | ! |
| $C$ = | 89 | 79 | 33 | 32 | 128 | 85 | 130 | 132 | 82 | 131 | 79 | 128 | 33 |




$D$ =

| Code | String |
|---|---|
| ... | |
| 32 | ␣ |
| 33 | ! |
| ... | |
| 79 | O |
| ... | |
| 82 | R |
| ... | |
| 85 | U |
| ... | |
| 89 | Y |
| ... | |

| Code | String |
|---|---|
| 128 | YO |
| 129 | O! |
| 130 | !␣ |
| 131 | ␣Y |
| 132 | YOU |
| 133 | U! |
| 134 | !␣Y |
| 135 | YOUR |
| 136 | R␣ |
| 137 | ␣YO |
| 138 | OY |
| 139 | YO! |

# LZW encoding – Code

**procedure** LZWencode($S[0..n)$)
    $x := \varepsilon$ *// previous phrase, initially empty*
    $C := \varepsilon$ *// output, initially empty* 13
    $D :=$ dictionary, initialized with codes for $c \in \Sigma_S$ *// stored as trie* ($\rightsquigarrow$ Unit ~~8~~)
    $k := |\Sigma_S|$ *// next free codeword*
    **for** $i := 0, \ldots, n-1$ **do**
        $c := S[i]$
        **if** $D$.containsKey($xc$) **then**
            $x := xc$
        **else**
            $C := C \cdot D$.get($x$) *// append codeword for $x$*
            $D$.put($xc, k$) *// add $xc$ to $D$, assigning next free codeword*
            $k := k + 1$;  $x := c$
    **end for**
    $C := C \cdot D$.get($x$)
    **return** $C$