# 9 Range-Minimum Queries

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# Learning Outcomes

1. Know the *RMQ problem* and its *connection* to longest common extensions in strings.
2. Know and understand trivial RMQ solutions and *sparse tables*.
3. Know and understand the *Cartesian trees* data structure.
4. Know and understand the *exhaustive-tabulation technique* for RMQ with linear-time preprocessing.

**Unit 9:** ***Range-Minimum Queries***

# Outline

## 9 Range-Minimum Queries

## 9.1 Introduction

# Range-minimum queries (RMQ)

array/numbers don't change

- **Given:** Static array $A[0..n)$ of numbers
- **Goal:** Find minimum in a range;
  $A$ known in advance and can be preprocessed

$\mathrm{RMQ}(7, 15) = 10$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
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- **Nitpicks:**
  - Report *index* of minimum, not its value
  - Report *leftmost* position in case of ties

# Clicker Question

Given the array from the slides, what is $\mathrm{RMQ}_A(1,6)$

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# Rules of the Game

- comparison-based $\rightsquigarrow$ values don't matter, only relative order

- Two main quantities of interest:

  $\rightsquigarrow$ space usage $\leq P(n)$

  1. **Preprocessing time**: Running time $P(n)$ of the preprocessing step
  2. **Query time**: Running time $Q(n)$ of one query (using precomputed data)

- Write $\langle P(n), Q(n) \rangle$ **time solution** for short

# Clicker Question

What do you think, what running times can we achieve? For a $\langle P(n), Q(n) \rangle$ time solution, enter “`<P(n),Q(n)>`”.

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## 9.2 RMQ, LCP, LCE, LCA — WTF?

# Recall Unit 6

## Application 4: Longest Common Extensions

- We implicitly used a special case of a more general, versatile idea:

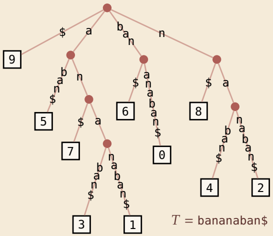
  The ***longest common extension (LCE)*** data structure:

  - **Given:** String $T[0..n-1]$
  - **Goal:** Answer LCE queries, i. e.,
    given positions $i, j$ in $T$,
    how far can we read the same text from there?
    formally: $\mathrm{LCE}(i,j) = \max\{\ell : T[i..i+\ell) = T[j..j+\ell)\}$

$\rightsquigarrow$ use suffix tree of $T$!

longest common prefix of $i$th and $j$th suffix

- In $\mathcal{T}$: $\mathrm{LCE}(i,j) = \mathrm{LCP}(T_i, T_j)$ $\rightsquigarrow$ same thing, different name!
  $=$ string depth of
  *lowest common ancester (LCA)* of
  leaves $\boxed{i}$ and $\boxed{j}$




- in short: $\boxed{\mathrm{LCE}(i,j) = \mathrm{LCP}(T_i, T_j) = \mathrm{stringDepth}\big(\mathrm{LCA}(\boxed{i}, \boxed{j})\big)}$

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# Recall Unit 6

## Efficient LCA

How to find lowest common ancestors?

- Could walk up the tree to find LCA $\rightsquigarrow$ $\Theta(n)$ worst case
- Could store all LCAs in big table $\rightsquigarrow$ $\Theta(n^2)$ space and preprocessing

**Amazing result:** Can compute data structure in $\Theta(n)$ time and space that finds any LCA is **constant(!) time**.

- a bit tricky to understand
- but a theoretical breakthrough
- and useful in practice

and suffix tree construction inside . . .

$\rightsquigarrow$ for now, use $O(1)$ LCA as black box.

$\rightsquigarrow$ After linear preprocessing (time & space), we can find LCEs in $O(1)$ time.

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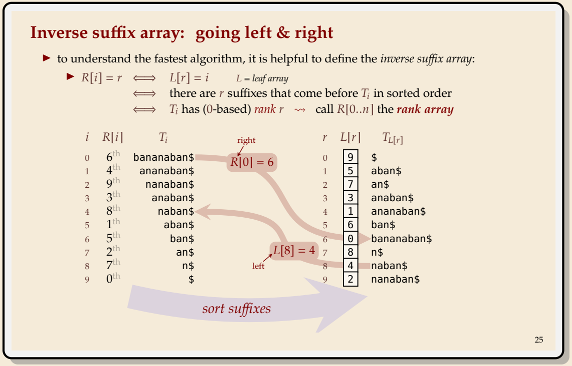
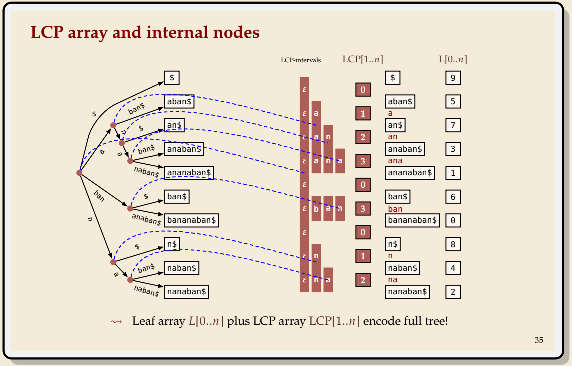
# Finally: Longest common extensions

- In Unit 6: Left question open how to compute LCA in suffix trees
- But: Enhanced Suffix Array makes life easier!

$$\boxed{\mathrm{LCE}(i,j) \;=\; \mathrm{LCP}\Big[\mathrm{RMQ}_{\mathrm{LCP}}\big(\min\{R[i],R[j]\}+1,\ \max\{R[i],R[j]\}\big)\Big]}$$



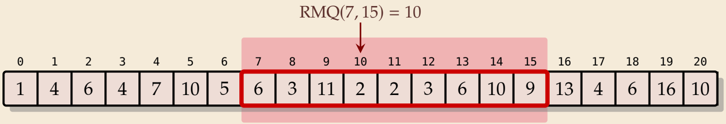
# RMQ Implications for LCE

- Recall: Can compute (inverse) suffix array and LCP array in $O(n)$ time

$\rightsquigarrow$ A $\langle P(n), Q(n) \rangle$ time RMQ data structure implies a $\langle P(n), Q(n) \rangle$ time solution for longest-common extensions

## 9.3 Trivial Solutions & Sparse Tables

# Trivial Solutions




- Two easy solutions show extreme ends of scale:

# Trivial Solutions

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- Two easy solutions show extreme ends of scale:

1. **Scan on demand**
    - no preprocessing at all
    - answer $\mathrm{RMQ}(i, j)$ by scanning through $A[i..j]$, keeping track of min

  $\rightsquigarrow$ $\langle O(1), O(n) \rangle$

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2. **Precompute all**
   - Precompute all answers in a big 2D array $M[0..n)[0..n)$
   - queries simple: $\mathrm{RMQ}(i, j) = M[i][j]$

   $\rightsquigarrow$ $\langle O(n^3), O(1) \rangle$

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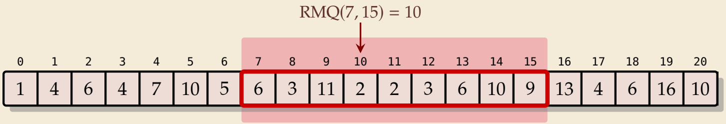


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   - Preprocessing can reuse partial results $\rightsquigarrow$ $\langle O(n^2), O(1)\rangle$

# Sparse Table

- **Idea:** Like "precompute-all", but keep only some entries
- store $M[i][j]$ iff $\ell = j - i + 1$ is $2^k$.
  - $\rightsquigarrow$ $\le n \cdot \lg n$ entries
  - $\rightsquigarrow$ Can be stored as $M'[i][k]$

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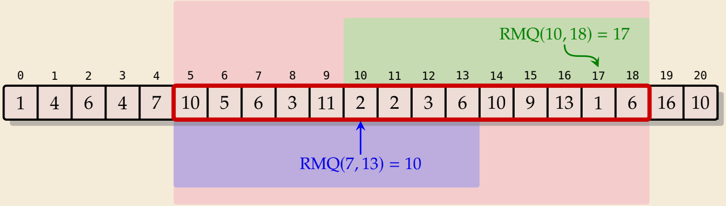


1. Find $k$ with $\ell/2 \le 2^k \le \ell$
2. Cover range $[i..j]$ by
   $2^k$ positions right from $i$ and
   $2^k$ positions left from $j$
3. $\mathrm{RMQ}(i,j) =$
   $\arg\min\{A[rmq_1], A[rmq_2]\}$

   with $rmq_1 = \mathrm{RMQ}(i, i + 2^k - 1)$
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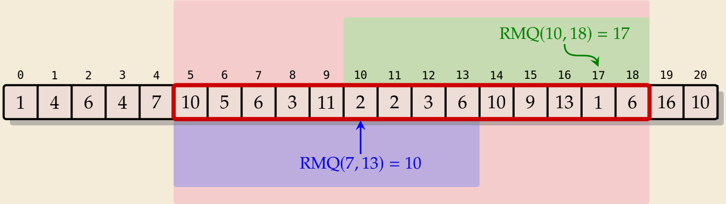


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- Preprocessing can be done in $O(n \log n)$ times

$\leadsto \langle O(n \log n), O(1) \rangle$ time solution!

## 9.4 Cartesian Trees

# RMQ & LCA

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
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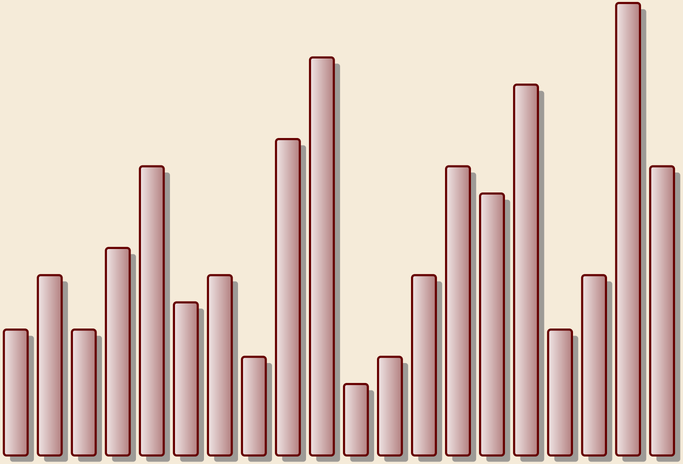

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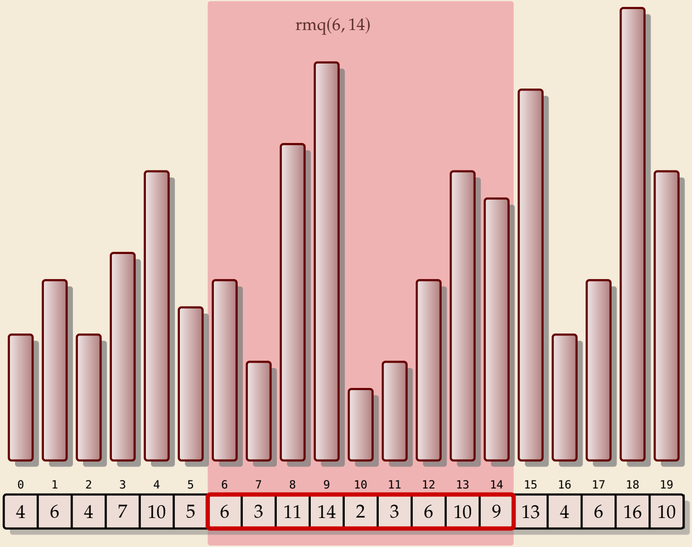
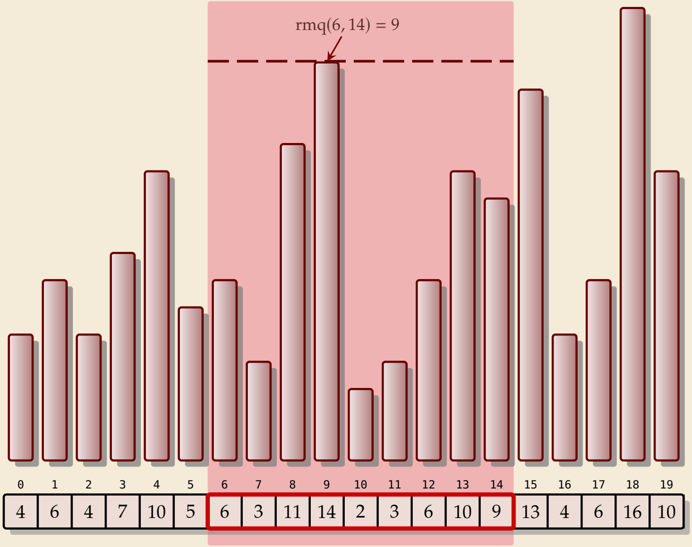



- **Range-max queries** on array $A$:
  $$\mathrm{rmq}_A(i, j) = \underset{i \le k \le j}{\arg\max}\, A[k]$$
  $$= \textit{index} \text{ of max}$$

# RMQ & LCA

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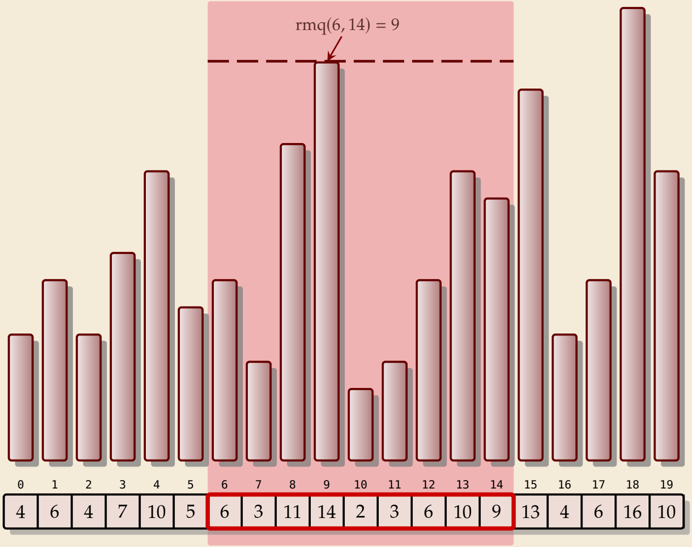


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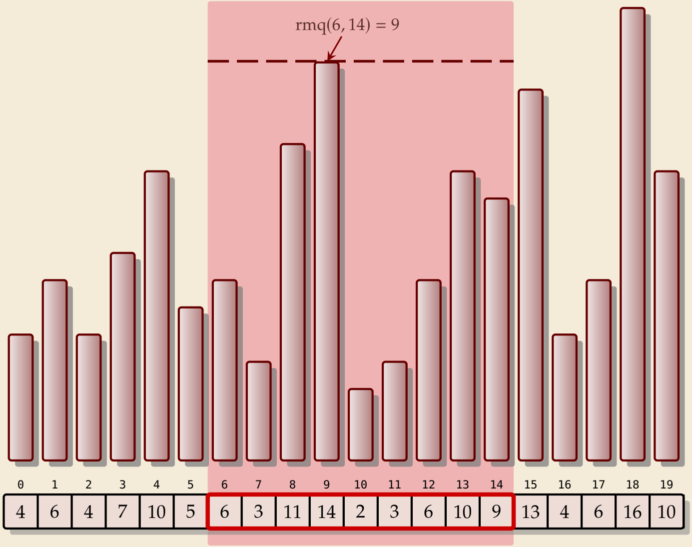


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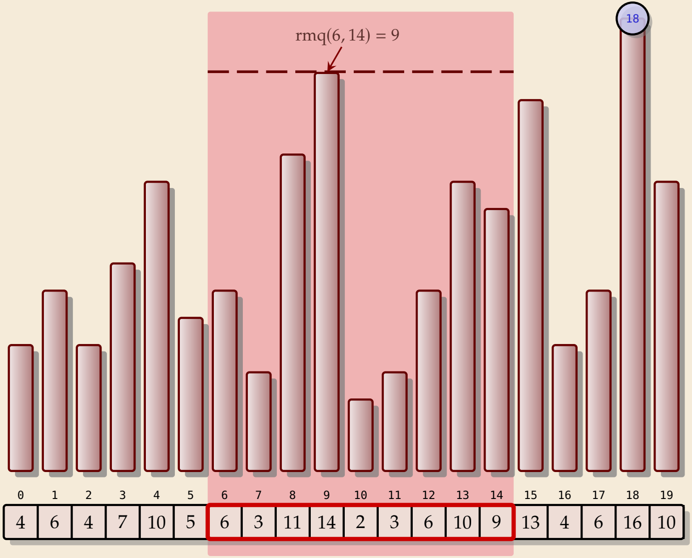
# RMQ & LCA

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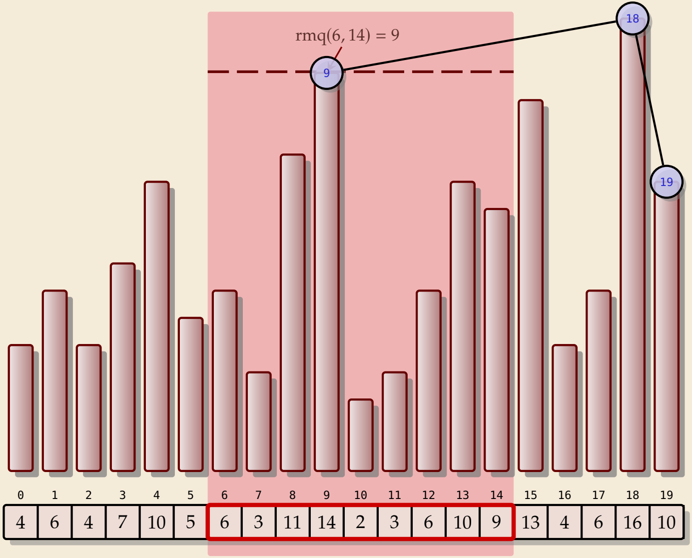


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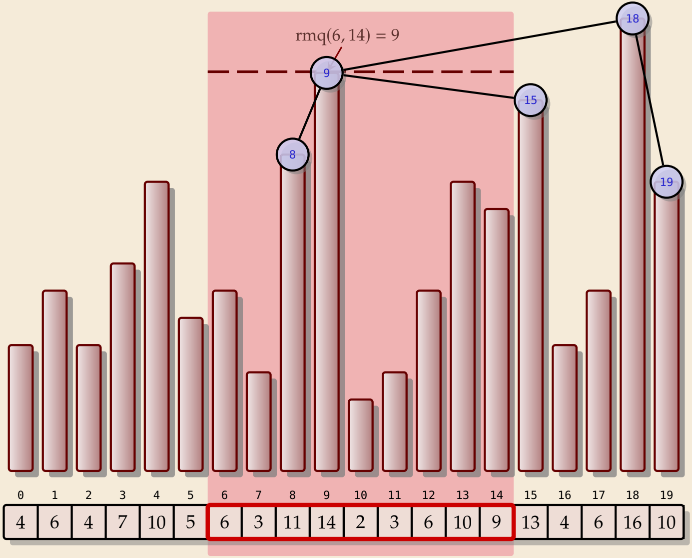


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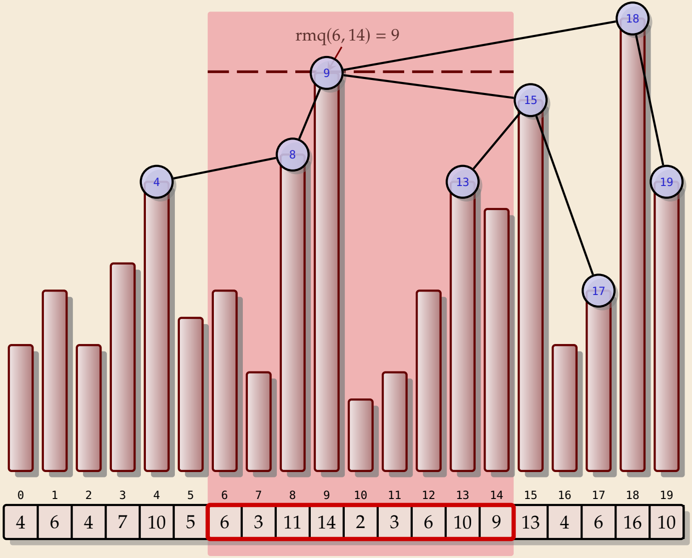


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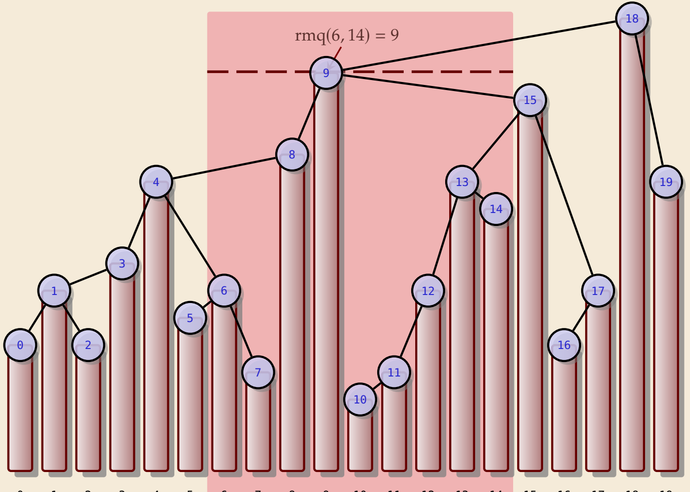


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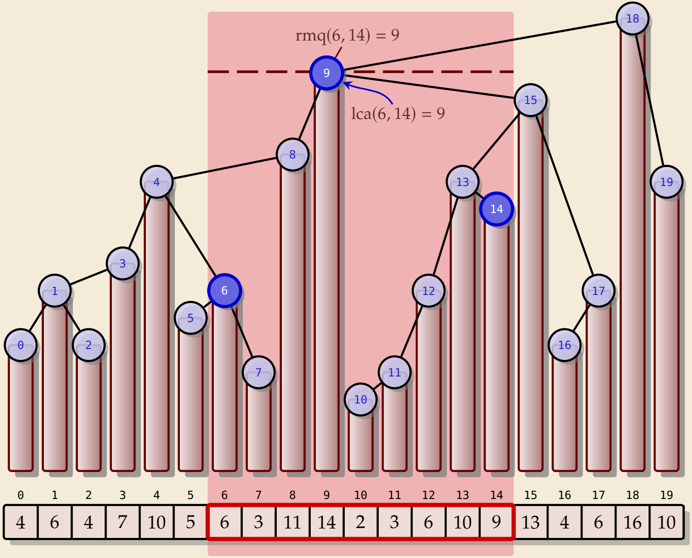


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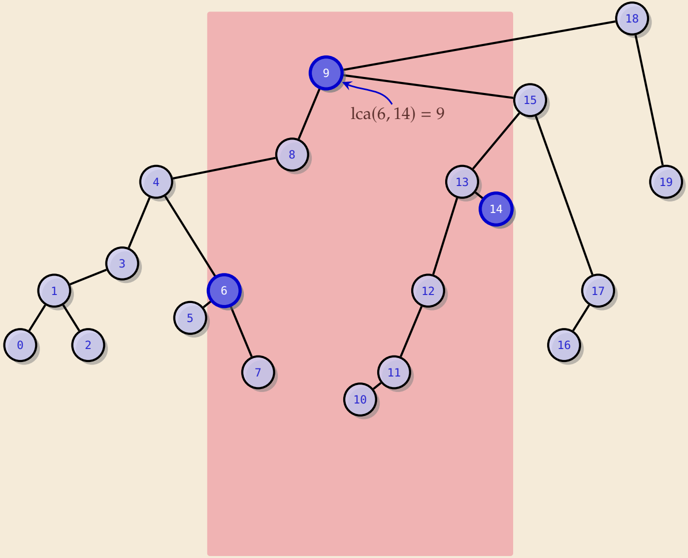
- $\mathrm{rmq}(i, j) =$
  **lowest common ancestor** (LCA)

# RMQ & LCA

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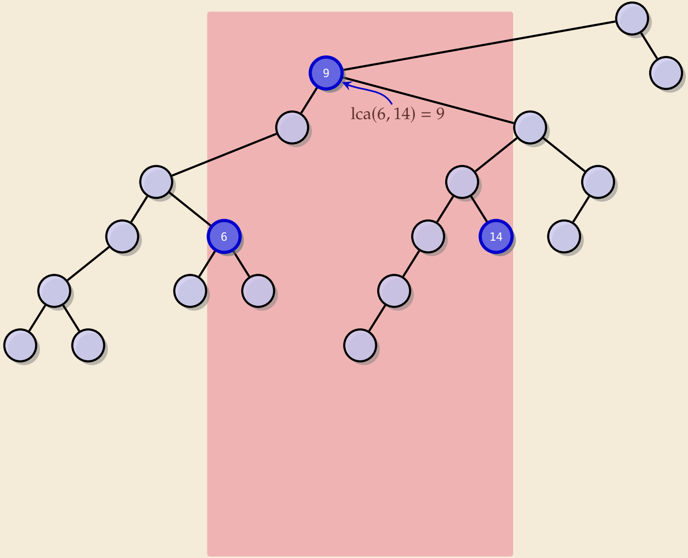
# RMQ & LCA

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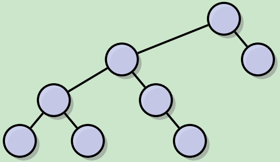
- $\mathrm{rmq}(i, j)$ = inorder of
  **lowest common ancestor** (LCA)
  of $i$th and $j$th node in inorder

## Clicker Question



Given the (max-oriented) Cartesian tree for $A$ on the left, what is $\mathrm{RMQ}_A(1,5)$?

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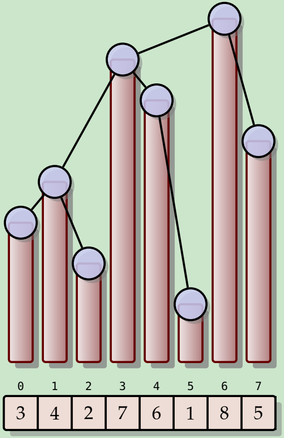


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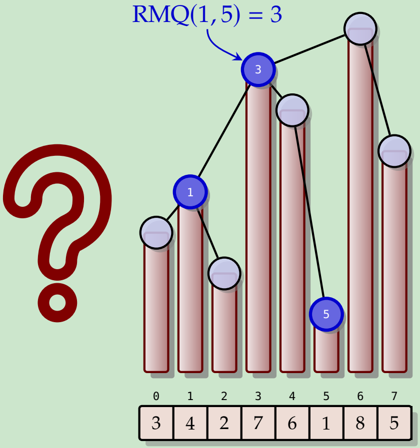


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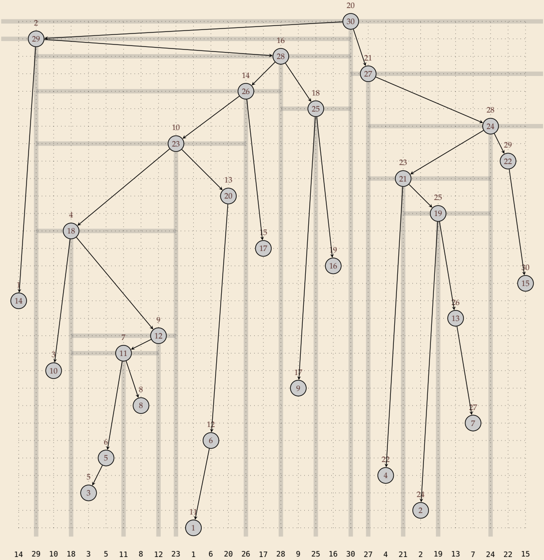
# Cartesian Tree – Larger Example



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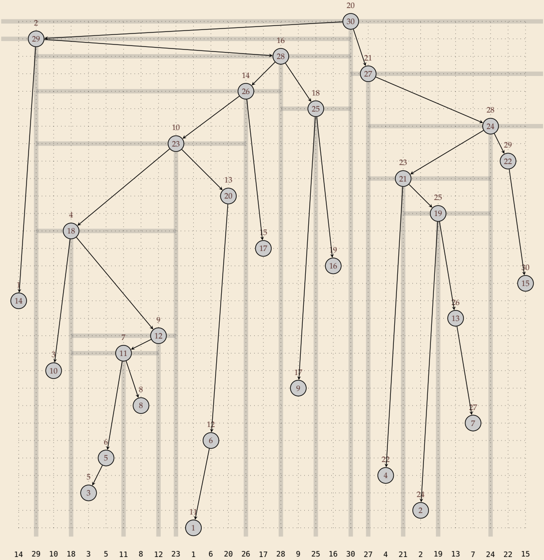


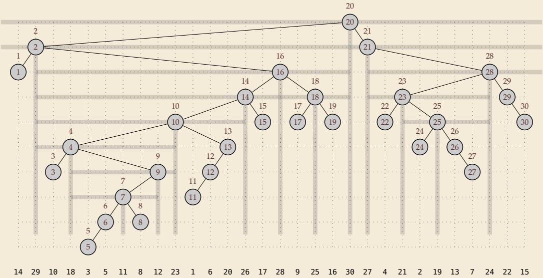

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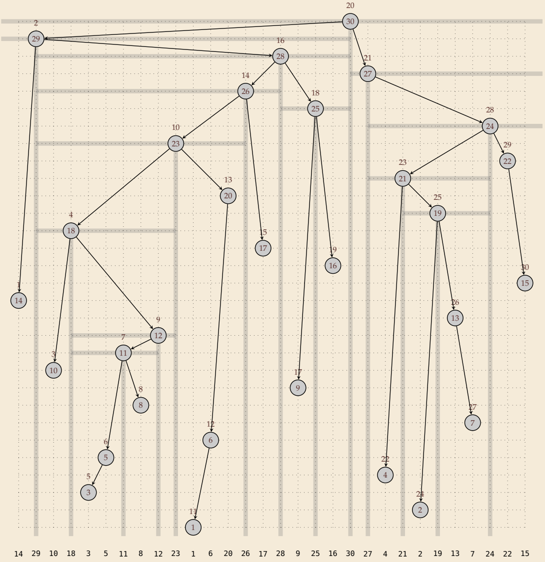
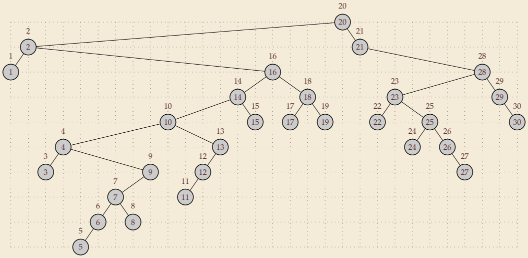
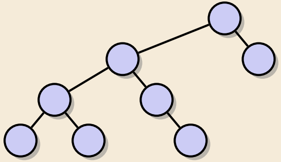


# Counting binary trees



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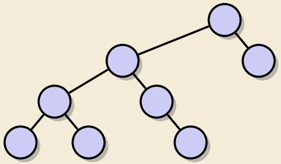


- Given the Cartesian tree,
  all RMQ answers are determined

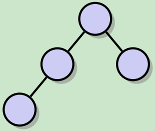
  and vice versa!

- How many different Cartesian trees are there for arrays of length $n$?
  - known result: *Catalan numbers* $\dfrac{1}{n+1}\dbinom{2n}{n}$
  - easy to see: $\le 2^{2n}$

$\rightsquigarrow$ many arrays will give rise to the same Cartesian tree

*Can we exploit that?*

# Clicker Question



What binary string corresponds to the tree shown on the left?

(using the encoding just discussed)

→ sli.do/comp526

## 9.5 Exhaustive Tabulation

# Four Russians?

The exhaustive-tabulation technique to follow is often called “Four Russians trick” . . .

- The algorithmic technique was published 1970 by
  V. L. Arlazarov, E. A. Dinitz, M. A. Kronrod, and I. A. Faradžev
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- American authors coined the slightly derogatory “Method of Four Russians”
  . . . name in widespread use

# Bootstrapping

- We know a $\langle O(n \log n), O(1) \rangle$ time solution
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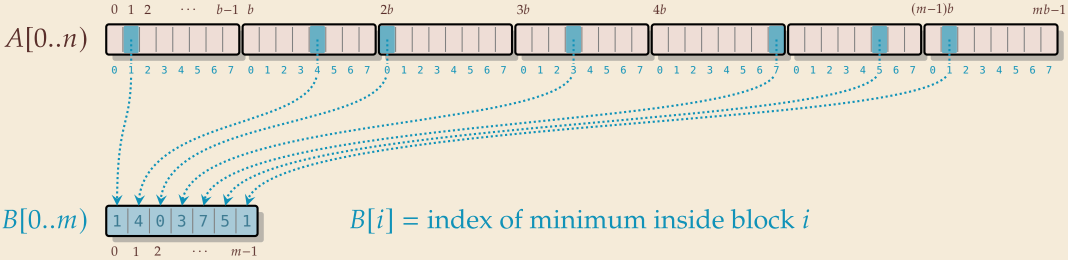

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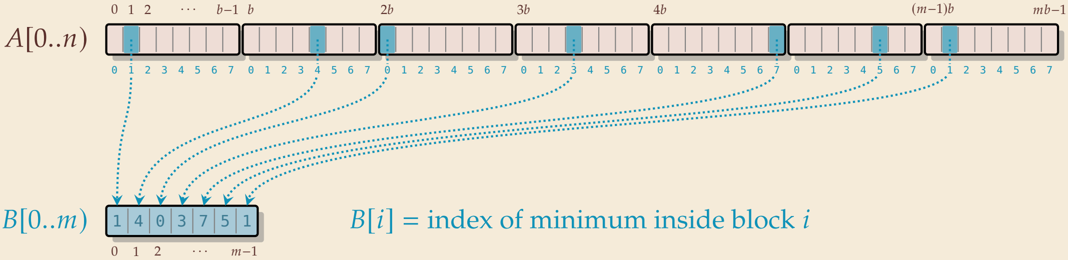


$\rightsquigarrow$ Use sparse tables for $B$.

$\rightsquigarrow$ Can solve RMQs in $B[0..m)$ in $\langle O(n), O(1) \rangle$ time

# Query decomposition

- Query $\mathrm{RMQ}_A(i, j)$ covers
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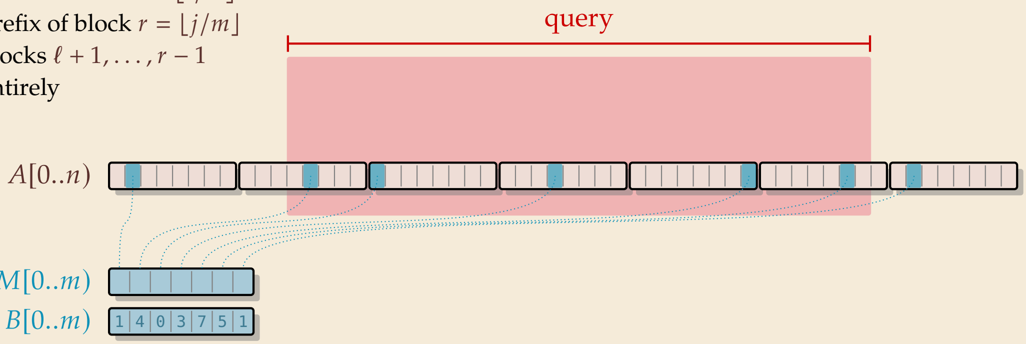

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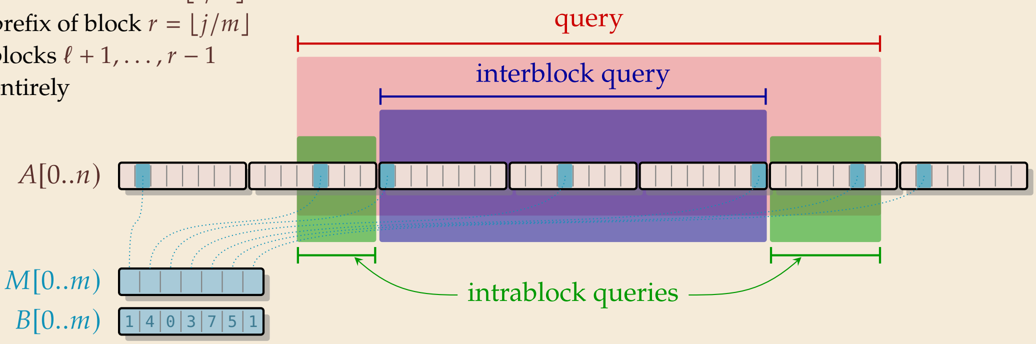

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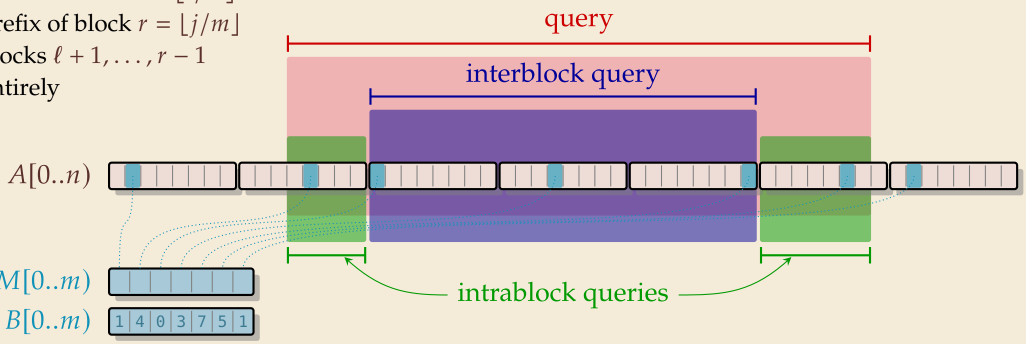


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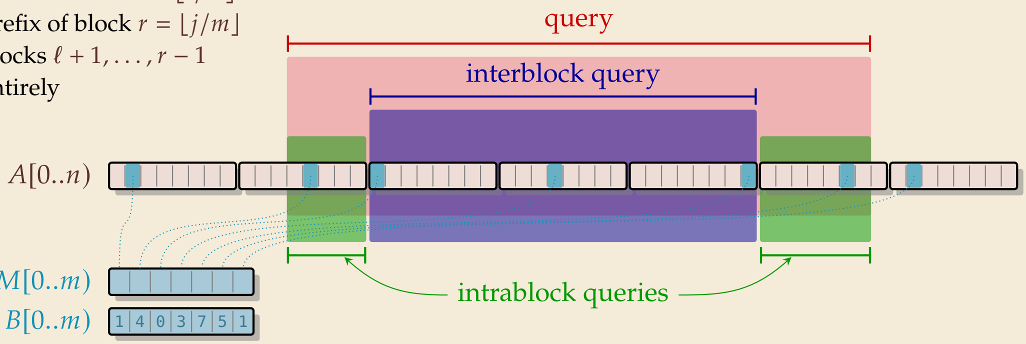


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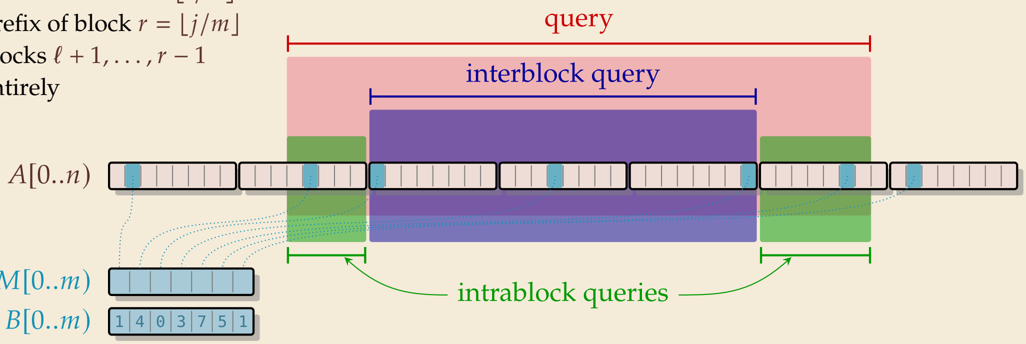


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$\rightsquigarrow$ ***Exhaustive Tabulation Technique:***

1. represent each subproblem by storing its *type* (here: encoding of Cartesian tree)
2. *enumerate* all possible subproblem types and their solutions
3. use type as index in a large *lookup table*

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|---|---|---|---|
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| $\vdots$ | | | |

- $\le \sqrt{n}$ block types
- $\le b^2$ combinations for $i$ and $j$

$\rightsquigarrow$ $\Theta\left(\sqrt{n} \cdot \log^2 n\right)$ rows

- each row can be computed in $O(\log n)$ time

$\rightsquigarrow$ overall preprocessing: $O(n)$ time!

# Discussion

- $\langle O(n), O(1) \rangle$ time solution for RMQ

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**Research questions:**

- Reduce the space usage
- Avoid access to $A$ at query time