10 Parallel Algorithms

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Learning Outcomes

Unit 10: ***Parallel Algorithms***

1. Know and apply *parallelization strategies* for embarrassingly parallel problems.
2. Identify *limits of parallel speedups*.
3. Understand and use the *parallel random-access-machine* model in its different variants.
4. Be able to *analyze* and compare simple shared-memory parallel algorithms by determining *parallel time and work*.
5. Understand efficient parallel *prefix sum* algorithms.
6. Be able to devise high-level description of *parallel quicksort and mergesort* methods.

Outline

10 Parallel Algorithms

10.1 Parallel Computation
10.2 Parallel String Matching
10.3 Parallel Primitives
10.4 Parallel Sorting

10.1 Parallel Computation

Types of parallel computation

£££ can't buy you more time . . . but more computers!

$\rightsquigarrow$ Challenge: Algorithms for *parallel* computation.

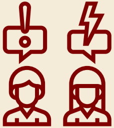
There are two main forms of parallelism:

1. **shared-memory parallel computer** ← *focus of today*
   - p *processing elements* (PEs, processors) working in parallel
   - **single** big memory, **accessible from every PE**
   - communication via shared memory
   - think: a big server, 128 CPU cores, terabyte of main memory
2. **distributed computing**
   - p PEs working in parallel
   - each PE has **private** memory
   - communication by sending **messages** via a network
   - think: a cluster of individual machines

PRAM – Parallel RAM

- extension of the RAM model (recall Unit 2)
- the p PEs are identified by ids $0, \ldots, p-1$
  - like w (the word size), p is a parameter of the model that can grow with n
  - $p = \Theta(n)$ is not unusual   maaany processors!
- the PEs all **independently** run the same RAM-style program (they can use their id there)
- each PE has its own registers, but `MEM` is shared among all PEs
- computation runs in **synchronous** steps: in each time step, every PE executes one instruction
- As for RAM:
  - assume a basic "operating system"
  - $\rightsquigarrow$ write algorithms in pseudocode instead of RAM assembly
  - **NEW:** loops and commands can be run **"in parallel"** (examples coming up)

PRAM – Conflict management



Problem: What if several PEs simultaneously overwrite a memory cell?

- **EREW**-PRAM (exclusive read, exclusive write)
  any **parallel access** to same memory cell is **forbidden** (crash if happens)
- **CREW**-PRAM (concurrent read, exclusive write)
  parallel **write** access to same memory cell is **forbidden**, *but reading is fine*
- **CRCW**-PRAM (concurrent read, concurrent write)
  concurrent access is allowed,
  need a rule for write conflicts:
  - common CRCW-PRAM:
    all concurrent writes to same cell must write **same** value
  - arbitrary CRCW-PRAM:
    some unspecified concurrent write wins
  - (more exist . . .)
- no single model is always adequate, but our default is CREW

PRAM – Execution costs

Cost metrics in PRAMs

- **space:** total amount of accessed memory
- **time:** number of steps till all PEs finish assuming sufficiently many PEs!
  sometimes called *depth* or *span*
- **work:** total #instructions executed on **all** PEs

Holy grail of PRAM algorithms:

- minimal time (=span)
- work (asymptotically) no worse than running time of best sequential algorithm
  ⇝ *"work-efficient"* algorithm: work in same Θ-class as best sequential

The number of processors

Hold on, my computer does not have $\Theta(n)$ processors! Why should I care for span and work!?

Theorem 10.1 (Brent's Theorem)

If an algorithm has span T and work W (for an arbitrarily large number of processors), it can be run on a PRAM with p PEs in time $O(T + \frac{W}{p})$ (and using $O(W)$ work). ◀

Proof: schedule parallel steps in round-robin fashion on the p PEs.

$\rightsquigarrow$ span and work give guideline for *any* number of processors

10.2 Parallel String Matching

Embarrassingly Parallel

- A problem is called *"embarrassingly parallel"*
  if it can immediately be split into *many, small subtasks*
  that can be solved completely *independently* of each other
- Typical example: sum of two large matrices (all entries independent)

$\rightsquigarrow$ best case for parallel computation (simply assign each processor one subtask)

- Sorting is not embarrassingly parallel
  - no obvious way to define many *small* (= efficiently solvable) subproblems
  - but: some subtasks of our algorithms are (stay tuned ...)

Parallel string matching – Easy?

- We have seen a plethora of string matching methods in Unit 6
- But all efficient methods seem inherently sequential
  Indeed, they became efficient only after building on knowledge from previous steps!

Sounds like the *opposite* of parallel!

$\rightsquigarrow$ How well can we parallelize string matching?

Here: string matching = find *all* occurrences of P in T (more natural problem for parallel)
always assume $m \le n$

Subproblems in string matching:

- string matching = check all guesses $i = 0, \ldots, n - m - 1$
- checking one guess is a subtask!

Parallel string matching – Brute force

- Check all guesses in parallel

```
procedure parallelBruteForce(T[0..n), P[0..m)):
    for i := 0, ..., n − m − 1 do in parallel  ← only difference to normal brute force!
        for j := 0, ..., m − 1 do
            if T[i + j] ≠ P[j] then break inner loop
        if j == m then report match at i
    end parallel for
```

- PE k is executing the loop iteration where $i = k$.

  $\rightsquigarrow$ requires that all iterations can be done **independently**!

  - Different PEs work **in lockstep** (synchronized after each instruction)
  - similar to OpenMP `#pragma omp parallel for`

- checking whether *no* match was found by *any* PE more effort $\rightsquigarrow$ ... stay tuned

$\rightsquigarrow$ **Time**: $\Theta(m)$ using sequential checks
$\Theta(\log m)$ on CREW-PRAM ($\rightsquigarrow$ tutorials)
$\Theta(1)$ on CRCW-PRAM ($\rightsquigarrow$ tutorials)

Work: $\Theta((n - m)m)$ $\rightsquigarrow$ not great
... much more than best sequential

Parallel string matching – Blocking

Divide T into **overlapping** blocks of $2m-1$ characters:
$T[0..2m-1)$, $T[m..3m-1)$, $T[2m..4m-1)$, $T[3m..5m-1)\ldots$

- Search all blocks in parallel, each using efficient *sequential* method

```
procedure blockingStringMatching(T[0..n), P[0..m)):
    for b := 0,...,⌈n/m⌉ do in parallel
        result := KMP(T[bm .. (b+1)m − 1), P)
        if result ≠ NO_MATCH then report match at result
    end parallel for
```

⇝ **Time**:

- loop body has text of length $n' = 2m-1$ and pattern of length m

⇝ KPM runtime $\Theta(n'+m) = \Theta(m)$

⇝ **Work**: $\Theta(\frac{n}{m}\cdot m) = \Theta(n)$ ⇝ work efficient!

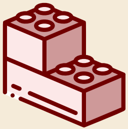
Parallel string matching – Discussion

- very simple methods
- could even run distributed with access to part of T
- parallel speedup only for $m \ll n$

- work-efficient methods with better parallel time possible?
  - $\rightsquigarrow$ must genuinely parallelize the matching process! (and the preprocessing of the pattern)
  - $\rightsquigarrow$ needs new ideas (much more complicated, but possible!)
- **Parallel string matching – State of the art:**
  - $O(\log m)$ time & work-efficient parallel string matching (very complicated)
    - this is optimal for CREW-PRAM
  - on CRCW-PRAM: matching part even in $O(1)$ time (easy)
    but preprocessing requires $\Theta(\log \log m)$ time (very complicated)

10.3 Parallel Primitives

Building blocks



- Most nontrivial problems need tricks to be parallelized
- Some versatile building blocks are known that help in many problems

$\rightsquigarrow$ We study some of them now, before we apply them to *parallel sorting*

The following problems might not look natural at first sight . . . but turn out to be good abstractions.
$\rightsquigarrow$ *bear with me*

Prefix sums

Prefix-sum problem (also: cumulative sums, running totals)

- Given: array $A[0..n)$ of numbers
- Goal: compute all prefix sums $A[0] + \cdots + A[i]$ for $i = 0, \ldots, n-1$
  may be done "in-place", i. e., by overwriting A

Example:

input:	3	0	0	5	7	0	0	2	0	0	0	4	0	8	0	1
output:	3	3	3	8	15	15	15	17	17	17	17	21	21	29	29	30

Prefix sums – Sequential

- sequential solution does $n-1$ additions
- but: cannot parallelize them!
  ϟ data dependencies!

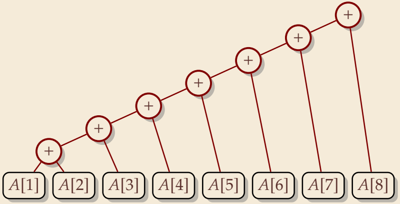
⇝ need a different approach

```
procedure prefixSum(A[0..n)):
    for i := 1, ..., n − 1 do
        A[i] := A[i − 1] + A[i]
```

Let's try a simpler problem first.

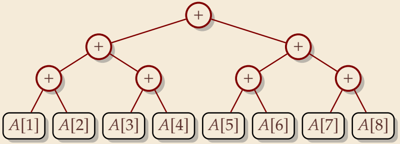
Excursion: Sum

- Given: array $A[0..n)$ of numbers
- Goal: compute $A[0] + A[1] + \cdots + A[n-1]$
  (solved by prefix sums)



Any algorithm *must* do $n-1$ binary additions

⇝ Height of tree = parallel time!



Parallel prefix sums

- Idea: Compute all prefix sums with balanced trees in parallel
  Remember partial results for reuse

input:	3	0	0	5	7	0	0	2	0	0	0	4	0	8	0	1
round 1:	3	3	0	5	12	7	0	2	2	0	0	4	4	8	8	1
round 2:	3	3	3	8	12	12	12	9	2	2	2	4	4	12	12	9
round 3:	3	3	3	8	15	15	15	17	14	14	14	13	6	14	14	13
round 4:	3	3	3	8	15	15	15	17	17	17	17	21	21	29	$\boxed{29}$	30

Parallel prefix sums – Code

- can be realized in-place (overwriting A)
- assumption: in each parallel step, all reads precede all writes

```
procedure parallelPrefixSums(A[0..n)):
    for r := 1, ... ⌈lg n⌉ do
        step := 2^(r-1)
        for i := step, ... n - 1 do in parallel
            x := A[i] + A[i - step]
            A[i] := x
        end parallel for
    end for
```

Parallel prefix sums – Analysis

- **Time:**
  - all additions of one round run in parallel
  - $\lceil \lg n \rceil$ rounds
  - $\rightsquigarrow$ $\Theta(\log n)$ time   best possible!
- **Work:**
  - $\geq \frac{n}{2}$ additions in all rounds (except maybe last round)
  - $\rightsquigarrow$ $\Theta(n \log n)$ work
  - more than the $\Theta(n)$ sequential algorithm!

- Typical trade-off: greater parallelism at the expense of more overall work

- For prefix sums:
  - can actually get $\Theta(n)$ work in *twice* that time!
  - $\rightsquigarrow$ algorithm is slightly more complicated
  - instead here: linear work in *thrice* the time using "blocking trick"

Work-efficient parallel prefix sums

recall string matching!

standard trick to improve work: compute small blocks sequentially

1. Set $b := \lceil \lg n \rceil$
2. For blocks of b consecutive indices, i. e., $A[0..b)$, $A[b..2b)$, ... **do in parallel**:
   - compute local prefix sums with fast **sequential** algorithm
3. Use previous work-inefficient parallel algorithm only on **rightmost elements** of blocks, i. e., to compute prefix sums of $A[b-1]$, $A[2b-1]$, $A[3b-1]$, ...
4. For blocks $A[0..b)$, $A[b..2b)$, ... do in parallel:
   Add block-prefix sums to local prefix sums

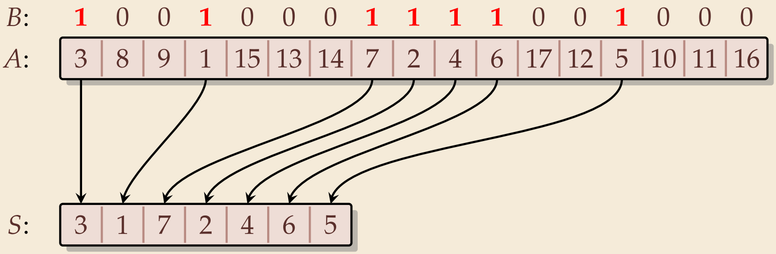
Analysis:

- **Time:**
  - 2. & 4.: $\Theta(b) = \Theta(\log n)$ time
  - 3. $\Theta(\log(n/b)) = \Theta(\log n)$ time
- **Work:**
  - 2. & 4.: $\Theta(b)$ per block $\times \lceil \frac{n}{b} \rceil$ blocks $\rightsquigarrow \Theta(n)$
  - 3. $\Theta\left(\frac{n}{b} \log(\frac{n}{b})\right) = \Theta(n)$

Compacting subsequences

How do prefix sums help with sorting? one more step to go . . .

Goal: *Compact* a subsequence of an array



Use prefix sums on bitvector B

$\rightsquigarrow$ offset of selected cells in S

$C := B$ // *deep copy of* B
parallelPrefixSums(C)
for $j := 0, \ldots, n-1$ **do in parallel**
  if $B[j] == 1$ **then** $S[C[j]-1] := A[j]$
end parallel for

10.4 Parallel Sorting

Parallel Mergesort

- Recursive calls can run in parallel (data independent)!

- how about merging sorted halves $A[l..m)$ and $A[m..r)$?
- Our pointer-based sequential method seems hard to parallelize

$\rightsquigarrow$ Must treat all elements independently.

#elements $\le x$

- correct position of x in sorted output = *rank* of x breaking ties by position in A
- $\#\,\text{elements} \le x \quad = \quad \#\,\text{elements from } A[l..m) \text{ that are} \le x$
  $\qquad\qquad\qquad\qquad + \#\,\text{elements from } A[m..r) \text{ that are} \le x$

- rank in **own run** is simply the **index** of x in that run!
- find rank in **other** run by *binary search*

$\rightsquigarrow$ can move x directly to correct position

Parallel Mergesort – Code

```
procedure parMergesort(A[l..r), buf):
    m := l + ⌊(r − l)/2⌋
    in parallel { parMergesort(A[l..m), buf), parMergesort(A[m..r), buf) }
    parallelMerge(A[l..m), A[m..r), buf)
    for i = l, ..., r − 1 do in parallel // copy back in parallel
        A[i] := buf[i]
    end parallel for

procedure parallelMerge(A[l..m), A[m..r), buf):
    for i = l, ..., m − 1 do in parallel
        r := (i − l) + binarySearch(A[m..r), A[i])  // binarySearch(A, x) returns #elements < x in A
        buf[r] = A[i]
    end parallel for
    for j = m, ..., r − 1 do in parallel
        r := binarySearch(A[l..m), A[j]) + (j − m)
        buf[r] = A[j]
    end parallel for
```

Parallel mergesort – Analysis

- **Time:**
  - merge: $\Theta(\log n)$ from binary search, rest $O(1)$
  - mergesort: depth of recursion tree is $\Theta(\log n)$

  $\rightsquigarrow$ total time $O(\log^2(n))$
- **Work:**
  - merge: n binary searches $\rightsquigarrow$ $\Theta(n \log n)$

  $\rightsquigarrow$ mergesort: $O(n \log^2(n))$ work

  - work can be reduced to $\Theta(n)$ for merge (complicated!)
    - do full binary searches only for regularly sampled elements
    - ranks of remaining elements are sandwiched between sampled ranks
    - use a sequential method for small blocks, treat blocks in parallel
    - (details omitted)

Parallel Quicksort

Let's try to parallelize Quicksort

- As for Mergesort, recursive calls can run in parallel ✓
- our sequential partitioning algorithm seems hard to parallelize
- but can split partitioning into *phases*:
  1. **comparisons:** compare all elements to pivot (in parallel), store result in bitvectors
  2. compute prefix sums of bit vectors (in parallel as above)
  3. **compact** subsequences of small and large elements (in parallel as above)

Parallel Quicksort – Code

```
procedure parQuicksort(A[l..r)):
    b := choosePivot(A[l..r))
    j := parallelPartition(A[l..r), b)
    in parallel { parQuicksort(A[l..j)), parQuicksort(A[j + 1..r)) }

procedure parallelPartition(A[0..n), b):
    swap(A[n − 1], A[b]);  p := A[n − 1]
    for i = 0, . . . , n − 2 do in parallel
        S[i] := [A[i] ≤ p]     // S[i] is 1 or 0
        L[i] := 1 − S[i]
    end parallel for
    in parallel { parallelPrefixSum(S[0..n − 2]);  parallelPrefixSum(L[0..n − 2]) }
    j := S[n − 2] + 1
    for i = 0, . . . , n − 2 do in parallel
        x := A[i]
        if x ≤ p then A[S[i] − 1] := x
        else A[j + L[i]] := x
    end parallel for
    A[j] := p
    return j
```

Parallel Quicksort – Analysis

- **Time:**
  - partition: all $O(1)$ time except prefix sums $\rightsquigarrow$ $\Theta(\log n)$ time
  - Quicksort: expected depth of recursion tree is $\Theta(\log n)$

  $\rightsquigarrow$ total time $O(\log^2(n))$ in expectation
- **Work:**
  - partition: $O(n)$ time except prefix sums $\rightsquigarrow$ $\Theta(n)$ work (with work-efficient prefix-sums algorithm)

  $\rightsquigarrow$ Quicksort $O(n \log(n))$ work in expectation
  - (expected) work-efficient parallel sorting!

Parallel sorting – State of the art

- more sophisticated methods can sort in $O(\log n)$ parallel time on CREW-PRAM (very complicated algorithm based on parallel mergesort with interleaved merges)
- practical challenge: small units of work add overhead
- need a lot of PEs to see improvement from $O(\log n)$ parallel time

$\rightsquigarrow$ implementations tend to use simpler methods above

- check the Java library sources for interesting examples! `java.util.Arrays.parallelSort(int[])`