The Balanced Pair Framework

A minimal outline of how to decompose any multiset of ± 1 tokens into *balanced pairs* and *leftovers*, with notation for measuring deviations and imbalances.

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1. Basic Objects

- **Multiset** $e \subset \{+1, -1\}^\infty$ - **Base pair**: $e_0 = \{+1, -1\}$ - **Singletons**: $e_{+1} = \{+1\}$, $e_{-1} = \{-1\}$

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2. Operators

1. **Forcing +1**:

$$w^+(e) \;=\; \bigl(e \,\cup\, e_0\bigr) \;\setminus\; e_{-1} \quad\Longrightarrow\quad \text{ensure } +1 \text{ is in, } -1 \text{ is out.}$$

2. **Forcing -1**:

$$w^-(e) \;=\; \bigl(e \,\cup\, e_0\bigr) \;\setminus\; e_{+1} \quad\Longrightarrow\quad \text{ensure } -1 \text{ is in, } +1 \text{ is out.}$$

3. **Removing matched pairs**:

$$w^{\%}(e) \;=\; e \;\setminus\; \bigl(e_0^k\bigr), \quad \text{where } e_0^k \text{ is } k \text{ copies of } \{+1, -1\}.$$

This *leftover* is the portion of e that cannot form ± 1 pairs.

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3. Principal Leftover Parts

- **Principal part**:

$$P(e) \;=\; e \;\setminus\; w^{\%}(e), \quad \text{(the portion of } e \text{ that *does* match into pairs).}$$

- **Number of pairs**:

$$n \;=\; \frac{|P(e)|}{2}.$$

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4. Imbalance Measure

Define

$$z(e) \;=\; \frac{|P(e)|_{(+1)} - |P(e)|_{(-1)}}{2^n}.$$

- $|P(e)|_{(+1)}$ = count of $+1$ in the principal part - $|P(e)|_{(-1)}$ = count of -1 in the principal part - $z(e)$ measures how *unbalanced* the *paired* portion is.

A related statement often given:

$$z(e) \;\subset\; \Bigl[\tfrac{1}{2^n},\, \tfrac{2^n - 1}{2n}\Bigr],$$

indicating bounds on the imbalance ratio.

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5. Quick Examples

Example A - Let $e = \{+1, -1, +1\}$. - **Matched portion**: one pair $\{+1, -1\}$ leaves a leftover $\{+1\}$.

$$w^{\%}(e) = \{+1\}, \quad P(e) = \{+1, -1\}.$$

- Then $|P(e)| = 2 \implies n = 1$. - $|P(e)|_{(+1)} = 1$, $|P(e)|_{(-1)} = 1$.

$$z(e) = \frac{1 - 1}{2^1} = 0.$$

Example B - Applying w^+ to $\{-1\}$:

$$w^+(\{-1\}) = \bigl(\{-1\} \cup \{+1, -1\}\bigr) \setminus \{-1\} = \{+1\}.$$

Forces the final set to $\{+1\}$ by "adding a pair then removing -1."

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In essence: any multiset of $\{+1, -1\}$ can be uniquely split into a "principal" (fully pairable) part and a "leftover" (unmatchable) part, with an imbalance ratio $z(e)$ quantifying how the matched portion leans toward +1 or -1. All of these operations and measures follow from the classical logical axioms (identity, non-contradiction, excluded middle).