Advanced Portfolio Theory: From Value at Risk to Merton's Framework

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Chapter 1

Introduction

1.1 Formulation of the Portfolio Management Problem

This course focuses on portfolio management involving mostly financial assets, with the possible inclusion of insurance products. It covers a broad spectrum of instruments: basic equity products such as spot stocks and futures, fixed income instruments including bonds and interest rate swaps, as well as commodity futures on metals and agricultural goods. The course also considers derivative products like stock options, interest rate derivatives, swaptions, and index-based derivatives, along with insurance and reinsurance contracts. The objectives are threefold: to provide a solid foundation in *Value-at-Risk (VaR)*, to introduce a unified portfolio management framework that incorporates all these asset classes through *utility maximization*, and to explore classical models such as the *Merton portfolio*, constant proportion strategies, and the *Mutual Fund Theorems*.

In all settings whether in discrete or continuous time the portfolio selection problem follows a common structure. Over time, the investor observes cash flows F_t, and allocates wealth across a set of N assets through a strategy $\theta_t = (\theta_t^1, \dots, \theta_t^N)$. The strategy evolves across a timeline that typically begins before time zero and continues until a terminal time T, with a possible final cash flow occurring at $T + \Delta$. The portfolio is rebalanced at discrete times $t = -2, -1, 0, \dots, T$, with each θ_t representing the chosen allocation at that point. After the last rebalancing at time T, the portfolio may still generate residual flows, leading to a *run-off* phase driven by underwriting or other long-term commitments. The overall objective is to determine the optimal sequence $\theta_0, \dots, \theta_T$ that maximizes utility or meets a given financial objective across this horizon.

In typical portfolio optimization problems, the main objective is to maximize the final discounted wealth $V_{T+T'}^{\text{disc}} = \sum_{t=1}^{T+T'} F_t^{\text{disc}}$, often through the expected utility $\mathbb{E}[U(V_{T+T'}^{\text{disc}})]$, where U is a utility function.
This optimization is subject to various constraints, such as:

- **Risk constraints:** the variance, semi-variance, Value-at-Risk, or other convex risk measures of $V_{T+T'}^{\text{disc}}$ should remain small;
- **Solvency constraints:** ensuring that the probability of solvency at terminal time is sufficiently high, e.g.,
$$\mathbb{P}(V_{T+T'}^{\text{disc}} + E \geq 0) \geq 1 - \varepsilon;$$

- **Return constraints:** lower bounds on expected returns, such as a minimum Return on Equity (e.g., ROE $\geq 10\%$);
- **Operational constraints:** such as prohibiting short positions or meeting regulatory requirements.

These constraints may be incorporated directly into the utility function. For example, in Markowitz's framework, one uses a quadratic utility function of the form $U(x) = -\frac{1}{2}x^2 + ax$ to penalize large deviations and implicitly control risk.

In quantitative portfolio management, it is essential to rely on mathematical models that enable a quantified trade-off between risk and return. The optimization problem can be formulated as follows: given an initial wealth $K > x$ at time $t = 0$, the goal is to find a self-financing portfolio $\hat{\theta}$ that maximizes the expected utility of the terminal wealth:

$$\sup_{\theta \text{ self-financing}, \ V_0(\theta)=K} \mathbb{E}[U(V_T(\theta))] = \mathbb{E}[U(V_T(\hat{\theta}))]$$

where U denotes a utility function. The output of the problem is a dynamic investment strategy represented by a sequence $\theta_0, \theta_1, \ldots, \theta_T$, where each θ_t is a $\mathcal{F}_t$-measurable random vector, reflecting the available information at time t.
In more complex settings involving multiple initial endowments $K_1, K_2, \ldots$, the problem also includes the allocation of equity across several portfolios $\theta^{(1)}, \theta^{(2)}, \ldots$, as well as the definition of a dividend strategy $D_1, D_2, \ldots$.

Chapter 2

Value at Risk

2.1 Motivations for VaR and other Risk Measures

Several major financial disasters in recent decades illustrate the dangers associated with the misuse or mismanagement of derivative products. One notable case is Metallgesellschaft (MG), which incurred a loss of approximately 1.3 billion USD due to a mismatch between long-term oil forward contracts (10 years) and a rolling hedge using short-term (3-month) oil futures. The discrepancy between short-term and long-term prices led to massive margin calls. Another famous case is Barings Bank, which collapsed following speculative positions on Nikkei 225 stock index futures. The trader took a 7 billion USD long position and added a short straddle strategy, effectively betting on the index's stability. When the Nikkei dropped from 19,000 to 17,000 in two months, the resulting 1.5 billion USD loss led to the bank's collapse. A third case involves Long-Term Capital Management (LTCM), a hedge fund founded by Nobel laureates Scholes and Merton. LTCM took massive leveraged positions on price spreads, with notional exposures of several hundred billion USD. The firm ultimately went bankrupt following extreme market movements, such as a sudden 13% drop in the USD/JPY exchange rate, which deviated drastically from assumptions based on a log-normal world.

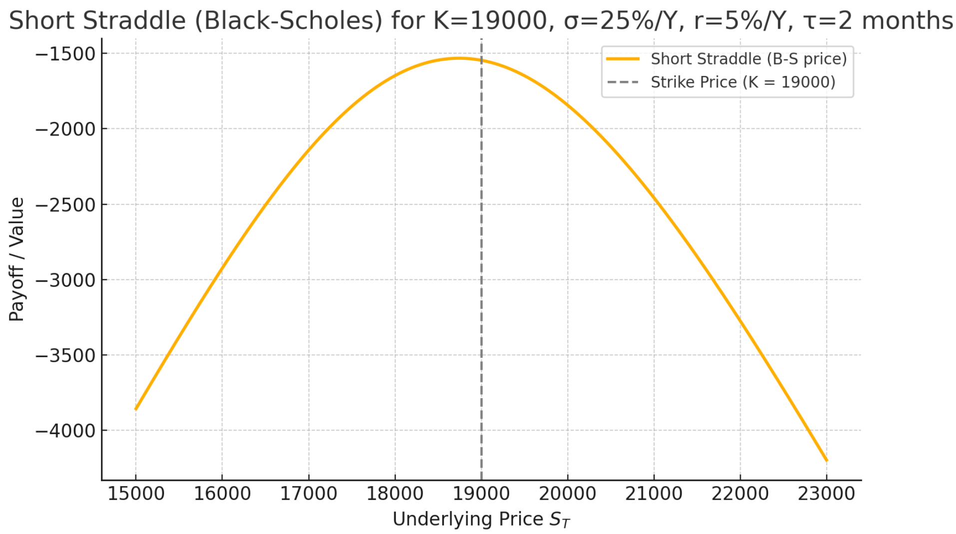

Figure 2.1: Short Straddle – Valeur Black-Scholes pour $K = 19000$, $\sigma = 25\%$ par an, $r = 5\%$ par an, $\tau = 2$ mois

In addition to the risks associated with derivatives, several major financial crises have arisen from other types of risk. Operational risk is illustrated by the case of Crédit Lyonnais, which suffered losses exceeding €10 billion due to a combination of poor management decisions and speculative real estate investments. Interest rate risk was a key factor in the Savings and Loans crisis in the United States, where long-term loans were financed by short-term deposits. When short-term interest rates rose, the cost of funding increased dramatically, leading to losses of around $150 billion. Finally, default risk has been central to numerous banking crises, notably in Japan, where the accumulation of non-performing loans led to estimated losses of around $500 billion. More recently, credit risk has been amplified through structured products such as Credit Default Swaps (CDS) and Collateralized Debt Obligations (CDOs), which played a significant role in the 2008 financial crisis.

In response to growing financial risks and several high-profile failures, both private institutions and regulators have developed frameworks to better assess and manage risk. Among private sector initiatives, the **G30 Group** a consortium of major financial institutions and academic experts published influential recommendations in 1993, advocating for the use of *market-based valuations* and *Value-at-Risk (VaR)* systems to quantify financial exposure. Notably, **J.P. Morgan** developed the widely used *RiskMetrics* methodology, and **Bankers Trust** introduced the *RAROC* (Risk-Adjusted Return on Capital) approach as early as the 1980s to link profitability with capital at risk.

On the regulatory side, the **Basel Committee** introduced the concept of a *Market Risk Charge (MRC)* to be applied on a daily basis. The MRC at time t is defined as:

$$\mathrm{MRC}_t = \max\left(k \cdot \frac{1}{60}\sum_{i=1}^{60} \mathrm{VaR}_{t-i},\ \mathrm{VaR}_{t-1}\right),$$

where k is a regulatory multiplier. This charge is designed to determine the capital buffers required to protect different trading portfolios against market volatility. These measures were formalized and expanded under **Basel II**, which provides a more comprehensive framework for banking supervision and risk-based capital adequacy.

The **Value at Risk** ($\mathrm{VaR}_\alpha(X)$) of a financial position X is defined as the threshold loss such that the probability of incurring a greater loss is at most α. In other words, with probability α, one expects to lose at least $\mathrm{VaR}_\alpha(X)$.

VaR is often favored in practice because it condenses the risk of a position into a single number making it simple to report and communicate. However, this apparent simplicity can be misleading. A major limitation is that VaR provides no information about the magnitude of losses *beyond* the threshold: it tells you how bad things can get with a given probability, but not how much worse they might be if that threshold is crossed.

This motivates the use of more informative risk measures such as the *shortfall distribution*, its moments, and coherent alternatives like the **Average Value at Risk (AVaR)**, which consider the entire tail of the loss distribution. These alternatives can be better aligned with utility-based approaches to risk assessment.

2.2 Definition of Value at Risk (VaR)

Let X be a P&L (Profit and Loss) random variable, i.e.,

$$X = \sum (\text{Discounted future cash flows}) - \text{Price at time } t = 0.$$

Example 2.1
For a time horizon T in the Black-Scholes model:

i) **Stock without dividends:**

$$X = e^{-rT} S_T - S_0$$

ii) **Stock with dividends:**

$$X = e^{-rT} S_T - S_0 + \sum_i e^{-rt_i} D_i$$

where D_i is the dividend paid at time t_i.

iii) **European Call option with strike K:**

$$X = e^{-rT}(S_T - K)^+ - C_0,$$

where C_0 is the price of the call option at time $t = 0$.

Let $F_X : \mathbb{R} \to [0, 1]$ be the cumulative distribution function (CDF) of a real-valued random variable X. By definition, $F_X(x) = \mathbb{P}(X \leq x)$ gives the probability that X takes a value less than or equal to x. It is a fundamental property that F_X is a non-decreasing function that is *cadlag*, meaning it is right-continuous with left limits existing at every point. These properties ensure that F_X captures the probabilistic behavior of X in a mathematically well-structured way.

Example 2.2
For a time horizon T in the Black-Scholes (B-S) model, consider the random variable X representing the profit and loss (P&L) of holding a European Call option with strike K. That is,

$$X = e^{-rT}(S_T - K)^+ - C_0,$$

where C_0 is the premium paid at time $t = 0$, and S_T is the underlying asset price at maturity.
The shape of the probability density function (pdf) f_X reflects the asymmetric nature of option payoffs:

- There is a positive probability mass (i.e., a Dirac peak) at $X = -C_0$, corresponding to the scenario where $S_T \leq K$, and the option expires worthless. In this case, the loss is exactly equal to the premium paid.

- For $S_T > K$, the payoff $(S_T - K)^+$ increases with S_T, so X becomes a strictly increasing function of S_T. Since $\log(S_T)$ is normally distributed in the B-S model, the right tail of f_X (i.e., when $X > -C_0$) is continuous and smooth, with a positively skewed shape.

Overall, the pdf f_X is a combination of a discrete part (Dirac mass at $-C_0$) and a continuous, positively skewed density on $(-C_0, \infty)$.

Definition 2.3 The lower and upper generalized inverses of a cumulative distribution function F_X are defined as follows:

$$F_X^-(y) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq y\}, \tag{2}$$
$$F_X^+(y) = \inf\{x \in \mathbb{R} \mid F_X(x) > y\}. \tag{3}$$

It follows from the right-continuity of F_X that:

$$F_X(F_X^-(y)) = F_X(F_X^+(y)) \geq y. \tag{4}$$

Example 2.4 For a distribution function F_X, we observe the following values:

$$F_X^-(a) = F_X^+(a) = x_a, \quad F_X^-(b) = x_b, \quad F_X^+(b) = y_b,$$

$$F_X^-(c) = F_X^+(c) = F_X^-(d) = F_X^+(d) = x_d.$$

It is important to note that in this example, for $\varepsilon = \pm$, neither of the compositions satisfies the identity:

$$F_X^\varepsilon \circ F_X \neq \mathrm{id}, \quad F_X \circ F_X^\varepsilon \neq \mathrm{id}.$$

The **Value at Risk** $\mathrm{VaR}_\alpha(X)$ represents the smallest amount of capital C that must be invested in a risk-free asset at time $t = 0$, in order to ensure that the probability of the position $X + C$ being strictly negative is no greater than α. In other words, the ruin probability is capped at level α.

Definition 2.5 The Value at Risk at level $\alpha \in]0, 1[$ for the P&L random variable X is defined as:

$$\mathrm{VaR}_\alpha(X) = \inf\left\{y \in \mathbb{R} \mid \mathbb{P}(X + y < 0) \leq \alpha\right\}. \tag{5}$$

In practice, typical confidence levels for α are 0.1%, 1%, or 5%.

Example: Compute $\mathrm{VaR}_\alpha(X)$ for a stock position subject to market risk.

To compute $\mathrm{VaR}_\alpha(X)$, the following result is particularly useful:

Theorem 2.6 For $\alpha \in]0, 1[$, the Value at Risk of X satisfies:

$$\mathrm{VaR}_\alpha(X) = -F_X^+(\alpha),$$

where F_X^+ denotes the upper generalized inverse of the cumulative distribution function of X.

Proof By definition,

$$F_X^+(\alpha) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) > \alpha\} = \sup\{x \in \mathbb{R} \mid \mathbb{P}(X < x) \leq \alpha\}.$$

Therefore,

$$-F_X^+(\alpha) = -\sup\{-y \in \mathbb{R} \mid \mathbb{P}(X < -y) \leq \alpha\} = \inf\{y \in \mathbb{R} \mid \mathbb{P}(X + y < 0) \leq \alpha\}.$$

Remark 2.7 One always has:

$$\mathbb{P}(X + \mathrm{VaR}_\alpha(X) < 0) \leq \alpha \leq \mathbb{P}(X + \mathrm{VaR}_\alpha(X) \leq 0).$$

Example 2.8 *Value at Risk for a profit/loss random variable* $X \sim \mathcal{N}(\mu, \sigma^2)$*:*

i) If $X \sim \mathcal{N}(0,1)$, then:

$$\mathrm{VaR}_\alpha(X) = \begin{cases} 1.645 & \text{if } \alpha = \frac{5}{100} \\ 2.326 & \text{if } \alpha = \frac{1}{100} \\ 3.090 & \text{if } \alpha = \frac{1}{1000} \end{cases}$$

ii) If $X \sim \mathcal{N}(\mu, \sigma^2)$, then by Theorem 2.6:

$$\mathrm{VaR}_\alpha(X) = |\sigma| \cdot \mathrm{VaR}_\alpha(Z) - \mu, \quad \text{where } Z \sim \mathcal{N}(0,1).$$

Thus:

$$\mathrm{VaR}_\alpha(X) = \begin{cases} 1.645|\sigma| - \mu & \text{if } \alpha = \frac{5}{100} \\ 2.326|\sigma| - \mu & \text{if } \alpha = \frac{1}{100} \\ 3.090|\sigma| - \mu & \text{if } \alpha = \frac{1}{1000} \end{cases}$$

iii) **Scaling property for normal models:** Let W_t be a standard Brownian motion and $X_t = \sigma W_t$. Then, since $X_t \sim \mathcal{N}(0, \sigma^2 t)$,

$$\mathrm{VaR}_\alpha(X_t) = \sqrt{t} \cdot |\sigma| \cdot \mathrm{VaR}_\alpha(Z), \quad \text{with } Z \sim \mathcal{N}(0,1).$$

Theorem 2.9 Let f be a strictly increasing and continuous function. Then the Value at Risk satisfies:

$$\mathrm{VaR}_\alpha(f(X)) = -f\big(-\mathrm{VaR}_\alpha(X)\big).$$

Example 2.10

i) If $a \geq 0$ and $b \in \mathbb{R}$, then:

$$\mathrm{VaR}_\alpha(aX + b) = a \cdot \mathrm{VaR}_\alpha(X) - b.$$

This follows directly from Theorem 2.9 when $a > 0$, and the case $a = 0$ is trivial since $aX + b = b$ is constant.

ii) For the exponential transformation:

$$\mathrm{VaR}_\alpha(e^X) = -e^{-\mathrm{VaR}_\alpha(X)}.$$

Example 2.11
You invest, with a time horizon of 1 year, a total of 3 million euros in a portfolio as follows:

- 1 million euros in a risk-free bank account,
- 1 million euros in stock A,
- 1 million euros in stock B.

The interest rate is $r = 5\%$, and at time $t = 0$, both stocks A and B have a price of 100 euros. At time $t = 1$,

- Stock A has a price $X_A \sim \mathcal{N}(100, 25)$,
- Stock B has a price $X_B \sim \mathcal{N}(120, 75)$,

and the two stocks are independent.
Let G be the P&L (profit and loss) of the portfolio at time $t = 1$. Then:

$$G = 0 + 10{,}000\left(\frac{X_A}{1+r} - 100\right) + 10{,}000\left(\frac{X_B}{1+r} - 100\right)$$

Since X_A and X_B are independent and normally distributed, G is also normally distributed:

$$G \sim \mathcal{N}(\mu, \sigma^2),$$

where:

$$\mu = 10{,}000\left(\frac{100}{1.05} - 100 + \frac{120}{1.05} - 100\right) = 10{,}000 \cdot \left(\frac{220}{1.05} - 200\right) \approx 106{,}190.48,$$

$$\sigma^2 = \left(\frac{10{,}000}{1.05}\right)^2 \cdot (25 + 75) = \left(\frac{10{,}000}{1.05}\right)^2 \cdot 100.$$

Thus,

$$\sigma \approx \frac{10{,}000}{1.05} \cdot \sqrt{100} = \frac{10{,}000}{1.05} \cdot 10 \approx 95{,}238.$$

Using the 1% quantile of the standard normal distribution $z_{1\%} \approx 2.326$, we compute:

$$\mathrm{VaR}_{1\%}(G) \approx 2.326 \cdot \sigma - \mu \approx 2.326 \cdot 95{,}238 - 106{,}190.48 \approx 126{,}000 \text{ euros}.$$

Conclusion: With 1% probability, the portfolio incurs a loss of approximately 126,000 euros or more over one year.

2.3 Problems and limitations of VaR

Problem 1: Lack of information beyond the VaR
The Value at Risk $\mathrm{VaR}_\alpha(X)$ only provides a threshold loss that is not exceeded with probability $1 - \alpha$. However, it gives no information about the magnitude of losses that may occur *beyond* this threshold. In other words, once a loss greater than $\mathrm{VaR}_\alpha(X)$ occurs, the VaR says nothing about how large that loss could be. There is theoretically no upper bound, and this limitation makes VaR unsuitable for evaluating tail risk or extreme events. This motivates the use of complementary measures, such as the Conditional Value at Risk (CVaR) or Average VaR, which account for the expected loss *given* that $X \leq -\mathrm{VaR}_\alpha(X)$.

Problem 2: Lack of sub-additivity of VaR

The Value at Risk is not always sub-additive. That is, it may happen that:

$$\mathrm{VaR}_\alpha(X+Y) > \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y). \tag{8}$$

When this inequality is violated, the aggregated portfolio requires more capital than the sum of the capitals required for its components. From the perspective of VaR, this means that diversification may appear *worse* than holding separate sub-portfolios, which contradicts fundamental principles of risk management. This failure of sub-additivity is one of the main criticisms of VaR and motivates the use of coherent risk measures like AVaR.

Example 2.12 (Illustration of Problem 1)
Consider a portfolio in a standard Black-Scholes market composed of L binary options and one European Call option, all written on the same stock with strike K and maturity T. The payoff of one binary option is given by:

$$H(S_T - K), \quad \text{where } H(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Assume a zero interest rate $r = 0$, and let P_0 be the value of the portfolio at time $t = 0$. The profit and loss X at maturity is:

$$X = (S_T - K)^+ + L \cdot H(S_T - K) - P_0.$$

This portfolio is designed to highlight the shortcomings of VaR: the P&L distribution is highly discontinuous due to the binary components. Above a certain loss level (i.e., beyond $\mathrm{VaR}_\alpha(X)$), the potential for large jumps in losses exists. Studying the conditional losses $X \mid X \leq -\mathrm{VaR}_\alpha(X)$ reveals that VaR fails to capture these extreme outcomes.

Example 2.13 (Illustration of Problem 2: Lack of Sub-additivity)
Consider a one-period market with two stocks, A and B, and a zero interest rate ($r = 0$). There are three possible states of the world: $\omega_1, \omega_2, \omega_3$, occurring with probabilities $p, p, 1-2p$, respectively, where $p = 0.006$. At time $t = 0$, both stocks are priced at 56 euros. At the end of the period, their values are:

$$\begin{aligned} S_A(\omega_1) &= 30, & \qquad S_B(\omega_1) &= 63, \\ S_A(\omega_2) &= 63, & \qquad S_B(\omega_2) &= 30, \\ S_A(\omega_3) &= 75, & \qquad S_B(\omega_3) &= 75. \end{aligned}$$

Let us compute the Value at Risk at level $\alpha = 1\%$ for the following positions:

a) **VaR_α for holding one unit of stock A:**

The worst case for stock A is in state ω_1, where the loss is $56 - 30 = 26$. Since $\mathbb{P}(\omega_1) = 0.006 < 0.01$, we must include both ω_1 and ω_2 in the computation. In ω_2, the price is 63, leading to a gain of 7. Thus, the 1%-quantile corresponds to the loss in ω_1, and:

$$\mathrm{VaR}_{1\%}(A) = -7.$$

b) **VaR_α for holding one unit of stock B:**

Similarly, stock B has its worst loss in ω_2, also equal to $56 - 30 = 26$. Since the probabilities are symmetric, we again get:

$$\text{VaR}_{1\%}(B) = -7.$$

c) **VaR_α for the portfolio consisting of one share of A and one share of B:**

The portfolio value at time $t = 0$ is $2 \times 56 = 112$. The end-of-period values are:

$$\begin{aligned}
\omega_1 : \quad & S_A + S_B = 30 + 63 = 93 \Rightarrow \text{Loss} = 112 - 93 = 19, \\
\omega_2 : \quad & S_A + S_B = 63 + 30 = 93 \Rightarrow \text{Loss} = 112 - 93 = 19, \\
\omega_3 : \quad & S_A + S_B = 75 + 75 = 150 \Rightarrow \text{Gain} = 150 - 112 = 38.
\end{aligned}$$

Since $\mathbb{P}(\omega_1) + \mathbb{P}(\omega_2) = 0.012 > 0.01$, we must include both ω_1 and ω_2 to satisfy the 1% risk level. Therefore:

$$\text{VaR}_{1\%}(\text{Portfolio}) = 19.$$

Discussion: In this case, both individual assets have a Value at Risk of 7, but the VaR of the aggregated portfolio is 19. This result violates the *sub-additivity* property, since:

$$\text{VaR}_\alpha(A + B) > \text{VaR}_\alpha(A) + \text{VaR}_\alpha(B).$$

From the perspective of Value at Risk, this implies that diversification appears to *increase* the overall risk, which contradicts the fundamental principle that diversification should reduce risk. This illustrates a well-known limitation of VaR: it is *not a coherent risk measure*, and may provide misleading signals when aggregating portfolios.

Average Value at Risk (AVaR)

- The **Average Value at Risk** of a random variable X at level $\alpha \in]0, 1]$ is defined as:

$$\text{AVaR}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\lambda(X)\, d\lambda. \tag{9}$$

- AVaR is a **sub-additive** risk measure, meaning:

$$\text{AVaR}_\alpha(X + Y) \leq \text{AVaR}_\alpha(X) + \text{AVaR}_\alpha(Y),$$

  which ensures that diversification does not increase risk.

- Under certain regularity conditions, one can prove that AVaR is the **smallest convex risk measure** that dominates VaR. That is, it provides a conservative yet coherent extension of VaR, accounting for the magnitude of extreme losses beyond the VaR threshold.

Example 2.14

Consider the same setting as in Example 2.13, but replace VaR_α by AVaR_α. Instead of focusing on the worst losses up to the α-quantile, we now average all losses in the lower tail. This approach captures the expected shortfall in the worst $\alpha \times 100\%$ of cases and better reflects tail risk. In such a setting, diversification always reduces or maintains risk, in contrast to what may happen under VaR.

Chapter 3

Mono-Period Market: Recall of Probabilistic Model

We consider a market model defined over a finite time horizon with two trading dates $T = \{0, T\}$. The uncertainty is modeled on a probability space $(\Omega, \mathbb{P}, \mathcal{F})$, where Ω is the set of elementary events, $\mathbb{P}$ is a prior probability measure, and $\mathcal{F}$ is a sigma-algebra representing all observable events.
In most cases, the sample space is finite, i.e., $\Omega = \{\omega_1, \ldots, \omega_K\}$, where K denotes the total number of possible scenarios, and each event has strictly positive probability: $\mathbb{P}(\omega_i) = p_i > 0$. The sigma-algebra is then $\mathcal{F} = \mathcal{P}(\Omega)$, the set of all subsets of Ω.
The market consists of N risky assets with prices $S^1, \ldots, S^N$, referred to as quoted spot prices, and one risk-free asset with deterministic price S^0. At initial time $t = 0$, the vector of spot prices is:

$$S_0 = \left(S_0^0, S_0^1, \ldots, S_0^N\right) \in \mathbb{R}^{N+1},$$

with $S_0^0 > 0$. By default, we assume $S_0^0 = 1$, unless specified otherwise.

At maturity T, the vector of spot prices is denoted by

$$S_T = \left(S_T^0, S_T^1, \ldots, S_T^N\right) \in \mathbb{R}^{N+1},$$

where $S_T^0 > 0$.
The market includes a deterministic risk-free asset with interest rate r, such that:

$$S_T^0 = (1 + r)S_0^0, \quad \text{so that } 1 + r > 0.$$

A portfolio θ consists of θ_i units of the i-th asset, for $i = 0, \ldots, N$, and is represented as:

$$\theta = (\theta_0, \theta_1, \ldots, \theta_N) \in \mathbb{R}^{N+1}.$$

The value of the portfolio θ is:

- at time $t = 0$: $V_0(\theta) = \sum_i \theta_i S_0^i = \theta \cdot S_0 \in \mathbb{R}$,
- at time $t = T$: $V_T(\theta) = \sum_i \theta_i S_T^i = \theta \cdot S_T \in \mathbb{R}$, which is a random variable.

The gain over the period $[0, T]$ from the investment $V_0(\theta)$ in portfolio θ is given by:

$$G(\theta) = V_T(\theta) - V_0(\theta) = \theta \cdot (S_T - S_0),$$

which is a random variable in $\mathbb{R}$.

The return on the investment in portfolio θ is:

$$R(\theta) = \frac{V_T(\theta)}{V_0(\theta)} \quad \text{if } V_0(\theta) \neq 0.$$

We define the discounted quantities by dividing all asset prices by the risk-free asset price S_t^0. For each asset i, the discounted price is:

$$\bar{S}_t^i = \frac{S_t^i}{S_t^0}.$$

The discounted value of a portfolio θ at time t is:

$$\bar{V}_t(\theta) = \theta \cdot \bar{S}_t.$$

The discounted gain over the period $[0, T]$ is:

$$\bar{G}(\theta) = \bar{V}_T(\theta) - \bar{V}_0(\theta).$$

The discounted return is:

$$\bar{R}(\theta) = \frac{\bar{V}_T(\theta)}{\bar{V}_0(\theta)} \quad \text{if } \bar{V}_0(\theta) \neq 0.$$

Representation when Ω is finite:
Assume the probability space is finite, $\Omega = \{\omega_1, \dots, \omega_K\}$. Then the discounted portfolio value at time T is a random variable given by the vector:

$$\bar{V}_T(\theta) = \begin{pmatrix} \bar{V}_T(\theta)(\omega_1) \\ \bar{V}_T(\theta)(\omega_2) \\ \vdots \\ \bar{V}_T(\theta)(\omega_K) \end{pmatrix}, \quad \text{with initial value } \bar{V}_0(\theta).$$

Matrix notation for finite probability spaces
Assume $\Omega = \{\omega_1, \dots, \omega_K\}$ is a finite probability space.

- Let S and R be matrices defined as follows:
  - If a risk-free asset S^0 exists, then S and R are $K \times (1 + N)$ matrices.
  - If no risk-free asset exists, they are $K \times N$ matrices.
  - The elements of S are defined by $S_{ij} = S_T^j(\omega_i)$, i.e., the price of asset j in state ω_i at time T.
  - The matrix R is the matrix of returns, with entries $R_{ij} = \frac{S_T^j(\omega_i)}{S_0^j}$.
- For a given portfolio θ such that $V_0(\theta) \neq 0$, define the following:
  - The portfolio vector $\Theta \in \mathbb{R}^{1+N}$ (or $\mathbb{R}^N$ if no risk-free asset), with components $\Theta_i = \theta_i$.
  - The normalized portfolio weights $\vartheta \in \mathbb{R}^{1+N}$ (or $\mathbb{R}^N$ if no risk-free asset), defined by:

$$\vartheta_i = \frac{\theta_i S_0^i}{V_0(\theta)}.$$

- Let $V \in \mathbb{R}^K$ be the vector of terminal portfolio values, with entries $V_i = V_T(\theta)(\omega_i)$.
- Then we have the following matrix relations:

$$V = S\Theta, \quad \text{and} \quad R(\theta)(\omega_i) = (R\vartheta)_i, \quad \text{for } 1 \leq i \leq K.$$

Chapter 4

Mono-Period Portfolio Management

Utility function and admissibility conditions
A utility function is a mapping $U : \mathbb{R} \to \{-\infty\} \cup \mathbb{R}$. We define the following condition for admissibility:
Let $\underline{x} \in \{-\infty\} \cup]-\infty, 0]$, representing the *subsistence level.*

The utility function U is said to be **admissible** if:

U is increasing, strictly concave, and upper semi-continuous on $\mathbb{R}$,

$U \in \mathcal{C}^1(]\underline{x}, +\infty[)$,

$U(x) = -\infty$ for all $x < \underline{x}$,

$U'(x) = +\infty$ as $x \to \underline{x}^+$, and $U'(\infty) = 0$ (Inada conditions).

Examples:

- With $\underline{x} = 0$:

$$U(x) = \ln(x), \quad U(x) = \frac{x^a}{a}, \quad \text{with } a \in]-\infty, 0[\cup]0, 1[$$

- With $\underline{x} = -\infty$ (defined on the whole real line):

$$U(x) = -\exp(-ax), \quad a > 0$$

- Affine transformation of an admissible utility function: If U_0 is admissible and $U(x) = U_0(ax + b)$ with $a > 0$, $b \geq 0$, then U is also admissible.

- **Non-admissible example:** the quadratic utility function used in the Markowitz model:
$$U(x) = -\frac{1}{2}x^2 + ax$$
is not admissible because it is not concave on all $\mathbb{R}$, and not bounded below by $-\infty$.

Optimization problem
Given an initial wealth $K \in \mathbb{R}$ at time $t = 0$, the goal is to find an optimal portfolio $\hat{\theta}$ such that:
$$V_0(\hat{\theta}) = K \quad \text{and} \quad \sup_{\theta,\, V_0(\theta)=K} \mathbb{E}\left[U(V_T(\theta))\right] = \mathbb{E}\left[U(V_T(\hat{\theta}))\right]. \tag{10}$$

Two methods of solution:

1. *The direct method:* optimize over θ directly.
2. *The final wealth method:*
   (a) First, determine the optimal final wealth $\hat{X} = V_T(\hat{\theta})$,
   (b) Then, find a hedging portfolio $\hat{\theta}$ such that:
   $$V_T(\hat{\theta}) = \hat{X}.$$

In this course, we will use the **final wealth method**, starting with the case of a *complete market*, and then extending to the case of an *incomplete market*.

4.1 General case of a mono-period complete market

Suppose that there exists an **absence of arbitrage opportunity (AOA)** in the market $\mathcal{M}$, and that $\mathcal{M}$ is **complete**. Then, there exists a **unique equivalent martingale measure (e.m.m.)** $\mathbb{Q}$.
Let $S_0^0 = 1$, and recall that $\mathbb{E}[X]$ denotes the expectation of a random variable X with respect to the probability measure $\mathbb{P}$.

For a contingent claim (or derivative product) X, and a hedging portfolio θ such that the initial wealth is $K = V_0(\theta)$, we have:

$$K = \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S_T^0}\right].$$

Since the market is complete, we obtain the following equivalence:

$$\{V_T(\theta) \mid V_0(\theta) = K\} = \left\{X \,\middle|\, \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S_T^0}\right] = K\right\}.$$

Consequently, the optimization problem can be rewritten as:

$$\sup_{\theta,\, V_0(\theta)=K} \mathbb{E}\left[U(V_T(\theta))\right] = \sup_{X,\, \mathbb{E}^{\mathbb{Q}}\left[X/S_T^0\right]=K} \mathbb{E}\left[U(X)\right].$$

This shows that one can solve problem (10) in two steps:

1. Find an optimal terminal wealth $\hat{X}$ such that:
$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\hat{X}}{S_T^0}\right] = K \quad \text{and} \quad \sup_{X,\, \mathbb{E}^{\mathbb{Q}}[X/S_T^0]=K} \mathbb{E}[U(X)] = \mathbb{E}[U(\hat{X})]. \tag{11}$$
2. Find a hedging portfolio $\hat{\theta}$ that replicates $\hat{X}$, i.e.,
$$V_T(\hat{\theta}) = \hat{X}. \tag{12}$$

Then $\hat{\theta}$ is a solution to the original optimization problem (10).

Solution of Step 1: Optimization problem (11)

We solve the problem using a Lagrangian method with multiplier λ. The Lagrangian is defined as:

$$\mathcal{L}(X;\lambda) = \mathbb{E}\left[U(X)\right] - \lambda\left(\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S_T^0}\right] - K\right).$$

Let $\xi = \frac{d\mathbb{Q}}{d\mathbb{P}}$ be the Radon-Nikodym derivative. Then we can rewrite:

$$\mathcal{L}(X) = \mathbb{E}\left[U(X) - \lambda\left(\frac{\xi X}{S_T^0} - K\right)\right].$$

The associated optimization problem becomes: Find X^λ such that

$$\sup_X \mathcal{L}(X;\lambda) = \mathcal{L}(X^\lambda;\lambda). \tag{13}$$

Let $I:]0,+\infty[\to]\underline{x},+\infty[$ be the inverse function of U' restricted to the domain $]\underline{x},+\infty[$, i.e.,

$$I(y) = x \quad \Leftrightarrow \quad U'(x) = y. \tag{14}$$

Note that for a given $y > 0$, the function $x \mapsto U(x) - yx$ attains its maximum at $x = x_M$ satisfying:

$$U'(x_M) = y.$$

Thus:

$$x_M = I(y) \quad \text{and} \quad U(x) - yx \leq U(I(y)) - yI(y), \quad \forall x \in \mathbb{R}.$$

We define the optimal terminal wealth as:

$$\hat{X}^\lambda = I\left(\lambda\frac{\xi}{S_T^0}\right), \tag{15}$$

where I is the inverse of the marginal utility U', and $\xi = \frac{d\mathbb{Q}}{d\mathbb{P}}$ is the Radon-Nikodym derivative.

It follows that X^λ satisfies the optimality condition (13), since:

$$\mathcal{L}(X;\lambda) = \mathbb{E}\left[U(X) - \lambda\left(\frac{\xi X}{S_T^0}\right)\right] + \lambda K \leq \mathcal{L}(X^\lambda;\lambda).$$

The multiplier $\hat{\lambda}$ is then determined by the constraint:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{X^{\hat{\lambda}}}{S_T^0}\right] = \mathbb{E}\left[\xi \cdot I\left(\hat{\lambda}\frac{\xi}{S_T^0}\right)\right] = K. \tag{16}$$

Depending on the utility function U, one or the other expression of this equation may be easier to compute in practice.

Finally, if the Euler-Lagrange equation $\nabla\mathcal{L}(X^\lambda) = 0$ makes sense, then X^λ from (15) satisfies:

$$\nabla\mathcal{L}(X^\lambda) = U'\left(X^\lambda\right) - \lambda\frac{\xi}{S_T^0} = 0. \tag{17}$$

Step 2: Finding a hedging portfolio

A portfolio $\hat{\theta}$ is said to hedge the optimal final wealth $\hat{X}$ if it satisfies:

$$V_T(\hat{\theta}) = \hat{X}.$$

In the case where Ω is a finite set, one can use matrix notation. Let $S_{ij} = S^j(\omega_i)$ denote the price of asset j in state ω_i. Then the system becomes:

$$S \cdot \begin{pmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \\ \vdots \\ \hat{\theta}_N \end{pmatrix} = \begin{pmatrix} \hat{X}(\omega_1) \\ \hat{X}(\omega_2) \\ \vdots \\ \hat{X}(\omega_K) \end{pmatrix}. \tag{18}$$

In the case where Ω is not finite (e.g., continuous models), one must solve the equation

$$V_T(\hat{\theta}) = \hat{X}$$

directly, which generally involves more advanced tools such as stochastic integrals or replication techniques in continuous-time finance.

Example 4.2.
Let the utility function be $U(x) = \ln(x)$, and assume the probability space is finite:

$$\Omega = \{\omega_1, \omega_2, \omega_3\}, \quad \text{with initial wealth } K > 0.$$

The probabilities of the elementary events are given by:

$$\mathbb{P}(\{\omega_1\}) = p_1 = \frac{1}{8}, \quad \mathbb{P}(\{\omega_2\}) = p_2 = \frac{3}{8}, \quad \mathbb{P}(\{\omega_3\}) = p_3 = \frac{4}{8}.$$

We consider three assets: one risk-free asset S^0, and two risky assets S^1 and S^2.

We consider three assets:
- Risk-free asset S^0 with $r = 5\% \Rightarrow S^0_T = 1 + r$,
- Two risky assets S^1 and S^2, with prices:

$$S = \begin{pmatrix} 21/20 & 42/31 & 21/124 \\ 21/20 & 21/31 & 42/31 \\ 21/20 & 21/62 & 168/31 \end{pmatrix} \tag{19}$$

Step 1: Find an equivalent martingale measure (e.m.m.)
We seek $\mathbb{Q} = (q_1, q_2, q_3)$ such that:

$$S^t \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = (1 + r) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{20}$$

The solution is:

$$q_1 = \frac{3}{5}, \quad q_2 = \frac{3}{10}, \quad q_3 = \frac{1}{10} \tag{21}$$

Step 2: Compute the Radon-Nikodym derivative $\xi = \frac{d\mathbb{Q}}{d\mathbb{P}}$:

$$\xi_1 = \frac{q_1}{p_1} = \frac{3/5}{1/8} = \frac{24}{5}, \quad \xi_2 = \frac{q_2}{p_2} = \frac{3/10}{3/8} = \frac{4}{5}, \quad \xi_3 = \frac{q_3}{p_3} = \frac{1/10}{4/8} = \frac{1}{5}$$

(Verification: $\mathbb{E}[\xi] = 1$, OK.)

Step 3: Optimization problem
We solve:

$$\sup_{X : \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{1+r}\right]=K} \mathbb{E}[\ln X] = \mathbb{E}[\ln \hat{X}] \tag{22}$$

$$\Leftrightarrow$$

$$\sup_{X : \mathbb{E}\left[\xi \cdot \frac{X}{1+r}\right]=K} \mathbb{E}[\ln X] = \mathbb{E}[\ln \hat{X}] \tag{23}$$

Lagrangian:

$$\mathcal{L}(X) = \mathbb{E}[\ln X] - \lambda \left(\mathbb{E}\left[\frac{\xi X}{1+r} \right] - K \right) \tag{24}$$

Inverse marginal utility: $I(y) = \frac{1}{y}$. Then from (15):

$$\hat{X} = \frac{1+r}{\lambda \xi}$$

Determine λ from the constraint:

$$K = \mathbb{E}\left[\xi \cdot \hat{X}/(1+r)\right] = \mathbb{E}\left[\frac{1}{\hat{\lambda}}\right] \Rightarrow \hat{\lambda} = \frac{1}{K} \Rightarrow \hat{X} = K(1+r) \cdot \frac{1}{\xi} \tag{25}$$

Thus:

$$\hat{X}(\omega_1) = K(1+r) \cdot \frac{1}{\xi_1} = K \cdot \frac{7}{32}, \hat{X}(\omega_2) = K(1+r) \cdot \frac{1}{\xi_2} = K \cdot \frac{21}{16}, \hat{X}(\omega_3) = K(1+r) \cdot \frac{1}{\xi_3} = K \cdot \frac{21}{4}$$

Step 4: Hedging portfolio
We solve:

$$S \cdot \begin{pmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix} = K \cdot \begin{pmatrix} 7/32 \\ 21/16 \\ 21/4 \end{pmatrix} \Rightarrow \begin{cases} \hat{\theta}_0 = -K \cdot \frac{35}{492} \\ \hat{\theta}_1 = K \cdot \frac{31}{328} \\ \hat{\theta}_2 = K \cdot \frac{961}{984} \end{cases} \tag{26}$$

Remark: For future reference:

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\xi} - 1\right] = 0, \quad \hat{X} = K(1+r) \cdot \frac{1}{\xi}, \quad \hat{\theta} \cdot \Delta \bar{S} = \sum_{i=1}^{2} H^{M_i} \cdot \Delta \bar{S}_i \tag{27}$$

where $\bar{S} = \frac{S_T}{S_T^0}$, and hedge ratios $H^{M_1} = \frac{31}{984}, \quad H^{M_2} = \frac{961}{984}$.
End of Example 4.2

4.2 General case of mono-period incomplete market

Even when the market is incomplete, one can still solve the optimization problem (10) in two steps:

1. Find an optimal terminal wealth $\hat{X}$ such that:

$$\forall \mathbb{Q} \in \mathcal{M}_e, \quad \mathbb{E}^{\mathbb{Q}}\left[\frac{\hat{X}}{S_T^0}\right] = K, \tag{28}$$

$$\mathbb{E}[U(\hat{X})] = \sup_{X\,:\,\forall \mathbb{Q} \in \mathcal{M}_e,\, \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S_T^0}\right] = K} \mathbb{E}[U(X)]. \tag{29}$$

2. Find a hedging portfolio $\hat{\theta}$ that replicates $\hat{X}$, i.e.,

$$V_T(\hat{\theta}) = \hat{X}. \tag{30}$$

Then, under certain general conditions (which we do not detail here), $\hat{\theta}$ is a solution to the original optimization problem (10).

Solution of the optimization problem in (29)
For $\lambda > 0$ and $\mathbb{Q} \in \mathcal{M}_e$, define the Lagrangian functional:

$$\mathcal{L}(X; \lambda, \mathbb{Q}) = \mathbb{E}[U(X)] - \lambda \left(\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S_T^0}\right] - K \right).$$

Let $\xi = \frac{d\mathbb{Q}}{d\mathbb{P}}$ be the Radon-Nikodym derivative. Then the Lagrangian becomes:

$$\mathcal{L}(X; \lambda, \mathbb{Q}) = \mathbb{E}\left[U(X) - \lambda \left(\frac{\xi X}{S_T^0} - K \right) \right].$$

Direct optimization: Find $X^{\lambda,\mathbb{Q}}$ such that

$$\sup_X \mathcal{L}(X; \lambda, \mathbb{Q}) = \mathcal{L}(X^{\lambda,\mathbb{Q}}; \lambda, \mathbb{Q}). \tag{31}$$

Dual problem: Find the pair $(\hat{\lambda}, \hat{\mathbb{Q}})$, called the *dual minimizer*, that solves:

$$\inf_{\lambda > 0, \mathbb{Q} \in \mathcal{M}_e} \mathcal{L}(X^{\lambda,\mathbb{Q}}; \lambda, \mathbb{Q}).$$

Define $\hat{\xi} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}$. Then the solution of the primal problem (28)–(29) is given by:

$$\hat{X} := X^{\hat{\lambda},\hat{\mathbb{Q}}} = I\left(\hat{\lambda} \cdot \frac{\hat{\xi}}{S_T^0} \right), \tag{33}$$

where $I = (U')^{-1}$ is the inverse of the marginal utility function.

4.3 Cass and Stiglitz, Mutual Fund Theorem

Let S denote an arbitrage-free one-period market. For a given initial wealth $x \in \mathbb{R}$ and a utility function U, denote by $\hat{\theta}(x, U)$ the optimal portfolio.

We ask the following question: *What are the admissible collections of utility functions $\mathcal{U} = \{U_1, \ldots, U_m\}$ such that the Mutual Fund Theorem (MFT) holds true in all markets S and for all initial wealths x?*

More precisely, we want to characterize the collections $\mathcal{U}$ of admissible utility functions for which there exist two portfolios $\Theta(A)$ and $\Theta(B)$ such that the following property holds:
For all $U \in \mathcal{U}$ and all $x \in \mathbb{R}$, there exist scalars $A(x, U), B(x, U) \in \mathbb{R}$ such that:

$$\hat{\theta}(x, U) = A(x, U)\Theta^{(A)} + B(x, U)\Theta^{(B)}. \tag{34}$$

Theorem 4.3 (MFT in Mono-Period Model; Arbitrary Market S)
The following two statements are equivalent:

- For any arbitrage-free one-period market model S, there exist two mutual funds such that each of the m agents can, independently of their initial capital x, construct their optimal portfolio by investing only in these two funds.
- All the utility functions $U_i \in \mathcal{U}$ are affine transformations of one and the same base utility function U, where U is of one of the following types:

$$U(x) = \ln(x), \quad U(x) = \frac{x^{1-p}}{1-p} \quad \text{with } p > 0,\, p \neq 1, \quad \text{or } U(x) = -e^{-x}.$$

Moreover, one can always choose the risk-free asset as one of the two mutual funds.

Reformulation: If the marginal utilities U_i' are not proportional (i.e., not affine transformations of one another) within the exponential, logarithmic, or power utility classes, then there exists a one-period model S such that the Mutual Fund Theorem does not hold with respect to the family $\mathcal{U}$.
This result generalizes the classical mean-variance case to broader families of utility functions: exponential, logarithmic, and power.

Example 4.4
Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with probability distribution:

$$\mathbb{P}(\omega_1) = \frac{1}{8}, \quad \mathbb{P}(\omega_2) = \frac{3}{8}, \quad \mathbb{P}(\omega_3) = \frac{4}{8}.$$

We consider the same assets S^0, S^1, S^2 as in Example 4.2.
Let the utility function be $U(x) = \ln(ax + b)$, with $a > 0$, $b \geq 0$, i.e., the logarithm composed with an affine transformation. The subsistence level is given by $x = -\frac{b}{a}$.
Let $\hat{X}(x, U)$ denote the optimal terminal wealth corresponding to initial capital x and utility function U.
Following the reasoning in Example 4.2, we obtain:

$$\hat{X}(x, U) = \frac{1+r}{\lambda\xi} - \frac{b}{a},$$

where the optimal value of λ corresponds to:

$$\lambda = \frac{1}{(1+r)(x + \frac{b}{a})}.$$

Thus, the final wealth becomes:

$$\hat{X}(x, U) = \frac{1+r}{\lambda\xi} - \frac{b}{a}.$$

Using the identity $\lambda = \frac{1}{(1+r)(x+\frac{b}{a})}$, this becomes:

$$\hat{X}(x,U) = \left(x + \frac{b}{a}\right) \cdot \frac{1+r}{\xi} - \frac{b}{a}.$$

We can express this as:

$$\hat{X}(x,U) = x + \left(x + \frac{b}{a}\right)\left(\frac{1+r}{\xi} - 1\right).$$

From equation (27), we know that:

$$\frac{1+r}{\xi} - 1 = \sum_{i=1}^{2} H^{M_i} \cdot \Delta\bar{S}_i,$$

where:

$$H^{M_1} = \frac{31}{328}, \quad H^{M_2} = \frac{961}{984}.$$

Therefore, the optimal portfolio is implemented by investing:

$$\frac{x + \frac{b}{a}}{1+r}$$

units in the hedge fund H^M, which replicates the payoff:

$$\sum_{i=1}^{2} H^{M_i} \cdot \Delta\bar{S}_i,$$

and investing the rest of the wealth in the risk-free asset.

Example 4.5
Let the financial market be as in Example 4.4, and let the utility function be:

$$U(x) = -\frac{1}{a}e^{-ax}, \quad \text{with } a > 0.$$

Following the approach from Examples 4.2 and 4.4, the optimal final wealth $\hat{X}(x,U)$ satisfies:

$$\frac{\hat{X}(x,U)}{1+r} = x + \frac{1}{(1+r)a}\left(\mathbb{E}[\xi \ln \xi] - \ln \xi\right).$$

Hence, we can write:

$$\hat{X}(x,U) = x + \frac{1}{(1+r)a}\left(\mathbb{E}[\xi \ln \xi] - \ln \xi\right) = x + \frac{1}{a}\sum_{i=1}^{2} h^{M_i} \cdot \Delta\bar{S}_i,$$

where:

$$\sum_{i=1}^{2} h^{M_i} \cdot \Delta\bar{S}_i = \frac{1}{1+r}\left(\mathbb{E}[\xi \ln \xi] - \ln \xi\right),$$

for some portfolio vector $(0, h^{M_1}, h^{M_2})$, since:

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}[\xi \ln \xi] - \ln \xi\right] = 0.$$

Conclusion:

- The optimal portfolio is constructed by investing $\frac{1}{a}$ units in the fund h^M, and the remaining wealth in the risk-free asset.
- The number of units $\frac{1}{a}$ is **independent of the initial wealth** x.
- The mutual funds H^M (from Example 4.4) and h^M are **not proportional**, since $\mathbb{E}[\xi \ln \xi] - \ln \xi$ and $\frac{1}{\xi}$ are not proportional.
- Therefore, the "log-optimal" and "exponential-optimal" portfolios cannot be realized using the same mutual fund.

Chapter 5

Discrete Time Markets: Recall of Probabilistic Model

Market Setup: Filtered Probability Framework

We consider a discrete-time financial market with a finite horizon. Trading dates are indexed by the set $\mathcal{T} = \{0, 1, \ldots, T\}$, where $T \in \mathbb{N}$ is a fixed terminal time.
The model is defined on a complete filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}})$, where:

- Ω is the set of elementary events (finite or infinite),
- $\mathbb{P}$ is a prior probability measure on $\mathcal{F}$,
- $\mathcal{F}$ is a σ-algebra of measurable events at time T,
- $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ is a filtration of sub-σ-algebras, satisfying $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leq s \leq t$, and $\mathcal{F}_T = \mathcal{F}$.

If Ω is finite, then $\mathcal{F}_0 = \{\Omega, \emptyset\}$. If Ω is infinite, then $\mathcal{F}_0$ is the σ-algebra generated by Ω and the $\mathbb{P}$-null sets.

The market consists of $1 + N$ assets: one risk-free asset (denoted $i = 0$) and N general risky assets. At each time $t \in \mathcal{T}$:

- S_t^i denotes the price of asset i, for $0 \leq i \leq N$. It is a real-valued $\mathcal{F}_t$-measurable random variable, meaning its value is known at time t.
- $S_t = (S_t^0, S_t^1, \ldots, S_t^N)$ is the price vector at time t.
- $S = (S_0, S_1, \ldots, S_T)$ is the price process, adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$.
- The process S^0 represents the risk-free asset (e.g., a bond), and the processes S^i for $1 \leq i \leq N$ represent general risky assets (e.g., stocks). It is typically assumed that $S_t^i > 0$.

Interest Rate and Discounted Prices

We make the following assumptions regarding the risk-free asset:

(i) The risk-free asset is strictly positive at all times: $S_t^0 > 0$ for all $t \in \mathcal{T}$.

(ii) The process (S_t^0) is predictable, meaning that S_{t+1}^0 is $\mathcal{F}_t$-measurable.

(iii) By convention, the initial value is normalized to $S_0^0 = 1$, unless stated otherwise.

The (spot) interest rate process $r = (r_t)_{t\in\mathcal{T}}$ is defined by:

$$1 + r_t = \frac{S_{t+1}^0}{S_t^0}, \quad \text{for each } t \in \mathcal{T}.$$

This rate is known at the beginning of the period, i.e., r_t is $\mathcal{F}_t$-measurable. Thus, r is an adapted stochastic process.
The evolution of the risk-free asset over time is given by:

$$S_t^0 = S_0^0 \cdot (1 + r_0)(1 + r_1) \cdots (1 + r_{t-1}).$$

This corresponds to the amount in a bank account at time t, resulting from an initial investment of S_0^0 that accumulates interest over the periods $[0, t-1]$. The interest rates r_t may be either deterministic or stochastic.

For each asset $i \in \{0, \dots, N\}$, we define the *discounted price process* by:

$$\bar{S}_t^i = \frac{S_t^i}{S_t^0}. \tag{35}$$

In particular, the discounted price of the risk-free asset is always $\bar{S}_t^0 = 1$ for all $t \in \mathcal{T}$. The full discounted price process is denoted by $\bar{S} = (\bar{S}_t)_{t\in\mathcal{T}}$.
We denote by $\mathcal{M}^e(S)$, or simply $\mathcal{M}^e$, the set of *equivalent martingale measures* (e.m.m.) for the price process S.
For each measure $\mathbb{Q} \in \mathcal{M}^e(S)$, we define its *Radon-Nikodym density process* with respect to the original probability measure $\mathbb{P}$ by:

$$\xi_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right], \quad t \in \mathcal{T}. \tag{36}$$

The set of terminal densities is denoted:

$$\mathcal{D}^e(S) = \left\{\xi_T \in L^\infty \,\middle|\, \mathbb{Q} \in \mathcal{M}^e(S)\right\}.$$

By definition of an equivalent martingale measure, the discounted price process $\bar{S}$ is a martingale under $\mathbb{P}$ when weighted by the density process ξ. More precisely, the following holds for all $t \in \mathcal{T}$:

$$\xi_t \bar{S}_t = \mathbb{E}\left[\xi_T \bar{S}_T \,\middle|\, \mathcal{F}_t\right]. \tag{37}$$

This condition ensures that $\bar{S}$ is a $\mathbb{Q}$-martingale under any measure $\mathbb{Q} \in \mathcal{M}^e(S)$.

Chapter 6

Portfolio Management in Discrete Time Markets

Optimization Problem and Legendre-Fenchel Transform

Let S be the price process of a discrete-time market satisfying the Absence of Arbitrage Opportunities (AOA).
For an initial wealth $K \in \mathbb{R}$, we define $\mathcal{A}(K)$ as the set of all self-financing portfolios θ such that $V_0(\theta) = K$.
Let U be an admissible utility function satisfying Condition 4.1, and let $\check{U}$ denote the Legendre-Fenchel transform of the function $x \mapsto -U(-x)$ defined on $\mathbb{R}$. That is,

$$\check{U}(y) = \sup_{x\in\mathbb{R}} (xy - (-U(-x))) = \sup_{x\in\mathbb{R}} (U(x) - xy). \tag{38}$$

The function $\check{U}$ is convex and lower semicontinuous on $\mathbb{R}$, and it is finite on the interval $]0, \infty[$. Moreover, for $y > 0$, we have:

$$\check{U}(y) = U(I(y)) - yI(y),$$

and for $y < 0$, $\check{U}(y) = \infty$.
The derivative of $\check{U}$ satisfies:

$$\check{U}'(y) = -I(y), \quad \text{for } y > 0. \tag{39}$$

Consequently, for all $x \in \mathbb{R}$ and $y > 0$, we have the inequality:

$$U(x) - xy \leq \check{U}(y) = U(I(y)) - yI(y). \tag{40}$$

The associated optimization problem consists in finding a portfolio $\hat{\theta} \in \mathcal{A}(K)$ such that:

$$u(K) = \mathbb{E}\left[U\left(V_T(\hat{\theta})\right)\right],$$

where

$$u(K) = \sup_{\theta\in\mathcal{A}(K)} \mathbb{E}\left[U\left(V_T(\theta)\right)\right]. \tag{41}$$

Solution in Two Steps: General Case

The solution to the utility maximization problem can still be approached in two steps, similar to the mono-period setting:

1. **Step 1:** Find the optimal final wealth $\hat{X}$ that is hedgeable and satisfies the pricing condition
$$\Pi_0(\hat{X}) = K,$$
where $\Pi_0(X)$ is the price at time $t = 0$ of the contingent claim X. The value function becomes:
$$u(K) = \sup_{\substack{X \text{ hedgeable} \\ \Pi_0(X)=K}} \mathbb{E}[U(X)] = \mathbb{E}[U(\hat{X})]. \tag{42}$$

2. **Step 2:** Find a self-financing hedging portfolio $\hat{\theta}$ such that
$$V_T(\hat{\theta}) = \hat{X}.$$

Alternatively, we may consider the superhedging price $\Pi_0^*(X)$, which leads to a relaxed constraint:
$$\Pi_0^*(X) \leq K,$$
i.e., X is only superhedgeable at cost less than or equal to K. This weaker constraint implies:
$$u(K) \leq \sup_{\Pi_0^*(X) \leq K} \mathbb{E}[U(X)].$$

Lagrangian Formulation.
We define the set of density processes:
$$\mathcal{Z} = \left\{ Z = \frac{Z_0\,\xi}{S_0} \;\middle|\; Z_0 > 0,\ \xi_T \in \mathcal{D}^e(S) \right\}. \tag{43}$$

Let $\bar{\mathcal{Z}}$ denote the closure of $\mathcal{Z}$ in L^∞.
The associated Lagrangian is a function:
$$\mathcal{L} : L^0(\Omega, \mathcal{F}, \mathbb{P}) \times \bar{\mathcal{Z}} \to [-\infty, \infty[$$
defined by:
$$\mathcal{L}(X, Z) = \mathbb{E}\left[U(X) - Z_T X\right] + Z_0 K. \tag{44}$$

We now express the utility maximization problem under a superhedging constraint using a Lagrangian approach.
1. By the definition of $\mathcal{Z}$ and the Lagrangian function $\mathcal{L}$, we have:
$$\inf_{Z \in \bar{\mathcal{Z}}} \mathcal{L}(X, Z) = \begin{cases} \mathbb{E}[U(X)] & \text{if } Z_0 K \geq \mathbb{E}[Z_T X] \quad \text{for all } Z \in \bar{\mathcal{Z}}, \\ -\infty & \text{otherwise.} \end{cases}$$

When the superhedging condition $\Pi_0^*(X) \leq K$ holds, we obtain the duality relation:
$$\sup_{\Pi_0^*(X) \leq K} \mathbb{E}[U(X)] = \sup_{\Pi_0^*(X) \leq K} \inf_{Z \in \bar{\mathcal{Z}}} \mathcal{L}(X, Z). \tag{45}$$

2. Using the definition of the convex dual function $\check{U}$ and substituting $x = X(\omega)$ and $y = Z_T(\omega)$ into inequality (40), we obtain for all $Z \in \bar{\mathcal{Z}}$ and $X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$:

$$U(X) - Z_T X \leq \check{U}(Z_T) = U(I(Z_T)) - Z_T I(Z_T).$$

Taking the expectation on both sides yields:

$$\mathcal{L}(X, Z) = \mathbb{E}[U(X) - Z_T X] + Z_0 K \leq \mathbb{E}[\check{U}(Z_T)] + Z_0 K = \mathcal{L}(I(Z_T), Z). \tag{46}$$

Therefore, we have:

$$\inf_{Z \in \bar{\mathcal{Z}}} \mathcal{L}(X, Z) \leq \inf_{Z \in \bar{\mathcal{Z}}} \mathcal{L}(I(Z_T), Z).$$

Applying formula (45), this leads to the inequality:

$$\sup_{\Pi_0^*(X) \leq K} \mathbb{E}[U(X)] \leq \inf_{Z \in \bar{\mathcal{Z}}} \left(\mathbb{E}[\check{U}(Z_T)] + Z_0 K\right) = \inf_{Z \in \bar{\mathcal{Z}}} \mathcal{L}(I(Z_T), Z). \tag{47}$$

3. Suppose that the dual problem admits a solution, i.e., there exists $\hat{Z} \in \bar{\mathcal{Z}}$ such that:

$$\inf_{Z \in \bar{\mathcal{Z}}} \mathcal{L}(I(Z_T), Z) = \mathcal{L}(I(\hat{Z}_T), \hat{Z}). \tag{48}$$

Then the terminal wealth $\hat{X} := I(\hat{Z}_T)$ is hedgeable and has cost K at time $t = 0$. Using equations (42), (45), (47), and (48), we obtain the identity:

$$u(K) = \sup_{\Pi_0^*(X) \leq K} \mathbb{E}[U(X)] = \inf_{Z \in \bar{\mathcal{Z}}} \mathcal{L}(I(Z_T), Z) = \mathcal{L}(I(\hat{Z}_T), \hat{Z}) = \mathbb{E}[U(\hat{X})]. \tag{49}$$

Theorem 6.1. Let U be an admissible utility function satisfying Condition 4.1, and suppose Ω is finite and $\mathcal{M}^e \neq \emptyset$. Let $K > 0$ be such that for all $\mathbb{Q} \in \mathcal{M}^e$, the following inequality holds:

$$K > \underline{x}\, \mathbb{E}\left[\frac{\xi_T^{\mathbb{Q}}}{S_T^0}\right].$$

Then the optimization problem (42) admits a unique solution $\hat{X}$. In particular, $\hat{X}$ is hedgeable and costs K at time $t = 0$.

Chapter 7

Continuous time market: 1-dim standard Black-Scholes model (Recall)

We consider a standard Black-Scholes market model with the following setup:

- Time horizon T and trading dates $t \in \mathbb{T} = [0, T]$.
- A complete filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}})$, where the filtration is generated by a one-dimensional standard Brownian motion W.
- One risk-free asset with price process S_t^0 evolving according to the differential equation:
$$dS_t^0 = rS_t^0\,dt, \quad S_0^0 = 1,$$
where r is the constant interest rate.
- One risky asset with price process S_t^1 governed by:
$$dS_t^1 = S_t^1(\mu\,dt + \sigma\,dW_t), \quad S_0^1 > 0,$$
where μ is the drift and $\sigma > 0$ is the volatility.
- The market price of risk is given by:
$$\gamma = \frac{\mu - r}{\sigma}.$$

We define the discounted price process $\bar{S}_t^i = \dfrac{S_t^i}{S_t^0}$. In particular, we have:

$$\bar{S}_t^0 = 1 \quad \text{and} \quad d\bar{S}_t^1 = \bar{S}_t^1((\mu - r)dt + \sigma dW_t) = \bar{S}_t^1\sigma(\gamma\,dt + dW_t), \quad t \in \mathbb{T}.$$

Define the process ξ by:

$$\xi_t = \exp\left(-\gamma W_t - \frac{1}{2}\gamma^2 t\right). \tag{58}$$

The process ξ is a $\mathbb{P}$-martingale and satisfies the stochastic differential equation:

$$d\xi_t = -\gamma\xi_t\,dW_t.$$

According to Girsanov's Theorem, the process $\bar{W}_t = W_t + \gamma t$ is a Brownian motion under a new probability measure $\mathbb{Q}$ defined by:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi_T. \tag{59}$$

Substituting into the dynamics, we obtain:

$$d\bar{S}^1_t = \bar{S}^1_t \sigma \, d\bar{W}_t. \tag{60}$$

Hence, under $\mathbb{Q}$, the discounted price process is a martingale:

$$\mathbb{E}^{\mathbb{Q}} \left[\bar{S}_T \mid \mathcal{F}_t \right] = \bar{S}_t, \quad \forall t \in \mathbb{T}. \tag{61}$$

A portfolio $\theta = (\theta^0, \theta^1)$ is defined as follows: θ^0_t is the number of units of the risk-free asset held at time t, and θ^1_t is the number of units of the risky asset held at time t. The vector $\theta_t = (\theta^0_t, \theta^1_t)$ represents the (instantaneous) portfolio at time t and is assumed to be $\mathcal{F}_t$-measurable. That is, the portfolio is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{T}}$.
The wealth process $V(\theta)$ associated with the portfolio θ is defined by:

$$V_t(\theta) = \theta_t \cdot S_t, \quad \text{for } t \in \mathbb{T}. \tag{62}$$

The gains process $G(\theta)$ of the portfolio is given by:

$$G_t(\theta) = \int_0^t \theta_s \cdot dS_s, \quad \text{for } t \in \mathbb{T}. \tag{63}$$

The discounted wealth and gains processes $\bar{V}(\theta)$ and $\bar{G}(\theta)$ are respectively defined as:

$$\bar{V}_t(\theta) = \theta_t \cdot \bar{S}_t, \qquad \bar{G}_t(\theta) = \int_0^t \theta_s \cdot d\bar{S}_s, \quad \text{for } t \in \mathbb{T}. \tag{64}$$

Attention: If $r \neq 0$, then there may exist portfolios θ such that $\bar{G}(\theta) \neq G(\theta)/S^0$. The wealth process of a self-financed portfolio θ satisfies:

$$V_t(\theta) = V_0(\theta) + G_t(\theta), \quad \forall t \in \mathbb{T}. \tag{65}$$

Equivalently, a portfolio θ is self-financed if and only if the discounted wealth process satisfies:

$$\bar{V}_t(\theta) = \bar{V}_0(\theta) + \bar{G}_t(\theta), \quad \forall t \in \mathbb{T}. \tag{66}$$

Since $d\bar{S}^0_t = 0$, it follows from (66) that θ is self-financed if and only if:

$$\bar{V}_t(\theta) = \bar{V}_0(\theta) + \int_0^t \theta^1_s \bar{S}^1_s \sigma \, d\bar{W}_s, \quad \forall t \in \mathbb{T}. \tag{67}$$

Let X be a derivative (i.e., a contingent claim) with maturity T. A *hedging portfolio* (or *replicating portfolio*) of X is a self-financed portfolio θ such that $V_T(\theta) = X$. Equivalently:

$$\frac{X}{e^{rT}} = \mathbb{E}^{\mathbb{Q}} \left[\frac{X}{e^{rT}} \right] + \int_0^T \theta^1_t \bar{S}^1_t \sigma \, d\bar{W}_t. \tag{68}$$

Important Remark: In order to ensure the *Absence of Arbitrage Opportunities (AOA)*, one imposes an *admissibility condition* on portfolios, such as $V_t(\theta)(\omega) > C$ almost everywhere for some $C \in \mathbb{R}$, or

$$\int_0^T \left(|\theta^0_t|^2 + (1 + |S^0_t|^2)|\theta^1_t|^2 \right) \, dt < \infty.$$

Admissible derivatives are also assumed to satisfy such conditions. These restrictions can be chosen so that the Black-Scholes market is arbitrage-free and complete.

Chapter 8

Merton's Portfolio: Case 1 risky asset

Setup: Consider a one-dimensional Black-Scholes market as described in Section 7. Let U be an admissible utility function as defined in Condition (4.1).

Optimization Problem: Given an initial wealth $K > x$ at time $t = 0$, find a self-financed portfolio θ such that:

$$\sup_{\theta \text{ self-financed},\ V_0(\theta)=K} \mathbb{E}\left[U\left(V_T(\theta)\right)\right] = \mathbb{E}\left[U\left(V_T\left(\hat{\theta}\right)\right)\right]. \tag{69}$$

Solution Strategy: As in the mono-period case, the problem is solved in two steps:

1. Find the optimal terminal wealth $\hat{X}$ such that:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\hat{X}}{S_T^0}\right] = K \quad \text{and} \quad \sup_{\mathbb{E}^{\mathbb{Q}}\left[X/S_T^0\right]=K} \mathbb{E}[U(X)] = \mathbb{E}[U(\hat{X})]. \tag{70}$$

2. Find a self-financed hedging portfolio $\hat{\theta}$ that replicates $\hat{X}$:

$$V_T(\hat{\theta}) = \hat{X}, \quad \text{with } \hat{\theta} \text{ self-financed.} \tag{71}$$

Optimal Final Wealth: The optimal terminal wealth $\hat{X}$, i.e., the solution of (69), is given by:

$$\hat{X} = I\left(\hat{\lambda} e^{rT} \xi_T\right), \tag{72}$$

where I is the inverse function of U' (see equation (14)).

According to equations (58) and (59), the density process ξ satisfies:

$$\xi_t = \exp\left(-\gamma W_t - \frac{1}{2}\gamma^2 t\right) = \exp\left(-\gamma \bar{W}_t + \frac{1}{2}\gamma^2 t\right),$$

where $\bar{W}_t$ is a Brownian motion under the risk-neutral measure $\mathbb{Q}$.

The multiplier $\hat{\lambda}$ is the unique solution of the equation:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\hat{X}}{S_T^0}\right] = K, \quad \text{or explicitly:} \quad \mathbb{E}\left[\xi_T \cdot \frac{1}{e^{rT}} \cdot I\left(\hat{\lambda} \xi_T e^{rT}\right)\right] = K. \tag{73}$$

Proof Outline:
We define the Lagrangian function as:

$$\mathcal{L}(X;\lambda) = \mathbb{E}[U(X)] - \lambda\left(\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{e^{rT}}\right] - K\right) = \mathbb{E}\left[U(X) - \lambda\left(\frac{\xi_T X}{e^{rT}} - K\right)\right].$$

Optimization Problem: Find X^λ such that:

$$\sup_X \mathcal{L}(X;\lambda) = \mathcal{L}(X^\lambda;\lambda). \tag{74}$$

For a given $y > 0$, the function $x \mapsto U(x) - yx$ achieves its maximum at $x = x_M$ satisfying:

$$U'(x_M) = y.$$

Hence,

$$x_M = I(y), \quad \text{and} \quad U(x) - yx \leq U(I(y)) - yI(y), \quad \forall x \in \mathbb{R}.$$

We define:

$$X^\lambda = I\left(\frac{\lambda}{e^{rT}\xi_T}\right).$$

It then follows that X^λ satisfies the optimality condition (74), and:

$$\mathcal{L}(X;\lambda) = \mathbb{E}\left[U(X) - \left(\frac{\lambda\xi_T}{e^{rT}}\right)X\right] + \lambda K \leq \mathcal{L}(X^\lambda;\lambda).$$

Finally, the multiplier $\hat{\lambda}$ is determined by the constraint:

$$\mathbb{E}\left[\frac{\xi_T}{e^{rT}} \cdot I\left(\frac{\hat{\lambda}\xi_T}{e^{rT}}\right)\right] = K.$$

So the optimal final wealth is:

$$\hat{X} = I\left(\frac{\hat{\lambda}\xi_T}{e^{rT}}\right).$$

Hedging portfolio $\hat{\theta}$ of $\hat{X}$:
The optimal final wealth $\hat{X}$ discounted by e^{rT} can be written as a stochastic integral (as will be seen in examples):

$$\frac{\hat{X}}{e^{rT}} = K + \int_0^T a_t\, d\bar{W}_t, \tag{75}$$

for some integrand a. Finding a is the difficult part of the portfolio selection problem. The risky part of the hedging portfolio is now given by the representations in the previous formulas. The discounted investment in the risky asset at time t is:

$$\hat{\theta}_t^1\, \bar{S}_t^1\, \sigma = a_t \quad \Rightarrow \quad \hat{\theta}_t^1\, \bar{S}_t^1 = \frac{a_t}{\sigma}. \tag{76}$$

The discounted wealth at time t is given by:

$$\bar{Y}_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{\hat{X}}{e^{rT}}\,\middle|\, \mathcal{F}_t\right].$$

Hence, the risk-free part of the hedging portfolio is:

$$\hat{\theta}_t^0 = \bar{Y}_t - \hat{\theta}_t^1\, \bar{S}_t^1 = \bar{Y}_t - \frac{a_t}{\sigma}. \tag{77}$$

Chapter 9

Continuous time market: Multi-dimensional standard Black-Scholes model (Recall)

We recall here the multi-dimensional standard Black-Scholes model with constant drift vector μ and an invertible volatility matrix σ. Since the formalism is similar to that of the one-dimensional Black-Scholes model, we shall only highlight the differences.
Let $W = (W^1, \ldots, W^N)$ be an N-dimensional standard Brownian motion generating a complete filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}})$.
There are N risky assets with prices $S_t^1, \ldots, S_t^N$, which evolve according to the following dynamics:

$$dS_t^i = S_t^i \left(\mu_i \, dt + \sum_{j=1}^{N} \sigma_{ij} \, dW_t^j \right), \quad S_0^i > 0, \quad \text{for } i = 1, \ldots, N,$$

where $\mu \in \mathbb{R}^N$ is the drift vector and $\sigma \in \mathbb{R}^{N \times N}$ is the volatility matrix, which is assumed to be invertible.
The market price of risk vector $\gamma \in \mathbb{R}^N$ is defined as the unique solution to the linear system:

$$\sigma\gamma = \mu - r\mathbf{1},$$

where $\mathbf{1}$ is the vector in $\mathbb{R}^N$ whose components are all equal to 1.
Define the process ξ by:

$$\xi_t = \exp\left(-(\gamma, W_t) - \frac{1}{2}\|\gamma\|^2 t \right), \tag{78}$$

where (x, y) denotes the scalar product of $x, y \in \mathbb{R}^N$.
The process ξ is a $\mathbb{P}$-martingale and satisfies the stochastic differential equation:

$$d\xi_t = -\xi_t \, (\gamma, dW_t).$$

According to Girsanov's Theorem, the process $\bar{W}_t := W_t + \gamma t$ is a standard N-dimensional Brownian motion under a new probability measure $\mathbb{Q}$ defined by the Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi_T. \tag{79}$$

$\mathbb{Q}$ is an equivalent martingale measure, i.e., equation (61) is satisfied. In fact, the dynamics of the discounted price process $\bar{S}_t^i$ is given by:

$$d\bar{S}_t^i = \bar{S}_t^i \sum_j \sigma_{ij}\, d\bar{W}_t^j.$$

Consider a portfolio $\theta = (\theta^0, \theta^1, \ldots, \theta^N)$. The associated wealth process $V(\theta)$, gains process $G(\theta)$, and their discounted counterparts $\bar{V}(\theta)$ and $\bar{G}(\theta)$ are defined as in equations (62), (63), and (64).
Let $D(\bar{S}_t)$ denote the $N\times N$ diagonal random matrix whose diagonal entries are $(D(\bar{S}_t))_{ii} = \bar{S}_t^i$.
Writing the portfolio as $\theta = (\theta^0, H)$, where $H = (\theta^1, \ldots, \theta^N)$ represents the risky part of the portfolio, it follows (cf. equation (67)) that θ is self-financed if and only if:

$$\bar{V}_t(\theta) = \bar{V}_0(\theta) + \int_0^t (H_s, D(\bar{S}_s)\sigma\, d\bar{W}_s)_{\mathbb{R}^N}, \quad \forall t \in \mathbb{T}. \tag{80}$$

Let X be a derivative with maturity T. Then $\theta = (\theta^0, H)$ is a hedging portfolio of X if and only if, in addition:

$$\frac{X}{e^{rT}} = \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{e^{rT}}\right] + \int_0^T (H_t, D(\bar{S}_t)\sigma\, d\bar{W}_t)_{\mathbb{R}^N}. \tag{81}$$

Chapter 10

Merton's Portfolio with N risky assets and the Mutual Fund Theorem

Merton's Portfolio with N Risky Assets

We consider the N-dimensional Black-Scholes market as described in Section 9. Let U be an admissible utility function, satisfying Condition (4.1).

We recall the optimization problem: given an initial wealth $K > x$ at time $t = 0$, find a self-financed portfolio θ such that

$$\sup_{\substack{\theta \text{ self-financed} \\ V_0(\theta)=K}} \mathbb{E}\left[U\left(V_T(\theta)\right)\right] = \mathbb{E}\left[U\left(V_T(\hat{\theta})\right)\right]. \tag{82}$$

As before, the resolution of this problem proceeds in two steps:

1. First, find the optimal final wealth $\hat{X}$;
2. Then, find a hedging portfolio $\hat{\theta}$ that replicates $\hat{X}$.

As in the case of one risky asset, the optimal final wealth $\hat{X}$, i.e., the solution of (70), is once more given by

$$\hat{X} = I\left(\frac{\hat{\lambda}\xi_T}{e^{rT}}\right), \tag{83}$$

where

$$\xi_t = \exp\left(-(\gamma, W_t)_{\mathbb{R}^N} - \frac{1}{2}\|\gamma\|^2 t\right) = \exp\left(-(\gamma, \bar{W}_t)_{\mathbb{R}^N} + \frac{1}{2}\|\gamma\|^2 t\right).$$

The optimal wealth $\hat{X}/e^{rT}$ can be written as a stochastic integral, and it is proved in the course that:

$$\frac{\hat{X}}{e^{rT}} = K + \int_0^T b_t\,(-\gamma, d\bar{W}_t)_{\mathbb{R}^N}, \tag{84}$$

where

$$b_t = \frac{1}{e^{rT}}\,\mathbb{E}^{\mathbb{Q}}\left[\hat{\lambda}\frac{1\xi_T}{e^{rT}} I'\left(\frac{\hat{\lambda}\xi_T}{e^{rT}}\right)\Big|\mathcal{F}_t\right].$$

The risky part $\hat{H}$ of the hedging portfolio is now given by (81) and (84):

$$\sigma' D(\bar{S}_t)\hat{H}_t = -b_t\gamma \quad \Rightarrow \quad \hat{H}_t = -b_t \left(D(\bar{S}_t)\right)^{-1} (\sigma')^{-1} \gamma \tag{85}$$

The discounted investment at time t in the i-th risky asset is the i-th coordinate of the random vector $v_t \equiv D(\bar{S}_t)\hat{H}_t$:

$$v_t = -b_t \, (\sigma')^{-1} \gamma$$

The discounted wealth at time t is denoted

$$\bar{Y}_t = \mathbb{E}^{\mathbb{Q}} \left[\frac{\hat{X}}{e^{rT}} \middle| \mathcal{F}_t \right]$$

The risk-free part of the hedging portfolio is then given by:

$$\hat{\theta}_t^0 = \bar{Y}_t - \sum_{i=1}^{N} \hat{H}_t^i \bar{S}_t^i \tag{86}$$

Merton's Mutual Fund Theorem
Let $\mathcal{U}_{\text{ALL}}$ be the set of all admissible utility functions, according to Condition 4.1. Merton's Mutual Fund Theorem applies with respect to $\mathcal{U}_{\text{ALL}}$:

Theorem 10.3 (Two-Fund Separation Theorem; Arbitrary Utility Function)
Let μ, σ be deterministic, and suppose the market S is arbitrage-free. Then there exists a mutual fund M such that for each initial wealth x and each admissible utility function $\hat{U} \in \mathcal{U}_{\text{ALL}}$, the optimal portfolio $(\hat{\theta}^0, \hat{H})$ is a time-dependent linear combination of the risk-free fund B and the mutual fund M.

In this setting, the market S may be incomplete.

One can choose the numéraire portfolio (i.e., the case $U = \log$ and $x = 1$) as the mutual fund.

There exist examples of more general markets with stochastic γ for which the Merton Mutual Fund Theorem does not hold for arbitrary utility functions.

Proof outline:
We only consider the setting of the multi-dimensional Black-Scholes model described in Section 9.
For initial wealth $x = 1$ and a logarithmic utility function, the optimal wealth process is denoted by N, with discounted value $\bar{N}_t$ satisfying:

$$\bar{N}_t = 1 + \int_0^t \bar{N}_s(\gamma, d\bar{W}_s)$$

and

$$\bar{N}_T = 1 + \int_0^T \bar{N}_s(\gamma, d\bar{W}_s)$$

The hedging portfolio $\theta^{(\text{Num})} = (\theta_0^{(\text{Num})}, H^{(\text{Num})})$ associated with this optimal wealth process satisfies:

$$\sigma' D(\bar{S}_t) H_t^{(\mathrm{Num})} = \bar{N}_t \gamma$$

It now follows from formula (85) that the risky part $H_t(x, U)$ of the optimal portfolio $\theta_t(x, U)$, for arbitrary initial wealth x and utility function $U \in \mathcal{U}_{\mathrm{ALL}}$, satisfies:

$$H_t(x, U) = k_t(x, U) H_t^{(\mathrm{Num})}$$

for some real $\mathcal{F}_t$-measurable random variable $k_t(x, U)$.
So, $\theta_t(x, U)$ can be realized by taking $k_t(x, U)$ shares of the numéraire portfolio N, and investing the remaining wealth in the risk-free asset.