Notes on Kleinian Groups

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Abstract

Some notes in the course of reading and coding up Grandma's recipes from the book Indra's Pearls by Mumford, Series and Wright[1].

1 Möbius transformations

A Möbius transformation is specified by four complex parameters,

$$Tz = \frac{a\,z + b}{c\,z + d}$$

They have the extraordinary feature that they map the set of all circles and straight lines onto the set of all circles and straight lines, known collectively as *clines*. Möbius transformations also preserve angles and their orientations, that is to say they are *conformal* maps, whilst freely distorting other shapes.

2 The Riemann Sphere and $\hat{\mathbb{C}}$

The extended complex plane and its model the *Riemann sphere*[1] is the complex plane with one point at infinity. The extended plane represents the extended complex numbers

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

It worth noting that there is only one such point, as *complex infinity* has infinite magnitude but undefined phase; having undefined phase is more obvious in the case of complex zero. Any rational function with complex coefficients can be extended to a holomorphic function on the Riemann sphere. The extended complex numbers have extended arithmetic but do not form a *field* due to not having additive or multiplicative inverses, however the following operations are defined:

for all nonzero z	$z + \infty = \infty$	$z \times \infty = \infty$	$\infty \times \infty = \infty$	$z/0 = \infty$	$z/\infty = \infty$	$\infty/0 = 0$	$0/\infty = 0$
undefined	$\infty + \infty$	$\infty - \infty$	$0 \times \infty$	$0/0$	∞/∞		

The complex plane is mapped to the unit sphere with its south pole at the origin and it's north pole represents the point at inifinity. The points on the complex plane are mapped to the sphere by a ray from the point at to the north pole to where this intersects the surface of the sphere; this is a complex projective space and can be thought of as the complex projective line $\boldsymbol{P}^1(\boldsymbol{C})$.

Among those rules some other equations regarding the value of a transformation at infinity is give on page 70, the authors omit the real story here and the relevant equation should have been stated:

$$T(\infty) = \lim_{z \to \infty} \frac{a\,z + b}{c\,z + d} = \lim_{z \to \infty} \frac{a + b/\infty}{c + d/\infty} = \frac{a}{c}$$

1. Named after the great German mathematician Bernhard Riemann (1826-1866) who among many other profound contributions to mathematics made pioneering contributions to differential geometry and created the field of Riemannian geometry with impacts on group theory, analysis and algebraic and differential topology.

3 The Kleinian Group

It turns out that the group of automorphism of the Riemann sphere, (which are invertible conformal maps), are exactly the Möbius transformations with non-zero determinants. The Möbius transformations are *homographies* of the complex projective line. Two matrices yield the same transformation iff they differ by a non-zero factor. The group of Möbius transformations is the *projective linear group* $\mathrm{PGL}(2, C)$. The Kleinian group is a discrete subgroup of the group of orienteation preserving isometries of hyperbolic 3-space $\boldsymbol{H}^3$, this is identifiable with $\mathrm{PSL}(2, C)$ which is the quotient group of the 2 by 2 matrices of determinant 1 by their *centre*. Thus a Kleinian group can be defined as a subgroup Γ of $\mathrm{PGL}(2, C)$ acting on one of these spaces.

Bibliography

[1] David Mumford, Caroline Series, and David Wright. *Indra's Pearls: The Vision of Felix Klein*. Cambridge University Press, 2002.