Applied Game Theory

Lecture 2

Sebastian Fest

Beliefs

- A player forms beliefs about the other players' strategies.
- These beliefs can be expressed as probability distributions.

Beliefs: Example

1 \ 2	L	M	R
U	8, 1	0, 2	4, 0
C	3, 3	1, 2	0, 0
D	5, 0	2, 3	8, 1

- Assume that player 1 thinks that player 2 plays L with probability $1/2$, M with probability $1/4$ and R with probability $1/4$.
- That is: $\theta_2(L) = 1/2$, $\theta_2(M) = 1/4$, $\theta_2(R) = 1/4$, which is often written $(1/2, 1/4, 1/4)$.
- If player 1 plays U, then expected payoff is:

$$u_1(U, \theta_2) = (1/2)8 + (1/4)0 + (1/4)4 = 5$$

Beliefs: Concepts I

- A probability distribution for player i over the other players' strategies is written θ_{-i} , where $\theta_{-i} \in \Delta S_{-i}$ and where ΔS_{-i} is the set of probability distributions over the strategies of all the players, except for player i.
- With two players, player 1's belief about the behavior of player 2 is a function $\theta_2 \in \Delta S_2$ such that, for a single strategy $s_2 \in S_2$, $\theta_2(s_2)$ is interpreted as the probability that player 1 thinks that player 2 plays the strategy s_2.

Beliefs: Concepts II

- One has that $\theta_2(s_j) \geq 0$ for each $s_j \in S_j$ and $\sum_{s_j \in S_j} \theta_2(s_j) = 1$.
- When we use probability distributions, payoffs are uncertain, and are thus expressed as expected *payoff:*

$$u_i(s_i, \theta_{-i}) = \sum_{s_{-i} \in S_{-i}} \theta_{-i}(s_{-i}) u_i(s_i, s_{-i})$$

Mixed strategies

- A player can select a strategy according to a probability distribution. This is called a *mixed strategy*.
- Example: You flip a coin a play U with probability $1/2$ and D with probability $1/2$.
- A mixed strategy for player i is written $\sigma_i \in \Delta S_i$
- A mixed strategy which puts all probability weight on a single strategy is called a *pure strategy*.

Mixed strategies: Real world examples

- Hitter in baseball
- Tennis serves
- Penalty kick in football

Mixed strategies: example

1 \ 2	L	M	R
U	8, 1	0, 2	4, 0
C	3, 3	1, 2	0, 0
D	5, 0	2, 3	8, 1

- Assume again that $\theta_2 = (1/2, 1/4, 1/4)$
- $u_1(U, \theta_2) = (1/2)8 + (1/4)0 + (1/4)4 = 5$
  $u_1(C, \theta_2) = (1/2)3 + (1/4)1 + (1/4)0 = 7/4$
  $u_1(D, \theta_2) = (1/2)5 + (1/4)2 + (1/4)8 = 5$
- A mixed strategy $\sigma_1 = (1/2, 0, 1/2)$ will then yield $u_1(\sigma, \theta_2) = (1/2)5 + (0)7/4 + (1/2)5 = 5$ to player 1.

General Assumptions

- Rationality: Each player will select the strategy that leads to the outcome he most prefers i.e. each player acts to maximize its expected payoff.
    - Note, this does not rule out that players incorporate a concern for others.
- Common knowledge: All players have a common understanding of the game that is played.
    - A particular fact F is said to be common knowledge between the players if each player knows F, each player knows that the others know F, each player knows that every other player knows that each player knows F, ad infinitum.

Prisoners' dilemma

Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of communicating with the other. The prosecutors lack sufficient evidence to convict the pair on the principal charge, but they have enough to convict both on a lesser charge. Simultaneously, the prosecutors offer each prisoner a bargain. Each prisoner is given the opportunity either to betray the other by testifying that the other committed the crime, or to cooperate with the other by remaining silent. The offer is:

- If A and B each betray the other, each of them serves two years in prison.
- If A betrays B but B remains silent, A will be set free and B will serve three years in prison (and vice versa).
- If A and B both remain silent, both of them will only serve one year in prison (on the lesser charge).

Prisoners' dilemma

1 \ 2	C	D
C	2, 2	0, 3
D	3, 0	1, 1

- Assume each player ranks outcomes accordingly:
  Free $\succ$ *One year* $\succ$ *Two years* $\succ$ *Three years*
- Strategy C is *dominated* by strategy D since it is always better to play D than C.

Dominance

- Formally: A pure strategy s_i of player i is dominated if there is a strategy (pure or mixed) $\sigma_i \in \Delta S_i$ such that $u(\sigma_i, s_{-i}) > u(s_i, s_{-i})$ for all strategy profiles $s_{-i} \in S_{-i}$ of the other players.
- Rationality implies that no player plays dominated strategies.

Dominance & efficiency

1 \ 2	C	D
C	2, 2	0, 3
D	3, 0	1, 1

- Note that individual rationality does not necessarily lead to efficient outcomes.
- Often, we find a clash between individual and joint interests (strategic tension).
- A strategy profile is Pareto efficient if there does not exist any other strategy profile where at least one player achieves a higher payoff while at the same time no player achieves a lower payoff.

Dominance: Examples

1 \ 2	L	R
U	2, 3	5, 0
D	1, 0	4, 3

(a)

1 \ 2	L	C	R
U	8, 3	0, 4	4, 4
M	4, 2	1, 5	5, 3
D	3, 7	0, 1	2, 0

(b)

1 \ 2	L	R
U	4, 1	0, 2
M	0, 0	4, 0
D	1, 3	1, 2

(c)

Best response I

1 \ 2	L	C	R
U	2, 6	0, 4	4, 4
M	3, 3	0, 0	1, 5
D	1, 1	3, 5	2, 3

- Suppose player 1 holds the following beliefs over player 2's strategies: $(1/3, 1/2, 1/6)$
- If player 1 chooses the strategy U, then the expected payoff is $(1/3)2 + (1/2)0 + (1/6)4 = 8/6$
- The expected payoff is $7/6$ if M is chosen and $13/6$ if D is chosen.
- Strategy D is *best response*: $BR_1(1/3, 1/2, 1/6) = \{D\}$

Best response II

1 \ 2	L	C	R
U	2, 6	0, 4	4, 4
M	3, 3	0, 0	1, 5
D	1, 1	3, 5	2, 3

- If player 2 holds the belief $(1/2, 1/4, 1/4)$ over player 1's strategies, then player 2 expects 4 if he chooses L, 13/4 if he chooses C and 4 if he chooses R.
- Both strategy L and R are then best responses: $BR_2(1/2, 1/4, 1/4) = \{L, R\}$

Best response III

- A strategy is best response if it maximizes the expected payoff, given the beliefs about the other players' strategies.
- Formally: Suppose player i has a belief $\theta_{-i} \in \Delta S_{-i}$ about the strategies played by the other players. Player i's strategy $s_i \in S_i$ is a *best response* if:

$$u_i(s_i, \theta_{-i}) \geq u_i(s_i', \theta_{-i}) \, \forall \, s_i' \in S_i.$$

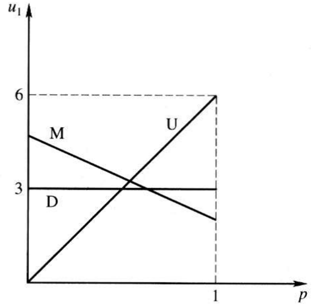
- For any belief θ_{-i} of player i, we denote the set of best responses by $BR_i(\theta_{-i})$.

Best response & dominance

- Let B_i be the set of strategies for player i that are best responses, over all the possible beliefs of player i.
- Let UD_i be the set of strategies for player i that are not strictly dominated.
- In a game with only two players, then $B_i = UD_i$, i.e. strategies can only be best responses if they are not dominated.

Best response & dominance: Example

1 \ 2	(p) L	(1 − p) R
U	6, 3	0, 1
M	2, 1	5, 0
D	3, 2	3, 1



Iterative dominance I

1 \ 2	X	Y	Z
A	3, 3	0, 5	0, 4
B	0, 0	3, 1	1, 2

Iterative dominance II

1 \ 2	~~X~~	Y	Z
A	~~3, 3~~	0, 5	0, 4
B	~~0, 0~~	3, 1	1, 2

(a)

1 \ 2	~~X~~	Y	Z
A	~~3, 3~~	~~0, 5~~	~~0, 4~~
B	~~0, 0~~	3, 1	1, 2

(b)

1 \ 2	~~X~~	Y	Z
A	~~3, 3~~	~~0, 5~~	~~0, 4~~
B	~~0, 0~~	3, 1	1, 2

(c)

Iterative dominance III

- "*Iterated removal of strictly dominated strategies*":
    - *Delete all dominated strategies so that we have a reduced game.*
    - *Continue deleting all dominated strategies in the reduced game, repeat.*
- We assume that when player i forms beliefs about the other players' strategies, player i assumes that the other players do not play dominated strategies.
- The set of strategies that survive iterated dominance is called *rationalizable strategies*.

Iterative dominance and mixed strategies

1 \ 2	L	C	R
U	5, 1	0, 4	1, 0
M	3, 1	0, 0	3, 5
D	3, 3	4, 4	2, 5

- One also has to consider mixed strategies.
- For player 2 L is dominated by the mixed strategy $(0, 1/2, 1/2)$.
- Through iterated dominance we are left with the rationalizable strategy $\{M, R\}$.

Limits to rationalization

1 \ 2	Stag	Hare
Stag	5, 5	0, 4
Hare	4, 0	4, 4

- Rationalizability does not always lead to a unique strategy profile.
- Even if beliefs are rational, they are not necessarily correct.
- This leads to *strategic uncertainty*