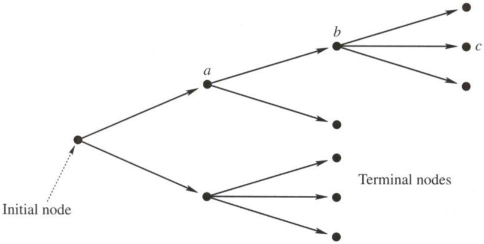
Applied Game Theory

Lecture 5

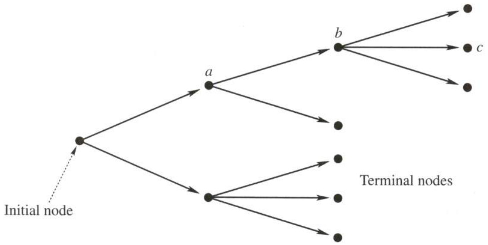
Sebastian Fest

Extensive form I



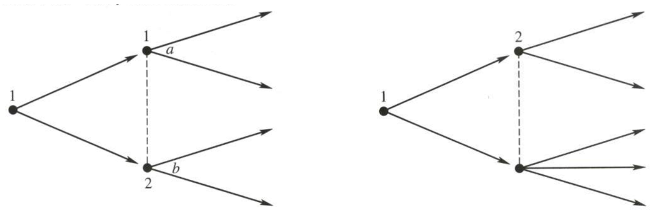
- Successors = nodes that can be reached from a given node.
- Immediate successors = the nodes that follows immediately after a given node.
- Predecessors and immediate predecessors can be traced backwards from a given node.

Extensive form II



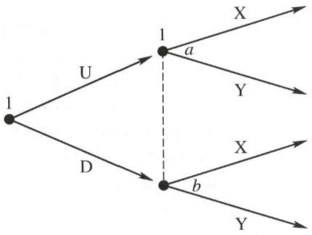
- Every node is a successor of the initial node, and the initial node is the only one with this property.
- Each node except the initial node has exactly one immediate predecessor.
- Multiple branches extending from the same node have different action labels.

Extensive form III



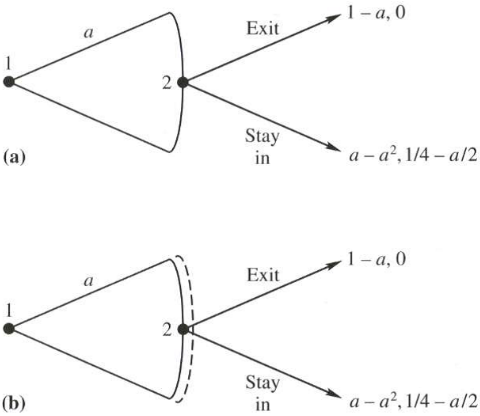
- Each information set contains decision nodes for only one of the players.
- All nodes in a given information set must have the same number of immediate successors and they must have the same set of action lables on the branches leading to these successors.

Extensive form IV



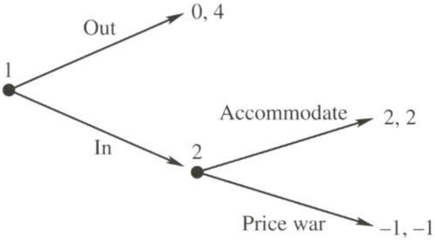
- We assume that players have perfect recall, i.e. they recall own and others' observed previous choices.
- Games with *perfect information*: All players know where they are in the game tree, i.e. there is only one node per information set.
- Games with *imperfect information*: There is at least one contingency in which the player who has to move does not know exactly where he is in the game tree.

Extensive form V



- In many games players can choose from an infinite number of actions (e.g. a number between 0 and 1). In such cases we draw an arch to represent choices over an interval.

The "price war" game

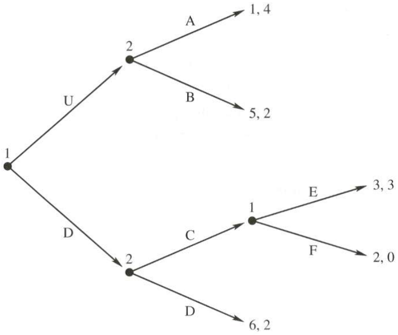


1 \ 2	A	P
I	2, 2	–1, –1
O	0, 4	0, 4

Sequential rationality

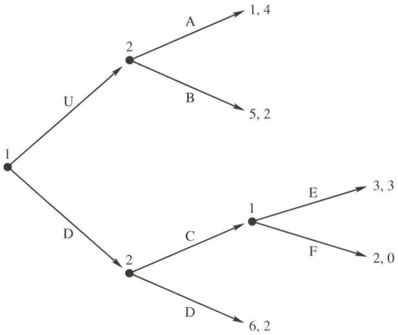
- *Sequential rationality:* An optimal strategy for a player should maximize his or her expected payoff, conditional on every information set at which this player has the move. That is, player i's strategy should specify an optimal action from each of player i's information set, even those that player i does not believe (ex ante) will be reached in the game.
- Sequential rationality implies not playing *conditionally dominated* strategies.
- A strategy s_i for player i is conditionally dominated if, contingent on reaching some information set of player i, there is another strategy σ_i that strictly dominates s_i.

Backward induction I



- *Backward induction:* Analyze the game from the end to the beginning. At each decision node, delete any actions that are dominated, given the terminal nodes that can be reached through the play of the actions identified at successor nodes.

Backward induction II

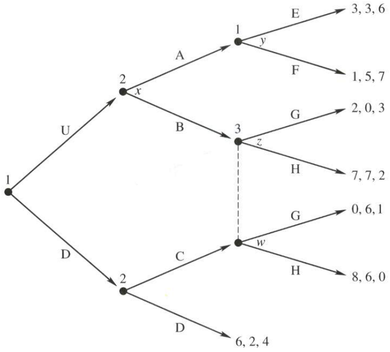


- (DE, AC) is the only sequentially rational strategy profile.
- (DE, AC) is also a Nash equilibrium: DE is best response for player 1 to player 2's strategy AC. And for player 2, AC is best response to DE.
- All (finite) games with perfect information can be solved by backward induction, and backward induction identifies an equilibrium.

Subgame

- Given an extensive-form game, a node x in the tree is said to initiate a subgame if neither x nor any of its successors are in an information set that contains nodes that are not successors of x. A subgame is the tree structure defined by such a node x and its successors.
- If there is a node y that is not a successor of x, but is connected to x or one of its successors by a dashed line, then x does not initiate a subgame.
- That is, when players are in a subgame, then it is common knowledge between them that they are inside the subgame.
- Proper subgame: Subgame that does not start with the initial node.

Subgame: Example

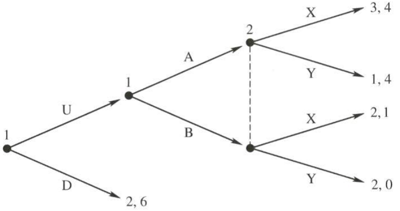


- Two subgames in this game: The whole game, and the game that starts at node y.

Subgame perfect Nash Equilibrium

- A strategy profile is called a subgame perfect Nash equilibrium if it specifies a Nash equilibrium in every subgame of the original game.
- Subgame perfection is a *refinement* of the Nash equilibrium.
- In games with perfect information, backwards induction always gives subgame perfect equilibria.

Subgame perfect Nash Equilibrium: Example



1 \ 2	X	Y
UA	3, 4	1, 4
UB	2, 1	2, 0
DA	2, 6	2, 6
DB	2, 6	2, 6

1 \ 2	X	Y
A	3, 4	1, 4
B	2, 1	2, 0

- $(UA, X), (DA, Y)$ and (DB, Y) are Nash equilibria in the game.
- One *proper subgame*, where only (A, X) is a Nash equilibrium.
- (UA, X) is thus the only subgame perfect Nash equilibrium.

Advertising and competition I

- Firm 1 selects an advertising level a at $2a^3/81$ costs per a.
- This affects the price function $p = a - q_1 - q_2$, where q_1, q_2 is the output of firm 1 and 2, respectively.
- The two firms set outputs simultaneously and independently. Production costs are zero.

Advertising and competition II

- For a given a firm 1 selects q_1 in order to maximize profits $(a - q_1 - q_2)q_1 - 2a^3/81$. Firm 2 selects q_2 to maximize profits $(a - q_1 - q_2)q_2$.
- This gives best response functions $BR_1(q_2) = \frac{a-q_2}{2}$ and $BR_2(q_1) = \frac{a-q_1}{2}$ and thus $q_1 = q_2 = a/3$.
- Firm 1 puts $a/3 = q_1 = q_2$ into the profit function and thus gets $z_1(a) = \frac{a^2}{9} - \frac{2a^3}{81}$.
- Optimal a satisfies $z_1'(a) = 0$ i.e. $\frac{2a}{9} - \frac{6a^2}{81} = 0$ such that $a^* = 3$.
- The strategy profile $a^* = 3$, $q_1(a) = a/3$ and $q_2(a) = a/3$ is a subgame perfect Nash equilibrium.

A model of limit capacity I

- Two firms meet the inverse demand function $p = 900 - q_1 - q_2$, where p is price and q_1, q_2 is output for firm 1 and 2, respectively. Production costs $= 0$
- The firms must choose between constrained or unconstrained capacity, where *constrained capacity* cost 50 (thousand) and gives max 100 units, while unconstrained capacity costs 175.
- Step 1: Firm 1 chooses between i) staying out of the industry, ii) build small capacity and iii) build large capacity.
- Step 2: Firm 2 observes firm 1s choice and chooses between the same alternatives.
- Step 3: The firms simultaneously and independently select output quantity.

A model of limit capacity II

- Solve by backward induction. Begin with step 3:
- If only firm i is in the market it selects q_i to maximize $(900 - q_i)q_i$.
  - Under unconstrained capacity, this yields $q_i = 450$ and revenue $202, 5$ (thousand) and thus profit $202.5 - 175 = 27.5$.
  - Under constrained capacity (100) the firm obtains revenue $(900 - 100)100 = 80$ (thousand). This yields profit $80 - 50 = 30$.
- If both firms enter the industry without capacity constraints, we have a Cournot game where $(900 - q_i - q_j)q_i$ is maximized, yielding best response functions $BR_i(q_j) = 450 - q_j/2$. This gives Nash equilibrium in the subgame $q_1 = q_2 = 300$. Revenue is 90 and profit $90 - 175 = -85$.

A model of limit capacity III

- If both firms have constrained capacity, they produce 100 each. Income is $(900 - 100 - 100)100 = 70$ (thousand) and thus profit $70 - 50 = 20$.
- If only one firm has constrained capacity it will produce 100 , while the other firm produce best response to 100, i.e. $450 - 100/2 = 400$. This yields $p = 400$ and profits $40 - 50 = -10$ and$160 - 175 = -15$,respectively.
- Step 2 and 1:
- Subgame perfect equilibrium: Firm 1 plays L (large capacity), while firm 2 plays $SS'N''$, i.e. does not enter the market.

A model of limit capacity IV