Applied Game Theory

Lecture 7

Sebastian Fest

Repeated games I

- People and organizations often interact in ongoing relationships:
  - employment relationships
  - international trade
  - competitions and cooperations
- These relationships can be analyzed as repeated games.
- In a repeated game, *history of play* can affect future behavior.

Repeated games II

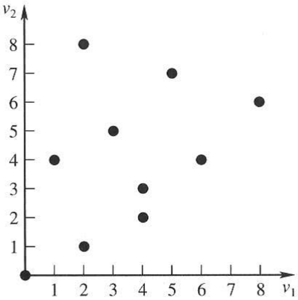
- A repeated game is played over a discrete number of periods (period 1, period 2, etc.).
- We let t denote a given period, and T denote total number of periods.
- In each period $t = 1, 2, ...T$ players play a *stage game*.

Repeated games III

- The stage game is a collection $\langle N, A, u \rangle$, where N is the number of players, $A = A_1 * A_2 * ... * A_n$ is the set of action profiles and $u_i(a)$ is player i's stage game payoff when a is played.
- The stage game is played in all periods. For each period t, the players have observed the history of play, i.e. all actions from first period to period $t - 1$.
- The payoff for the whole game is the sum of the stage game payoffs from period 1 to T.
- When necessary, future payoffs are discounted with a discount factor δ.

Repeated games: Example I

1 \ 2	X	Y	Z
A	4, 3	0, 0	1, 4
B	0, 0	2, 1	0, 0



- Suppose the stage game is played twice ($T = 2$).
- We have 6 different action profiles in $t = 1$, yielding 6 different information sets in $t = 2$.
- Since the stage game is also played in $t = 2$, we have 36 different strategy profiles and payoff vectors for the whole game.

Repeated games: Example II

1 \ 2	X	Y	Z
A	4, 3	0, 0	1, 4
B	0, 0	2, 1	0, 0

1 \ 2	X	Y	Z
A	5, 7	1, 4	2, 8
B	1, 4	3, 5	1, 4

- Subgame perfection requires equilibrium play in every subgame.
- The matrix to the right shows the subgame following (A, Z) in the first stage, yielding the payoff $(1, 4)$ in $t = 1$.

Repeated games: Example III

1 \ 2	X	Y	Z
A	4, 3	0, 0	1, 4
B	0, 0	2, 1	0, 0

1 \ 2	X	Y	Z
A	5, 7	1, 4	2, 8
B	1, 4	3, 5	1, 4

- A player's preference over actions in the subgame starting at $t = 2$ are the same as for the stage game. Hence, the Nash equilibria for the subgame and the stage game are identical.
- Sequential rationality requires players to play a Nash equilibrium in the last period.
- We can then have a subgame perfect Nash equilibrium for the two period game if players also play a Nash equilibrium in the first period.

Repeated games IV

- For any repeated game, any sequence of stage Nash profiles can be supported as the outcome of a subgame perfect Nash equilibrium.
- In addition, other equilibria might exist that don't stipulate the play of stage Nash profiles. Consider the following strategy profile:
    - Select (A,X) in the first period and then, as long as player 2 did not deviate from X in the first period, select (A,Z) in the second period
    - If player 2 deviated by playing Y or Z in the first period, then play (B,Y) in the second period.

Repeated games V

1 \ 2	X	Y	Z
A	4, 3	0, 0	1, 4
B	0, 0	2, 1	0, 0

- No-one has incentives to deviate from this strategy profile in any of the subgames, hence we have a subgame perfect Nash equilibrium.
- If player 2 deviates in $t = 1$, he is a "cheater" and lost reputation. This yields a lower payoff in $t = 2$ (and a lower payoff overall).
- Hence, in the two period game, subgame perfection requires *stage Nash profiles* in period 2, but there may exist "reputational equilibria" where one does not play stage Nash profile in $t = 1$.

Infinitely repeated games I

1 \ 2	C	D
C	2, 2	0, 3
D	3, 0	1, 1

- An infinitely repeated game has an infinite horizon, i.e. $T = \infty$, which is often useful to model long term relationships.
- A subgame perfect equilibrium can often be achieved through the play of simple strategies, e.g. *trigger strategies*:
  - Play (C,C) (cooperative profile) in each period. If someone deviates, play (D,D) (punishment profile) forever after.

Infinitely repeated games II

1 \ 2	C	D
C	2, 2	0, 3
D	3, 0	1, 1

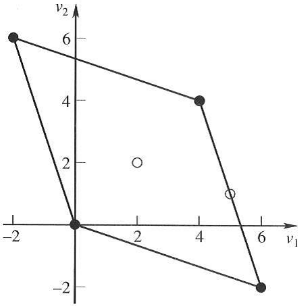
- The players will not deviate from this strategy if the present value from cooperation is greater then the present value from deviating:
- $2 + 2\delta + 2\delta^2 + ... = \frac{2}{1-\delta} \geq 3 + \delta + \delta^2 + ... = 3 + \frac{\delta}{1-\delta}$, i.e. $\delta \geq \frac{1}{2}$

Infinitely repeated games III

1 \ 2	C	D
C	4, 4	–2, 6
D	6, –2	0, 0

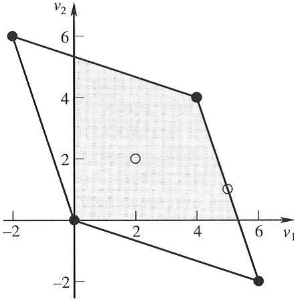
- In this game, the trigger strategy constitutes a subgame perfect equilibrium if $\frac{4}{1-\delta} \geq 6$, i.e. $\delta \geq \frac{1}{3}$.
- Note that this holds true for all periods, i.e. whether $t = 1$ or $t = 1000$ etc.
- The typical case in repeated games: Short term gain from cooperation vs. long term loss from deviation.
- The higher δ, the larger is the long term loss, and the easier it is to sustain cooperation in equilibrium.

Infinitely repeated games IV



- All payoff vectors within the diamond can be obtained as an average stage payoff.
- All payoff vectors within the diamond can be supported in a subgame perfect equilibrium as long as each player gets more than zero (payoff from (D,D)), and the discount factor is sufficiently high.

Infinitely repeated games V



- The folk theorem: Consider any infinitely repeated game. Suppose there is a stage Nash profile that yields payoff vector w (w_i for player $i, i = 1, 2, ..., n$). Let v be any feasible average per period payoff such that $v_i > w_i$. The vector v can be supported arbitrarily closely by a subgame perfect Nash equilibrium if δ is sufficiently close to 1.

Infinitely repeated games: Cournot

- Consider a Cournot game with $p = 1 - q_1 - q_2$. Production costs is 0, so that each player obtain payoff $(1 - q_i - q_j)q_i$
- This yields Nash equilibrium $q_1 = q_2 = \frac{1}{3}$ and profits $\frac{1}{9}$ to each firm.
- Collusion on monopoly quantity where each produces $\frac{1}{4}$ yields a payoff of $\frac{1}{8}$ to each. But this cannot be sustained as an equilibrium in the stage game.

Infinitely repeated games: Cournot

- In an infinitely repeated game, trigger strategies can constitute an equilibrium. Play $\frac{1}{4}$ in each period as long as both played this in the past. If someone deviates, play stage Nash profil$\frac{1}{3}$ for ever.
- By deviating firm i maximizes $(1 - \frac{1}{4} - q_i)q_i$ which yields $q_i = \frac{3}{8}$ and payoff $\frac{9}{64}$ in the first deviation period. But thereafter it obtains $\frac{1}{9}$ forever.
- Collusion can be sustained as a subgame perfect Nash equilibrium if $\frac{1}{8(1-\delta)} \geq \frac{9}{64} + \frac{\delta}{9(1-\delta)}$, i.e. $\delta \geq \frac{9}{17}$.

Self-enforcing contracts

- Collusion is an example of a self-enforcing contract.
- Since collusion is illegal, and thus external enforcement impossible, the parties must rely on self-enforcement.
- Sometimes external enforcement is desirable, but not possible, e.g.:
    - employment contracts (hard to verify relevant variables).
    - international trade (no supranational enforcement).