Applied Game Theory

Lecture 4

Sebastian Fest

Nash equilibrium in mixed strategies I

1 \ 2	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

Matching Pennies

- There are no pure strategiy Nash equilibria in the MP game.
- Do any equilibria exist at all?
    - Check for mixed strategy Nash equilibria.

Nash equilibrium in mixed strategies II

1 \ 2	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

Matching Pennies

- A mixed strategy can be a best response if it yields the same expected payoff for both pure strategies.
- Let q denote the probability that P2 plays H. P1 then gets $q(1) + (1 - q)(-1) = 2q - 1$ from playing H and $q(-1) + (1 - q)(1) = 1 - 2q$ from playing T.
  - Expected payoff is the same if $2q - 1 = 1 - 2q$, i.e. $q = 1/2$.

Nash equilibrium in mixed strategies III

1 \ 2	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

Matching Pennies

- Let p denote the probability that P1 plays H. P2 then gets $1 - 2p$ from playing H and $2p - 1$ from playing T.
  - Expected payoff is the same if $1 - 2p = 2p - 1$, i.e. $p = 1/2$.
- The strategy profile $((1/2, 1/2), (1/2, 1/2))$ is thus a *Nash equilibrium in mixed strategies.*

Nash equilibrium in mixed strategies IV

- The strategy profile $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$, where $\sigma_i \in \Delta S_i$ for each player i, is a Nash equilibrium in mixed strategies if and only if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ for each $s_i' \in S_i$ for each player i. That is, σ_i is a best response to σ_{-i} for each player i.
- All games with a finite number of players and strategies have at least one Nash equilibrium in pure or mixed strategies.

Nash equilibrium in mixed strategies V

- Procedure for finding mixed strategy equilibria:
  - Calculate the set of rationalizable strategies by performing the iterated dominance procedure.
  - For each player, set up equations in order to determine for which mixed strategy of the other player, the player becomes indifferent between choosing pure strategies.
  - Solve these equations to determine equilibrium mixing probabilities.

Nash equilibrium in mixed strategies: Examples

- Evidence for MNE in practice:
  - Penalty kicks in football (Palacios-Huerta, 2003, "Professionals play minimax")
  - Serves in tennis (Walker and Wooders, 2001, "Minimax Play at Wimbledon")

Contracts I

- Contracts and institutions (like the legal system) help us achieving efficient equilibria:
  - Contracts mitigate conflicts between common interests and individual incentives.
  - Contracts reduce the probability of inefficient coordination.
  - Contracts reduce strategic uncertainty.

Contracts II

- A contract is an agreement about behavior that is intended to be enforced.
- We have a contractual relationship if the parties, with some deliberation, work together to set the terms of their relationship. We distinguish between:
  - the **contracting phase**, in which players set the terms of the contract.
  - the **implementation phase**, in which the contract is carried out and enforced.

Enforcement I

- A contract is **self-enforced** if the players have the individual incentives to abide by the terms of the contract.
- A contract is **externally enforced** if the players are motivated to behave by the actions of a third party (like the court), who can enforce the contract on the basis of verifiable information.
- A contract is **automatically enforced** if it is honored at the same moment that it is agreed upon.

Enforcement II

1 \ 2	I	N
I	z_1, z_2	y_1, x_2
N	x_1, y_2	$0, 0$

- Two players agree upon a contract on how much to invest in a common project. I=Invest (honor the contract), N= not invest (breach the contract).
- Total surplus is maximized when both invests (I, I), i.e. $z_1 + z_2 > x_1 + y_2$, $z_1 + z_2 > x_2 + y_1$, $z_1 + z_2 > 0$
- Can the players make an enforceable contract which ensures that (I, I) is played?
- Yes, if it is a Nash equilibrium, i.e. $z_1 \geq x_1$, and $z_2 \geq x_2$. The contract is then self-enforcing.

Transfers

1 \ 2	I	N
I	z_1, z_2	$y_1 + \beta, x_2 - \beta$
N	$x_1 + \alpha, y_2 - \alpha$	$\gamma, -\gamma$

- If (I, I) is not a Nash equilibrium, then the parties can agree on transfer m from player 2 to 1 (i.e. $m < 0$ if transfer from player 1 to 2) in case (I, I) is not played.
- $m = \alpha$ when (N, I) is played, $m = \beta$ when (I, N) is played, and $m = \gamma$ when (N, N) is played.
- The **underlying game** is when $m = 0$. When $m \neq 0$ we call it the **induced game**.

Complete contract

- A third party (court) can enforce the transfers: External enforcement transforms the game from the underlying game to the induced game.
- Suppose the court is able to enforce α, β, γ.
- (I,I) can now be made a Nash equilibrium as long as $z_1 > x_1 + \alpha$ and $z_2 > x_2 - \beta$.

Complete contract: Example

1 \ 2	I	N
I	8, 8	−4, 4
N	10, −2	0, 0

- We can have (I, I) as a Nash equilibrium by setting e.g. $\alpha = -3, \beta = 0, \gamma = 0$.

1 \ 2	I	N
I	8, 8	−4, 4
N	7, 1	0, 0

- *Full verifiability:* If the court can verify all strategy profiles, then the parties can always make an enforceable contract that yields the efficient outcome, i.e. that maximizes the players' total payoff.

Incomplete contract

- Suppose that it is not possible to write a complete contract. If the contract was breached, the court cannot identify which strategy profile has been played due to *limited verifiability*.

1 \ 2	I	N
I	z_1, z_2	$y_1 + \alpha, x_2 - \alpha$
N	$x_1 + \alpha, y_2 - \alpha$	$\alpha, -\alpha$

- In order to achieve (I, I) as a Nash equilibrium, one needs $z_1 > x_1 + \alpha$ and $z_2 > x_2 - \alpha$, i.e. $z_1 - x_1 > \alpha > x_2 - z_2$.

Incomplete contract: Example I

1 \ 2	I	N
I	8, 8	–4, 4
N	10, –2	0, 0

- (I, I) is achieved with $\alpha = -3$.

1 \ 2	I	N
I	8, 8	–7, 7
N	7, 1	–3, 3

Incomplete contract: Example II

1 \ 2	I	N
I	10, 10	−4, 12
N	12, −4	0, 0

- $10 - 12 > \alpha \Leftrightarrow \alpha < -2$ and $12 - 10 < \alpha \Leftrightarrow \alpha > 2$
- There is no α that makes (I, I) a Nash equilibrium, i.e. (I, I) cannot be enforced.

Breach remedies

- It is costly to write complete contracts and *breach remedies* can help reducing these costs.
- *Expectation damages*: the court forces the defendant to transfer to the plaintiff the sum of money needed to give the plaintiff the payoff that he or she would have received had the contract been fulfilled.
- *Reliance damages*: the court imposes a transfer that returns the plaintiff to the state in which he or she would have been if the contract had never been signed.
- *Restitution damages:* the breacher must compensate any unjust enrichment that the defendant obtained by breaching the contract.

Expectation damages I

1 \ 2	I	N
I	z_1, z_2	$y_1 + \beta, x_2 - \beta$
N	$x_1 + \alpha, y_2 - \alpha$	$\gamma, -\gamma$

1 \ 2	I	N
I	z_1, z_2	$z_1, x_2 + y_1 - z_1$
N	$x_1 + y_2 - z_2, z_2$	$0, 0$

- If player 1 breaches (N, I), then player 2 shall end up with z_2, i.e. $y_2 - \alpha = z_2$ such that $\alpha = y_2 - z_2$.And if player 2 breaches, then $y_1 + \beta = z_1$, i.e. $\beta = z_1 - y_1$. (Assume $\gamma = 0$)
- (I, I) is a Nash equilibrium if and only if $z_1 \geq x_1 + y_2 - z_2$ and $z_2 \geq x_2 + y_1 - z_1$
  - i.e. $z_1 + z_2 \geq x_1 + y_2$ and $z_1 + z_2 \geq x_2 + y_1$ which define conditions under which (I, I) is efficient.

Expectation damages II

1 \ 2	I	N
I	z_1, z_2	$y_1 + \beta, x_2 - \beta$
N	$x_1 + \alpha, y_2 - \alpha$	$\gamma, -\gamma$

1 \ 2	I	N
I	z_1, z_2	$z_1, x_2 + y_1 - z_1$
N	$x_1 + y_2 - z_2, z_2$	$0, 0$

- Under expectation damages (I, I) is enforced only if (I, I) is efficient.
- Expectation damages is an optimal breach remedy but it requires a great deal of information.

Reliance damages I

1 \ 2	I	N
I	z_1, z_2	$y_1 + \beta, x_2 - \beta$
N	$x_1 + \alpha, y_2 - \alpha$	$\gamma, -\gamma$

1 \ 2	I	N
I	z_1, z_2	$0, x_2 + y_1$
N	$x_1 + y_2, 0$	$0, 0$

- Suppose the court cannot verify z_i, but that it knows y_i which is the players' payoff after the opponent has breached the contract.
- Assume $y_i \leq 0$ and that (N, N) is a Nash equilibrium ($\gamma = 0$).
- Reliance damages now implies $\alpha = y_2$ and $\beta = -y_1$

Reliance damages II

1 \ 2	I	N
I	z_1, z_2	$y_1 + \beta, x_2 - \beta$
N	$x_1 + \alpha, y_2 - \alpha$	$\gamma, -\gamma$

1 \ 2	I	N
I	z_1, z_2	$0, x_2 + y_1$
N	$x_1 + y_2, 0$	$0, 0$

- We then get (I, I) as a Nash equilibrium if and only if $z_1 \geq x_1 + y_2$ and $z_2 \geq x_2 + y_1$.
- Unless the defendant's damage y_i is sufficiently large, reliance awards may not support the efficient outcome.

Restitution damages

1 \ 2	I	N
I	z_1, z_2	$y_1 + \beta, x_2 - \beta$
N	$x_1 + \alpha, y_2 - \alpha$	$\gamma, -\gamma$

1 \ 2	I	N
I	z_1, z_2	$x_2 + y_1, 0$
N	$0, x_1 + y_2$	$0, 0$

- The payoff from breaching a contract is $x_i \geq 0$.
- Restitution damages then yields $\alpha = -x_1$ and $\beta = x_2$
- (I, I) is a Nash equilibrium if and only if $z_1 \geq 0$ and $z_2 \geq 0$.
- Restitution damages can give (I, I) as a Nash equilibrium in cases where (I, I) is not efficient (i.e. where $z_1 + z_2 > x_1 + y_2$ or $z_1 + z_2 > x_2 + y_1$ is not satisfied)