Exercise 1

Please explain the following:

a) What is a rationalizable strategy profile?

Answer: A rationalizable strategy profile is the set of all strategy profiles that remain after one iteratively removes those strategies from the strategy space that are never best responses. Similarly, the set of strategies that survive iterated dominance can be called rationalizable.

b) What is a Nash equilibrium?

Answer: A strategy profile $s \in S$ is a Nash equilibrium if and only if $s_i \in BR_i(s_{-i})$ for each player *i*. That is, $u_i(s_i s_{-i}) \geq u_i(s_i' s_{-i})$ for every $s_i' \in S_i$ and each player *i*.

c) Can there be more than a single Nash equilibrium if the game only has a single rationalizable strategy profile?

Answer: No. The set of Nash equilibria is a subset of the set of rationalizable strategy profiles. In other words, strategies that are never best responses cannot be a member of the set of Nash equilibria.

d) Is a rationalizable strategy profile always a Nash equilibria?

Answer: No. A Nash equilibrium strategy profile requires that the strategy for each player are mutual best responses, i.e. for each player there is no other strategy that yields a higher payoff to that player. The set of rationalizable strategy profile can be larger than the set of Nash equilibria.

Exercise 2

a) Compute the mixed-strategy equilibria of the following games:

1 \ 2	A	B
A	2, 4	0, 0
B	1, 6	3, 7

1 \ 2	L	M	R
U	8, 3	3, 5	6, 3
C	3, 3	5, 5	4, 8
D	5, 2	3, 7	4, 9

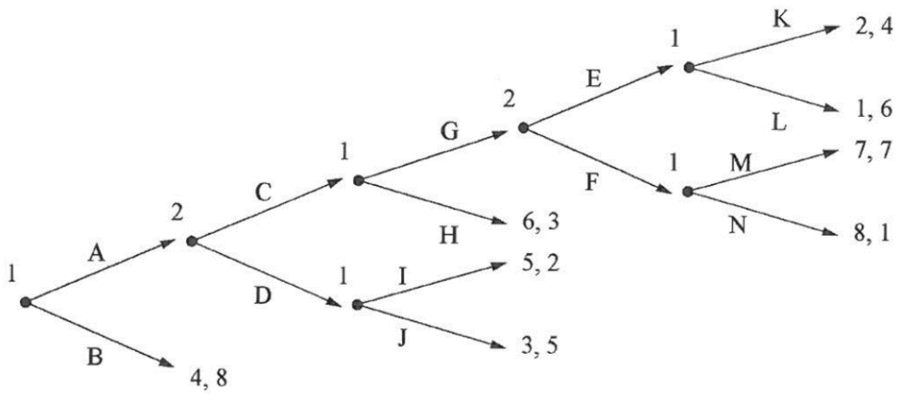
Answer:

Game to the left: MSNE = $\{\left(\frac{1}{5}, \frac{4}{5}\right), \left(\frac{3}{4}, \frac{1}{4}\right)\}$

Game to the right: MSNE = $\{\left(\frac{3}{5}, \frac{2}{5}, 0\right), \left(0, \frac{1}{2}, \frac{1}{2}\right)\}$

Exercise 3

a) Solve the following sequential game by using the concept of backward induction.

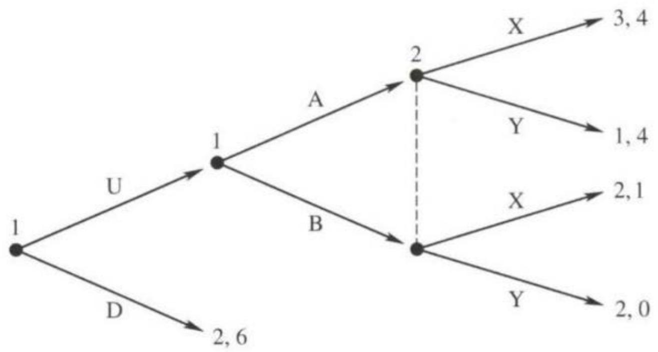


Answer: SPNE = {(AHIKN), (CE)}

b) How many subgames does this game have?

Answer: This game has seven subgames. The whole game and six proper subgames.

c) Find all Nash equilibria and subgame perfect Nash equilibria in this game:



Answer: NE = {(UA, X), (DA,Y), (DB,Y)}, SPNE = {(UA, X)}

Exercise 4

Consider the following two-player one-shot game where each player can either invest (I) or not invest (N) in a project.

1 \ 2	I	N
I	6, 8	0, 9
N	7, 0	0, 0

a) What is the Nash equilibrium of this game?

Answer: NE = {(I, N), (N, N), (N, I)}

Assume now that players can write a contract that stipulates that each player has to invest, and assume in addition that the courthouse can observe players' behavior.

b) Write down the contract and the induced game for the case in which the contract follows the rule of 'expectation damages' in case the contract is breached. Is the strategy profile (I, I) enforceable?

Answer:

Under expectation damages, the following transfer rules are applied in the induced game:

$$\begin{array}{ccc} {}^{1}/_{2} & I & N \\ I & z_1, z_2 & z_1, x_2 + y_1 - z_1 \\ N & x_1 + y_2 - z_2,\ z_2 & 0, 0 \end{array}$$

The contract is as follows:

$$m = \begin{cases} \alpha = -8 \\ \beta = 6 \\ \gamma = 0 \end{cases}$$

The induced game is:

$$\begin{array}{ccc} {}^{1}/_{2} & I & N \\ I & 6{,}8 & 6{,}3 \\ N & -1{,}8 & 0{,}0 \end{array}$$

In the induced game (I, I) is a NE and thus enforceable. Since (I, I) is the efficient outcome of the underlying game, under expectation damages, (I, I) is enforceable.

c) Write down the contract and the induced game for the case in which the contract follows the rule of 'reliance damages' in case the contract is breached. Is the strategy profile (I, I) enforceable?

Answer:

Under reliance, the following transfer rules are applied in the induced game:

$1/2$	I	N
I	z_1, z_2	$0, x_2 + y_1$
N	$x_1 + y_2, 0$	$0, 0$

The contract is as follows:

$$m = \begin{cases} \alpha = -0 \\ \beta = 0 \\ \gamma = 0 \end{cases}$$

The induced game is:

$1/2$	I	N
I	6,8	0,9
N	7,0	0,0

(I, I) is not enforceable. The damage caused by a breach on each party is not sufficiently large enough.

d) Assume again that the court applies 'expectation damages' in case the contract is breached but assume further that the court can only identify that the contract has been breached but not by which party ('limited verifiability'). Is there a contract under such conditions such that the strategy profile (I, I) is enforceable?

Answer:

Under limited verifiability, the court can only impose one transfer level. The induced game looks as follows:

$1/2$	I	N
I	$6, 8$	$0 + \alpha, 9 - \alpha$
N	$7 + \alpha, 0 - \alpha$	0,0

For (I, I) to be enforceable it must hold that $7 + \alpha \leq 6$ and $9 - \alpha \leq 8$. There exists no α to fulfill both conditions. Hence, the contract is not enforceable under limited verifiability. These are the necessary conditions. Because the court wants to apply expectation damages, and since it cannot distinguish which side breached the contract, the reference for the expectation damage rule would be 8 (the maximum payoff if (I, I) is played. Since 8 is larger than 6, the sufficient condition to impose an enforceable contract under limited verifiability is not satisfied either.

Exercise 5

Dwight is a sales expert as well as a skilled leader, and now he is negotiating the possibility of a new position with Michael, the CEO of the company. The contract specifies the salary t, as well as what Dwight is going to work with: Sales or management. If Dwight works as a Salesperson, he gets a payoff of t - 10,000 and the company gets 100,000 - t. If Dwight instead becomes department manager, he gets a payoff of t - 40,000 while the company gets x - t. If Dwight and Michael do not agree, the company gets 0 while Dwight gets w. Assume that x > 150,000 and w < 90,000. Dwight's negotiation power is given by π_D while Michael's negotiating power is given by π_M. Solve the negotiation problem using the Nash bargaining solution: What should Dwight work with and what will he be paid? (Determine the salary t as a function of $\pi_D, \pi_M, x \wedge w$).

Answer:

If Dwight works as a salesperson, the joint value of their agreement is as follows:

$$[v_D(0) + t] + [v_M(0) - t] = [-10000 + t] + [100000 - t] = 90000$$

The surplus created through this agreement is:

$$90000 - w - 0 = 90000 - w$$

If Dwight works as a manager, the joint value of their agreement is as follows:

$$[v_D(1) + t] + [v_M(1) - t] = [-40000 + t] + [x - t] = x - 40000$$

The surplus created through this agreement is:

$$x - 40000 - w - 0 = x - 40000 - w$$

For x > 130 000 the joined value (and surplus) is maximized if Dwight works as a manager:

$$90000 - w = x - 40000 - w$$

$$130000 \leq x$$

Since we assume that x > 150 000 it is best if Dwight works as a manager.

The standard bargaining solution then implies:

$$U_D = w + \pi_D(x - 40000 - w) = (1 - \pi_D)w + \pi_D(x - 40000)$$

$$U_M = \pi_M(x - 40000 - w)$$

Since $U_D = -40000 + t$, $U_M = x - t$ and $\pi_D + \pi_M = 1$, we can simplify and yield the following function for t:

$$t(\pi_M, \pi_D, w, x) = \pi_M(w + 40000) + \pi_D$$

Exercise 6

a) Assume that the following stage game is played twice ($T = 2$) and that that there is no discounting. Describe a subgame perfect Nash equilibrium for which the strategy profile (U, L) is being played in the first period ($t = 1$)

1 \ 2	L	M	R
U	8, 8	0, 9	0, 0
C	9, 0	0, 0	3, 1
D	0, 0	1, 3	3, 3

Answer:

The game has three Nash equilibria (NE = {(D, M), (C, R), (D, R)}). The efficient strategy profile which is not a Nash equilibrium is (U, L). Because there is more than one Nash equilibrium in this game, reputational equilibria might exist in which the efficient strategy profile is played in the first period. Notice the following strategy description for example:

Strategy for player 1: Play U in t = 1 and D in t = 2 if player 2 played L in t = 1, otherwise play C in t = 2.

Strategy for player 2: Play L in t = 1 and R in t = 2 if player 1 played U in t = 1, otherwise play M in t = 2.

To see that this constitutes a SPNE, notice that if each player adheres to the described strategy, his or her payoff from the game is equal to 8 + 3 = 11. If either player decides to deviate in t = 1 and play his or her best response in the first period, then the payoff from the game equals 9 + 1 = 10, which is strictly less than adhering to the prescribed strategy. Thus, (UD, LR) is a subgame perfect Nash equilibrium.

b) Assume now that the following stage game is played over two rounds ($T = 2$). Is there a subgame perfect Nash equilibrium for which (C, C) is played in the first period? Explain.

1 \ 2	C	D
C	2, 2	0, 3
D	3, 0	1, 1

(D, D) is the only one Nash equilibrium in this prisoners' dilemma (stage) game. Since the only Nash equilibrium has to be played in the final round, (D, D) will also be played in the first (previous) period. This can be easily seen by noticing that, if one player plays C in the first period, the other player can gain 3 by playing D in the same period. This strategy yields a higher final payoff than playing C as well (2 + 1 = 3 by playing C in t=1 and D in t = 2; 3 + 1 = 4 by playing D in both periods). Cooperation cannot be sustained in the finitely repeated prisoners' dilemma game because there are no other Nash equilibria that either player can induce after observing that the other player did not cooperate.

c) How would your answer in b) change if the horizon of the game is extended to ten periods ($T = 10$).

Answer:

Changing the horizon of the game to ten periods has no effect on the possibility to sustain mutual cooperation (C, C) in the prisoners dilemma game. The explanation given in b) is also valid for the last two periods of the ten period game. By the logic of backward induction, deviation will then also occur in the 8th period, and subsequently in all previous periods such that mutual defection is the outcome for each round.

Exercise 7

Consider the following strategic setting. Every fall, two neighboring elementary schools raise money for field trips and playground equipment by selling giant candy bars. Suppose that individuals in the surrounding communities love candy bars and care about helping the children from both schools but have a slight preference for purchase of candy from the closest school. (In other words, candy bars from the two schools are imperfect substitutes.) Demand for school i's candy, in hundreds of bars, is given by$q_i = 24 - 2p_i + p_j$, where p_i is the price charged by school i, and p_j is the price charged by the other school j. Assume that the candy bars are donated to the school and there are no costs of selling the candy. The schools simultaneously set prices and sell the number of candy bars demanded, so school 1's payoff is the revenue $p_1 * q_1$ and school 2's payoff is the revenue$p_2 * q_2$.

a) Compute the schools' best-response functions and the Nash equilibrium prices. How much money (in hundreds) does each school raise?

Answer:

$$(1)\ \Pi_1 = q_1 * p_1 = (24 - 2p_1 + p_2) * p_1 = (24p_1 - 2p_1^2 + p_1 * p_2)$$

$$(2)\ \Pi_2 = q_2 * p_2 = (24 - 2p_2 + p_1) * p_2 = (24p_2 - 2p_2^2 + p_1 * p_2)$$

Maximizing yields:

$$(3)\ \frac{\partial \Pi_1}{\partial p_1} = 0 = 24 - 4p_1 + p_2 \xrightarrow{\text{yields}} BR_1(p_2) = 6 + \frac{p_2}{4}$$

$$(4)\ \frac{\partial \Pi_2}{\partial p_2} = 0 = 24 - 4p_2 + p_1 \xrightarrow{\text{yields}} BR_2(p_1) = 6 + \frac{p_1}{4}$$

Putting (3) in (4) yields:

$$(5)\ p_1 = p_2 = 8$$

The associated profit is:

$$(6)\Pi_1(8{,}8) = 128 \text{ and } (7)\Pi_2(8{,}8) = 128$$

Each school raises 12 800 with a price per candy bar of 8.

b) In an effort to raise more money, the schools decide to meet and work together to set a common price for the candy bars sold at both schools. What price should the schools charge to maximize their joint fundraising revenues? How much money (in hundreds) would each school raise if they charge this price?

Answer:

The profit function of the 'monopolist', assuming ($p_1 = p_2 = p$) is given by:

$$(8)\ \Pi = q * p = (q_1 + q_2) * p = (48 - 2p) * p = (48p - 2p^2)$$

Maximizing (8) yields:

$$(9)\ \frac{d\Pi}{dp} = 0 = (48 - 4p) \xrightarrow{\text{yields}} p^* = 12$$

The associated profit is:

$$(10)\ \Pi_1(12{,}12) = 144 \text{ and } (11)\ \Pi_2(12{,}12) = 144$$

If both schools agree to set a common price, then they should set the price equal to 12. Each school will then raise 14 400.

c) Suppose that there is no way to externally enforce the price-fixing agreement, so the schools must rely on repeated interaction and reputations to sustain cooperation. If the schools anticipate holding the same fundraiser each fall for 5 years (and no longer), will they be able to maintain the price obtained in part (b)? Explain how or why not.

Answer:

In order to see if a price fixing regime is self-sustainable, we need to calculate how much each school could raise if it deviates from the price fixing agreement. In particular:

The best response to a price of 12 is to set the price equal to:

$$(11)BR_i(12) = 6 + \frac{12}{4} = 9 \text{ for each school.}$$

The associated profit of deviating is then:

$$(12)\ \Pi_i(9{,}12) = 162$$

The associated profit when the other player deviates is:

$$(12)\ \Pi_j(12{,}9) = 108$$

This yields the following normal form game representation:

$^1/_2$	C	D
C	144,144	108,162
D	162,108	128,128

The fundraiser game is a prisoners' dilemma game. In the final period, each school has an incentive to deviate from the price agreement and undercut the price of the other school, which in return increases the own profit, while lowering the other school's profit. By the logic of backward induction, it is therefore best for each school to break the price agreement one period before the other school breaks the agreement. Therefore, the only rationalizable strategy for both schools is to set prices equal to 8 in each period.

d) Now suppose that the schools anticipate holding the same fundraiser every year forever. Define δ as the schools' discount factor for periods of a year. Derive a condition on δ that guarantees the schools will be able to sustain a cooperative agreement to sell candy bars at the price obtained in part (b).

Answer:

Assuming that each school plays a simple grim trigger strategy, a price setting regime with a grim trigger strategy is sustainable if the discount factor fulfills the following condition:

$$\delta \geq \frac{(\Pi_i^D - \Pi_i^C)}{(\Pi_i^D - \Pi_i^P)} = \frac{(162-144)}{(162-128)} = \frac{9}{17}$$