Student names: . . . (please update)

Instructions: Update this file (or recreate a similar one, e.g. in Word) to prepare your answers to the questions. Feel free to add text, equations and figures as needed. Hand-written notes, e.g. for the development of equations, can also be included e.g. as pictures (from your cell phone or from a scanner). ***This lab is not graded. However, the lab exercises are meant as a way to familiarise with dynamical systems and to study them using Python to prepare you for the final project.*** *This file does not need to be submitted and is provided for your own benefit. The graded exercises will have a similar format.*

The file `lab#.py` is provided to run all exercises in Python. When a file is run, message logs will be printed to indicate information such as what is currently being run and and what is left to be implemented. All warning messages are only present to guide you in the implementation, and can be deleted whenever the corresponding code has been implemented correctly.

In this exercise, you will explore the different modeling techniques that can be used to control a single joint and segment. We initially start by exploring a single joint controlled by a pair of antagonist spring like muscles and then extend the model by adding dampers to it. These only represent the passive dynamics observed in a real musculoskeletal system. To make the behavior more realistic we then study more complex hill muscle model in detail.

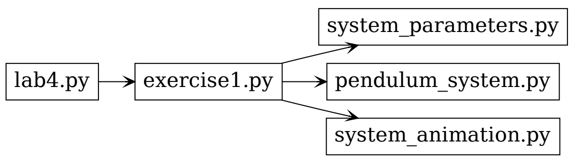


Figure 1: Exercise files dependencies. In this lab, you will be modifying `exercise1.py` and `pendulum_system.py`

Exercise 1 : Pendulum model with passive elements

Mechanical behavior of muscle tissue can be approximated by simple passive elements such as springs and dampers. These elements, when combined properly, allow to study the behavior of muscle under compressive and tensile loads.

Consider the following equation describing the motion of simple pendulum with an external torque T_{ext},

$$I\ddot{\theta} = -mgLsin(\theta) + T_{ext} \tag{1}$$

Considering Inertia $I = mL^2$, the equation of the pendulum can be written as,

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} \tag{2}$$

Consider the system only for the pendulum range $\theta = [-\pi/2, \pi/2]$

Explore the pendulum model with two antagonist spring elements

In this question the goal is to add two antagonist springs to the pendulum model which you are already familiar with from lab 2 exercises. For simplicity we assume the springs directly apply a torsional force on to the pendulum. Use equation 3 to develop the spring model.

Note *: The springs can only produce force in one-direction like the muscles. That is, they can only apply a pulling force and apply a zero force when compressed. In terms of torsion this translates to, spring S1 can exert only clockwise torque and spring S2 can exert only counter-clockwise torque. You need to accommodate for this condition in the equations shown below.*

The setup for the pendulum with a pair of antagonist springs is as shown in figure 2. Use `exercise1.py`, `pendulum_system.py` and `system_parameters.py` files to complete the exercise.

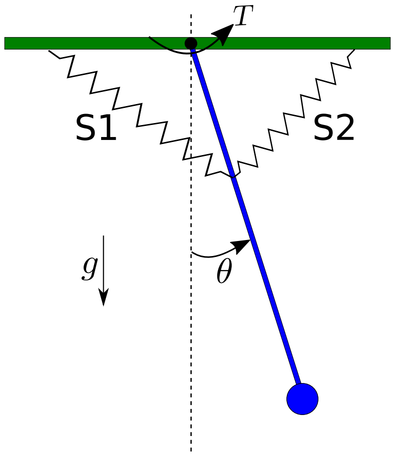


Figure 2: Pendulum model with two springs S1 and S2.
T - Positive torque direction.
g - Gravity.
θ - Angle made by the pendulum

$$T_S = k \cdot (\theta_{ref} - \theta) \tag{3}$$

Where,

- T_S : Torsional Spring force
- k : Spring Constant
- θ_{ref} : Spring reference angle
- θ : pendulum angle

Substituting the above in 2,

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} + \frac{T_S}{I} \tag{4}$$

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} + \frac{k \cdot (\theta_{ref} - \theta)}{I} \tag{5}$$

Use the generalized form of the spring equation described in 5 to extend it to both the antagonist springs S1 and S2 with the necessary conditions to make sure springs do not produce when compressed.

Extending the above equation to both springs,

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + min(\frac{k1 \cdot (\theta_{ref1} - \theta)}{I}, 0) + max(\frac{k2 \cdot (\theta_{ref2} - \theta)}{I}, 0) + \frac{T_{ext}}{I} \tag{6}$$

For all questions the initial conditions used are,

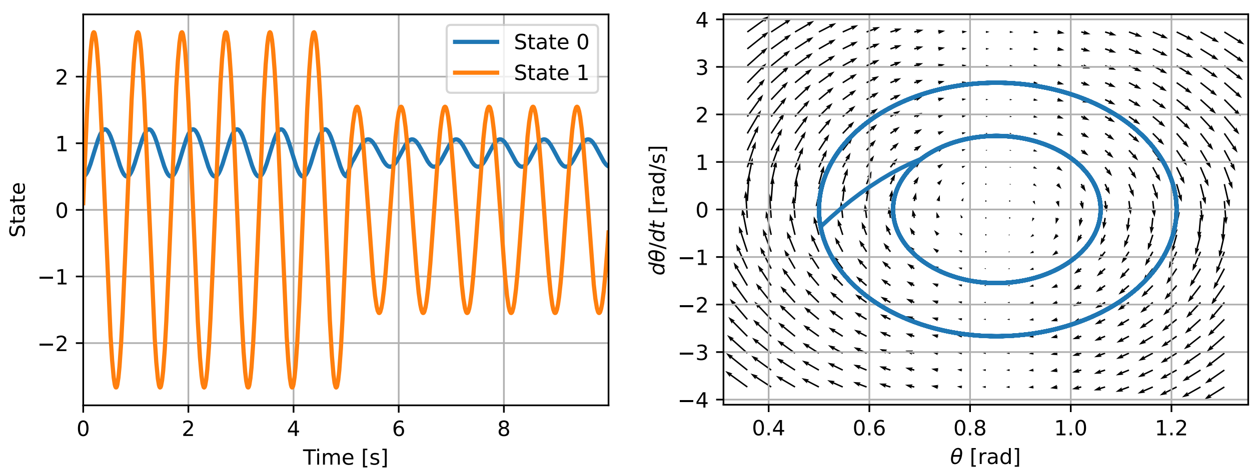
$$\theta = 0.5$$

$$\dot{\theta} = 0.1$$

unless explicity specificied otherwise. Students may use different set of initial conditions

1.a Implement the dynamic equations of the pendulum with springs using equations described above in the function `pendulum_system.py::pendulum_equation`. Does the system have a stable limit cycle behavior? Describe and run an experiment to support your answer. You can use the function `exercise1.py::pendulum_perturbation` to perturb the pendulum either by changing states or applying an external torque. Use the class `system_animation.py::SystemAnimation` to visualize the pendulum. Example code can be found in `exercise1.py::exercise1`

The first requirement for a limit cycle is that the system should have a closed trajectory. The pendulum system with springs does exhibit a closed trajectory behavior. But, in order to have a stable limit cycle the system should converge to a single trajectory as time tends to either positive/negative infinity. One solution to check for stable limit cycle behavior is to use the state and phase plot like shown in figure 3 under perturbations to show that there is no stable limit cycle behavior. At t=5s a pertubation is applied to the velocity of the system ($\dot{\theta} = 2.0$). This pushes the trajectory to a new trajectory and it never returns to the original trajectory. This shows that the system does not have a stable limit cycle. Alternatively students may also use poincare map to show that there is no stable limit cycle. *Students should clearly detail the pertubation they used.*



(a) State of pendulum with spring under perturbations *(b) Phase of pendulum with spring under perturbations*

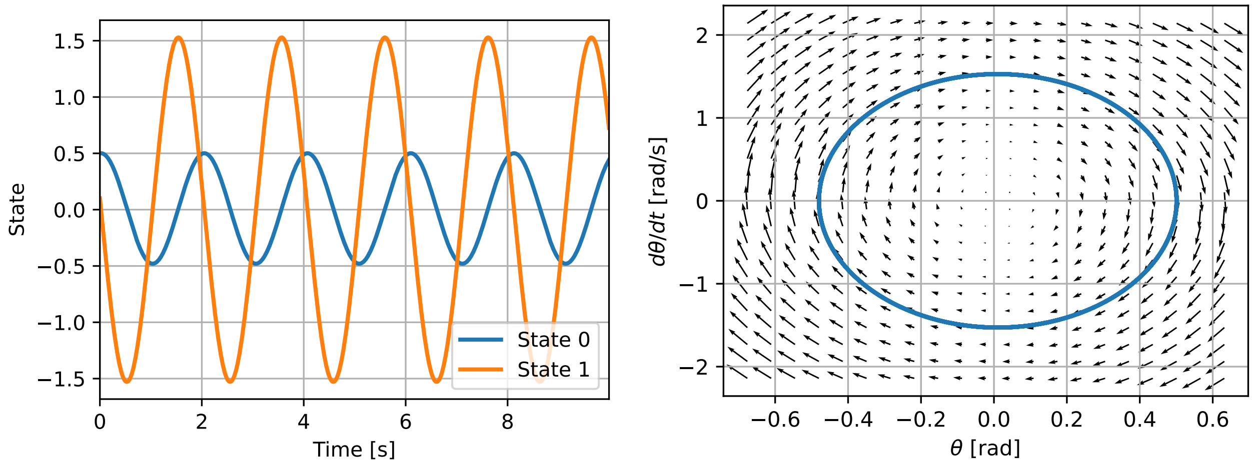
Figure 3: Pertubation approach to check to system limit cycle behavior

1.b Explore the role of spring constant (k) and spring reference angle (θ_{ref}) in terms of range of motion, amplitude and frequency of pendulum. Keep the constants equal, i.e $k_1 = k_2$ and $\theta_{ref1} = \theta_{ref2}$
Refer to `exercise1.py::exercise1` for an example

Spring constant (k): Dictates the magnitude and rate at which the pendulum oscillates. The larger the constant the faster the system oscillates. This can also be seen as the responsiveness of the system. Figure 4 shows the reponse of the pendulum with a small spring constant of $k_1 = k_2 = 0.1$. Figure 5 shows the reponse of the pendulum with a large spring constant of $k_1 = k_2 = 100$.

- For both low/high spring constant, the amplitude of the θ remains the same while the amplitude of $\dot{\theta}$ increases with increase in spring constant magnitude.
- The frequency of both θ and $\dot{\theta}$ increases with higher spring constant and vice-versa.

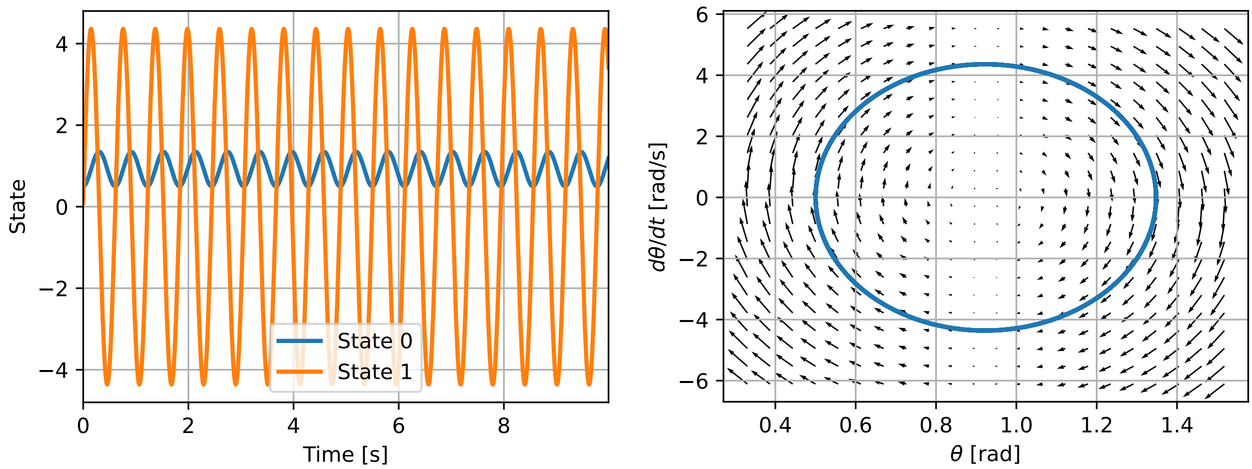
Reference angle: The resting angle for the spring. Since the spring like muscles act only in one direction, the resting angle dictates the angular position of the pendulum at which springs start to act. But having a symmetric spring reference angle for both springs leads to no change in amplitude, range of motion or frequency for a given set of initial conditions. Figures 6 and 7 show the state and phase plot of the system with spring references close to reference ($\theta_{ref1} = -10°$ & $\theta_{ref1} = 10°$) and far from reference ($\theta_{ref1} = -75°$ & $\theta_{ref2} = 75°$) respectively. With reference being $\theta = 0.0°$



(a) State of pendulum with high spring constant

(b) Phase of pendulum with high spring constant

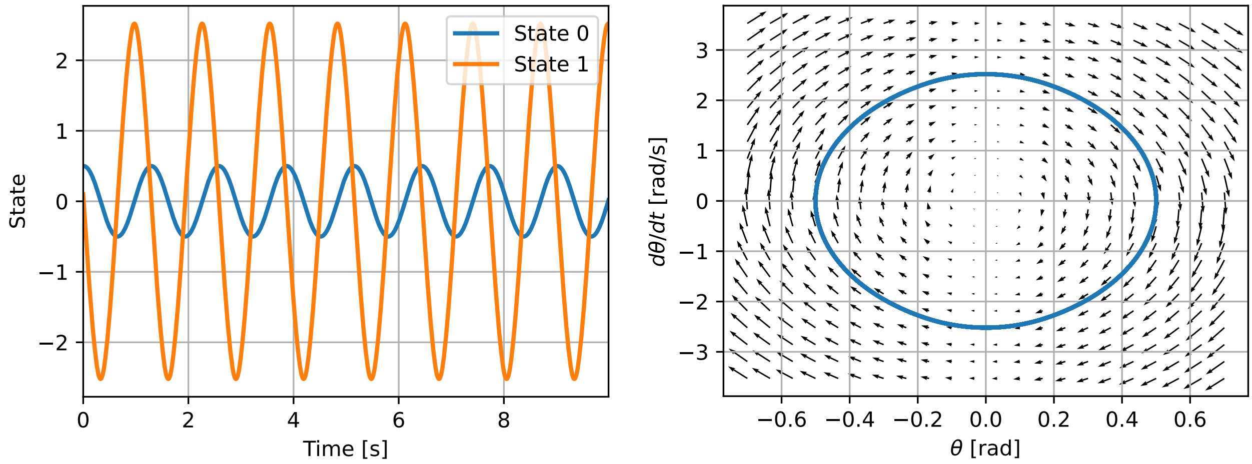
Figure 4: State of pendulum with spring to study the effect of spring constant



(a) State of pendulum with low spring constant

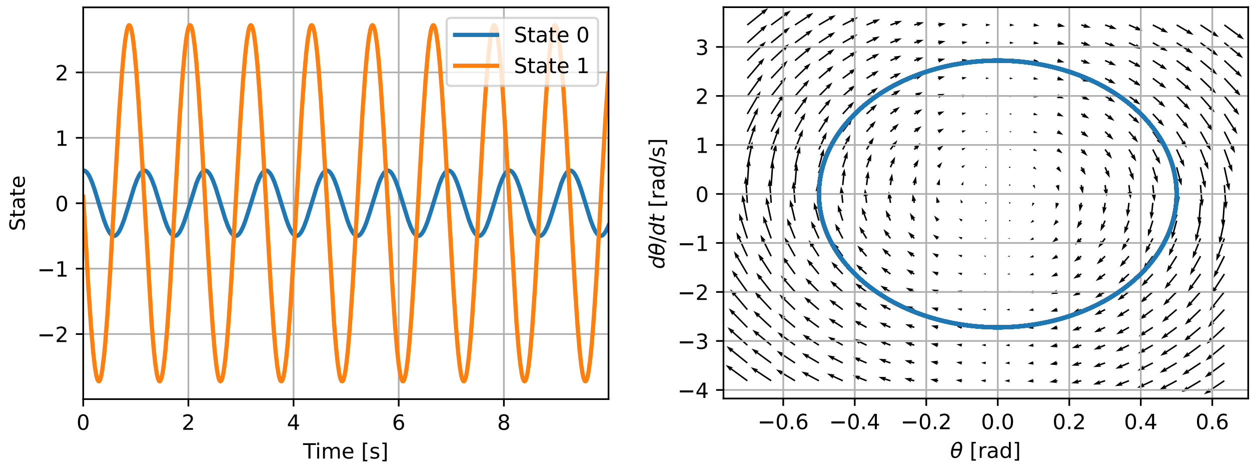
(b) Phase of pendulum with low spring constant

Figure 5: State of pendulum with spring to study the effect of spring constant



(a) State of pendulum with reference close to pendulum rest position *(b) Phase of pendulum with reference close to pendulum rest position*

Figure 6: State of pendulum with spring to study the effect of spring reference



(a) State of pendulum with reference far to pendulum rest position *(b) Phase of pendulum with reference far to pendulum rest position*

Figure 7: State of pendulum with spring to study the effect of spring reference

1.c Explain the behavior of the model when you have asymmetric spring constants (k) and spring reference angles (θ_{ref}), i.e. $k_1 \neq k_2$ and $\theta_{ref1} \neq \theta_{ref2}$ Support your responses with relevant plots

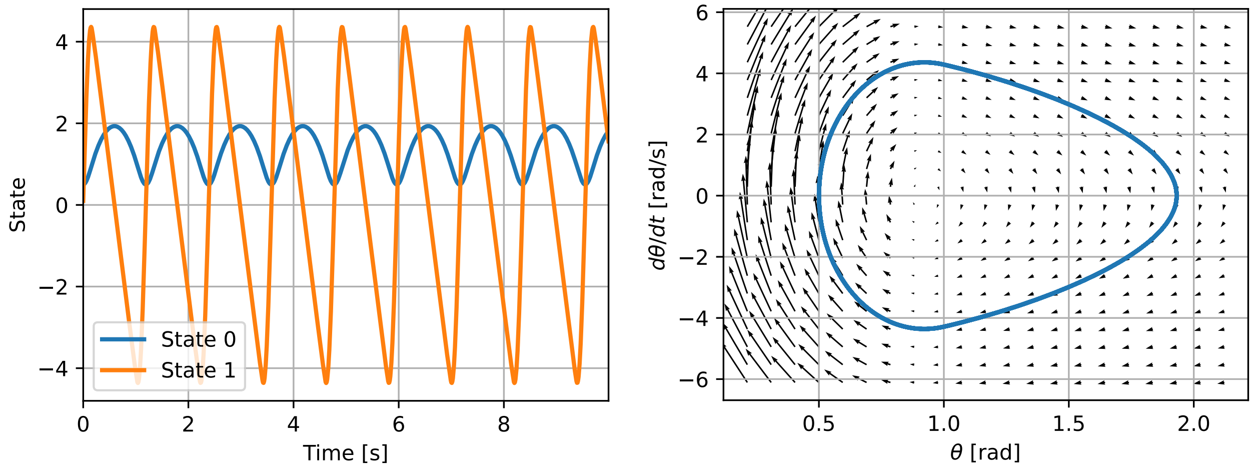
As we saw the previous question, changing the spring constant and reference angle yielded different behaviors. Here we introduce assymetry in the system and change parameters individually.

Variable Spring Constant (k): In figure 8 the spring constants are set to $k_1 = 1.0$ and $k_2 = 100.$ and both spring references set to $\theta_{ref1} = \theta_{ref2} = 0.0°$. With these values, it is clear from the phase plot 8b that variable spring constants introduces assymetry in the shape of the closed trajectory. The side with higher spring constant pulls the pendulum back to the reference faster. While the slower spring side is dominated more by the pendulum dynamics rather than of the spring forces.

Variable Spring reference (θ_{ref}): In figure 9 the spring references are set to $\theta_{ref1} = 0.0°$ and

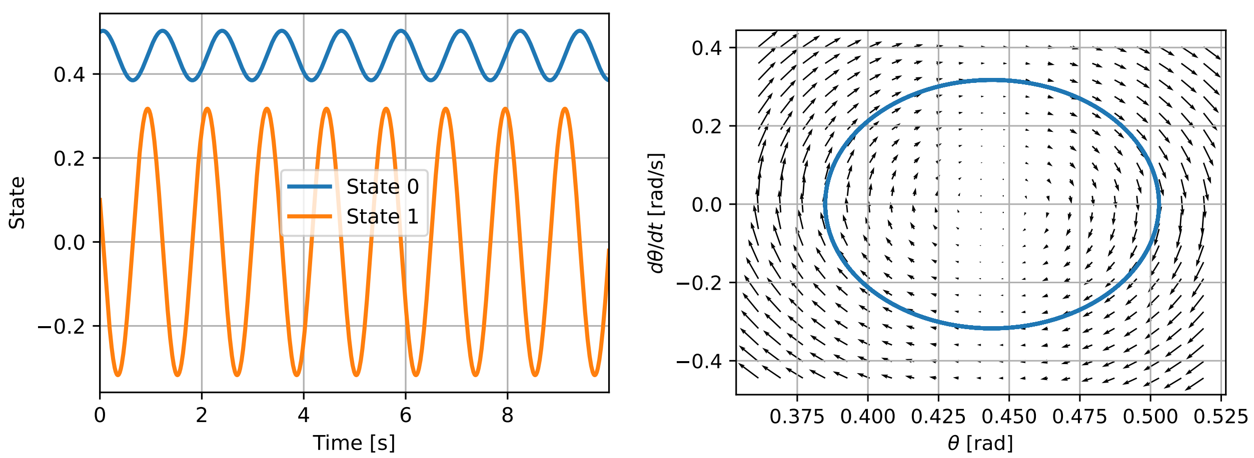
$\theta_{ref2} = 75.^\circ$ and both spring constants set to $k_1 = k_2 = 10.0$. With these values, the it is clear from the phase plot 9b that variable spring references changes the center of the closed trajectory. That is, the pendulum system now oscillates around a non-zero point.

Thus by having unssymetric spring values, we can produce more complex closed loop trajectories in a simple system.



(a) State of pendulum with Variable spring constant *(b) Phase of pendulum with Variable spring constant*

Figure 8: State of pendulum with spring to study the effect of Variable spring constant



(a) State of pendulum with Variable spring reference *(b) Phase of pendulum with Variable spring reference*

Figure 9: State of pendulum with spring to study the effect of variable spring reference

Explore the pendulum model with two antagonist spring and damper elements

Over time muscles lose energy while doing work. In order to account for this property, let us now add a damper in parallel to the spring model. Use equation 7 to develop the damper model.

***Note** : Like the previous springs, the springs in spring-dampers can only produce a force in one-direction. However, the damper terms do not have this limitation and each damper can exert a force in both directions.*

Again use `exercise1.py`, `pendulum_system.py` and `system_parameters.py` files to complete the exercise. The setup for the pendulum model with a pair of antagonist spring and dampers in parallel is as shown in figure 10.

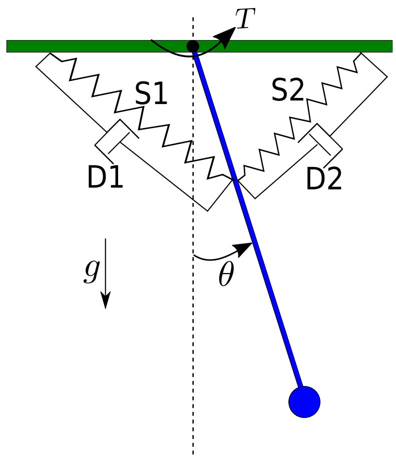


Figure 10: Pendulum model with two springs S1 and S2 and two dampers b1 and b2
T - Positive torque direction.
g - Gravity.
θ - Angle made by the pendulum

$$T_B = b \cdot \dot{\theta} \tag{7}$$

Where,

- T_B : Torsional Damper force
- b : Damping Constant
- $\dot{\theta}$: pendulum angular velocity

The combined spring damper torque is given by,

$$T_S - T_B = k \cdot (\theta_{ref} - \theta) - b \cdot \dot{\theta} \tag{8}$$

The minus for the damper comes from the fact that damper is acting against the work done by the spring.

Substituting the above in 2

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} + \frac{T_S - T_B}{I} \tag{9}$$

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} + (\frac{k \cdot (\theta_{ref} - \theta) - b \cdot \dot{\theta}}{I}) \tag{10}$$

Use the generalized form of the spring equation described in 10 to extend it to both the antagonist spring-damper systems (S1-D1) and (S2-D2).

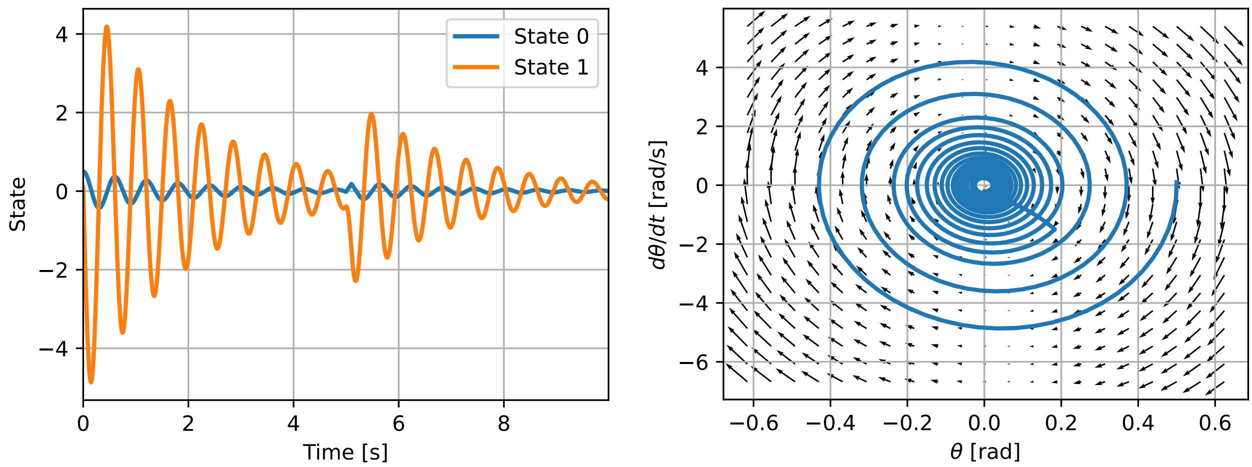
Extending the above equation for both spring and dampers,

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} + min\left(\frac{k1 \cdot (\theta_{ref1} - \theta)}{I}, 0\right) - \frac{b1 \cdot \dot{\theta}}{I} + max\left(\frac{k2 \cdot (\theta_{ref2} - \theta)}{I}, 0\right) - \frac{b2 \cdot \dot{\theta}}{I} \quad (11)$$

1.d Implement the dynamics equations of the pendulum to now include the damping using the equations described above. Modify `pendulum_system.py::pendulum_equation`. How does the behavior now change compared to the pendulum without dampers? Briefly explain and support your responses with relevant plots

In questions 1a-1c observed a closed loop trajectory. By adding dampers to the system introduces a fixed point behavior. The system now loses energy over time and converges to a single position. Even when the system is perturbed the pendulum is returns to the same fixed point showing that there is only one stable fixed point in the system. Figure 11 shows the behavior of the system with the following system parameters,

- $k_1 = 50.0$
- $k_2 = 50.0$
- $b_1 = 0.5$
- $b_2 = 0.5$
- $\theta_{ref1} = -45°$
- $\theta_{ref2} = 45°$



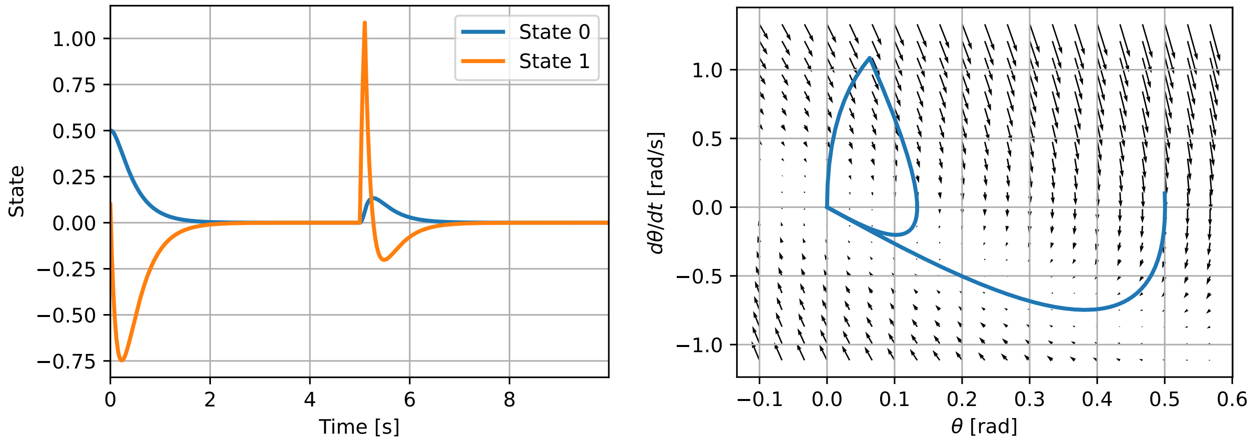
(a) State of pendulum with spring and damper *(b) Phase of pendulum with spring and damper*

Figure 11: Pendulum setup with spring and damper

Note that, like you observed in Lab 2, the nature of the fixed point might change depending on the magnitude of the damping term. Figure 12 shows the behavior of the system with the following system parameters,

- $k_1 = 5.0$
- $k_2 = 5.0$

- $b_1 = 5.0$
- $b_2 = 5.0$
- $\theta_{ref1} = -45°$
- $\theta_{ref2} = 45°$



(a) State of pendulum with spring and large damping *(b) Phase of pendulum with spring and large damping*

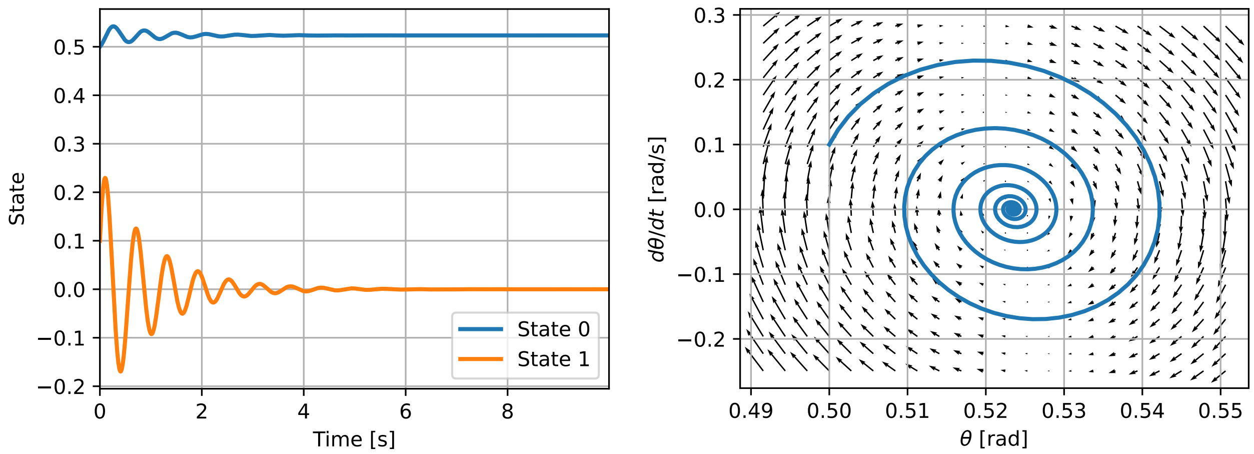
Figure 12: Pendulum setup with spring and large damping term. The fixed point now shows an overdamped behavior

1.e Can you find a combination of spring constants (k), damping constants (b) and spring reference angles (θ_{ref}) that makes the pendulum rest in a stable equilibrium at ($\theta = \pi/6$) radians? Describe how you arrive at the necessary parameters and support your response with relevant plots.

The following parameters set the pendulum at $\pi/6$

$$b1 = 1.$$
$$b2 = 1.$$
$$k1 = 50.0$$
$$k2 = 50.0$$
$$s_{theta\_ref1} = np.deg2rad(0.0)$$
$$s_{theta\_ref2} = np.deg2rad(65.6)$$

Using the knowledge from previous questions and setting system parameters assymetrically we obtain a pendulum convergence point at $\theta = \pi/6$ as shown in figure 13.



(a) State of pendulum with spring and damper for a given set point

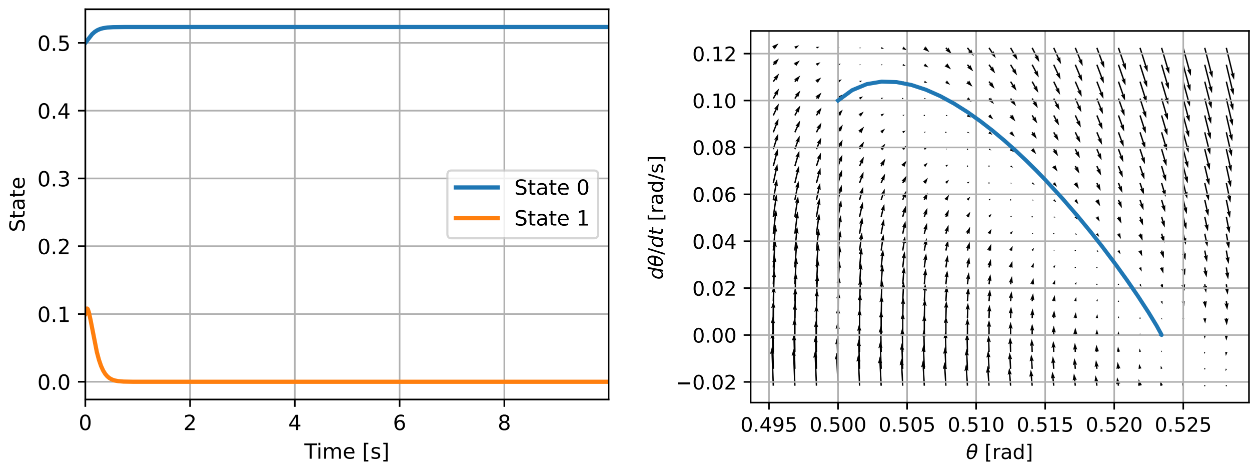
(b) Phase of pendulum with spring and damper for a given set point

Figure 13: Pendulum setup with spring and damper for a given set point

Note that changing the damping term will not influence the position of the equilibrium point of the system. Consider the following parameters:

$$
\begin{aligned}
b1 &= 1. \\
b2 &= 1. \\
k1 &= 50.0 \\
k2 &= 50.0 \\
s_{theta\_ref1} &= np.deg2rad(0.0) \\
s_{theta\_ref2} &= np.deg2rad(65.6)
\end{aligned}
$$

The system still approaches the convergence point in $\theta = \pi/6$, this time without damped oscillations, as shown in figure14.



(a) State of pendulum with spring and large damping for a given set point

(b) Phase of pendulum with spring and large damping for a given set point

Figure 14: Pendulum setup with spring and large damping term for a given set point

1.f What is the missing component between a real muscle and the muscle model with passive components that you just explored? What behavior's do you lack because of this missing component?

The missing component between a real muscle the muscle model with passive components is the active contractile element. The active contractile element can contract and produce force upon receiving an external activation. Having an active element allows for an external control to switch the behavior of the pendulum from a fixed point behavior to oscillatory and even stable limit cycle behaviors