A Two-Step Method for Solving Two-Asset Models

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1 Introduction

Heterogeneous-agent models with multiple assets have become increasingly common in macroeconomics. For example, models with illiquid capital and liquid bonds, such as Greg Kaplan, Benjamin Moll and Giovanni L Violante (2018) or Christian Bayer, Ralph Lütticke, Lien Pham-Dao and Volker Tjaden (2019), or models with durable consumption goods, such as David Berger and Joseph Vavra (2015) or Alisdair McKay and Johannes F Wieland (2019).

The solution of such models is complicated by the multi-dimensional maximization problem that agents face. In this paper, I show that in a two-asset model the two-dimensional maximization problem can be split into two sequential one-dimensional problems. First, I solve the problem assuming that the agent doesn't adjust one of the assets. I then use this to solve the problem assuming that the agent can adjust that asset.

This method has benefits in terms of speed, generality, and simplicity. In cases where a two-asset version of the endogenous grid method can be applied, the two-step method is faster than that alternative. There are also many models where an endogenous grid method cannot be applied, for example if one asset is subject to kinked adjustment costs. The two-step method presented here can still be applied in such cases. Finally, this method is

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simple to implement, as it is based on value function iteration and one-dimensional maximization.

2 A Two-Asset Model

Consider a simple two-asset model. Agents receive income z, which follows a Markov process, and choose their level of consumption and holdings of two assets. One asset, b, can be adjusted costlessly each period. The other asset, k, is subject to adjustment costs. In particular, if the agent chooses k' for the next period, they pay adjustment cost $g(k, k')$. Assume that $g(k, k) = 0$. The agents face borrowing constraints on both assets. The problem satisfies the following Bellman equation:

$$\begin{aligned} V(b,k,z) &= \max_{b',k',c} \frac{c^{1-\gamma}-1}{1-\gamma} + \beta \mathbb{E}_{z'}[V(b',k',z')] \qquad (2.1) \\ &\text{subject to} \\ c + k' + b' &= R_k k + R_b b + z - g(k,k') \\ b' &\geqslant \underline{\mathrm{b}} \\ k' &\geqslant \underline{\mathrm{k}} \\ z' &= \Gamma(z) \end{aligned}$$

2.1 A Two-Step Method

The solution to the above problem is complicated due to its two-dimensional nature. Agents face not only a consumption-savings decision, but also an asset portfolio decision. The two-step approach outlined below splits the problem into two simpler one-dimensional problems.

Step one is to solve the problem of an agent that leaves their illiquid asset holdings unchanged. Define the problem of such an agent as:

$$V^{NA}(b,k,z) = \max_{b'^{NA},c^{NA}} \frac{(c^{NA})^{1-\gamma}-1}{1-\gamma} + \beta \mathbb{E}_{z'}[V(b'^{NA},k,z')] \qquad (2.2)$$

subject to

$$
\begin{aligned}
c^{NA} + k + b'^{NA} &= R_k k + R_b b + z \\
b'^{NA} &\geqslant \underline{\mathrm{b}} \\
z' &= \Gamma(z)
\end{aligned}
$$

As this agent faces a simple consumption-savings problem, the optimal value of $b'^{NA}(b, k, z)$ can easily be solved using, for example, golden-section search.

Step two is then to solve the problem of an agent that is adjusting their illiquid asset holdings. Define $\tilde{V}(b, k, z; k')$ as the intermediate value for an agent that is free to choose their liquid assets but must adjust their illiquid assets to k':

$$
\begin{aligned}
\tilde{V}(b, k, z; k') &= \max_{\tilde{b}', \tilde{c}} \frac{\tilde{c}^{1-\gamma} - 1}{1-\gamma} + \beta \mathbb{E}_{z'}[V(\tilde{b}', k', z')] \\
&\text{subject to} \\
\tilde{c} + k' + \tilde{b}' &= R_k k + R_b b + z - g(k, k') \\
\tilde{b}' &\geqslant \underline{\mathrm{b}} \\
z' &= \Gamma(z)
\end{aligned}
\tag{2.3}
$$

The solution to the intermediate problem is closely related to that already found in the case of no illiquid asset adjustment:

$$
\begin{aligned}
\tilde{V}(b, k, z; k') &= V^{NA}(b^*, k', z) \\
&\text{subject to} \\
b^* &= b + \frac{R^k}{R^b}(k - k') - \frac{g(k', k)}{R^b}
\end{aligned}
\tag{2.4}
$$

The value for an agent that begins the period with asset holdings (b, k) and will choose illiquid asset holdings k' is equal to that of an agent who begins the period with asset holdings (b^*, k') and will not adjust their illiquid asset holdings.

The key to this method is to note that the costs associated with adjusting illiquid asset holdings from k to k' can be accounted for by adjusting liquid asset holdings, from b to b^*. The budget constraint for an agent with current state variables b^* and k' who will not adjust the illiquid asset is:

$$
c + k' + b' = R_k k' + R_b b^* + z \tag{2.5}
$$

$$
\begin{aligned}
&= R_k k' + R_b \left(b + \frac{R^k}{R^b}(k - k') - \frac{g(k', k)}{R^b} \right) + z \\
&= R_k k' + R_b b + R_k(k - k') - g(k', k) + z \\
&= R_k k + R_b b + z - g(k, k')
\end{aligned}
$$

This is exactly the budget constraint of the agent facing the intermediate problem. Consequently the solution of these problems is identical, implying that:

$$
\tilde{b}'(b, k, z; k') = b'^{NA}(b^*, k, z) \tag{2.6}
$$

The second step of the method uses this to solve for the choice of illiquid assets as follows:

$$
\begin{aligned}
V(b, k, z) = \max_{k', c} \frac{c^{1-\gamma} - 1}{1 - \gamma} + \beta \mathbb{E}_{z'}[V(b', k', z')] \qquad (2.7) \\
\text{subject to} \\
b' &= b'^{NA}(b^*, k', z) \\
b^* &= b + \frac{R^k}{R^b}(k - k') - \frac{g(k', k)}{R^b} \\
k' &\geqslant \underline{\text{k}}
\end{aligned}
$$

As the agent in this step faces a one-dimensional problem, the optimal value of k' can also be solved using golden-section search.

3 A Comparison with Alternative Methods

To show the potential of this method, in this section I compare it against a number of alternative approaches for two different specifications of the adjustment cost, $g(k, k')$.[1] The first specification uses convex adjustment costs:

$$
g(k, k') = \frac{\alpha}{2} \left(\frac{k' - k}{k} \right)^2 k \tag{3.1}
$$

[1]The two-step method can also be used in cases of (possibly random) adjustment costs denoted in terms of utility. I use this method to solve the model in Sebastian Graves (2020).

Table 1: Parameter Values

Model	β	γ	ρ_Z	σ_Z	R^b	R^k	$\underline{b}$	$\underline{k}$	f	α
Convex Costs: $g(k,k') = \frac{\alpha}{2}\left(\frac{k'-k}{k}\right)^2 k$	0.9	1	0.8	0.1	1.01	1.02	0	1	N/A	0.05
Linear Costs: $g(k,k') = f\lvert k'-k\rvert$	0.9	1	0.8	0.1	1.01	1.02	0	0	0.0075	N/A

In this case, it is possible to solve the problem using the two-asset endogenous grid method developed by Thomas Hintermaier and Winfried Koeniger (2010).[2] Appendix A outlines the algorithm as applied here. In the second specification I use kinked linear adjustment costs:

$$g(k,k') = f|k' - k| \tag{3.2}$$

In this case, it is not possible to use the two-asset endogenous grid method as $g(k,k')$ is non-differentiable when $k' = k$. Consequently, I will also compare the two-step method to two other approaches that do work in this case. The first is a nested golden-section search method. In an outer loop the agent maximizes over one of the assets, while in an inner loop they maximize over the other asset, conditional on the choice of the outer-loop asset.[3]

The final alternative approach is to discretize the problem entirely and restrict an agent's asset choices to the grid-points. When the set of possible future asset holdings is discretized, it is possible to simply calculate the value of every possible (b', k') pair and then choose the optimum.

Regardless of the adjustment cost specification, I assume that the log of idiosyncratic productivity follows an AR(1) process:

$$\log z' = \rho_z \log z + \epsilon_z \tag{3.3}$$

where $\epsilon_z \sim N(0, \sigma_z^2)$. I use the method of George Tauchen (1986) to discretize the productivity process on a grid with 5 points. The parameter values that I use are shown in Table 1.

For both adjustment cost specifications, I solve the problem using each possible solution method and vary the number of grid points used for the liquid and illiquid asset, N_b and

[2] Bayer et al. (2019) is an example of an application of this two-asset endogenous grid method.

[3] For both this approach and the two-step method I use cubic spline interpolation to evaluate the value function in between grid-points.

Table 2: Comparison of Solution Methods

Grid Points	Two-Step	Endogenous Grid Method	Nested Golden-Section Search	Discretized
		Convex Costs: $g(k,k') = \frac{\alpha}{2}\left(\frac{k'-k}{k}\right)^2 k$		
$N_b = N_k = 40$	4	7	46	39
$N_b = N_k = 60$	8	21	101	207
$N_b = N_k = 80$	14	49	182	2073
		Linear Costs: $g(k,k') = f\|k' - k\|$		
$N_b = N_k = 40$	4	N/A	52	40
$N_b = N_k = 60$	8	N/A	115	199
$N_b = N_k = 80$	13	N/A	185	2210

Notes: Table shows the time in seconds to solve the agents' problem using each approach in combination with value function iteration. N_b and N_k denote the number of grid points used for the liquid and illiquid asset, respectively. In each case the maximum value on the grid for b is 10 and for k is 95, with grid points concentrated towards the borrowing constraints. I discretize the income process using the Tauchen (1986) method on a 5 point grid.

N_k.[4] Table 2 shows the time required to solve the problem using each approach and each number of grid points.

The two-step method is around two to three times as fast as the endogenous grid method in the case with convex adjustment costs.[5] Both these methods are significantly faster than nested golden-section search or discretization. The latter fares particularly poorly as the number of grid points (and consequently the number of possible asset choices) rises. The relative speeds are basically the same in the case of linear adjustment costs. The key difference in this case is that here it is not possible to use the endogenous grid method. Consequently, the two-step method is by far the fastest solution technique for such a model.

[4] For all solution methods except the endogenous grid method, I approximate the expected value function $V^e(b,k,z) = \mathbb{E}_{z'}[V(b,k,z')]$. I deem that the value functions have converged when $\frac{\sum_s |V_i^e(s) - V_{i-1}^e(s)|}{\sum_s |V_{i-1}^e(s)|} < 10^{-6}$. $V_i^e(s)$ is the expected value function in iteration i at grid point s. I initialize the expected value function at the value consistent with $b' = b$ and $k' = k$. For the endogenous grid method, I initialize the policy functions at the same starting point and iterate until $\frac{\sum_s |c_i(s) - c_{i-1}(s)|}{\sum_s |c_{i-1}(s)|} + \frac{\sum_s |k_i'(s) - k_{i-1}'(s)|}{\sum_s |k_{i-1}'(s)|} < 10^{-6}$.

[5] Both methods take almost exactly the same number of iterations to converge.

4 Conclusion

This paper outlines a new method for solving the agent's problem in a two-asset model. This method involves splitting the two-dimensional maximization problem into two sequential one-dimensional problems. The key insight is that the problem for an agent who is adjusting their holdings of one asset to a new level and then considering their holdings of the second asset can be shown to be equivalent to that of a particular agent who already had the new level of the first asset but is not going to adjust it.

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Appendices

A Two-Asset Endogenous Grid Method

The first-order conditions for the full problem are:

$$u'(c(b,k,z)) \geqslant \beta \mathbb{E}_{z'}[u'(c(b',k',z')R^b] \tag{A.1}$$

$$u'(c(b,k,z))(1+g_2(k,k')) \geqslant \beta \mathbb{E}_{z'}[u'(c(b',k',z')(R^k - g_1(k',k'')] \tag{A.2}$$

where $g_1(k,k')$ and $g_2(k,k')$ are the derivatives of the adjustment cost with respect to the first and second argument. The first FOC holds with equality if $b'(b,k,z) > \underline{\mathrm{b}}$ and the second holds with equality if $k'(b,k,z) > \underline{\mathrm{k}}$. The algorithm for the two-asset endogenous grid method is as follows.

Provide an initial guess of $c(b,k,z)$ and $k'(b,k,z)$. Iterate on the following until the policy functions have converged:

1. For each possible (k,k',z) solve the first-order conditions to find the implied value of b':
$$\mathbb{E}_{z'}[u'(c(b',k',z')R^b] = \frac{\mathbb{E}_{z'}[u'(c(b',k',z')(R^k - g_1(k',k'(b',k',z')]}{1+g_2(k,k')} \tag{A.3}$$
If the left-hand side of the above is greater than the right-hand side for all $b' \in [\underline{\mathrm{b}}, \bar{b}]$, set $b'(b,k,z) = \bar{b}$. If the opposite is true, set $b'(b,k,z) = \underline{\mathrm{b}}$.

2. Use $b'(k,k',z)$ and the second first-order condition to find $c(k,k',z)$ as follows:
$$c(k,k',z) = u'^{-1}\left\{\left(\frac{\beta \mathbb{E}_{z'}[u'(c(b'(k,k',z),k',z')(R^k - g_1(k',k'(b,k',z))]}{1+g_2(k,k')}\right)\right\} \tag{A.4}$$

3. Use the budget constraint to find the level of b that is consistent with this:
$$b(k,k',z) = \frac{c(k,k',z) + b'(k,k',z) + k' + g(k,k') - z - R^k k}{R^b} \tag{A.5}$$
This gives us the endogenous grid for liquid asset holdings.

4. Interpolate to recover the policy functions $c(b,k,z)$ and $k'(b,k,z)$.

5. For cases where $b < b(k, \underline{\mathrm{k}}, z)$ the lower bound on k' is binding. In this case, set $k'(b, k, z) = \underline{\mathrm{k}}$. To find b' in such cases use the standard endogenous grid method:

   (a) Use the first first-order condition to find $c_{\underline{\mathrm{k}}}(b', k, z)$

   $$c_{\underline{\mathrm{k}}}(b', k, z) = u'^{-1}\left\{\left(\beta \mathbb{E}_{z'}[u'(c(b', \underline{\mathrm{k}}, z')R^b]\right)\right\} \tag{A.6}$$

   (b) Use the budget constraint to find the level of b that is consistent with this:

   $$b_{\underline{\mathrm{k}}}(b', k, z) = \frac{c_{\underline{\mathrm{k}}}(b', k, z) + b' + \underline{\mathrm{k}} + g(k, \underline{\mathrm{k}}) - z - R^k k}{R^b} \tag{A.7}$$

   This gives us the endogenous grid for liuqid asset holdings for $k' = \underline{\mathrm{k}}$ cases.

   (c) Interpolate to recover the policy functions $b'(b, k, z)$ and $c(b, k, z)$ for these cases.

   (d) For the cases where $b < b_{\underline{\mathrm{k}}}(\underline{\mathrm{b}}, k, z)$, the lower bound on b' is also binding. Set $b' = \underline{\mathrm{b}}$ and recover c from the budget constraint.

6. Update guesses of $c(b, k, z)$ and $k'(b, k, z)$ and return to step 1.