Temă integrale duble (1) → „Prin dairetuni"

1) $I = \iint_D (x+3y)\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2,\ x^2+y^2 \le 4,\ y \ge \sqrt{3}x,\ y \ge -\sqrt{3}x\}$

2) $I = \iint_D (x^2-2y^2)\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2;\ y \ge 2x,\ y \le 2x+3,\ x \le 1,\ y \ge -1\}$

3) $I = \iint_D (1+x)\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2;\ x^2+y^2 \le 2y,\ y \le -x+2\}$

4) $I = \iint_D 2xy\,dx\,dy$, D = Domeniul determ. de trapezul $ABCD$ cu $A(1,0)$, $B(1,4)$, $C(3,4)$, $D(5,0)$

5) $I = \iint_M (4+xy^2)\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2 /\ 1 \le x \le 2,\ y \ge 2x,\ y \le 4x\}$

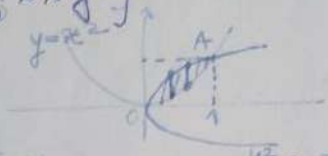
6) $I = \iint_M (x^2y+xy)\,dx\,dy$, $M = \{(x,y) \in \mathbb{R}^2 /\ y-x \le 1,\ x+y \le 3,\ x \ge 0,\ y \ge 0\}$

7) Aria $(M) = ?$ $M = \{(x,y) \in \mathbb{R}^2 /\ x^2+y^2 \le 4,\ y \le \sqrt{3}x\}$

8) Aria $(M) = ?$ $M = \{(x,y) \in \mathbb{R}^2 /\ y \ge x^2-1,\ y \le -x^2+1\}$

$\iint_D x \, dx \, dy$ D = domeniul cuprins între curbele de ec $\begin{cases} y = x^2 \\ x = y^2 \end{cases}$

i.e.: $D = \{ y \geq x^2 ; x \geq y^2 \}$



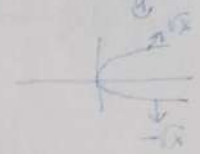
Sol $A = ?$ $\begin{cases} y = x^2 \\ x = y^2 \end{cases} \Rightarrow x^4 = x \Rightarrow$

$\Rightarrow x(x-1)(x^2+x+1) = 0$

$\Rightarrow x_1 = 0 \Rightarrow y_1 = 0 \Rightarrow O(0,0)$

$x_2 = 1 \Rightarrow y_2 = 1 \Rightarrow A(1,1)$

$y^2 = x \Rightarrow y = \pm\sqrt{x}$



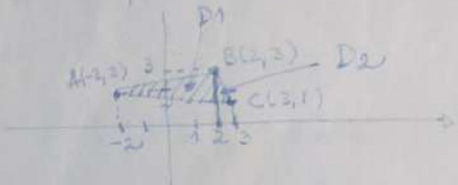
$pr_{Ox}[D] = [0,1]$

$D[x] = [x^2, \sqrt{x}]$

$\Rightarrow \iint_D dx\,dy = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} x \, dy \right) dx = \int_0^1 xy \Big|_{x^2}^{\sqrt{x}} dx =$

$= \int_0^1 (x\sqrt{x} - x^3) dx = \left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{x^4}{4} \right) \Big|_0^1 = \frac{1}{\frac{5}{2}} - \frac{1}{4} = \frac{2}{5} - \frac{1}{4} = \frac{3}{20}.$

$\iint_D 2 \, dx \, dy$: D = triunghiul det. de pct. A(-2,2), B(2,3), C(3,1)



Sol: ec. dr. AB, AC, BC:

$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$

AB: $y - 2 = \frac{3-2}{2+2}(x+2) \Leftrightarrow x - 4y + 10 = 0 \Leftrightarrow y = \frac{x+10}{4}.$

AC: $y - 2 = \frac{1-2}{3+2}(x+2) \Leftrightarrow x + 5y - 8 = 0 \Leftrightarrow y = \frac{-x+8}{5}.$

BC: $y - 3 = \frac{1-3}{3-2}(x-2) \Leftrightarrow y - 3 = -2x + 4 \Leftrightarrow y = -2x + 7.$

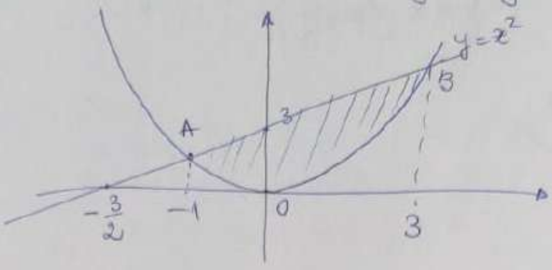
$D = D_1 \cup D_2$

$pr_{Ox} D_1 = [-2, 2]$; $D_1[x] = \left[\frac{-x+8}{5}, \frac{x+10}{4} \right]$

$pr_{Ox} D_2 = [2,3]$ $D_2[x] = \left[\frac{-x+8}{5}, -2x+7 \right].$

$\Rightarrow \iint_D f \, dx \, dy = \iint_{D_1} f \, dx \, dy + \iint_{D_2} f \, dx \, dy = \dots$

$I = \iint_D xy\,dx\,dy$, D domeniul delimitat de $y = x^2$, $y = 2x+3$



$A, B = ?$

$$\begin{cases} y = x^2 \\ y = 2x+3 \end{cases}$$

$x^2 = 2x+3$

$x^2 - 2x - 3 = 0$

$$x_{1,2} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} \begin{matrix} \nearrow 3 \\ \searrow -1 \end{matrix}$$

$$I = \int_{-1}^{3} \left(\int_{x^2}^{2x+3} xy\,dy \right) dx = \int_{-1}^{3} \frac{xy^2}{2} \Big|_{x^2}^{2x+3} dx =$$

$$= \frac{1}{2} \int_{-1}^{3} \left(x(2x+3) - x^3 \right) dx = \frac{1}{2} \int_{-1}^{3} \left(2x^2 + 3x - x^3 \right) dx =$$

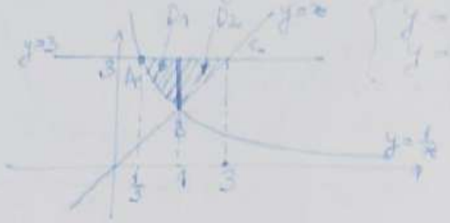
$$= \frac{1}{2} \left[\left(2\frac{x^3}{3} + \frac{3x^2}{2} - \frac{x^4}{4} \right) \right] \Big|_{-1}^{3} = \frac{1}{2} \left[\left(\frac{2 \cdot 3^{\cancel{3}}}{\cancel{3}} + 3 \cdot \frac{9}{2} - \frac{3^4}{4} \right) - \left(-\frac{2}{3} + \frac{3}{2} - \frac{1}{4} \right) \right] =$$

$$= \frac{1}{2} \left(\left(18 + \frac{27}{2} - \frac{81}{4} \right) + \frac{2}{3} - \frac{3}{2} + \frac{1}{4} \right) = \frac{1}{2} \left(18 + 12 - 20 + \frac{2}{3} \right) =$$

$$= \frac{1}{2} \left(10 + \frac{2}{3} \right) \overset{3}{=} 5 + \frac{1}{3} = \cancel{\frac{16}{3}}.$$

Corect: $53 + \frac{1}{3}$

$\iint_D x^2 dxdy$ $\quad$ D = domeniul mărginit de $\begin{cases} y = \frac{1}{x} \\ y = x \\ y = 3 \end{cases}$



A=? B=? C=?

A: $\begin{cases} y=3 \\ y=\frac{1}{x} \end{cases} \Rightarrow A\left(\frac{1}{3}, 3\right)$

B: $\begin{cases} y=x \\ y=\frac{1}{x} \end{cases} \Rightarrow x=\frac{1}{x} \Rightarrow x^2=1 \Rightarrow x=\pm 1 \underset{x\geq 0}{\Rightarrow} x=1 \Rightarrow y=1 \Rightarrow B(1,1)$

C: $\begin{cases} y=3 \\ y=x \end{cases} \Rightarrow C(3,3)$

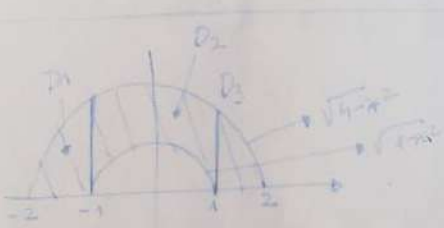
$D = D_1 \cup D_2$; $pr_{Ox} D_1 = \left[\frac{1}{3}, 1\right]$ $\quad D_1[x] = \left[\frac{1}{x}, 3\right]$

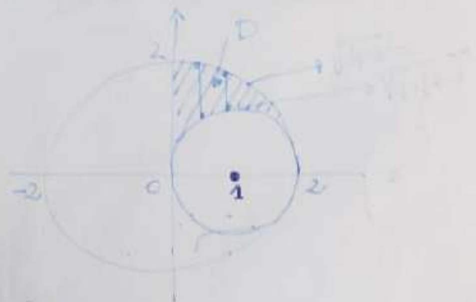
$pr_{Ox} D_2 = [1, 3]$ $\quad D_2[x] = [x, 3]$.

$\iint_D x^2 dxdy = \int_{\frac{1}{3}}^{1} \left(\int_{\frac{1}{x}}^{3} x^2 dy \right) dx + \int_{1}^{3} \left(\int_{x}^{3} x^2 dy \right) dx =$

$= \ldots dx + \int_1^3 x^2 y \Big|_x^3 dx = \int_{\frac{1}{3}}^{1} (3x^2 - x) dx + \int_1^3 (3x^2 - x^3) dx$

$= x^3 - \frac{x^2}{2} \Big|_{\frac{1}{3}} + \left(x^3 - \frac{x^4}{4}\right) \Big|_1^3 = \left(1 - \frac{1}{2}\right) - \left(\frac{1}{27} - \frac{1}{18}\right) + \left(27 - \frac{81}{4}\right) \ldots$





$D = \{ x^2+y^2 \geq 1 ; x^2+y^2 \leq 4, y \geq 0 \}$

$D = D_1 \cup D_2 \cup D_3$

$pr_{Ox} D_1 = [-2, -1]$; $D_1[x] = [0; \sqrt{4-x^2}]$

$pr_{Ox} D_2 = [-1, 1]$; $D_2[x] = [\sqrt{1-x^2}, \sqrt{4-x^2}]$

$pr_{Ox} D_3 = [1, 2]$; $D_3[x] = [0, \sqrt{4-x^2}]$

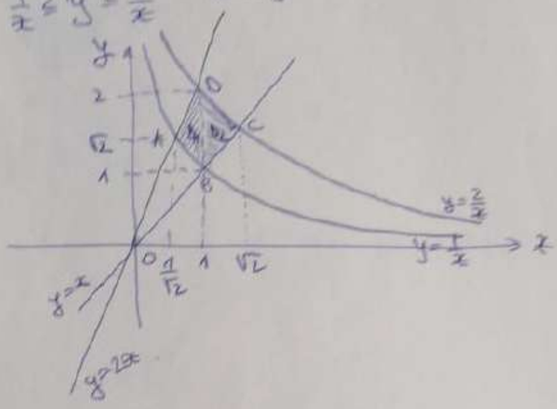
$D = \{ x^2+y^2 \leq 4 ; (x-1)^2+y^2 \geq 1 , x \geq \ldots$

$pr_{Ox} D = [0, 2]$

$D[x] = [\sqrt{1-x^2}, \sqrt{1-(x-1)^2} \ldots]$

$$I = \iint_D x\,dx\,dy\ ,\quad D = \{(x,y)\in\mathbb{R}^2 ;\ \underbrace{1 \le xy \le 2}_{\Updownarrow} ;\ \underbrace{1 \le \frac{y}{x} \le 2}_{\Updownarrow},\ x>0\}$$

$$\frac{1}{x} \le y \le \frac{2}{x} \qquad x \le y \le 2x.$$



$$A: \begin{cases} y = 2x \\ y = \frac{1}{x} \end{cases} \Rightarrow x = \frac{1}{\sqrt{2}} ;\quad B: \begin{cases} y = x \\ y = \frac{1}{x} \end{cases} \Rightarrow x = 1$$

$$C: \begin{cases} y = \frac{2}{x} \\ y = x \end{cases} \Rightarrow x = \sqrt{2}$$

$$D: \begin{cases} y = \frac{2}{x} \\ y = 2x \end{cases} \Rightarrow x = 1$$

$$pr_{Ox} D_1 = [\tfrac{1}{\sqrt{2}}, 1] ;\quad pr_{Ox} D_2 = [1, \sqrt{2}]$$

$$D_1[x] = [\tfrac{1}{x} ; 2x] ;\quad D_2[x] = [x, \tfrac{2}{x}].$$

$$I = \iint_{D_1} + \iint_{D_2} = \int_{\frac{1}{\sqrt{2}}}^{1} \left(\int_{\frac{1}{x}}^{2x} x\,dy \right) dx + \int_{1}^{\sqrt{2}} \left(\int_{x}^{\frac{2}{x}} x\,dy \right) dx =$$

$$= \int_{\frac{1}{\sqrt{2}}}^{1} xy \Big|_{\frac{1}{x}}^{2x} dx + \int_{1}^{\sqrt{2}} xy \Big|_{x}^{\frac{2}{x}} dx = \int_{\frac{1}{\sqrt{2}}}^{1} (2x^2 - 1)\,dx + \int_{1}^{\sqrt{2}} (2 - x^2)\,dx =$$

$$= \left(\frac{2x^3}{3} - x\right)\Big|_{\frac{1}{\sqrt{2}}}^{1} + \left(2x - \frac{x^3}{3}\right)\Big|_{1}^{\sqrt{2}} = \left(\frac{2}{3} - 1\right) - \left(\frac{\frac{1}{2\sqrt{2}}}{3} - \frac{1}{\sqrt{2}}\right) + \left(2\sqrt{2} - \frac{2\sqrt{2}}{3}\right) - \left(2 - \frac{1}{3}\right) =$$

$$\cancel{\tfrac{2}{3}}\ \cancel{-1} - \frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{2}} + 2\sqrt{2} - \frac{2\sqrt{2}}{3} - 2 + \frac{1}{3} = \cancel{\ldots}$$

$$= \frac{-1 - 3 + 12 - 4 - 6\sqrt{2}}{3\sqrt{2}} = \frac{10 - 6\sqrt{2}}{3\sqrt{2}} = \frac{5\sqrt{2} - 6}{3}.$$

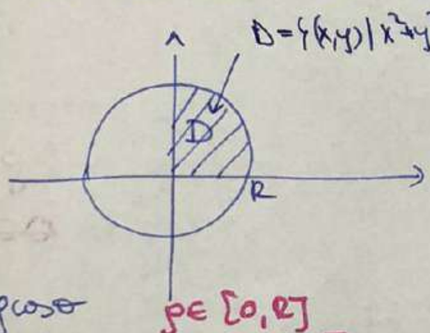
Atunci:

$$\iint_D f(x,y)\,dx\,dy = \iint_{\underbrace{[\dots,\dots]}_{\rho}\times\underbrace{[\dots,\dots]}_{\theta}} f(x(\rho,\theta), y(\rho,\theta))\cdot J \, d\rho\, d\theta .$$

Exemple de sch de variabilă:

! Se vor face modificări doar asupra intervalelor lui ρ și θ.

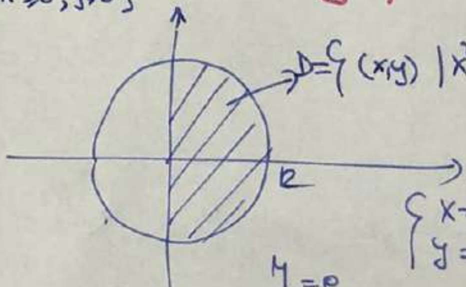
$D=\{(x,y) \mid x^2+y^2\le R^2, x\ge 0, y\ge 0\}$



$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$ $\quad \rho\in[0,R]$, $\theta\in[0,\frac{\pi}{2}]$

$J=\rho$

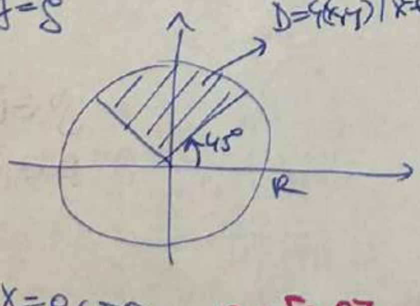
$D=\{(x,y) \mid x^2+y^2\le R^2, x\ge 0\}$



$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$ $\quad \rho\in[0,R]$, $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$J=\rho$

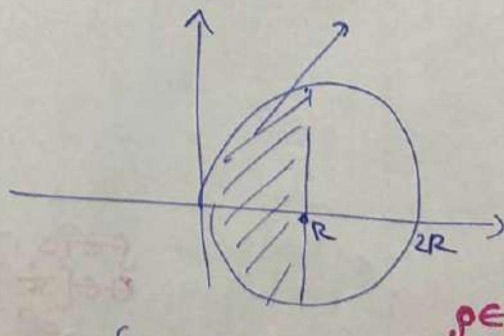
$D=\{(x,y) \mid x^2+y^2\le R^2, y\ge x, y\ge -x\}$



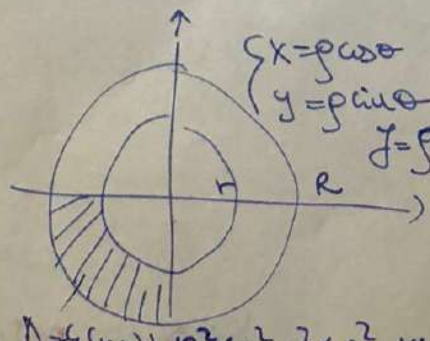
$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$ $\quad \rho\in[0,R]$, $\theta\in[\frac{\pi}{4},\frac{3\pi}{4}]$

$J=\rho$

$D=\{(x,y) \mid x^2+y^2\le R^2, x\le R\}$



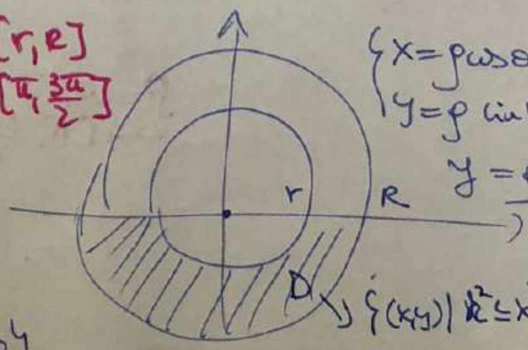
$\begin{cases} x=R+\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$ $\quad J=\rho$ $\quad \rho\in[0,R]$, $\theta\in[\frac{\pi}{2},\frac{3\pi}{2}]$



$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$ $\quad \rho\in[r,R]$, $\theta\in[\pi,\frac{3\pi}{2}]$

$J=\rho$

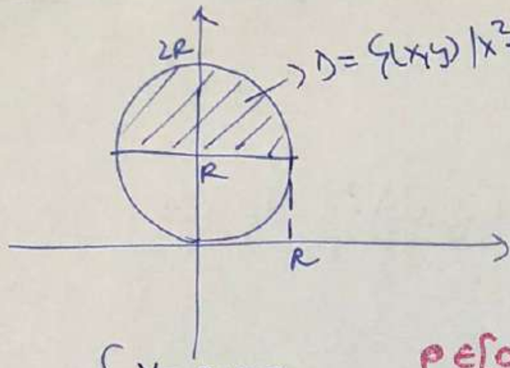
$D=\{(x,y) \mid r^2\le x^2+y^2\le R^2, x\le 0, y\le 0\}$



$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$ $\quad \rho\in[r,R]$, $\theta\in[\pi,2\pi]$

$J=\rho$

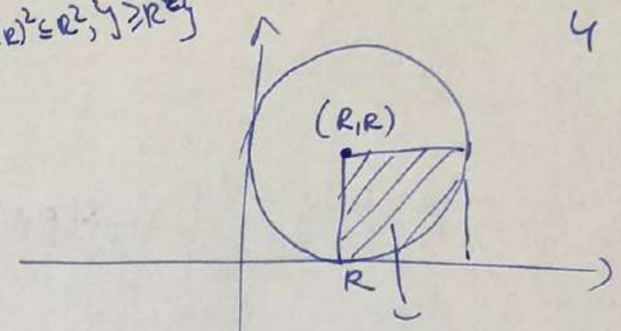
$D=\{(x,y) \mid r^2\le x^2+y^2\le R^2, y\le 0\}$



$D=\{(x,y) \mid x^2+(y-R)^2\le R^2,\ y\ge R\}$

$\begin{cases} x=\rho\cos\theta \\ y=R+\rho\sin\theta \end{cases}$ $\rho\in[0,R]$, $\theta\in[0,\pi]$

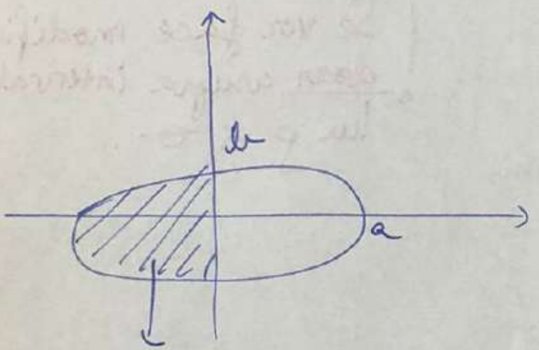
$J=\rho$



$D=\{(x,y) \mid (x-R)^2+(y-R)^2\le R^2,\ x\ge R,\ y\le R\}$

$\begin{cases} x=R+\rho\cos\theta \\ y=R+\rho\sin\theta \end{cases}$ $\rho\in[0,R]$, $\theta\in\left[-\frac{\pi}{2},0\right]$

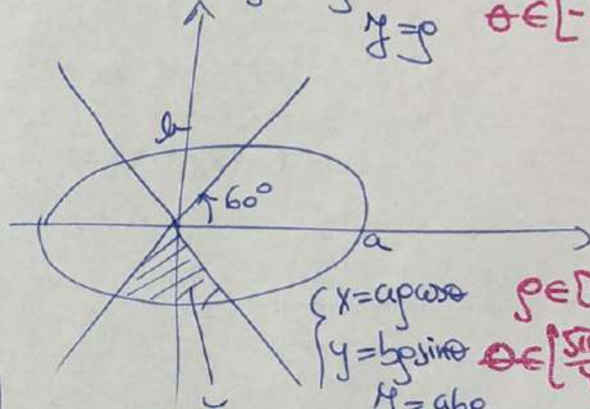
$J=\rho$



$D=\{(x,y) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ x\le 0\}$

$\begin{cases} x=a\rho\cos\theta \\ y=b\rho\sin\theta \end{cases}$ $\rho\in[0,1]$, $\theta\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$

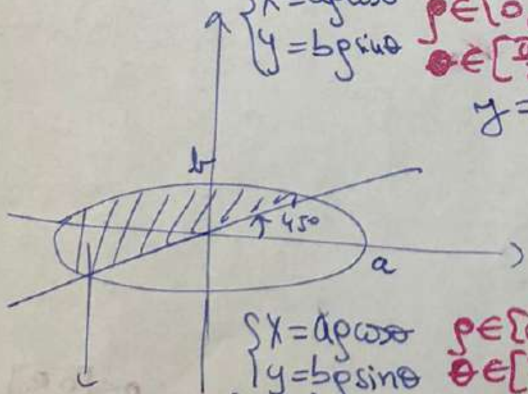
$J=ab\rho$



$\begin{cases} x=a\rho\cos\theta \\ y=b\rho\sin\theta \end{cases}$ $\rho\in[0,1]$, $\theta\in\left[\frac{5\pi}{4},\frac{7\pi}{4}\right]$

$J=ab\rho$

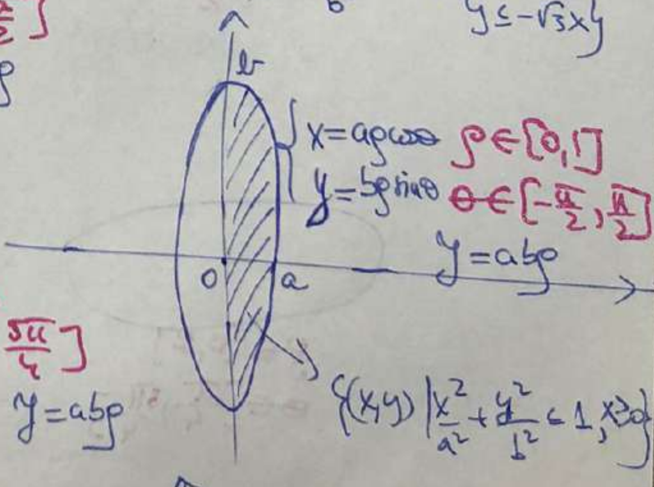
$D=\{(x,y) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ y\le\sqrt{3}x,\ y\le-\sqrt{3}x\}$



$\begin{cases} x=a\rho\cos\theta \\ y=b\rho\sin\theta \end{cases}$ $\rho\in[0,1]$, $\theta\in\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$

$D=\{(x,y) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ y\ge x\}$

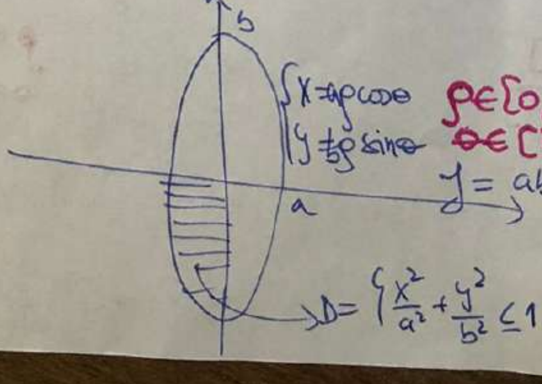
$J=ab\rho$



$\begin{cases} x=a\rho\cos\theta \\ y=b\rho\sin\theta \end{cases}$ $\rho\in[0,1]$, $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$J=ab\rho$

$\{(x,y) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ x\ge 0\}$

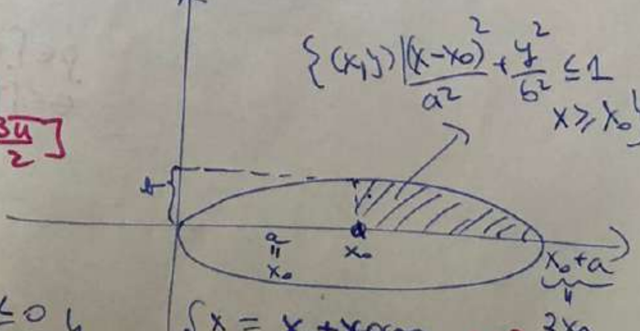


$\begin{cases} x=a\rho\cos\theta \\ y=b\rho\sin\theta \end{cases}$ $\rho\in[0,1]$, $\theta\in\left[\pi,\frac{3\pi}{2}\right]$

$J=ab\rho$

$D=\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ x\le 0,\ y\le 0\}$

$\{(x,y) \mid \frac{(x-x_0)^2}{a^2}+\frac{y^2}{b^2}\le 1,\ x\ge x_0\}$



$\begin{cases} x=x_0+x_0\rho\cos\theta \\ y=b\rho\sin\theta \end{cases}$ $\rho\in[0,1]$, $\theta\in\left[0,\frac{\pi}{2}\right]$