Temă curs

1)
$$\begin{cases} x_2 + x_3 = 3 \\ 2x_1 + x_2 + 5x_3 = 5 \\ 4x_1 + 2x_2 + x_3 = 1 \end{cases} \Rightarrow A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 5 \\ 4 & 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$$

(LU - GPP) $\qquad w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(k=1):
$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 5 \\ 4 & 2 & 1 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 4 & 2 & 1 \\ 2 & 1 & 5 \\ 0 & 1 & 1 \end{pmatrix} \overset{L_2 - \frac{1}{2}L_1}{\sim} \begin{pmatrix} 4 & 2 & 1 \\ 0 & 0 & \frac{9}{2} \\ 0 & 1 & 1 \end{pmatrix}$$

(k=2)

$$w = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\overset{L_3 \sim L_2}{\sim} \underbrace{\begin{pmatrix} 4 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \frac{9}{2} \end{pmatrix}}_{U} \Rightarrow w = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \text{ și } l_{21} \leftrightarrow l_{31} \Bigg\} \Rightarrow$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \left(m_{ik} = \frac{a_{ik}}{a_{kk}} \right)$$

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

$$Ax = b \Leftrightarrow L\underbrace{Ux}_{y} = b' \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$Ly = b' \Rightarrow \begin{cases} y_1 = 1 \\ y_2 = 3 \\ \frac{1}{2}y_1 + y_3 = 5 \Rightarrow y_3 = 5 - \frac{1}{2} = \frac{9}{2} \end{cases}$$

$$\Rightarrow y = \begin{pmatrix} 1 \\ 3 \\ \frac{9}{2} \end{pmatrix}$$

$$Ux = y \Rightarrow \begin{pmatrix} 4 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ \frac{9}{2} \end{pmatrix}$$

$$\Rightarrow \begin{cases} 4x_1 + 2x_2 + x_3 = 1 \\ x_2 + x_3 = 3 \\ \frac{9}{2}x_3 = \frac{9}{2} \Rightarrow x_3 = 1 \end{cases} \Rightarrow x_2 = 2 \Rightarrow x_1 = -1$$

$$\Rightarrow X = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

2)

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 10 & 4 \\ 2 & 4 & 6 \end{pmatrix}$$

a) A - simetrică și poz. def.

$$A^T = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 10 & 4 \\ 2 & 4 & 6 \end{pmatrix} = A \Rightarrow A \text{ - simetrică}$$

$$\Delta_1 = 4 > 0$$

$$\Delta_2 = \begin{vmatrix} 4 & 2 \\ 2 & 10 \end{vmatrix} = 40 - 4 > 0$$

$$\Delta_3 = \begin{vmatrix} 4 & 2 & 2 \\ 2 & 10 & 4 \\ 2 & 4 & 6 \end{vmatrix} = 240 + 8 \cdot 2 + 16 - 40 - 64 - 24 = 144 > 0$$

$\Delta_1, \Delta_2, \Delta_3 > 0 \Rightarrow A$ - poz. def.

b) factorizarea Cholesky

$L \cdot L^T = A$

$$\Rightarrow \begin{pmatrix} l_1 & 0 & 0 \\ l_2 & l_3 & 0 \\ l_4 & l_5 & l_6 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_4 \\ 0 & l_3 & l_5 \\ 0 & 0 & l_6 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 10 & 4 \\ 2 & 4 & 6 \end{pmatrix}$$

$$\Rightarrow \begin{cases} l_1^2 = 4 \Rightarrow l_1 = \pm 2 \quad \text{(iau val. pozitivă)} \Rightarrow l_1 = 2 \\ \underline{l_1 l_2 = 2} \Rightarrow l_2 = 1 \\ \underline{l_1 l_4 = 2} \Rightarrow l_4 = 1 \\ \underline{l_1 l_2 = 2} \\ l_2^2 + l_3^2 = 10 \Rightarrow l_3 = \pm\sqrt{10-4} \Rightarrow l_3 = 3 \\ \underline{l_2 l_4 + l_3 l_5 = 4} \Rightarrow l_5 = \frac{4-1}{3} \Rightarrow l_5 = 1 \\ \underline{l_1 l_4 = 2} \\ \underline{l_2 l_4 + l_3 l_5 = 4} \\ l_4^2 + l_5^2 + l_6^2 = 6 \Rightarrow l_6 = \sqrt{6-1-1} \Rightarrow l_6 = 2 \end{cases}$$

$$\Rightarrow L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

c) $Ax = b$ $\qquad b = \begin{pmatrix} 12 \\ 30 \\ 10 \end{pmatrix}$

$\overset{\wedge}{L \cdot L^T}$

$L\underbrace{L^T x}_{y} = b$

$$Ly = b \iff \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 30 \\ 10 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2y_1 = 12 & \Rightarrow y_1 = 6 \\ y_1 + 3y_2 = 30 & \Rightarrow y_2 = \frac{24}{3} = 8 \\ y_1 + y_2 + 2y_3 = 10 \end{cases}$$

$$y_3 = \frac{10 - 6 - 8}{2} \Rightarrow y_3 = -2$$

$$\Rightarrow y = \begin{pmatrix} 6 \\ 8 \\ -2 \end{pmatrix}$$

$$L^T x = y \iff \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ -2 \end{pmatrix}$$

$$\begin{cases} 2x_1 + x_2 + x_3 = 6 \\ 3x_2 + x_3 = 8 \\ 2x_3 = -2 \Rightarrow x_3 = -1 \end{cases} \Rightarrow x_2 = 3 \Rightarrow x_1 = 2$$

$$\Rightarrow x = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

3)
$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

a) factorizarea QR a matricei A : $A = Q \cdot R$

$$\underset{\substack{\swarrow\ \downarrow \\ j\ \ i}}{a_{31}} \neq 0 \Rightarrow R^{(13)} = \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix}$$

$$c = \frac{a_{ii}}{\sqrt{a_{ii}^2 + a_{ji}^2}} = \frac{1}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}$$

$$s = \frac{a_{ji}}{\sqrt{a_{ii}^2 + a_{ji}^2}} = \frac{1}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow R^{(13)} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$R^{(13)} \cdot A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \\ 0 & -\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & 1 & 1 \\ 0 & -\sqrt{2} & \frac{\sqrt{2}}{2} \end{pmatrix}}_{\text{noul } A}$$

$$a_{32} \neq 0 \Rightarrow R^{(23)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}$$

$$c = \frac{1}{\sqrt{1^2+(-\sqrt{2})^2}} = \frac{1}{\sqrt{3}}$$

$$s = \frac{-\sqrt{2}}{\sqrt{1^2+(-\sqrt{2})^2}} = -\frac{\sqrt{2}}{\sqrt{3}} \left(= -\sqrt{\frac{2}{3}}\right)$$

$$R^{(23)} \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & 1 & 1 \\ 0 & -\sqrt{2} & \frac{\sqrt{2}}{2} \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} \sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{pmatrix}}_{\text{supr } \triangle \; = R}$$

$$Q^T = R^{(23)} \cdot R^{(13)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

b) $Ax = b \iff QRx = b \;\Big|\; \cdot Q^T \iff Rx = Q^T b$

$$\Rightarrow \begin{pmatrix} \sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{2\sqrt{2}}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\Rightarrow \begin{cases} \sqrt{2}\, x_1 + \sqrt{2}\, x_2 + \frac{\sqrt{2}}{2} x_3 = \frac{\sqrt{2}}{2} \Rightarrow x_1 = \dfrac{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{6} - \frac{2\sqrt{2}}{3}}{\sqrt{2}} \\ \sqrt{3}\, x_2 = \frac{2\sqrt{3}}{3} \Rightarrow x_2 = \frac{2}{3} \\ \frac{\sqrt{6}}{2} x_3 = \frac{\sqrt{6}}{6} \Rightarrow x_3 = \frac{1}{3} \end{cases}$$

$$\Rightarrow x_1 = \frac{3\sqrt{2} - \sqrt{2} - 4\sqrt{2}}{6\sqrt{2}} \Rightarrow x_1 = -\frac{1}{3}$$

$$\Rightarrow X = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$