Def O pereche $((x_n)_{n\geq p}, (y_n)_{n\geq p})$ unde

$$y_n = \sum_{k=p}^{n} x_k$$ s.n. serie. $$\sum_{n\geq p} x_n = \sum_{n=p} x_n$$

x_n termenii seriei

y_n - sumele parțiale (s_n)

Dacă $\exists \lim_{n\to\infty} y_n$ spunem $\sum_{n\geq p} x_n$ are limită

și $$\lim_{n\to\infty} y_n = \sum_{n\geq p} x_n$$

Dacă $\sum_{n\geq p} x_n \in \mathbb{R}$ spunem că seria este convergentă.

$$\sum_{n\geq 1} \frac{1}{n(n+1)} \qquad x_n = \frac{1}{n(n+1)}$$

$$y_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{n+1} \nearrow 1$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3} + \ldots$$

$$\sum_{n\geq 1} \frac{1}{n(n+1)} = 1$$

Ex 2 $\displaystyle\sum_{n\geq 0} a^n$

c1) $|a|<1$ $\quad y_n = \displaystyle\sum_{k=0}^{n} a^k = 1+a+\dots+a^n = \frac{1-a^{n+1}}{1-a}$, $\quad a^{n+1} \to 0$

c2) $a=1$ $\quad y_n = \displaystyle\sum_{k=0}^{n} 1 = n+1 \to \infty$

c3) $a>1$ $\quad y_n = \displaystyle\sum_{k=0}^{n} a^k \geq \sum_{k=0}^{n} 1 \to \infty$

1) $\displaystyle\sum_{n\geq 1} n a^{n-1}$ $\qquad$ 2) $\displaystyle\sum_{n\geq 1} \frac{1}{n(n+2)}$

3) $\displaystyle\sum_{n\geq 2} \frac{1}{n^2-1}$ $\qquad$ 3) $\displaystyle\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$

4) $\displaystyle\sum_{n\geq 1} \frac{1}{n(n+1)(n+3)}$ $\qquad$ 5) $\displaystyle\sum_{n\geq 1} \frac{1}{n!(n+2)}$

$$\sum_{n\geq 2} \frac{1}{n^3}$$

$$y_{n+1} - y_n = \frac{1}{(n+1)^3} \geq 0 \Rightarrow y_n \nearrow$$

$\Rightarrow (y_n)_n$ este conv $\Leftrightarrow$ este mărginit

$(x_n \geq 0)$

$$y_n = \sum_{k=2}^{n} \frac{1}{k^3} \leq \sum_{k=2}^{n} \frac{1}{k(k-1)} = \sum_{k=2}^{n} \frac{1}{k-1} - \frac{1}{k} = 1 - \frac{1}{n} \leq 1$$

$$\sum_{n\geq 1} \frac{1}{n^3} \text{ conv}$$

$$\sum_{n\geq 2} \frac{1}{n^3} \leq \sum_{n\geq 2} \frac{1}{n(n-1)} = 1$$

<u>Crit comp</u> Fie $\sum_{n\geq 1} a_n$ și $\sum_{n\geq 1} b_n$

cu $a_n, b_n > 0$. Dacă $\exists \lim_{n\to\infty} \frac{a_n}{b_n} \in (0,\infty)$

$$\Rightarrow \sum_{n\geq 1} a_n \sim \sum_{n\geq 1} b_n$$

$$\boxed{\sum_{n\geq 1} \frac{1}{n^{\alpha}} \quad \text{conv } \alpha > 1 \quad \text{div } \alpha \leq 1}$$

$$\frac{1}{2} - 1 = -\frac{1}{2}$$

$$\sum_{n\geq 1} \underbrace{\frac{\sqrt{n}}{2n+1}}_{a_n} \sim \sum_{n\geq 1} \underbrace{\frac{1}{\sqrt{n}}}_{b_n}$$

sus $\frac{1}{2}$

jos 1

$$\alpha = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\frac{a_n}{b_n} \to \frac{1}{2}$$

$$\sum_{n\geq 1} \frac{\sqrt{n}}{n^2+1} \sim \sum \frac{1}{n\sqrt{n}} \quad \text{conv}$$

$$\frac{\text{grad } \frac{1}{2}}{\text{grad } 2} \qquad \frac{1}{1+\frac{1}{2}} \qquad \alpha = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\sum_{n\geq 1} \sin^2 \frac{1}{\sqrt{n}} \sim \sum_{n\geq 1} \frac{1}{n\sqrt{n}} \quad \text{conv}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{n\to\infty} \frac{\left(\sin\frac{1}{\sqrt{n}}\right)^3}{\left(\frac{1}{\sqrt{n}}\right)^3} = 1$$

$$\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$$

$$a_n \qquad a_{n+1}$$

$$\frac{1}{2}\,\frac{n+2-n}{n(n+1)(n+2)} = \frac{1}{2}\left(\underbrace{\frac{1}{n(n+1)}}_{a_n} - \underbrace{\frac{1}{(n+1)(n+2)}}_{a_{n+1}}\right)$$

$$y_n = \sum_{k=1}^{n} \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\sum_{k=1}^{n} a_k - a_{k+1}\right) = \frac{1}{2}\left(a_1 - a_{n+1}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1\cdot 2} - \underbrace{\frac{1}{(n+1)(n+2)}}_{\to 0}\right) = \frac{1}{4}$$

$$\sum_{n\geq 2}\frac{1}{n^2-1} = \sum_{n\geq 2}\frac{1}{2}\Big(\underbrace{\frac{1}{n-1}-\frac{1}{n+1}}_{a_{n-1}-a_{n+1}}\Big) \qquad a_n = \frac{1}{n}$$

$$=\frac{1}{2}\sum_{n\geq 2} 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{4} \dots$$

$$= \frac{1}{2}\left(1+\frac{1}{2}\right)$$

$$y_n = \sum_{k=2}^{n}\frac{1}{k^2-1} = \frac{1}{2}\Big(1+\frac{1}{2}-\overset{\nearrow 0}{\frac{1}{n+1}}+\overset{\nearrow 0}{\frac{1}{n+1}}\Big) = \frac{3}{4}$$

$$\sum_{n\geq 1}\frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{n} \sim \sum_{n\geq 1}\frac{\ln n}{n} \geq \sum \frac{1}{n} = \infty.$$

$$* \lim_{n\to\infty}\frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n} = \lim_{n\to\infty}\frac{\overbrace{\cancel{1+\frac{1}{2}+\dots+\frac{1}{n}}+\frac{1}{n+1}}^{a_{n+1}}\cancel{-1-\dots-\frac{1}{n}}}{\ln(n+1)-\ln n}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim\frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$

$b_n \nearrow \infty$ $\quad$ $(\overset{\nearrow}{7})$

$$= \lim_{n\to\infty} \frac{1}{(n+1)\ln\left(\frac{n+1}{n}\right)} = \lim_{n\to\infty} \frac{1}{\ln \underbrace{\left(1+\frac{1}{n}\right)^{n+1}}_{e}} = \frac{1}{\ln e} = \frac{1}{1}$$

$$\lim_{n\to\infty} 1 + \frac{1}{2} + \dots + \frac{1}{n} = \infty$$

$\alpha > 1$ con

$\alpha = 1$

$\alpha \leq 1$ div

$$\boxed{\sum_{n\geq 1} \frac{1}{n} = \infty}$$

$$\sum_{1} \frac{1}{n^\alpha}$$

$$\sum \frac{1}{n^2} < \infty$$

$$\sum_{n\geq 1} \left(\sqrt{n^4+3n+1} - n^2\right) =$$

$$\sum_{n\geq 1} \frac{\cancel{n^4}+3n+1-\cancel{n^4}}{\underbrace{\sqrt{n^4+3n+1}+n^2}_{a_n}} \sim \sum_{n\geq 1} \frac{n}{n^2} = \sum_{n\geq 1} \frac{1}{n}$$

l_n

div

grad $3n = 1$

grad 2

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{3n+1}{n^2\left(\sqrt{1+3/n^3+1/n^4}+1\right)}}{\dfrac{1}{\cancel{n}}} \to 2$$

$$= \lim_{n\to\infty} \frac{3n+1}{2n} = \frac{3}{2} \in (0,\infty)$$

$$\sum_{n\geq 1} 2^n \sin\frac{\pi}{4^n} \sim \sum_{n\geq 1} 2^n \cdot \frac{\pi}{4^n} = \pi \sum_{n\geq 1} \overset{1,?}{\frac{1}{2^n}} - \pi$$

$$\frac{\pi}{4^n} \longrightarrow 0 \qquad \frac{\sin x}{x} \xrightarrow[x\to 0]{} 1 \qquad \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \to \frac{1}{1-a}$$

ρ_{m_0}

$$\lim_{n\to\infty} \frac{\sin\frac{\pi}{4^n}}{\frac{\pi}{4^n}} = 1$$

Crit raportului: $\sum_{n\geq 1} x_n \quad x_n > 0$

$$l = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} \quad \begin{cases} > 1 & (x_n \to \infty) \text{ div} \\ < 1 & (x_n \to 0) \text{ conv} \end{cases}$$

Crit radicalului: $\sum_{n\geq 1} x_n \quad x_n \geq 0$

$$l_1 = \lim_{n\to\infty} \sqrt[n]{x_n} \quad \begin{cases} l > 1 & \text{div} \\ l < 1 & \text{conv} \end{cases}$$

$$\sum_{n\geq 1} x^n \frac{\sqrt{n}}{n^4+1} \qquad x > 0$$

$$a_n = x^n \cdot \frac{\sqrt{n}}{n^4+1}$$

$$\frac{a_{n+1}}{a_n} = \cancel{x^{n+1}}\, x \cdot \frac{\sqrt{n+1}}{(n+1)^4+1} \cdot \frac{n^4+1}{\cancel{x^n}\left(\sqrt{n}\right)}$$

$$= x \cdot \underbrace{\frac{\sqrt{n+1}}{\sqrt{n}}}_{\to 1} \cdot \underbrace{\frac{n^4+1}{(n+1)^4+1}}_{\to 1} = x$$

$x > 1$ ∩ div

$x < 1$ ∩ · conv

$$4 - \frac{1}{2} = 3 + \frac{1}{2}$$

$$x = 1 \quad \sum_{n \geq 1} \underbrace{\frac{\sqrt{n}}{n^4+1}}_{b_n} \sim \sum_{n \geq 1} \underbrace{\frac{1}{n^3\sqrt{n}}}_{c_n} \text{ conv}$$

$$\frac{b_n}{c_n} \to 1$$