Să se determine domeniul de convergență pentru următoarele serii:

1) $\sum\limits_{n\geq 1} \dfrac{x^n}{n^2\cdot 3^n}$ 2) $\sum\limits_{n\geq 1} \dfrac{x^n}{\sqrt{n}} \left(\dfrac{n+1}{2n+1}\right)^n$

3) $\sum\limits_{n\geq 1} \sqrt[n]{(2n)!}\cdot x^n$ 4) $\sum\limits_{n\geq 1} \dfrac{x^n\cdot n!}{(2n)^{2n}}$

5) $\sum\limits_{n\geq 1} \dfrac{1}{n!\,x^n}$ 6) $\sum\limits_{n\geq 1} \dfrac{x^n}{3^n+5^n}$

7) $\sum\limits_{n\geq 1} \dfrac{1\cdot 4\cdot 7\cdots(3n-2)}{3\cdot 6\cdot 9\cdots(3n)}\cdot x^n$

8) $\sum\limits_{n\geq 1} \dfrac{n+3}{n\sqrt{n}+1}(x-3)^n$ 9) $\sum\limits_{n\geq 1} \dfrac{n}{n\sqrt{n}+1}\left(\dfrac{x-1}{x-4}\right)^n$

10) $\sum\limits_{n\geq 1} n^n x^n$

1) $a_n = \frac{1}{n^2 \cdot 3^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2 \cdot 3^n}} = \frac{1}{3}$$

$$\rho = \frac{1}{\lim \sqrt[n]{|a_n|}} = 3. \Rightarrow$$ (ρ - raza de conv).

$(-3, 3) \subset D \subset [-3, 3]$.

$x = 3$ $\sum_{n\geq 1} \frac{1}{n^2} \cdot \frac{3^n}{3^n} = \sum_{n\geq 1} \frac{1}{n^2}$ - conv

$|x| = 3$

$\Rightarrow D = [-3, 3]$ (s abs conv pe D).

2) $a_n = \frac{1}{\sqrt{n}} \cdot \left(\frac{n+1}{2n+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\sqrt[2n]{n}} \cdot \frac{n+1}{2n+1} = \frac{1}{2} \Rightarrow \rho = 2 \Rightarrow$$

$(-2, 2) \subset D \subset [-2, 2]$. $|x| = 2 \Rightarrow \sum_{n\geq 1} \frac{2^n}{\sqrt{n}} \cdot \left(\frac{n+1}{2n+1}\right)^n$

$(x = 2)$

$$\left(\frac{2n+2}{2n+1}\right)^n = \left(1 + \frac{1}{2n+1}\right)^n \to e^{\frac{1}{2}} \Rightarrow \sum_{n\geq 1} \frac{2^n}{\sqrt{n}} \cdot \left(\frac{n+1}{2n+1}\right)^n \sim$$

$\sim \sum_{n\geq 1} \frac{1}{\sqrt{n}}$ divergentă

$x=-2 \quad \sum_{n\geq 1} (-1)^n \cdot \frac{1}{\sqrt{n}} \left(\frac{2n+2}{2n+1}\right)^n$

$b_n = \frac{1}{\sqrt{n}} \left(\frac{2n+2}{2n+1}\right)^n \to 0$

$\frac{b_{n+1}}{b_n} = \sqrt{\frac{n}{n+1}} \cdot \left(\frac{2n+4}{2n+3}\right)^n \cdot \left(\frac{2n+1}{2n+2}\right)^n$

$= \sqrt{\frac{n}{n+1}} \cdot \left(\frac{4n^2+10n+4}{4n^2+10n+6}\right)^n < 1$

$\Rightarrow b_n \downarrow 0 \Rightarrow x=-2$ s. serie convergentă

$D = [-2, 2)$.

4) $a_n = \frac{n!}{(2n)^{2n}}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{(2n)^{2n}}} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} =$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{(2n+2)^{2n+2}} \cdot \frac{(2n)^{2n}}{\cancel{n!}} =$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{2n}{2n+2}\right)^{2n}}_{e} \cdot \frac{n+1}{(2n+2)^2} \to 0$$

$$\Rightarrow \rho = \frac{1}{0} = \infty \Rightarrow D = \mathbb{R}$$

10) $\lim \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{n^n} = \infty \Rightarrow \rho = \frac{1}{\infty} = 0$

$\Rightarrow D = \{0\}$.

5) $\sum_{n\geq 1} \frac{1}{n!\, x^n}$ Notăm $\frac{1}{x} = y$ și

considerăm seria de puteri $s_2(y) = \sum_{n\geq 1} \frac{y^n}{n!}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = 0 \quad \rho = \frac{1}{0} = \infty$$

$D_2 = \mathbb{R} \Rightarrow D = \mathbb{R}^*$

7) $a_n = \frac{1\cdot 4 \cdots 3n-2}{3\cdot 6 \cdots 3n}$ Raabe conv

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{3n+1}{3n+3} \to 1 \Rightarrow \rho = 1$$

$(-1, 1) \subset D \subset [-1, 1]$

$|x| = 1 \quad (x = 1)$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{3n+3}{3n+1} - 1\right) = \frac{+2n}{3n+3} \to +\frac{2}{3}$$

$\frac{2}{3} < 1$ s div

$0 < \frac{2}{3} < 1 \Rightarrow a_n \to 0$, $\frac{a_{n+1}}{a_n} < 1 \Rightarrow a_n \downarrow 0 \Rightarrow$ semi conv pentru $x = -1$

$D = [-1, 1)$.

g) $\sum\limits_{n\geq 1} \frac{n}{n\sqrt{n}+1}\left(\frac{x-1}{x-4}\right)^n$

Notăm $\frac{x-1}{x-4} =: y$ şi considerăm $s_1(y) = \sum\limits_{n\geq 1} \frac{n}{n\sqrt{n}+1} \cdot y^n$

$\lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{n\sqrt{n}+1}} = 1 \Rightarrow \rho = 1$ (raza de conv) $\Rightarrow$

$(-1, 1) \subset D_1 \subset [-1, 1]$

$|y| = 1$ ($y = 1$) $\sum\limits_{n\geq 1} \frac{n}{n\sqrt{n}+1} \sim \sum \frac{1}{\sqrt{n}}$ div

$y = -1$ $\sum\limits_{n\geq 1} (-1)^n \frac{n}{n\sqrt{n}+1}$ $\frac{n}{n\sqrt{n}+1} \to 0$

$f(x) = \frac{x}{x^{\frac{3}{2}}+1}$ $f'(x) = \frac{(x^{\frac{3}{2}}+1) - x \cdot \frac{3}{2} x^{\frac{1}{2}}}{(x^{\frac{3}{2}}+1)^2} =$

$= \frac{-\frac{1}{2} x^{\frac{3}{2}} + 1}{(x^{\frac{3}{2}}+1)^2} \leq 0$ $x \geq 2^{\frac{2}{3}} = x_0 \Rightarrow f \downarrow$ pe (x_0, ∞)

$\Rightarrow$ s este semiconv $\Rightarrow$ $D_1 = [-1, 1)$

$\frac{x-1}{x-4} \in [-1, 1) \Leftrightarrow -1 \leq \frac{x-1}{x-4} \leq 1$

$x > 4 \Rightarrow \dfrac{x-1}{x-4} \leq 1 \Rightarrow x-1 \leq x-4 \Rightarrow 3 \leq 0$

fals.

$x < 4 \Rightarrow$

$4 - x \geq x - 1 \geq \cancel{x} - \cancel{4}$

$\Leftrightarrow 5 - 2x \geq 0 \geq -3 \qquad \underbrace{x \leq \frac{5}{2}}$

$D = \left(-\infty, \frac{5}{2}\right).$

Să se dezvolte în serie de puteri următoarele funcții

1) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = e^x$ în jurul $a=0$

2) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = a^x$ unde $a>0$ în jurul lui $c=0$

3) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \sin x$ în jurul lui $c=0$

4) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \cos x$ în jurul lui $c=0$

5) $f: (-1,1) \to \mathbb{R}$ $f(x) = \frac{1}{1+x}$ în jurul lui $c=0$

6) $f: (-1,1) \to \mathbb{R}$ $f(x) = (1+x)^2$ în jurul lui $c=0$

7) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \cos^2 x$ —//—

8) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \operatorname{arctg} x$ —//—

9) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \arcsin x$ —//—

$c=0$

10) $f(x) = \frac{1}{1-x^2}$ în

1)

$$f(x) = e^x \Rightarrow f^{(n)}(x) = e^x \Rightarrow f^{(n)}(a) = 1$$

$$\Rightarrow e^x = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{T_{f,n,a}(x)} + \underbrace{\frac{f^{(n+1)}(\alpha)(x-a)^{n+1}}{(n+1)!}}_{R_{f,n,a}(x) = R^n}$$

$|x| \leq M$

$$|R^n| = \left| \frac{e^{\alpha} \cdot x^{n+1}}{(n+1)!} \right| \leq \underbrace{\frac{e^M \cdot M^{n+1}}{(n+1)!}}_{a_n} \to 0$$

$$\frac{a_{n+1}}{a_n} = \frac{M}{n+2} \to 0$$

$$e^x = \sum_{k=0}^{n} \frac{1}{k!} x^k + \underbrace{R^n(x)}_{\to 0 \; n \to \infty}$$

$$\Rightarrow e^x = \sum_{n \geq 0} \frac{x^n}{n!}$$

2) $$a^x = e^{x \ln a} = \sum_{n \geq 0} \frac{(\ln a)^n x^n}{n!}$$

3) $f(x) = \sin x \qquad f'(x) = \cos x \qquad f''(x) = -\sin x$

$f^{(III)}(x) = -\cos x \qquad f^{IV}(x) = f$

$$f^{(n+4)} = f^{(n)}$$

$f^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right)$ (se verifică pentru $n = 0, 1, 2, 3$)

$$\sin x = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \underbrace{\frac{f^{(n+1)}(\alpha)}{(n+1)!}(x-a)^{n+1}}_{r_n}$$

$|x| \le M$

$$r_n = \left| \frac{\overset{\sin}{\cancel{\cos}}\left(\alpha + (n+1)\frac{\pi}{2}\right)(x-a)^{n+1}}{(n+1)!} \right| \le \frac{M^{n+1}}{(n+1)!} \to 0$$

$$\sin x = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + r_n$$

$\downarrow \qquad\qquad\qquad \searrow 0$

$$\sum_{n \ge 0} \frac{\sin\left(n\frac{\pi}{2}\right)}{n!} x^n \quad \Rightarrow$$

$n = 2k+1 \qquad n = 2k \Rightarrow \sin 2k \cdot \frac{\pi}{2} = 0$

$$\sin x = \sum_{k \geq 0} \frac{\sin\left(k\pi + \frac{\pi}{2}\right)}{(2k+1)!} x^{2k+1} = \sum_{k \geq 0} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

5) $f(x) = \dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$

$$= \sum_{n \geq 0} (-1)^n x^n \qquad |x| < 1$$

6) $f(x) = (1+x)^\alpha \qquad f'(x) = \alpha (1+x)^{\alpha - 1}$

$$f^{(k)}(x) = \alpha(\alpha-1) \dots (\alpha - k + 1)(1+x)^{\alpha - k}$$

$$f^{(k)}(0) = \alpha(\alpha-1) \dots (\alpha-k+1)$$

$$(1+x)^\alpha = \sum_{k=0}^{n} \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} x^k + r_n$$

$$|r_n| = \left| \frac{\alpha(\alpha-1) \dots (\alpha - \cancel{n}) (1+y_n)^{\alpha - \cancel{n} - 1}}{(n+1)!} \cdot x^n \right|$$

$$= \left| \frac{\alpha(\alpha-1) \dots (\alpha - n)}{(n+1)!} \cdot \left(\frac{x}{1+y_n}\right)^n \cdot (1+y_n)^\alpha \right|$$

$$\leq 2^\alpha \left| \frac{\alpha(\alpha-1)\dots(\alpha-n)}{(n+1)!} \cdot \left(\frac{x}{1+y_n}\right)^n \right|$$

$|x| \geq n$

Dacă $\left|\frac{x}{1+y_n}\right| < u < 1$

$$\Rightarrow |\lambda_n| \le C \left|\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}\right| \cdot u^n = b_n$$

$$\frac{b_{n+1}}{b_n} = \left|\frac{\alpha-n-1}{n+2}\right| \cdot u \to u < 1$$

$$y_n \in (0, x) \Rightarrow \left|\frac{x}{1+y_n}\right| \le \frac{|x|}{1-|y_n|} \le \frac{|x|}{1-|x|} < u < 1$$

$$|x| < u - u|x| \qquad (1+u)|x| < u \qquad |x| < \frac{u}{1+u} < \frac{1}{2}$$

$$\forall x \in \left(-\frac{u}{1+u}, \frac{u}{1+u}\right)$$

$$(1+x)^\alpha = \sum_{n\ge 0} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

Obs $\quad \underline{x \ge 0} \Rightarrow \left|\frac{x}{1+y_n}\right| \le |x|$ și $\quad \forall x \in (0,1)$.

$\Downarrow$

$y_n > 0$

În realitate $\forall x \in (-1,1) \Rightarrow$

$$(1+x)^\alpha = \sum_{n\ge 0} \frac{\alpha(\alpha+1)\cdots(\alpha-n+1)}{n!} = s(x)$$

Raza de conv. a lui s este 1.

Să se arate că

1) Seria $s = \sum_{n\geq 1} \frac{1}{n^4} \cos nx$ unde $x \in \mathbb{R}$ este de clasă C^2

2) Seria $s = \sum_{n\geq 1} \frac{1}{n^5} \sin nx$ —//— —//— C^3

3) Seria $s = \sum_{n\geq 1} \frac{1}{n^6} e^{nx}$ unde $x \in (0,1)$ —//— C^4

4) Seria $s = \sum_{n\geq 1} \frac{1}{n^2} \cos \frac{1}{n} x$ unde $x \in \mathbb{R}$ —//— C^∞

5) Seria $s = \sum_{n\geq 1} \frac{1}{n^x}$ cu $x > 0$ este de clasă C^∞

6) Seria $s = \sum_{n\geq 1} \frac{1}{n^2+x}$ cu $x \in (0,1)$ —// —

7) Seria $s = \sum_{n\geq 1} \frac{1}{n^2+x^2}$ cu $x \in (0,1)$ este de clasă C^2.

8) Seria $s = \sum_{n\geq 1} \frac{1}{n^4} \ln(1+nx)$ cu $x > 0$ —//— C^2.

Să se arate că funcțiile de la punctele 4) și 6) se pot desvolta în serie de puteri în jurul fiecărui punct din domeniul de definiție