; TUTORIAT 3 E
ANALIZĂ I

1. $\exists \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}$, $n \geq 1$ ⟶ nu converge.

SOLUȚIE

Fie $\varepsilon = 1$. Oricare $N \in \mathbb{N}^*$ este $\forall n \geq N$, $\forall p \in \mathbb{N}$ a.î. avem

$$|x_{n+p} - x_n| < \varepsilon.$$

$$\left|\sum_{k=n+1}^{n+p} \frac{1}{k}\right| = \sum_{k=n+1}^{n+p} \frac{1}{k} \geq \left|\sum_{k=n+1}^{n+p} \frac{1}{k}\right| = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p} \geq \frac{p}{n+p} \geq \frac{1}{2} \geq 1 = \varepsilon$$

Folosim inegalitatea:

$$\frac{1}{(n+1)^2} \leq \frac{1}{(n+1)(n+2)} = \frac{1}{n+2} - \frac{1}{n+1}, \quad n \geq 1.$$

$$\frac{1}{(n+2)^2} \leq \frac{1}{n+1} - \frac{1}{(n+2)^2}$$

$$\frac{1}{(n+3)^2} \leq \frac{1}{(n+2)^2} - \frac{1}{(n+3)^2}$$

$$\frac{1}{(n+4)^2} \leq \frac{1}{(n+3)^2} - \frac{1}{(n+4)^2}$$

$$\frac{1}{(n+p)^2} \leq \frac{1}{(n+p-1)^2} - \frac{1}{(n+p)^2}$$

$$\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(n+p)^2} \leq \frac{1}{n+1} - \frac{1}{(n+p)^2} \leq \frac{1}{n+1} \leq \frac{1}{2}$$

deci $N_\varepsilon = \left[\frac{1}{\varepsilon}\right] + 1$. □

2. $(x_n)_n$,

$$\begin{cases} x_1 = \frac{1}{2} \\ x_{n+1} = \frac{x_n^2}{2} - 2x_n + 2, \ \forall n \geq 1. \end{cases}$$

$x \in [1,3]$

$(x_n)_n$ convergentă

$\lim_{n\to\infty} x_n = ?$

SOLUȚIE

$$x_{n+1} = (x_n^2 - 2x_n + 4) \cdot \frac{1}{2} = \frac{(x_n - 1)^2}{2} + 1.$$

Arătăm că $x_n \in [1,3]$, $\forall n \geq 1$.

P.M.I.: $x_1 \in [1,3]$, verificăm.

$P(n) \to P(n+1)$: $x_n \in [1,3]$, arătăm $x_{n+1} \in [1,3]$.

$$x_{n+1} = \frac{(x_n - 1)^2}{2} + 1.$$

$x_n \in [1,3] \Rightarrow x_n - 1 \in [0,2] \Rightarrow (x_n - 1)^2 \in [0,4] \Rightarrow$

$\Rightarrow 1 + \frac{(x_n - 1)^2}{2} \in [1,3] \Rightarrow x_{n+1} \in [1,3]$, $\forall n \geq 1$.

Deci $x_n \in [1,3]$, $\forall n \geq 1$ $\Rightarrow$ $(x_n)_n$ - mărginșit.

$$x_{n+1} - x_n = (x_n - 1)^2 + 1 - x_n = (x_n - 1)(x_n - 2) =$$

$$= \underbrace{(x_n - 1)}_{\geq 0}\underbrace{(x_n - 2)}_{\leq 0} \leq 0 \Rightarrow x_{n+1} \leq x_n, \ \forall n \geq 1 \Rightarrow$$

$\Rightarrow (x_n)_n$ este monotonă descrescătoare.

Wnioskujemy $\Rightarrow (x_n)_n$ - este convergentă.

Fie $l = \lim_{n\to\infty} x_n \in \mathbb{R}$.

$$l = \frac{l^2}{2} - 2l + 2 \Rightarrow l^2 - 4l + 4 = 0 \Rightarrow (l-2)^2 = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} x_n = 2.$$

$$\begin{cases} x_{n+1} = \frac{x_n^2}{2} - 2x_n + 2 \\ l = \frac{l^2}{2} - 2l + 2 \end{cases}$$

□

3. $(a_n)_{n\geq 1}$, $a_n = \sum_{k=1}^{n} \frac{k}{n^2+n}$.

$\lim_{n\to\infty} a_n = ?$

SOLUȚIE

$$a_n = \frac{1}{n^2+1} + \frac{2}{n^2+2} + \dots + \frac{n}{n^2+n}.$$

$$\frac{1}{n^2+n} \leq \frac{1}{n^2+1}$$

$$\frac{2}{n^2+n} \leq \frac{2}{n^2+2}$$

$$\frac{n}{n^2+n} \leq \frac{n}{n^2+1}$$

$$\frac{1+2+\dots+n}{n^2+n} \leq a_n \leq \frac{1+2+\dots+n}{n^2+1}.$$

Criteriul 3

Conform $\Rightarrow \boxed{\lim_{n\to\infty} a_n = \frac{1}{2}}$.

□

4. $\lim_{n\to\infty} \left(\sqrt{n+1} + \sqrt{n+2} - \sqrt{n}\right)$.

SOLUȚIE

$x_n = n + \sqrt{n+1} + \dots + \sqrt{n}$ $(x_n)_{n\geq 1}$ - crescătoare

$y_n = n\sqrt{n}$

I) $(y_n)_{n\geq 1}$ - strict crescătoare.

$$\begin{cases} y_{n+1} > y_n, \ \forall n \geq 1 \\ \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} n\sqrt{n} = \lim_{n\to\infty} y_n, \ \forall n \geq 1 \end{cases}$$

II) $\lim_{n\to\infty} y_n = \infty$

$\lim_{n\to\infty} n\sqrt{n} = \infty \cdot \infty = \infty$

III) Calculăm limita $\exists \lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$ $\frac{\infty}{\infty}$

$$\frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \frac{\sqrt{n+1} + \sqrt{n+2} - \sqrt{n}}{(n+1)\sqrt{n+1} - n\sqrt{n}} \cdot \frac{(\sqrt{n+1} + \sqrt{n+2} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})}{(\sqrt{n+1} + \sqrt{n+2} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})} =$$

$$= \frac{\sqrt{n+1}(\sqrt{n+1} + \sqrt{n+2} + \sqrt{n})}{n+1 - n^2\sqrt{n}\sqrt{n+1}}$$

$$\exists \lim_{n\to\infty} \frac{\sqrt{n+1}(\sqrt{n+1} + \sqrt{n+2} + \sqrt{n})}{n+1 - n^2\sqrt{n}\sqrt{n+1}} = \frac{3}{2}$$

$$\Rightarrow \exists \lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \frac{3}{2} \Rightarrow \exists \lim_{n\to\infty} x_n = \infty$$

Teorema Cesaro $\Rightarrow \exists \lim_{n\to\infty} \frac{x_n}{y_n} = \frac{3}{2}$

□

5. Studiați natura seriei $(x_n)_{n\geq 1}$ cu x_n pe care este

$$|x_{n+2} - x_{n+1}| \leq \frac{1}{4^{n+1}} |x_{n+1} - x_n|, \ \forall n \geq 1.$$

SOLUȚIE: Din Cauchy

$$|x_{n+2} - x_{n+1}| \leq \frac{1}{4^{n+1}} |x_{n+1} - x_n|, \ \forall n \geq 1.$$

$$\frac{\sin n}{n+1} \leq \frac{1}{4^{n+1}}, \ \forall n \geq 1$$

$\sin n - \cos n = \sqrt{2}\sin(n - \frac{\pi}{4})$, este

$$\Rightarrow |x_{n+2} - x_{n+1}| \leq \frac{1}{4^{n+1}} |x_{n+1} - x_n|, \ \forall n \geq 1.$$

$$|x_{n+1} - x_n| \leq \frac{1}{4^n} |x_n - x_{n-1}|$$

$$|x_{n+1} - x_n| \leq \frac{1}{4^n} |x_n - x_{n-1}|$$

$$|x_{n+1} - x_n| \leq \frac{1}{4^n} \frac{1}{4^{n-1}} |x_{n-1} - x_{n-2}|$$

$$|x_{n+1} - x_n| \leq \frac{1}{4^n} \frac{1}{4^{n-1}} \frac{1}{4^{n-2}} |x_{n-2} - x_{n-3}|$$

$$|x_{n+1} - x_n| \leq \frac{1}{4^n} \frac{1}{4^{n-1}} \frac{1}{4^{n-2}} \dots \frac{1}{4^2} |x_2 - x_1|$$

$$0 \leq |x_{n+2} - x_{n+1}| \leq \frac{1}{4^{n+1}} |x_2 - x_1| \xrightarrow[n\to\infty]{} 0$$

Conform $\Rightarrow \exists \lim_{n\to\infty} |x_{n+2} - x_{n+1}| = 0$.

Fie $\varepsilon > 0$. Considerăm $N_\varepsilon \in \mathbb{N}$ este $\forall n \geq N_\varepsilon$, $\forall p \in \mathbb{N}$,

$$|x_{n+p} - x_n| \leq \varepsilon$$

$$|x_{n+p} - x_n| = |x_{n+p} - x_{n+p-1} + x_{n+p-1} - x_{n+p-2} + \dots + x_{n+1} - x_n| \leq$$

$$\leq |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + \dots + |x_{n+1} - x_n|$$

$$\leq \varepsilon \Rightarrow |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + \dots + |x_{n+1} - x_n| \leq \varepsilon$$

$$|x_{n+2} - x_{n+1}| \leq \frac{1}{4^n} |x_{n+1} - x_n|, \ \forall n \geq 1.$$

$$|x_{n+1} - x_n| \leq \frac{1}{4^n} |x_n - x_{n-1}|$$

$$|x_{n+1} - x_n| \leq \frac{1}{4^n} \frac{1}{4^{n-1}} |x_{n-1} - x_{n-2}|$$

$$|x_{n+1} - x_n| \leq \frac{1}{4^n} \frac{1}{4^{n-1}} \frac{1}{4^{n-2}} \dots \frac{1}{4^2} |x_2 - x_1|$$

$$|x_{n+1} - x_n| + |x_{n+p} - x_{n+p-1}| + \dots + |x_{n+1} - x_n| \leq \frac{1}{4^n} |x_2 - x_1| \left(1 + \frac{1}{4} + \dots + \frac{1}{4^{p-1}}\right) \leq$$

$$\leq \frac{1}{4^n} |x_2 - x_1| \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{3} |x_2 - x_1| \leq \varepsilon$$

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0 \Rightarrow \lim_{n\to\infty} |x_n - x_{n-1}| = 0$$

$\exists (\varepsilon > 0)$ este $|x_n - x_{n-1}| \leq \frac{\varepsilon}{3}$

$\exists N_\varepsilon \in \mathbb{N}$ este $|x_{n+p} - x_n| \leq \frac{\varepsilon}{3}$, $\forall n \geq N_\varepsilon, \forall p \in \mathbb{N}$

$\Rightarrow (x_n)_n$ fiind Cauchy $\Rightarrow (x_n)_n$ - este convergentă.

□

$\lim_{n\to\infty} x_n = \frac{3 \cdot \sin n}{n+1}$

(a) $x_n = \frac{\ln(\frac{n}{2})}{n}$, $n \in \mathbb{N}^*$ $\frac{\infty}{\infty}$, $n \geq 1$.

SOLUȚIE

I. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\ln(\frac{n}{2})}{n} = \lim_{n\to\infty} \frac{\ln n - \ln 2}{n} =$

$$= \lim_{n\to\infty} \frac{\ln n}{n} - \lim_{n\to\infty} \frac{\ln 2}{n} = 0 - 0 = 0$$

II. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3 \cdot \sin n}{n+1} = \lim_{n\to\infty} \frac{3 \cdot \sin n}{n+1} = 0$

$$\Rightarrow \lim_{n\to\infty} x_n = \frac{3}{2} \cdot \lim_{n\to\infty} \frac{3 \cdot \sin n}{n+1} = 0$$

$$\Rightarrow \lim_{n\to\infty} x_n = \frac{3}{2} \cdot 0 = 0$$

□

(b) $x_n = \left(\frac{n+1}{2n}\right)^{\sin n}$, $n \geq 1$

SOLUȚIE

$$\sin\frac{n}{2} \cdot \sin n = \sin^2\frac{n}{2} \cdot \frac{\sin n}{\sin\frac{n}{2}} = \sin^2\frac{n}{2} \cdot \frac{2\sin\frac{n}{2}\cos\frac{n}{2}}{\sin\frac{n}{2}} = 2\sin^2\frac{n}{2}\cos\frac{n}{2}$$

$m = k, m = 2k+1, m = 4k+3$

I. $m = 2k$ sau $m = 4k+2$, $(m \geq 2k, k \geq 1)$

$$\Rightarrow x_{2k} = \left(1 + \frac{1}{2k}\right)^{\sin k} = \left(1 + \frac{1}{2k}\right)^{\sin 2\sin^2 k} =$$

$$= \left(1 + \frac{1}{2k}\right)^{2\sin^2 k} \xrightarrow[k\to\infty]{} e^{\lim_{k\to\infty} \frac{2\sin^2 k}{2k}} = e^0 = 1$$

II. $m = 4k+1$ $(m \geq 1)$

$$\Rightarrow x_{4k+1} = \left(1 + \frac{1}{4k+1}\right)^{\sin(4k+1)} = \left(1 + \frac{1}{4k+1}\right)^{\sin 2\sin^2 2k\cos 2k} =$$

$$= \left(1 + \frac{1}{4k+1}\right)^{4\sin^2 2k\cos 2k} = \left(1 + \frac{1}{4k+1}\right)^{\frac{4\sin^2 2k}{2k} \cdot 2k\cos 2k} \xrightarrow[k\to\infty]{} e^{\lim_{k\to\infty} \frac{4\sin^2 2k}{2k} \cdot 2k\cos 2k} = e^0 = 1$$

III. $m = 4k+3$ $(m \geq 3)$

$$\Rightarrow x_{4k+3} = \left(1 + \frac{1}{4k+3}\right)^{\sin(4k+3)} = \left(1 + \frac{1}{4k+3}\right)^{\sin 2\sin^2 2k\cos 2k} =$$

$$= \left(1 + \frac{1}{4k+3}\right)^{4\sin^2 2k\cos 2k} = \left(1 + \frac{1}{4k+3}\right)^{\frac{4\sin^2 2k}{2k} \cdot 2k\cos 2k} \xrightarrow[k\to\infty]{} e^{\lim_{k\to\infty} \frac{4\sin^2 2k}{2k} \cdot 2k\cos 2k} = e^0 = 1$$

$$\lim_{n\to\infty} x_n = e^0 = 1$$

$$\lim_{n\to\infty} x_{2n} = e^0 = 1, \ \lim_{n\to\infty} x_{2n+1} = e^0 = 1$$

□