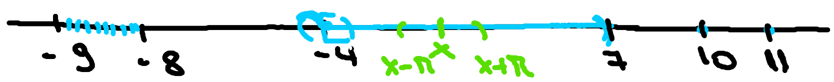
TUTORIAT 8 ANALIZA I

Friday, December 10, 2021 6:01 PM

(g) $A = [-4,7) \cup \{10,11\} \cup [(-9,-8) \cap \mathbb{Q}]$



$\mathring{A}$, A', $\bar{A}$, $\mathrm{Fr}A$, A închisă
deschisă
mărg.
compactă
conexă

• $\mathring{A} = (-4,7)$ $\qquad \bar{A} = \underline{A} \cup A'$

1°. $\underbrace{(-4,7)}_{\text{"}}$ - m.d

$\forall x \in (-4,7), \exists r = \min\{x+4, 7-x\}$
a.î. $B(x,r) = (x-r, x+r) \subseteq A$.

2°. $\left.\begin{array}{l}\exists x \in \mathring{A}, x \notin (-4,7) \\ \mathring{A} \subseteq A\end{array}\right\} \Rightarrow x \in \underline{\{-4,10,11\} \cup [(-9,-8) \cap \mathbb{Q}]}$

$x = -4 \in \mathring{A} \Rightarrow \exists r > 0$ a.î. $(-4-r, -4+r) \subseteq A$

Aleg $x_0 = -4 - \frac{r}{2} \in (-4-r, -4+r)$, $\notin A$ ↯

Analog $10, 11 \notin \mathring{A}$.

Dacă $x \in (-9,-8) \cap \mathbb{Q} \in \mathring{A} \Rightarrow$

$\Rightarrow \exists r > 0$ a.î. $\underline{(x-r, x+r) \subseteq A} \Rightarrow$

$\Rightarrow \underbrace{(x-r, x+r) \cap (-9,-8)}_{\text{conține nr. din } \mathbb{R}\setminus\mathbb{Q}} \subseteq \underbrace{A \cap (-9,-8)}_{(-9,-8)\cap\mathbb{Q}}$

$\underbrace{(-9,-8)\cap\mathbb{Q}}_{\text{conține doar nr. din } \mathbb{Q}}$

↯

$\Rightarrow \mathring{A} = (-4,7)$.

• $A' = [-4,7] \cup [-9,-8]$

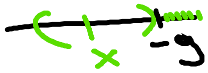
$x \in A' \overset{\text{def}}{\Longleftrightarrow} \forall V \in \mathcal{V}_x$, $V \cap A$ = ∞ de el.

$x \in A' \Leftrightarrow \forall V \in \mathcal{V}_x,\ V \cap A = \infty$ de el.

1°. $\forall x \in [-4,7] \cup [-9,-8]$, $\forall V \in \mathcal{V}_x$, $V \cap A$ are ∞ de el.

2°. $x \notin [-4,7] \cup [-9,-8] \Rightarrow x \notin A'$

- $x \in (-\infty,-9)$: găsim $V \in \mathcal{V}_x$ a.î. $V \cap A$ să nu aibă ∞ de el.



$V = (x-\varepsilon, \underbrace{x+\varepsilon}) \cap A \neq \infty$ de el.

$\varepsilon = -9 - x \Rightarrow V = (2x+9, -9)$

$V \cap A = \emptyset$

- $x \in (-8,-4)$: găsim $V \in \mathcal{V}_x$ a.î. $V \cap A \neq \infty$ de el.

$V = (x-\varepsilon, x+\varepsilon)$

$\varepsilon = \min\{x+8, -4-x\}$

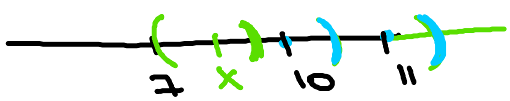
$V \cap A = \emptyset$

- $x \in (7,\infty)$: găsim $V \in \mathcal{V}_x$ a.î. $V \cap A \neq \infty$ de el.

$V = (x-\varepsilon, x+\varepsilon)$

$\varepsilon = x - 7$

$V = (7, 2x-7)$



$V \cap A$ are el. mult 2 el.

$\Rightarrow A' = [-4,7] \cup [-9,-8]$.

- $\bar{A} = A' \cup A = \overbrace{[-4,7] \cup [-9,-8]}^{A'} \cup \overbrace{[-4,7) \cup \{10,11\} \cup [[-9,-8] \cap \mathbb{Q}]}^{A} =$
$= [-4,7] \cup [-9,-8] \cup \{10,11\}$

- $\mathrm{Fr}A = \bar{A} \setminus \mathring{A} = [-4,7] \cup [-9,-8] \cup \{10,11\} \setminus (-4,7) =$
$= \{-4,7,10,11\} \cup [-9,-8]$

- A – m.d.: NU ($A \neq \mathring{A}$)
- A – m.î.: NU ($A \neq \bar{A}$)
- A – mărg.: DA: $A \subseteq B(0,12)$
- A – compactă: NU (A nu e închisă)
- A – conexă: NU (A nu e interval)

$\square$

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CONTINUITATE. DERIVABILITATE

(a) $f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases} x\,\mathrm{arctg}\,\frac{1}{x}, & x\neq 0\\ 0, & x=0. \end{cases}$$

Continuitate:

- f continuă pe $\mathbb{R}\setminus\{0\}$

($x\cdot\mathrm{arctg}\,\frac{1}{x}$ este obținută prin operații algebrice și de compunere cu funcții elementare continue)

- Studiem cont. în 0:

$$\lim_{x\nearrow 0} f(x)=\lim_{\substack{x\to 0\\ x<0}} f(x)=\lim_{\substack{x\to 0\\ x<0}} x\,\mathrm{arctg}\,\frac{1}{x}=0\cdot\mathrm{arctg}(-\infty)=0\cdot\left(-\frac{\pi}{2}\right)=0$$

$$\lim_{x\searrow 0} f(x)=\lim_{\substack{x\to 0\\ x>0}} f(x)=\lim_{\substack{x\to 0\\ x>0}} x\,\mathrm{arctg}\,\frac{1}{x}=0\cdot\mathrm{arctg}(\infty)=0\cdot\frac{\pi}{2}=0$$

$\Rightarrow$

$f(0)=0$

$\Rightarrow f$ este continuă în $0\Rightarrow$

$\Rightarrow f$ este continuă pe $\mathbb{R}$.

f este cont în $x_0 \Leftrightarrow \lim_{x\to x_0} f(x)=f(x_0)$

Derivabilitate:

- f derivabilă pe $\mathbb{R}\setminus\{0\}$ (......)
- $x\neq 0$: $f'(x)=\left(x\,\mathrm{arctg}\,\frac{1}{x}\right)'=$

$$=\mathrm{arctg}\,\frac{1}{x}+\cancel{x}\cdot\frac{1}{1+\frac{1}{x^2}}\cdot\left(-\frac{1}{x^{\cancel{2}}}\right)=$$

$$=\mathrm{arctg}\,\frac{1}{x}-\frac{x}{x^2+1}.$$

$= \operatorname{arctg}\frac{1}{x} - \frac{x}{x^2+1}.$

• studiem deriv. in 0:

$$f_s'(0) = \lim_{\substack{x\to 0\\ x<0}} \frac{f(x)-f(0)^{\nearrow 0}}{x-0} =$$

$$\lim_{\substack{h\to 0\\ (h \lessgtr 0)}} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{\substack{x\to 0\\ x<0}} \frac{f(x)}{x} = \lim_{\substack{x\to 0\\ x<0}} \operatorname{arctg}\frac{1}{x} = -\frac{\pi}{2}.$$

$$f_d'(0) = \lim_{\substack{x\to 0\\ x>0}} \frac{f(x)-f(0)}{x-0} = \lim_{\substack{x\to 0\\ x>0}} \frac{f(x)}{x} =$$

$$= \lim_{\substack{x\to 0\\ x>0}} \operatorname{arctg}\frac{1}{x} = \frac{\pi}{2}.$$

f e derivabilă in $x_0 \Leftrightarrow f_s'(x_0) = f_d'(x_0) \in \mathbb{R}$.

$f_s'(0) \neq f_d'(0) \Rightarrow f$ nu e derivabilă in 0.

(c) $f : [0,\infty) \to \mathbb{R}$

$$f(x) = \begin{cases} x\cos\frac{1}{x} + \frac{\ln(x^2+x+1)}{2x}, & x>0 \\ \frac{1}{2}, & x=0. \end{cases}$$

SOLUȚIE:

Continuitate:

- f continuă pe $(0,\infty)$ (...)
- Studiem cont. în 0:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(x\cos\frac{1}{x} + \frac{\ln(x^2+x+1)}{2x}\right)$$

$$\lim_{x\to 0} x\cos\frac{1}{x} = 0 \text{ (0·mărg)}$$

$$\lim_{x\to 0} \frac{\ln(x^2+x+1)}{2x} \overset{L'H}{\underset{\frac{0}{0}}{=}}$$

$$= \lim_{x\to 0} \frac{\frac{1}{x^2+x+1}\cdot(2x+1)}{2} =$$

$$= \lim_{x\to 0} \frac{2x+1}{2(x^2+x+1)} = \frac{1}{2}.$$

$$\Rightarrow \lim_{x\to 0} f(x) = 0 + \frac{1}{2} = \frac{1}{2} = f(0) \Rightarrow$$

$\Rightarrow f$ e cont. în 0 $\Rightarrow$

$\Rightarrow f$ e cont pe $[0,\infty)$.

Derivabilitatea:

- f e deriv. pe $(0,\infty)$ (.....)
- Studiem deriv în 0:

$$f_d'(0) = \lim_{\substack{x\to 0\\ x>0}} \frac{f(x)-f(0)}{x-0} = \lim_{\substack{x\to 0\\ x>0}} \frac{f(x)-\frac{1}{2}}{x} =$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{x\cos\frac{1}{x} + \frac{\ln(x^2+x+1)}{2x} - \frac{1}{2}}{x} =$$

$$= \lim_{x\to 0}\left(\cos\frac{1}{x} + \frac{\frac{\ln(x^2+x+1)}{2x} - \frac{1}{2}}{x}\right)$$

$$= \lim_{\substack{x\to 0\\ x>0}} \left(\cos\frac{1}{x} + \frac{\frac{\ln(x^2+x+1)}{2x} - \frac{1}{2}}{x} \right)$$

$$L = \lim_{\substack{x\to 0\\ x>0}} \frac{\frac{\ln(x^2+x+1)}{2x} - \frac{1}{2}}{x} \overset{L'H}{\underset{0/0}{=}}$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{\frac{2x+1}{x^2+x+1}\cdot \cancel{2}x - \ln(x^2+x+1)\cdot \cancel{2}}{4x^2\ 2x^2+x} =$$

$$= \frac{1}{2}\lim_{\substack{x\to 0\\ x>0}} \frac{(2x+1)\cdot x - \ln(x^2+x+1)\cdot(x^2+x+1)}{x^2(x^2+x+1)} =$$

$$= \frac{1}{2}\cdot\lim_{\substack{x\to 0\\ x>0}} \left(2 + \frac{x - \ln(x^2+x+1)(x^2+x+1)}{x^2}\right)$$

$$\lim_{\substack{x\to 0\\ x>0}} \frac{x - \ln(x^2+x+1)(x^2+x+1)}{x^2} \overset{L'H}{\underset{0/0}{=}}$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{1 - \frac{2x+1}{\cancel{x^2+x+1}}\cdot(\cancel{x^2+x+1}) - \ln(x^2+x+1)\cdot(2x+1)}{2x}$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{-2x - \ln(x^2+x+1)\cdot(2x+1)}{2x} =$$

$$= -1 - \frac{1}{2}\lim_{\substack{x\to 0\\ x>0}} \frac{\ln(x^2+x+1)(2x+1)}{x}$$

$$= -1 - \frac{1}{2}\lim_{\substack{x\to 0\\ x>0}} \frac{\ln(x^2+x+1)}{x} =$$

$$= -1 - \frac{1}{2}\cdot\lim_{\substack{x\to 0\\ x>0}} \frac{2x+1}{x^2+x+1} = -1 - \frac{1}{2} = -\frac{3}{2}.$$

$$L = \frac{1}{2}\left(2 - \frac{3}{2}\right) = \frac{1}{4} \in \mathbb{R}.$$

$$\nexists \lim_{\substack{x\to 0\\ x>0}} \cos\frac{1}{x} \quad \left(\begin{array}{l} x = \frac{1}{2n\pi} \to 0 \\ x = \frac{1}{(2n+1)\pi} \to 0 \end{array} \right)$$

$$\Rightarrow \nexists \lim_{\substack{x\to 0\\ x>0}} \frac{f(x)-f(0)}{x-0} \Rightarrow f \text{ nu e deriv în } 0.$$

$f \circ d \circ i \circ w \circ q \circ e(o\,\rho\alpha).$

$\square$

2. Fie $a, b \in \mathbb{R}, a<b$. Studiați dacă $\exists$ funcții bijective $f:[a,b] \to \mathbb{R}$ și care au P.D. (prop. lui Darboux).

SOLUȚIE:

$f:[a,b] \to \mathbb{R}$

f — bijectivă
— P.D.

- P.D $\Leftrightarrow$ duce intervale în intervale.
- monotonă + PD $\Rightarrow$ continuă.
- injectivă + PD $\Rightarrow$ monotonă (strict) $\Rightarrow$ continuă.

Pp. că $\exists f:[a,b] \to \mathbb{R}$ bijectivă cu PD.

$\left.\begin{array}{l} f \text{ - bijectivă} \Rightarrow f \text{ injectivă} \\ f \text{ are PD} \end{array}\right\} \Rightarrow f$ este strict monotonă și continuă.

Pp. f este strict crescătoare.

$a<b \Rightarrow f(a)<f(b) \Rightarrow f([a,b])=[f(a),f(b)]$

f este bij. $\Rightarrow f$ este surjectivă $\Rightarrow \underbrace{\operatorname{Im} f}_{f([a,b])}=\mathbb{R} \Rightarrow$

$\Rightarrow \underbrace{[f(a), f(b)]=\mathbb{R}}$, contradicție

$\Rightarrow \nexists f:[a,b] \to \mathbb{R}$ bij. cu P.D.

$\square$

3. Fie $f,g:[a,b]\to\mathbb{R}$, f,g cont. pe $[a,b]$, deriv. pe (a,b). Știind că $f(a)=f(b)=0$, arătați că $\exists\, c\in(a,b)$ a.î. $f'(c)+f(c)\cdot g'(c)=0$.

$h'(x)$

Soluție:

Fie $h:[a,b]\to\mathbb{R}$

$h(x)=f(x)\cdot e^{g(x)}$

$$\left\{\begin{array}{l} h \text{ cont pe } [a,b] \\ \text{ deriv pe } (a,b) \\ h(a)=h(b)=0.\ (f(a)=f(b)=0) \end{array}\right.$$

$\xRightarrow{\text{Th. Rolle}} \exists\, c\in(a,b)$ a.î. $h'(c)=0$.

$$h'(x)=f'(x)\cdot e^{g(x)}+f(x)\cdot e^{g(x)}\cdot g'(x)=$$
$$=e^{g(x)}\underbrace{\left(f'(x)+f(x)\cdot g'(x)\right)}$$

$$h'(c)=0\Rightarrow \underset{\neq 0}{e^{g(c)}}\left(f'(c)+f(c)\cdot g'(c)\right)=0\Rightarrow$$
$$\Rightarrow f'(c)+f(c)\cdot g'(c)=0.$$

$\square$

<u>Th Lagrange</u>: $f:[a,b]\to\mathbb{R}$

$$\left.\begin{array}{l} \cdot\ f \text{ cont pe } [a,b] \\ \cdot\ f \text{ deriv pe } (a,b) \end{array}\right\}\Rightarrow \exists\, c\in(a,b) \text{ a.î. } f'(c)=\frac{f(b)-f(a)}{b-a}.$$

<u>Th. Rolle (consecință)</u> $f:[a,b]\to\mathbb{R}$

$$\left.\begin{array}{l} \bullet\ f \text{ cont pe } [a,b] \\ \bullet\ f \text{ deriv pe } (a,b) \\ \bullet\ f(a)=f(b) \end{array}\right\}\Rightarrow \exists\, c\in(a,b) \text{ a.î. } f'(c)=0.$$

4. $f : \mathbb{R} \to \mathbb{R}$
$f(x) = \sqrt{x^2+3}$.
$f(x+1) - f(x) \le 1, \forall x \in \mathbb{R}$.

<u>SOLUȚIE:</u>

Fie $x \in \mathbb{R}$. Aplic Th. Lagrange pe $[x, x+1]$.
① f cont pe $[x, x+1]$
② f deriv pe $(x, x+1)$

$\Rightarrow \exists c_x \in (x, x+1)$ a.î. $f'(c_x) = \dfrac{f(x+1)-f(x)}{x+1-x} \Rightarrow$ $x+1-x = 1$

$\Rightarrow \overset{\le 1.}{f'(c_x)} = f(x+1) - f(x)$

$f'(x) = \dfrac{1}{\cancel{2}\sqrt{x^2+3}} \cdot \cancel{2}x = \dfrac{x}{\sqrt{x^2+3}}$.

$f'(c_x) = \dfrac{c_x}{\sqrt{c_x^2+3}} \overset{??}{\le} 1$.

$c_x^2+3 > c_x^2 \Rightarrow \underbrace{\sqrt{c_x^2+3}}_{>0} > |c_x| \ge c_x \Rightarrow$

$\Rightarrow \dfrac{c_x}{\sqrt{c_x^2+3}} < 1. \Rightarrow f'(c_x) \underset{(<)}{\le} 1. \Rightarrow$

$\Rightarrow f(x+1) - f(x) \le 1. \Rightarrow$

$\Rightarrow f(x+1) - f(x) \le 1, \forall x \in \mathbb{R}$.

$\square$

$\sqrt{(x+1)^2+3} - \sqrt{x^2+3} \le 1, \forall x \in \mathbb{R}$

$g(x) = \sqrt{(x+1)^2+3} - \sqrt{x^2+3}$.

$g'(x) = \dfrac{1}{\cancel{2}\sqrt{(x+1)^2+3}} \cdot \cancel{2}(x+1) - \dfrac{1}{\cancel{2}\sqrt{x^2+3}} \cdot \cancel{2}x = 0$

$\dfrac{x+1}{\quad} = \dfrac{x}{\quad}$

$$\frac{x+1}{\sqrt{(x+1)^2+3}} = \frac{x}{\sqrt{x^2+3}}$$

$$1 = \frac{(x+1)^2+3-3}{(x+1)^2+3} = \frac{x^2+3-3}{x^2+3}$$

$$\frac{1}{(x+1)^2+3} = \frac{1}{x^2+3} \Rightarrow x^2+\cancel{3} = (x+1)^2+\cancel{3} \Rightarrow$$

$$\Rightarrow 2x+1=0$$

$$x=-\frac{1}{2}.$$

$$\frac{\frac{1}{2}}{\sqrt{\left(\frac{1}{2}\right)^2+3}} \neq \frac{-\frac{1}{2}}{\sqrt{\left(\frac{1}{2}\right)^2+3}}$$

$\Rightarrow g'(x)=0$ nu are sol.

$$g'(x) = \frac{x+1}{\sqrt{(x+1)^2+3}} - \frac{x}{\sqrt{x^2+3}} \overset{?}{>} 0$$

x	$-\infty$		∞
g'		?	
g			

$$f(x+1)-f(x) \leq 1 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{(x+1)^2+3} - \sqrt{x^2+3} \leq 1 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{(x+1)^2+3} \leq 1+\sqrt{x^2+3} \Leftrightarrow$$

$$\Leftrightarrow \cancel{x^2}+2x+\cancel{1}+\cancel{3} \leq \cancel{1}+\cancel{x^2}+\cancel{3}+2\sqrt{x^2+3} \Leftrightarrow$$

$$\Leftrightarrow \cancel{2}x \leq \cancel{2}\sqrt{x^2+3} \Leftrightarrow x \leq \sqrt{x^2+3},$$ adevărat $\forall x \in \mathbb{R}$.

$\boxed{z}$

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Th. Cauchy: $f, g : [a,b] \to \mathbb{R}$.

① f, g cont. pe $[a,b]$

② f, g deriv pe (a,b).

③ $g(a) \neq g(b)$, $g'(x) \neq 0, \forall x \in (a,b)$

$\Rightarrow \exists c \in (a,b)$ a.î. $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$.

Cauchy $\Rightarrow$ Lagrange $\Rightarrow$ Rolle

$(g(x)=x)$ $(f(a)=f(b))$