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Grupa 164

Examen

- Algebră și geometrie -

Subiectul I:

1. Notez submulțimea $\{(t+1, 2(t+1), -(t+1)) \mid t \in \mathbb{R}\}$ cu A

~~Fie $w_1 = (a+1, 2(a+1)) - (a$~~

$t+1+2t+2-t-1 = 2t+2 \in \mathbb{R} \; (t \in \mathbb{R})$

$(\forall)\, a \in \mathbb{R}, t \in \mathbb{R} \quad a(t+1) \in \mathbb{R};\ a \cdot 2(t+1) \in \mathbb{R};\ -a(t+1) \in \mathbb{R}$

$\Rightarrow$ (Adevărat) subsp. vect. în $\mathbb{R}^3$

4. Adevărat

3. $A = (0,0,0)$

$B = (-3,1,-3)$

$C = (-4,-1,-4)$

$D = (-1,-2,-1)$

$d(A,B) = \sqrt{(0+3)^2+(0-1)^2+(0+3)^2} = \cancel{\sqrt{20}}\ \sqrt{19}$

$d(A,D) = \sqrt{1^2+2^2+1^2} = \sqrt{6}$

$d(B,C) = \sqrt{(-3+4)^2+(1+1)^2+(-3+4)^2} = \sqrt{6}$

$d(D,C) = \sqrt{(-1+4)^2+(-2+1)^2+(-1+4)^2} = \sqrt{19}$

$\Rightarrow$

$\Rightarrow$ patrulaterul cu vârfurile în A, B, C, D este dreptunghi (Adevărat), deoarece $d(A,B) = d(D,C)$ și $d(A,D) = d(B,C)$

5. Fals, deoarece matricea ${}^tA \cdot A \notin \mathcal{M}_{m,n}(\mathbb{R})$

Subiectul II.

1. a) $\begin{cases} x_1 - 4x_3 + 10x_4 - 15x_5 \\ x_1 + x_2 + 2x_3 \\ -x_1 + x_2 + 11x_3 - 22x_4 + 33x_5 \\ x_1 + x_2 + 2x_3 \end{cases}$

$$\bar{A} = \begin{pmatrix} 1 & 0 & -4 & 10 & -15 & -2 \\ 1 & 1 & 2 & 0 & 0 & +5 \\ -1 & 1 & 11 & -22 & 33 & +10 \\ 1 & 1 & 2 & 0 & 0 & \alpha \end{pmatrix}$$

Facem forma eșalon

$$\xrightarrow{C_1' = C_1 - C_2} \begin{pmatrix} 1 & 0 & -4 & 10 & -15 & -2 \\ 0 & 1 & 2 & 0 & 0 & +5 \\ -2 & 1 & 11 & -22 & 33 & +10 \\ 0 & 1 & 2 & 0 & 0 & \alpha \end{pmatrix}$$

$$\xrightarrow{L_3 + 2L_1} \begin{pmatrix} 1 & 0 & -4 & 10 & -15 & -2 \\ 0 & 1 & 2 & 0 & 0 & 5 \\ 0 & 1 & 3 & -2 & 3 & 6 \\ 0 & 1 & 2 & 0 & 0 & \alpha \end{pmatrix} \xrightarrow{C_3' = L_3 - L_2}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -4 & 10 & -15 & -2 \\ 0 & 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 & 0 & \alpha \end{pmatrix} \xrightarrow{L_4 - L_2}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -4 & 10 & -15 & -2 \\ 0 & 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \alpha - 5 \end{pmatrix}$$

sistem comp det $\Leftrightarrow \alpha - 5 = 0 \Rightarrow$

$\Rightarrow \alpha = 5$

5) Noul sistem

$$\begin{cases} x_1 - 4x_3 + 16x_4 - 15x_5 = -2 \\ x_2 + 2x_3 = 5 \\ x_3 - 2x_4 + 3x_5 = 1 \end{cases}$$

Necunoscutele principale: x_1, x_2, x_3
secundare: x_4, x_5

$x_3 = 1 + 2x_4 - 3x_5$

$x_2 = 5 - 2x_3 = 5 - 2 - 4x_4 - 6x_5$

$x_2 = 3 - 4x_4 - 6x_5$

$x_1 = -2 + 4x_3 - 16x_4 + 15x_5$

$x_1 = -2 + 4 + 8x_4 - 12x_5 - 16x_4 + 15x_5$

$x_1 = 2 - 2x_4 + 3x_5$

$$\Rightarrow \begin{cases} x_1 = 2 - 2\alpha + 3\beta \\ x_2 = 3 - 4\alpha - 6\beta \\ x_3 = 1 + 2\alpha - 3\beta \\ x_4 = \alpha \\ x_5 = \beta \end{cases}$$

$\alpha, \beta \in \mathbb{R}$

5I.

2) Fals, deoarece $rg\begin{pmatrix} -3 & 2 & -1 \\ -5 & 2 & -3 \\ -1 & 2 & 2 \\ -5 & 6 & 1 \end{pmatrix} = 3$

~~tot~~ Nu poate fi scris ca o combinație liniară de cei 2 vectori.

2. $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (-4y+4z, x+4y-3z, x+2y-z)$

a) $f(x,y,z) = \begin{pmatrix} 0 & -4 & 4 \\ 1 & 4 & -3 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

f e liniară $\Rightarrow A = [f]_{B_0} = \begin{pmatrix} 0 & -4 & 4 \\ 1 & 4 & -3 \\ 1 & 2 & -1 \end{pmatrix}$

b) $\operatorname{Ker} f = \{ v \in \mathbb{R}^3 \mid f(v) = 0 \}$

$\operatorname{Im} f = \{ f(v) \mid v \in \mathbb{R}^3 \}$

$\operatorname{Ker} f = \{ (x,y,z) \in \mathbb{R}^3 \mid (-4y+4z, x+4y-3z, x+2y-z) =$
$= (0,0,0) \} =$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} -4y+4z = 0 \\ x + 4y - 3z = 0 \\ x + 2y - z = 0 \end{cases} \right.$

$\begin{cases} -4y+4z = 0 \\ x + 4y - 3z = 0 \\ x + 2y - z = 0 \end{cases}$

$$\begin{pmatrix} 0 & -4 & 4 & 0 \\ 1 & 4 & -3 & 0 \\ 1 & 2 & -1 & 0 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & 4 & -3 & 0 \\ 0 & -4 & 4 & 0 \\ 1 & 2 & -1 & 0 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_1}{\sim}$$

$$\sim \begin{pmatrix} 1 & 4 & -3 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & -2 & 2 & 0 \end{pmatrix} \overset{L_2 \leftarrow L_2 \cdot (-1)}{\sim} \begin{pmatrix} 1 & 4 & -3 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & -2 & 2 & 0 \end{pmatrix} \sim$$

$$\overset{L_2 \leftarrow \frac{1}{4} L_2}{\sim} \begin{pmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{pmatrix} \overset{L_1 \leftarrow L_1 - 4L_2}{\sim}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{pmatrix} \overset{L_3 \leftarrow L_3 + 2 L_2}{\sim} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$z = \alpha \Rightarrow x = -\alpha \ ; \ y = \alpha$

$\ker f = \{(-\alpha, \alpha, \alpha) \mid \alpha \in \mathbb{R}\} = \langle(-1, 1, 1)\rangle$ este bază $\Rightarrow$

$\Rightarrow \dim \ker f = 1$

Th. rang-defect

$\dim \overset{1}{\ker f} + \dim \operatorname{Im} f = \overset{3}{\dim \mathbb{R}^3} \Rightarrow \dim \operatorname{Im} f = 3 - 1 = 2 \Rightarrow$

$\Rightarrow \dim \operatorname{Im} f = 2$

f surjectivă $\Leftrightarrow \dim \operatorname{Im} f = \dim \mathbb{R}^3$

$\left.\begin{array}{l} \dim \operatorname{Im} f = 2 \\ \dim \mathbb{R}^3 = 3 \end{array}\right| \Rightarrow 2 \neq 3 \Rightarrow$

$\Rightarrow f$ nu e surjectivă

c) Notez matricea $A = \begin{pmatrix} 0 & -4 & 4 \\ 1 & 4 & -3 \\ 1 & 2 & -1 \end{pmatrix}$

Calculez ~~poligon~~ polinomul caracteristic

$$\det(A - \lambda I_3) = \begin{vmatrix} -\lambda & -4 & 4 \\ 1 & 4-\lambda & -3 \\ 1 & 2 & -1-\lambda \end{vmatrix} = (-\lambda)(-\lambda+4)\cdot(\lambda-1) +$$

$$+(-4)\cdot(-3)\cdot 1 + 8 - 1\cdot(-\cancel{\lambda}+4)\cdot 4 - 2\cdot(-3)\cdot(-\cancel{\lambda}) - (-\cancel{\lambda}-1)\cdot 1\cdot(-4) =$$

$$= (\lambda^2 - 4\lambda)(\lambda - 1) + 20 + 4\lambda - 16 + 6\lambda - 4\lambda - 4 =$$

$$= -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda^2 - 3\lambda + 2)$$

$$-\lambda(\lambda^2 - 3\lambda + 2) = 0 \Rightarrow$$

$$\Rightarrow \lambda_1 = 0$$

$$\lambda^2_3 - 3\lambda + 2 = 0$$

$$\left.\begin{matrix} a+b=-3 \\ ab = 2 \end{matrix}\right| \Rightarrow \left.\begin{matrix} a=-2 \\ b=-1 \end{matrix}\right| \Rightarrow \lambda^2_2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) \Rightarrow$$

$$\Rightarrow \lambda_2 = 2$$

$$\Rightarrow \lambda_3 = 1$$

$\lambda_1 = 0$, $m_a(\lambda_1) = 1$

$\lambda_2 = 2$, $m_a(\lambda_2) = 1$

$\lambda_3 = 1$, $m_a(\lambda_3) = 1$

Toate valorile proprii sunt reale

Obs.: $1 \le m_g(\lambda_1) \le m_a(\lambda_1) = 1 \Rightarrow m_g(\lambda_1) = 1$

$1 \le m_g(\lambda_2) \le m_a(\lambda_2) = 1 \Rightarrow m_g(\lambda_2) = 1$

$1 \le m_g(\lambda_3) \le m_a(\lambda_3) = 1 \Rightarrow m_g(\lambda_3) = 1$ $\Bigg| \Rightarrow$

$\Rightarrow$ Deci f este diagonalizabilă

$$\lambda_1 = 0 \Rightarrow V_{\lambda_1} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 0 & -4 & 4 \\ 1 & 4 & -3 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\}$$

$$V_{\lambda_1}: \begin{cases} -4y + 4z = 0 \\ x + 4y - 3z = 0 \\ x + 2y - z = 0 \end{cases} \quad m_g(\lambda_1) = 1 \Rightarrow \text{rg} \begin{pmatrix} 0 & -4 & 4 \\ 1 & 4 & -3 \\ 1 & 2 & -1 \end{pmatrix} = 2$$

Nec. secundare $z = \alpha \Rightarrow$

$$\Rightarrow \begin{cases} 4\alpha = 4y \\ x + 4y = 3\alpha \end{cases} \Leftrightarrow \begin{cases} y = \alpha \\ x + 4\alpha = 3\alpha \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = \alpha \\ x = -\alpha \end{cases} \Rightarrow$$

$\Rightarrow V_{\lambda_1} = \{(-\alpha, \alpha, \alpha) \mid \alpha \in \mathbb{R}\} = \langle -1, 1, 6) \Rightarrow$

$\Rightarrow B_1 = \{(-1, 1, 6)\}$ bază în V_{λ_1}

$\lambda_2 = 2$

$$V_{\lambda_2} = \left\{ \begin{pmatrix} x\\ y\\ z\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix}0 & -4 & 4\\ 1 & 4 & -3\\ 1 & 2 & -1\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix} = 2\begin{pmatrix} x\\ y\\ z\end{pmatrix}\right\} \Rightarrow$$

$$\Rightarrow V_{\lambda_2} : \begin{cases} -4y + 4z = 2x \\ x + 4y - 3z = 2y \\ x + 2y - z = 2z \end{cases} \Rightarrow$$

$$\begin{cases} -2x - 4y + 4z = 0 \\ x + 2y - 3z = 0 \\ x + 2y - 3z = 0 \end{cases} \Bigg| \Rightarrow rg(\lambda_1) = 1 \Rightarrow$$

$$\Rightarrow rg\begin{pmatrix} -2 & -4 & 4 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \end{pmatrix} = 2 \Rightarrow$$

Nec ne $z = \lambda \Rightarrow \begin{cases} -2x - 4y = -4\lambda \\ x + 2y = 3\lambda \end{cases} \Bigg| \Rightarrow$

$$\Rightarrow \begin{cases} x + 2y = 2\lambda \\ x + 2y = 3\lambda \quad (-) \end{cases}$$

$$0 = -\lambda \Rightarrow \lambda = 0 \Rightarrow x = 0,\ y = 0$$

$$V_{\lambda_2} = \{(0,0,0)\} \mid Lin4 = \langle (0,0,0) \rangle \Rightarrow$$

$$\Rightarrow B_2 = \{0, 0, 0\}$$

$\lambda_3 = 1 \Rightarrow V_{\lambda_3} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 0 & -4 & 4 \\ 1 & 4 & -3 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\}$

$V_{\lambda_3} : \begin{cases} -4y + 4z = x \\ x + 4y - 3z = y \\ x + 2y - z = z \end{cases} \Rightarrow$

$\Rightarrow V_{\lambda_3} : \begin{cases} -x - 4y + 4z = 0 \\ x + 3y - 3z = 0 \\ x + 2y - 2z = 0 \end{cases}$ $mg(\lambda_1) = 1 \Rightarrow$

$\Rightarrow rg \begin{pmatrix} -1 & -4 & 4 \\ 1 & 3 & -3 \\ 1 & 2 & -2 \end{pmatrix} = 2$

Nec. sec. $z = \alpha \Rightarrow \begin{cases} -x - 4y = -4\alpha \\ x + 3y = 3\alpha \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x + 4y = 4\alpha \\ x + 3y = 3\alpha \quad (-) \end{cases} \Rightarrow \begin{cases} y = \alpha \\ x + 3y = 3\alpha \end{cases} \Rightarrow$

$\Rightarrow x = 0$

$V_{\lambda_3} = \{ (0, 1, 1) \alpha \mid \alpha \in \mathbb{R} \} = \langle (0, 1, 1) \rangle \Rightarrow$

$\Rightarrow B_3 = \{ (0, 1, 1) \}$

O bază în care f e diag

$B = \{(-1,1,0), (0,0,0), (0,1,1)\}$

$[f]_B = \begin{pmatrix} 0 & & \\ & 2 & \\ & & 1 \end{pmatrix}$ și

$$\begin{pmatrix} 0 & -4 & 4 \\ 1 & 4 & -3 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

3.

$Q(x_1,x_2,x_3) = 4x_1^2 + 2x_2^2 + 2x_3^2 - 4x_1x_2 - 2x_2x_3$

a) $Q(x_1,x_2,x_3) = \left(4x_1^2 - 4x_1x_2\right) + 2x_2^2 - 2x_2x_3 + 2x_3^2$

$Q(x_1,x_2,x_3) = (2x_1 - x_2)^2 - x_2^2 + 2x_2^2 - 2x_2x_3 + 2x_3^2$

$Q(x_1,x_2,x_3) = (2x_1 - x_2)^2 + x_2^2 - 2x_2x_3 + 2x_3^2$

S.V.: $\begin{cases} y_1 = 2x_1 - x_2 \\ y_2 = x_2 \\ y_3 = x_3 \end{cases}$

$Q(y_1,y_2,y_3) = y_1^2 + y_2^2 - 2y_2y_3 + 2y_3^2$

$Q(y_1,y_2,y_3) = y_1^2 + (y_2^2 - 2y_2y_3) + 2y_3^2$

S.V.:

$Q(y_1,y_2,y_3) = y_1^2 + (y_2 - y_3)^2 - y_3^2 + 2y_3^2$

$Q(y_1,y_2,y_3) = y_1^2 + (y_2 - y_3)^2 + y_3^2$

S.V.: $\begin{cases} z_1 = y_1 \\ z_2 = y_2 - y_3 \\ z_3 = y_3 \end{cases}$ $\Rightarrow$

$\Rightarrow Q(z_1,z_2,z_3) = z_1^2 + z_2^2 + z_3^2$

$\begin{cases} z_1 = y_1 = 2x_1 - x_2 \\ z_2 = y_2 - y_3 = x_2 - x_3 \\ z_3 = y_3 = x_3 \end{cases}$

ne ~~[illegible]~~ [illegible]

b) $\underline{\text{signa}(Q) =}$

$$Q(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2$$

signa $(Q) = 1 + 1 + 1 = 3 \Rightarrow$ poz. definita, dar și nedegenerată

c)

Continuare

$$\Rightarrow \underbrace{\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}}_{\text{coord în } B} = \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{M_{B_0, B}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\text{coord în } B_0}$$

și B apare pe coloanele lui $M_{B, B_0} = (M_{B_0, B})^{-1}$

$$(M_{B_0, B})^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \left\{ \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{pmatrix} \right\}$$

4. $A \in M_3(\mathbb{C})$ $Tr(A) = Tr(A^2) = Tr(A^3) = 0$

$A^3 = O_3$

$$A = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}\begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix} = \begin{pmatrix} ac+be & bf & ad \\ de & ac+df & bc \\ cf & ae & ab+fd \end{pmatrix}$$

$ac+be = ac+df = ab+fd = 0$

$$A^3 = \begin{pmatrix} 0 & bf & ad \\ de & 0 & bc \\ cf & ae & 0 \end{pmatrix}\begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix} = \begin{pmatrix} bfc+ade & adf & bfd \\ bce & ade+bcf & bde \\ ace & acf & bcf+ade \end{pmatrix}$$

$$bcf + ade = 0 \Rightarrow A^3 = \begin{pmatrix} 0 & adf & bfd \\ bce & 0 & bde \\ ace & acf & 0 \end{pmatrix}$$

$\Downarrow$

$\left.\begin{array}{l} bcf = 0 \\ ade = 0 \end{array}\right| \Rightarrow \boxed{A^3 = O_3}$