Scientific Visualization

Chapter 10:
Tensor Field Visualization

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Summer Term 2017

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Contents

- Overview
- Diffusion tensor
- Glyph-based mapping techniques
- Hue-balls and lit-tensors
- Hyperstreamlines and tensorlines
- Tensor field topology

Literature

Reading

- ***The Visualization Handbook***:
  - Chapter 15 (Oriented Tensor Reconstruction)
  - Chapter 16 (Diffusion Tensor MRI Visualization)

Tensor Field Visualization

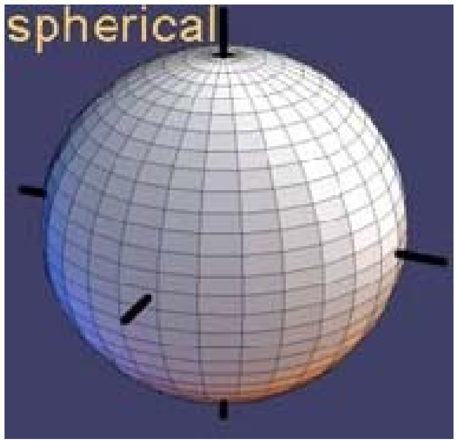
- Tensor: extension of concept of scalar and vector
- Tensor data
  for a tensor of level k is given by $t_{i1,i2,\dots,ik}(x_1,\dots,x_n)$
  - Zero-order tensor: scalar
  - First-order tensor: vector
  - Second-order tensor: matrix
  - Third-order tensor: “3D matrix”, …
- Examples for tensors:
  - Diffusion tensor (from medical imaging, see later)
  - Material properties (material sciences):
    - Conductivity tensor
    - Dielectric susceptibility
    - Magnetic permittivity
    - Stress tensor

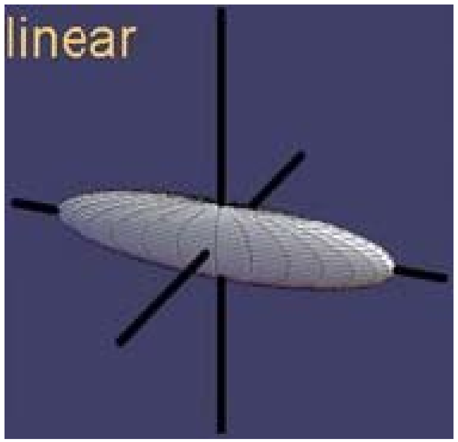
Diffusion Tensor

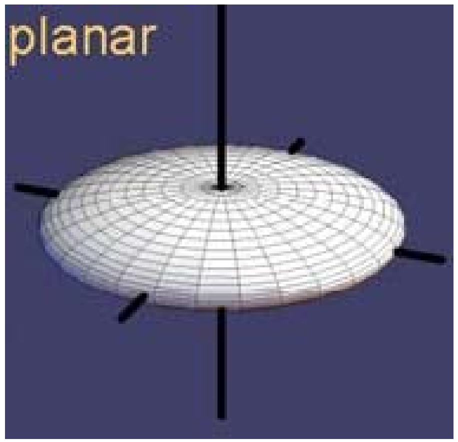
- Typical second-order tensor: diffusion tensor
  - Diffusion: based on motion of fluid particles on microscopic level
  - Probabilistic phenomenon
  - Based on particle's Brownian motion
  - Measurements by modern MR (magnetic resonance) scanners
  - Diffusion tensor describes diffusion rate into different directions via symmetric tensor (probability density distribution)
  - In 3D: representation via 3×3 symmetric matrix

Diffusion Tensor

- *Symmetric* matrices diagonalized:
  - Real eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$
  - Eigenvectors are perpendicular
- Isotropy/anisotropy:
  - Spherical: $\lambda_1 = \lambda_2 = \lambda_3$
  - Linear: $\lambda_2 \approx \lambda_3 \approx 0$
  - Planar: $\lambda_1 \approx \lambda_2$ and $\lambda_3 \approx 0$

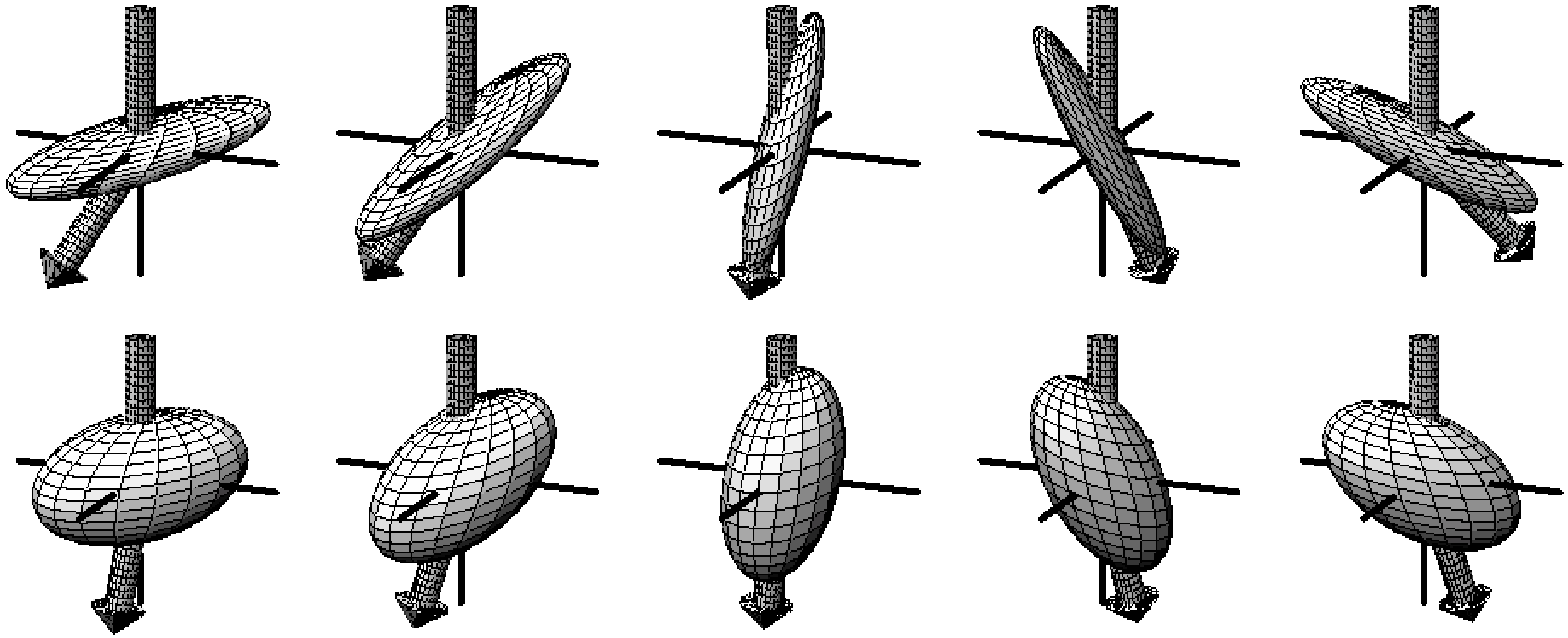






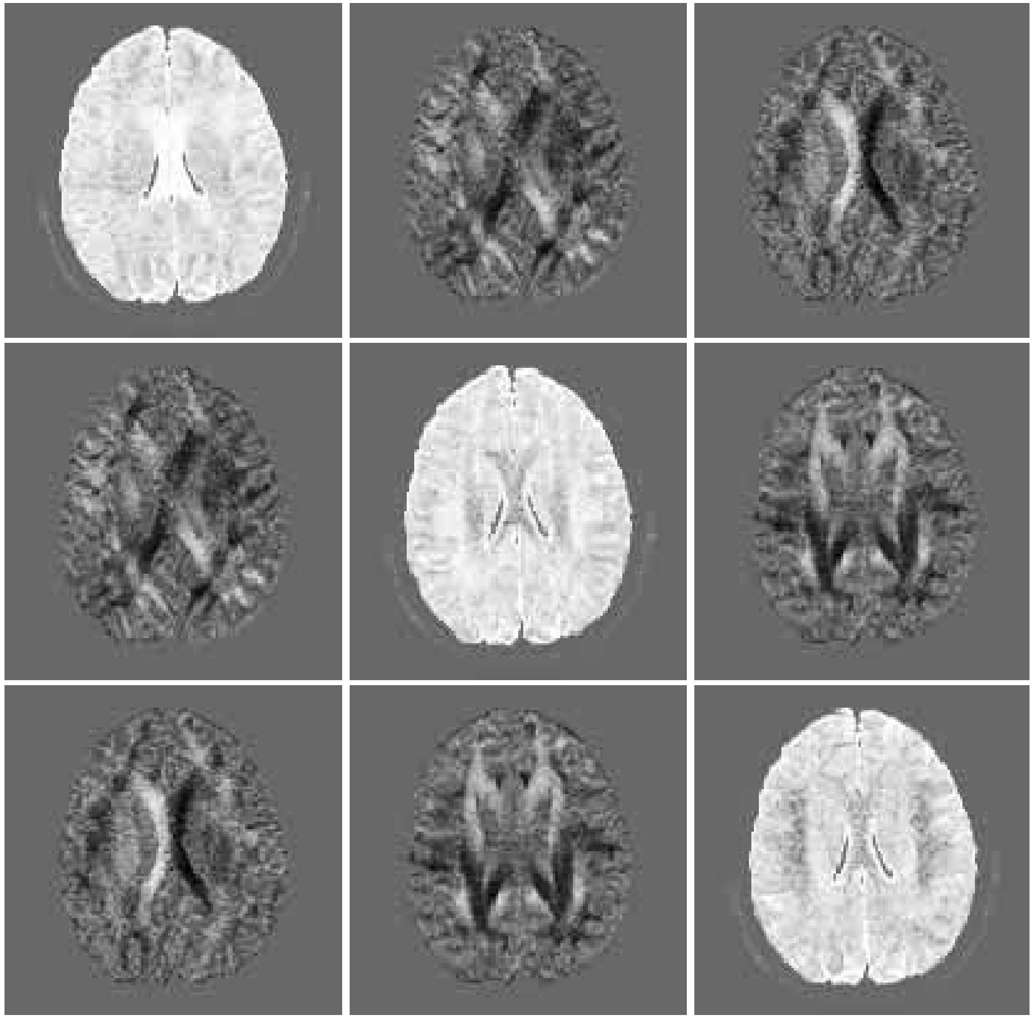
Diffusion Tensor

- Arbitrary input vectors are generally deflected after matrix multiplication (here, arrow depicts output vector resulting from vertical input vector)
- Deflection into direction of major eigenvector (largest eigenvalue)



Glyph-Based Mapping Techniques

- Matrix of images
  - Slices through volume
  - Each image shows one component of the matrix



Glyph-Based Mapping Techniques

Symmetric positive semidefinite ($\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$) tensors

$\rightarrow$ Represent by ellipsoids, with half axes being eigenvalues/eigenvectors

Three types of anisotropy:

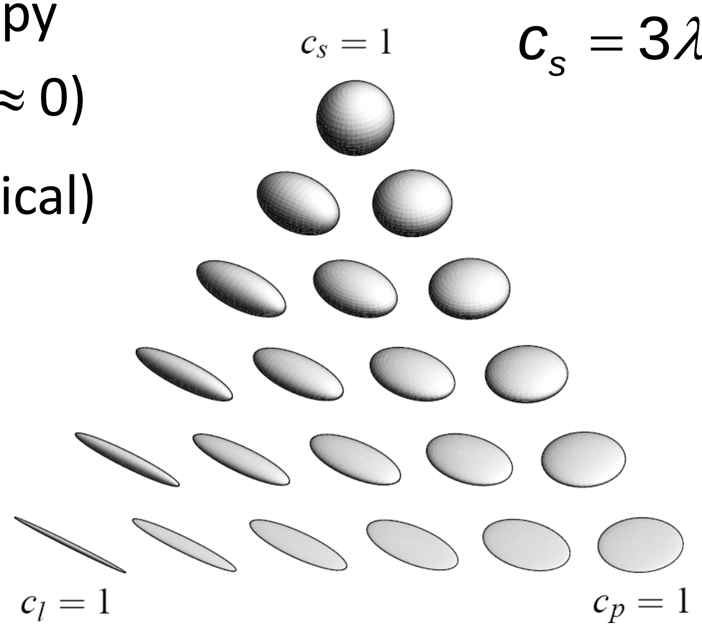
- linear anisotropy
  ($\lambda_2 \approx \lambda_3 \approx 0$)
- planar anisotropy
  ($\lambda_1 \approx \lambda_2$ and $\lambda_3 \approx 0$)
- isotropy (spherical)
  ($\lambda_1 = \lambda_2 = \lambda_3$)

Anisotropy measure:

$$c_l = (\lambda_1 - \lambda_2)/(\lambda_1 + \lambda_2 + \lambda_3)$$

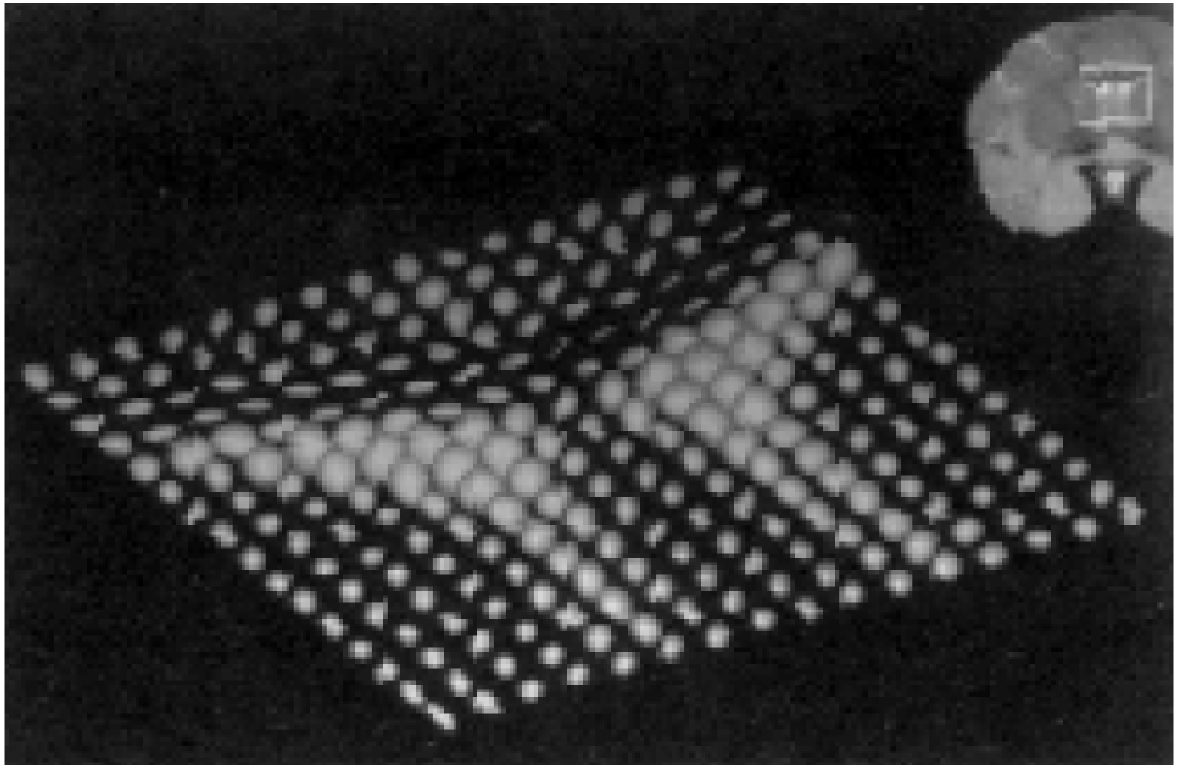
$$c_p = 2(\lambda_2 - \lambda_3)/(\lambda_1 + \lambda_2 + \lambda_3)$$

$$c_s = 3\lambda_3/(\lambda_1 + \lambda_2 + \lambda_3)$$



Glyph-Based Mapping Techniques

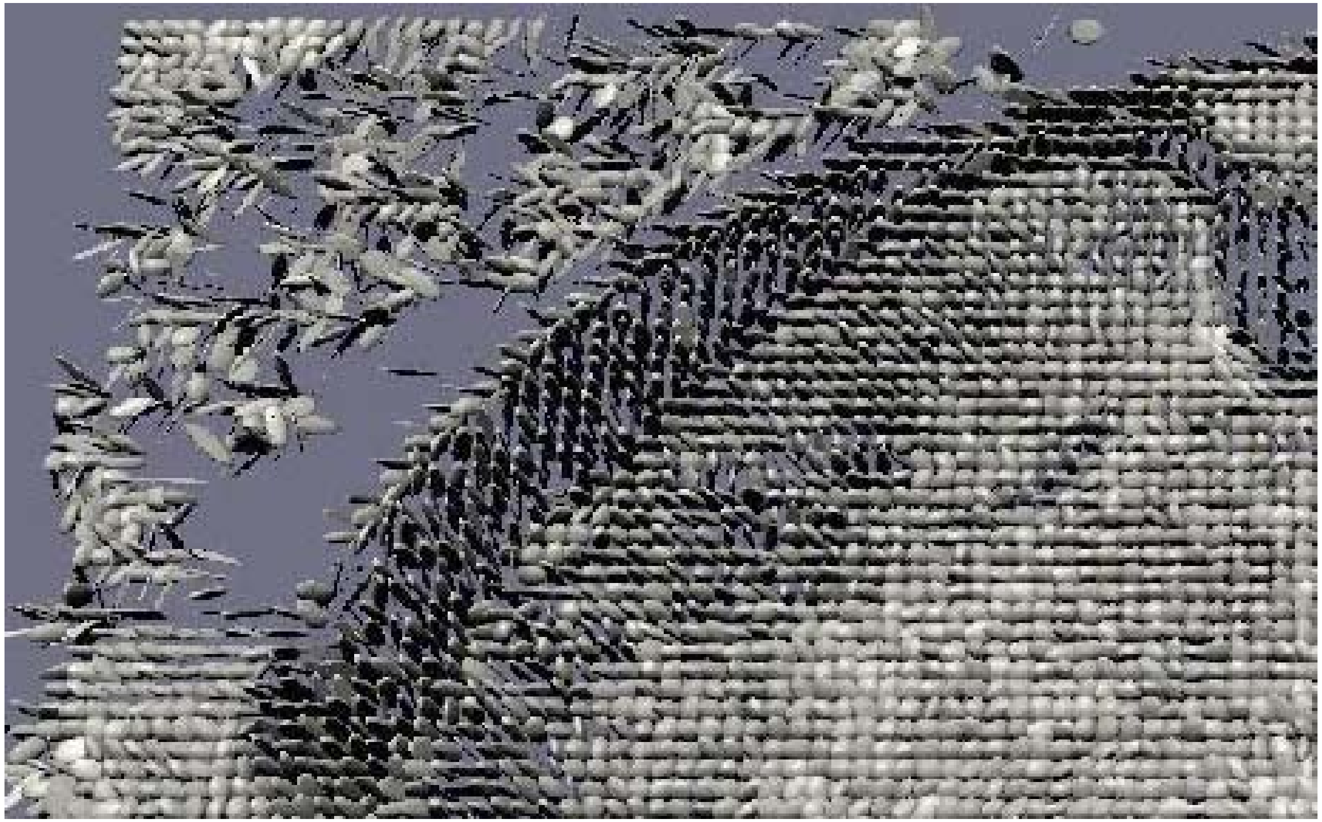
- Uniform grid of ellipsoids
  - Second-order symmetric tensor mapped to ellipsoid
  - Sliced volume



[Pierpaoli et al. 1996]

Glyph-Based Mapping Techniques

- Uniform grid of ellipsoids
  - Normalized sizes of the ellipsoids

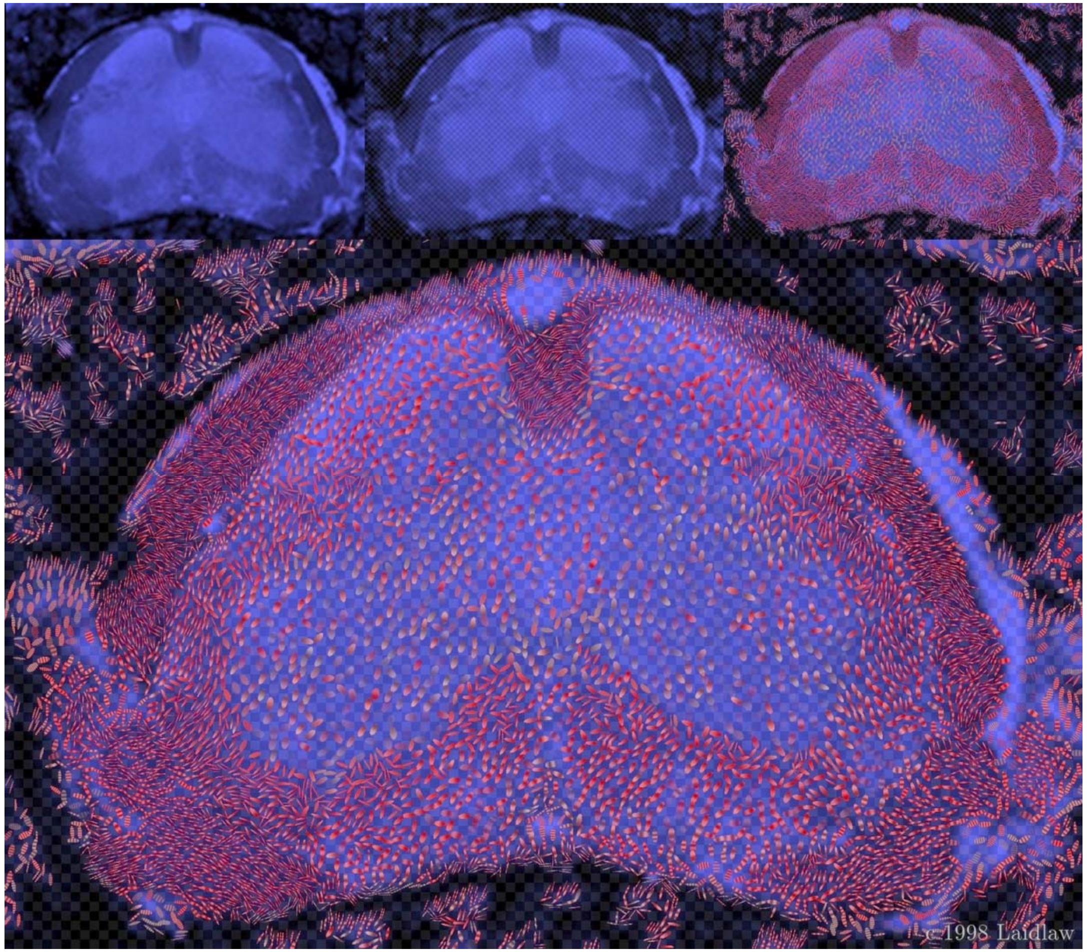


[Laidlaw et al. 1998]

Glyph-Based Mapping Techniques

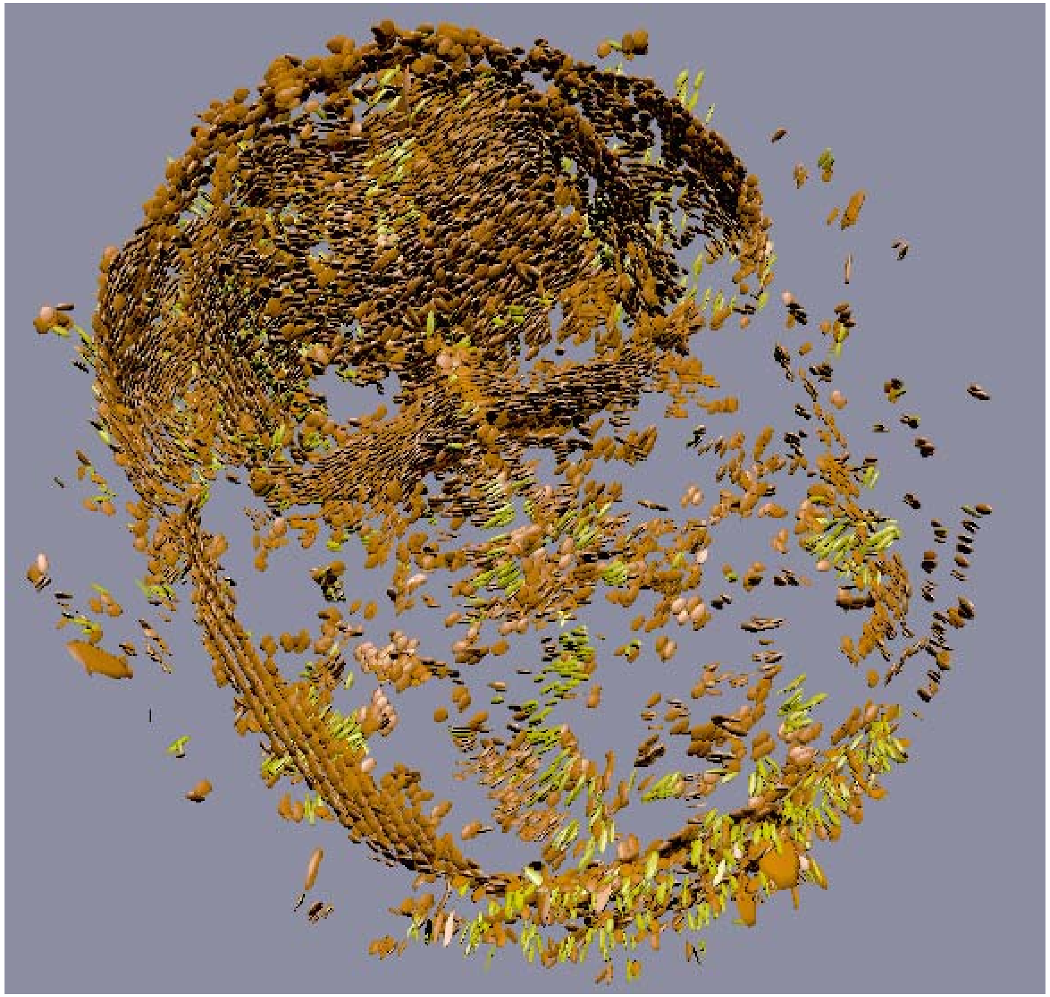
- Brushstrokes
  [Laidlaw et al. 1998]
  - Influenced by paintings
  - Multivalued data
  - Scalar intensity
  - Sampling rate
  - Diffusion tensor
  - Textured strokes

scalar sampling rate tensor



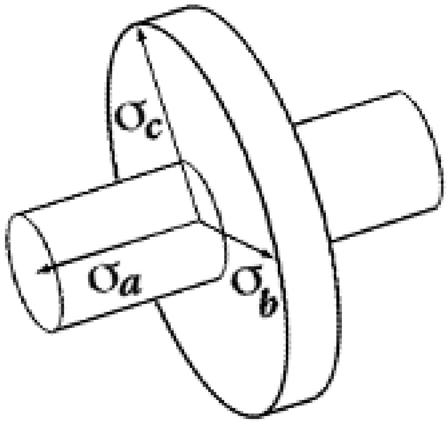
Glyph-Based Mapping Techniques

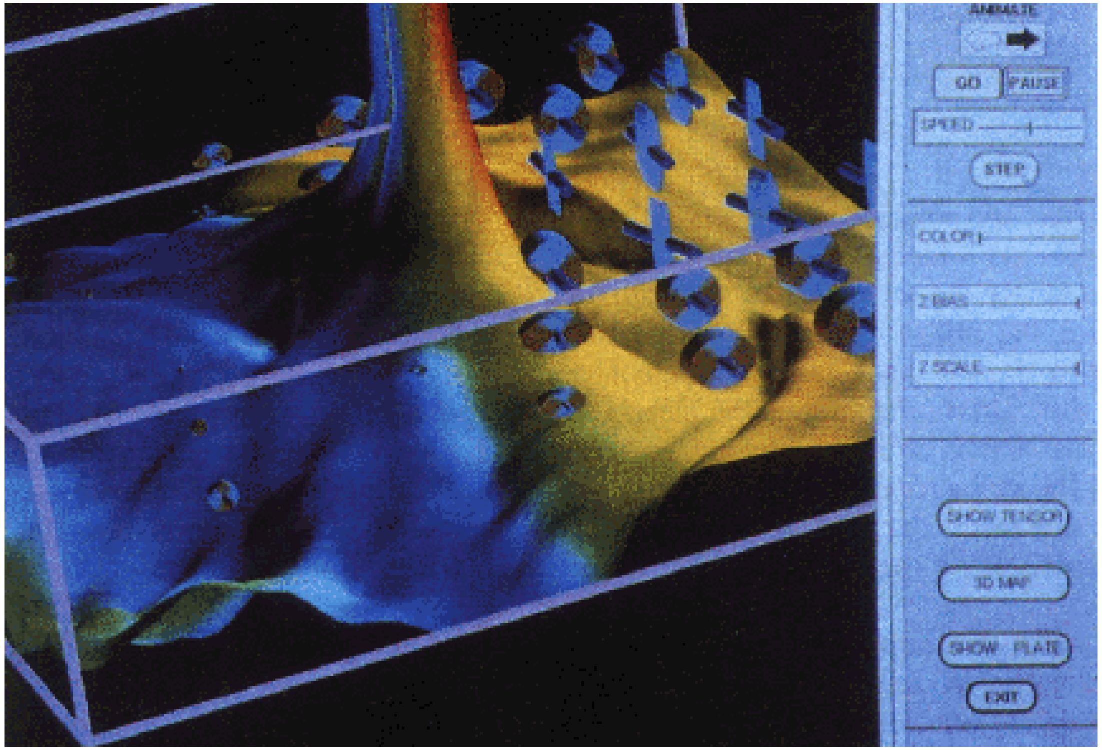
- Ellipsoids in 3D
- Problems:
  - Occlusion
  - Missing continuity



Glyph-Based Mapping Techniques

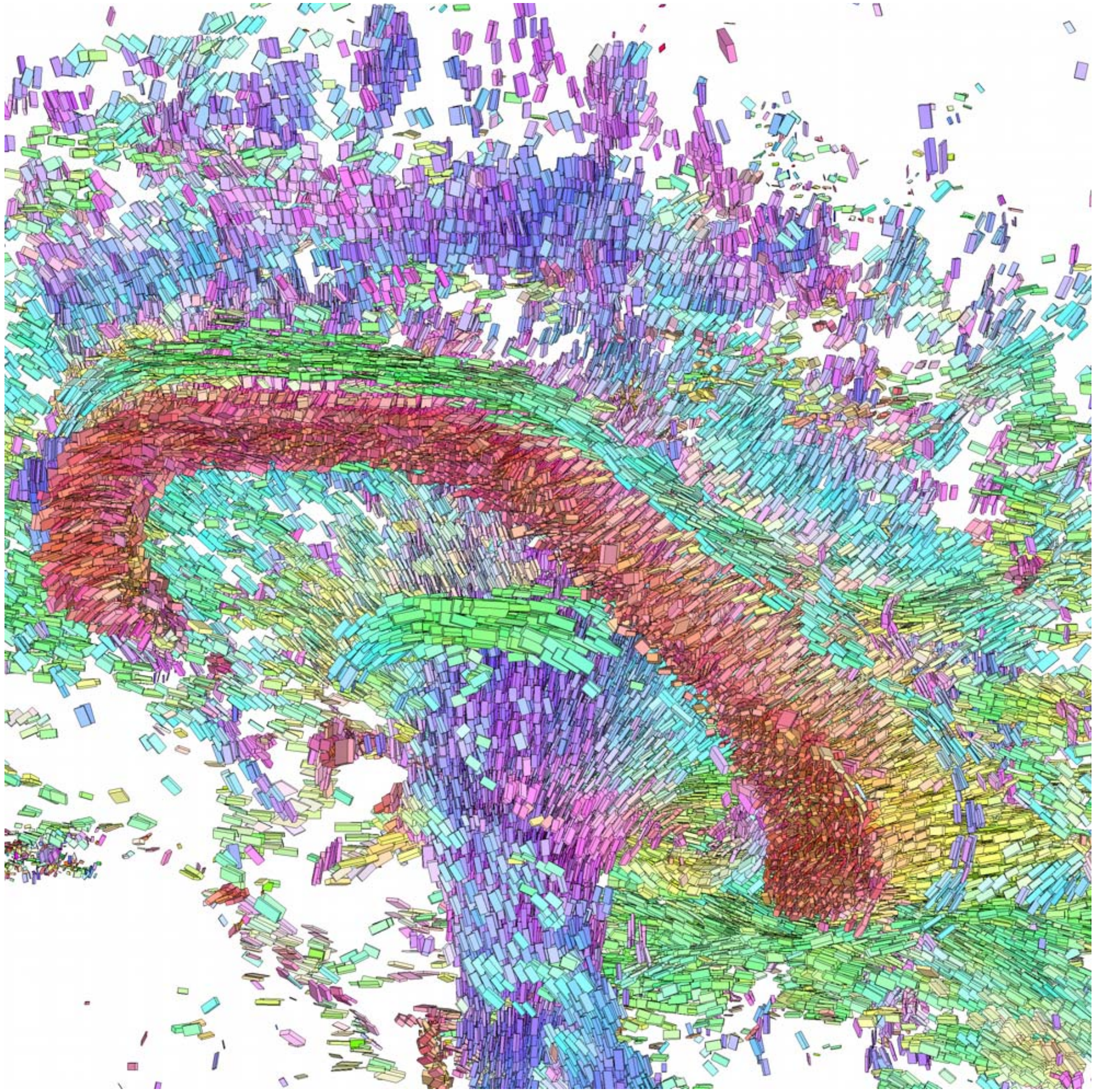
- Haber glyphs [Haber 1990]
  - Rod and elliptical disk
  - Better suited to visualize magnitudes of the tensor and principal axis





Glyph-Based Mapping Techniques

- Box glyphs
  [Johnson et al. 2001]



Glyph-Based Mapping Techniques

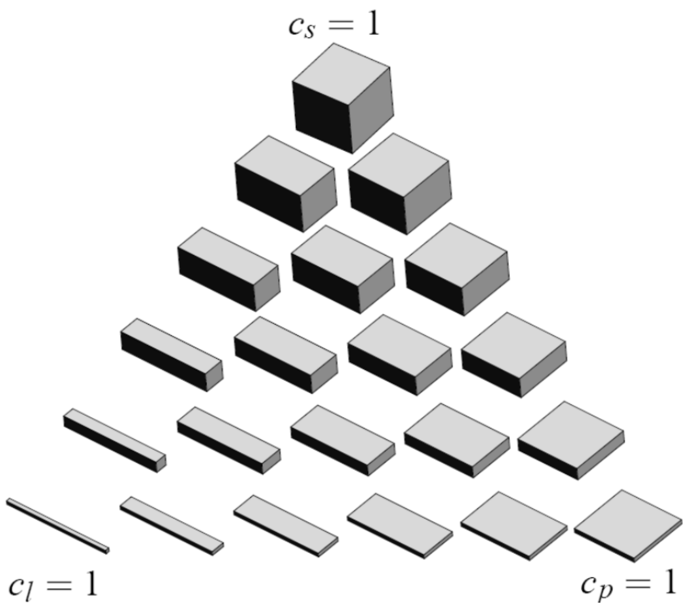
only box glyphs and superquadrics

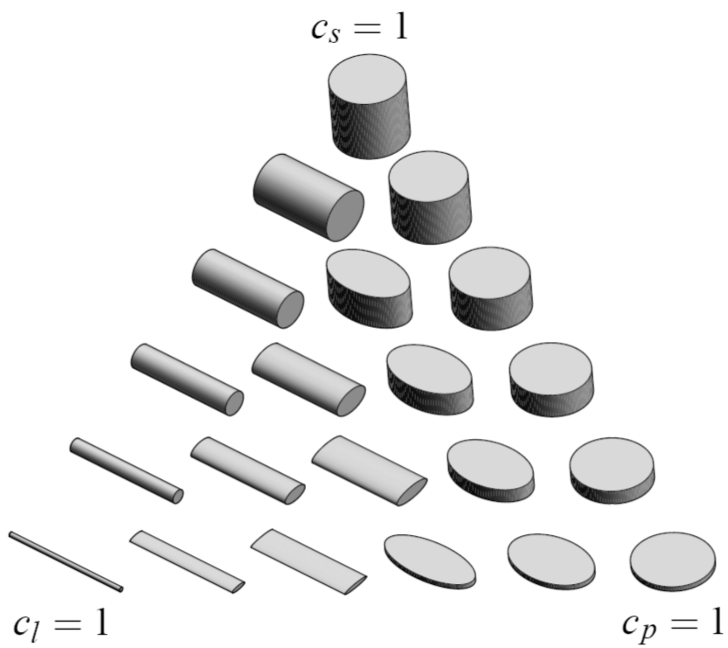
Problem of cuboid glyphs:

- small differences in eigenvalues are over-emphasized

Problems of cylinder glyphs:

- discontinuity at $c_l = c_p$
- artificial orientation at $c_s = 1$





Glyph-Based Mapping Techniques

Combining advantages: superquadrics

Superquadrics with *z* as primary axis:

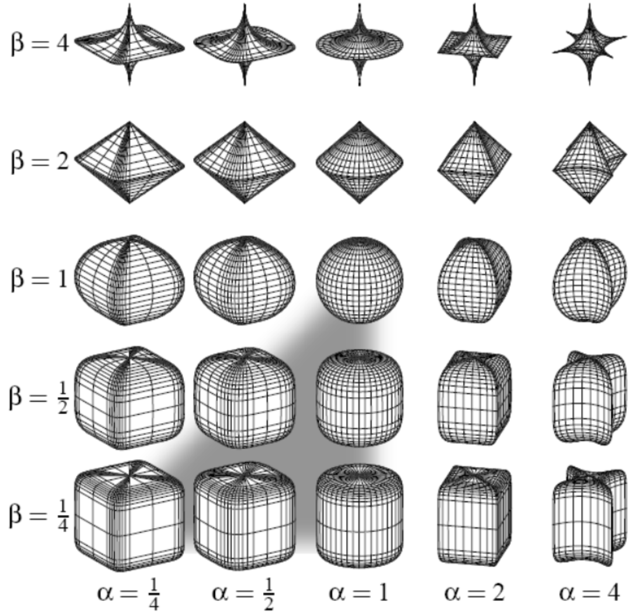
$$\mathbf{q}_z(\theta,\phi) = \begin{pmatrix} \cos^\alpha\theta\sin^\beta\phi \\ \sin^\alpha\theta\sin^\beta\phi \\ \cos^\beta\phi \end{pmatrix}$$

$$0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$$

with $\cos^\alpha\theta$ used as shorthand for

$$|\cos\theta|^\alpha \operatorname{sgn}(\cos\theta)$$

motivation for
superquadrics



Superquadrics for some pairs (α,β)
Shaded: subrange used for glyphs

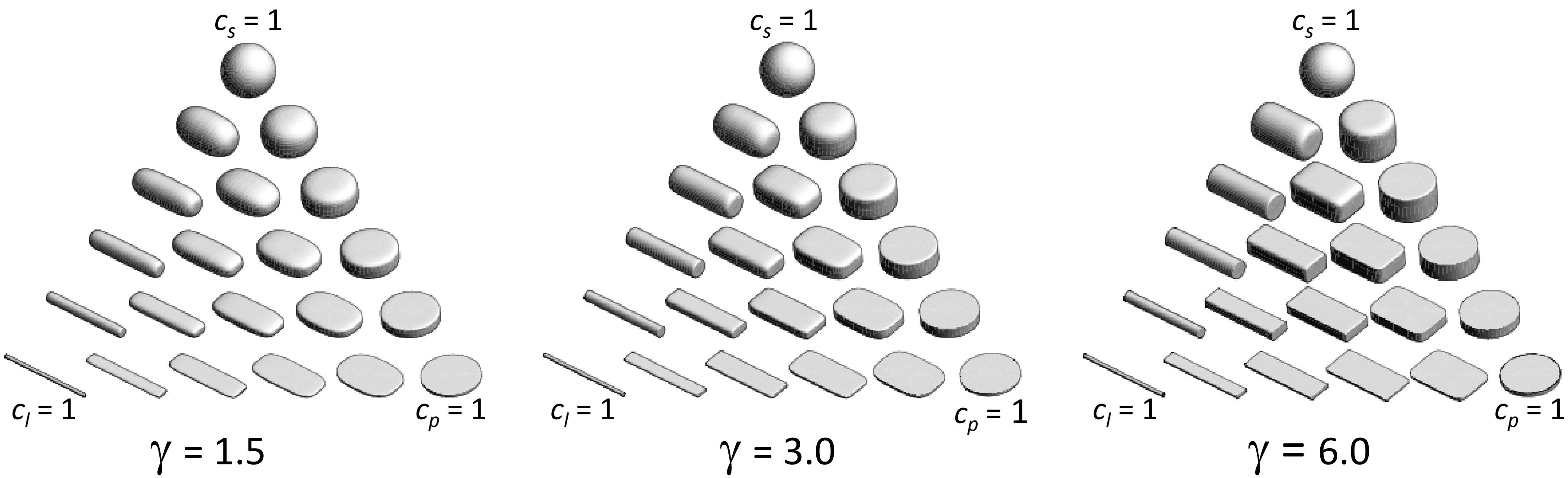
Glyph-Based Mapping Techniques

Superquadric glyphs [Kindlmann 2004]: Given c_l , c_p , c_s

- Compute a base superquadric using a sharpness value γ:

$$q(\theta,\phi) = \begin{cases} \text{if } c_l \geq c_p : \; q_z(\theta,\phi) \text{ with } \alpha = (1-c_p)^\gamma \text{ and } \beta = (1-c_l)^\gamma \\ \text{if } c_l < c_p : \; q_x(\theta,\phi) \text{ with } \alpha = (1-c_l)^\gamma \text{ and } \beta = (1-c_p)^\gamma \end{cases}$$

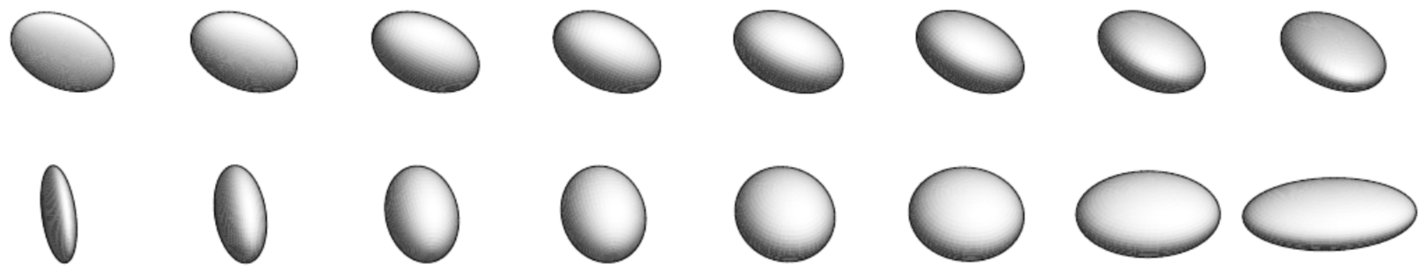
- Rotate into eigenvector frame and scale with λ_1, λ_2, λ_3



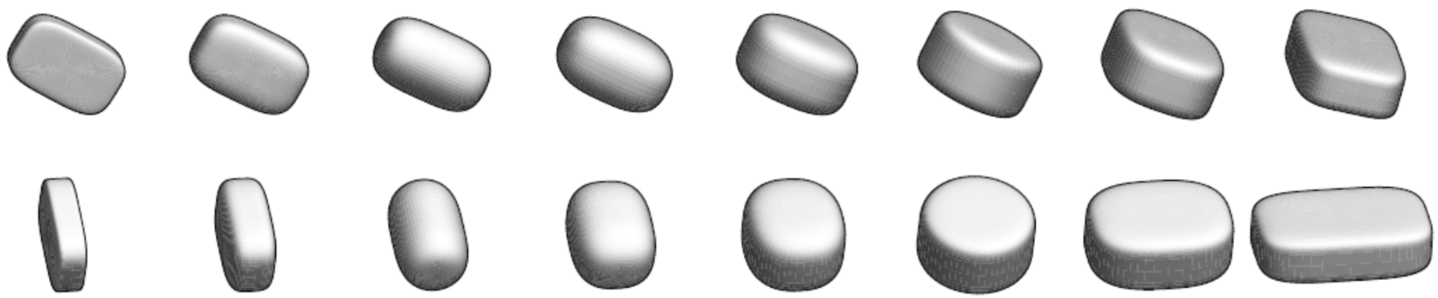
Glyph-Based Mapping Techniques

Comparison of shape perception

- Ellipsoid glyphs

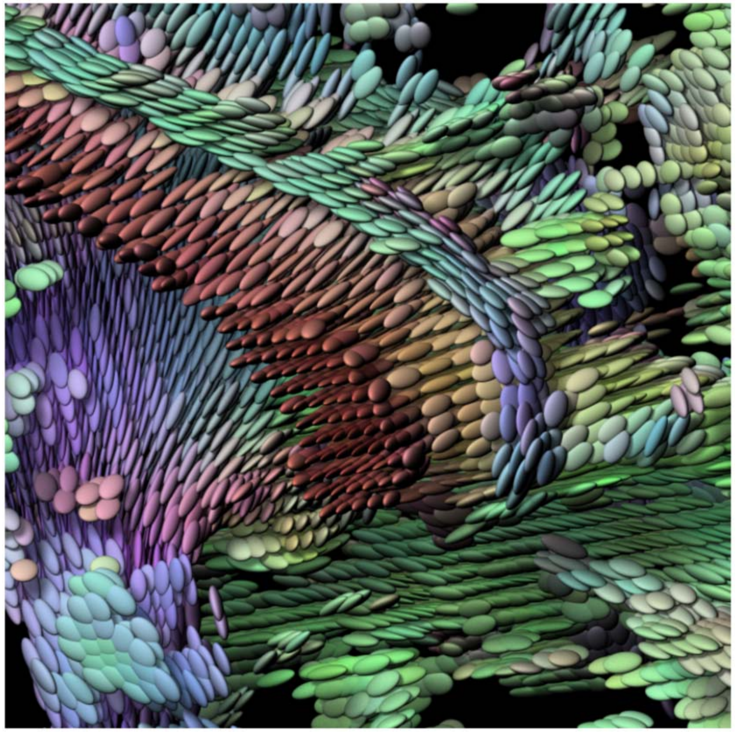


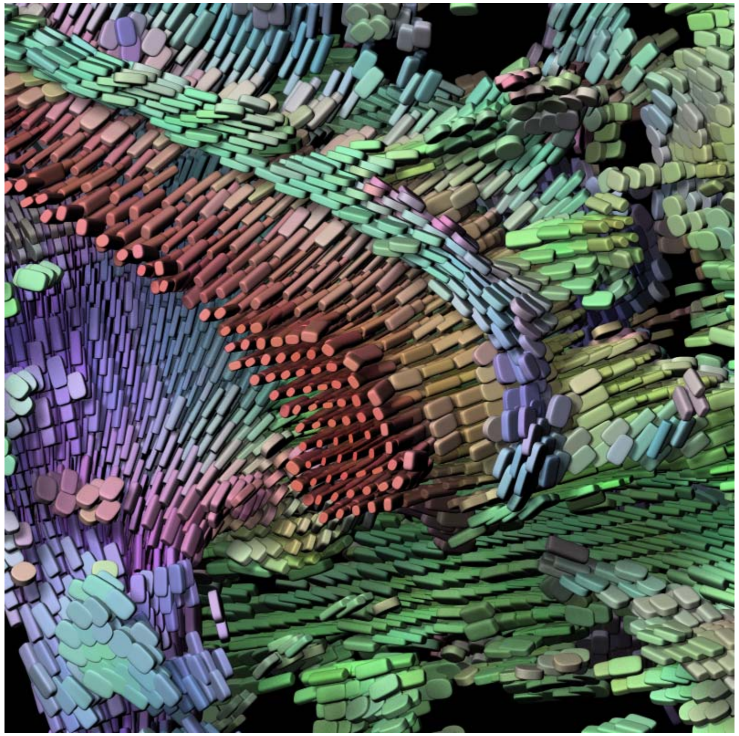
- Superquadric glyphs



Glyph-Based Mapping Techniques

Comparison: Ellipsoids vs. superquadrics

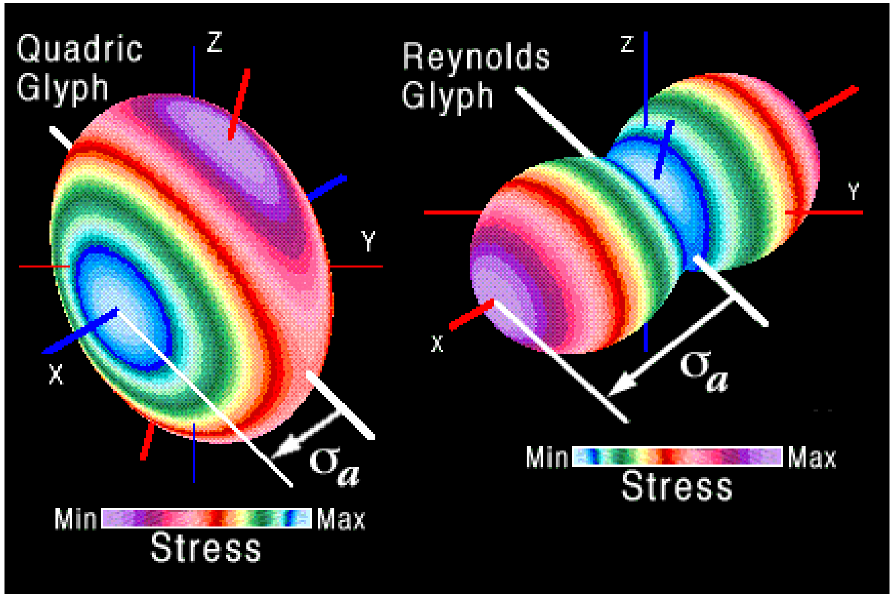


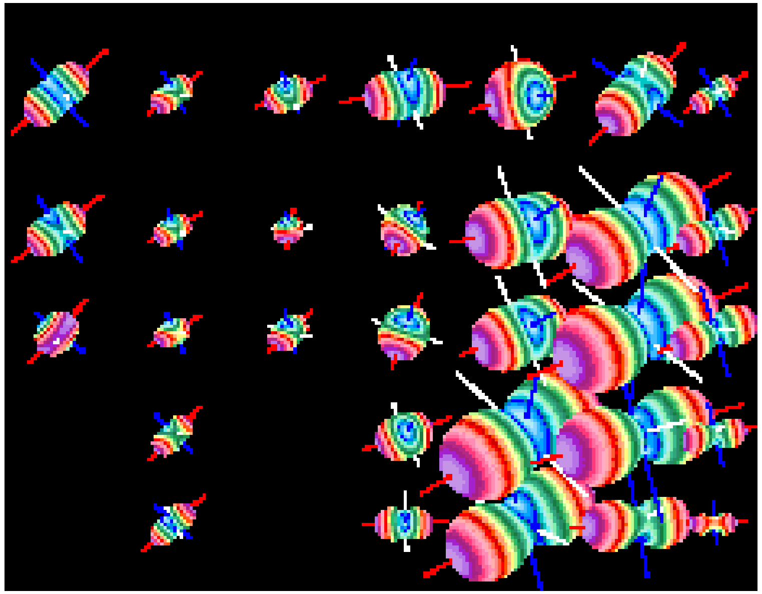


color map: $$\begin{pmatrix} R \\ G \\ B \end{pmatrix} = c_l \begin{pmatrix} |e_x^1| \\ |e_y^1| \\ |e_z^1| \end{pmatrix} + (1 - c_l) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$ (with e^1 = major eigenvector)

Glyph-Based Mapping Techniques

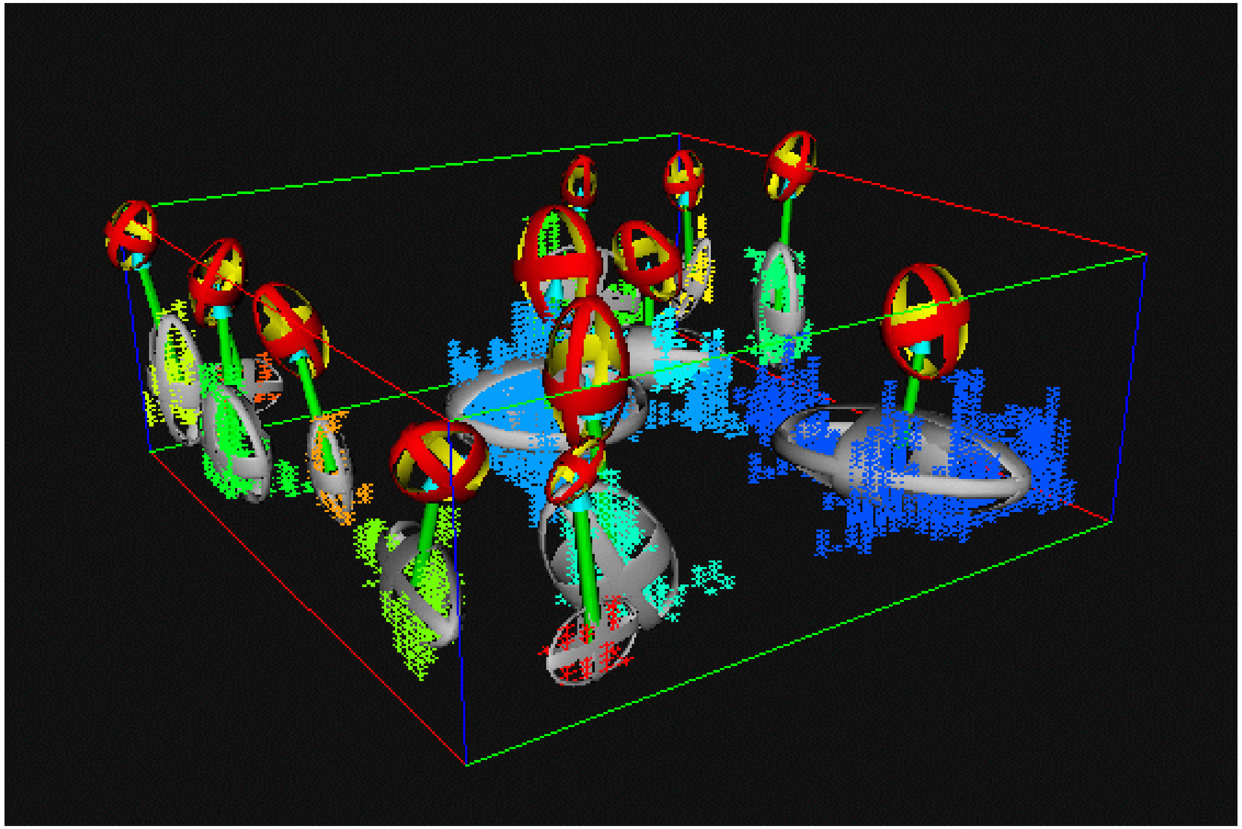
- Reynolds glyph [Moore et al. 1994]





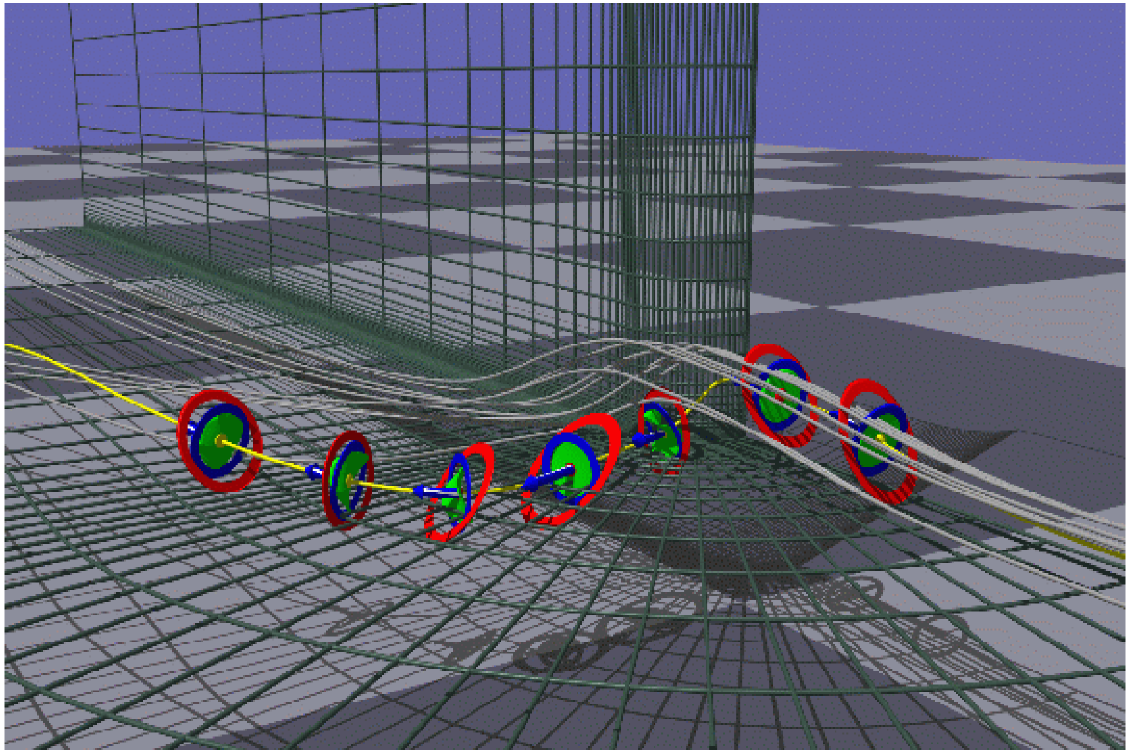
Glyph-Based Mapping Techniques

- Generic iconic techniques for feature visualization [Post et al. 1995]



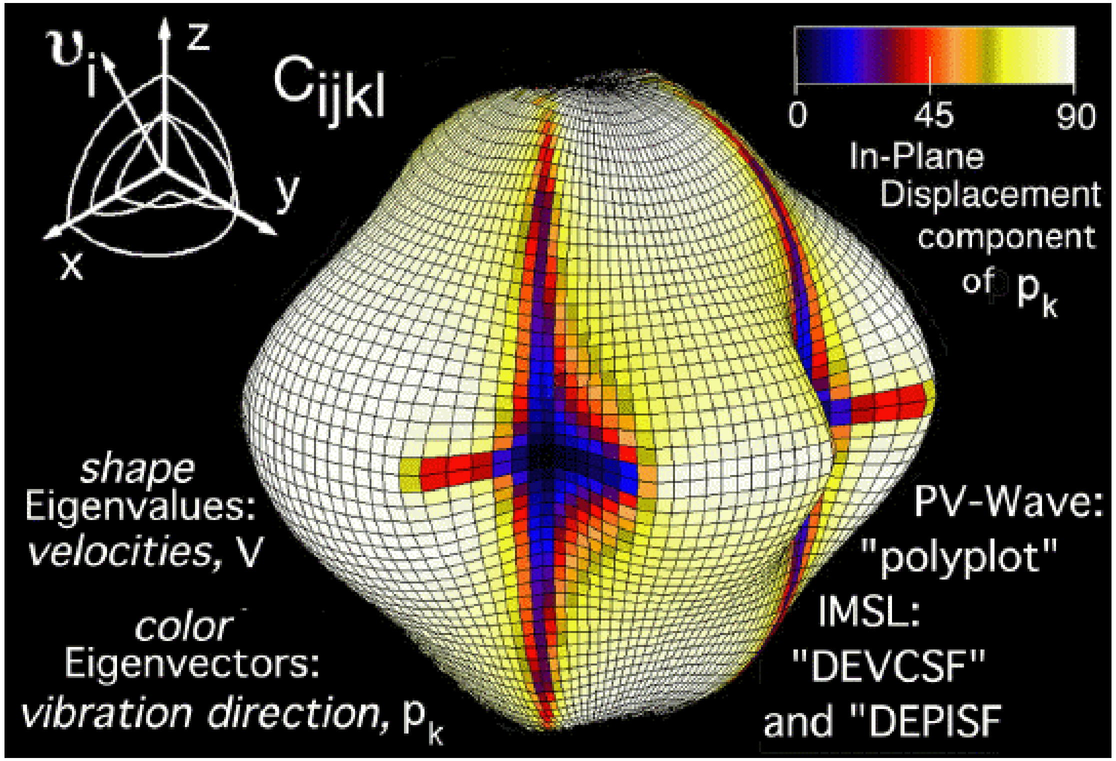
Glyph-Based Mapping Techniques

- Glyph probe for local flow field visualization [Leeuw, Wijk 1993]
  - Arrow: particle path
  - Green cap: tangential acceleration
  - Orange ring: shear (with respect to gray ring)



Glyph-Based Mapping Techniques

- Glyph for fourth-order tensor
  (wave propagation in crystals)



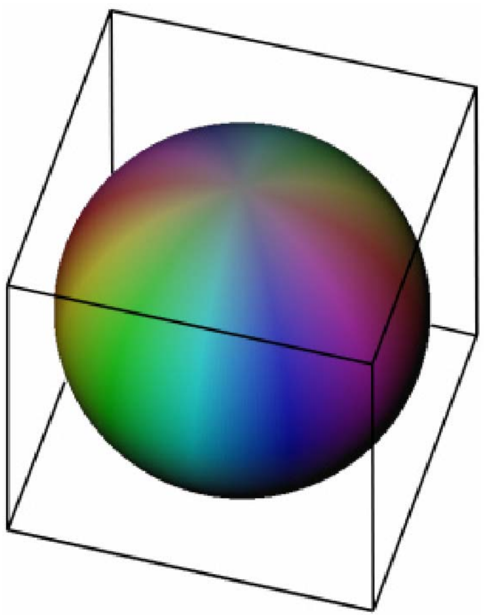
Hue-Balls and Lit-Tensors

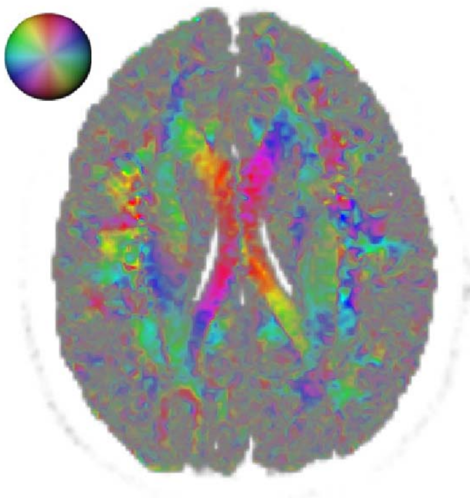
- Hue-balls and lit-tensors [Kindlmann, Weinstein 1999]
- Ideas and elements
  - Visualize anisotropy (relevant, e.g., in biological applications)
  - Color coding
  - Opacity function
  - Illumination
  - Volume rendering

Hue-Balls and Lit-Tensors

- Color coding (hue-ball)
  - Fixed, yet arbitrary input vector (e.g., user specified)
  - Color coding for output vector (after multiplication with tensor)
  - Coding on sphere

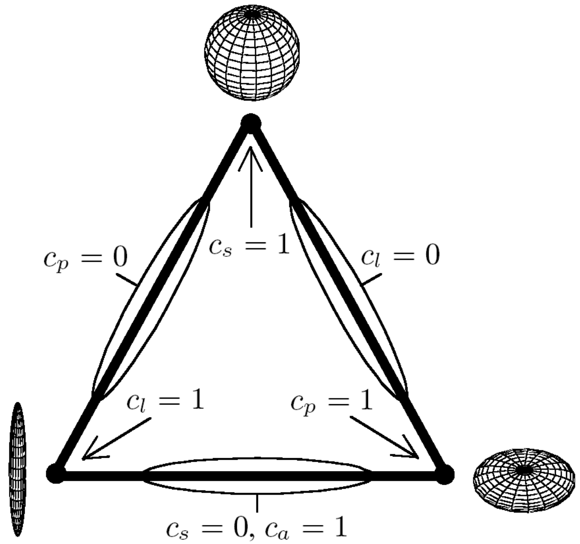
- Idea:
  - Deflection is strongly coupled with anisotropy





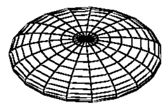
Hue-Balls and Lit-Tensors

- Barycentric opacity mapping
  - Emphasize important features
  - Make unimportant regions transparent
  - Can define 3 barycentric coordinates c_l, c_p, c_s

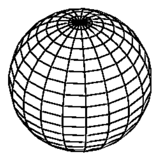


$$c_l = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$c_p = \frac{2(\lambda_2 - \lambda_3)}{\lambda_1 + \lambda_2 + \lambda_3}$$

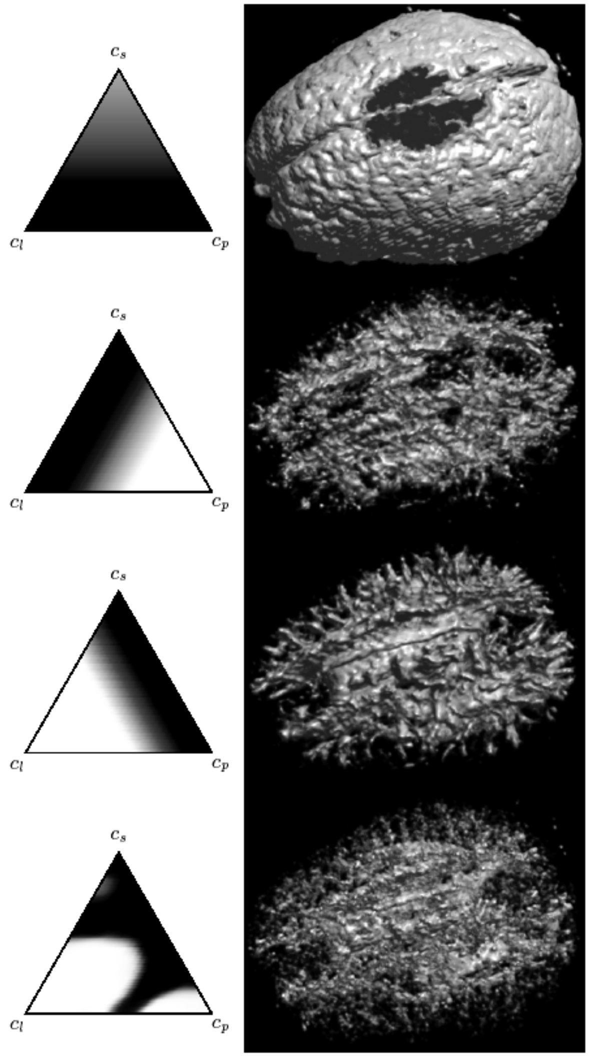


$$c_s = \frac{3\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}$$



Hue-Balls and Lit-Tensors

- Barycentric opacity mapping *(cont.)*
  - Examples for transfer functions



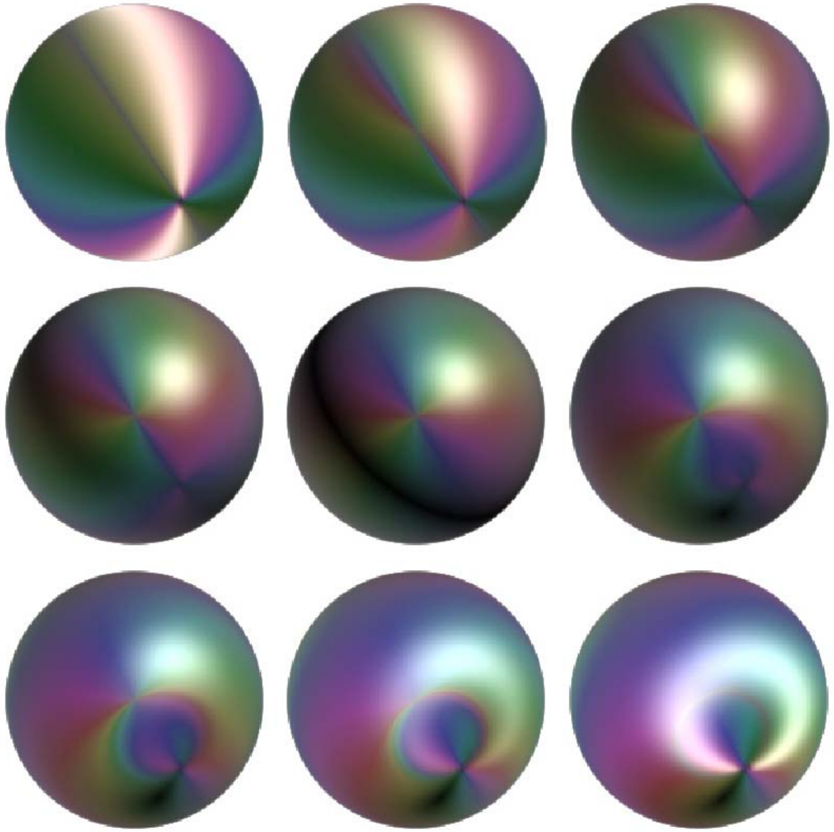
Hue-Balls and Lit-Tensors

- Lit-tensors
  - Similar to illuminated streamlines
  - Illumination of tensor representations
  - Provide information on direction and curvature

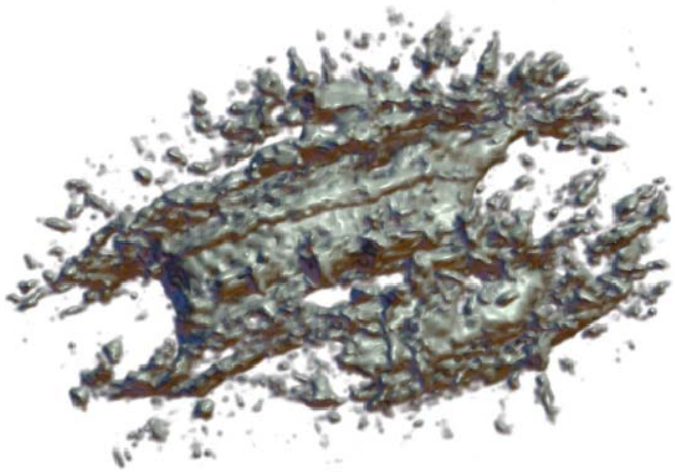
- Cases
  - Linear anisotropy: same as illuminated streamlines
  - Planar anisotropy: surface shading
  - Other cases: smooth interpolation between these two extremes

Hue-Balls and Lit-Tensors

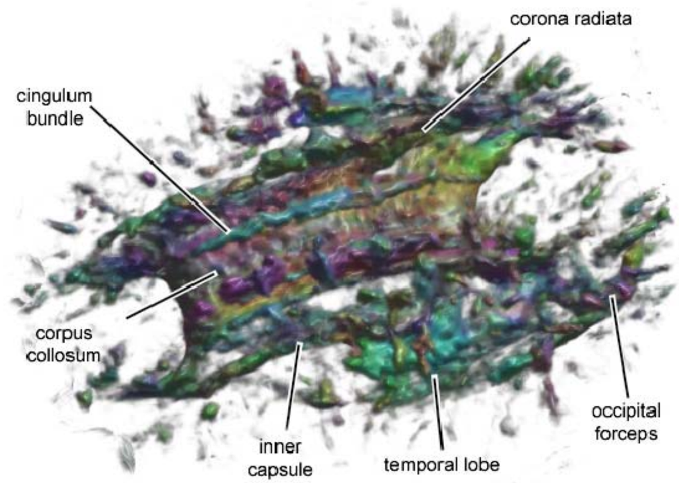
- Lit-tensors (cont.)
  - Example



Hue-Balls and Lit-Tensors

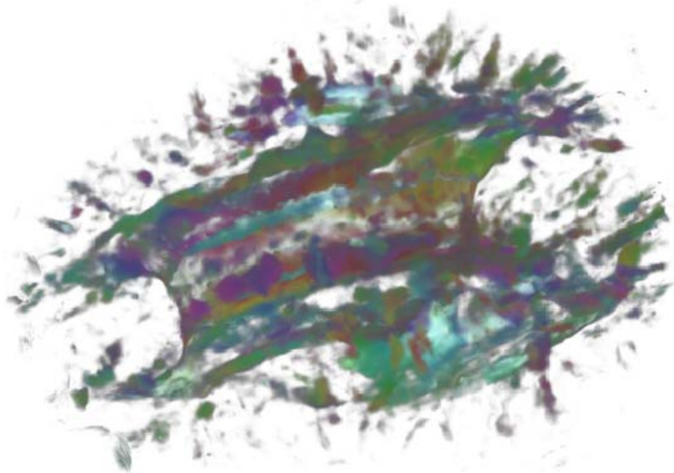


(a) Scalar volume, colored lights



(b) Scalar volume, hue-ball coloring

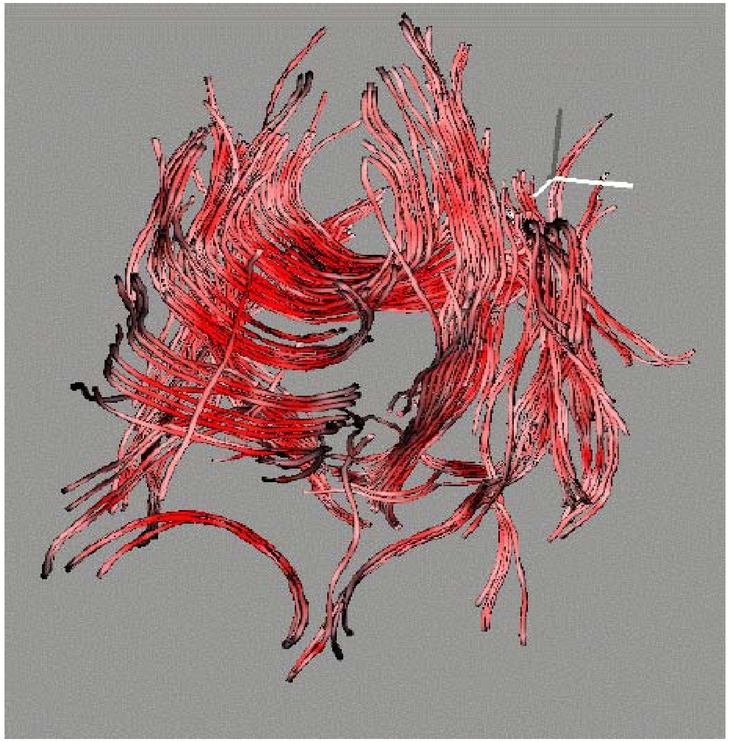
(c) Tensor volume, colored lights



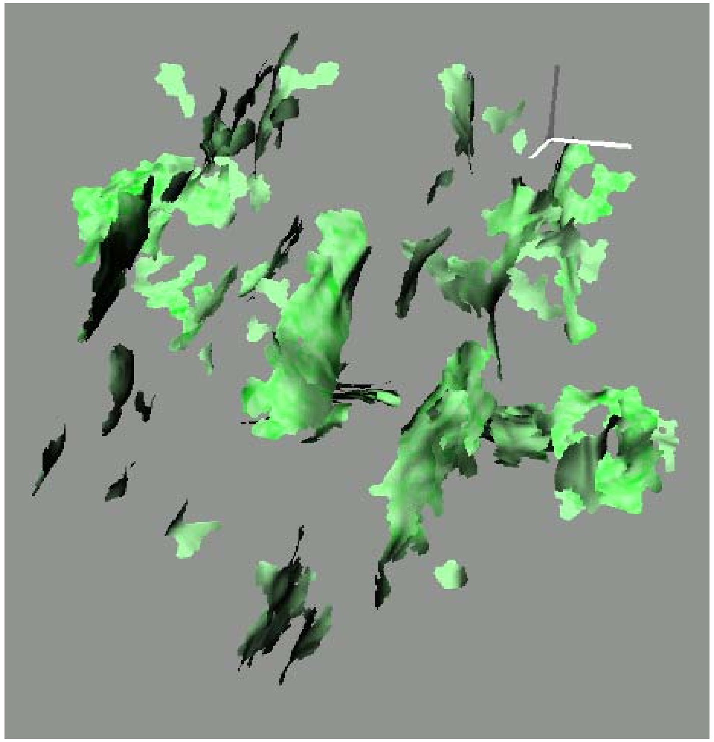
(d) Tensor volume, hue-ball coloring

Hue-Balls and Lit-Tensors

- Variation: streamtubes and streamsurfaces [Zhang et al. 2000]
  - Streamtubes: linear anisotropic regions
  - Streamsurfaces: planar anisotropic surfaces



linear



planar

Tensor Field Lines

- Let $t_{i1,i2,\ldots,ik}(x_1,\ldots,x_n)$ be a (second-order) symmetric tensor field

$\rightarrow$ Real eigenvalues, orthogonal eigenvectors

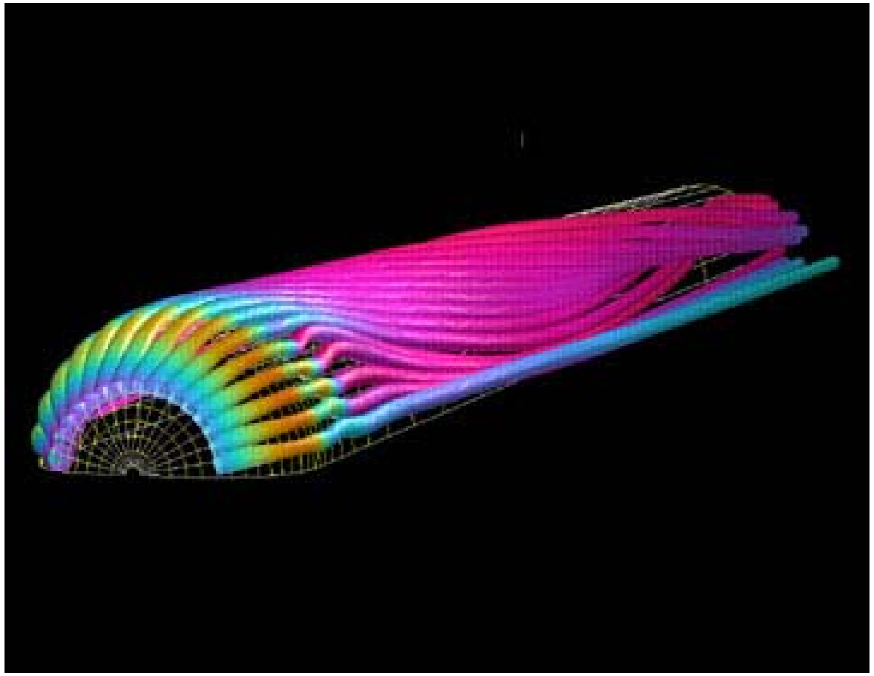
- Tensor field line: integrate along one of the eigenvectors

- Important: eigenvector fields are not vector fields!
  - Eigenvectors have no magnitude and no orientation (are bidirectional)
  - The choice of the eigenvector (minor, medium, major) is unambiguous only as long as all eigenvalues are different
  - Tensor field lines can meet (only) at points where two or more eigenvalues are equal, so-called degenerate points (see later)

Hyperstreamlines and Tensorlines

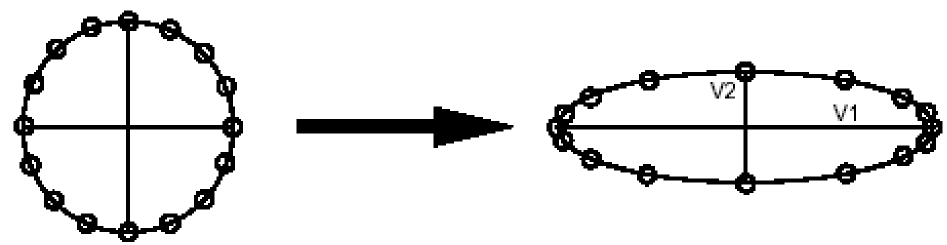
important

- Hyperstreamlines [Delmarcelle, Hesselink 1992/93]
  - Representation of tensor field lines with tubes
  - Elliptic cross section, radii proportional to other two eigenvalues

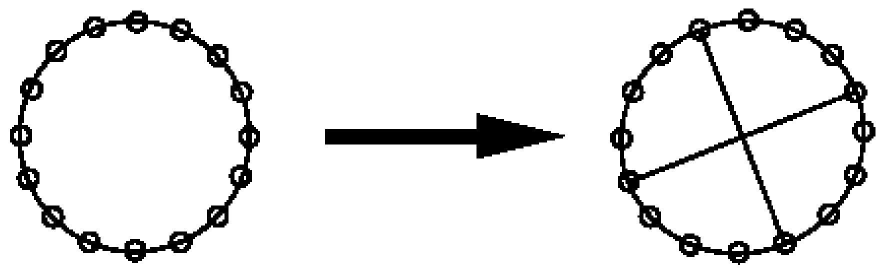


Hyperstreamlines and Tensorlines

- Idea of tensor field lines / hyperstreamlines:
  - Major eigenvector describes direction of diffusion with highest probability density

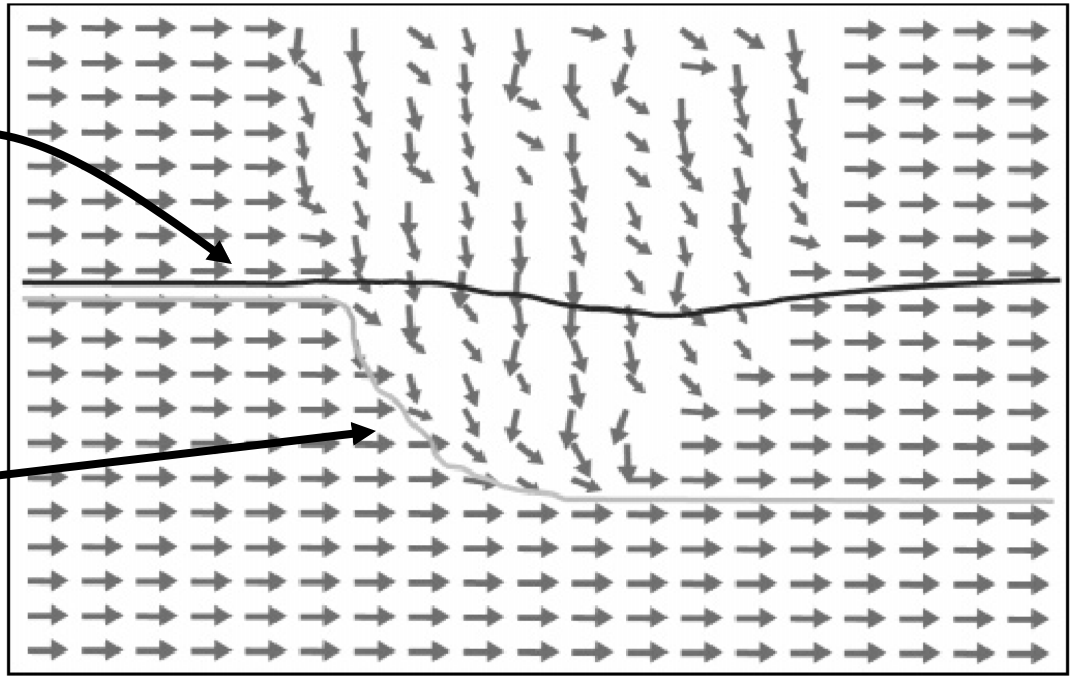


  - Ambiguity for (nearly) isotropic case



Hyperstreamlines and Tensorlines

- Problems of tensor field lines / hyperstreamlines
  - Ambiguity in (nearly) isotropic regions:
  - Partial volume effect, especially in low resolution images (MR images)
  - Noise in data
  - Solution: tensorlines



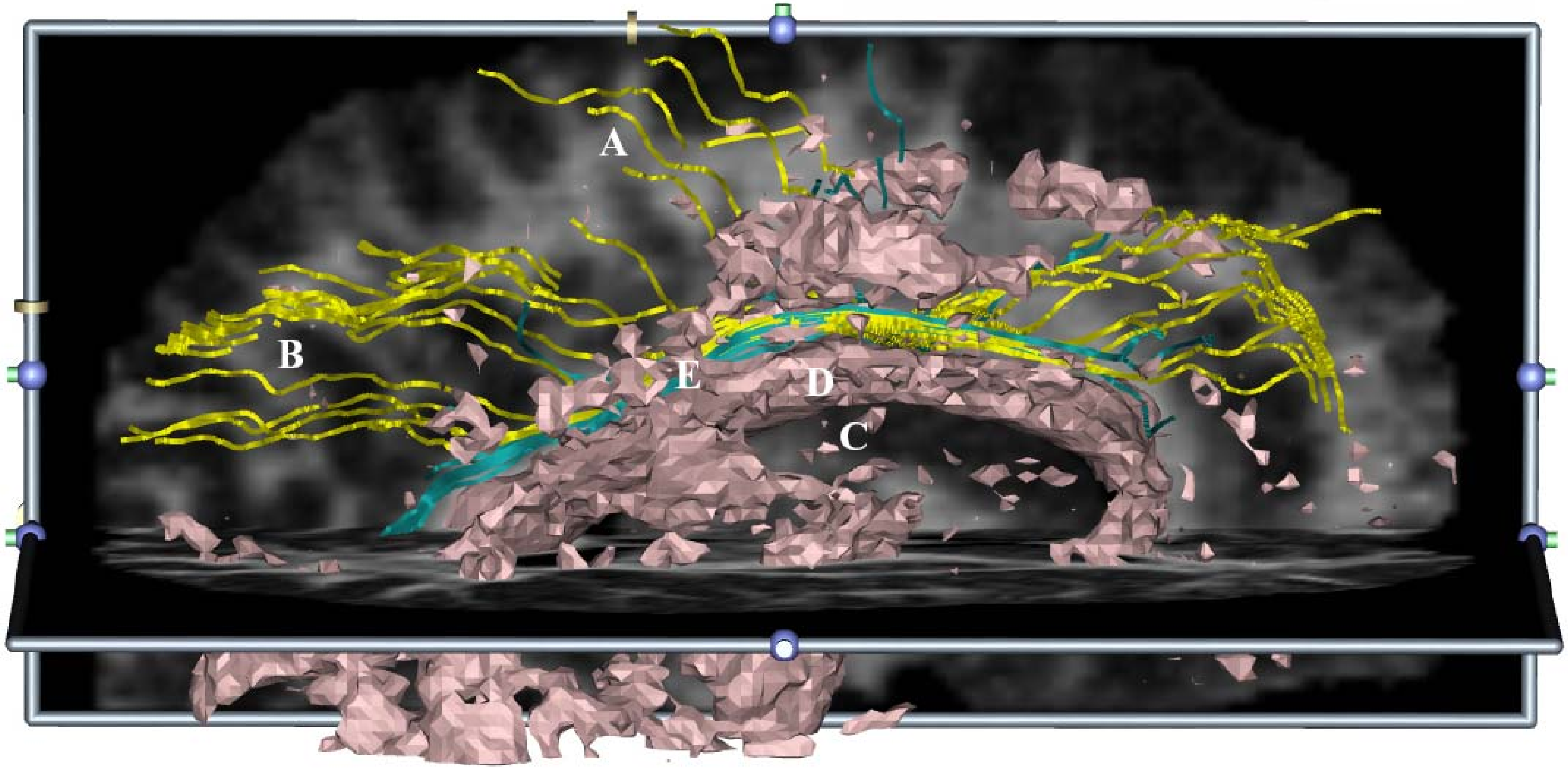
- Tensorline
- Hyperstreamline
- Arrows:
  major eigenvector

Hyperstreamlines and Tensorlines

- Tensorlines [Weinstein, Kindlmann 1999]
  - Advection vector
  - Stabilization of propagation by considering
    - Input velocity vector
    - Output velocity vector (after application of tensor operation)
    - Vector along major eigenvector
  - Weighting of the three components depends on anisotropy at specific position:
    - Linear anisotropy: only along major eigenvector
    - Other cases: input or output vector

Hyperstreamlines and Tensorlines

- Tensorlines



yellow: tensorlines, blue: tensor field lines, cutting planes: linear anisotropy

Tensor Field Topology

- In analogy to vector field topology, a tensor field topology for symmetric second-order tensor fields $\mathbf{T}(\mathbf{x})$ (with $\mathbf{T}_{ij} = \mathbf{T}_{ji}$) can be defined based on tensor field lines
- For simplicity, we only study the 2D case
- Degenerate points play the role of critical points:
  - A point $\mathbf{x}$ is degenerate iff both eigenvalues of $\mathbf{T}(\mathbf{x})$ are equal, i.e. $\lambda_1 = \lambda_2 = \lambda$
  - At degenerate points, infinitely many directions (of eigenvectors) exist
- Hence, at a degenerate point, **T** has (in **any** coordinate frame, as can be shown) the form

$$\mathbf{T} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda \mathbf{I}$$

- Hence, degenerate points are found by solving the equations

$$\mathbf{T}_{11}(\mathbf{x}) - \mathbf{T}_{22}(\mathbf{x}) = 0\,, \quad \mathbf{T}_{12}(\mathbf{x}) = 0$$

Tensor Field Topology

- The topological type of the degenerate point depends on

$$\delta = ad - bc$$

where

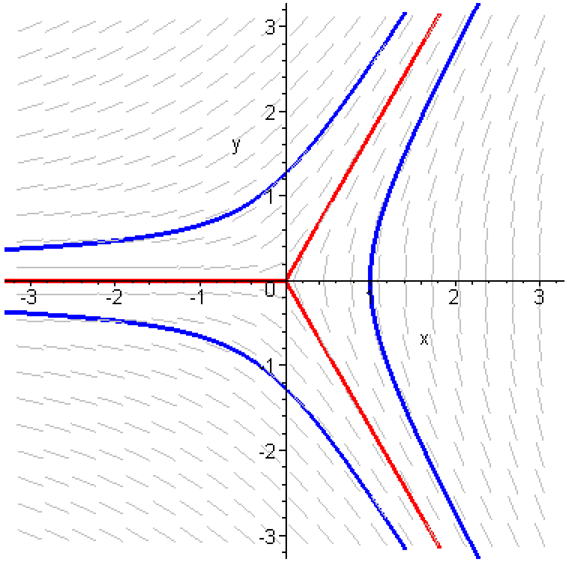
$$a = \frac{1}{2}\frac{\partial(\mathbf{T}_{11} - \mathbf{T}_{22})}{\partial x}, \quad b = \frac{1}{2}\frac{\partial(\mathbf{T}_{11} - \mathbf{T}_{22})}{\partial y}$$

$$c = \frac{1}{2}\frac{\partial \mathbf{T}_{12}}{\partial x}, \quad d = \frac{1}{2}\frac{\partial \mathbf{T}_{12}}{\partial y}$$

- For $\delta < 0$ the type is: trisector
- For $\delta > 0$ the type is: wedge
- For $\delta = 0$ the case is structurally unstable

Tensor Field Topology

- Types of degenerate points, illustrated with linear tensor fields

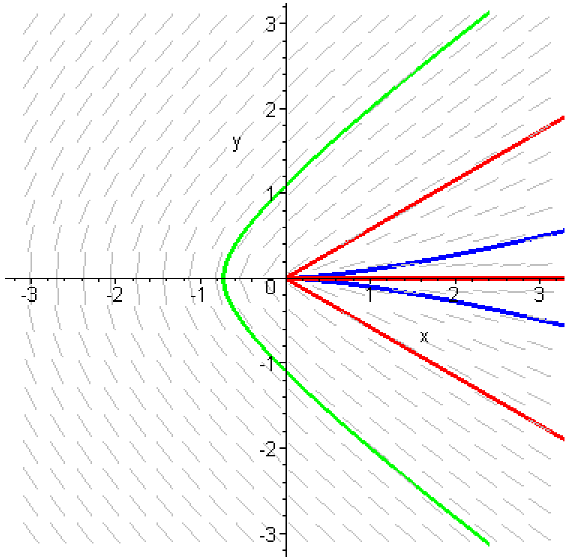


trisector

$$\mathbf{T} = \begin{pmatrix} 1 - 2x & y \\ y & 1 \end{pmatrix}$$

$$\mathbf{e} = \begin{pmatrix} \sqrt{x^2 + y^2} - x \\ y \end{pmatrix}$$

$$\delta = -1$$

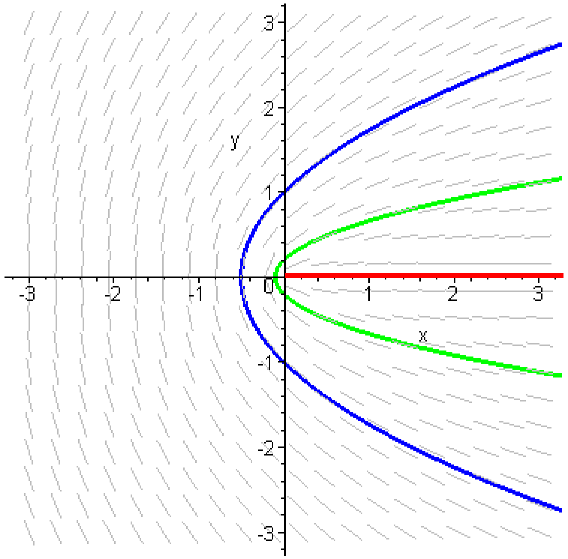


double wedge

$$\mathbf{T} = \begin{pmatrix} 1 + 2x/3 & y \\ y & 1 \end{pmatrix}$$

$$\mathbf{e} = \begin{pmatrix} x + \sqrt{x^2 + 9y^2} \\ 3y \end{pmatrix}$$

$$\delta = 1/3$$



single wedge

$$\mathbf{T} = \begin{pmatrix} 1 + x & y \\ y & 1 - x \end{pmatrix}$$

$$\mathbf{e} = \begin{pmatrix} y \\ \sqrt{x^2 + y^2} - x \end{pmatrix}$$

$$\delta = 1$$

Tensor Field Topology

- Separatrices are tensor field lines converging to the degenerate points of type trisector
- They are straight lines in the special case of linear tensor fields

- Double wedges have one “hidden separatrix” and two other separatrices which separate regions of different field line behavior
- Single wedges have just one separatrix

Tensor Field Topology

- The angles of the separatrices are obtained by solving

$$dm^3 + (c + 2b)m^2 + (2a - d)m - c = 0$$

- If $m \in \mathbb{R}$, the two angles

$$\theta = \pm \arctan m$$

are the angles of a separatrix. The two choices of signs correspond to the two choices of tensor field lines (minor and major eigenvalue)

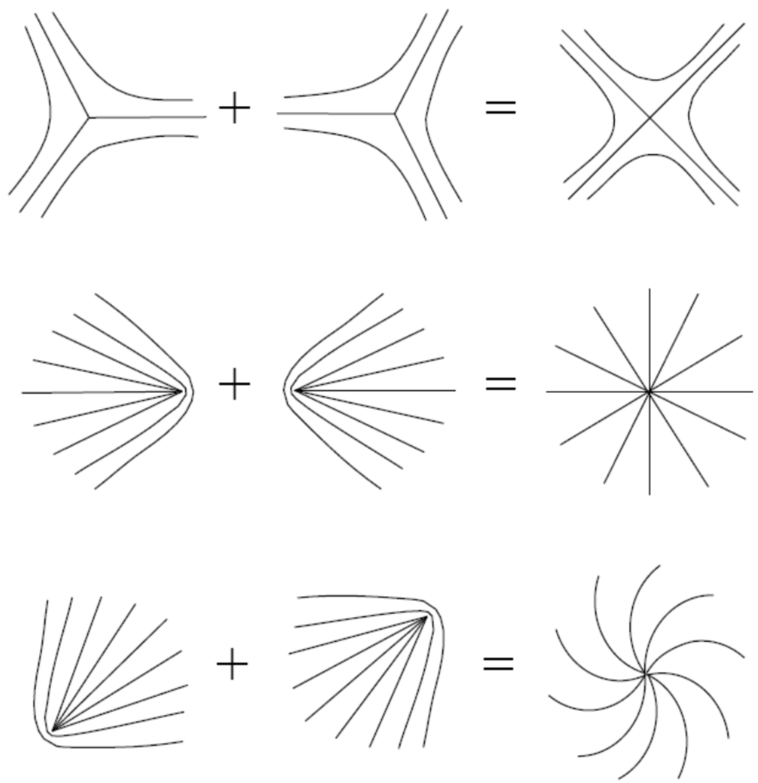
- If $d = 0$, an additional solution is

$$\theta = \pm 90°$$

- There are in general 1 or 3 real solutions:
  - 3 separatrices for trisector and double wedge
  - 1 separatrix for single wedge

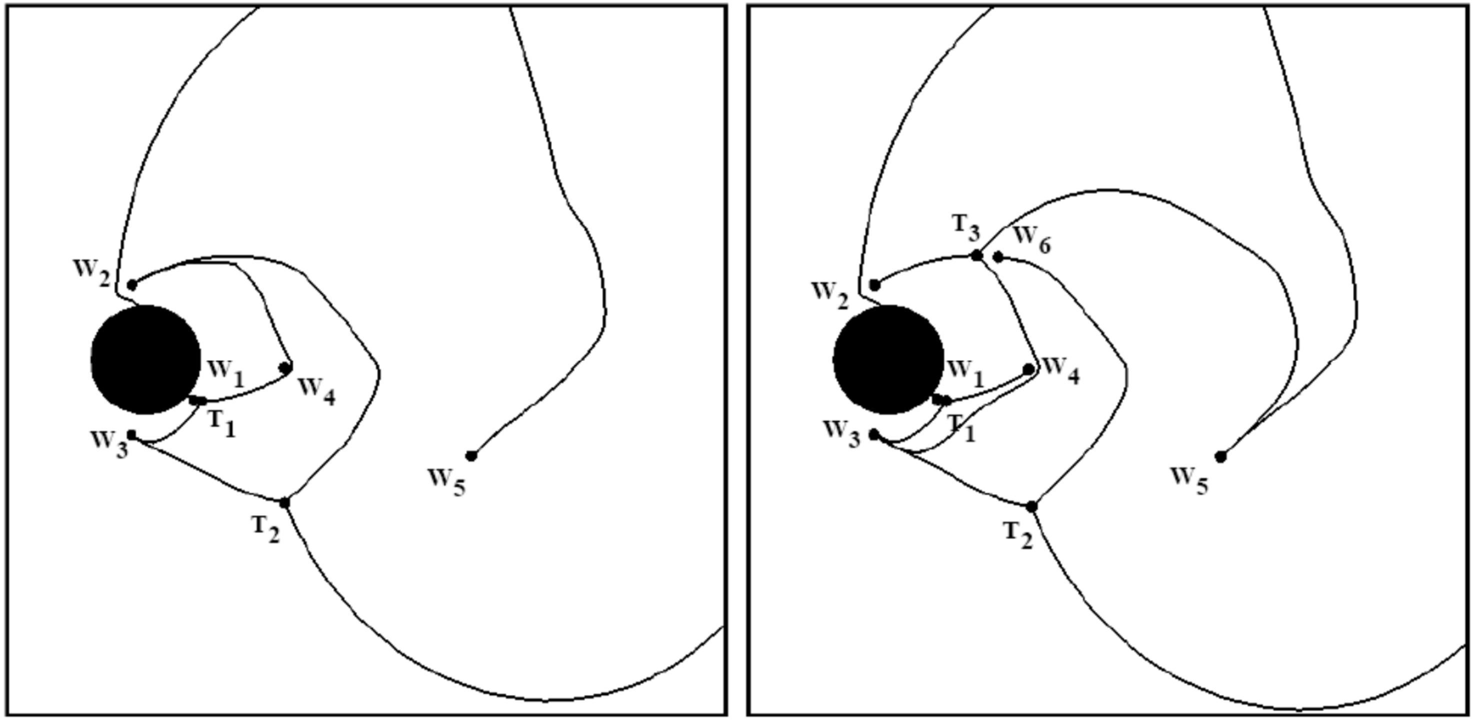
Tensor Field Topology

- Saddles, nodes, and foci can exist as nonelementary (higher-order) degenerate points with $\delta = 0$.
- They are created by merging trisectors or wedges
- They are not structurally stable, i.e. break up in elements if perturbed



Tensor Field Topology

- The topological skeleton is defined as the set of separatrices of trisector points
- Example: topological transition of the stress tensor field of a flow past a cylinder



[Delmarcelle and Hesselink 1994]