Level 3: Small-Signal Linearization

From Nonlinear Averaged Model to Linear Transfer Functions

Why Linearize? Unlocking Linear Control Theory

Challenge: The averaged model is still nonlinear due to:

- CPL load: $f_{CPL} = -P/(C0 \cdot vC0)$ (nonlinear in vC0)
- Duty cycle dependencies in $A_{avg}(d1, d2)$

Solution: Linearize around a DC operating point to use powerful linear control design tools:

- Bode plots for frequency response analysis
- Root locus for pole placement
- Standard PI/PID controller tuning methods

Step 1: Perturbation About Operating Point

$$\text{Nonlinear averaged model: } \dot{x} = f(x, d1, d2, V_{in})$$

$$\text{Decompose each variable: } x = x_0 + \Delta x, \quad d1 = d_{10} + \Delta d1, \quad \text{etc.}$$

$$\text{At steady state: } 0 = f(x_0, d_{10}, d_{20}, V_{in0})$$

Operating point (x_0, d_{10}, d_{20}, V_{in0}) found via fsolve (MATLAB) or Newton-Raphson

Step 2: Jacobian Linearization

Taylor series expansion: $f(x_0+\Delta x, d_{10}+\Delta d1, \ldots) \approx f(x_0, d_{10}, \ldots) + \boldsymbol{\partial f/\partial x \cdot \Delta x + \partial f/\partial d1 \cdot \Delta d1} + \ldots$

State Jacobian (A_{linear}):

$$A_{linear} = \partial f/\partial x \,|_{(x0,d10,d20,Vin0)} = A_{avg}(d_{10},d_{20}) + \partial f_{CPL}/\partial x$$

CPL Term Derivative (critical for stability):

$$f_{CPL}(vC0) = -P/(C0 \cdot vC0) \quad \rightarrow \quad \text{row 9 of } f$$

$$\partial f_{CPL}/\partial vC0 = \partial/\partial vC0[-P/(C0 \cdot vC0)] = -P/(C0) \cdot (-1/vC0^2) = \boldsymbol{+P/(C0 \cdot vC0^2)}$$

$$\text{Therefore: } A_{linear}(9,9) = 0 + P/(C0 \cdot vC0_0^2) \quad \text{(POSITIVE term} \rightarrow \text{destabilizing)}$$

Step 3: Control Input Jacobian (B_d)

B_d represents how duty cycle perturbations affect state derivatives

$$B_d = [\partial f/\partial d1 \mid \partial f/\partial d2] \,|_{(x0,d10,d20,Vin0)} \quad (9\times 2 \text{ matrix})$$

Chain rule (duty cycles affect A_{avg} and $B_{in,avg}$):

$$\partial f/\partial d1 = (\partial A_{avg}/\partial d1) \cdot x_0 + (\partial B_{in,avg}/\partial d1) \cdot V_{in0}$$
$$\partial f/\partial d2 = (\partial A_{avg}/\partial d2) \cdot x_0 + (\partial B_{in,avg}/\partial d2) \cdot V_{in0}$$

(Derivatives of w_k w.r.t. d1, d2 propagate through weighted sum)

Step 4: Linearized Model and Transfer Functions

Small-signal model:

$$\Delta \dot{x} = A_{linear} \cdot \Delta x + B_d \cdot [\Delta d1; \Delta d2] + B_{in} \cdot \Delta V_{in}$$

Transfer functions (Laplace domain):

$$G(s) = C \cdot (sI - A_{linear})^{-1} \cdot B_d$$

$$\text{Example: } G_{vd1}(s) = \text{output voltage to duty d1} \rightarrow C = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1]$$

Control Design Applications

- $G_{id}(s)$: duty → input current (inner loop for PFC, tracks AC reference)
- $G_{vd}(s)$: duty → bus voltage (outer loop for regulation)
- PI/PID tuning via Bode plots, phase/gain margins
- Stability assessment: check poles of $(sI - A_{linear})^{-1}$ for LHP (stable)

Critical Insight: The CPL term $+P/(C0 \cdot vC0_0^2)$ in $A_{linear}(9,9)$ acts as negative incremental resistance, potentially pushing poles toward RHP (instability). Control design must provide sufficient damping to counteract this destabilizing effect, especially at high power loads.