$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ on which we have

tradable asset price processes

$S = (S_t^0, S_t^1, \dots, S_t^d)_{t\geq 0}$

$\llcorner$ numeraire asset ($S_t^0 > 0$ a.s. $\forall\, t \geq 0$)

$\mathcal{F}_t = \sigma((S_u^0, S_u^1, \dots, S_u^d)_{u\leq t})$

S will satisfy some SDE driven by a collection of Brownian motions

$W = (W_t^0, W_t^1, \dots, W_t^d)_{t\geq 0}$

Def: A self-financing trading strategy is

process $\alpha := (\alpha_t^0, \alpha_t^1, \dots, \alpha_t^d)$, s.t. α_t is $\mathcal{F}_t$-measurable $\forall\, t \geq 0$

and its corresponding value process

$V^\alpha := (V_t^\alpha)_{t\geq 0}$ s.t. $V_t^\alpha = \alpha_t \cdot S_t$

& $dV_t^\alpha = \alpha_t \cdot dS_t$

this is key.

$d(g_t f_t) = g_t\, df_t + f_t\, dg_t + d[f, g]_t$

self-financing $S_t \cdot d\alpha_t + \sum_k d[\alpha^k, S^k]_t = 0$

* shorting assets is allowed, i.e. $\alpha^k_t < 0$
* arbitrary positions are allowed, i.e., $\alpha^k_t \in \mathbb{R}$
* there are no transaction costs.

Def: An arbitrage strategy α is a self-financing strategy s.t.

a) $V_0^\alpha = 0$ (costs nothing)

b) $\exists\, t > 0$ s.t. i) $\mathbb{P}(V_t^\alpha \geq 0) = 1$ (never lose)

ii) $\mathbb{P}(V_t^\alpha > 0) > 0$ (sometimes win)

FTAP:

The market admits no arbitrage strategies

$\Longleftrightarrow$

$$\exists\, \mathbb{Q} \sim \mathbb{P} \text{ s.t. } \mathbb{E}[\tilde{S}^k_u \mid \mathcal{F}_t] = \tilde{S}^k_t, \quad \forall\, u \geq t \geq 0, \quad k = 0, 1, \ldots, d$$

$$\& \quad \tilde{S}^k_t := \frac{S^k_t}{S^0_t}.$$

Black-Scholes dynamic hedging argument

Generalized Black-Scholes PDE

* one source of risk $W = (W_t)_{t \geq 0}$ Brownian motion

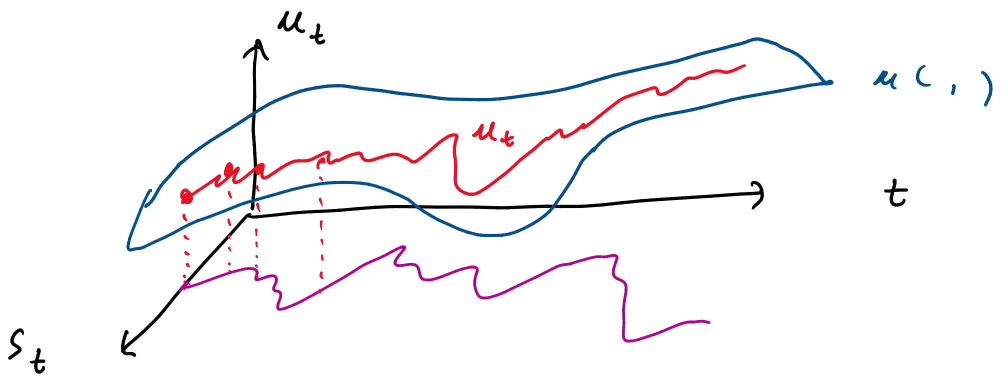
* risky asset $S = (S_t)_{t \geq 0}$ satisfies an SDE

$$dS_t = S_t \overbrace{\mu(t, S_t)}^{\mu_t} \, dt + S_t \overbrace{\sigma(t, S_t)}^{\sigma_t} \, dW_t$$

μ_t - drift process $\leftrightarrow$ $\mu(\,,\,) : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ drift function

σ_t - volatility process $\leftrightarrow$ $\sigma(\,,\,) : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}_+$ voltility "

Black-Scholes model is when $\mu(\,,\,) = \text{const.}$
$\sigma(\,,\,) = \text{const.}$



B-S : $\quad dS_t = S_t \mu \, dt + S_t \sigma \, dW_t$

$$y_t = g(S_t) \quad , \; g(s) = \log s \; , \; g'(s) = \frac{1}{s} , \; g''(s) = -\frac{1}{s^2}$$

$$dy_t = g'(S_t) \, dS_t + \tfrac{1}{2} g''(S_t) \, d[S, S]_t$$

$$= \frac{1}{S_t} \left(S_t \mu \, dt + S_t \sigma \, dW_t \right) + \tfrac{1}{2} \left(-\frac{1}{S_t^2} \right) S_t^2 \sigma^2 \, dt$$

$$= \left(\mu - \tfrac{1}{2} \sigma^2 \right) dt + \sigma \, dW_t$$

$$= (\mu - \tfrac{1}{2}\sigma^2)\,dt + \sigma\, dW_t$$

$$y_t - y_0 = (\mu - \tfrac{1}{2}\sigma^2)\, t + \sigma (W_t - W_0)$$

$$\Rightarrow \quad \frac{e^{y_t}}{e^{y_0}} = e^{(\mu - \frac{1}{2}\sigma^2) t + \sigma W_t}$$

$$\Rightarrow \quad S_t = S_0\, e^{(\mu - \frac{1}{2}\sigma^2) t + \sigma W_t}$$

Geometric Brownian motion

If we were to simulate to solve the SDE

$$dS_t = S_t\, \mu(t, S_t)\, dt + S_t\, \sigma(t, S_t)\, dW_t$$

$$S_{t+\Delta t} - S_t = S_t\, \mu(t, S_t)\, \Delta t + S_t\, \sigma(t, S_t) \underbrace{(W_{t+\Delta t} - W_t)}_{\substack{\overset{d}{=} \sqrt{\Delta t}\, Z_t \\ \text{iid } N(0,1)}}$$

* Bank account that is risk-free $B = (B_t)_{t \geq 0}$

$$B_t = e^{rt}, \quad \text{i.e.} \quad dB_t = r e^{rt}\, dt = r B_t\, dt$$

* European option price process $g = (g_t)_{t \in [0,T]}$

s.t. $g_T = G(S_T)$, e.g. $G(S) = \max(S - K, 0)$ call

$\max(K - S, 0)$ put

$\mathbb{1}_{S > K}$ digital

* What is $g_t = ?$

Solve this using a hedging argument + no arbitrage + self-financing + SDE...

suppose you sold g hold (α, β) in (S, B)

i.e. our strategy $\theta = ((\alpha_t, \gamma_t, -1))_{t \in [0,T]}$

pos in S pos in B pos in g

Associated value process $V^\theta = (V_t^\theta)_{t \in [0,T]}$

$V_t^\theta = \alpha_t S_t + \gamma_t B_t - g_t$, choose α_0, γ_0 s.t. $V_0^\theta = 0$

$dV_t^\theta \underset{\text{self-financing}}{=} \alpha_t \, dS_t + \gamma_t \, dB_t - 1 \cdot dg_t$

assume g_t admits a Markov structure, i.e. $\exists$ a $g(\,,\,)$, s.t. $g_t = g(t, S_t)$

moreover assume $g(\,,\,) \in C^{1,2}$

$$
\begin{aligned}
dg_t &= \partial_2 g(t, S_t)\, dS_t + \partial_1 g(t, S_t)\, dt \\
&\quad + \tfrac{1}{2} \partial_{22} g(t, S_t)\, d[S, S]_t \\
&= \partial_2 g(t, S_t)(S_t \mu_t \, dt + S_t \sigma_t \, dW_t) + \partial_1 g(t, S_t)\, dt \\
&\quad + \tfrac{1}{2} \partial_{22} g(t, S_t)\, S_t^2 \sigma_t^2 \, dt \\
&= \underbrace{\{ \partial_1 g(t, S_t) + \mu_t S_t \partial_2 g(t, S_t) + \tfrac{1}{2} \partial_{22} g(t, S_t) \}}_{\mu_t^g g_t} dt \\
&\quad + \underbrace{\sigma_t S_t \partial_2 g(t, S_t)}_{\sigma_t^g g_t} dW_t \\
&= g_t \mu_t^g \, dt + g_t \sigma_t^g \, dW_t
\end{aligned}
$$

$$= g_t \mu_t^g \, dt + g_t \sigma_t^g \, dW_t$$

$$dV_t^\theta = \alpha_t (S_t \mu_t \, dt + S_t \sigma_t \, dW_t)$$

$$+ \gamma_t \, r B_t \, dt$$

$$- (g_t \mu_t^g \, dt + g_t \sigma_t^g \, dW_t)$$

$$= \underbrace{(\alpha_t S_t \mu_t + \gamma_t r B_t - g_t \mu_t^g)}_{A_t} dt$$

$$+ \underbrace{(\alpha_t S_t \sigma_t - g_t \sigma_t^g)}\, dW_t$$

Choosing $\alpha_t = \dfrac{g_t \sigma_t^g}{S_t \sigma_t}$ removes the risk locally.

$$\Rightarrow \quad dV_t^\theta = A_t dt + 0 \, dW_t$$

As $A_t \in \mathcal{F}_t$, and there is no uncertainty

$A_t \neq 0 \Rightarrow$ an arbitrage.

$\therefore \quad A_t = 0$.

$$\Leftrightarrow \quad \alpha_t S_t \mu_t + r (\gamma_t B_t) - g_t \mu_t^g = 0$$

recall $V_0^\theta = 0$ & $dV_t^\theta = 0 \Rightarrow V_t^\theta = 0$

$$\Rightarrow \quad \alpha_t S_t + (\gamma_t B_t) - g_t = 0$$

$$\Rightarrow \gamma_t B_t = g_t - \alpha_t S_t$$

$$\Rightarrow \alpha_t S_t \mu_t + r(g_t - \alpha_t S_t) - g_t \mu_t^g = 0$$

$$\Rightarrow \alpha_t S_t (\mu_t - r) - g_t(\mu_t^g - r) = 0$$

$$\Rightarrow g_t \frac{\sigma_t^g}{\sigma_t}(\mu_t - r) - g_t(\mu_t^g - r) = 0$$

$$\Rightarrow g_t \sigma_t^g \Bigg(\underbrace{\frac{\mu_t - r}{\sigma_t} - \frac{\mu_t^g - r}{\sigma_t^g}}_{=0 \ \forall g!!!} \Bigg) = 0$$

Thus

$$\frac{\mu_t^g - r}{\sigma_t^g} = \frac{\mu_t - r}{\sigma_t}$$ is a market property

market-price of risk

Sharpe ratios of assets & options on that asset are equal!

$$\sigma_t(\mu_t^g - r) = (\mu_t - r)\sigma_t^g$$

$$\sigma_t\left((g_t \mu_t^g) - r g_t\right) = (\mu_t - r)(\sigma_t^g g_t)$$

$$\cancel{\sigma_t}\Big(\big(\partial_1 g(t,S_t) + \cancel{\mu_t S_t \partial_2 g(t,S_t)} + \tfrac{1}{2}\overset{S_t^2\sigma_t^2}{\downarrow}\partial_{22} g(t,S_t)\big) - r g_t \Big)$$

$$= (\cancel{\mu_t} - r)(\cancel{\sigma_t} S_t \partial_2 g(t,S_t))$$

$$\Rightarrow \partial_1 g(t,S_t) + r S_t \partial_2 g(t,S_t)$$

$$\Rightarrow \partial_1 g(t, S_t) + r S_t \partial_2 g(t, S_t) + \frac{1}{2} S_t^2 \sigma^2(t, S_t) \partial_{22} g(t, S_t) = r g(t, S_t)$$

must hold for all paths of $(S_t)_{t \in [0,T]}$!

$$\therefore \boxed{\begin{gathered} \partial_1 g(t,s) + r s \partial_2 g(t,s) + \frac{1}{2} \sigma^2(t,s) s^2 \partial_{22} g(t,s) = r g(t,s) \\ \forall \quad s > 0, \quad t \in [0,T) \\ g(T,s) = G(s) \end{gathered}}$$

Generalised Black-Scholes PDE !

Solving BS PDE

$$\boxed{\begin{aligned} &\partial_1 g(t,S) + rS\,\partial_2 g(t,S) + \tfrac{1}{2}\sigma^2(t,S)\,S^2\,\partial_{22} g(t,S) = r g(t,S) \\ &\forall \quad S>0, \quad t\in[0,T) \\ &\qquad g(T,S) = G(S) \end{aligned}}$$

e.g. I. $G(S) = 0$, $g(t,S) = 0$

II. $G(S) = 1$, think $g(t,S) = e^{-r(T-t)}$

$$\partial_1 g(t,S) = r e^{-r(T-t)} = r g(t,S)$$

$$\& \quad g(T,S) = e^{-r(T-T)} = 1$$

III. $G(S) = S$ think $g(t,S) = S$

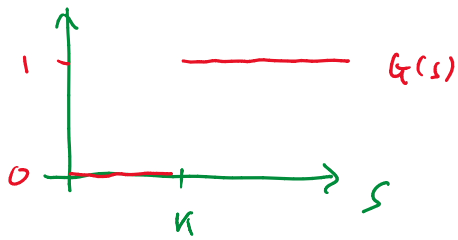
$$\partial_1 g(t,S) = 0$$

$$\partial_2 g(t,S) = 1, \quad \partial_{22} g(t,S) = 0$$

$$0 + rS\cdot 1 + \tfrac{1}{2}\sigma^2(t,S)\cdot 0 = rS$$

$$g(T,S) = S = G(S)$$

IV. $G(S) = \mathbb{1}_{S>K}$ digital option



$$g(t,s) = \overset{\mathbb{Q}}{\cancel{\mathbb{P}}}(S_T > K \mid S_t = s)\, e^{-r(T-t)}$$

FTAP: $\Rightarrow \exists\ \mathbb{Q} \sim \mathbb{P}$ s.t.

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{g_T}{B_T} \,\Big|\, \mathcal{F}_t \right] = \frac{g_t}{B_t}$$

$$\Rightarrow \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{1}_{S_T>K}}{e^{rT}} \,\Big|\, \mathcal{F}_t \right] = \frac{g_t}{e^{rt}}$$

$$\Rightarrow g_t = e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{S_T>K} \mid \mathcal{F}_t]$$

$$\overset{g(t,S_t)}{\parallel} \qquad = e^{-r(T-t)}\, \underbrace{\mathbb{Q}(S_T > K \mid \mathcal{F}_t)}_{\mathbb{Q}(S_T > K \mid \sigma(S_t))}$$

assume $\log\left(\frac{S_T}{S_t}\right) \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t),\ \sigma^2 (T-t) \right)$

$\sigma_t = \text{const } \sigma$

$$\mathbb{Q}(S_T > K \mid \sigma(S_t))$$

$$= \mathbb{Q}\left(S_t\, e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\, Z} > K \mid \sigma(S_t) \right), \quad Z \overset{\mathbb{Q}}{\sim} N(0,1)$$

$$= Q\left(Z > \frac{\log(K/S_t) - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \,\middle|\, \sigma(S_t) \right)$$

$$= \Phi\left(\frac{\log(S_t/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right)$$

Feynman-Kac I

Wednesday, October 18, 2023 3:03 PM

Let's consider the PDE

$$\begin{cases} \partial_t g(t,x) + \frac{1}{2}\partial_{xx} g(t,x) = 0 \\ g(T,x) = G(x) \end{cases}$$

heat equation

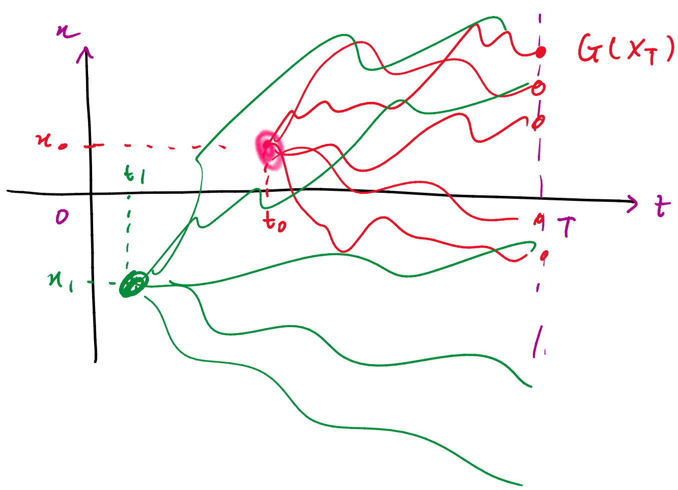
Feynman-Kac Theorem:

$g(t,x)$ admits the following representation

$$g(t,x) = \mathbb{E}^{\mathbb{M}}[G(X_T) \mid X_t = x]$$

where $X = (X_t)_{t\geq 0}$ that is a $\mathbb{M}$-Brownian motion.

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{M})$$



$g(t_0, x_0) = \mathbb{E}[G(X_T) \mid X_{t_0} = x_0]$

$g(t_1, x_1) = \mathbb{E}[G(X_T) \mid X_{t_1} = x_1]$

e.g. I suppose that $G(x) = x$

$$\begin{aligned} g(t,x) &= \mathbb{E}[G(X_T) \mid X_t = x] \\ &= \mathbb{E}[X_T \mid X_t = x] \\ &= \mathbb{E}[\underbrace{(X_T - X_t)} + X_t \mid X_t = x] \\ &= x \end{aligned}$$

recall

$X_T - X_t \big|_{\mathcal{F}_t} \sim \mathcal{N}(0, T-t)$

$\partial_t g(t,x) = 0, \quad \frac{1}{2}\partial_{xx} g(t,x) = 0$

$\partial_t g(t,x) = 0, \quad \frac{1}{2}\partial_{xx} g(t,x) = 0$

so PDE ✓ terminal condition ✓

$(g(T,x) = x = G(x))$

II. $G(x) = x^2$

$$g(t,x) = \mathbb{E}[\, G(X_T) \mid X_t = x]$$
$$= \mathbb{E}[(X_T)^2 \mid X_t = x]$$
$$= \mathbb{E}[((X_T - X_t) + X_t)^2 \mid X_t = x]$$
$$= \mathbb{E}[(X_T - X_t)^2 + 2X_t(X_T - X_t) + X_t^2 \mid X_t = x]$$
$$= (T-t) + 0 + x^2$$

$\partial_t g(t,x) = -1$

$\partial_{xx} g(t,x) = 2$

$\partial_t g + \frac{1}{2}\partial_{xx} g = -1 + \frac{1}{2} 2 = 0$ ✓

$g(T,x) = x^2 = G(x)$ ✓

III: $G(x) = \mathbb{1}_{x>K}$

$$g(t,x) = \mathbb{E}[\, G(X_T) \mid X_t = x]$$
$$= \mathbb{E}[\, \mathbb{1}_{X_T > K} \mid X_t = x]$$
$$= \mathbb{E}[\, \mathbb{1}_{(X_T - X_t) > K - X_t} \mid X_t = x]$$
$$= \mathbb{M}(\underbrace{(X_T - X_t)}_{\stackrel{d}{=} \sqrt{T-t}\, Z,\ Z \overset{\mathbb{M}}{\sim} \mathcal{N}(0,1)} > K - X_t \mid X_t = x)$$
$$= \mathbb{M}\left(Z > \frac{K - X_t}{\sqrt{T-t}} \mid X_t = x \right)$$

b/c Z is .

$$\stackrel{\rightarrow}{=} \mathbb{M}(Z < X_t - K \mid X_t = x)$$

s/c z is symmetric

$$= \mathbb{M}\left(Z < \frac{X_t - K}{\sqrt{T-t}} \,\middle|\, X_t = x \right)$$

$$= \Phi\left(\frac{x-K}{\sqrt{T-t}} \right)$$

cdf of std. normal

$$\partial_t g(t,x) = \Phi'\left(\frac{x-K}{\sqrt{T-t}} \right) \partial_t (T-t)^{-1/2} (x-K)$$

$$= \phi\left(\frac{x-K}{\sqrt{T-t}} \right) (T-t)^{-3/2} (x-K) \frac{1}{2}$$

$$\partial_x g(t,x) = \phi\left(\frac{x-K}{\sqrt{T-t}} \right) \frac{1}{\sqrt{T-t}}$$

$$\partial_{xx} g(t,x) = \phi'\left(\frac{x-K}{\sqrt{T-t}} \right) \frac{1}{T-t} = -\phi\left(\frac{x-K}{\sqrt{T-t}} \right) \frac{(x-K)}{\sqrt{T-t}} \frac{1}{T-t}$$

$$\phi(y) = \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}, \quad \phi'(y) = -y \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$$

$$= -y\,\phi(y)$$

$$\partial_t g + \frac{1}{2} \partial_{xx} g = 0 \quad \checkmark$$

$$g(T,x) = \lim_{t \to T} \Phi\left(\frac{x-K}{\sqrt{T-t}} \right) = \begin{cases} 1, & x > K \\ \frac{1}{2}, & x = K \\ 0, & x < K \end{cases}$$

$$= \mathbb{1}_{x>K} + \underline{\underline{\frac{1}{2} \mathbb{1}_{x=K}}}$$

?

up to measure 0 ✓

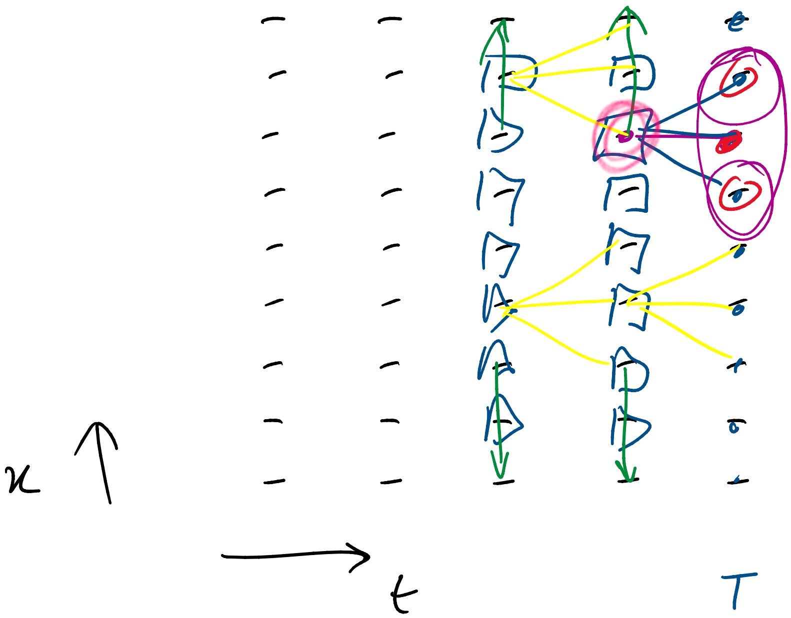
V: $G(x) = (x)_+ = \max(x, 0)$

...

Finite Difference Scheme

Wednesday, October 18, 2023 3:48 PM

Finite Difference Scheme (Explicit)



$g(T,x) = G(x)$

$$\partial_t g \sim \frac{g_{i+1,n} - g_{i,n}}{\Delta t}$$

$$\partial_{xx} g \sim \frac{g_{i+1,n+1} - 2 g_{i+1,n} + g_{i+1,n-1}}{\Delta x^2}$$

$$\partial_t g + \frac{1}{2} \partial_{xx} g = 0$$

$$\Rightarrow \frac{g_{i+1,n} - g_{i,n}}{\Delta t} + \frac{1}{2} \frac{g_{i+1,n+1} - 2 g_{i+1,n} + g_{i+1,n-1}}{\Delta x^2} = 0$$

$$\Rightarrow g_{i,n} = g_{i+1,n} + \frac{1}{2} \frac{\Delta t}{\Delta x^2} \left(g_{i+1,n+1} - 2 g_{i+1,n} + g_{i+1,n-1} \right)$$

g_{-1}, g_{-2}, g_{-3} $\qquad$ g_2, g_1, g_0

$$g_1 - (g_2 - g_1) = g_0$$

$$g_0 = 2 g_1 - g_2$$

$$g_{-1} = g_{-2} + g_{-2} - g_{-3}$$