BS – PDE

$$\partial_t g(t,x) + r x\, \partial_x g(t,x) + \tfrac{1}{2} x^2 \sigma^2(t,x)\, \partial_{xx} g(t,x) = r g(t,x)$$
$$g(T,x) = G(x)$$

suppose $g(t,x)$ satisfies the PDE

$$\partial_t g(t,x) + a(t,x)\, \partial_x g(t,x) + \tfrac{1}{2} b^2(t,x)\, \partial_{xx} g(t,x) = c(t,x)\, g(t,x)$$
$$g(T,x) = G(x)$$

Feynman-Kac Thm:

The solution to the above admits the representation

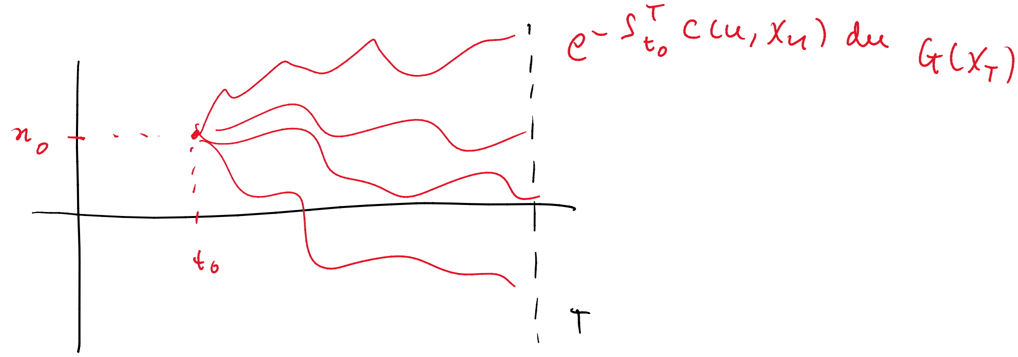
$$g(t,x) = \mathbb{E}\left[e^{-\int_t^T c(u,X_u)\,du}\, G(X_T) \mid X_t = x \right]$$

where $X = (X_t)_{t\geq 0}$ is a process on

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{M})$ s.t. X satisfies the

SDE: $\quad dX_t = a(t,X_t)\,dt + b(t,X_t)\,dW_t$

& $W = (W_t)_{t\geq 0}$ is a $\mathbb{M}$-Brownian motion.



solution to the BS-PDE can be represented as

$$g(t,x) = \mathbb{E}^{\mathbb{M}}\left[e^{-r(T-t)}\, G(X_T) \mid X_t = x \right]$$

where $X = (X_t)_{t\geq 0}$ is a s.p. on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{M})$

s.t. $dX_t = r X_t\, dt + X_t\, \sigma(t, X_t)\, dW_t$

$W = (W_t)_{t\geq 0}$ is a $\mathbb{M}$-Brownian motion.

s.t. $dX_t = rX_t\,dt + X_t\,\sigma(t,X_t)\,dW_t$

$W=(W_t)_{t\geq 0}$ is a $\mathbb{M}$-Brownian motion.

$$g(t,x) = \mathbb{E}^{\mathbb{M}}\left[\frac{B_t}{B_T}\, g_T \;\middle|\; X_t = x\right]$$

$$\frac{g(t,x)}{B_t} = \mathbb{E}^{\mathbb{M}}\left[\frac{g_T}{B_T} \;\middle|\; X_t = x\right]$$

$$\frac{g_t}{B_t} = \frac{g(t,X_t)}{B_t} = \mathbb{E}^{\mathbb{M}}\left[\frac{g_T}{B_T} \;\middle|\; \mathcal{F}_t\right]$$

$$\hat{g}_t = \mathbb{E}^{\mathbb{M}}\left[\hat{g}_T \mid \mathcal{F}_t\right], \qquad \hat{g}_t = \frac{g_t}{B_t}$$

X - price process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

① $dX_t = X_t\,\mu(t,X_t)\,dt + X_t\,\sigma(t,X_t)\,dW_t$ ← $\mathbb{P}$-B.mtn

$\downarrow$ self-financing, hedging, no-arbitrage

$$(\partial_t + \mathcal{L})\,g(t,x) = r\,g(t,x) \quad \& \quad g(T,x) = G(x)$$

↳ $rx\partial_x + \frac{1}{2}x^2\sigma^2(t,x)\,\partial_{xx}$

$\downarrow$ Feynman-Kac

$$g(t,x) = \mathbb{E}^{\mathbb{M}}\left[e^{-r(T-t)}\,G(X_T) \mid X_t = x\right]$$

$(\Omega', \mathcal{F}', (\mathcal{F}_t')_{t\geq 0}, \mathbb{M})$

② s.t. $dX_t = X_t\,r\,dt + X_t\,\sigma(t,X_t)\,dW_t'$ ← $\mathbb{M}$-B.mtn

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = dW_t + \lambda_t\,dt$

$W_t' = W_t + \int_0^t \lambda_s\,ds \quad \Leftarrow$

¿ s.t. ① = ② ?

$\stackrel{?}{\exists}$ s.t. ① = ② ?

$$\Leftrightarrow \quad \lambda_t X_t \sigma(t, X_t) + X_t r = X_t \mu(t, X_t)$$

$$\Leftrightarrow \quad X_t \left[r + \lambda_t \sigma(t, X_t) - \mu(t, X_t) \right] = 0$$

$$\Leftrightarrow \quad X_t \left[\lambda_t - \frac{\mu(t, X_t) - r}{\sigma(t, X_t)} \right] \sigma(t, X_t) = 0$$

$$\lambda_t = \frac{\mu(t, X_t) - r}{\sigma(t, X_t)}$$

Sharpe ratio, <u>or</u> Market price of risk !!!

Measure Changes for RVs

$$(\Omega, \mathcal{F}, \mathbb{P}) \qquad \text{r.v. } Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0, a^2)$$

$$\downarrow$$

$$\mathbb{P}^*$$

$$\omega \in \Omega, \qquad \mathbb{P}^*(\omega) := e^{-\frac{a^2}{2}\ell^2 - \ell Z(\omega)} \, \mathbb{P}(\omega)$$

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\frac{a^2}{2}\ell^2 - \ell Z}$$

$$\cancel{\frac{d\mathbb{P}^*}{d\mathbb{P}} = Z}$$

$$\mathbb{E}^{\mathbb{P}^*}[G] = \sum_{\omega} G(\omega)\, \mathbb{P}^*(\omega)$$

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}(\omega)$$

$$= \sum_{\omega} \left(G(\omega) \left(\frac{\mathbb{P}^*(\omega)}{\mathbb{P}(\omega)} \right) \right) \mathbb{P}(\omega)$$

$$= \mathbb{E}^{\mathbb{P}}\left[G \, \frac{d\mathbb{P}^*}{d\mathbb{P}} \right]$$

└ Radon-Nikodym (RN) derivative

R-N derivatives $\frac{d\mathbb{P}^*}{d\mathbb{P}}$ must

i) > 0 a.s.

ii) $\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}} \right] = 1$

iii) $\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}} \right] = 1$

¿ check is $\dfrac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\frac{1}{2} l^2 a^2 - l Z}$

a bonified R-N derivative? $\qquad Z \sim \mathcal{N}(0; a^2)$

$$\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}} \right] = \mathbb{E}^{\mathbb{P}}\left[e^{-\frac{1}{2} l^2 a^2 - l Z} \right]$$

$$= e^{-\frac{1}{2} l^2 a^2} \, \mathbb{E}^{\mathbb{P}}\left[e^{-l Z} \right]$$

$$= e^{-\frac{1}{2} l^2 a^2} \, e^{\frac{1}{2} l^2 a^2} = 1$$

[in general if $X \overset{\mathbb{P}}{\sim} \mathcal{N}(m; v)$

$$\mathbb{E}^{\mathbb{P}}\left[e^{u X} \right] = e^{m u + \frac{1}{2} u^2 v}$$]

¿ What is Z in terms $\mathbb{P}^*$?

we will compute its $\mathbb{P}^*$ - distribution...

$$\mathbb{E}^{\mathbb{P}^*}\left[e^{u Z} \right] = \mathbb{E}^{\mathbb{P}}\left[e^{u Z} \frac{d\mathbb{P}^*}{d\mathbb{P}} \right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{u \underline{Z}} \, e^{-\frac{1}{2} l^2 a^2 - l \underline{Z}} \right]$$

$$= e^{-\frac{1}{2} l^2 a^2} \, \mathbb{E}^{\mathbb{P}}\left[e^{(u - l) Z} \right]$$

$$= e \quad \mathbb{E}[\, e \,]$$

$$= e^{-\frac{1}{2} l^2 a^2} \quad e^{\frac{1}{2}(u-l)^2 a^2}$$

$$= e^{\cancel{-\frac{1}{2} l^2} a^2 + \frac{1}{2} u^2 a^2 + \cancel{\frac{1}{2} l^2} a^2 - u l a^2}$$

$$= e^{-u l a^2 + \frac{1}{2} u^2 a^2}$$

$$\Rightarrow \quad Z \overset{\mathbb{P}^*}{\sim} \mathcal{N}(-l a^2 \,;\, a^2)$$

$$\& \qquad \overset{\mathbb{P}}{\sim} \mathcal{N}(0 \,;\, a^2)$$

define $Z^* := Z + l a^2 \overset{\mathbb{P}^*}{\sim} \mathcal{N}(0 \,;\, a^2)$

Girsanov's Theorem:

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ on which we have a $\mathbb{P}$-B.mtn $W = (W_t)_{t\geq 0}$.

Define a Radon-Nikodym derivative

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} := e^{-\frac{1}{2}\int_0^T \ell_s^2\, ds - \int_0^T \ell_s\, dW_s}.$$

[$\ell_s = \text{const} \rightarrow e^{-\frac{1}{2}\ell^2 T - \ell W_T}$ $\quad Z := W_T \overset{\mathbb{P}}{\sim} N(0; T)$]

Then $W^* = (W_t^*)_{t\geq 0}$,

$$W_t^* := W_t + \int_0^t \ell_s\, ds \quad \text{is a } \mathbb{P}^*\text{-B.mtn.}$$

When $\ell_t = \lambda_t = \dfrac{\mu(t, X_t) - r}{\sigma(t, X_t)}$

$\mathbb{P}^*$ is the martingale measure that allows us to write

$$g(t,x) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-r(T-t)} G(X_T) \mid X_t = x \right]$$

$$\begin{aligned} dX_t &= X_t\, \mu(t, X_t)\, dt + X_t\, \sigma(t, X_t)\, dW_t \\ &= X_t\, r\, dt + X_t\, \sigma(t, X_t)\left(dW_t + \lambda_t\, dt\right) \\ &= X_t\, r\, dt + X_t\, \sigma(t, X_t)\, dW_t^* \end{aligned}$$

$$g_t = g(t, X_t) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-r(T-t)} g_T \mid \mathcal{F}_t \right]$$

$$g_t = g(t, X_t) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-r(T-t)} g_T \mid \mathcal{F}_t \right]$$

$$\Rightarrow \quad \frac{g_t}{B_t} = \mathbb{E}^{\mathbb{P}^*}\left[\frac{g_T}{B_T} \mid \mathcal{F}_t \right]$$

<u>FTAP</u>: the market admits no arbitrage

$$\Leftrightarrow$$

$$\exists\, \mathbb{P}^* \sim \mathbb{P} \quad \text{s.t.} \quad \mathbb{E}^{\mathbb{P}^*}[\hat{g}_u \mid \mathcal{F}_t] = \hat{g}_t \quad \forall\ 0 \le t < u \le T$$

$$\hat{g} = (\hat{g}_t)_{t \ge 0}, \quad \hat{g}_t = \frac{g_t}{B_t} \quad \forall \text{ all claims } g_T.$$

$$\mathbb{E}[A \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[A \mid \mathcal{F}_s] \mid \mathcal{F}_t] \qquad \mathcal{F}_t \subseteq \mathcal{F}_s$$

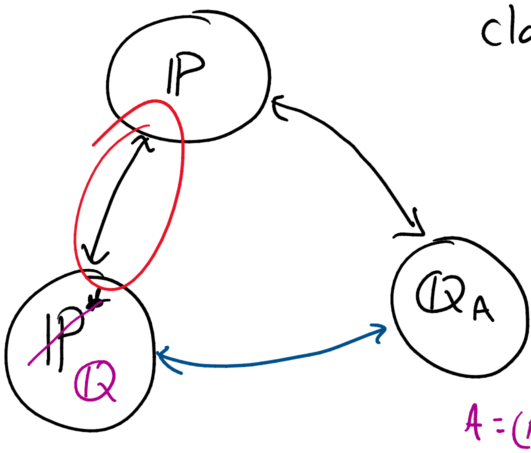
i.e. $s \ge t$

$$\hat{g}_u = \mathbb{E}^{\mathbb{P}^*}[\hat{g}_T \mid \mathcal{F}_u]$$

$$\Rightarrow \mathbb{E}^{\mathbb{P}^*}[\hat{g}_u \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}^*}[\mathbb{E}^{\mathbb{P}^*}[\hat{g}_T \mid \mathcal{F}_u] \mid \mathcal{F}_t]$$

$$= \mathbb{E}^{\mathbb{P}^*}[\hat{g}_T \mid \mathcal{F}_t] = \hat{g}_t$$

Measure Change to another Numeraire



bank account as numeraire asset

$A:=(A_t)_{t\geq 0}$ as numeraire asset ($A_t > 0$ a.s. $\forall t$)

claim $\dfrac{dQ_A}{dQ} = \dfrac{A_T/A_0}{B_T/B_0}$

we have that

$$\hat{g}_t = \mathbb{E}^{Q}[\hat{g}_u \mid \mathcal{F}_t] \quad \forall\, 0\leq t\leq u\leq T$$

hopefully!

$$\frac{g_t}{A_t} = \mathbb{E}^{Q_A}\left[\frac{g_u}{A_u} \,\middle|\, \mathcal{F}_t\right]$$

$$\mathbb{E}^{Q_A}\left[\frac{g_u}{A_u} \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}^{Q}\left[\frac{g_u}{A_u}\cdot\frac{dQ_A}{dQ} \,\middle|\, \mathcal{F}_t\right] \Big/ \mathbb{E}^{Q}\left[\frac{dQ_A}{dQ} \,\middle|\, \mathcal{F}_t\right]$$

$$\mathbb{E}^{P^*}[G \mid \mathcal{F}_t] = \frac{\mathbb{E}^{P}\left[G\,\frac{dP^*}{dP} \,\middle|\, \mathcal{F}_t\right]}{\mathbb{E}^{P}\left[\frac{dP^*}{dP} \,\middle|\, \mathcal{F}_t\right]}$$

$$= \underbrace{\mathbb{E}^{Q}\left[\frac{g_u}{A_u}\,\frac{A_T}{B_T} \,\middle|\, \mathcal{F}_t\right]} \Big/ \underbrace{\mathbb{E}^{Q}\left[\frac{A_T}{B_T} \,\middle|\, \mathcal{F}_t\right]}_{\frac{A_t}{B_t} \text{ as } \frac{A}{B} \text{ is a } Q\text{-mtg.}}$$

$$= \mathbb{E}^{Q}\left[\mathbb{E}^{Q}\left[\left(\frac{g_u}{A_u}\right)\frac{A_T}{B_T} \,\middle|\, \mathcal{F}_u\right] \,\middle|\, \mathcal{F}_t\right]$$

$$= \mathbb{E}^{Q}\left[\frac{g_u}{A_u}\cdot\underbrace{\mathbb{E}^{Q}\left[\frac{A_T}{B_T} \,\middle|\, \mathcal{F}_u\right]}_{\frac{A_u}{B_u} \text{ as } \frac{A}{B} \text{ is a } Q\text{-mtg}} \,\middle|\, \mathcal{F}_t\right]$$

$$\frac{A_u}{B_u} \quad \text{as } \frac{A}{B} \text{ is a } \mathbb{Q}\text{-mtg}$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{g_u}{B_u} \,\middle|\, \mathcal{F}_t \right] = \frac{g_t}{B_t} \quad \text{as } \frac{g}{B} \text{ is a } \mathbb{Q}\text{-mtg}$$

$$= \frac{g_t}{B_t} \Big/ \frac{A_t}{B_t} = \frac{g_t}{A_t}$$

B-S : $(X_t)_{t\geq 0}$ satisfies SDE

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

$(W_t)_{t\geq 0}$ is $\mathbb{P}$-B.motion.

$$\frac{g_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{g_T}{B_T} \mid \mathcal{F}_t\right]$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{1}{2}\lambda^2 T - \lambda W_T}$$

where $\lambda = \frac{\mu - r}{\sigma}$

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \lambda_s ds$$

$$= W_t + \lambda t \quad \Rightarrow \quad dW_t^{\mathbb{Q}} = dW_t + \lambda dt$$

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

$$= \mu X_t dt + \sigma X_t \left(dW_t^{\mathbb{Q}} - \frac{\mu - r}{\sigma} dt\right)$$

$$= r X_t dt + \sigma X_t dW_t^{\mathbb{Q}}$$

let's take $G(x) = \mathbb{1}_{x > K}$, i.e.

$$g_T = \mathbb{1}_{X_T > K} \quad \text{digital call option}$$

$$\frac{g_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{g_T}{B_T} \mid \mathcal{F}_t \right]$$

$$\Rightarrow \quad g_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{X_T > K} \mid \mathcal{F}_t]$$

$$= e^{-r(T-t)} \mathbb{Q}(X_T > K \mid \mathcal{F}_t)$$

Find $\mathbb{Q}$-distribution of $\log\left(\frac{X_T}{X_t}\right) \Big| \mathcal{F}_t$

Hint: solve the SDE:

$$dX_t = rX_t\, dt + \underline{\underline{\sigma X_t}}\, dW_t^{\mathbb{Q}}$$

even simpler find $\log X_t =: Y_t = f(X_t)$

$f(x) = \log(x)$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$

$$dY_t = f'(X_t)\, dX_t + \tfrac{1}{2} f''(X_t)\, d[X,X]_t$$

$$= \frac{1}{X_t} dX_t - \frac{1}{2} \frac{1}{X_t^2} \underbrace{d[X,X]_t}_{\sigma^2 X_t^2\, dt}$$

$$= r\, dt + \sigma\, dW_t^{\mathbb{Q}} - \tfrac{1}{2}\sigma^2\, dt$$

$$= (r - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dW_t^{\mathbb{Q}}$$

$\int_t^T \cdots = \int_t^T \cdots$

$$\Rightarrow \quad Y_T - Y_t = (r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma (W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}})$$

$$\Rightarrow \quad \underbrace{Y_T - Y_t}_{\log\left(\frac{X_T}{X_t}\right)} = (r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma \underbrace{(W_T^{Q} - W_t^{Q})}_{\stackrel{d}{=} \sqrt{T-t}\, Z}$$

$$Z \stackrel{Q}{\sim} N(0,1)$$

$$\therefore \log\left(\frac{X_T}{X_t}\right) \stackrel{d}{=} (r - \tfrac{1}{2}\sigma^2)\overbrace{(T-t)}^{\tau} + \sigma\sqrt{T-t}\, Z$$

$$\text{i.e. } \log\left(\frac{X_T}{X_t}\right) \stackrel{Q}{\sim} N\left((r - \tfrac{1}{2}\sigma^2)\tau;\ \sigma^2\tau\right)$$

$$g_t = e^{-r\tau} Q(X_T > K \mid \mathcal{F}_t)$$

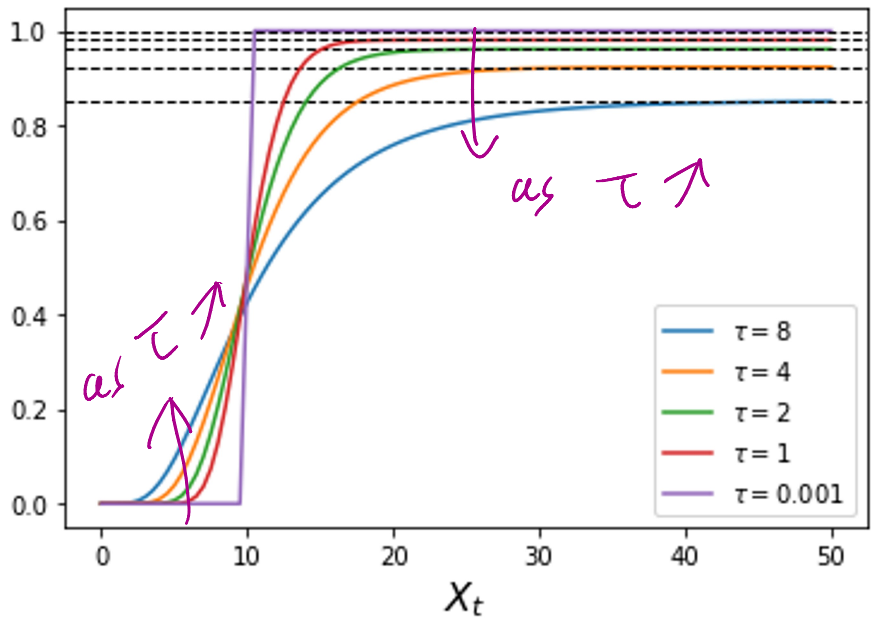
$$= e^{-r\tau} Q\left(\log\left(\frac{X_T}{X_t}\right) > \log\left(\frac{K}{X_t}\right) \mid \mathcal{F}_t\right)$$

$$= e^{-r\tau} Q\left((r - \tfrac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\, Z > \log\left(\frac{K}{X_t}\right) \mid \mathcal{F}_t\right)$$

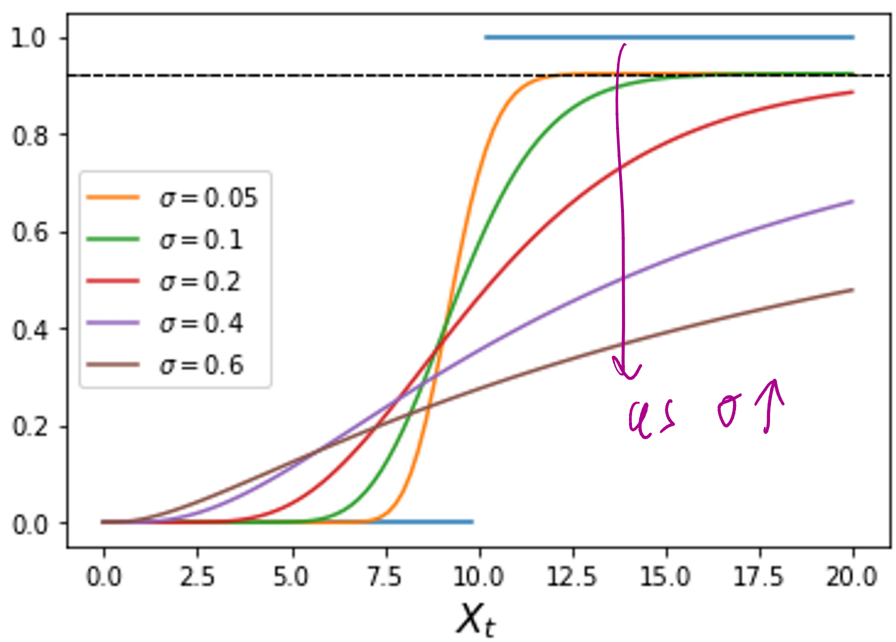
$$= e^{-r\tau} Q\left(Z > \frac{\log\left(\frac{K}{X_t}\right) - (r - \tfrac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \mid \mathcal{F}_t\right)$$

$$= e^{-r\tau} Q\left(Z < \frac{\log(X_t/K) + (r - \tfrac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \mid \mathcal{F}_t\right)$$

$$\boxed{g_t = e^{-r\tau}\, \Phi\left(\frac{\log(X_t/K) + (r - \tfrac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)}$$



1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50
X_t
$\tau = 8$
$\tau = 4$
$\tau = 2$
$\tau = 1$
$\tau = 0.001$
as $\tau \uparrow$
as $\tau \uparrow$



1.0
0.8
0.6
0.4
0.2
0.0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
X_t
$\sigma = 0.05$
$\sigma = 0.1$
$\sigma = 0.2$
$\sigma = 0.4$
$\sigma = 0.6$
as $\sigma \uparrow$

take $G(x) = (x-K)_+$, i.e. a call option

so $g_T = (X_T - K)_+$

$$g_t = e^{-r\tau}\, \mathbb{E}^{\mathbb{Q}}\left[(X_T - K)_+ \mid \mathcal{F}_t \right]$$

$$\log\left(\frac{X_T}{X_t}\right) \overset{d}{=} (r - \tfrac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\, Z, \quad Z \overset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$X_T \overset{d}{=} X_t\, e^{\overbrace{(r-\frac{1}{2}\sigma^2)}^{\alpha}\tau + \sigma\sqrt{\tau}\, Z}$$

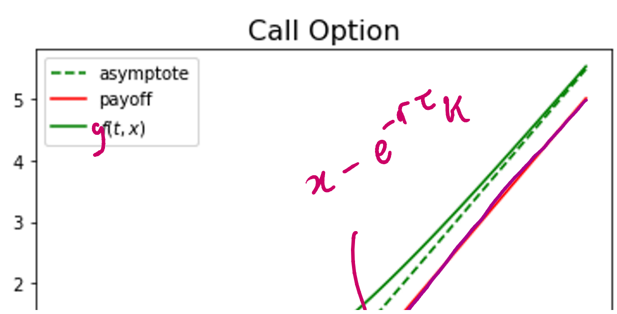
$$= e^{-r\tau}\, \mathbb{E}^{\mathbb{Q}}\left[(X_t\, e^{\alpha\tau + \sigma\sqrt{\tau}\, Z} - K)_+ \mid \mathcal{F}_t \right]$$

$$= e^{-r\tau} \int_{-\infty}^{\infty} (X_t\, e^{\alpha\tau + \sigma\sqrt{\tau}\, z} - K)_+ \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\, dz$$

$\vdots$

$$\boxed{\begin{aligned} g_t &= X_t\, \Phi(d_+) - K e^{-r\tau}\, \Phi(d_-) \\ d_\pm &= \frac{\log(X_t/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \end{aligned}}$$

BS formula for a call option

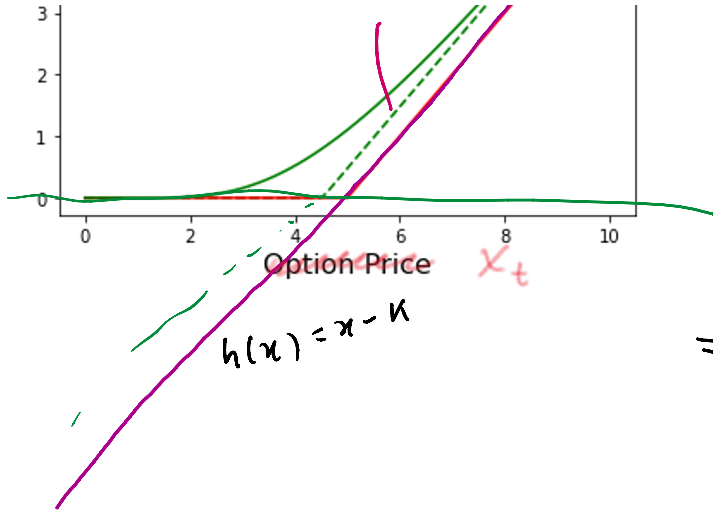


$$g(T,x) \ge \overset{\overset{x-K}{\shortparallel}}{h(x)}$$

$\hookrightarrow h(t,x)$

$\Downarrow$

$g \ge h$ ...



$$\Downarrow \quad g_T \ge h_T \quad \text{a.s.}$$

$$\Rightarrow \quad e^{-r(T-t)} g_T \ge e^{-r(T-t)} h_T$$

$$\Rightarrow \quad \mathbb{E}^{Q}[e^{-r(T-t)} g_T \mid \mathcal{F}_t] \ge \mathbb{E}^{Q}[e^{-r(T-t)} h_T \mid \mathcal{F}_t]$$

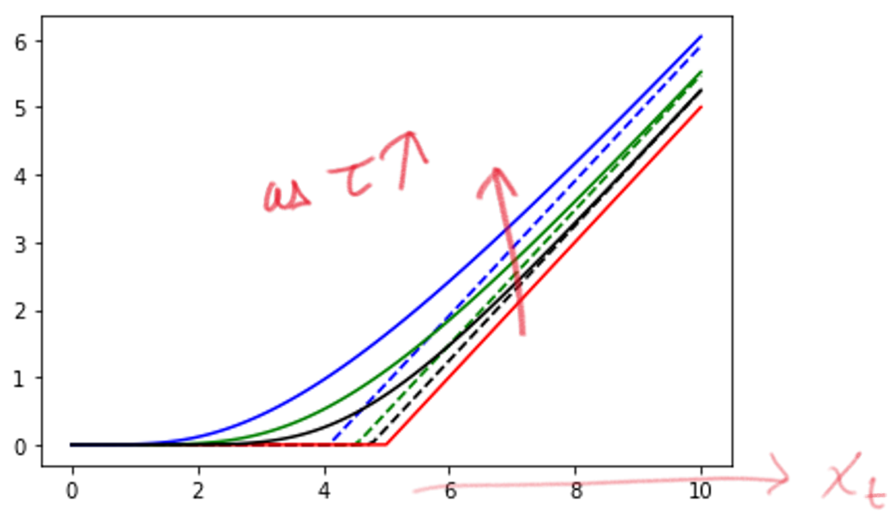
$$\Rightarrow \quad g_t \ge h_t \quad \text{a.s.}$$

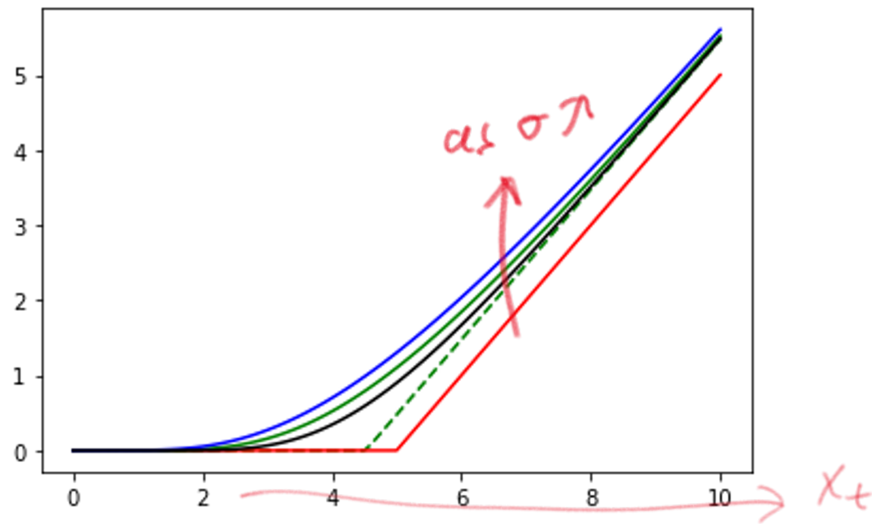
$$g(t,x) \ge h(t,x)$$

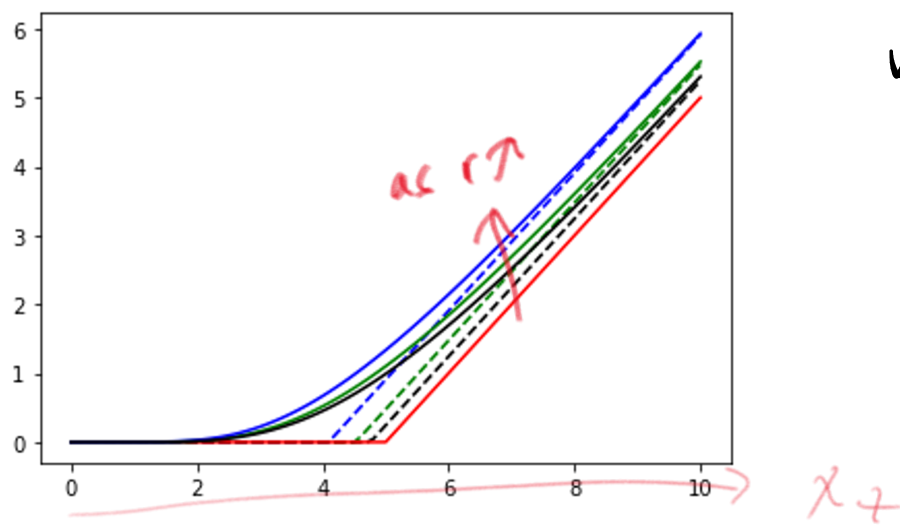
$$h_t = \mathbb{E}^{Q}[e^{-r(T-t)}(X_T - K) \mid \mathcal{F}_t]$$

$$= \underbrace{\mathbb{E}^{Q}[e^{-r\tau} X_T \mid \mathcal{F}_t]}_{= X_t \text{ by mtg-ppty.}} - K e^{-r\tau}$$

$$h(t,x) = x - K e^{-r\tau} \quad \left(= e^{-r\tau}(x e^{r\tau} - K) \right)$$







$h(t,x) = x - K e^{-r\tau}$

Deriving BS Call Option Price

$\mathbb{P}$ - model $\quad dX_t = \mu X_t\,dt + \sigma X_t\,dW_t$

$\mathbb{Q}$ - model $\quad\quad\quad = r X_t\,dt + \sigma X_t\,dW_t^{\mathbb{Q}}$

$$\frac{g_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{g_T}{B_T} \,\middle|\, \mathcal{F}_t\right],$$

& $g_T = (X_T - K)_+ = (X_T - K)\,\mathbb{1}_{X_T > K}$

$$= \underbrace{X_T\,\mathbb{1}_{X_T > K}}_{\substack{H_T \\ \text{is an} \\ \text{asset-or-nothing} \\ \text{option payoff}}} - \underbrace{K\,\mathbb{1}_{X_T > K}}_{\substack{P_T \text{ is a} \\ \text{digital call} \\ \text{option payoff}}}$$

$$p_t = K e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{X_T > K} \mid \mathcal{F}_t]$$

$$= K e^{-r\tau}\,\mathbb{Q}(X_T > K \mid \mathcal{F}_t) = K e^{-r\tau}\,\Phi(d_-)$$

$$h_t = e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}[X_T\,\mathbb{1}_{X_T > K} \mid \mathcal{F}_t]$$

also:

$$\frac{h_t}{X_t} = \mathbb{E}^{\mathbb{Q}_X}\left[\frac{h_T}{X_T} \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_X}\left[\frac{X_T\,\mathbb{1}_{X_T > K}}{X_T} \,\middle|\, \mathcal{F}_t\right]$$

$$= \mathbb{E}^{\mathbb{Q}_X}[\mathbb{1}_{X_T > K} \mid \mathcal{F}_t]$$

$$= \mathbb{Q}_X (X_T > K \mid \mathcal{F}_t)$$

$$\Rightarrow \quad h_t = X_t \, \mathbb{Q}_X (X_T > K \mid \mathcal{F}_t)$$

$$\boxed{g_t = X_t \, \mathbb{Q}_X (X_T > K \mid \mathcal{F}_t) - K e^{-r\tau} \, \mathbb{Q} (X_T > K \mid \mathcal{F}_t)}$$

need to find $\mathbb{Q}_X$ - distribution of X_T

recall: $$\frac{d\mathbb{Q}_X}{d\mathbb{Q}} = \frac{X_T / X_0}{B_T / B_0} = \frac{X_0 e^{(r - \frac{1}{2}\sigma^2) T + \sigma W_T^{\mathbb{Q}}} / X_0}{B_0 e^{rT} / B_0}$$

$$= e^{-\frac{1}{2}\sigma^2 T + \sigma W_T^{\mathbb{Q}}} \quad = e^{-\frac{1}{2}\int_0^T \lambda_s^2 ds - \int_0^T \lambda_s dW_s^{\mathbb{Q}}}$$

from Girsanov's Theorem

$$W_t^{\mathbb{Q}_X} = W_t^{\mathbb{Q}} - \int_0^t \sigma \, ds = W_t^{\mathbb{Q}} - \sigma t$$

is a $\mathbb{Q}_X$ - B.mtn.

$$dX_t = r X_t \, dt + \sigma X_t \, dW_t^{\mathbb{Q}}$$

$$= r X_t \, dt + \sigma X_t (dW_t^{\mathbb{Q}_X} + \sigma \, dt)$$

$$= (r + \sigma^2) X_t \, dt + \sigma X_t \, dW_t^{\mathbb{Q}_X}$$

$$\Rightarrow \quad \log\left(\frac{X_T}{X_t}\right) = (r + \tfrac{1}{2}\sigma^2)\tau + \sigma \underbrace{(W_T^{\mathbb{Q}_X} - W_t^{\mathbb{Q}_X})}_{\stackrel{d}{=} \sqrt{\tau} Z}, \quad Z \stackrel{\mathbb{Q}_X}{\sim} N(0,1)$$

$\therefore$ (using old result)

$\therefore$ (using old result)

$$\mathbb{Q}_X(X_T > K \mid \mathcal{F}_t) = \Phi\left(\frac{\log(X_t/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right)$$

$$\therefore \quad \boxed{g_t = X_t \, \Phi(d_+) - K e^{-r\tau} \, \Phi(d_-)}$$