Quantum Groups in Mathematical Finance

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Abstract

Mathematical finance explores the consistency relationships between the prices of securities imposed by elementary economic principles. Commonplace among these are the absence of arbitrage and the equivalence of expectation and price, both essentially algebraic constraints on the valuation map. The principles that govern pricing are here reviewed in the context of the stochastic and functional calculus of quantum processes. Framed in terms of the duality between *states*, the arbitrage-free valuation maps, and *observables*, the contractual settlements of securities, quantum groups are central to the approach. Translating the economic principles into this framework, a link is made between option pricing and von Neumann algebras that is illuminating in both directions. The essay concludes with the construction of interest rate models from the irreducible representations of semisimple Lie algebras, demonstrating their application in the pricing of European and Bermudan swaptions.

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1 Introduction

The data associated with a system is presented in this essay as a pair of complementary $*$-algebras, comprising the test states and observables that are combined in the investigation of the system. The state space M and the observable space W are paired by a bilinear map:

$$\bullet : \mathsf{M} \times \mathsf{W} \to \mathbb{C} \tag{1}$$

that sends the test state $\mathsf{z} \in \mathsf{M}$ and the test observable $\mathsf{a} \in \mathsf{W}$ to their valuation $\mathsf{z} \bullet \mathsf{a} \in \mathbb{C}$. With the emphasis on empirical determination, these spaces are completed as locally convex topological spaces that include all the experiments uniquely identified by their test valuations.

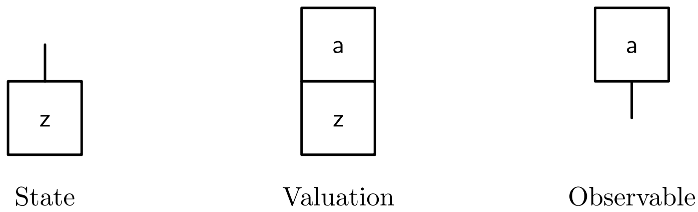
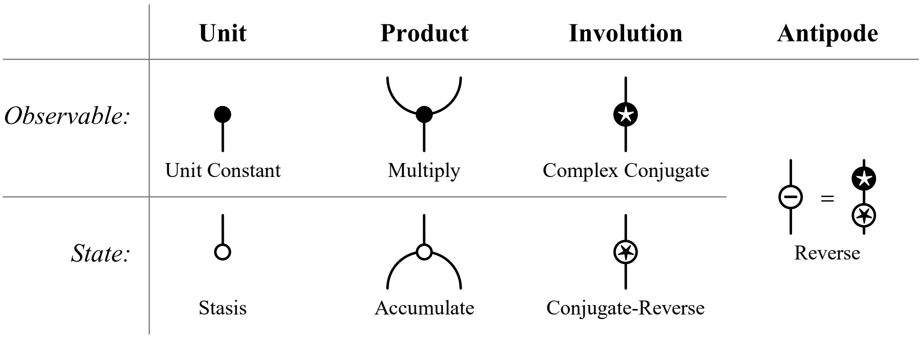


Figure 1: Visualise the pairing $\mathsf{z} \bullet \mathsf{a}$ of the state z with the observable a as the stacking of two boxes, where the lower box represents the state and the upper box represents the observable. The dual interpretation of the information model is generated by reversing this diagram.

While the definition is symmetric between these dual spaces, their naming indicates the distinct roles they play in the model of information. On one side, the $*$-algebra models the accumulation of states, and is a prerequisite for the development of the stochastic calculus. On the opposite side, the $*$-algebra models the multiplication of observables, and is a prerequisite for the development of the functional calculus. The operations of the $*$-algebra – the unit, product and involution – are then interpreted according to their application.



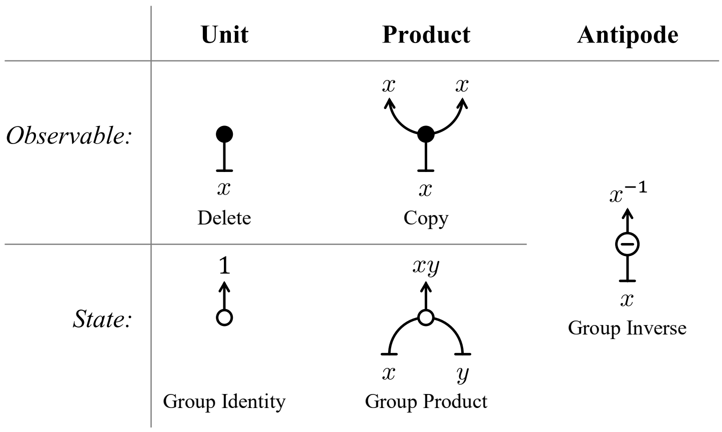
These are the building blocks for the string diagrams of valuations, symbolising operations that act upwards on states and downwards on observables. Duality is the statement that these interpretations can be reversed, furnishing two inequivalent information models from the same pairing of $*$-algebras.

In the application to finance, the economy is empirically investigated by price testing: *state* is the economic model used to assess the present value of future cashflows and *observable* is the contractual termsheet that prescribes the settlement of the derivative security. Reducing the economic data to a single underlying price, the Black-Scholes model combines log-Brownian diffusion with the terminal payoff of the European call option. Extensions of this seminal model follow the same pattern, adding complexity in the variables of the economic state and the terms of the settlement observable. In each case, the state generates the equilibrium equation satisfied by the derivative price and the observable provides its boundary conditions, completing the specification of the price problem. Stochastic calculus accommodates the evolution of the economy over time and functional calculus supports the convex relationships among derivative payoffs. Understanding the duality between these two structures, and how this is utilised in pricing, is the main objective of this essay.

The prototypical example is the classical group, wherein test states are elements of the group and test observables are subsets of the group, respectively identified with economic events and the Arrow-Debreu securities contingent on them. Abstracting this structure, the rules of the quantum group extract the essential properties of classical groups that enable the stochastic and functional calculus. Applied to economies, founding principles can then be developed as axioms within a model of information established on purely algebraic grounds, encompassing the elementary economic assumptions that govern pricing such as the absence of arbitrage and the equivalence of expectation and price.

Classical stochastic calculus

Taking the states to be elements of a classical group, the operations that facilitate the stochastic calculus are defined as the following maps on states:



where the unit state is the group identity, the product of two states is the group product, and the antipode of the state is the group inverse. In the finance application, the element of the group represents an economic event. The operations then enable the accumulation of events over time.

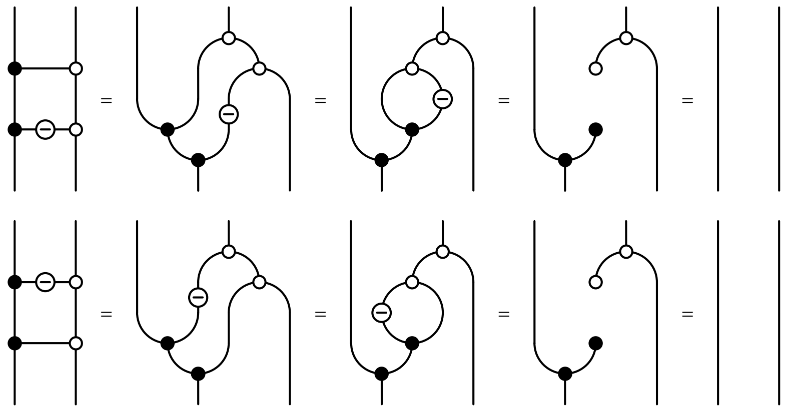
Stochastic calculus relies on the ability to switch between integral and differential perspectives for the evolution of the system. Over a discrete schedule, the evolution is equivalently described by its endpoints $(x_1, \ldots, x_n)$ or its increments $(y_1, \ldots, y_n)$, related by:

$$x_i = y_1 \ldots y_i \qquad\qquad y_i = x_{i-1}^{-1} x_i \tag{2}$$

for $i = 1, \ldots, n$, where the path is initialised with $x_0 = 1$. The first map integrates the increments to generate the endpoints and the second map differentiates the endpoints to generate the increments. The operations of the group define the integrator and differentiator maps:

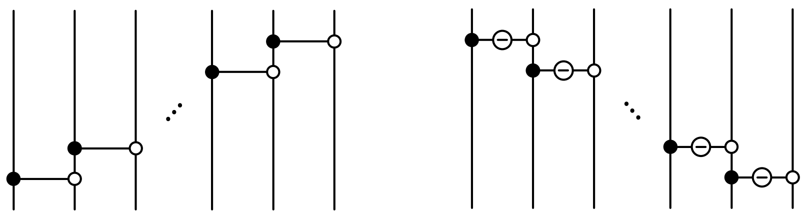
$(y_1, y_2) \mapsto (y_1, y_1 y_2)$ $\qquad$ $(x_1, x_2) \mapsto (x_1, x_1^{-1} x_2)$

These maps are inverse to each other, a property that is derived from the group axioms in the string diagrams:



Of the three steps in this derivation, the first exploits associativity and the last exploits unitality. The derivation is then completed by using the inverse property in the middle step.

Integration and differentiation intermediate between observables modelled on the endpoints and states modelled on the increments, repeated over multiple time steps as the operations:

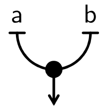


$(y_i) \mapsto (y_1 \dots y_i)$ $\qquad$ $(x_i) \mapsto (x_{i-1}^{-1} x_i)$

Valuation proceeds as the composition of operations that implement the settlement observable of the derivative security with the action of the pricing state that computes its present value, interchanging along the way between the endpoint and increment representations for the evolution.

Classical functional calculus

Taking the observables to be scalar-valued functions on a classical set, the operations that facilitate the functional calculus are defined as the following maps on observables:

	Unit	**Product**	**Involution**
Observable:	 1 Unit Constant	a b  ab Multiply	a  a^* Complex Conjugate

where the unit observable is the unit constant, the product of two observables is the pointwise product, and the involution of the observable is the pointwise complex conjugate. In the finance application, the scalar-valued function represents the value of a derivative security whose settlement is contingent on the events in the set. The operations then enable convex relationships between derivative payoffs.

Functional calculus expands the space of observables beyond polynomial combinations via topological closure, utilising a background state to measure convergence. Two remarkable theorems from functional analysis underpin this expansion: the Gelfand-Naimark-Segal construction represents the observables as operators; and the Radon-Nikodym theorem represents the states as operators that commute with the observables. Topological closure then generates the empirical states and observables respectively as the commutant W' and double-commutant W'' of the test observables W. The $*$-algebras of states and observables are in this way identified with dual von Neumann algebras acting on a Hilbert space.

These constructions have natural interpretation in mathematical finance, emerging from the relationship between expectation and price. Guiding economic principles relate the pricing state z_c, mapping observables to their present values in currency c, to the economic state z_e, mapping observables to their expected values. Both these states are positive maps on observables: positivity of the economic state ensures that the probabilities of events are positive, and positivity of the pricing state ensures that the state prices of events are positive. They are furthermore assumed to be equivalent, in the sense that the price of an Arrow-Debreu security is non-zero if and only if the probability of the event it indicates is non-zero. These properties of the pricing state are asserted in the following principles.

Principle of No-Arbitrage: The pricing state does not permit arbitrage.

$$\mathsf{z}_c \bullet \mathsf{aa}^* \geq 0 \tag{3}$$

Principle of Equivalence: The pricing and economic states are equivalent.

$$\mathsf{z}_c \bullet \mathsf{aa}^* \neq 0 \iff \mathsf{z}_e \bullet \mathsf{aa}^* \neq 0 \tag{4}$$

A stronger principle that applies equivalence uniformly across observables is considered in this essay:

$$m_-(\mathsf{z}_e \bullet \mathsf{aa}^*) \leq \mathsf{z}_c \bullet \mathsf{aa}^* \leq m_+(\mathsf{z}_e \bullet \mathsf{aa}^*) \tag{5}$$

for scalars m_- and m_+ satisfying $0 < m_- \leq m_+ < \infty$. State prices are thus confined within a finite range of the corresponding probabilities. Implicit in these expressions is a concept of positivity defined algebraically using only the product and involution: the observable b is positive if it can be written in the form $\mathsf{b} = \mathsf{aa}^*$ for an observable a.

The economic state creates an inner product on the observables a and b:

$$\langle \mathsf{a} | \mathsf{b} \rangle := \mathsf{z}_e \bullet \mathsf{ab}^* \tag{6}$$

and the observable b acts as an operator $[\mathsf{b}]$ on the Hilbert space completion of the resulting inner product space via multiplication on the right:

$$\langle \mathsf{a} | [\mathsf{b}] := \langle \mathsf{ab} | \tag{7}$$

These definitions, which are rigidified by removing degeneracies from the observables and completing the operators, are the content of the Gelfand-Naimark-Segal construction. Strong equivalence of the economic and pricing states then derives the Radon-Nikodym theorem:

$$\mathsf{z}_c \bullet \mathsf{a} = \langle 1|[\mathsf{z}_c][\mathsf{a}]|1\rangle \tag{8}$$

representing the state z_c as an operator $[\mathsf{z}_c]$ on the Hilbert space that commutes with all the observables:

$$[\mathsf{z}_c][\mathsf{a}] = [\mathsf{a}][\mathsf{z}_c] \tag{9}$$

In this picture, the economic state is uniquely identified by its representation as the identity operator, $[\mathsf{z}_e] = 1$. The operator $[\mathsf{z}_c]$ that represents the pricing state scales probabilities to Arrow-Debreu prices, and positivity of this operator is sufficient to ensure the absence of arbitrage.

With these economic principles, states and observables are generated from a von Neumann algebra. The economic model, assumed a priori, creates a Hilbert space via the Gelfand-Naimark-Segal construction. Empirical states and observables are then identified, thanks to the Radon-Nikodym theorem, with operators in the commutant and double-commutant of the test observables. Representing the economic and pricing states in this way enables operator methods in mathematical finance.

Quantum economics

None of the constructions in the stochastic and functional calculus require commutativity of the product for either states or observables, and the novel contribution of this presentation is in the extension to noncommutativity. The founding principles of mathematical finance effectively identify the theories of arbitrage-free pricing models and von Neumann algebras, an observation that is instructive for both disciplines as noncommutativity impacts the price of convexity in ways that characterise the algebra.

The bulk of this essay is devoted to the development of the quantum framework for mathematical finance, justifying the approach on the grounds that it both contains and extends the classical framework. That the extension presents novel and useful behaviour is easily demonstrated in the discrete model, in which the observable is represented as a self-adjoint complex matrix with a finite set of real eigenvalues. The classical variant assumes that all observables commute, and without loss of generality are represented as diagonal matrices with their eigenvalues on the diagonal. The quantum variant drops this commutativity condition. Since the observable is evaluated by its eigenvalues – the roots of its characteristic polynomial – the transition from classical to quantum introduces complexity and thereby enriches the phenomenology.

As a toy example, consider the binomial model of two underlying payoffs P and R that both have eigenvalues 0 and 1. The option to receive one unit of R in exchange for $k \geq 0$ units of P has price:

$$o[k] := \mathrm{tr}[(R - kP)^+] \tag{10}$$

in this model, computed as the sum of the positive eigenvalues of the bracketed matrix. The trace acts as the expectation operator, assuming for convenience that the weights of the pricing distribution are absorbed in the payoff matrices. Classicality imposes commutativity of the matrices, leaving only two possible cases. The first is:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{11}$$

for which $o[k] = (1-k)^+$, and the second is:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{12}$$

for which $o[k] = 1$. These are respectively the minimum and maximum possible prices for the options. The quantum model discovers prices between these bounds by rotating the diagonalising basis of R:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} \cos[\theta]^2 & \cos[\theta]\sin[\theta] \\ \cos[\theta]\sin[\theta] & \sin[\theta]^2 \end{bmatrix} \tag{13}$$

for angles in the range $0 \leq \theta \leq \pi/2$, leading to the option price:

$$o[k] = \frac{1}{2}(1-k) + \frac{1}{2}\sqrt{1 - 2k\cos[2\theta] + k^2} \tag{14}$$

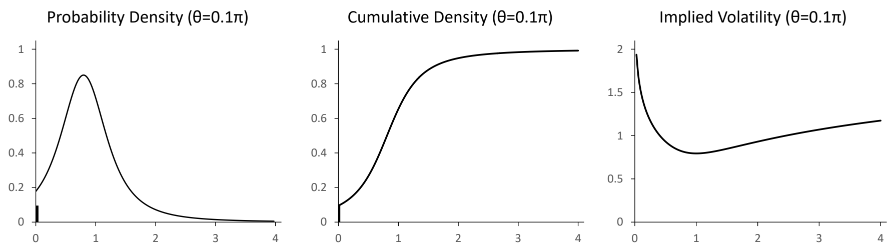
given by the only positive eigenvalue. The price is recreated as the expected value of the option expressed as a function of the swap rate s with the definition:

$$\int_{s=0}^{\infty} (s-k)^+ \,\mathrm{pdf}[s]\, ds := o[k] \tag{15}$$

for the implied probability density $\mathrm{pdf}[s]$, deriving the expression:

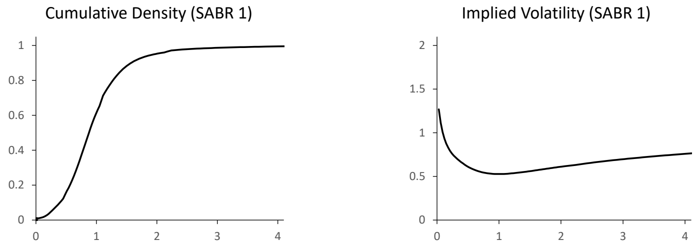
$$\mathrm{cdf}[s] = \frac{1}{2}\left(1 + \frac{s - \cos[2\theta]}{\sqrt{1 - 2s\cos[2\theta] + s^2}}\right) \tag{16}$$

for the implied cumulative density $\mathrm{cdf}[s]$ on the half line $0 \leq s < \infty$.



As these graphs show for the case $\theta = \pi/10$, the quantum extension magically creates an implied pricing distribution with continuous support from a model that has only two discrete eigenstates.

Increasing the number of eigenstates rapidly expands the range of implied pricing distributions generated by the quantum multinomial model. The examples on the next page calibrate the model with five eigenstates to four cases of the SABR model, and the graphs that follow are calibrations to the Black-Scholes



P Eigenvalues

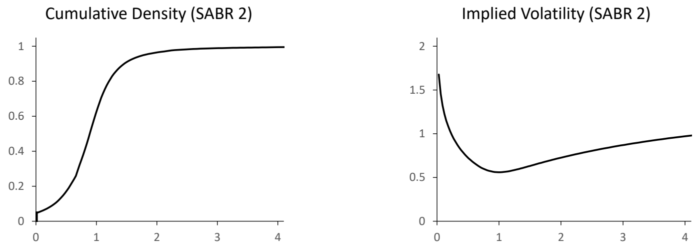
0.6635
0.2919
0.0305
0.0141
0.0000

R Eigenvalues

0.6647
0.2823
0.0389
0.0140
0.0001

R Rotation

-0.9105	-0.4028	-0.0558	-0.0744	-0.0069
-0.4012	0.8597	-0.0473	0.2987	-0.0922
0.0439	0.0775	-0.9464	-0.2629	0.1652
0.0900	-0.2840	-0.3139	0.7220	-0.5398
-0.0024	-0.1093	-0.0217	0.5611	0.8202



P Eigenvalues

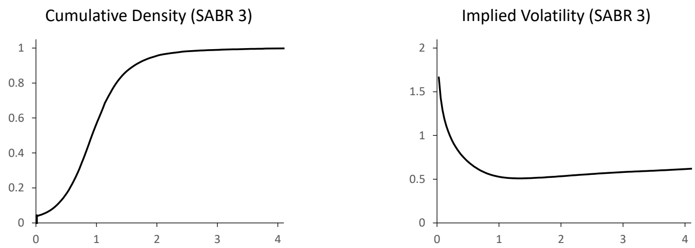
0.6952
0.2997
0.0046
0.0006
0.0000

R Eigenvalues

0.7059
0.2763
0.0149
0.0029
0.0000

R Rotation

-0.9332	-0.3439	-0.0564	-0.0098	-0.0872
-0.3455	0.8604	-0.0590	-0.1065	0.3542
0.0422	0.0622	-0.9118	0.3753	-0.1489
0.0894	-0.2850	-0.3904	-0.7129	0.5002
0.0033	-0.2373	0.0979	0.5827	0.7711



P Eigenvalues

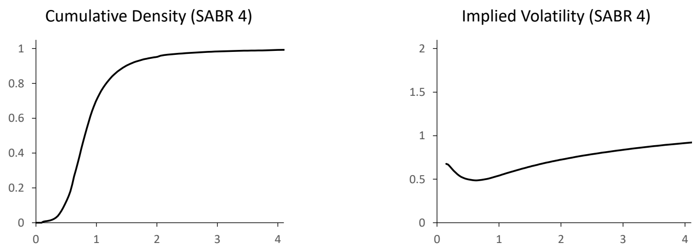
0.8820
0.0878
0.0229
0.0073
0.0000

R Eigenvalues

0.8880
0.0927
0.0193
0.0003
0.0000

R Rotation

-0.9706	-0.2080	-0.0492	-0.0049	-0.1106
-0.2391	0.8456	0.0874	0.0210	0.4688
0.0239	0.2314	-0.9286	0.1621	-0.2393
0.0129	-0.4191	-0.3407	-0.1276	0.8318
-0.0020	-0.1122	0.1073	0.9783	0.1375



P Eigenvalues

0.6999
0.2803
0.0146
0.0051
0.0000

R Eigenvalues

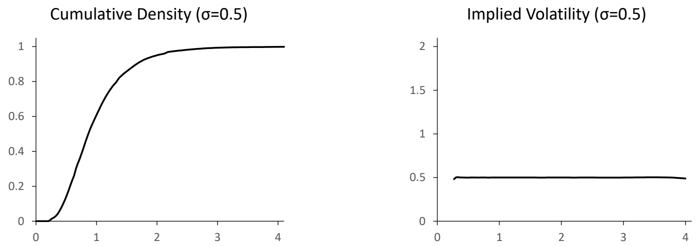
0.6388
0.2976
0.0401
0.0228
0.0000

R Rotation

-0.9226	-0.3758	-0.0545	-0.0678	-0.0088
-0.3719	0.8740	-0.0735	0.2886	-0.0959
0.0488	0.0722	-0.9237	-0.3413	0.1505
0.0903	-0.2782	-0.3602	0.6774	-0.5709
0.0005	-0.1113	-0.0926	0.5804	0.8014

Figure 2: The implied cumulative density and (non-annualised) lognormal volatility in the quantum multinomial model with five eigenstates calibrated to four examples of the SABR model with parameters $(t, \sigma, \alpha, \beta, \rho)$ given by $(4, 0.25, 0.4, 1, 0)$, $(4, 0.25, 0.6, 1, 0)$, $(4, 0.25, 0.4, 0.5, 0)$, $(4, 0.25, 0.4, 1, 0.5)$.

model. These examples demonstrate the novel and useful phenomenology unlocked by admitting noncommutativity in the information model.

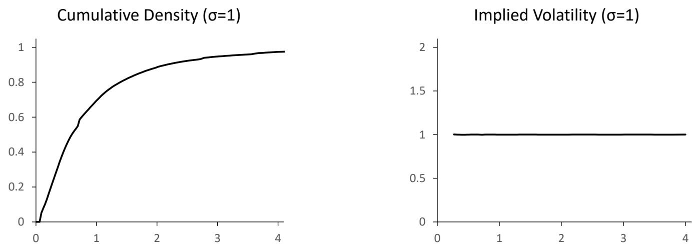


P Eigenvalues
0.5738
0.3055
0.1076
0.0123
0.0009

R Eigenvalues
0.6544
0.1767
0.1519
0.0157
0.0014

R Rotation

-0.9760	-0.0765	-0.1701	-0.0987	-0.0530
0.0317	-0.9331	0.3290	-0.1358	-0.0413
0.1828	-0.3268	-0.9248	0.0580	-0.0344
-0.1060	-0.1193	0.0817	0.9833	0.0323
-0.0408	-0.0502	-0.0300	-0.0407	0.9966



P Eigenvalues
0.4986
0.4464
0.0512
0.0035
0.0003

R Eigenvalues
0.7239
0.1523
0.1148
0.0076
0.0013

R Rotation

-0.9324	-0.0145	-0.3284	-0.1130	-0.0990
0.0276	-0.9623	0.0777	-0.2355	-0.1081
0.3243	-0.0847	-0.9349	0.1015	-0.0568
-0.1532	-0.2531	0.0550	0.9216	0.2451
-0.0348	-0.0504	-0.0948	-0.2685	0.9567

All the components of the information model are located in the complementary $*$-algebraic structures of states and observables and their empirical pairing. In the following, these components are woven into a coherent whole in the definition of the quantum group, with duality as the unifying principle.

2 The information category

At its most abstract, the information model resides within a symmetric monoidal category whose objects represent systems and whose morphisms represent the processes that act between them. The category offers a convenient mathematical shorthand for the combinatorics of process building, capturing the commonalities of systems across many contexts. It is beyond the scope of this essay to explore their coherence rules in detail; instead, a diagrammatic language is presented, noting that sensible transformations performed on string diagrams identify with well-defined relations within the category.

In this setting, the principal lesson of category theory is that systems are best understood in terms of the algebraic properties of their connecting processes. As a minimal base for empirical investigation, the system is here associated with a collection of experiments, decomposing as the pairing of state and observable, whose measurements are the only source of knowledge regarding the system. This framework supports all the algebraic operations needed in the developments that follow, and includes within its dominion important dynamical systems such as statistical and quantum mechanics.

2.1 Process building

The binary operands of the symmetric monoidal category, *concatenation* $\otimes$ and *composition* $\circ$, are respectively the parallel and serial combination of processes, and the rules they adhere to mirror the properties of string diagrams. This is sufficient to support a model that includes a natural notion of state and observable along with a consistent definition of valuation.

The systems K and L are concatenated in the combined system $K \otimes L$, a method of system construction that is assumed to be unital and associative up to natural isomorphisms:

$$\begin{gathered} 1 \otimes K \cong K \cong K \otimes 1 \\ (K \otimes L) \otimes M \cong K \otimes (L \otimes M) \end{gathered} \tag{17}$$

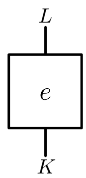
where 1 is the empty system, the monoidal unit of the category. The identifications in these expressions are implemented by the unitor and associator natural isomorphisms of the category, which for notational convenience are omitted in the following where they are understood from the context.

The variant of the category considered in this essay assumes that the hom-set of processes between the systems K and L decomposes as the (not necessarily disjoint) union of hom-subsets:

$$\mathrm{Hom}[K, L] = \bigcup\nolimits_{\alpha \in \Gamma} \mathrm{Hom}^{\alpha}[K, L] \tag{18}$$

parametrised by a signature group Γ associated with the category. In this decomposition, the process e in the hom-set $\mathrm{Hom}^{\alpha}[K, L]$ is tagged with domain $\mathrm{dom}[e] = K$, codomain $\mathrm{cod}[e] = L$ and signature $\mathrm{sig}[e] = \alpha$. This metadata determines the compatibility of processes for concatenation and composition, the two methods of process building in the category.

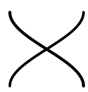
Introducing the diagrammatic language, the process e with signature α between the systems K and L is visualised as a box:



$$e \in \mathrm{Hom}^{\alpha}[K, L] \tag{19}$$

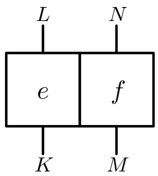
Systems are represented in the diagram as legs attached to the box; by convention, the leg is omitted when it represents the empty system. The primitive processes of the symmetric monoidal category are the *identity* $\imath$ and *braid* τ, which enable the rearrangement of concatenated systems:

Identity **Braid**



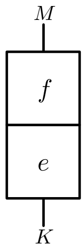
$\imath \in \mathrm{Hom}^{1}[K, K]$ $\tau \in \mathrm{Hom}^{1}[K \otimes L, L \otimes K]$

Concatenation is the parallel combination of compatible processes, visualised in string diagrams as the horizontal stacking of boxes:



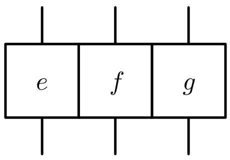
$$e \in \mathrm{Hom}^{\alpha}[K, L], f \in \mathrm{Hom}^{\alpha}[M, N] \mapsto e \otimes f \in \mathrm{Hom}^{\alpha}[K \otimes M, L \otimes N] \tag{20}$$

Composition is the serial combination of compatible processes, visualised in string diagrams as the vertical stacking of boxes:

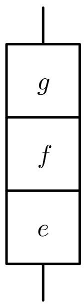


$$e \in \mathrm{Hom}^{\alpha}[K, L], f \in \mathrm{Hom}^{\beta}[L, M] \mapsto e \circ f \in \mathrm{Hom}^{\alpha\beta}[K, M] \tag{21}$$

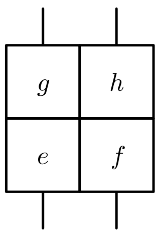
Well-formed diagrams built from these combinations are determined, up to the natural isomorphisms of the category, according to the associativity rules of concatenation and composition and the interchange rule between them:



$(e \otimes f) \otimes g = e \otimes (f \otimes g)$



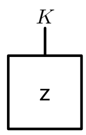
$(e \circ f) \circ g = e \circ (f \circ g)$



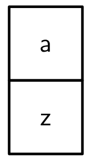
$(e \otimes f) \circ (g \otimes h) = (e \circ g) \otimes (f \circ h)$

These stacking rules, encoding the universal characteristics of process building, are validated by the coherence axioms of the category. Combinations of the identity and braid generate arbitrary permutations of concatenated systems. Beyond these primitive processes, additional processes in the hom-sets express structural properties of the systems they connect.

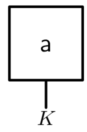
Completing the diagrammatic language of the category, states and observables are identified respectively as the processes from and to the empty system:



$\mathsf{z} \in \mathrm{Hom}^{\alpha}[1, K]$

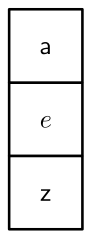


$\mathsf{z} \circ \mathsf{a} \in \mathrm{Hom}^{\alpha\beta}[1, 1]$



$\mathsf{a} \in \mathrm{Hom}^{\beta}[K, 1]$

With these identifications, valuation becomes the composition of state and observable, interpreting the hom-set $\mathrm{Hom}[1, 1]$ as the scalars for the information model. Processes can then be interpreted as maps on states (the Schrödinger picture) or as maps on observables (the Heisenberg picture). These interpretations are consistent with valuation thanks to the associativity of composition:



$$(\mathsf{z} \circ e) \circ \mathsf{a} = \mathsf{z} \circ (e \circ \mathsf{a}) \tag{22}$$

Any monoidal category generates in this way two models of information: one obtained by reading diagrams upwards, and one obtained by reading diagrams downwards. These dual interpretations exchange the roles of state and observable but evaluate within the same set of scalars. Additional processes in the hom-sets then dictate the nature of the scalars and the method of evaluation for states and observables.

2.2 Empirical systems and processes

In proposing the category used throughout this essay as the locale for the information model, the foremost consideration is its empirical interpretation. The system is represented as a pair of vector spaces, the test states and observables, combined in experiments whose measurements are given by their pairing. With this setup, a process is equivalently defined by a semilinear map between the state spaces or the observable spaces of its domain and codomain systems, imposing consistency of these Schrödinger and Heisenberg perspectives through the duality of their associated maps. Adhering to the empirical principle, two processes are then identified if they are indistinguishable by test experiments.

The information category is thus defined to be the set of all such bilinear pairings, with morphisms expressed as adjointable semilinear maps.

The information category is created from the field $\mathbb{F}$ that provides its space of measurements. The automorphism group $\Gamma = \mathrm{aut}[\mathbb{F}]$ acts as signature group for the processes, in recognition of the internal symmetries in the measurement system presented by the automorphisms. Semilinearity as a concept is then defined for maps between vector spaces, and the process is associated with semilinear maps that are compatible with the algebraic and topological features of its measurement field.

Empirical systems

The system K is associated with the data $(\mathsf{M}[K], \mathsf{W}[K], \bullet)$, where the state space $\mathsf{M}[K]$ and the observable space $\mathsf{W}[K]$ are vector spaces, and the pairing:

$$\bullet : \mathsf{M}[K] \times \mathsf{W}[K] \to \mathbb{F} \tag{23}$$

is a bilinear map that combines the test state $\mathsf{z} \in \mathsf{M}[K]$ with the test observable $\mathsf{a} \in \mathsf{W}[K]$ to generate their valuation $\mathsf{z} \bullet \mathsf{a} \in \mathbb{F}$. Systems are thus identified with the experiments that can be performed on them, and all further constructions are defined in terms of these measurements.

The empty system 1 has data given by the scalars:

$$\begin{aligned} \mathsf{M}[1] &:= \mathbb{F} \\ \mathsf{W}[1] &:= \mathbb{F} \end{aligned} \tag{24}$$

with the pairing on the empty system defined by:

$$\lambda \bullet \mu := \lambda\mu \tag{25}$$

for the test state $\lambda \in \mathsf{M}[1]$ and the test observable $\mu \in \mathsf{W}[1]$. This trivial definition for the pairing is, up to a scale factor, the only one supported by the state and observable spaces of the empty system. Valuations of this system have no information content.

The concatenated system $K \otimes L$ that combines the systems K and L has data given by the algebraic tensor products:

$$\begin{aligned} \mathsf{M}[K \otimes L] &:= \mathsf{M}[K] \otimes \mathsf{M}[L] \\ \mathsf{W}[K \otimes L] &:= \mathsf{W}[K] \otimes \mathsf{W}[L] \end{aligned} \tag{26}$$

with the pairing on the concatenated system defined by:

$$(\mathsf{z} \otimes \mathsf{y}) \bullet (\mathsf{a} \otimes \mathsf{b}) := (\mathsf{z} \bullet \mathsf{a})(\mathsf{y} \bullet \mathsf{b}) \tag{27}$$

for the test states $\mathsf{z} \in \mathsf{M}[K]$ and $\mathsf{y} \in \mathsf{M}[L]$ and the test observables $\mathsf{a} \in \mathsf{W}[K]$ and $\mathsf{b} \in \mathsf{W}[L]$. Experiments in the concatenated system are linear combinations of products of experiments available in the component systems.

Empirical processes

Imposing the empirical principle, the process is identified by its contractions with test states and observables, encapsulated in its valuation maps.

Schrödinger picture: The process e acts on the state z and is paired with the observable a to generate its valuation $\mathsf{z} \circ e \bullet \mathsf{a}$.

Heisenberg picture: The process e acts on the observable a and is paired with the state z to generate its valuation $\mathsf{z} \bullet e \circ \mathsf{a}$.

In these dual perspectives, the process is defined by its association with the state map $\mathsf{z} \mapsto \mathsf{z} \circ e$ and the observable map $\mathsf{a} \mapsto e \circ \mathsf{a}$, which are required to be semilinear and must generate consistent contractions. For a process with signature automorphism α, the state map is assumed to be α-linear and the observable map is assumed to be α^{-1}-linear, meaning that both maps commute with addition, and commutation with scalar multiplication takes the form:

$$(\lambda \mathsf{z}) \circ e = \lambda^{\alpha}(\mathsf{z} \circ e) \qquad\qquad e \circ (\lambda \mathsf{a}) = \lambda^{\alpha^{-1}}(e \circ \mathsf{a}) \tag{28}$$

where the action of the automorphism on scalars is denoted by the superscript. Consistency of the Schrödinger and Heisenberg pictures is then imposed by the following condition on the valuation maps:

$$\mathsf{z} \circ e \bullet \mathsf{a} = (\mathsf{z} \bullet e \circ \mathsf{a})^{\alpha} \tag{29}$$

asserting that the state and observable maps are mutually adjoint.

For an arbitrary field $\mathbb{F}$, the process maps are defined directly on the test states and observables.

Definition 1 (Process). *The process $e \in \hom^{\alpha}[K, L]$ is associated with the data $(\mathsf{M}[e], \mathsf{W}[e])$, where the state map $\mathsf{M}[e]$ and the observable map $\mathsf{W}[e]$ are respectively α-linear and α^{-1}-linear maps:*

$$\begin{aligned} \mathsf{M}[e] &: \mathsf{z} \in \mathsf{M}[K] \mapsto \mathsf{z} \circ e \in \mathsf{M}[L] \\ \mathsf{W}[e] &: \mathsf{a} \in \mathsf{W}[L] \mapsto e \circ \mathsf{a} \in \mathsf{W}[K] \end{aligned} \tag{30}$$

The valuation maps generated by the process are then defined by:

$$\begin{aligned} \mathsf{z} \circ e \bullet \mathsf{a} &:= (\mathsf{z} \circ e) \bullet \mathsf{a} \\ \mathsf{z} \bullet e \circ \mathsf{a} &:= \mathsf{z} \bullet (e \circ \mathsf{a}) \end{aligned} \tag{31}$$

for the test state $\mathsf{z} \in \mathsf{M}[K]$ and the test observable $\mathsf{a} \in \mathsf{W}[L]$. Imposing consistency, the process maps are required to be mutually adjoint, satisfying:

$$\mathsf{z} \circ e \bullet \mathsf{a} = (\mathsf{z} \bullet e \circ \mathsf{a})^{\alpha} \tag{32}$$

An alternative definition of the process is available for the complex field $\mathbb{C}$ that is cognisant of the topological as well as algebraic structure of the field. The signature group is in this case restricted to the group of continuous automorphisms $\Gamma = \{1, *\}$ comprising the identity and complex conjugation, and the pairing generates families of seminorms $\{n_{\mathsf{a}} : \mathsf{a} \in \mathsf{W}[K]\}$ and $\{n_{\mathsf{z}} : \mathsf{z} \in \mathsf{M}[K]\}$ that implement locally convex topologies on the test states and observables associated with the system K:

$$n_{\mathsf{a}}[\mathsf{z}] := |\mathsf{z} \bullet \mathsf{a}| =: n_{\mathsf{z}}[\mathsf{a}] \tag{33}$$

The completed state space $\overline{\mathsf{M}}[K]$ is defined to be the space of convergent sequences of test states modulo the sequences whose valuations converge to zero, and the completed observable space $\overline{\mathsf{W}}[K]$ is defined to be the space of convergent sequences of test observables modulo the sequences whose valuations converge to zero. The maps:

$$\begin{aligned} \mathsf{z} \in \mathsf{M}[K] &\mapsto (\mathsf{z}) \in \overline{\mathsf{M}}[K] \\ \mathsf{a} \in \mathsf{W}[K] &\mapsto (\mathsf{a}) \in \overline{\mathsf{W}}[K] \end{aligned} \tag{34}$$

that include the test spaces in their completions are not necessarily injective: the kernels comprise the test states and observables that are topologically indistinguishable from zero. Using these completed spaces, process maps can now be defined with the additional condition of continuity.

Definition 2 (Process)**.** *The process $e \in \mathrm{Hom}^{\alpha}[K, L]$ is associated with the data $(\overline{\mathsf{M}}[e], \overline{\mathsf{W}}[e])$, where the state map $\overline{\mathsf{M}}[e]$ and the observable map $\overline{\mathsf{W}}[e]$ are respectively α-linear and α^{-1}-linear continuous maps:*

$$\begin{aligned} \overline{\mathsf{M}}[e] : \mathsf{z} \in \overline{\mathsf{M}}[K] &\mapsto \mathsf{z} \circ e \in \overline{\mathsf{M}}[L] \\ \overline{\mathsf{W}}[e] : \mathsf{a} \in \overline{\mathsf{W}}[L] &\mapsto e \circ \mathsf{a} \in \overline{\mathsf{W}}[K] \end{aligned} \tag{35}$$

The valuation maps generated by the process are then defined by:

$$\begin{aligned} \mathsf{z} \circ e \bullet \mathsf{a} &:= \lim_{n \to \infty} ((\mathsf{z} \circ e)_n \bullet \mathsf{a}) \\ \mathsf{z} \bullet e \circ \mathsf{a} &:= \lim_{n \to \infty} (\mathsf{z} \bullet (e \circ \mathsf{a})_n) \end{aligned} \tag{36}$$

for the test state $\mathsf{z} \in \mathsf{M}[K]$ and the test observable $\mathsf{a} \in \mathsf{W}[L]$. Imposing consistency, the process maps are required to be mutually adjoint, satisfying:

$$\mathsf{z} \circ e \bullet \mathsf{a} = (\mathsf{z} \bullet e \circ \mathsf{a})^{\alpha} \tag{37}$$

This definition reflects that, in practice, measurements extrapolate from refinements of the experimental apparatus, an operation that only makes sense if the process maps are continuous.

The universal constructions of the category – the identity and braid processes and the binary operands of concatenation and composition – are defined on states and observables.

	State	**Observable**
Identity:	$\mathsf{z} \circ \imath := \mathsf{z}$	$\imath \circ \mathsf{a} := \mathsf{a}$
Braid:	$(\mathsf{z} \otimes \mathsf{y}) \circ \tau := \mathsf{y} \otimes \mathsf{z}$	$\tau \circ (\mathsf{a} \otimes \mathsf{b}) := \mathsf{b} \otimes \mathsf{a}$
Concatenation:	$(\mathsf{z} \otimes \mathsf{y}) \circ (e \otimes f) :=$ $(\mathsf{z} \circ e) \otimes (\mathsf{y} \circ f)$	$(e \otimes f) \circ (\mathsf{a} \otimes \mathsf{b}) :=$ $(e \circ \mathsf{a}) \otimes (f \circ \mathsf{b})$
Composition:	$\mathsf{z} \circ (e \circ f) := (\mathsf{z} \circ e) \circ f$	$(e \circ f) \circ \mathsf{a} := e \circ (f \circ \mathsf{a})$

The following developments assume that these universal constructions are well defined and satisfy the coherence rules of the symmetric monoidal category; in particular, that topological completion is natural, as expressed in the commutative diagrams:

$$\begin{tikzcd} \overline{\mathsf{M}}[L] \otimes \overline{\mathsf{M}}[N] \arrow[r, hook] & \overline{\mathsf{M}}[L \otimes N] \\ \overline{\mathsf{M}}[K] \otimes \overline{\mathsf{M}}[M] \arrow[r, hook] \arrow[u, "\overline{\mathsf{M}}[e] \otimes \overline{\mathsf{M}}[f]"] & \overline{\mathsf{M}}[K \otimes M] \arrow[u, "\overline{\mathsf{M}}[e \otimes f]"'] \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathsf{W}}[L] \otimes \overline{\mathsf{W}}[N] \arrow[r, hook] & \overline{\mathsf{W}}[L \otimes N] \\ \overline{\mathsf{W}}[K] \otimes \overline{\mathsf{W}}[M] \arrow[r, hook] \arrow[u, "\overline{\mathsf{W}}[e] \otimes \overline{\mathsf{W}}[f]"] & \overline{\mathsf{W}}[K \otimes M] \arrow[u, "\overline{\mathsf{W}}[e \otimes f]"'] \end{tikzcd}$$

Technical challenges emerge from the gap between the tensor product of completed topological spaces and the completed tensor product of topological spaces. Continuity uniquely and naturally extends the tensor product of process maps, filling the gap with appropriate limits. Confirmation of this statement is beyond the scope of this essay, here noting only that topological completion is trivial when the state and observable spaces are finite dimensional.

Valuation model

In the valuation model generated by the information category, scalars are represented by the hom-space $\mathrm{Hom}[1,1]$, and states and observables of the system K are represented respectively by the hom-spaces $\mathrm{Hom}[1,K]$ and $\mathrm{Hom}[K,1]$.

The automorphism $\alpha \in \Gamma$ is identified with the process $\alpha \in \mathrm{Hom}^{\alpha}[1,1]$ with:

$$\lambda \circ \alpha := \lambda^{\alpha} \qquad\qquad \alpha \circ \lambda := \lambda^{\alpha^{-1}} \tag{38}$$

and the space of scalars with signature α is generated from this process. More generally, the hom-space $\mathrm{Hom}^{\alpha}[K,L]$ of processes with signature α between the systems K and L has the structure of a topological vector space with:

$$\begin{aligned} \mathsf{z} \circ (\lambda e) &:= \lambda^{\alpha}(\mathsf{z} \circ e) \qquad\qquad & (\lambda e) \circ \mathsf{a} &:= \lambda(e \circ \mathsf{a}) \\ \mathsf{z} \circ (e\lambda) &:= \lambda(\mathsf{z} \circ e) & (e\lambda) \circ \mathsf{a} &:= \lambda^{\alpha^{-1}}(e \circ \mathsf{a}) \end{aligned} \tag{39}$$

for scalar multiplication on both sides of the process, endowed with the locally convex topology of pointwise convergence generated by the family of seminorms $\{n_{\mathsf{z},\mathsf{a}} : \mathsf{z} \in \mathsf{M}[K], \mathsf{a} \in \mathsf{W}[L]\}$:

$$|\mathsf{z} \circ e \bullet \mathsf{a}| =: n_{\mathsf{z},\mathsf{a}}[e] := |\mathsf{z} \bullet e \circ \mathsf{a}| \tag{40}$$

where the definitions on both sides match thanks to duality. The scalar then transforms by the signature as it passes through the process:

$$\lambda^{\alpha^{-1}}(e \circ f) = (\lambda^{\alpha^{-1}} e) \circ f = (e\lambda) \circ f = e \circ (\lambda f) = e \circ (f \lambda^{\beta}) = (e \circ f)\lambda^{\beta} \tag{41}$$

for the processes $e \in \mathrm{Hom}^{\alpha}[K,L]$ and $f \in \mathrm{Hom}^{\beta}[L,M]$.

The bilinear pairing is implemented on states and observables via:

$$(1 \bullet \mathsf{z} \circ \mathsf{a} \circ 1)^{\alpha} =: \mathsf{z} \bullet \mathsf{a} := (1 \circ \mathsf{z} \circ \mathsf{a} \bullet 1)^{\beta^{-1}} \tag{42}$$

for the state $\mathsf{z} \in \mathrm{Hom}^{\alpha}[1,K]$ and observable $\mathsf{a} \in \mathrm{Hom}^{\beta}[K,1]$. Bilinearity in this definition means that the pairing commutes with scalar multiplication as:

$$(\mathsf{z}\lambda) \bullet \mathsf{a} = \lambda(\mathsf{z} \bullet \mathsf{a}) = \mathsf{z} \bullet (\lambda \mathsf{a}) \tag{43}$$

This extends the valuation model to the hom-spaces of states and observables.

Empirical subsystems

For the subsets $\mathsf{N} \subset \mathrm{Hom}^{1}[1,K]$ and $\mathsf{И} \subset \mathrm{Hom}^{1}[L,1]$, define the annihilators:

$$\begin{aligned} \mathsf{N}^0 &= \{\mathsf{a} \in \mathrm{Hom}^1[K,1] : \mathsf{N} \bullet \mathsf{a} = \{0\}\} \\ \mathsf{И}^0 &= \{\mathsf{z} \in \mathrm{Hom}^1[1,L] \ : \mathsf{z} \bullet \mathsf{И} = \{0\}\} \end{aligned} \tag{44}$$

The annihilators frame the limits of empirical determination for the subsets. The space N^0 is the closed subspace of observables that are indistinguishable from zero under pairing with the states of N, and the space $\mathsf{И}^0$ is the closed subspace of states that are indistinguishable from zero under pairing with the observables of $\mathsf{И}$. Annihilation is compounded with the following results:

$$\begin{aligned} &\mathsf{N} \subset \overline{\mathsf{N}} \subset \mathsf{N}^{00} && \mathsf{И} \subset \overline{\mathsf{И}} \subset \mathsf{И}^{00} \\ &\mathsf{N}^0 = \overline{\mathsf{N}}^0 = \mathsf{N}^{000} && \mathsf{И}^0 = \overline{\mathsf{И}}^0 = \mathsf{И}^{000} \end{aligned} \tag{45}$$

for the double and triple annihilations, where $\overline{\mathsf{N}}$ is the topological closure of the linear span of N and $\overline{\mathsf{И}}$ is the topological closure of the linear span of $\mathsf{И}$.

The boundary cases for annihilation are:

$$\begin{aligned} &\mathrm{Hom}[1, K]^0 = \{0\} && \mathrm{Hom}[L, 1]^0 = \{0\} \\ &\mathrm{Hom}[1, K] \ \ = \{0\}^0 && \mathrm{Hom}[L, 1] \ \ = \{0\}^0 \end{aligned} \tag{46}$$

so that pairings with the hom-spaces are sufficient to discriminate their duals. Furthermore, for the process $e \in \mathrm{Hom}^1[K, L]$, annihilation commutes with the process in the form:

$$(\mathsf{N} \circ e)^0 = e^{-1} \circ \mathsf{N}^0 \qquad\qquad (e \circ \mathsf{И})^0 = \mathsf{И}^0 \circ e^{-1} \tag{47}$$

thanks to the duality of the state and observable maps. The degree of compatibility these relationships express between annihilation and the algebra and topology of processes assists the development of enhanced structures within the information category.

The states $\mathrm{Hom}^1[1, K]$ and observables $\mathrm{Hom}^1[K, 1]$, together with their pairing, constitute the assumed bounds of investigability for the system K. For a participant whose access to the system is further restricted to the subset of states $\mathsf{N} \subset \mathrm{Hom}^1[1, K]$ and the subset of observables $\mathsf{И} \subset \mathrm{Hom}^1[K, 1]$, the system effectively reduces to the subsystem $K^{\mathsf{И}}_{\mathsf{N}}$ with the definitions:

$$\begin{aligned} \mathsf{M}[K^{\mathsf{И}}_{\mathsf{N}}] &:= \mathsf{N}^{00}/(\mathsf{N}^{00} \cap \mathsf{И}^0) \cong (\mathsf{N}^{00} + \mathsf{И}^0)/\mathsf{И}^0 \\ \mathsf{W}[K^{\mathsf{И}}_{\mathsf{N}}] &:= \mathsf{И}^{00}/(\mathsf{И}^{00} \cap \mathsf{N}^0) \cong (\mathsf{И}^{00} + \mathsf{N}^0)/\mathsf{N}^0 \end{aligned} \tag{48}$$

for the effective states and observables, and the definition:

$$(\mathsf{z} + (\mathsf{N}^{00} \cap \mathsf{И}^0)) \bullet (\mathsf{a} + (\mathsf{И}^{00} \cap \mathsf{N}^0)) := \mathsf{z} \bullet \mathsf{a} \tag{49}$$

for the pairing of the state $\mathsf{z} \in \mathsf{N}^{00}$ and the observable $\mathsf{a} \in \mathsf{И}^{00}$.

By restricting the states and observables, the subsystem captures the empirical reality of the participant. Effective states are restricted to those available in the subset of states, and further reduced by identifying those that are indistinguishable by pairing with the subset of observables. Effective observables are restricted to those available in the subset of observables, and further reduced by identifying those that are indistinguishable by pairing with the subset of states.

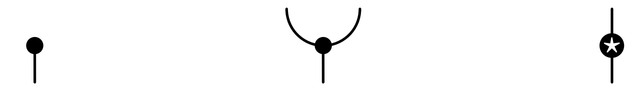
The subsystem thus exists within the a priori confines of the larger system, with its scope of investigation then determined a posteriori according to the empirical constraints of the participant. This facilitates the development of mathematical finance, as quotients of the observable space are utilised in the operator representation of states and observables, implementing the concept of time from its space of ideals. These ideas are expanded in later sections.

2.3 Quantum groups

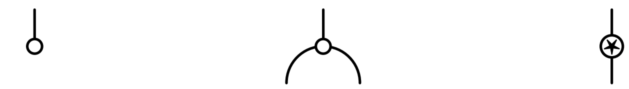
With the apparatus of the information category in place, the quantum group is characterised by its extended lexicon of processes and the grammar they satisfy. The definition is motivated by the properties of classical groups, and is essentially equivalent to the definition of the classical group in the case when the product of observables is commutative. Removing this ugly asymmetry from the otherwise pristine duality between state and observable is the principal abstract achievement of the quantum group. The variance this generates has numerous applications, including as the origin of indeterminism in quantum mechanics and as a source of value for convexity in the pricing of derivative securities.

A system K is a quantum group if its hom-spaces include two families of processes acting as unit, product and involution on the states and observables:

Unit **Product** **Involution**



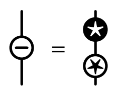
$\eta \in \mathrm{Hom}^1[K, 1]$ $\nabla \in \mathrm{Hom}^1[K, K \otimes K]$ $\star \in \mathrm{Hom}^*[K, K]$



$\iota \in \mathrm{Hom}^1[1, K]$ $\Delta \in \mathrm{Hom}^1[K \otimes K, K]$ $\ast \in \mathrm{Hom}^*[K, K]$

The axioms that relate these structural processes of the quantum group, presented in the figures on the next page, fall into three categories. The first set of axioms implies that the states and observables separately have the structure of $*$-algebras. By requiring them to commute, the second set of axioms combines the two $*$-algebras into a $*$-bialgebra. The final axiom then enables the creation of an antipode that generates the structure of a Hopf $*$-algebra on the system:

Antipode



$s := \ast \circ \star \in \mathrm{Hom}^1[K, K]$

These processes, with the rules they satisfy, are sufficient for the application of the quantum group to mathematical finance. As will be demonstrated later, the state and observable algebras separately create stochastic and functional calculus, the bialgebra properties ensure they can be implemented consistently, and the Hopf axiom relates integration and differentiation on the dynamics.

The standard notation for the $*$-algebra expresses the product by juxtaposition and the involution by superscript:

	State	**Observable**
Unit:	$1 := 1 \circ \iota$	$1 := \eta \circ 1$
Product:	$\mathsf{zy} := (\mathsf{z} \otimes \mathsf{y}) \circ \Delta$	$\mathsf{ab} := \nabla \circ (\mathsf{a} \otimes \mathsf{b})$
Involution:	$\mathsf{z}^* := \mathsf{z} \circ \ast$	$\mathsf{a}^* := \star \circ \mathsf{a}$

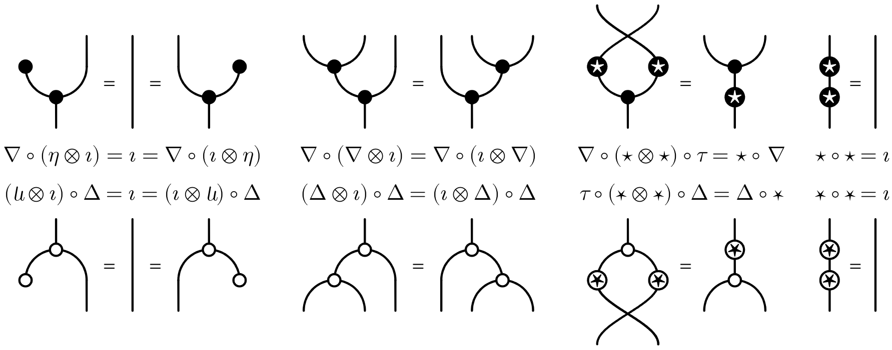


Figure 3: The axioms of $*$-algebra: The states and observables separately support unit, product and involution processes that are unital, associative, antiautomorphic and involutive.

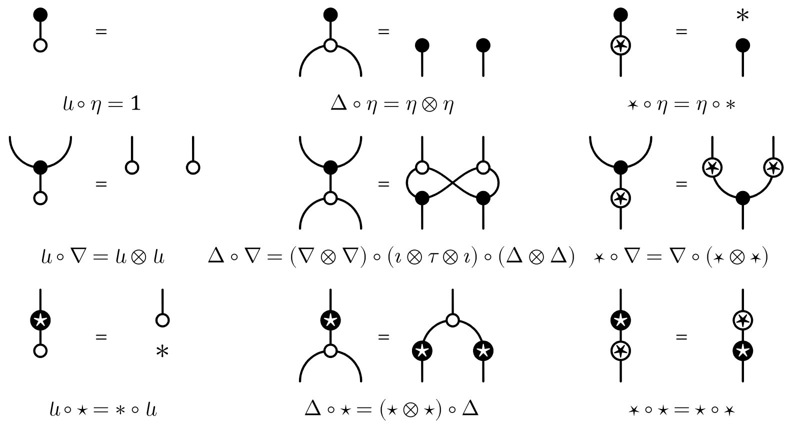


Figure 4: The axioms of $*$-bialgebra: Complementarity of the $*$-algebras of states and observables is expressed by these commutation relations between their structural processes.

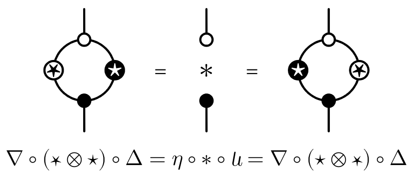


Figure 5: The axiom of Hopf $*$-algebra: This axiom relates all the structural processes of the $*$-bialgebra. The antipode, defined as the composition of the two involutions, then satisfies the Hopf axiom.

With this notation, the axioms of $*$-algebra are more familiarly expressed as:

	State	**Observable**
Unital:	$1\mathsf{z} = \mathsf{z} = \mathsf{z}1$	$1\mathsf{a} = \mathsf{a} = \mathsf{a}1$
Associative:	$(\mathsf{zy})\mathsf{x} = \mathsf{z}(\mathsf{yx})$	$(\mathsf{ab})\mathsf{c} = \mathsf{a}(\mathsf{bc})$
Antiautomorphic:	$\mathsf{y}^*\mathsf{z}^* = (\mathsf{zy})^*$	$\mathsf{b}^*\mathsf{a}^* = (\mathsf{ab})^*$
Involutive:	$\mathsf{z}^{**} = \mathsf{z}$	$\mathsf{a}^{**} = \mathsf{a}$

Wherever the distinction needs to be made, the dual operations to the unit, product and involution are referred to as the counit, coproduct and coinvolution respectively.

Quantum groups are closed under concatenation with the definitions:

	State	**Observable**
Unit:	$\reflectbox{$\eta$} := \reflectbox{$\eta$} \otimes \reflectbox{$\eta$}$	$\eta := \eta \otimes \eta$
Product:	$\Delta := (\imath \otimes \tau \otimes \imath) \circ (\Delta \otimes \Delta)$	$\nabla := (\nabla \otimes \nabla) \circ (\imath \otimes \tau \otimes \imath)$
Involution:	$\ast := \ast \otimes \ast$	$\star := \star \otimes \star$

where the process for the concatenated system, appearing on the left in these expressions, is defined on the right in terms of the corresponding processes for its component systems. Concatenated quantum groups thus inherit the structural processes from their component systems, enabling the extension of the functional calculus to combinations of independent systems, and supporting the development of the stochastic calculus on discrete schedules.

3 Mathematical finance

Mathematical finance studies the relationship between the economic state z_e and the pricing state z_c, respectively quantifying the expected and present values of economic variables. Applied to undetermined observables, the economic state is inherently subjective, providing an assessment of economic conditions yet to be revealed. In contrast, the pricing state is marked to observed market prices, incorporating the market consensus and external factors such as liquidity and funding, and relegating subjective expectations to the unhedged convexities.

A sound platform for mathematical finance thus needs to establish the essential links between the economic and pricing states without being overly constraining. The constraints considered here prevent arbitrage and impose a loose relationship that allows both subjective and market expectations to influence price. Markets are not obliged to obey these principles, but they are satisfied to a reasonable approximation in normal conditions, and are a point of attraction even when markets are stressed.

3.1 Positivity and the functional calculus

The economic principles assert that the economic and pricing states are linear and positive, and are equivalent in the sense that they have the same space of null observables. Implemented in the information category, the structural processes of the observable algebra are leveraged for their algebraic properties: linearity and positivity are expressed as algebraic relationships applied separately to the states; equivalence then relates the two states.

Linearity: The economic and pricing states are linear.

$$\begin{aligned}\mathsf{z}_e \bullet (\lambda\mathsf{a} + \mu\mathsf{b}) &= \lambda(\mathsf{z}_e \bullet \mathsf{a}) + \mu(\mathsf{z}_e \bullet \mathsf{b}) \\ \mathsf{z}_c \bullet (\lambda\mathsf{a} + \mu\mathsf{b}) &= \lambda(\mathsf{z}_c \bullet \mathsf{a}) + \mu(\mathsf{z}_c \bullet \mathsf{b})\end{aligned} \tag{50}$$

Positivity: The economic and pricing states are real and positive.

$$\begin{aligned}\mathsf{z}_e \bullet \mathsf{a}^* &= (\mathsf{z}_e \bullet \mathsf{a})^* \qquad & \mathsf{z}_e \bullet \mathsf{aa}^* &\geq 0 \\ \mathsf{z}_c \bullet \mathsf{a}^* &= (\mathsf{z}_c \bullet \mathsf{a})^* \qquad & \mathsf{z}_c \bullet \mathsf{aa}^* &\geq 0\end{aligned} \tag{51}$$

Equivalence: The economic and pricing states are equivalent.

$$\mathsf{z}_e \bullet \mathsf{aa}^* \neq 0 \iff \mathsf{z}_c \bullet \mathsf{aa}^* \neq 0 \tag{52}$$

The observable algebra supports these principles by furnishing the notions of linearity and positivity. For the economic state, these are the defining properties of expectation, enforcing positivity of variance. Supplemented with the normalisation $\mathsf{z}_e \bullet 1 = 1$, the principles imply the state acts as a probability measure on families of commuting observables. For the pricing state, linearity means that the price of a portfolio is the weighted sum of the prices of its constituents, and positivity means that positive observables have positive prices, thereby preventing arbitrage. Equivalence then requires that a positive observable has non-zero price if and only if it has non-zero expectation, so that financial value is not attributed to economic events deemed to be impossible.

In the context of the information category, positivity immediately leads to the Cauchy-Schwarz inequality, which in turn enables operator representations of the states and observables. Defining a *copositive state* to be a state that satisfies the linearity and positivity principles, the inequality strengthens the condition of positivity, and in so doing opens a bridge from algebra to topology.

Theorem 1 (Cauchy-Schwarz inequality)**.** *The copositive state* z *satisfies the inequality:*

$$|\mathsf{z} \bullet \mathsf{ab}^*|^2 \leq (\mathsf{z} \bullet \mathsf{aa}^*)(\mathsf{z} \bullet \mathsf{bb}^*) \tag{53}$$

for the observables a *and* b*.*

Proof. This elementary result is a consequence of the following inequality applied with appropriate choices for the scalars α and β:

$$\begin{aligned}0 &\leq \mathsf{z} \bullet (\alpha\mathsf{a} - \beta\mathsf{b})(\alpha\mathsf{a} - \beta\mathsf{b})^* \\ &= \alpha\alpha^*\mathsf{z} \bullet \mathsf{aa}^* - \alpha\beta^*\mathsf{z} \bullet \mathsf{ab}^* - \beta\alpha^*\mathsf{z} \bullet \mathsf{ba}^* + \beta\beta^*\mathsf{z} \bullet \mathsf{bb}^*\end{aligned} \tag{54}$$

exploiting the reality and positivity properties of the copositive state. □

The Cauchy-Schwarz inequality initiates a chain of foundational results, taking in the Gelfand-Naimark-Segal construction and the Radon-Nikodym theorem, that terminates with the representation of states and observables as dual von Neumann algebras on a Hilbert space.

The Gelfand-Naimark-Segal construction

The observable a is, by definition, a null observable for the copositive state z_e if it satisfies $\mathsf{z}_e \bullet \mathsf{aa}^* = 0$. Thanks to the Cauchy-Schwarz inequality, this is equivalent to the condition $\mathsf{z}_e \bullet \mathsf{ab}^* = 0$ for all observables b. From this property, the set of null observables:

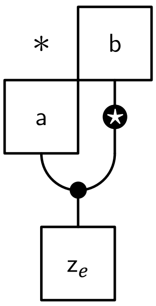
$$\text{И}_e = \{\mathsf{a} \in \mathsf{W} : \mathsf{z}_e \bullet \mathsf{aa}^* = 0\} \tag{55}$$

is a right ideal of the space W of test observables. The quotient space $\mathsf{W}/\text{И}_e$ is then equipped with an inner product:

$$\langle \mathsf{a} + \text{И}_e | \mathsf{b} + \text{И}_e \rangle := \mathsf{z}_e \bullet \mathsf{ab}^* \tag{56}$$

and the observable b is represented as an operator $[\mathsf{b}]$ on the quotient space with the definition:

$$\langle \mathsf{a} + \text{И}_e | [\mathsf{b}] := \langle \mathsf{ab} + \text{И}_e | \tag{57}$$



In this string diagram, the inner product of the observables a and b in the state z_e is given by:

$$\mathsf{z}_e \bullet \nabla \circ ((\mathsf{a} \circ *) \otimes (\star \circ \mathsf{b})) \tag{58}$$

using the identification of states and observables with processes from and to the empty system.

The representation on the inner product space respects the $*$-algebras of observables and operators, with the $*$-homomorphism properties:

$$\begin{aligned} [1] &= 1 \\ [\mathsf{ab}] &= [\mathsf{a}][\mathsf{b}] \\ [\mathsf{a}^*] &= [\mathsf{a}]^* \end{aligned} \tag{59}$$

The first two properties are trivial, and the final property follows from:

$$\langle \mathsf{b} + \text{И}_e | [\mathsf{a}] | \mathsf{c} + \text{И}_e \rangle^* = (\mathsf{z}_e \bullet \mathsf{bac}^*)^* = \mathsf{z}_e \bullet \mathsf{ca}^*\mathsf{b}^* = \langle \mathsf{c} + \text{И}_e | [\mathsf{a}^*] | \mathsf{b} + \text{И}_e \rangle \tag{60}$$

The operator $[\mathsf{a}]$ is thus adjointable with adjoint $[\mathsf{a}^*]$. The vector $\langle 1 + \text{И}_e|$ is generating for the representation, and the corresponding pure state satisfies:

$$\mathsf{z}_e \bullet \mathsf{a} = \langle 1 + \text{И}_e | [\mathsf{a}] | 1 + \text{И}_e \rangle \tag{61}$$

This reconstructs the copositive state from the representation.

Topological completion. This purely algebraic presentation is the algebraic content of the Gelfand-Naimark-Segal construction. Completing the inner product space $\mathsf{W}/\text{И}_e$ to the Hilbert space H_e, the operator $[\mathsf{a}] : \mathsf{W}/\text{И}_e \to \mathsf{W}/\text{И}_e$ extends to an operator $[\mathsf{a}] : \mathsf{H}_e \to \mathsf{H}_e$ if and only if it is bounded in the operator norm:

$$\|[\mathsf{a}]\| = \sqrt{\sup\{\mathsf{z}_e \bullet \mathsf{caa}^*\mathsf{c}^* : \mathsf{z}_e \bullet \mathsf{cc}^* = 1\}} < \infty \tag{62}$$

otherwise it is unbounded. Observables are represented on the Hilbert space as operators that may be unbounded but which are always defined on the common dense subspace $\mathsf{W}/\text{И}_e \subset \mathsf{H}_e$.

The Radon-Nikodym theorem

When there are two copositive states z_e and z_c, there are two inner product spaces W/N_e and W/N_c with inner products:

$$\begin{aligned}\langle \mathsf{a} + \mathsf{N}_e | \mathsf{b} + \mathsf{N}_e \rangle &= \mathsf{z}_e \bullet \mathsf{ab}^* \\ \langle \mathsf{a} + \mathsf{N}_c | \mathsf{b} + \mathsf{N}_c \rangle &= \mathsf{z}_c \bullet \mathsf{ab}^*\end{aligned} \tag{63}$$

Consider copositive states that satisfy the order relation:

$$\mathsf{z}_e \bullet \mathsf{aa}^* \neq 0 \implies \mathsf{z}_c \bullet \mathsf{aa}^* \neq 0 \tag{64}$$

In words, if an observable is null for z_e then it is null for z_c. The right ideals N_e and N_c satisfy $\mathsf{N}_e \subset \mathsf{N}_c$ in this case, and there is a surjective operator $\mathsf{m}_c : \mathsf{W}/\mathsf{N}_e \to \mathsf{W}/\mathsf{N}_c$ defined by:

$$\langle \mathsf{a} + \mathsf{N}_e | \mathsf{m}_c := \langle \mathsf{a} + \mathsf{N}_c | \tag{65}$$

Assuming for now that this operator has an adjoint $\mathsf{m}_c^* : \mathsf{W}/\mathsf{N}_c \to \mathsf{W}/\mathsf{N}_e$, define the positive operator:

$$[\mathsf{z}_c] = \mathsf{m}_c \mathsf{m}_c^* \tag{66}$$

on W/N_e. The operator satisfies:

$$\langle \mathsf{b} + \mathsf{N}_e | [\mathsf{z}_c][\mathsf{a}] | \mathsf{c} + \mathsf{N}_e \rangle = \mathsf{z}_c \bullet \mathsf{bac}^* = \langle \mathsf{b} + \mathsf{N}_e | [\mathsf{a}][\mathsf{z}_c] | \mathsf{c} + \mathsf{N}_e \rangle \tag{67}$$

The matrix elements of $[\mathsf{z}_c][\mathsf{a}]$ thus match the matrix elements of $[\mathsf{a}][\mathsf{z}_c]$. Both copositive states are reconstructed using the pure state associated with the unit in this representation:

$$\begin{aligned}\mathsf{z}_e \bullet \mathsf{a} &= \langle 1 + \mathsf{N}_e | [\mathsf{a}] | 1 + \mathsf{N}_e \rangle \\ \mathsf{z}_c \bullet \mathsf{a} &= \langle 1 + \mathsf{N}_e | [\mathsf{z}_c][\mathsf{a}] | 1 + \mathsf{N}_e \rangle = \langle 1 + \mathsf{N}_e | [\mathsf{a}][\mathsf{z}_c] | 1 + \mathsf{N}_e \rangle\end{aligned} \tag{68}$$

where the operator $[\mathsf{z}_c]$, the *Radon-Nikodym weight* of the copositive state z_c over the copositive state z_e, commutes with the operators $[\mathsf{a}]$ associated with observables, in the sense that the matrix elements of the commutator vanish.

Topological completion. This purely algebraic presentation is the algebraic content of the Radon-Nikodym theorem. The weak link in the argument is the assumption of adjointability, which relies on the existence of orthogonals. Completing the inner product spaces W/N_e and W/N_c respectively to the Hilbert spaces H_e and H_c, the operator $\mathsf{m}_c : \mathsf{W}/\mathsf{N}_e \to \mathsf{W}/\mathsf{N}_c$ extends to an operator $\mathsf{m}_c : \mathsf{H}_e \to \mathsf{H}_c$ if and only if it is bounded in the operator norm:

$$\|\mathsf{m}_c\| = \sqrt{\sup\{\mathsf{z}_c \bullet \mathsf{aa}^* : \mathsf{z}_e \bullet \mathsf{aa}^* = 1\}} < \infty \tag{69}$$

in which case the adjoint $\mathsf{m}_c^* : \mathsf{H}_c \to \mathsf{H}_e$ exists and the corresponding weight $[\mathsf{z}_c] : \mathsf{H}_e \to \mathsf{H}_e$ is a well-defined bounded operator. The Radon-Nikodym theorem is thus satisfied on the completion if the order relation between the copositive states is strengthened to:

$$\mathsf{z}_c \bullet \mathsf{aa}^* \leq \|\mathsf{m}_c\|^2 \,(\mathsf{z}_e \bullet \mathsf{aa}^*) \tag{70}$$

for the finite scalar $\|\mathsf{m}_c\|$, guaranteeing the existence of the operator representation for the copositive state by imposing the order relation uniformly across observables.

Existence of the adjoint is the reason for considering the completed inner product space. Orthogonals are only guaranteed to exist in the completion, and this is utilised in the definition of the adjoint of a bounded operator. For the application to mathematical finance, the Radon-Nikodym theorem in this form is therefore valid only when equivalence between the economic and pricing states is uniformly enforced.

3.2 Time and the stochastic calculus

The progression from static to dynamic system introduces the notion of time as a partially-ordered set. A discrete schedule is defined to be a finite ordered sequence of times:

$$P = (p_0, \ldots, p_n) \tag{71}$$

with $p_{i-1} \leq p_i$, where n is the number of intervals in the schedule. Introduce the following notation for the set, length, start and end of the schedule:

$$\begin{aligned} \{P\} &= \{p_0, \ldots, p_n\} \\ |P| &= n \\ P_- &= p_0 \\ P_+ &= p_n \end{aligned} \tag{72}$$

Two consecutive schedules P and Q satisfying $P_+ = Q_-$ are concatenated to create a new schedule $P \vee Q$:

$$P \vee Q = (p_0, \ldots, p_{|P|} = q_0, \ldots, q_{|Q|}) \tag{73}$$

with:

$$\begin{aligned} \{P \vee Q\} &= \{P\} \cup \{Q\} \\ |P \vee Q| &= |P| + |Q| \\ (P \vee Q)_- &= P_- \\ (P \vee Q)_+ &= Q_+ \end{aligned} \tag{74}$$

Refinement is then the order relation $P \supset Q$ for schedules P and Q satisfying:

$$\begin{aligned} \{P\} &\supset \{Q\} \\ P_- &= Q_- \\ P_+ &= Q_+ \end{aligned} \tag{75}$$

The two schedules start and end at the same times, but the schedule P refines the schedule Q as all the intervals from Q are mergers of intervals from P.

These definitions establish a bicategory structure on time, with 0-cells given by the times, 1-cells given by the schedules and 2-cells given by the refinement relations. Concatenation and composition are implemented on compatible refinements with the definitions:

$$\begin{aligned} (P \supset Q) \vee (R \supset S) &:= (P \vee R) \supset (Q \vee S) \\ (P \supset Q) \circ (Q \supset R) &:= P \supset R \end{aligned} \tag{76}$$

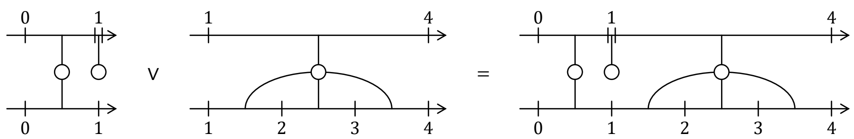
In the following, the time category is restricted to refinements with coarse schedules Q satisfying $|Q| \geq 1$. These refinements are generated via concatenation from the total refinements $P \supset (P_-, P_+)$ for each schedule P, equivalently via concatenation and composition from the elementary refinements:

$$(p) \supset (p,p) \tag{77}$$
$$(p,q,r) \supset (p,r)$$

and the trivial refinements $P \supset P$ for each schedule P. As will be demonstrated below, these elementary refinements generalise the algebraic unit and product, and restricting to the subcategory they generate removes refinements such as $(p,p) \supset (p)$ that do not map to algebraic operations.

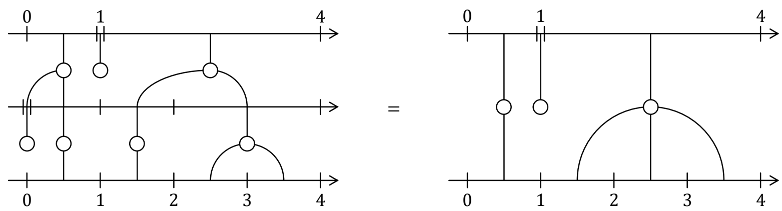
The string diagram for the refinement links each interval in the coarse schedule with the intervals it merges in the refined schedule. As examples, this string diagram presents the concatenation of two refinements:

$$\big((0,1) \supset (0,1,1)\big) \vee \big((1,2,3,4) \supset (1,4)\big) \quad = \quad \big((0,1,2,3,4) \supset (0,1,1,4)\big)$$



and this string diagram presents the composition of two refinements:

$$\big((0,1,2,3,4) \supset (0,0,1,2,4)\big) \circ \big((0,0,1,2,4) \supset (0,1,1,4)\big) \quad = \quad \big((0,1,2,3,4) \supset (0,1,1,4)\big)$$



The resemblance with string diagrams of the state algebra is intentional. Refinement is a generalisation of algebra, and the dynamic system is implemented as a functor from this generalised algebra.

Dynamic systems

A dynamic system is defined to be a monoidal functor from the time category to the information category. The functors are themselves objects of a monoidal category whose morphisms are the natural transformations between them. In this category, the monoidal unit 1 is the functor that maps all schedules to the empty system; a state of the dynamic system K is then a natural transformation $\mathsf{z} \in \mathrm{Hom}^1[1, K]$ between the functors.

Expanding the definition, the dynamic system is implemented by an accumulation functor K from the time category to the information category. The schedule P maps to the system $K[P]$, consistent with concatenation in the form:

$$K[(p)] = 1 \tag{78}$$
$$K[P \vee Q] = K[P] \otimes K[Q]$$

for compatible schedules P and Q satisfying $P_+ = Q_-$. Each refinement $P \supset Q$ maps to a process:

$$K[P \supset Q] \in \mathrm{Hom}^1[K[P], K[Q]] \tag{79}$$

that accumulates the state for each merger of intervals in the refinement. Functoriality is then expressed in the properties:

$$\begin{aligned} K[P \supset Q] \otimes K[R \supset S] &= K[(P \vee R) \supset (Q \vee S)] \\ K[P \supset Q] \circ K[Q \supset R] &= K[P \supset R] \end{aligned} \tag{80}$$

for compatible refinements, so that string diagrams in the time category are replicated as string diagrams in the information category.

The dynamic state is implemented by a natural transformation z from the unit functor to the accumulation functor. The schedule P maps to the state $\mathsf{z}[P] \in \mathrm{Hom}^1[1, K[P]]$, consistent with concatenation in the form:

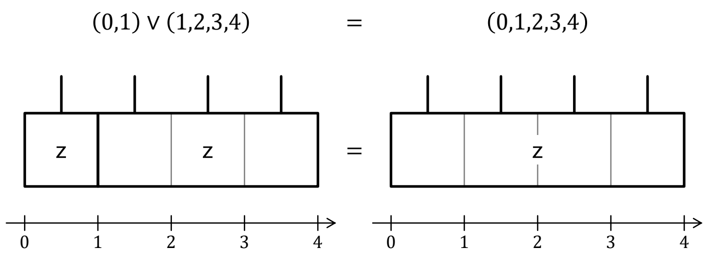
$$\mathsf{z}[P] \otimes \mathsf{z}[Q] = \mathsf{z}[P \vee Q] \tag{81}$$

for compatible schedules P and Q with $P_+ = Q_-$. Naturality is then expressed in the property:

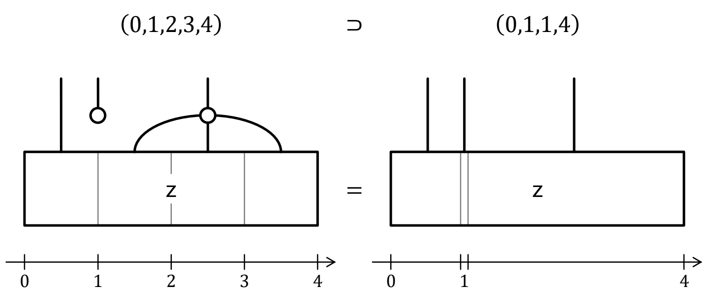
$$\mathsf{z}[P] \circ K[P \supset Q] = \mathsf{z}[Q] \tag{82}$$

for each refinement $P \supset Q$, ensuring consistency between the states.

String diagrams for the dynamic state reflect these properties. The following string diagram is an example of the compatibility of state with concatenation:



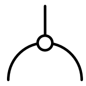
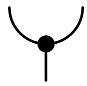
applied to the concatenation $(0,1) \vee (1,2,3,4) = (0,1,2,3,4)$, and the next string diagram is an example of the naturality property:



applied to the refinement $(0,1,2,3,4) \supset (0,1,1,4)$. Any string diagram in the time category maps to a string diagram in the information category. The dynamic state respects the concatenation and composition of refinements thanks to naturality and the functorial properties of the accumulation processes.

3.3 Valuation models inside quantum groups

Summarising the preceding sections, the stochastic and functional calculus is constructed from structural processes associated with schedules and refinements:

State:	*Observable:*
	
$\iota \in \mathrm{Hom}^1[K[(p)], K[(p,p)]]$	$\eta \in \mathrm{Hom}^1[K[(p,q)], 1]$
	
$\Delta \in \mathrm{Hom}^1[K[(p,q,r)], K[(p,r)]]$	$\nabla \in \mathrm{Hom}^1[K[(p,q)], K[(p,q)] \otimes K[(p,q)]]$
	
$\ast \in \mathrm{Hom}^*[K[(p,q)], K[(p,q)]]$	$\star \in \mathrm{Hom}^*[K[(p,q)], K[(p,q)]]$

where the processes in the first column implement accumulation of states for the elementary refinements $(p) \supset (p,p)$ and $(p,q,r) \supset (p,r)$ and the processes in the second column implement multiplication of observables on the interval (p,q). Observable involution engenders the concept of positivity and is used in the operator representation of states and observables. State involution has yet to be utilised, but this process is essential to the relationship between stochastic integration and differentiation, and is composed with observable involution to create the antipode for the reversible model of information dynamics.

In this section, the dynamic system is generated from the subsystems of a fixed quantum group K, expressing both the a priori bounds of investigability of the system and the a posteriori limitations of the participant. Time is the unavoidable limitation, and each interval (p,q) is associated with mutually annihilating subspaces $\mathsf{N}[p,q] \subset \mathrm{Hom}^1[1,K]$ and $\reflectbox{$\mathsf{N}$}[p,q] \subset \mathrm{Hom}^1[K,1]$ satisfying:

$$\mathsf{N}[p,q]^0 = \reflectbox{N}[p,q] \qquad \reflectbox{N}[p,q]^0 = \mathsf{N}[p,q] \tag{83}$$

comprising the available states that can be accumulated over the interval and the null observables that are indistinguishable from zero on the interval. The dynamic system then maps the interval to the subsystem:

$$K[(p,q)] := K_{\mathsf{N}[p,q]} \tag{84}$$

with state and observable spaces:

$$\begin{aligned} \mathsf{M}[K[(p,q)]] &= \mathsf{N}[p,q] \\ \mathsf{W}[K[(p,q)]] &= \mathrm{Hom}^1[K,1]/\reflectbox{N}[p,q] \end{aligned} \tag{85}$$

States are restricted to the available states and observables are quotiented by the null observables, and the association is extended to all schedules with the definitions:

$$\begin{aligned} K[(p)] &:= 1 \\ K[(p_0,\ldots,p_n)] &:= K_{\mathsf{N}[p_0,p_1]} \otimes \cdots \otimes K_{\mathsf{N}[p_{n-1},p_n]} \end{aligned} \tag{86}$$

Empowered with the structural processes of the quantum group, the available states and null observables on each interval are used to create dynamic systems that implement stochastic and functional calculus for the valuation model. The processes and axioms of the quantum group are precisely those that meet the criteria for this objective, with the state and observable algebras supporting accumulation of states and multiplication of observables, the rules of bialgebra ensuring consistency of these operations, and the Hopf axiom imposing computational reversibility. Structural processes are inherited directly from the quantum group, and are well defined provided that the state and observable subspaces satisfy specific closure conditions.

The dynamic system supports the accumulation of states when the available states form a semigroup in the state algebra of the quantum group, as per the conditions expressed in the following result.

Theorem 2 (State algebra)**.** *If the available states satisfy:*

$$\begin{aligned} \mathsf{N}[p,p] &= \mathbb{C} \circ \iota \\ \mathsf{N}[p,r] &= (\mathsf{N}[p,q] \otimes \mathsf{N}[q,r]) \circ \Delta \end{aligned} \tag{87}$$

then the processes:

$$\begin{aligned} \iota &\in \mathrm{Hom}^1[K[(p)], K[(p,p)]] \\ \Delta &\in \mathrm{Hom}^1[K[(p,q,r)], K[(p,r)]] \end{aligned} \tag{88}$$

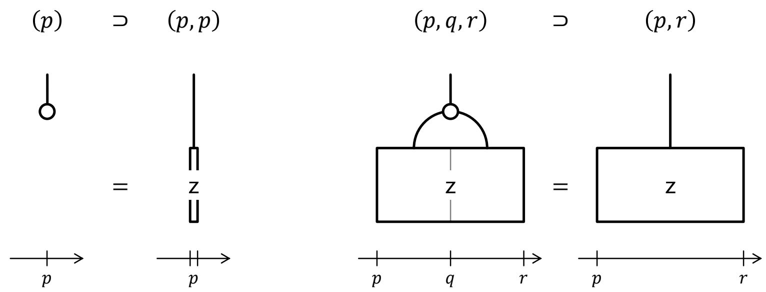
inherited from the state algebra of the quantum group K are well defined and create stochastic calculus on the dynamic system.

These claims are verified with the axioms of the state $*$-algebra. The conditions on the available states are readily interpreted: the first condition imposes that the trivial interval accumulates no non-trivial states, and the second condition imposes that states accumulated over merged intervals are the products of states accumulated over the subintervals.

The dynamic state z forms a semigroup with the properties:

$$\begin{aligned} \mathsf{z}[(p,p)] &= 1 \circ \iota \\ \mathsf{z}[(p,r)] &= (\mathsf{z}[(p,q)] \otimes \mathsf{z}[(q,r)]) \circ \Delta \end{aligned} \tag{89}$$

so that the state accumulates over merged intervals via the product of the state algebra. The semigroup properties of the dynamic state are expressed in the following string diagrams.



String diagrams of the dynamic system constructed from these elementary diagrams express the compatibility of the state with the partial ordering of time.

The dynamic system supports the multiplication of observables when the null observables form $*$-ideals in the observable algebra of the quantum group, as per the conditions expressed in the following result.

Theorem 3 (Observable algebra)**.** *If the null observables satisfy:*

$$\begin{aligned} \nabla \circ (\mathrm{Hom}^1[K,1] \otimes \mathsf{И}[p,q]) &\subset \mathsf{И}[p,q] \\ \nabla \circ (\mathsf{И}[p,q] \otimes \mathrm{Hom}^1[K,1]) &\subset \mathsf{И}[p,q] \\ \star \circ \mathsf{И}[p,q] &\subset \mathsf{И}[p,q] \end{aligned} \tag{90}$$

then the processes:

$$\begin{aligned} \eta &\in \mathrm{Hom}^1[K[(p,q)],1] \\ \nabla &\in \mathrm{Hom}^1[K[(p,q)], K[(p,q)] \otimes K[(p,q)]] \\ \star &\in \mathrm{Hom}^*[K[(p,q)], K[(p,q)]] \end{aligned} \tag{91}$$

inherited from the observable algebra of the quantum group K are well defined and create functional calculus on the dynamic system.

These claims are verified with the axioms of the observable $*$-algebra. The null observables are the observables on the interval that are indistinguishable from zero, and it makes sense to assume that these observables are closed under multiplication with arbitrary observables and under complex conjugation.

The empirical constraints of the participant are expressed in the mutually-annihilating pair of subspaces $(\mathsf{N}, \mathsf{И})$ of available states and null observables. By definition, the subspace N is a $*$-coideal if the subspace $\mathsf{И}$ is a $*$-ideal, and the conditions of the previous results require that the available states associated with each interval are $*$-coideals in the state coalgebra that collectively form a semigroup in the state algebra. Fortunately, these conditions are compatible thanks to the following result.

Theorem 4 (Bialgebra)**.** *The $*$-coideals of states are preserved by the structural processes of the state algebra.*

- *The subspace $\mathbb{C} \circ u$ of states is a $*$-coideal.*
- *If the subspaces N_1 and N_2 of states are $*$-coideals, then their product subspace $(\mathsf{N}_1 \otimes \mathsf{N}_2) \circ \Delta$ is also a $*$-coideal.*

These claims are verified with the axioms of the $*$-bialgebra. Given this compatibility with the state and observable algebras, assumed from hereon, the families of available states and null observables facilitate the stochastic and functional calculus on the dynamic system.

In the transition from discrete to continuous dynamics, the stochastic calculus approximates the path-dependent observable as a discrete integral and attempts to construct a well-defined continuous integral by refining the discrete schedule to the continuous schedule. The technical challenge in this construction lies in the transition between integral and differential perspectives, whose reversibility depends non-trivially on the state and observable algebras and on the antipode generated as the composition of their involutions.

For the schedule $P = (p_0, \ldots, p_n)$, the perspectives differ in the way they deconstruct the schedule into intervals:

$$\begin{aligned} &(p_0, p_1), (p_1, p_2), \ldots, (p_{n-1}, p_n) \\ &(p_0, p_1), (p_0, p_2), \ldots, (p_0, p_n) \end{aligned} \tag{92}$$

interchanged using integration and differentiation processes:

$$\begin{aligned} \mathcal{I} &\in \mathrm{Hom}^1[\textstyle\bigotimes_{i=1}^n K[(p_{i-1}, p_i)], \bigotimes_{i=1}^n K[(p_0, p_i)]] \\ \mathcal{D} &\in \mathrm{Hom}^1[\textstyle\bigotimes_{i=1}^n K[(p_0, p_i)], \bigotimes_{i=1}^n K[(p_{i-1}, p_i)]] \end{aligned} \tag{93}$$

Reversibility requires that the processes are inverses of each other:

$$\mathcal{D} \circ \mathcal{I} = \otimes^n \imath = \mathcal{I} \circ \mathcal{D} \tag{94}$$

implying that the two representations of the information on the schedule are equivalent. These processes need to be constructed from the structural processes of the quantum group.

The basis for the construction of integration and differentiation is the following result, imposing one more closure condition on the available states and null observables of the dynamic system.

Theorem 5 (Hopf algebra)**.** *If the available states and null observables satisfy:*

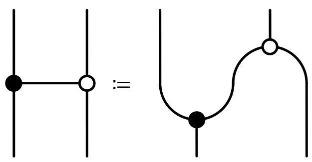
$$\begin{aligned} \mathsf{N}[p, q] \circ s &\subset \mathsf{N}[p, q] \\ s \circ \text{И}[p, q] &\subset \text{И}[p, q] \end{aligned} \tag{95}$$

then the process:

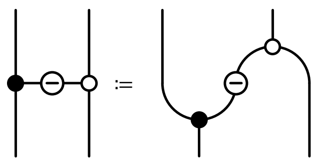
$$s \in \mathrm{Hom}^1[K[(p, q)], K[(p, q)]] \tag{96}$$

inherited from the Hopf algebra of the quantum group K is well defined.

The forward-starting and spot-starting deconstructions of the two-interval schedule are interchanged by the integration and differentiation processes:

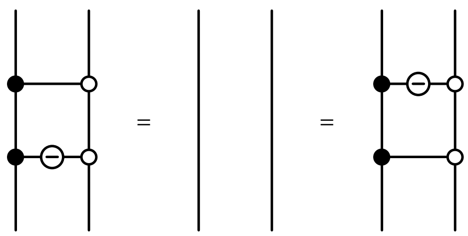


$\mathcal{I} := (\nabla \otimes \imath) \circ (\imath \otimes \Delta)$



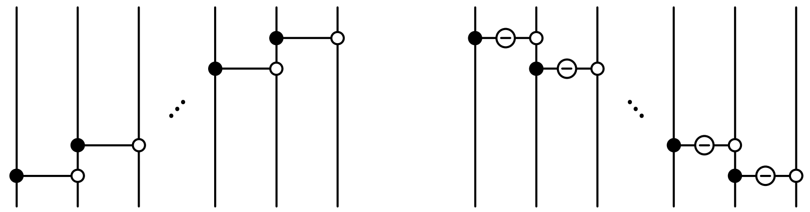
$\mathcal{D} := (\nabla \otimes \imath) \circ (\imath \otimes s \otimes \imath) \circ (\imath \otimes \Delta)$

Thanks to the Hopf $*$-algebra properties of the quantum group, these processes are mutually inverse:



$$\mathcal{D} \circ \mathcal{I} = \imath \otimes \imath = \mathcal{I} \circ \mathcal{D} \tag{97}$$

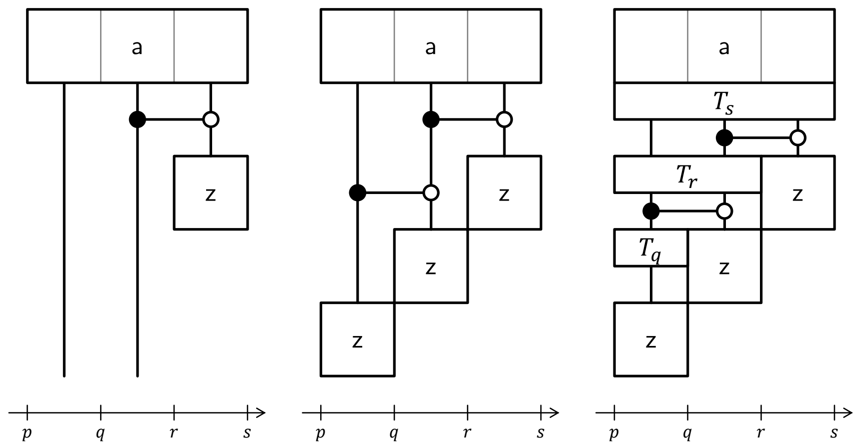
so that the two perspectives have the same information content. Using concatenation and composition, the processes integrate and differentiate over the discrete schedule:



Continuous integration is then obtained in the limit of refinement, making use of the reversibility of integration and differentiation.

The following string diagrams demonstrate the use of these processes in the evaluation of the path-dependent observable:

$$\mathsf{a} \in \mathrm{Hom}^1[K[(p,q)] \otimes K[(p,r)] \otimes K[(p,s)], 1] \tag{98}$$



The path-dependent observable a is contingent on the state accumulated from time p to times q, r and s, and the three examples demonstrate ways the observable can be evaluated by the incremental states of the valuation model z. The first example is conditional valuation:

$$(\imath \otimes ((\imath \otimes \mathsf{z}[(r,s)]) \circ \mathcal{I})) \circ \mathsf{a} \tag{99}$$

applying the state on the interval (r,s) to generate an observable contingent on the state accumulated to the times q and r. The second example completes the valuation:

$$\mathsf{z}[(p,q)] \bullet ((\imath \otimes \mathsf{z}[(q,r)]) \circ \mathcal{I}) \circ (\imath \otimes ((\imath \otimes \mathsf{z}[(r,s)]) \circ \mathcal{I})) \circ \mathsf{a} \tag{100}$$

by iterating the action of the state on the third, second and first intervals. In the third example, this sequential application of the state is interlaced:

$$\mathsf{z}[(p,q)] \bullet T_q[((\imath \otimes \mathsf{z}[(q,r)]) \circ \mathcal{I}) \circ T_r[(\imath \otimes ((\imath \otimes \mathsf{z}[(r,s)]) \circ \mathcal{I})) \circ T_s[\mathsf{a}]]] \tag{101}$$

with additional transformations acting on the observable at each stage.

As demonstrated in these examples, conditional valuation over the final interval in the schedule $P \vee (p, q)$ is the composition of integrator and state:

$$(\otimes^{|P|-1}\imath) \otimes ((\imath \otimes \mathsf{z}[(p, q)]) \circ \mathcal{I}) \tag{102}$$

Generalising the examples, a valuation scheme is defined that applies the transformation, the integrator and the state to create a sequence of observables $(\mathsf{a}_0, \dots, \mathsf{a}_n)$ on the schedule $(p_0, \dots, p_n)$ by iterating the actions:

$$\mathsf{a}_{i-1} = ((\otimes^{i-2}\imath) \otimes ((\imath \otimes \mathsf{z}[(p_{i-1}, p_i)]) \circ \mathcal{I})) \circ T_i[\mathsf{a}_i] \tag{103}$$

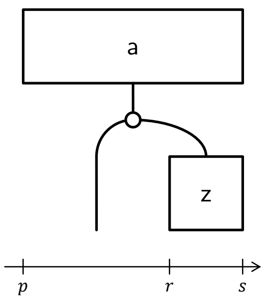
for $i = 2, \dots, n$, terminating the scheme with the valuation:

$$\mathsf{a}_0 = \mathsf{z}[(p_0, p_1)] \bullet T_1[\mathsf{a}_1] \tag{104}$$

Transformations are drawn from the functional calculus, polynomially combining the unit and product of the observable algebra, and extending beyond this using convergent limits in the operator representation of the Gelfand-Naimark-Segal construction. The integrator and state then combine to roll the valuation to the preceding time.

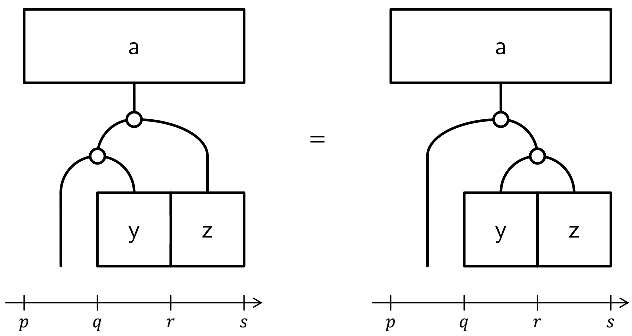
This is a typical sequence in the pricing of financial derivatives, where the termsheet prescribes both the terminal payoff and additional payments or termination clauses at earlier times. The structural processes of the quantum group are essential for this construction, consistently capturing the accumulation of states and the convexities between observables, and equipping the valuation model with a notion of integral and differential. The quantum group thus provisions the valuation model with all the operations necessary to express and evaluate the financial derivative.

The formalism simplifies when the observable depends only on the accumulated state from the start to the end of the interval, and does not depend on the path between these times. There are many useful problems where this assumption can be made, and the reduction in dimensionality it presents dramatically improves the performance of numerical methods. The property of path independence is preserved by conditional valuation, which is essential for the efficacy of the method. By definition, the conditional valuation $\mathsf{z} \triangleleft \mathsf{a}$ of the observable a by the state z is the observable:



$$\mathsf{z} \triangleleft \mathsf{a} := (\imath \otimes \mathsf{z}) \circ \Delta \circ \mathsf{a} \tag{105}$$

using the coproduct to express the observable as contingent on two intervals and then contracting with the state on the second interval. The compounded action of two states on the observable is generated by the product of the states:



$$\mathsf{y} \triangleleft (\mathsf{z} \triangleleft \mathsf{a}) = \mathsf{yz} \triangleleft \mathsf{a} \tag{106}$$

thanks to associativity. Conditional valuation thus defines an action of the semigroup of states on the observables. As in the path-dependent case, this action of states is interlaced with transformations on the observable to generate a valuation scheme that can be used in pricing.

4 Discrete states and observables

The most general valuation model with finite states and observables is developed from its representation as operators on a finite-dimensional space. This model is easy to evaluate and avoids technical pitfalls, and can be embedded within common algorithms that discretise the states and observables. None of these statements carry merit, though, if the resulting model does not exhibit novel properties useful for real applications. Fortunately, the transition from classical to quantum significantly extends the phenomenology of the model, even in the finite-dimensional case. As an example of the approach, option pricing – the most elementary challenge of mathematical finance – is enriched by the quantum extension in ways that characterise the underlying von Neumann algebra.

4.1 Finite-dimensional von Neumann algebras

The mutually-commutant von Neumann algebras of states and observables decompose as direct sums of von Neumann factors with trivial centres, each isomorphic to a matrix algebra. A convenient representation, used in the following, decomposes the Hilbert space:

$$\mathsf{H} = \bigoplus_{i=1}^{n} \mathsf{H}_i \tag{107}$$

where $\mathsf{H}_i = \mathbb{C}[M_i, W_i]$ is the space of complex matrices with M_i rows and W_i columns equipped with the Hilbert-Schmidt inner product:

$$\langle \phi_i | \psi_i \rangle := \mathrm{tr}[\phi_i \psi_i^*] \tag{108}$$

for the matrices $\phi_i, \psi_i \in \mathsf{H}_i$. The von Neumann algebras M of states and W of observables then decompose:

$$\mathsf{M} = \bigoplus_{i=1}^{n} \mathsf{M}_i \qquad\qquad \mathsf{W} = \bigoplus_{i=1}^{n} \mathsf{W}_i \tag{109}$$

where $\mathsf{M}_i = \mathbb{C}[M_i]$ is the algebra of complex square matrices with M_i rows and columns, and $\mathsf{W}_i = \mathbb{C}[W_i]$ is the algebra of complex square matrices with W_i rows and columns. These algebras are represented on the Hilbert space via left and right multiplication:

$$\langle\phi_i|[\mathsf{z}_i] := \langle\mathsf{z}_i^t\phi_i| \qquad\qquad \langle\phi_i|[\mathsf{a}_i] := \langle\phi_i\mathsf{a}_i| \tag{110}$$

for the state $\mathsf{z} = \mathsf{z}_1 \oplus \cdots \oplus \mathsf{z}_n \in \mathsf{M}$ and the observable $\mathsf{a} = \mathsf{a}_1 \oplus \cdots \oplus \mathsf{a}_n \in \mathsf{W}$. The representations commute:

$$\langle\phi_i|[\mathsf{z}_i][\mathsf{a}_i] = \langle\mathsf{z}_i^t\phi_i\mathsf{a}_i| = \langle\phi_i|[\mathsf{a}_i][\mathsf{z}_i] \tag{111}$$

so the states and observables are mutually commutant, $\mathsf{M}' = \mathsf{W}$ and $\mathsf{W}' = \mathsf{M}$.

With this representation, the valuation model is expressed as:

$$\mathsf{z} \bullet \mathsf{a} = \sum\nolimits_{i=1}^{n} \langle\omega_i|[\mathsf{z}_i][\mathsf{a}_i]|\omega_i\rangle = \sum\nolimits_{i=1}^{n} \mathrm{tr}[\mathsf{z}_i^t\omega_i\mathsf{a}_i\omega_i^*] \tag{112}$$

in terms of its weight matrix $\omega = \omega_1 \oplus \cdots \oplus \omega_n \in \mathsf{H}$. In the application to finance, the weight ω describes the economic state:

$$\mathsf{z}_e \bullet \mathsf{a} = \sum\nolimits_{i=1}^{n} \langle\omega_i|[\mathsf{a}_i]|\omega_i\rangle = \sum\nolimits_{i=1}^{n} \mathrm{tr}[\omega_i\mathsf{a}_i\omega_i^*] \tag{113}$$

which is positive by construction. A pricing state that is equivalent to the economic state is obtained by inserting its Radon-Nikodym matrix in the trace; the pricing state is then absent of arbitrage when this matrix is positive definite.

4.2 Quantum option pricing

The structural parameters of the valuation model are the pairs of strictly positive integers $((M_1, W_1), \ldots, (M_n, W_n))$ that dimension its von Neumann factors. Classical valuation sets each of these integers to one, reducing the model to evaluation against a discrete distribution with weights $(\mathsf{z}_1|\omega_1|^2, \ldots, \mathsf{z}_n|\omega_n|^2)$. If there is a novel contribution from quantum valuation, it must therefore emerge from the trace of noncommuting matrices in dimensions greater than one. In this section, the focus is restricted to the case of a single von Neumann factor, and the subscripts indexing the factor are dropped from expressions.

The multiple settlements associated with a typical financial derivative may occur at different times and in different currencies, and the present value of each of these settlements is evaluated using the state associated with its settlement conditions. In this general case, the derivative has present value:

$$u := \sum\nolimits_{c} \mathsf{z}_c \bullet \mathsf{a}_c = \mathrm{tr}\Big[\sum\nolimits_{c} \mathsf{z}_c^t\omega\mathsf{a}_c\omega^*\Big] \tag{114}$$

where the observable a_c is the payoff with settlement conditions c and the state z_c is the corresponding pricing state, accounting for discounting and the exchange rate of the settlement.

Overlaying this underlying with optionality, if the holder has the right but not the obligation to settle, the exercise decision needs to be incorporated into the option price:

$$o_{\mathsf{p}} := \sum\nolimits_{c} \mathsf{z}_c \bullet \mathsf{p}\mathsf{a}_c\mathsf{p} = \mathrm{tr}\Big[\sum\nolimits_{c} \mathsf{z}_c^t\omega\mathsf{p}\mathsf{a}_c\mathsf{p}\omega^*\Big] \tag{115}$$

Exercise is indicated by the projection observable p, a self-adjoint matrix with eigenvalues in $\{0,1\}$, expressed algebraically as the properties $\mathsf{p}^* = \mathsf{p} = \mathsf{p}^2$. Optimal exercise maximises the price of the option:

$$o := \sup_{\mathsf{p}} \left\{ \sum\nolimits_c \mathsf{z}_c \bullet \mathsf{p}\mathsf{a}_c\mathsf{p} \right\} = \sup_{\mathsf{p}} \left\{ \mathrm{tr}\left[\sum\nolimits_c \mathsf{z}_c^t \omega \mathsf{p}\mathsf{a}_c\mathsf{p}\omega^* \right] \right\} \tag{116}$$

taking the supremum over all projections. The projection that achieves this supremum represents the optimal exercise strategy for the option. Noncommutativity introduces complexity in this calculation, preventing the simultaneous diagonalisation of the matrices in the sum, which leads to an option price that cannot be replicated in a classical discrete model.

As an example where this supremum, and the projection that achieves it, can be calculated explicitly, consider the case where the settlement amounts are known in advance. In this case, the settlement observables are proportional to the identity, $\mathsf{a}_c = a_c 1$, and cyclicity of trace derives the price:

$$o_{\mathsf{p}} = \mathrm{tr}\left[\mathsf{p} \left(\sum\nolimits_c a_c Z_c \right) \mathsf{p} \right] \tag{117}$$

where:

$$Z_c = \omega^* \mathsf{z}_c^t \omega \tag{118}$$

The exercise decision that maximises this expression is the projection onto the direct sum of the eigenspaces with positive eigenvalues of the bracketed matrix, and the option price becomes:

$$o = \mathrm{tr}\left[\left(\sum\nolimits_c a_c Z_c \right)^+ \right] \tag{119}$$

where the trace on the right is the sum of the positive eigenvalues of the bracketed matrix. This evaluation of the option price thus involves the roots of the characteristic polynomial of the W-dimensional payoff matrix.

By the absence of arbitrage for the state z_c, the matrix Z_c is positive. The option price is thus evaluated as the sum of the positive eigenvalues for the difference of two positive matrices:

$$o = \mathrm{tr}[(R - P)^+] \tag{120}$$

paying the matrix P and receiving the matrix R defined by:

$$\begin{aligned} P &= -\sum_{a_c<0} a_c Z_c \\ R &= \sum_{a_c>0} a_c Z_c \end{aligned} \tag{121}$$

The underlying has the structure of a swap, exchanging the settlements with $a_c < 0$ for the settlements with $a_c > 0$, and the option confers the right but not the obligation to enter the swap. This expression can be used to model the prices of common derivatives, such as foreign exchange options and interest rate swaptions.

Slightly generalising the presentation, consider the option to receive one unit of the settlements with $a_c > 0$ in exchange for k units of the settlements with

$a_c < 0$, where $k \geq 0$ is the strike of the option. For strikes near zero, the received settlements dominate and the option is always exercised. For high strikes, the paid settlements dominate and the option is never exercised. Inbetween, there is a regime where it may or may not be optimal to exercise.

By introducing the strike, the option price can be expressed in terms of an implied probability density pdf[s] for the swap rate s:

$$\int_{s=0}^{\infty} (s-k)^+ \operatorname{pdf}[s]\, ds := \operatorname{tr}[(R-kP)^+]/\operatorname{tr}[P] \tag{122}$$

From this definition, the implied cumulative density cdf[s] is derived as:

$$\operatorname{cdf}[s] := 1 + \frac{d}{dk} \operatorname{tr}[(R-kP)^+]/\operatorname{tr}[P]\Big|_{k=s} \tag{123}$$

In the classical discrete model, the probability density is discrete, supported on at most $\mathcal{W}$ points, and the cumulative density is a step function. As will be demonstrated below, the probability density of the quantum discrete model has components with both discrete and continuous support.

To see how this happens, further simplify to the two-dimensional binomial model, $\mathcal{W} = 2$. Without loss of generality, the two matrices P and R are expressed as:

$$\begin{aligned} P &= \begin{bmatrix} p_+ & 0 \\ 0 & p_- \end{bmatrix} \\ R &= \begin{bmatrix} r_+ \cos[\theta]^2 + r_- \sin[\theta]^2 & (r_+ - r_-) e^{-i\phi} \cos[\theta] \sin[\theta] \\ (r_+ - r_-) e^{i\phi} \cos[\theta] \sin[\theta] & r_+ \sin[\theta]^2 + r_- \cos[\theta]^2 \end{bmatrix} \end{aligned} \tag{124}$$

where the eigenvalues $\{p_-, p_+\}$ of the matrix P and the eigenvalues $\{r_-, r_+\}$ of the matrix R are assumed to satisfy $0 \leq p_- \leq p_+$ and $0 \leq r_- \leq r_+$. The coordinate frame is chosen to diagonalise P, and the angles θ and ϕ then rotate to the coordinate frame that diagonalises R. It is these rotations that create the quantum features of the model.

The eigenvalues $\{u_-[k], u_+[k]\}$ of the matrix $R - kP$ are computed as the roots of its characteristic binomial:

$$u_\pm[k] = \frac{1}{2}(\bar{r} - k\bar{p}) \pm \frac{1}{2}\sqrt{\hat{r}^2 - 2k\hat{r}\hat{p}\cos[2\theta] + k^2\hat{p}^2} \tag{125}$$

where:

$$\begin{aligned} \bar{p} &= p_+ + p_- & \qquad \hat{p} &= p_+ - p_- \\ \bar{r} &= r_+ + r_- & \qquad \hat{r} &= r_+ - r_- \end{aligned} \tag{126}$$

The option price sums only the positive eigenvalues. Three regimes for the option price are delimited by the strikes:

$$k_\pm = \frac{\bar{r}\bar{p} - \hat{r}\hat{p}\cos[2\theta]}{\bar{p}^2 - \hat{p}^2} \left(1 \pm \sqrt{1 - \frac{(\bar{r}^2 - \hat{r}^2)(\bar{p}^2 - \hat{p}^2)}{(\bar{r}\bar{p} - \hat{r}\hat{p}\cos[2\theta])^2}} \right) \tag{127}$$

satisfying $0 \leq k_- \leq k_+$. If $p_- = 0$, these delimiting strikes are:

$$k_- = \frac{1}{2\bar{p}} \frac{\bar{r}^2 - \hat{r}^2}{\bar{r} - \hat{r}\cos[2\theta]} \qquad\qquad k_+ = \infty \tag{128}$$

and if $r_- = 0$, these delimiting strikes are:

$$k_- = 0 \qquad\qquad k_+ = 2\bar{r}\frac{\bar{p} - \hat{p}\cos[2\theta]}{\bar{p}^2 - \hat{p}^2} \tag{129}$$

In the low-strike regime $k \leq k_-$, both eigenvalues are positive and the option is always exercised. In the mid-strike regime $k_- \leq k \leq k_+$, the eigenvalues satisfy $u_-[k] \leq 0 \leq u_+[k]$ and the option price is the positive eigenvalue. In the high-strike regime $k_+ \leq k$, both eigenvalues are negative and the option is never exercised. The option price is thus:

$$\begin{aligned} o[k] &= \bar{r} - k\bar{p} && k < k_- \\ o[k] &= \frac{1}{2}(\bar{r} - k\bar{p}) + \frac{1}{2}\sqrt{\hat{r}^2 - 2k\hat{r}\hat{p}\cos[2\theta] + k^2\hat{p}^2} && k_- \leq k < k_+ \\ o[k] &= 0 && k_+ \leq k \end{aligned} \tag{130}$$

Differentiating the option price with respect to the strike generates the implied cumulative density:

$$\begin{aligned} \mathrm{cdf}[s] &= 0 && s < k_- \\ \mathrm{cdf}[s] &= \frac{1}{2}\left(1 + \frac{\hat{p}}{\bar{p}}\frac{s\hat{p} - \hat{r}\cos[2\theta]}{\sqrt{\hat{r}^2 - 2s\hat{r}\hat{p}\cos[2\theta] + s^2\hat{p}^2}}\right) && k_- \leq s < k_+ \\ \mathrm{cdf}[s] &= 1 && k_+ \leq s \end{aligned} \tag{131}$$

This is clearly not the cumulative density of a discrete distribution. The distribution has discrete probability density at the boundary points $s = k_\pm$, but also has continuous probability density inbetween. No classical discrete model is able to generate this distribution.

In the binomial model that generates the graphs on the next page, the matrices P and R both have eigenvalues $\{0.2, 0.8\}$, and the graphs present the implied probability and cumulative densities for six different angles of separation θ between the diagonalising eigenbases of the matrices. The swap rate R/P is well defined only when the two matrices are simultaneously diagonalisable, corresponding to the angles $\theta = 0$ and $\theta = \pi/2$. In the case $\theta = 0$, the matrices positively correlate and their simultaneous eigenvalues are $\{(0.2, 0.2), (0.8, 0.8)\}$, so the swap rate has a single eigenvalue $\{1\}$. In the case $\theta = \pi/2$, the matrices negatively correlate and their simultaneous eigenvalues are $\{(0.2, 0.8), (0.8, 0.2)\}$, so the swap rate has two eigenvalues $\{1/4, 4\}$. These eigenvalues for the swap rate are the discrete support of the implied density.

For other values of θ, the distribution has finite probabilities at two discrete points and a continuous density between these boundary points. The discrete points smoothly interpolate the bounding eigenvalues of the swap rate as the angle is varied. Quantum tunneling effectively leaks probability into the interval spanned by the boundary points, implying a distribution with continuous support from a binomial model whose pay and receive matrices have at most two discrete eigenvalues. Dialling the angle from the positively correlated case $\theta = 0$ to the negatively correlated case $\theta = \pi/2$ monotonically increases the variance of the resulting distribution.

The quantum binomial model has particularly interesting behaviour when the pay and receive matrices have zero eigenvalues, as the implied distribution

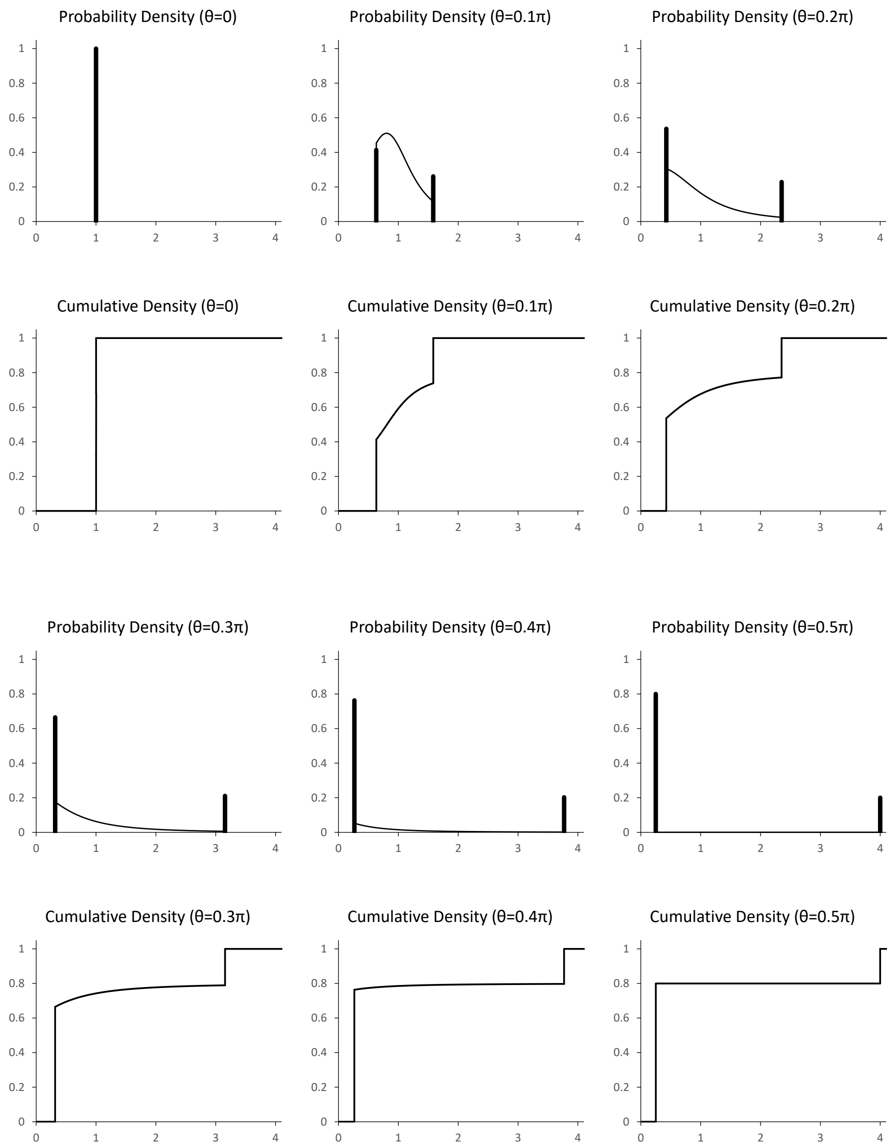
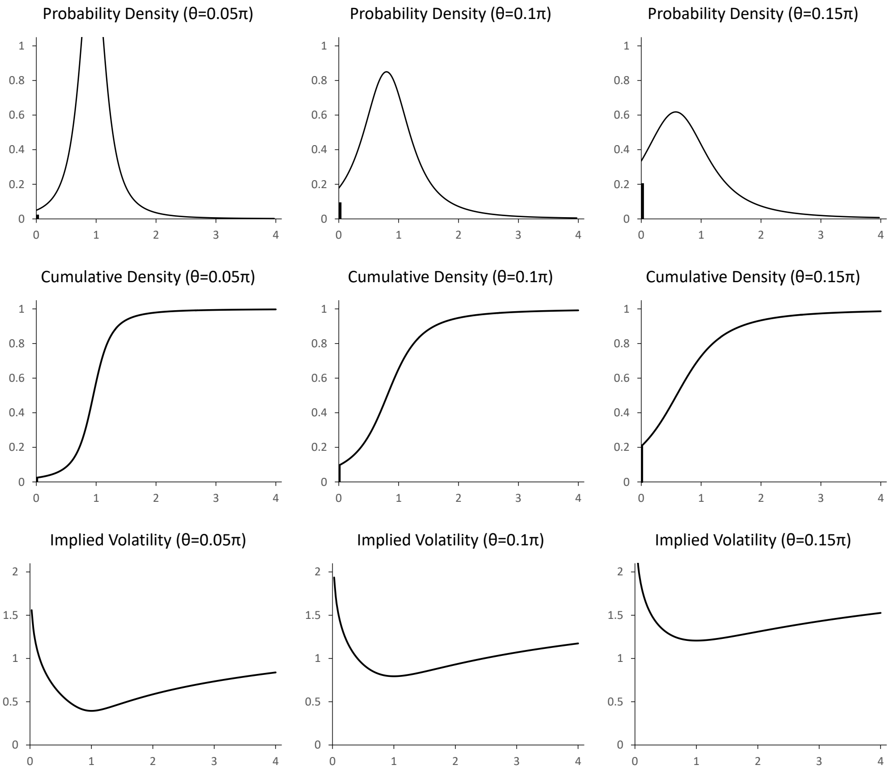


Figure 6: The implied probability and cumulative densities for option pricing in the quantum binomial model with eigenvalues 0.2 and 0.8 for the pay and receive matrices. The distribution changes as the angle θ between the diagonalising bases for the two matrices varies. In the cases $\theta = 0$ and $\theta = \pi/2$, the matrices are simultaneously diagonalisable and the result is a classical binomial model. As θ varies between these extremes, quantum effects create a continuous density between the two points of the discrete density.

is in this case supported on the upper half line, and the resulting option price is similar to that produced by popular stochastic volatility models. In the next example, both the matrices P and R are assumed to have eigenvalues $\{0, 1\}$.



The last set of graphs in this series represents the distribution by its implied volatility smile, which for each strike expresses the option price as the lognormal volatility used in the Black-Scholes formula to reproduce the price.

Higher dimensions generate more complex phenomenology from eigenvalues computed as the roots of the higher-order characteristic polynomial of the payoff matrix. As with the two-dimensional case, the price of the option is expressed as the sum of the positive roots of the polynomial, and complexity is generated in the valuation model thanks to the non-triviality of this exercise. The implied probability density is then a combination of a discrete distribution and a continuous distribution on its convex support, with structure characterised by the geometric relationship between the eigenbases of the pay and receive matrices.

Fundamental to the development of the price model is the existence of a complete set of projections representing the exercise strategies for the option. Beyond finite dimensions, the model is enriched by the theory of von Neumann factors. Utilising the trace to extend the method to type I_∞ and II factors, the results of this section remain valid subject to any necessary convergence conditions. For type III factors, the relationships between the state and observable algebras, encapsulated in the operators of Tomita-Takesaki theory, are leveraged to evaluate the option price. The distributions implied by the model characterise the von Neumann factor in the finite-dimensional case. Whether this statement holds more generally remains to be investigated.

5 Interest rate modelling

The preceding section considered the challenge of option pricing in the single-interval context, completely describing the discrete model for options with European exercise style. In this section, additional structure is introduced that supports development in the multi-interval context, subsequently applied to the pricing of options with Bermudan exercise style. The additional structure is provided by the finite-dimensional representations of a complex semisimple Lie algebra, exploiting the property that its universal enveloping algebra can be equipped with the operations of Hopf $*$-algebra.

5.1 Quantum groups from classical groups

For a classical group K, define the corresponding system to have states given by representative functions and observables given by formal linear combinations of group elements. The state $\mathsf{z} \in \mathsf{M}[K]$ is defined by a matrix $\mathsf{z} \in \mathrm{End}[V]$ and a representation $\omega : K \to \mathrm{End}[V]$ of the group on a finite-dimensional complex space V associated with the state, generating the scalar-valued function:

$$\mathsf{z} : x \mapsto \mathrm{tr}[\mathsf{z}\,\omega[x]] \tag{132}$$

on the group. The observable $\mathsf{a} \in \mathsf{W}[K]$ is defined to be a formal sum:

$$\mathsf{a} = \sum\nolimits_{x \in K} \mathsf{a}_x\, x \tag{133}$$

where only a finite number of the scalars a_x are non-zero. The pairing is then:

$$\mathsf{z} \bullet \mathsf{a} := \mathrm{tr}[\mathsf{z}\,\omega[\mathsf{a}]] \tag{134}$$

linearly extending the action of the representation. This system is a quantum group with the structural processes:

$$\begin{aligned} \iota \circ \mathsf{a} &:= \sum\nolimits_x \mathsf{a}_x & \eta \circ \lambda &:= \lambda\, 1 \\ \Delta \circ \mathsf{a} &:= \sum\nolimits_x \mathsf{a}_x\, x \otimes x & \nabla \circ (\mathsf{a} \otimes \mathsf{b}) &:= \sum\nolimits_{x,y} \mathsf{a}_x \mathsf{b}_y\, xy \\ \ast \circ \mathsf{a} &:= \sum\nolimits_x \mathsf{a}_x^*\, x & \star \circ \mathsf{a} &:= \sum\nolimits_x \mathsf{a}_x^*\, x^{-1} \end{aligned} \tag{135}$$

and antipode:

$$s \circ \mathsf{a} := \sum\nolimits_x \mathsf{a}_x\, x^{-1} \tag{136}$$

constructed as the composition of the involutions.

With dual interpretations available for the information model, the group product can be utilised either as accumulation of states or as multiplication of observables. The definitions above take the latter interpretation, which benefits option pricing through the availability of non-commuting observables. The representation ω creates a subsystem $K[\omega]$ with available states $\mathsf{N}[\omega]$ comprising all the representative functions generated by the representation and null observables $\mathsf{И}[\omega]$ given by the kernel of the representation. Dynamic systems are constructed from subsystems in this form, making a connection between dynamic modelling and the representation theory of groups.

Inspired by these observations, this idea is now applied to the complexified Lie algebra K of a compact Lie group. The observable space $\mathsf{W}[K]$ associated with the Lie algebra is taken to be its universal enveloping algebra, the free algebra generated by the elements of K modulo relations of the form:

$$\mathsf{ab} - \mathsf{ba} = [\mathsf{a}, \mathsf{b}] \tag{137}$$

for the elements a and b of the Lie algebra. This system is a quantum group with the structural processes:

$$\begin{aligned} u \circ \mathsf{a} &:= 0 & \eta \circ \lambda &:= \lambda \\ \Delta \circ \mathsf{a} &:= \mathsf{a} \otimes 1 + 1 \otimes \mathsf{a} & \nabla \circ (\mathsf{a} \otimes \mathsf{b}) &:= \mathsf{ab} \\ \ast \circ \mathsf{a} &:= \mathsf{a}_R - i\mathsf{a}_I & \star \circ \mathsf{a} &:= -\mathsf{a}_R + i\mathsf{a}_I \end{aligned} \tag{138}$$

and antipode:

$$s \circ \mathsf{a} := -\mathsf{a} \tag{139}$$

defined on the Lie algebra and extended to the universal enveloping algebra consistently with the axioms of $*$-bialgebra, where $\mathsf{a} = \mathsf{a}_R + i\mathsf{a}_I$ is the unique decomposition of the complexified element into its real and imaginary parts. The complexified Lie algebras of compact Lie groups are precisely the complex semisimple Lie algebras, with an elegant representation theory used in the following to create models for pricing interest rate derivatives.

5.2 Semisimple Lie algebras

A dynamic system is constructed from a semisimple Lie algebra K over the complex scalars. The construction exploits the weight lattice of its set of irreducible representations, wherein each positive weight p is associated with the simple module $K[p]$ generated by the action of the universal enveloping algebra on the eigenvector with highest weight. Option pricing in this model is rich in features and numerically efficient, reducing to the eigenvalue problem as outlined in the previous section. Remarkably, everything needed by the dynamic system, including time and the states and observables, can be uniquely identified from the Lie algebra.

Definition 3 (Lie dynamic system)**.** *The dynamic system generated by the semisimple Lie algebra K has time given by its weight lattice of irreducible representations, partially ordered by a choice of simple roots. For the schedule $P = (p_0, \ldots, p_n)$, define the module:*

$$K[P] := K[p_1 - p_0] \otimes \cdots \otimes K[p_n - p_{n-1}] \tag{140}$$

where $K[p]$ is the simple module with highest weight p. The states and observables associated with the schedule are the endomorphisms on this module:

$$\mathsf{M}[K[P]] := \mathrm{End}[K[P]] =: \mathsf{W}[K[P]] \tag{141}$$

with pairing defined by the trace:

$$\mathsf{z} \bullet \mathsf{a} := \mathrm{tr}[\mathsf{za}] \tag{142}$$

for the state $\mathsf{z} \in \mathsf{M}[K[P]]$ and the observable $\mathsf{a} \in \mathsf{W}[K[P]]$.

Clebsch-Gordan decomposition expresses the tensor product of simple modules $K[p_1] \otimes \cdots \otimes K[p_n]$ as the direct sum of simple modules. For the dynamic system, the key property needed from this decomposition is the inclusion of the simple submodule:

$$K[p_1 + \cdots + p_n] \subset K[p_1] \otimes \cdots \otimes K[p_n] \tag{143}$$

with multiplicity one, separating the data for the total weight from the additional data that describes the weight path. Use the decomposition to define the isometry and coisometry:

$$\begin{aligned} \Delta &: K[p_1 + \cdots + p_n] \to K[p_1] \otimes \cdots \otimes K[p_n] \\ \Delta^* &: K[p_1] \otimes \cdots \otimes K[p_n] \to K[p_1 + \cdots + p_n] \end{aligned} \tag{144}$$

satisfying the property:

$$\Delta\Delta^* = 1 \tag{145}$$

The existence of a unique contribution with highest weight to the decomposition of the tensor product supports the definition of accumulation in the dynamic system, which provides the conditions for the consistent expression of the state.

Definition 4 (Lie dynamic state)**.** *The accumulation functor associated with the semisimple Lie algebra K is defined on the refinement $P \supset (P_-, P_+)$ by:*

$$\begin{aligned} K[P \supset (P_-, P_+)] \circ \mathsf{a} &:= \Delta^* \mathsf{a} \Delta \\ \mathsf{z} \circ K[P \supset (P_-, P_+)] &:= \Delta \mathsf{z} \Delta^* \end{aligned} \tag{146}$$

for the state $\mathsf{z} \in \mathsf{M}[K[P]]$ and the observable $\mathsf{a} \in \mathsf{W}[K[(P_-, P_+)]]$, where:

$$\begin{aligned} \Delta &: K[(P_-, P_+)] \to K[P] \\ \Delta^* &: K[P] \to K[(P_-, P_+)] \end{aligned} \tag{147}$$

are the isometry and coisometry that identify the highest-weight submodule in the tensor product. The definition is extended to other refinements by concatenation.

The state $\mathsf{z} \in \mathsf{M}[K]$ associates the state $\mathsf{z}[P] \in \mathsf{M}[K[P]]$ with the schedule P. This association satisfies the naturality condition:

$$\Delta(\mathsf{z}[(p_0, p_1)] \otimes \cdots \otimes \mathsf{z}[(p_{n-1}, p_n)])\Delta^* = \mathsf{z}[(p_0, p_n)] \tag{148}$$

for the schedule $P = (p_0, \ldots, p_n)$, enforcing consistency of the state with refinement in the dynamic system.

5.3 European and Bermudan swaptions

By adding time to the information model, the dynamic system generated by the semisimple Lie algebra is able to price financial derivatives contingent on the state of the economy at multiple times, generating credible implied volatility smiles from a dynamic model with finite discrete eigenstates.

The interest rate model is simplified with the assumption that the floating rate matches the discount rate over its accrual period, so that the floating accrual can be replaced by notional exchange. This simplification is easily removed, but is retained here to avoid obscuring the contribution that the Lie algebra makes to the price of convexity.

Discounting in the interest rate model is marked to the price of the discount bond $\mathsf{d}[p]$ that returns the unit payoff at time p. In the pricing state z, the discount factor at earlier time e is:

$$\mathsf{d}[p][e] = \mathsf{z}[(e, p)] \triangleleft 1 \tag{149}$$

The discount curve d at time zero maps the positive time p to its discount factor $d[p]$, and the model is calibrated to this curve with the condition:

$$d[p] = \mathrm{tr}[\mathsf{z}[(0, p)]] \tag{150}$$

matching the trace of the pricing state with the discount curve. With this normalisation, the model consistently prices financial derivatives constructed as linear combinations of discount bonds.

The basic interest rate derivative is the swap $\mathsf{u}[P, \delta, k]$ parametrised by its accrual schedule $P = (p_0, \ldots, p_n)$, its accrual fractions $\delta = (\delta_1, \ldots, \delta_n)$ and its fixed rate k. The price of the swap at earlier time e is:

$$\mathsf{u}[P, \delta, k][e] = (\mathsf{d}[p_0][e] - \mathsf{d}[p_n][e]) - k \sum_{i=1}^{n} \mathsf{d}[p_i][e]\, \delta_i \tag{151}$$

where the first two terms price the floating leg, collapsed to initial and final notional exchange, and the last term prices the fixed leg. The model is calibrated to the par swap rate $f[P, \delta, k]$ at time zero with the condition:

$$f[P, \delta, k] = \frac{d[p_0] - d[p_n]}{\sum_{i=1}^{n} d[p_i]\, \delta_i} \tag{152}$$

that derives the par swap rate from the discount curve.

Interest rate options are priced in this model by incorporating the contribution from the exercise decision. The European swaption $\mathsf{o}[e, P, \delta, k]$ is the option to enter the swap $\mathsf{u}[P, \delta, k]$ exercisable at a single time e prior to the schedule of the swap. The price at time zero for the European swaption is then:

$$o[e, P, \delta, k] = \mathrm{tr}[\mathsf{z}[(0, e)]\mathsf{u}[P, \delta, k][e]^{+}] \tag{153}$$

The Bermudan swaption $\mathsf{b}[P, \delta, k]$ is the option to enter the swap $\mathsf{u}[P, \delta, k]$ exercisable at each time $p_i = p_0, \ldots, p_{n-1}$ during the schedule of the swap, where the price $\mathsf{u}[P, \delta, k][p_i]$ is assumed to include only the accruals on the remainder $(p_i, \ldots, p_n)$ of the schedule. The price of the Bermudan swaption thus satisfies the valuation scheme:

$$\begin{aligned} \mathsf{b}[P, \delta, k][p_n] &= 0 \\ \mathsf{b}[P, \delta, k][p_i\,] &= \mathsf{z}[(p_i, p_{i+1})] \triangleleft \mathsf{b}[P, \delta, k][p_{i+1}] \\ &\quad + (\mathsf{u}[P, \delta, k][p_i] - \mathsf{z}[(p_i, p_{i+1})] \triangleleft \mathsf{b}[P, \delta, k][p_{i+1}])^{+} \end{aligned} \tag{154}$$

for $i = 0, \ldots, n-1$, terminating with the price:

$$b[P, \delta, k] = \mathrm{tr}[\mathsf{z}[(0, p_0)]\mathsf{b}[P, \delta, k][p_0]] \tag{155}$$

at time zero. The first term in the iteration is the continuation value of the Bermudan swaption, and the second term is the additional value from the option to replace this with the underlying swap.

Both these types of interest rate option are evaluated from the weighted trace of the positive component of the payoff matrix, in a single step for the European case and embedded in the valuation scheme for the Bermudan case. Volatility emerges from the lack of simultaneous diagonalisability of the fixed and floating legs in the swap, creating a wide variety of realistic implied volatility smiles from the eigenvalue algorithm for the option price.

The price expressions involve the operation of conditional valuation constructed from the accumulation functor of the dynamic system, in turn derived from the irreducible representations of the Lie algebra. Semisimplicity ensures this construction remains within the confines of finite dimensions, and the weight lattice imposes a recombining structure that generalises the classical multinomial model. This is demonstrated below for the case of the two-dimensional special linear Lie algebra.

5.4 The quantum binomial model

Constructed from the representations of the semisimple Lie algebra, the dynamic system depends on the identification of the submodule with highest weight in the Clebsch-Gordan decomposition of the tensor product. While there are algorithms for determining this decomposition, it is highly non-trivial and has no simple expression in general.

In the following the Lie algebra $K = \mathrm{sl}[2]$, the special linear Lie algebra in two dimensions, is considered in depth. The decomposition is straightforward in this case, and the resulting dynamic system highlights the main attributes of representation theory that have utility in pricing. This discrete model combines the numerical efficiency of the classical binomial model with the more realistic marginal distributions implied by its quantum extension.

The Lie algebra K is spanned by the standard basis $\{\mathsf{h}, \mathsf{e}, \mathsf{f}\}$ satisfying the commutation relations:

$$\begin{aligned} [\mathsf{e}, \mathsf{h}] &= 2\mathsf{e} \\ [\mathsf{f}, \mathsf{h}] &= -2\mathsf{f} \\ [\mathsf{f}, \mathsf{e}] &= \mathsf{h} \end{aligned} \tag{156}$$

The weight lattice of the Lie algebra is the set of integers, and the simple module $K[p]$ generated by the positive weight p is $(p+1)$-dimensional. There is a basis $\{\langle p, q| : q = -p, -p+2, \ldots, p-2, p\}$ of eigenvectors of h with e acting as raising operator and f acting as lowering operator:

$$\begin{aligned} \langle p, q| \,\mathsf{h} &= q \,\langle p, q| \\ \langle p, q| \,\mathsf{e} &= \frac{1}{2}\sqrt{(p-q)(p+q+2)}\,\langle p, q+2| \\ \langle p, q| \,\mathsf{f} &= \frac{1}{2}\sqrt{(p+q)(p-q+2)}\,\langle p, q-2| \end{aligned} \tag{157}$$

The Clebsch-Gordan decomposition is derived from these relations. Separating out the contribution $K[p+1]$ from the tensor product $K[p] \otimes K[1]$, the dynamic system is generated from the mutually-adjoint linear maps:

$$\begin{aligned} \Delta &: K[p+1] \to K[p] \otimes K[1] \\ \Delta^* &: K[p] \otimes K[1] \to K[p+1] \end{aligned} \tag{158}$$

where the isometry is defined by:

$$\langle p+1, q|\,\Delta = \tag{159}$$
$$\sqrt{\frac{p+q+1}{2(p+1)}}\,\langle p, q-1| \otimes \langle 1, 1| + \sqrt{\frac{p-q+1}{2(p+1)}}\,\langle p, q+1| \otimes \langle 1, -1|$$

and the coisometry is defined by:

$$(\langle p, q| \otimes \langle 1, 1|)\Delta^* = \sqrt{\frac{p+q+2}{2(p+1)}}\,\langle p+1, q+1| \tag{160}$$
$$(\langle p, q| \otimes \langle 1, -1|)\Delta^* = \sqrt{\frac{p-q+2}{2(p+1)}}\,\langle p+1, q-1|$$

For the weights e and p satisfying $e \leq p$, conditional valuation of the observable a on the interval $(e, p+1)$ by the state z on the interval $(p, p+1)$ generates the observable $\mathsf{z} \triangleleft \mathsf{a}$ on the interval (e, p):

$$\mathsf{z} \triangleleft \mathsf{a} = \mathrm{tr}_{K[1]}[(1 \otimes \mathsf{z})\Delta^* \mathsf{a} \Delta] \tag{161}$$

This expression is evaluated by decomposing the accumulation map into its raising and lowering components:

$$\Delta_\pm : K[p+1] \to K[p] \tag{162}$$
$$\Delta^*_\pm : K[p] \to K[p+1]$$

defined by:

$$\langle p+1, q|\,\Delta_\pm = \sqrt{\frac{p \pm q+1}{2(p+1)}}\,\langle p, q \mp 1| \tag{163}$$
$$\langle p, q|\,\Delta^*_\pm = \sqrt{\frac{p \pm q+2}{2(p+1)}}\,\langle p+1, q \pm 1|$$

With these maps, the action of the state on the observable resolves to:

$$\mathsf{z} \triangleleft \mathsf{a} = z_{++}\Delta^*_+ \mathsf{a} \Delta_+ + z_{+-}\Delta^*_- \mathsf{a} \Delta_+ + z_{-+}\Delta^*_+ \mathsf{a} \Delta_- + z_{--}\Delta^*_- \mathsf{a} \Delta_- \tag{164}$$

where:

$$\mathsf{z} = \begin{bmatrix} z_{++} & z_{+-} \\ z_{-+} & z_{--} \end{bmatrix} \tag{165}$$

This explicit expression for conditional valuation in the case of the special linear Lie algebra in two dimensions combines with the earlier price expressions to create an efficient scheme for pricing interest rate options.

A comparable scheme is derived from the representations of any semisimple Lie algebra, albeit at the expense of increasing complexity in the Clebsch-Gordan decomposition of the tensor product. More generally, the approach relies on the recombining structure of the weight space of irreducible representations, and any algebraic construction that has a similar lattice of representations could be used in place. This includes Kac-Moody algebras, Lie superalgebras and quantum deformations of Lie algebras, all cases where the universal enveloping algebra is structured as a quantum group. Characteristic features of the source algebra are then related to the prices of options in the corresponding interest rate model.

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