The category of chord diagrams and its generic limit

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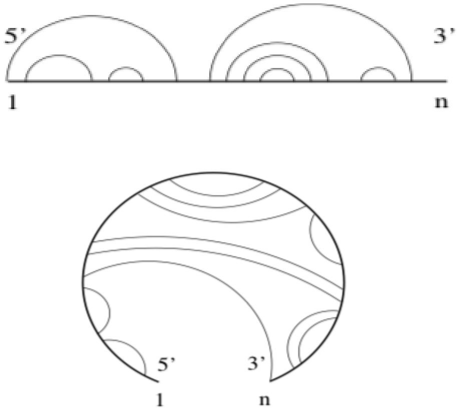
Motivation

Chord diagrams may be used to represent the inter-relationships between some objects.

What is a chord diagram ?

Let's consider a `collection of` $b \geqslant 1$ (pairwise, disjoint, oriented and labeled) `intervals` lying in the real line. We call them `backbones`.
By a `chord diagram` on backbones we understand a collection of $n \geqslant 0$ semi-circles (called `chords`) whose endpoints lie at distinct interior points of backbones.

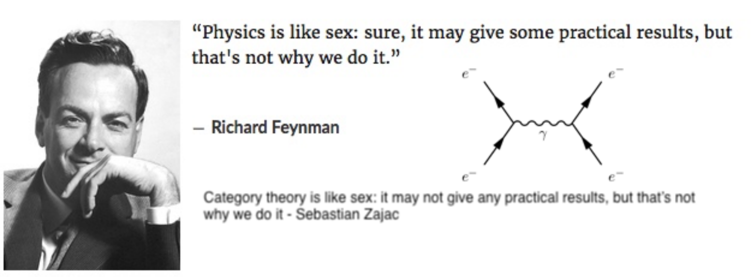
Graphical representation



T. Doslic "Secondary structures and some related combinatorial objects", DOI: https://doi.org/10.5592/CO/CCD.2016.02

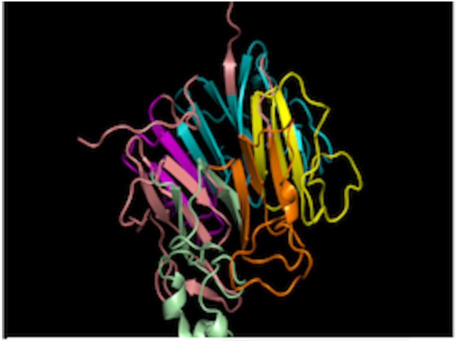
Where can you find it ?

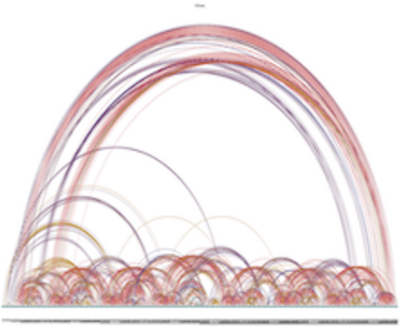
Physics

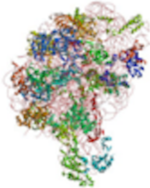


Math

- Matrix Field Theory (0-dim quantum field theory),
- geometry, topology, representation theory,
- moduli spaces of Riemann surfaces.





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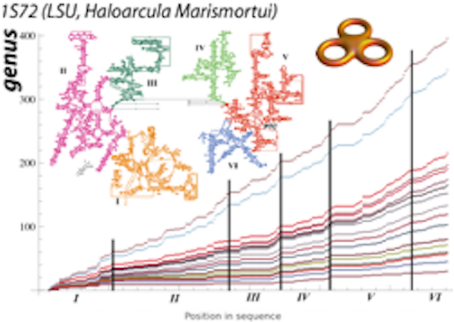
1S72 (LSU, Haloarcula Marismortui)
genus
400
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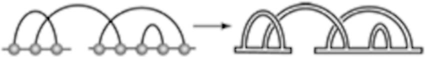
Position in sequence



Preliminaries

1. A natural number n is the set $\{0, 1, \ldots, n-1\}$.
2. Recall that a *linear ordering* on a set S is an irreflexive transitive relation $<$ satisfying $x < y$ or $y < x$ for every $x \neq y$ in S. In that case $\langle S, < \rangle$ is called a *linearly ordered* set.
3. If $\langle S, <^S \rangle$ and $\langle T, <^T \rangle$ are linearly ordered sets then a mapping $f \colon S \to T$ is called *increasing* if $f(x) <^T f(y)$ whenever $x <^S y$.
4. If A, B are subsets of a linearly ordered set $\langle S, < \rangle$ then we write $A < B$ if $a < b$ holds for every $a \in A$ and $b \in B$.
5. we denote by $\mathcal{P}(S)$ - power-set of S (the set of all subsets of S).
6. Given a natural number k, we denote by $[S]^k$ the family of all k-element subsets of S
7. A family $\mathcal{A} \subseteq \mathcal{P}(S)$ is called a *partition* of S if $S = \bigcup \mathcal{A}$ and $A_0 \cap A_1 = \emptyset$ for every $A_0 \neq A_1$ in $\mathcal{A}$.
8. A *graph* (more formally: a simple undirected graph) is a structure of the form $\langle S, E \rangle$, where $E \subseteq [S]^2$.

The category of chord diagrams

We fix three positive integers k, ℓ, and m and a multifunction

$$\varphi\colon \ell \times \ell \to m$$

The objects of $\mathfrak{C}_{k,\varphi}$ are structures of the form

$$\mathbb{S} = \langle S, <^S, \{B_i^S\}_{i<k}, \{N_i^S\}_{i<\ell}, \{E_i\}_{i<m} \rangle,$$

where:

(D1) $\langle S, <^S \rangle$ is a finite linearly ordered set.
(D2) $\{B_i^S\}_{i<k}$ and $\{N_i^S\}_{i<\ell}$ are partitions of S.
(D3) $B_{i_0} < B_{i_1}$ whenever $i_0 < i_1 < k$.
(D4) $\langle S, E_i \rangle$ is a graph for every $i < m$.
(D5) $E_{i_0} \cap E_{i_1} = \emptyset$ whenever $i_0 \neq i_1$.
(D6) If $x \in N_{i_0}$, $y \in N_{i_1}$ and $\langle x, y \rangle \in E_j$, then $j \in \varphi(i_0, i_1)$.

The sets B_i are called *backbones*, while the sets N_i are *types of nodes* and E_i are *types of edges*.

A $\mathfrak{C}_{k,\varphi}$-morphism from $\mathbb{S}$ to $\mathbb{T} = \langle T, <^T, \{B_i^T\}_{i<k}, \{N_i^T\}_{i<\ell}, \{E_i\}_{i<m}\rangle$ is a mapping $f\colon S \to T$ that preserves the linear orderings (that is, $x <^S y \implies f(x) <^T f(y)$) and satisfies for every $x, y \in S$:

(M1) $f(x) \in B_i^T \iff x \in B_i^S$

(M2) $f(x) \in N_i^T \iff x \in N_i^S$

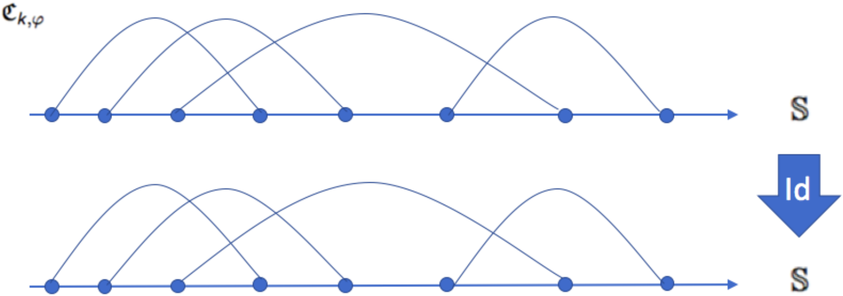
(M3) $\langle f(x), f(y)\rangle \in E_i^T \iff \langle x, y\rangle \in E_i^S$.

Informally, a $\mathfrak{C}_{k,\varphi}$-arrow is a mapping that preserves the structure of $\mathbb{S}$, "adding" new vertices and new edges of various types.

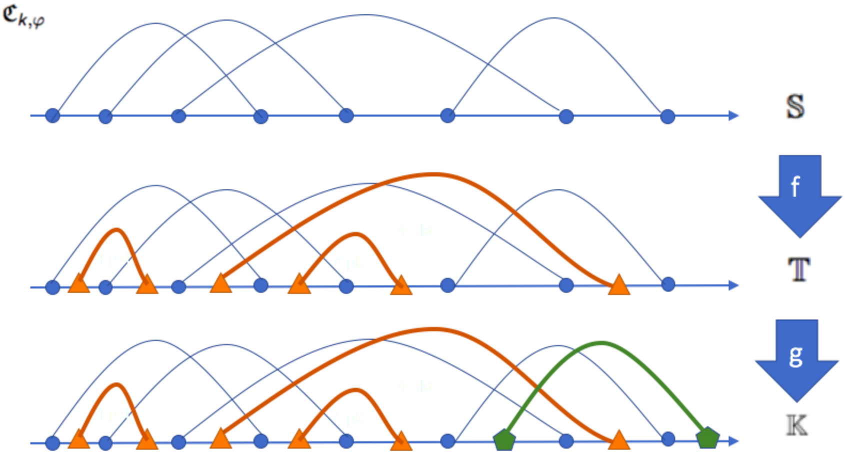
In the language of model theory, $\mathfrak{C}_{k,\varphi}$-arrows are called *embeddings*.

It is clear that $\mathfrak{C}_{k,\varphi}$ forms a category.

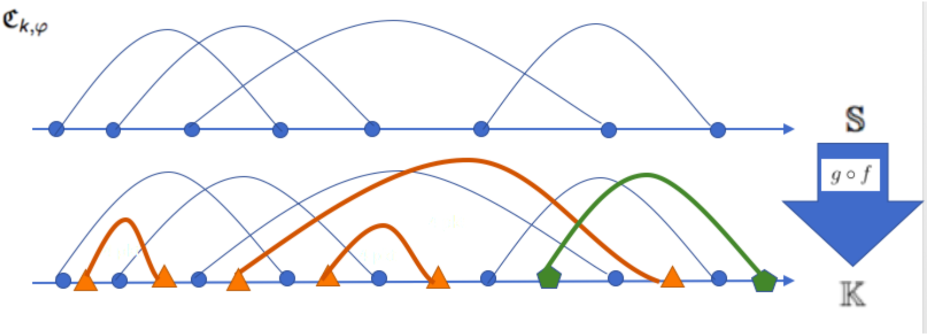
Identity



Morphisms - emmbedings



Composition of morphisms



The amalgamation property

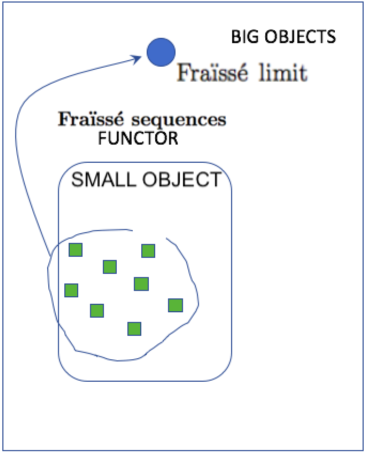
Given

$$f : S \to T,\ g : S \to K,$$

there should exist an object W and embeddings $f' : T \to W$ and $g' : K \to W$ satisfying

$$f' \circ f = g' \circ g$$

$$\begin{tikzcd} K \arrow[r, "g"] & W \\ S \arrow[u, "g"] \arrow[r, "f"'] & T \arrow[u, "f'"'] \end{tikzcd}$$



Universal chord diagram

In one backbone case Fraïsse limit is the Rado graph with a linear ordering isomorphic to the rationals.