Chord diagrams category and its limit

Category Theory in Physics, Mathematics and Philosophy

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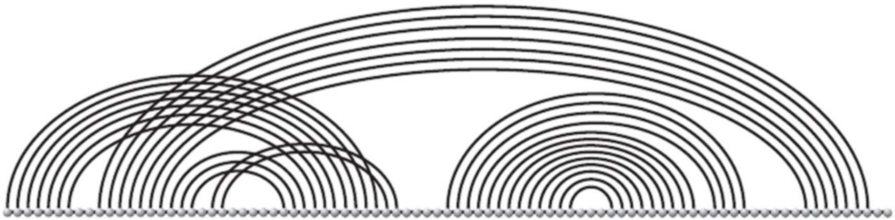
Motivation

What is a chord diagram ?

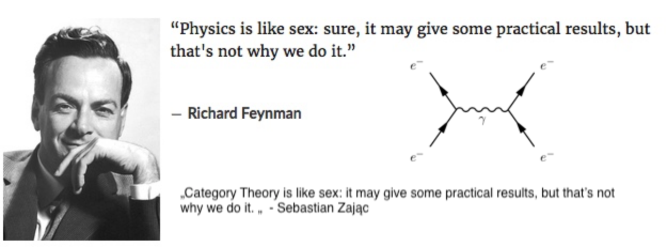
Let's consider a collection of $b \geqslant 1$ (pairwise, disjoint, oriented and labeled) intervals lying in real line. We call it *backbones*. By *chord diagram* on backbones we understand a collection of $n \geqslant 0$ semi-circles (called *chords*) whose endpoints lie at distinct interior points of backbones.

Chord diagrams may be used to represent the inter-relationships between some objects (data).



Where You can find it?

Physics



Math

- Matrix Field Theory (0-dim quantum field theory),
- geometry, topology, representation theory,
- moduli spaces of Riemann surfaces.

Philosophy

the influence of philosophers on one another in time.

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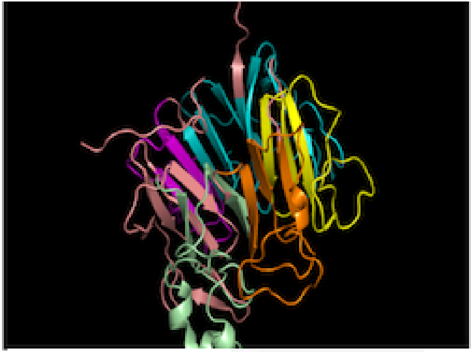

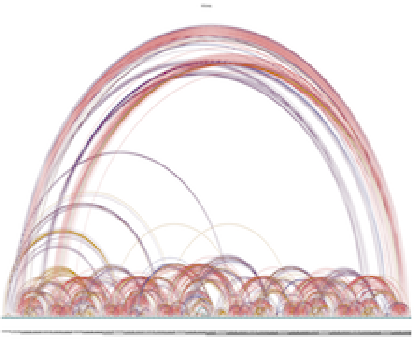

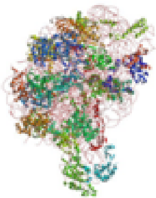

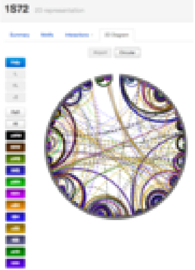

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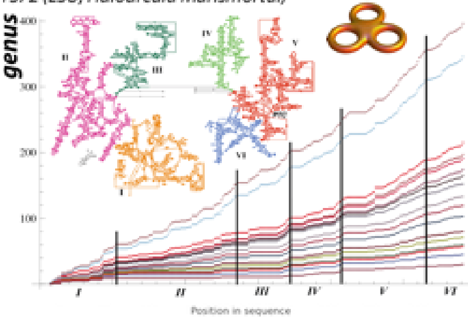
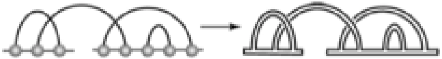




Preliminaries

1. Natural number n as the set $\{0, 1, \ldots, n-1\}$.
2. Recall that a *linear ordering* on a set S is an irreflexive transitive relation $<$ satisfying $x < y$ or $y < x$ for every $x \neq y$ in S. In that case $\langle S, < \rangle$ is called a *linearly ordered* set.
3. If $\langle S, <^S \rangle$ and $\langle T, <^T \rangle$ are linearly ordered sets then a mapping $f \colon S \to T$ is called *increasing* if $f(x) <^T f(y)$ whenever $x <^S y$.
4. If A, B are subsets of a linearly ordered set $\langle S, < \rangle$ then we write $A < B$ if $a < b$ holds for every $a \in A$ and $b \in B$.
5. by $\mathcal{P}(S)$ - power-set of S (the set of all subsets of S).
6. Given a natural number k, we denote by $[S]^k$ the family of all k-element subsets of S
7. A family $\mathcal{A} \subseteq \mathcal{P}(S)$ is called a *partition* of S if $S = \bigcup \mathcal{A}$ and $A_0 \cap A_1 = \emptyset$ for every $A_0 \neq A_1$ in $\mathcal{A}$.
8. A *graph* (more formally: a simple undirected graph) is a structure of the form $\langle S, E \rangle$, where $E \subseteq [S]^2$.

The category of chord diagrams

We fix three positive integers k, ℓ, and m and a multifunction

$$\varphi\colon \ell \times \ell \to m$$

The objects of $\mathfrak{C}_{k,\varphi}$ are structures of the form

$$\mathbb{S} = \langle S, <^S, \{B_i^S\}_{i<k}, \{N_i^S\}_{i<\ell}, \{E_i\}_{i<m}\rangle,$$

where:

(D1) $\langle S, <^S\rangle$ is a finite linearly ordered set.
(D2) $\{B_i^S\}_{i<k}$ and $\{N_i^S\}_{i<\ell}$ are partitions of S.
(D3) $B_{i_0} < B_{i_1}$ whenever $i_0 < i_1 < k$.
(D4) $\langle S, E_i\rangle$ is a graph for every $i < m$.
(D5) $E_{i_0} \cap E_{i_1} = \emptyset$ whenever $i_0 \neq i_1$.
(D6) If $x \in N_{i_0}$, $y \in N_{i_1}$ and $\langle x, y\rangle \in E_j$, then $j \in \varphi(i_0, i_1)$.

The sets B_i are called *backbones*, while the sets N_i are *types of nodes* and E_i are *types of edges*.

A $\mathfrak{C}_{k,\varphi}$-morphism from $\mathbb{S}$ to $\mathbb{T} = \langle T, <^T, \{B_i^T\}_{i<k}, \{N_i^T\}_{i<\ell}, \{E_i\}_{i<m}\rangle$ is a mapping $f\colon S \to T$ that preserves the linear orderings (that is, $x <^S y \implies f(x) <^T f(y)$) and satisfies for every $x, y \in S$:

(M1) $f(x) \in B_i^T \iff x \in B_i^S$

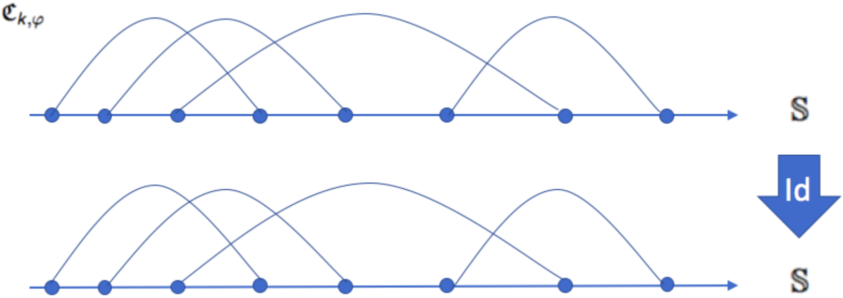
(M2) $f(x) \in N_i^T \iff x \in N_i^S$

(M3) $\langle f(x), f(y)\rangle \in E_i^T \iff \langle x, y\rangle \in E_i^S$.

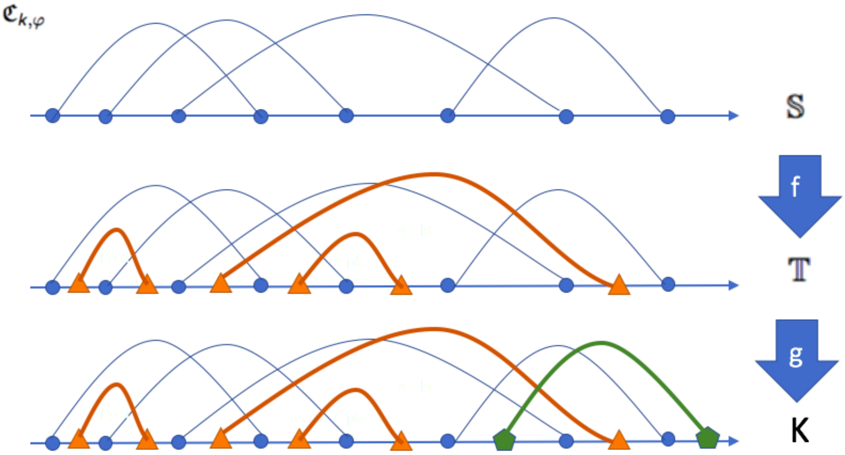
Informally, a $\mathfrak{C}_{k,\varphi}$-arrow is a mapping that preserves the structure of $\mathbb{S}$, "adding" new vertices and new edges of various types.
In the language of model theory, $\mathfrak{C}_{k,\varphi}$-arrows are called *embeddings*.

It is clear that $\mathfrak{C}_{k,\varphi}$ forms a category.

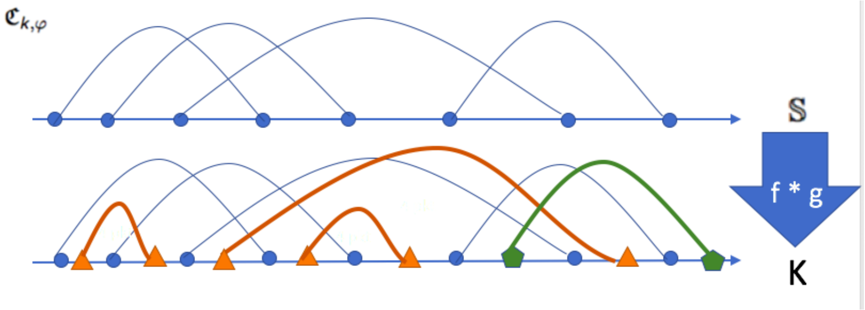
Identity



Morphisms - emmbedings



Composition of morphisms



Amalgamtion

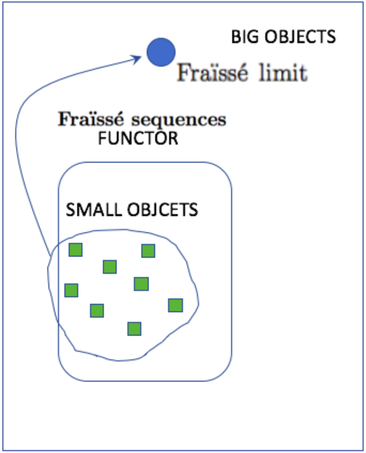
Each two embeddings can be extended to a further embedding !
For

$$f : S \to T, g : S \to K$$

There should exist object W and embeddings $f' : T \to W$ and $g' : K \to W$ satisfying

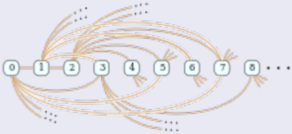
$$f' \circ f = g' \circ g$$

Chord category as a Fraïsse category example



Universal chord diagram

Fraïsse limit is the Rado graph with ordering



Adam - new Category Theory team member

he is 4 (month) and has a few questions in his internal language:

"Daddy, teach me Category theory "

Teaching

Category Theory as a new language at primary school (like computer languages)

Daddy, is the Category Theory kind of "update" of Set Theory ?

Categorification in physics, New version of General Relativity, Description of Automata etc. Set as special case of Category.

Daddy, You speak a lot about symmetry. Where is symmetry in category ? Is functor kind of symmetry ?

Daddy, In physics You change theory by some physical parameters. Why You don't do this in Category Theory?

Daddy, You told me about statistics and physics.
Where is "accident" in Category Theory ?