Recitation 7 - Topological Sort

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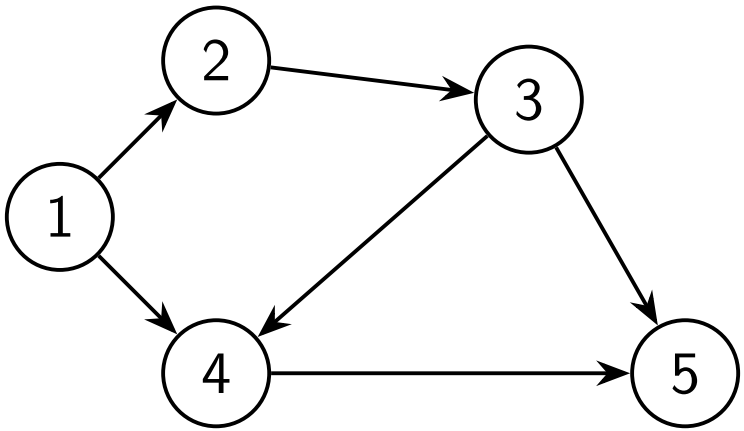
Topological sort

- Given a directed acyclic graph (DAG) $G = (V, E)$, we can perform a topological sort.
- A topological sort on a DAG is a linear ordering of all its vertices such that if (u, v) is an edge in G, then u appears before v in the ordering.
- Note that if we have a cycle in G, we cannot have such a linear ordering.

Example

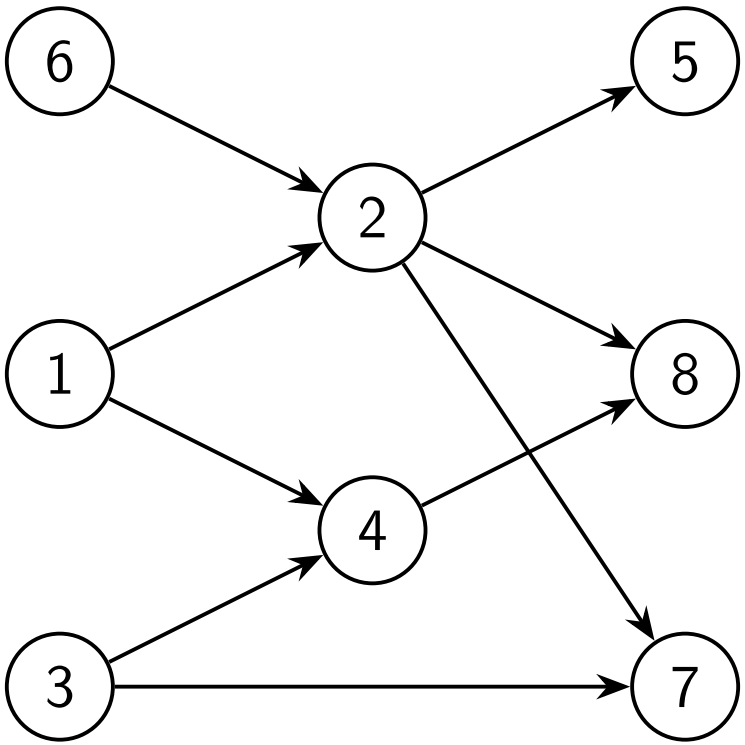
- Consider the following DAG:



- What would a valid topological ordering be for this DAG?
- How about $\{1, 2, 3, 4, 5\}$?

Your turn

- Give a topological sorting of the following DAG:



Example solutions

- $\{6, 1, 3, 2, 7, 4, 5, 8\}$
- $\{1, 6, 2, 3, 4, 5, 7, 8\}$
- ... and more?

Pseudocode

```
def TopologicalSort(G):
    DFS(G)  # get finishing times for each vertex
    when a vertex finishes, insert into linked list
    return the linked list
```

- Topologically sorted vertices appear in reverse order of their DFS finishing times.

Back to example (2)

- What is the DFS finishing time list for the previous example?
- Now reverse that list.
- Is that a valid topological ordering?

Complexity analysis

- Running time?
- DFS is $\mathcal{O}(V + E)$.
- Each of the $|V|$ insertions into the linked list takes $\mathcal{O}(1)$ time.
- Total running time of `TopologicalSort` is then $\mathcal{O}(V + E)$.

Theorem

- A directed graph G is acyclic if and only if a DFS on G produces no back edges.
- `TopologicalSort` produces a topological sort of an input DAG.
  - Run DFS on a DAG G. Want to show that for any distinct vertices $u, v \in V$, if G has an edge from u to v, then $v.f < u.f$.
  - Consider any edge (u, v) in G. At the time of exploration, v cannot be gray, otherwise v would be an ancestor of u and this would thus be a back edge, contradicting our assumption of G being a DAG.
  - Since v can't be gray, it must be either white or black.
  - If v is white, then v becomes a descendant of u, meaning that $v.f < u.f$.
  - If v is black, its finishing time $v.f$ has already been set. Since we have not finished exploring u, this means that $v.f < u.f$.
  - So for any edge (u, v) in G, we have that $v.f < u.f$, which means u will appear before $v \Rightarrow$ topologically sorted.