Recitation 1 - RAM model & Proofs

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January 19, 2018

Preliminaries

- Make sure to enroll in the course on Moodle.
- Problem Set 1 is out already and is due next Thursday (1/25).
  - Collaborate! (**but don't cheat!!**)
- Make graders (read: me) happy and please learn LaTeX.
- My office hours are by appointment only.
  - If you have any questions, try Piazza first, but certainly feel free to shoot me an email if needed.
- Piazza should be your go-to for questions.
  - Chances are someone else has the same question.
- I'll try and post these recitation slides on my github page.
  - More info on that probably next week.

Analyzing algorithms

- When analyzing algorithmic complexity (both space and time), we make some simplifying assumptions (RAM model):
  - Basic (atomic) operations take $\mathcal{O}(1)$ time.
  - All higher order operations consist of (possibly many) basic operations.
  - Basic (atomic) data types take $\mathcal{O}(1)$ space.
  - All higher order data types consist of many basic data types.
- We tend to abstract out a lot of computing specifics when analyzing algorithms.
- Thus, we don't tend to worry about things like paging or caching.

Counting operations

- Consider the following code for computing a dot product of two vectors ($\mathbf{a} \cdot \mathbf{b}$):

```
def dotProduct(a, b, n):
    dot = 0
    for i = 0:n
        dot += a[i] * b[i]
    return dot
```

- What atomic operations are being done? Atomic data types?
- High level operations / data types?
- Asymptotic complexity?

Induction

- What is it?
- A way to prove that something is true for all n, which we encounter often in algorithms.
- An induction proof needs three things:
  1. Base case - Show that the smallest (base) case is true.
  2. (Weak) Induction hypothesis - Assume statement you want to prove is true for some arbitrary $n = k$.
     (Strong) Induction hypothesis - Assume statement you want to prove is true for all $k \leq n$.
  3. Induction step - Using the fact that the statement is true for arbitrary k, show that it is true for $k + 1$ as well.

Weak induction Example

- Prove that

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}$$

  by (weak) induction.

Base case

- First check to see that the base case holds:

$$\prod_{i=2}^{2}\left(1-\frac{1}{i^2}\right) = 1-\frac{1}{4}$$
$$= \frac{3}{4}$$
$$= \frac{2+1}{2(2)}$$

- So the base case holds.

Induction hypothesis

- Assume that

$$\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)=\frac{k+1}{2k}$$

Inductive step

- Show that the equation holds for $k+1$:

$$
\begin{aligned}
\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right) &= \left(\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)\right)\left(1-\frac{1}{(k+1)^2}\right) \\
&= \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right) \\
&= \frac{(k+1)^2-1}{2k(k+1)} \\
&= \frac{k^2+2k}{2k(k+1)} \\
&= \frac{k+2}{2(k+1)}
\end{aligned}
$$

- . . . and we're done.

Strong induction example

- Prove that any integer $n > 1$ can be written as a product of primes.
- Use strong induction.

Base case

- Base case is $n = 2$, which can be written as a product of just one prime, namely itself.

Induction hypothesis

- Now we assume that all integers x (where $2 \leq x \leq k$) can be written as a product of primes.

Inductive step

- Want to show that $k+1$ can be written as a product of primes.
- Case 1: If $k+1$ is a prime, then just like in the base case we are done.
- Case 2: If $k+1$ is composite, it has some smallest prime factor p. So we can write $k+1$ as $p \cdot x$, where $x < k$. Since $x < k$, we can apply our induction hypothesis to say that x can be written as a product of primes. Since p is prime and x can be written as a product of primes, we can conclude that $p \cdot x = k+1$ can be written as a product of primes.

Proof by Contradiction

- If we want to show that $P => Q$, we can assume $!P$ and show that this leads to a contradiction, thus proving the original statement.
- Example: Prove that $\sqrt{2}$ is irrational.

Contradiction example

- To attempt to prove this by contradiction, let's assume that $\sqrt{2}$ is rational.
- That means we can write it as $\frac{x}{y}$ (in lowest terms, meaning no number greater than one divides both x and y), where x and y are positive integers.
- Squaring both sides gives us:

$$
\begin{aligned}
2 &= \frac{x^2}{y^2} \\
2y^2 &= x^2
\end{aligned}
$$

- Clearly, $2y^2$ is even, meaning that x^2 is also even. Thus, x must also be even.
- Since x is even, we can write it as $x = 2z$. This gives us

$$
\begin{aligned}
2y^2 &= 4z^2 \\
y^2 &= 2z^2
\end{aligned}
$$

Example, cont.

- This shows that y^2 is even (do you see why?), and thus y must be even as well.
- But this shows that $\frac{x}{y}$ is not in lowest terms, which contradicts our assumption of being able to express $\sqrt{2}$ as a rational number in lowest terms, and thus $\sqrt{2}$ must be irrational.

Proof by counterexample

- We can disprove a claim by providing a counterexample.
- Example: For all real numbers x and y, if $x^2 < y^2$, then $x < y$.
- Counterexample: $x = 3$ and $y = -4$.
- Clearly $3^2 < (-4)^2$, but $3 \not< -4$

That's it for now

- Any questions?
- Have a good weekend and see you next week.