ROBOTIC Fundamentals (UFMF4X-15-M)

Homogenous Transformation

Previously on ROBOTIC FUNDAMENTALS

- DOFs, Mobility, serial/parallel manipulators
- Types of joints, Kinematic chains
- Workspace, examples of serial manipulators
- Refreshment of maths representation of matrices
- Refreshment of your MATLAB skills

Questions?

Blackboard

Today’s Lecture

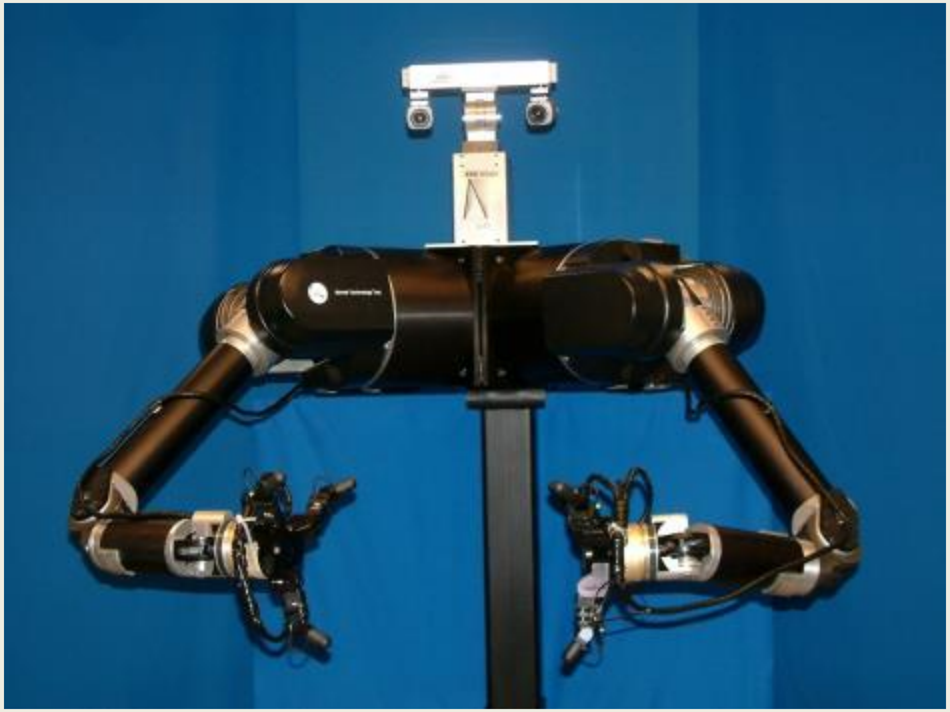
Frames for Forward Kinematics

Rotation & Translation of Frames

Homogenous transformation Matrix

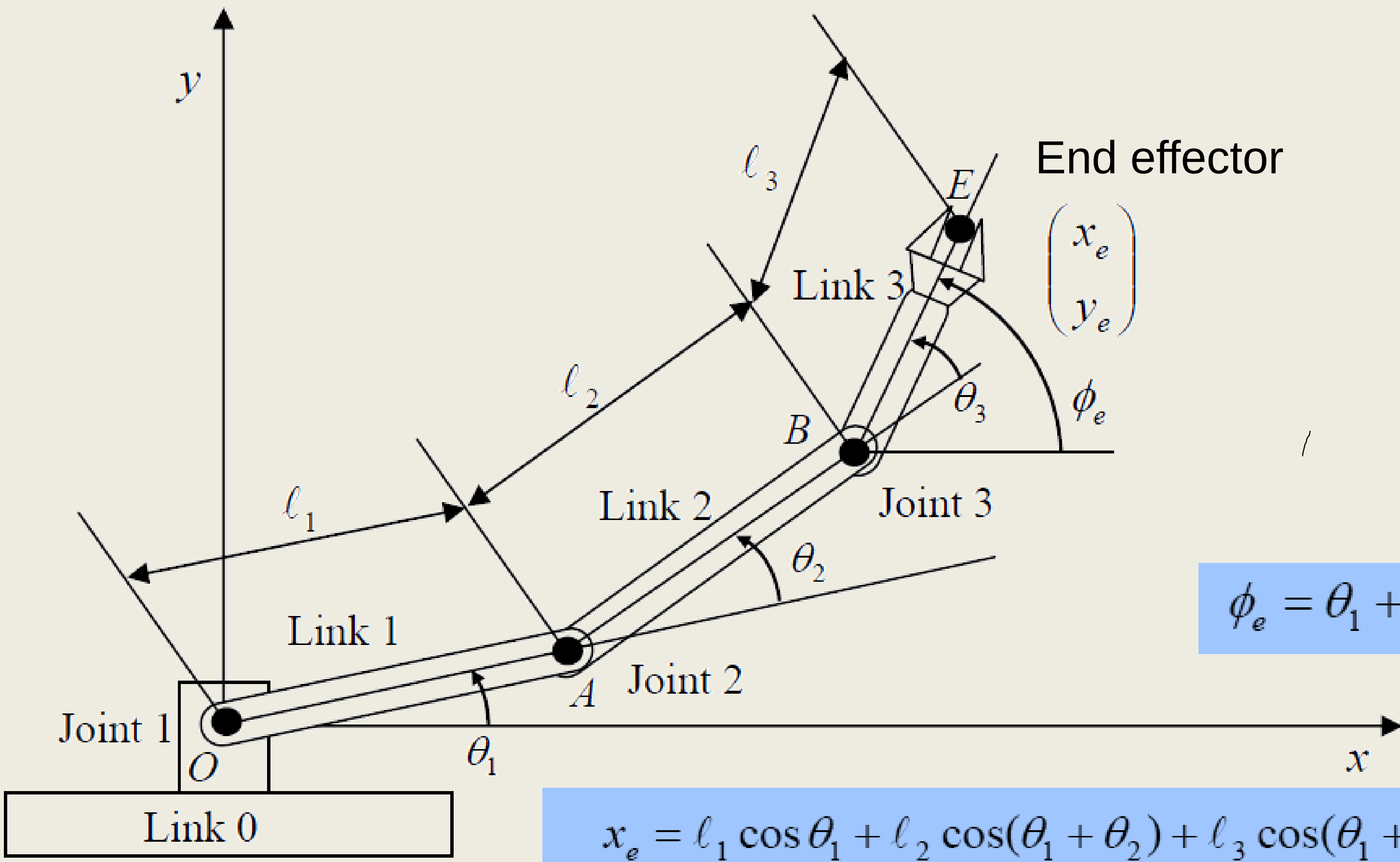
REFERENCE FRAMES AND FORWARD KINEMATICS

Forward Kinematics



Where is the end effector w.r.t. the "base" frame?

Planar RRR – Forward kinematics via Trigonometry



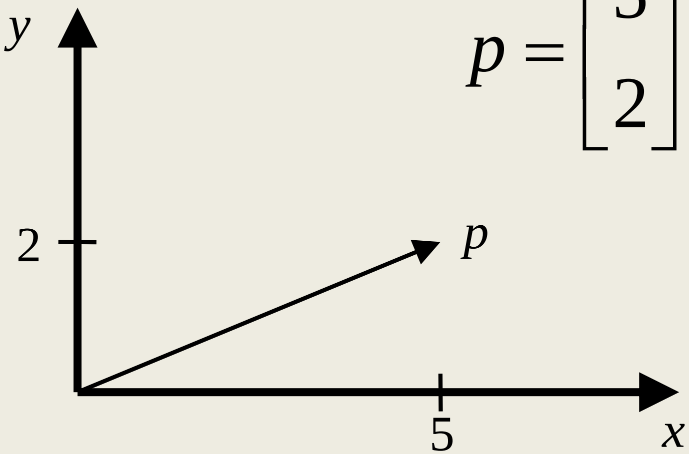
$$\phi_e = \theta_1 + \theta_2 + \theta_3$$

$$x_e = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2) + \ell_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$y_e = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1 + \theta_2) + \ell_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

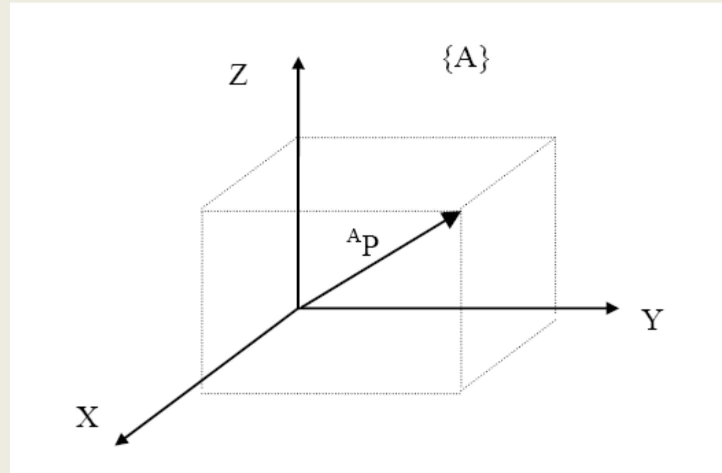
ie each component in x, y directions summed.

Vectors

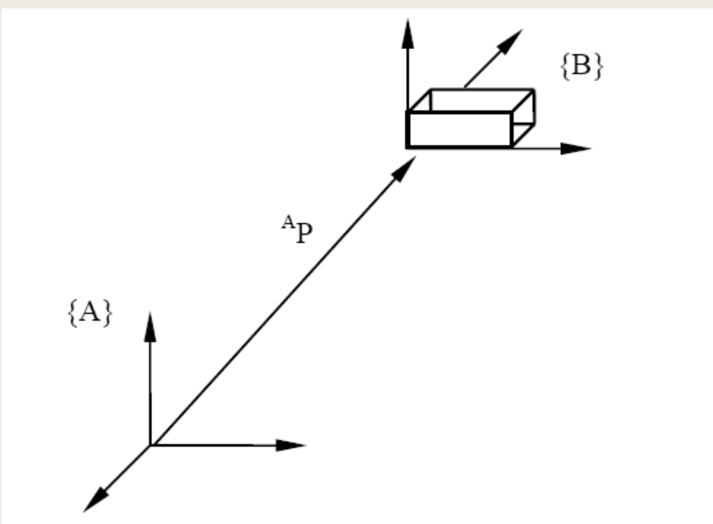
Components of the vector are distances along x, y and z axis



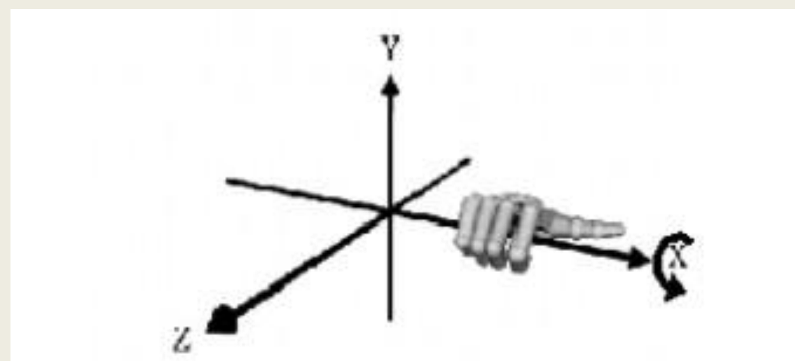
$$p = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$



$$p = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$



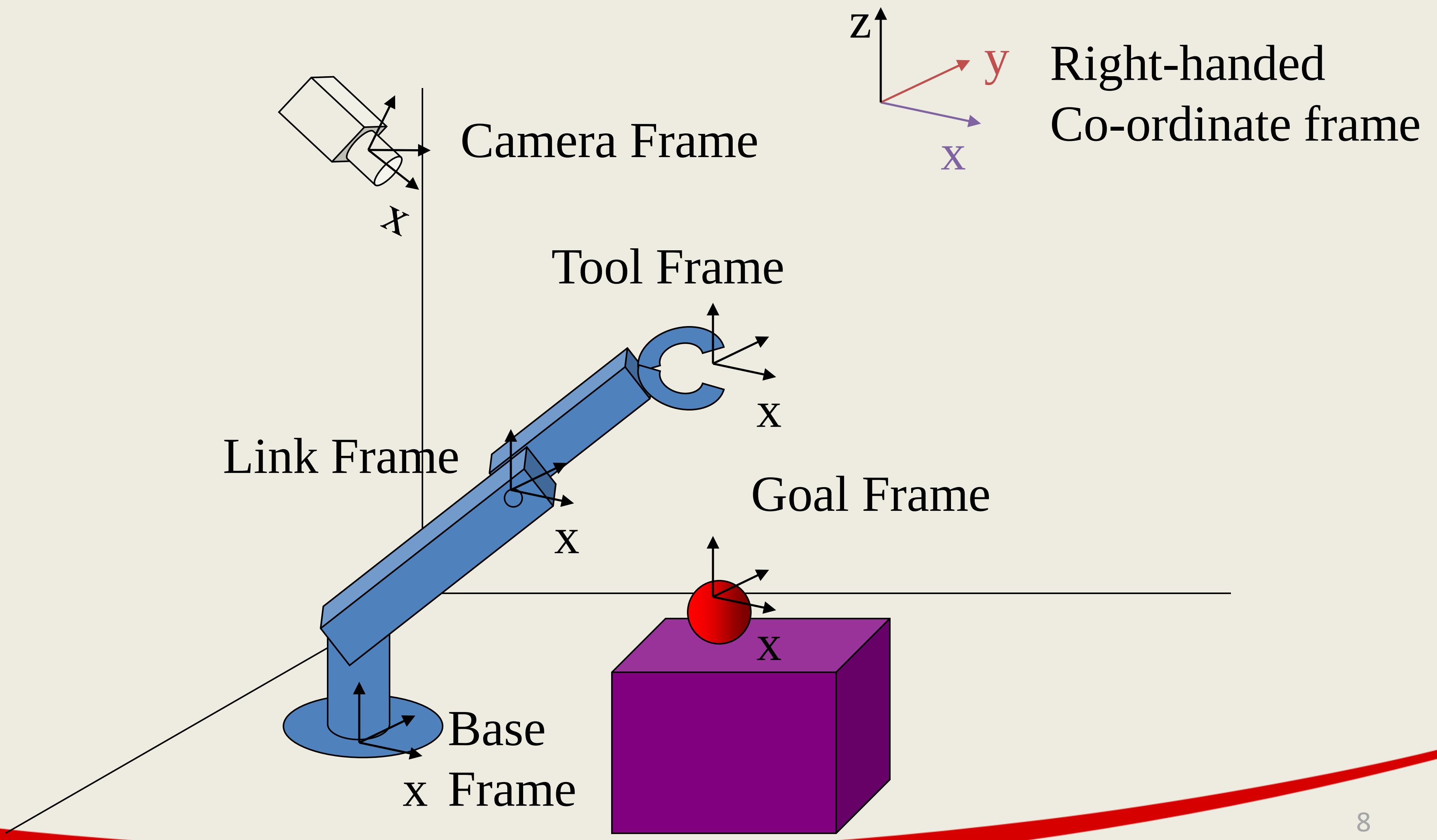
Convention: right hand frames



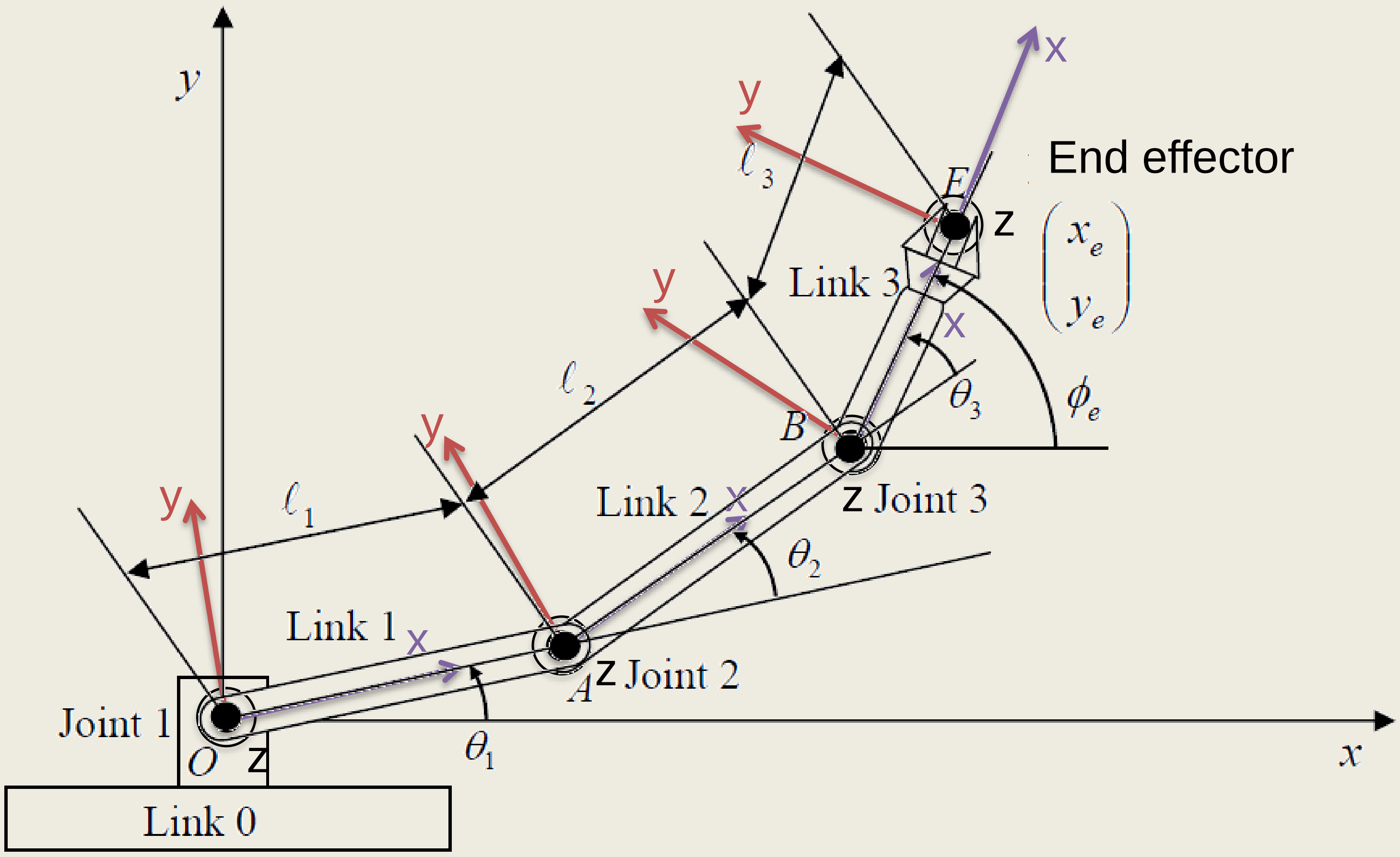
$x \to y \to z \to x$

So around z axis, x to y, around y axis, z to x etc.

Coordinate Frames

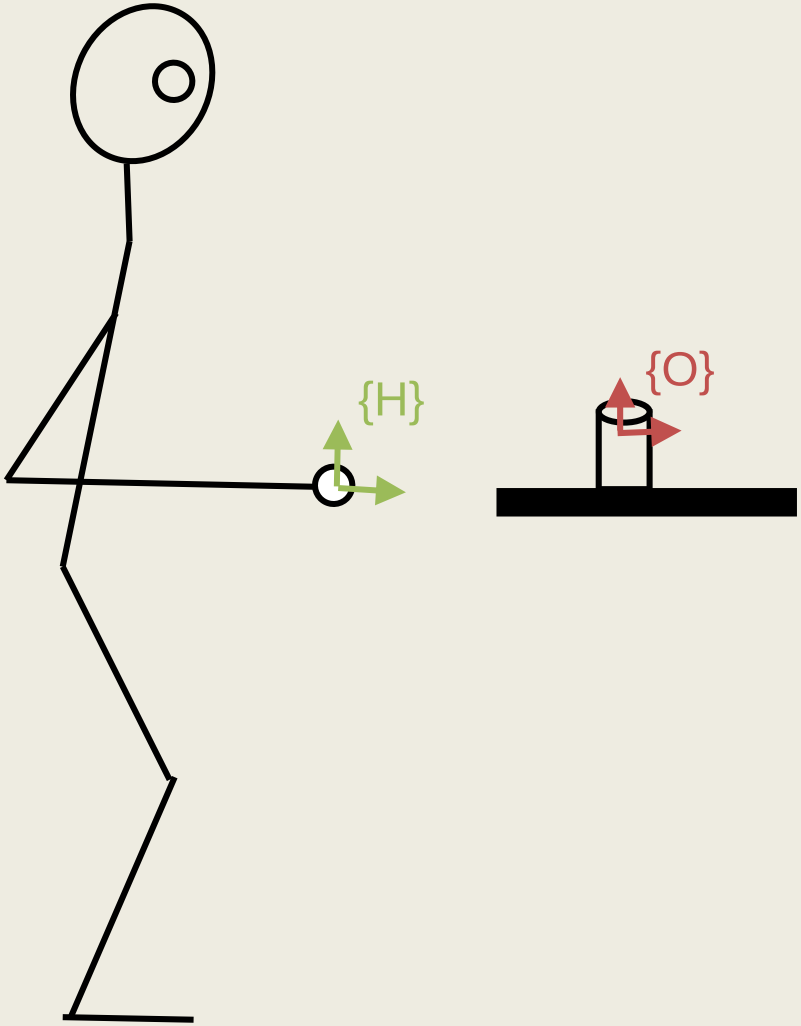


Example – Planar RRR



MOVING BETWEEN FRAMES

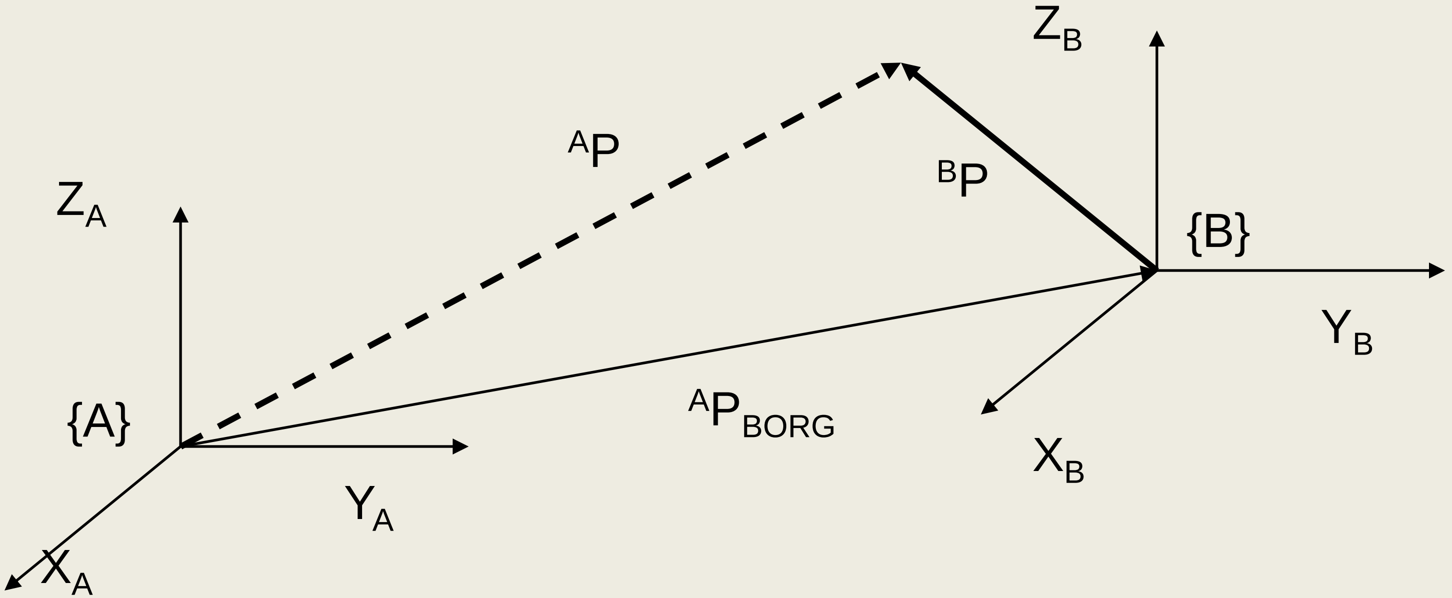
Representing Displacement: Translation Vector



The reference frame of the hand {H} and the object {O} are <u>spatially displaced</u>.

We want to represent this difference in a consistent way.

Mapping: from frame to frame

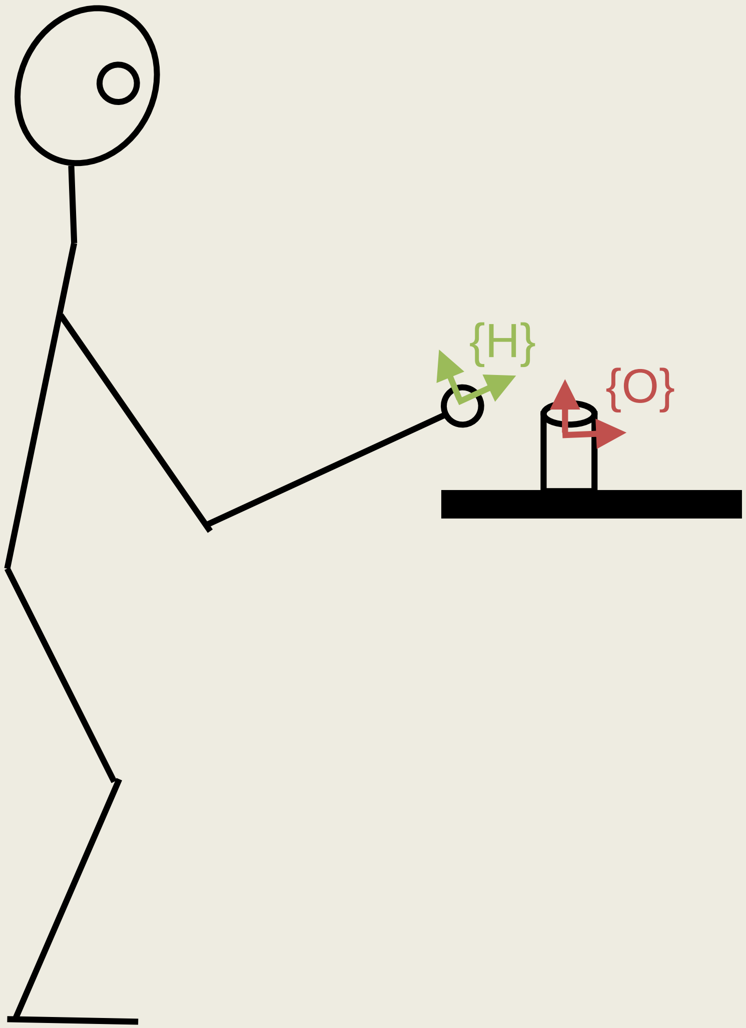


- If {A} has same orientation as {B}, then {B} differs from {A} in a translation: $^AP_{BORG}$

$$^AP = {}^BP + {}^AP_{BORG}$$

- Mapping: change of description from one frame to another. The vector $\mathbf{^AP_{BORG}}$ defines the mapping.

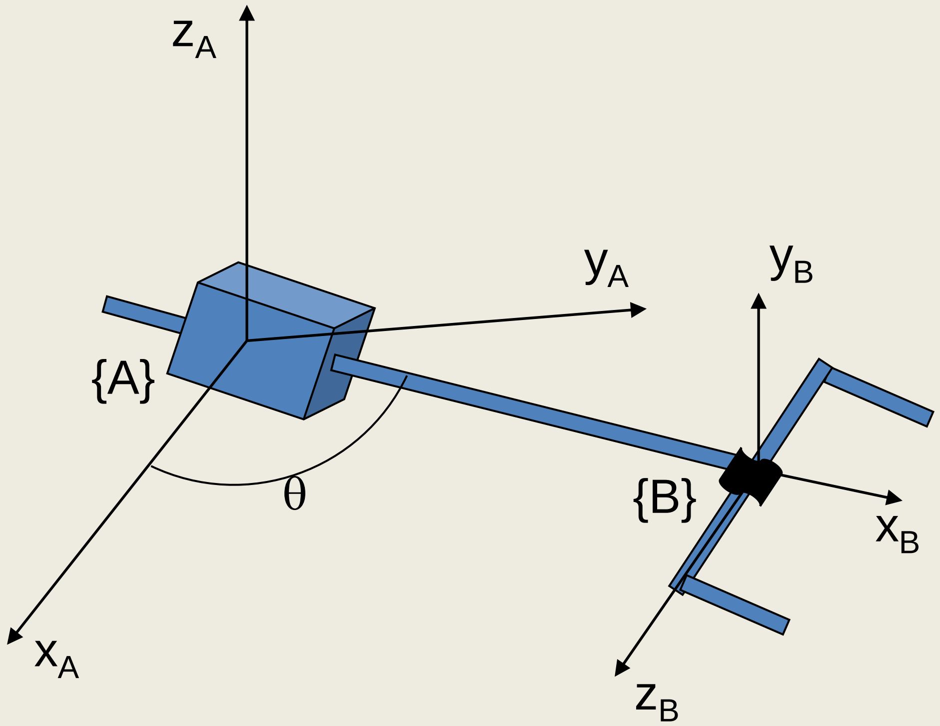
Representing Orientation: Rotation Matrices



The reference frame of the hand {H} and the object {O} have different <u>orientations</u>

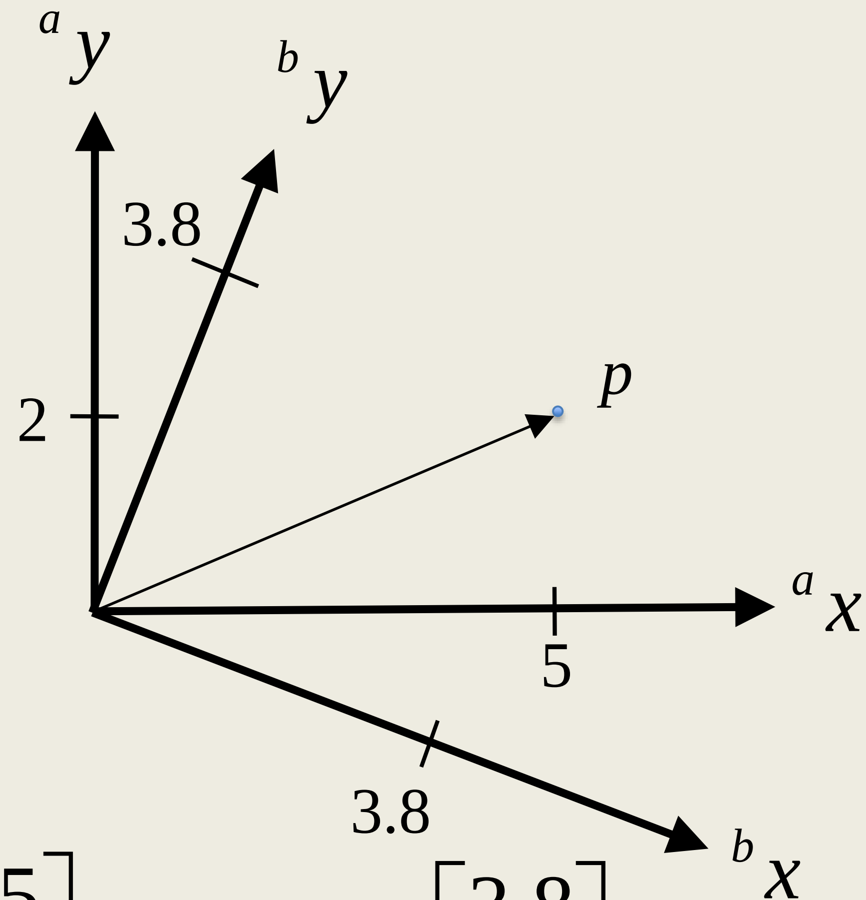
We want to represent different orientations in a consistent way, just like we did for positions…

Description of an Orientation



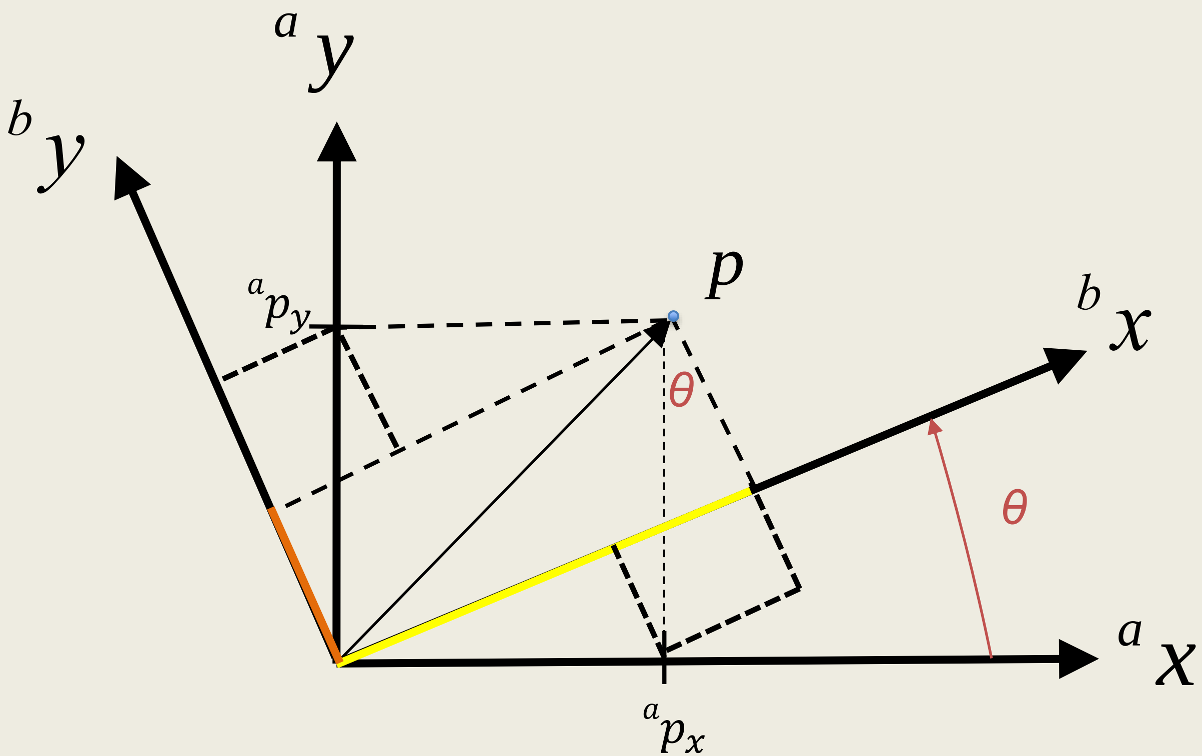
We need a description of the angles to move from axes in {A} to axes in {B}
Or in other words: a description of {B} with respect to {A}

Rotation between two frames



$$ {}^a p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad {}^b p = \begin{bmatrix} 3.8 \\ 3.8 \end{bmatrix} $$

Rotation between two frames



$${}^{b}p_y = -{}^{a}p_x \sin(\theta) + {}^{a}p_y \cos(\theta)$$

$${}^{b}p_x = {}^{a}p_x \cos(\theta) + {}^{a}p_y \sin(\theta)$$

Rotation Matrix

$$^{b}p_x = {}^{a}p_x \cos(\theta) + {}^{a}p_y \sin(\theta)$$

$$^{b}p_y = -{}^{a}p_x \sin(\theta) + {}^{a}p_y \cos(\theta)$$

This way, we had a point (or vector) which was in frame {A} and we have expressed it in {B}

To express a point (or vector) from {B} to {A}, the equations are:

$$^{a}p_x = {}^{b}p_x \cos(\theta) - {}^{b}p_y \sin(\theta)$$

$$^{a}p_y = {}^{b}p_x \sin(\theta) + {}^{b}p_y \cos(\theta)$$

Rotation Matrix

To express a point (or vector) from {B} to {A}, the equations are:

$$^{a}p_x = {}^{b}p_x \cos(\theta) - {}^{b}p_y \sin(\theta)$$

$$^{a}p_y = {}^{b}p_x \sin(\theta) + {}^{b}p_y \cos(\theta)$$

$$\begin{bmatrix} ^{a}p_x \\ ^{a}p_y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} ^{b}p_x \\ ^{b}p_y \end{bmatrix}$$

$$^{A}P = {}^{A}R_B {}^{B}P$$

(Using a different convention: $P_A = R_{AB} P_B \quad or \quad {}^{A}P = R_{AB} {}^{B}P$)

Rotation Matrix

$$\begin{bmatrix} {}^{a}p_x \\ {}^{a}p_y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} {}^{b}p_x \\ {}^{b}p_y \end{bmatrix}$$

$${}^{A}R_B = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

The rotation matrix ${}^{A}R_B$ is described as the *rotation matrix* for transforming from the frame {B} to the frame {A}.

Rotation Matrix: from 2D to 3D

$$^{a}p_x = {}^{b}p_x \cos(\theta) - {}^{b}p_y \sin(\theta)$$

$$^{a}p_y = {}^{b}p_x \sin(\theta) + {}^{b}p_y \cos(\theta)$$

$$^{a}p_z = {}^{b}p_z$$

The 3D rotation matrix $^{A}R_B$ is given as:

$$^{A}R_B = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or R_{AB} or ${}^{A}_{B}R$

Rotation matrices (rotation around x/y/z)

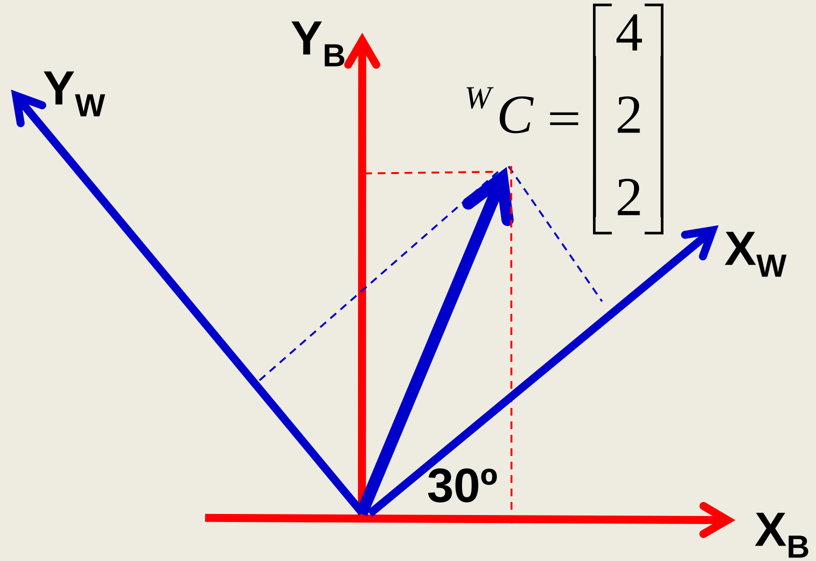
$$ROT(z,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\mathrm{c}_\theta & -s_\theta & 0\\ s_\theta & \mathrm{c}_\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$ROT(y,\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}=\begin{bmatrix}\mathrm{c}_\theta & 0 & s_\theta\\ 0 & 1 & 0\\ -s_\theta & 0 & \mathrm{c}_\theta\end{bmatrix}$$

$$ROT(x,\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \mathrm{c}_\theta & -s_\theta\\ 0 & s_\theta & \mathrm{c}_\theta\end{bmatrix}$$

where: $\begin{cases} s_\theta = \sin\theta \\ c_\theta = \cos\theta \end{cases}$

Example with Rotation Matrices



$$\textit{What is } {}^{B}C\,?$$

$$^{B}C = \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.46 \\ 3.73 \\ 2 \end{bmatrix}$$

Properties of Rotation Matrices to keep in mind

$$ {}^{A}R_{B} = \left({}^{B}R_{A}\right)^{T} $$

$$ {}^{A}R_{B}^{-1} = \left({}^{A}R_{B}\right)^{T} = {}^{B}R_{A} $$

$$ R^{-1} = R^{T} $$

$$ R^{T}R = RR^{T} = I $$

$$ \det(R) = +1 $$

$$ \left[Rot(i,\theta)\right]^{-1} = Rot(i,-\theta) $$

$$ Rot(i,\theta_1)Rot(i,\theta_2) = Rot(i,\theta_1+\theta_2) $$

Homework

- Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Can A be a rotation matrix?

- Find the values of the missing elements:

R= $\begin{bmatrix} r_{11} & 0 & -1 & -1 \\ r_{21} & 0 & 0 & 5 \\ r_{31} & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Putting it all together

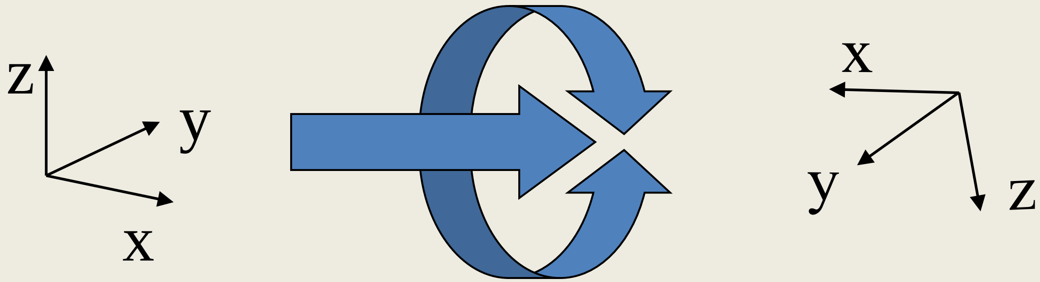
HOMOGENEOUS TRANSFORMATIONS

In summary

- Position of a point is represented by a vector
- Orientation of a body is represented by a matrix

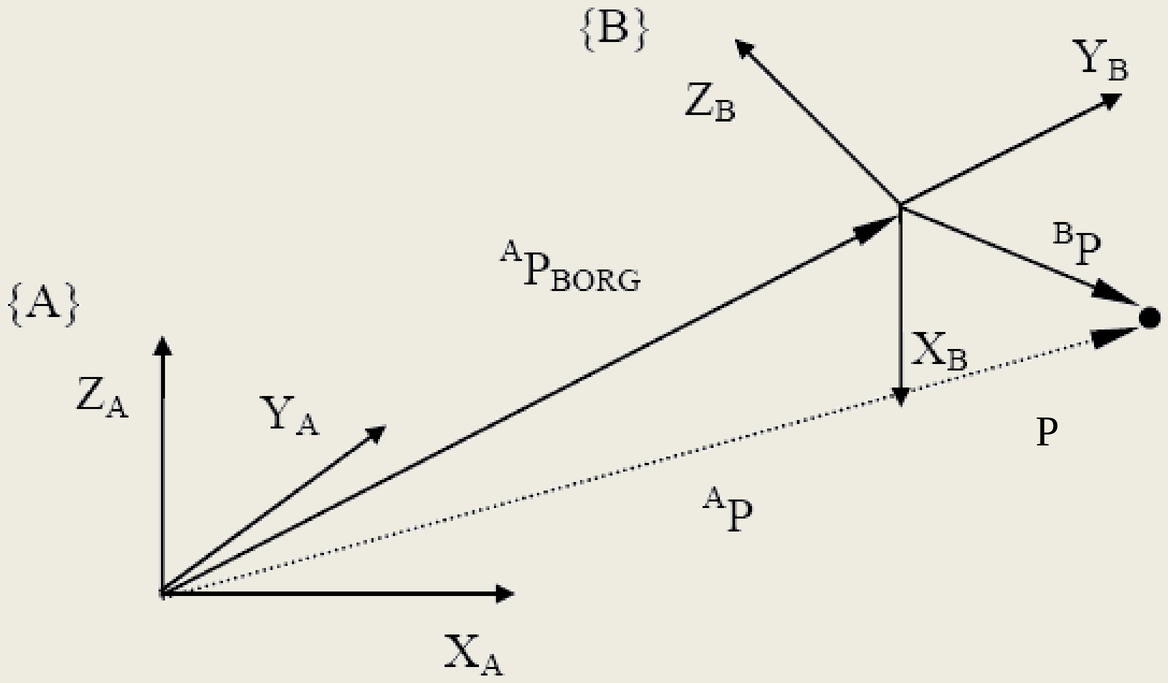
Kinematic Relationship

– Between two frames we have a *kinematic relationship* - translation and rotation.



– This relationship is mathematically represented by a 4 $\times$ 4 Homogeneous Transformation Matrix.

Mapping of the frames (Translation + Rotation)



$$ {}^A P = {}^A R_B \, {}^B P + {}^A P_{BORG} $$

Homogenous transformation matrix

$$
{}^{A}T_{B} \text{ or } T_{AB} \text{ or } {}^{A}_{B}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
T = \left[\begin{array}{ccc|c} & Rotation & & Translation \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

Homogeneous Coordinates

- Homogeneous coordinates: embed 3D vectors into 4D by adding a “1”
- More generally, the transformation matrix T has the form:

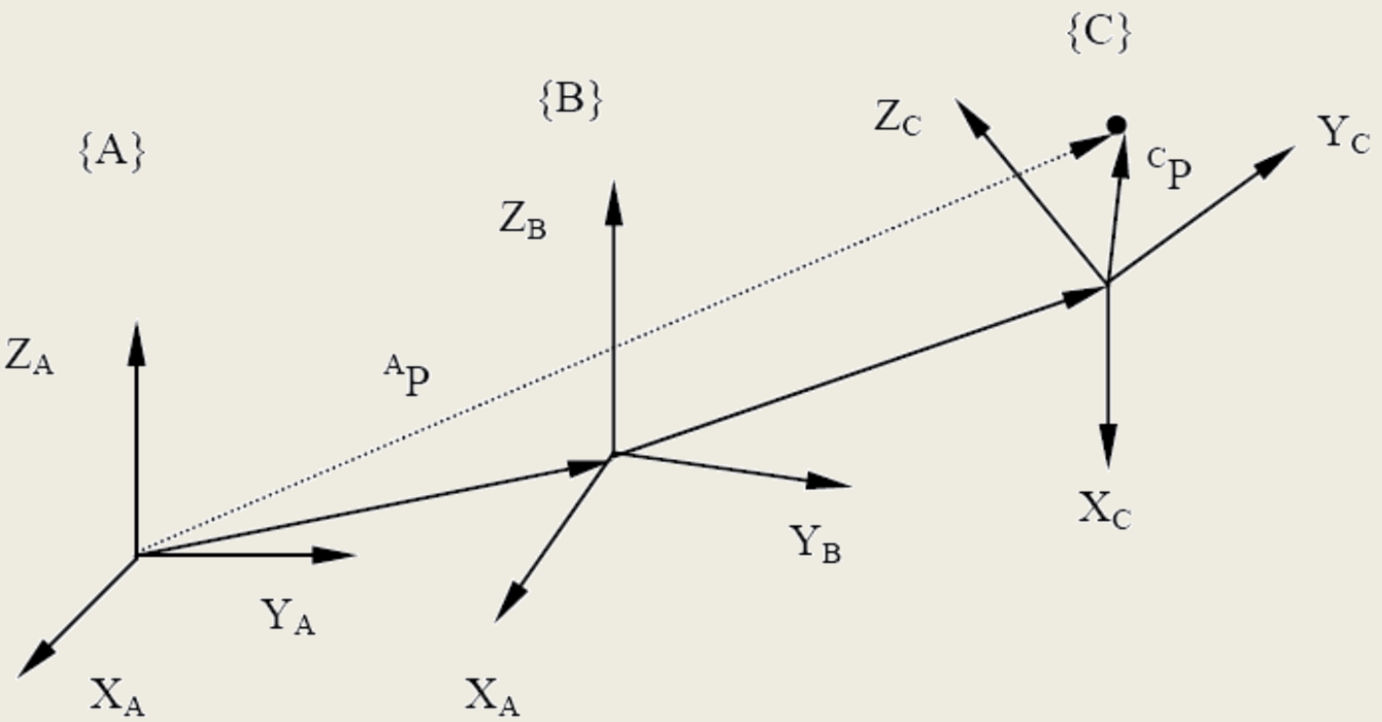
$$T = \begin{bmatrix} \text{Rot. Matrix} & \text{Trans. Vector} \\ \text{Perspect. Trans.} & \text{Scaling Factor} \end{bmatrix}$$

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \\ c_1 & c_2 & c_3 & sf \end{array}$$

Getting many of them together

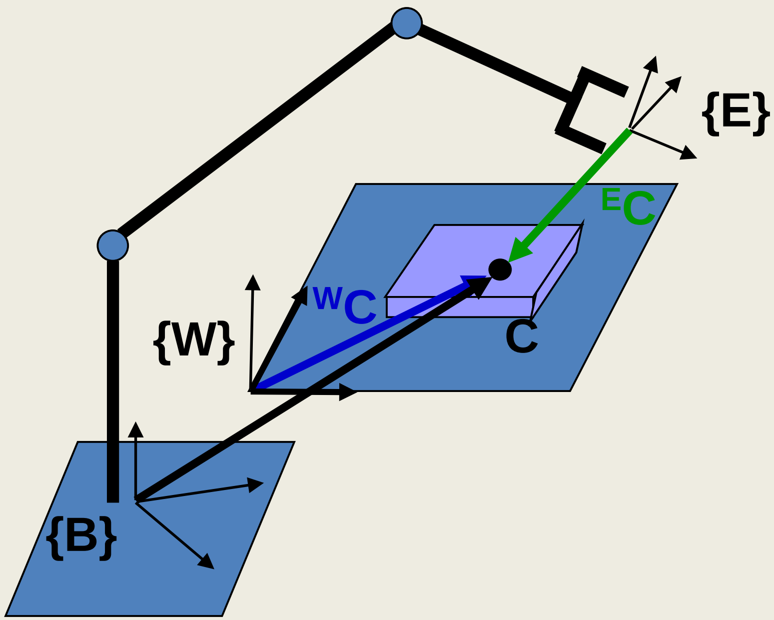
COMPOUND TRANSFORMATIONS

Compound Transformations



$$^{A}P = {}^{A}T_{B}\,{}^{B}T_{C}\,{}^{C}P$$

Compound Transformations



The vector ^{W}C may be known, but ^{E}C needs to be calculated. If the transformations representing the position and orientation of {W} w.r.t. {B} and {B} w.r.t. to {E} are known, the following calculations can be performed:

$$^{B}C = {}^{B}T_{W} \; {}^{W}C$$
$$^{E}C = {}^{E}T_{B} \; {}^{B}C$$

or these equations can be combined to give:

$$^{E}C = {}^{E}T_{B} \; {}^{B}T_{W} \; {}^{W}C$$

Combining the transformations we can define:

$$^{E}T_{W} = {}^{E}T_{B} \; {}^{B}T_{W}$$

Inverse transformations

In the previous slide we may know ${}^{B}T_{E}$ (description of the frame E relative to frame B) rather than ${}^{E}T_{B}$. Or it may be necessary to calculate the position of an object relative to the hand of a robot from its position in relation to the world co-ordinate system.

To do this we find an inverse transformation. In general:

$$
{}^{B}T_{A} = \begin{bmatrix} {}^{A}R_{B}^{T} & -{}^{A}R_{B}^{T}\ {}^{A}P_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}
$$

Example: Inverting a Transformation matrix

If the transformation matrix T is given by

$$T = \left[\begin{array}{ccc|c} 0.87 & -0.50 & 0 & 1 \\ 0.50 & 0.87 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$If\ \ T = \begin{bmatrix} & R & & P \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$then\quad T^{-1} = \begin{bmatrix} & R^T & & -R^T P \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \left[\begin{array}{ccc|c} 0.87 & 0.50 & 0 & -(0.87*1+0.5*2+0*4) \\ -0.50 & 0.87 & 0 & -(-0.5*1+0.87*2+0*4) \\ 0 & 0 & 1 & -(0*1+0*2+1*4) \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} 0.87 & 0.50 & 0 & -1.87 \\ -0.50 & 0.87 & 0 & -1.24 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

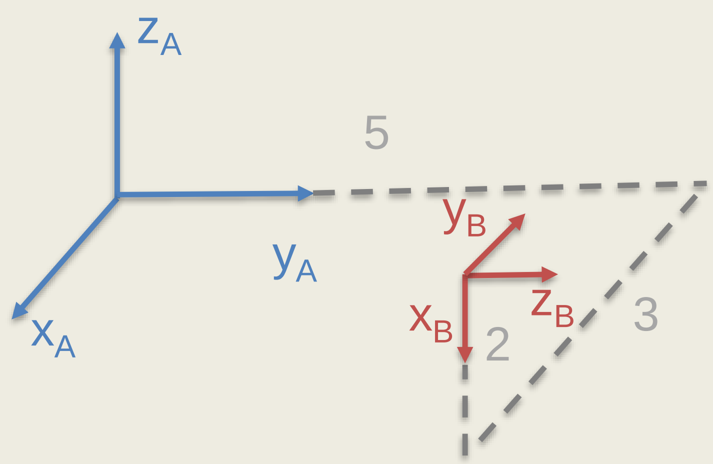
Prove: $T * T^{-1} = I$ where I is the identity matrix

Using transformations to describe frames

- Transformations can be used to move between frames
- Transformation required to move from frame {A} to frame {B} can be used as a description of the position and orientation of {B} relative to {A}
- The same transformation, ${}^{A}T_{B}$ or T_{AB} can be used to map a vector defined in frame {B} to frame {A}

Example

Figure 1 shows the positions and orientations of Frames A and B. Determine ${}^{A}T_{B}$



$${}^{A}T_{B} = Trans(3,5,2)Rot(x,-90)Rot(z,90)$$

Example

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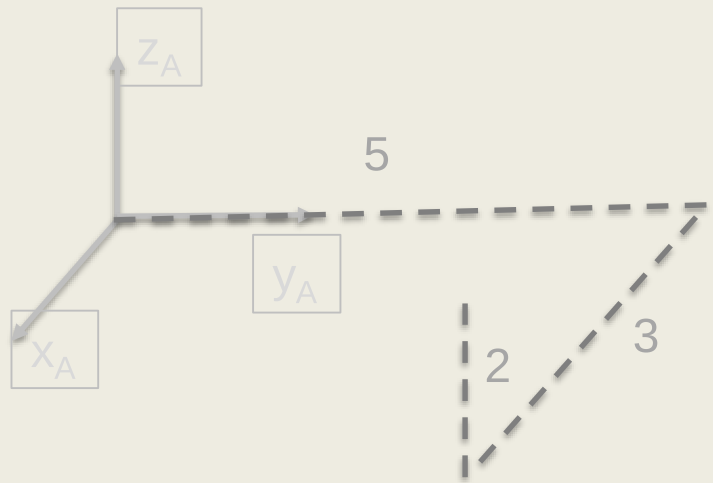

$$Trans(3,5,2)$$

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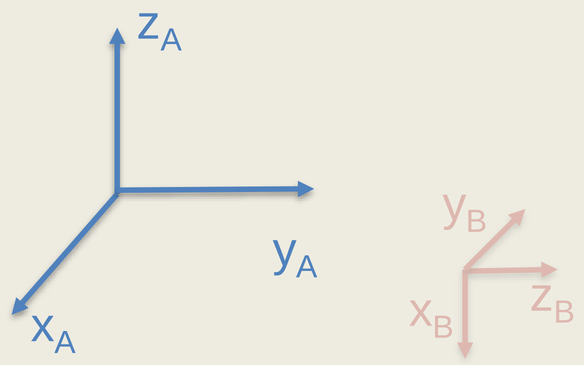

Example

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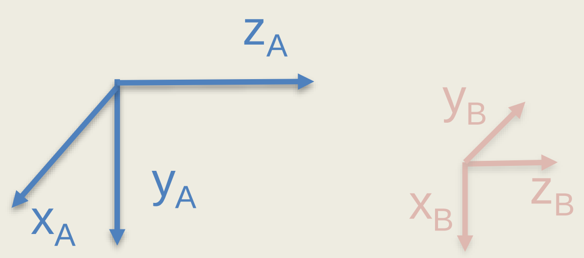
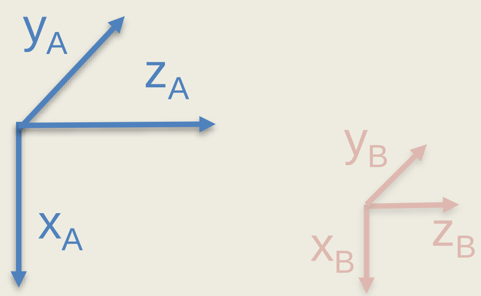


$$Rot(x, -90)$$

Example

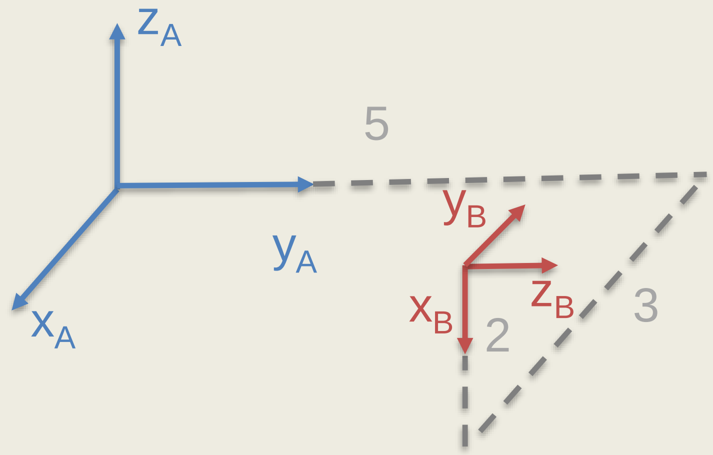
Figure 1 shows the positions and orientations of Frames A and B. Determine ${}^{A}T_{B}$



$$Rot(z, 90)$$

Example

Figure 1 shows the positions and orientations of Frames A and B. Determine ${}^{A}T_{B}$



$${}^{A}T_{B} = Trans(3,5,2)Rot(x,-90)Rot(z,90)$$

Example

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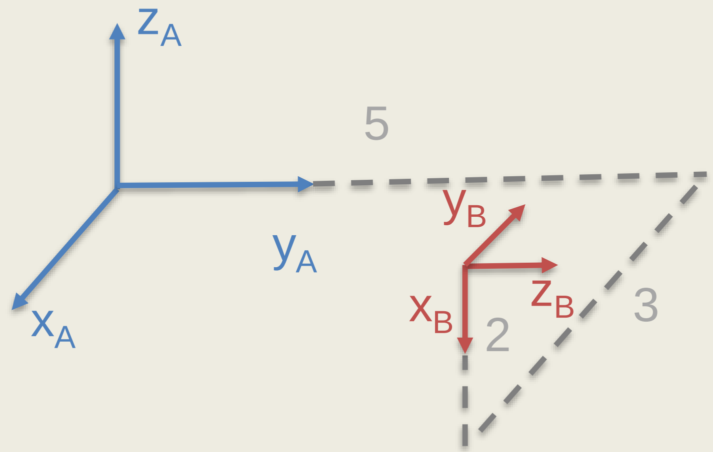
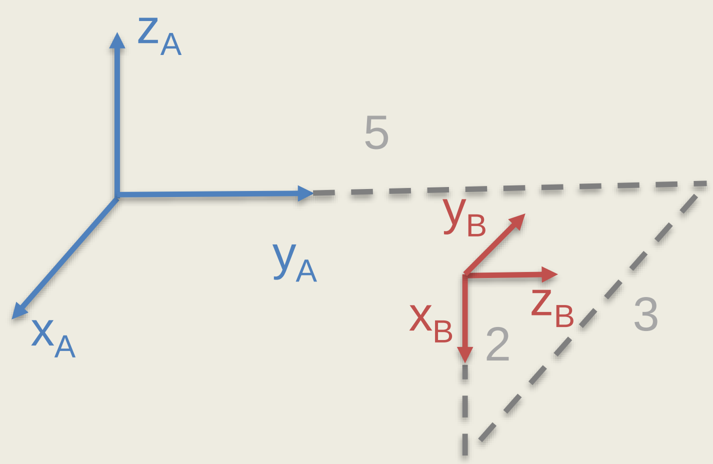


$$
{}^{A}T_{B} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Example

Figure 1 shows the positions and orientations of Frames A and B. Determine ${}^{A}T_{B}$



$$
{}^{A}T_{B} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Alternative Rotation Representation

- Unit Vectors
- Euler Angles
- Quaternions
- SVD (not going to see this)

Rotation Matrices using Unit Vectors

Dot product!

$$R_{ab} = \begin{bmatrix} x_{ab} & y_{ab} & y_{ab} \end{bmatrix} = \begin{bmatrix} x_a x_b & x_a y_b & x_a z_b \\ y_a x_b & y_a y_b & y_a z_b \\ z_a x_b & z_a y_b & z_a z_b \end{bmatrix}$$

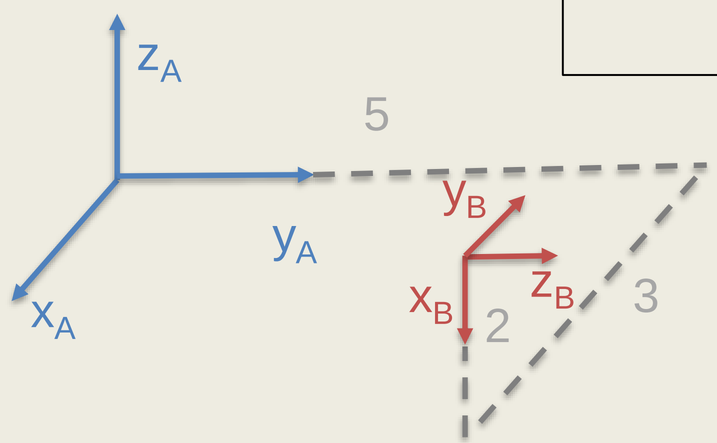
Especially for when we only have rotations of **90 degrees** around axes!

Let $v=\{x_a,y_a,z_a\}$ and $w=\{x_b,y_b,z_b\}$:

If $v||w \rightarrow v\cdot w=1$ if $v\perp w \rightarrow v\cdot w=0$ if $v||-w \rightarrow v\cdot w=-1$

In the previous example…

$$R_{ab} = \begin{bmatrix} x_a x_b & x_a y_b & x_a z_b \\ y_a x_b & y_a y_b & y_a z_b \\ z_a x_b & z_a y_b & z_a z_b \end{bmatrix}$$



$$R_{ab} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

Euler Angles

- Orientation represented as *a vector of 3 angles*
- Orientation frequently specified by a *sequence of rotations about the X, Y, and Z axes.*
- A sequence of rotations around principle axes is called an *Euler Angle Sequence*
- Minimal representation of orientation

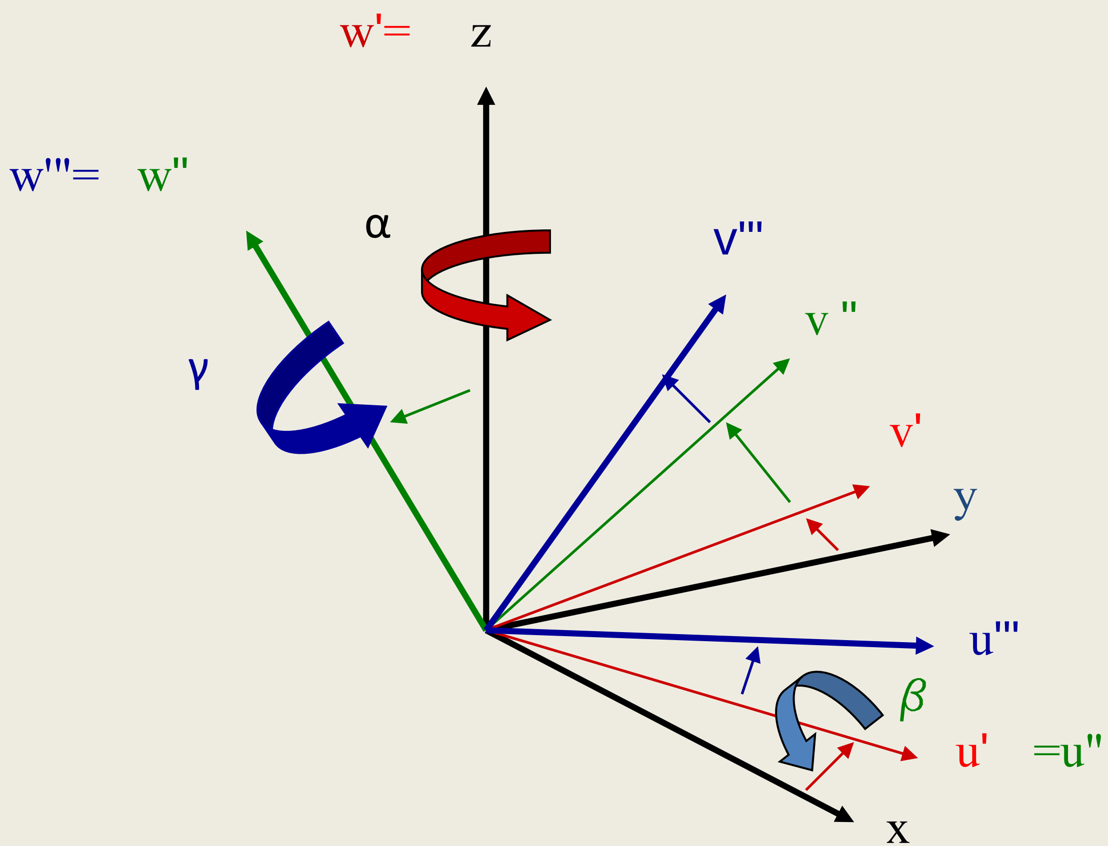
Euler Angles

- This gives us 12 redundant ways to store an orientation using Euler angles

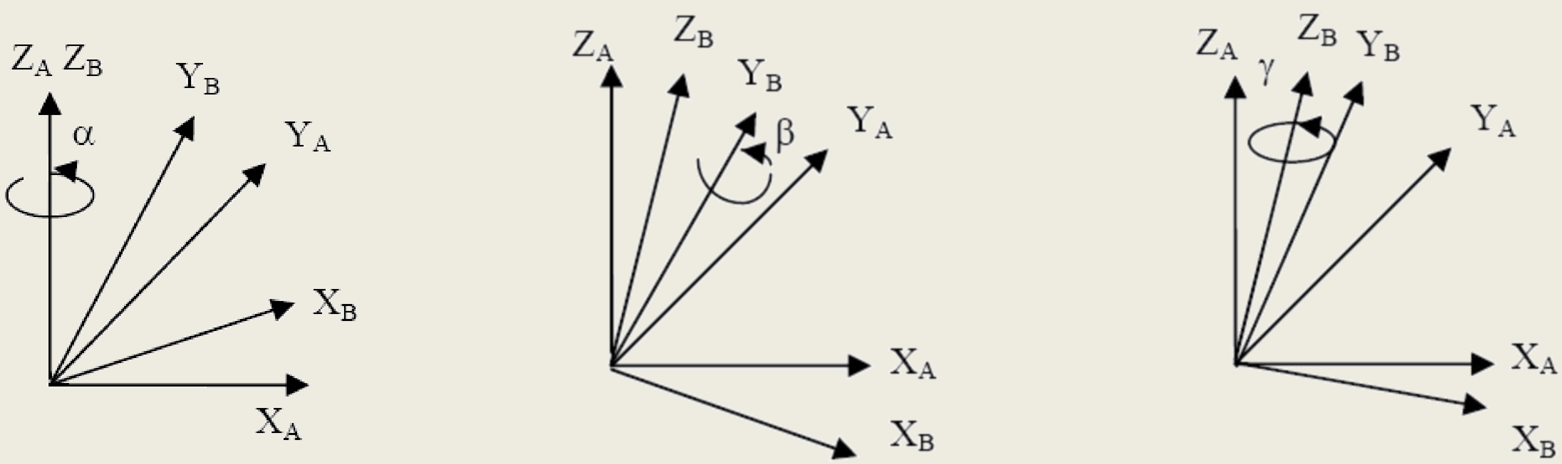
- Different industries use different conventions for handling Euler angles (or no conventions)

XYZ	XZY	XYX	XZX
YXZ	YZX	YXY	YZY
ZXY	ZYX	ZXZ	ZYZ

Euler Angles ZXZ, Animated



Euler Angles ZYZ – Rotation Matrix



$$ {}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = R(Z,\alpha)\ \ R(Y,\beta)\ \ R(Z,\gamma) = $$

$$\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

ZYZ from Homogenous matrix

$$\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix} \qquad \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

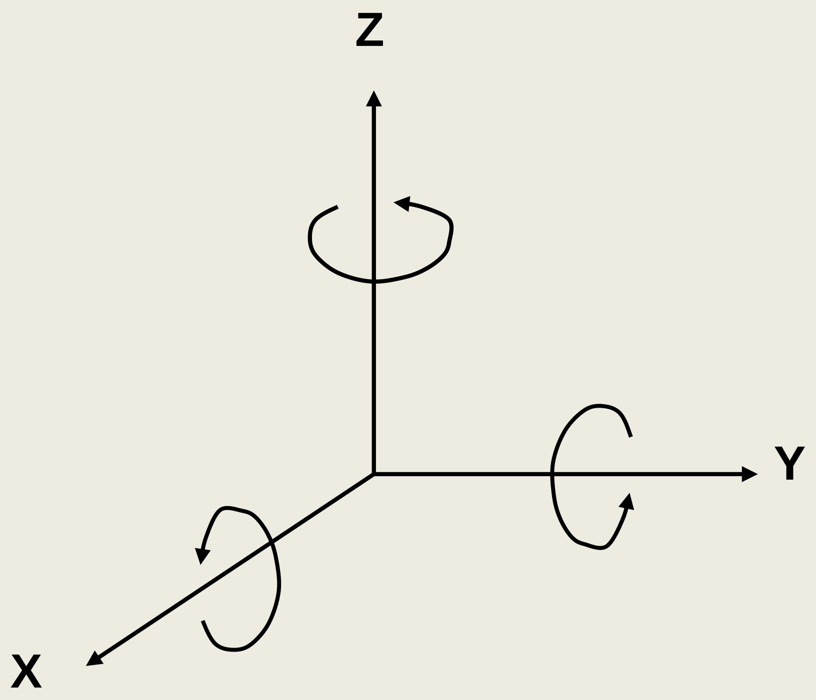
$$\beta = \boxed{A\tan 2}\left(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)$$

$$\alpha = A\tan 2\left(\frac{r_{23}}{s\beta}, \frac{r_{13}}{s\beta}\right)$$

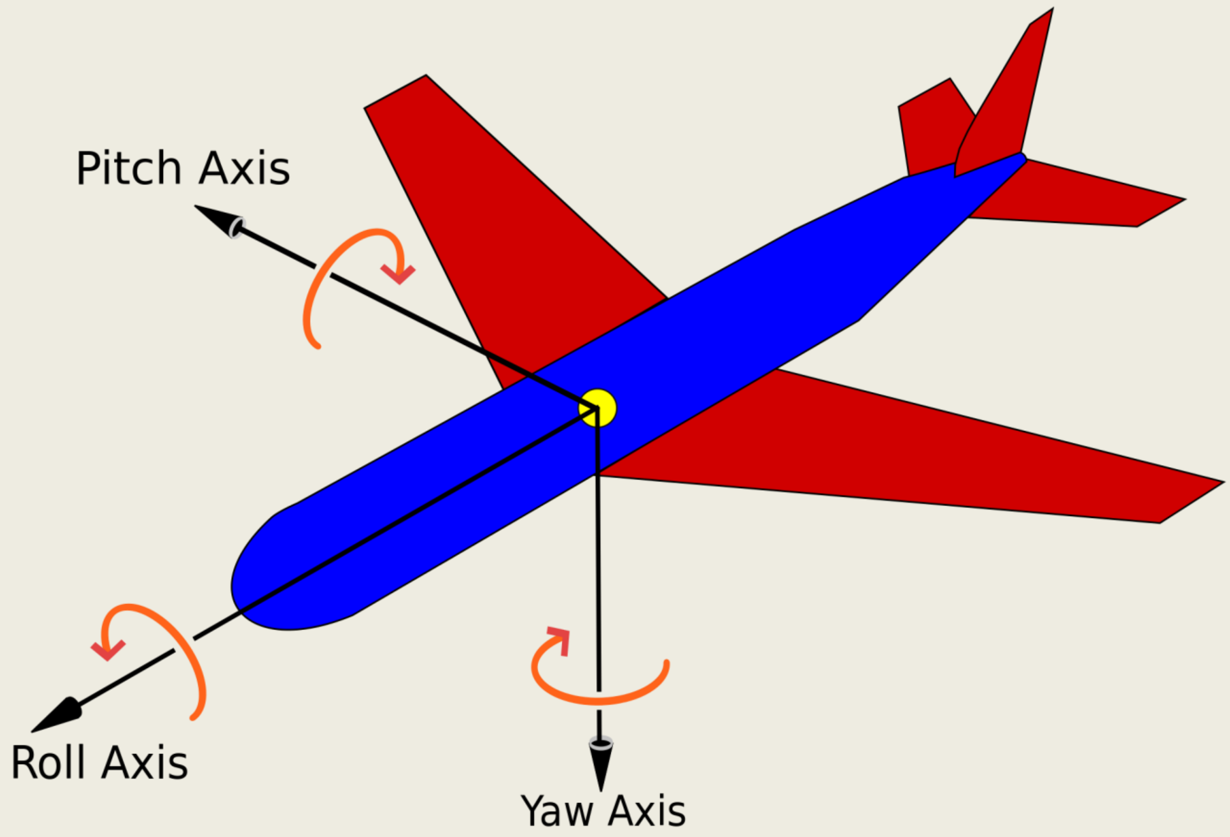
$$\gamma = A\tan 2\left(\frac{r_{32}}{s\beta}, -\frac{r_{31}}{s\beta}\right)$$

Singularity for the last two angles when β=0 or 180

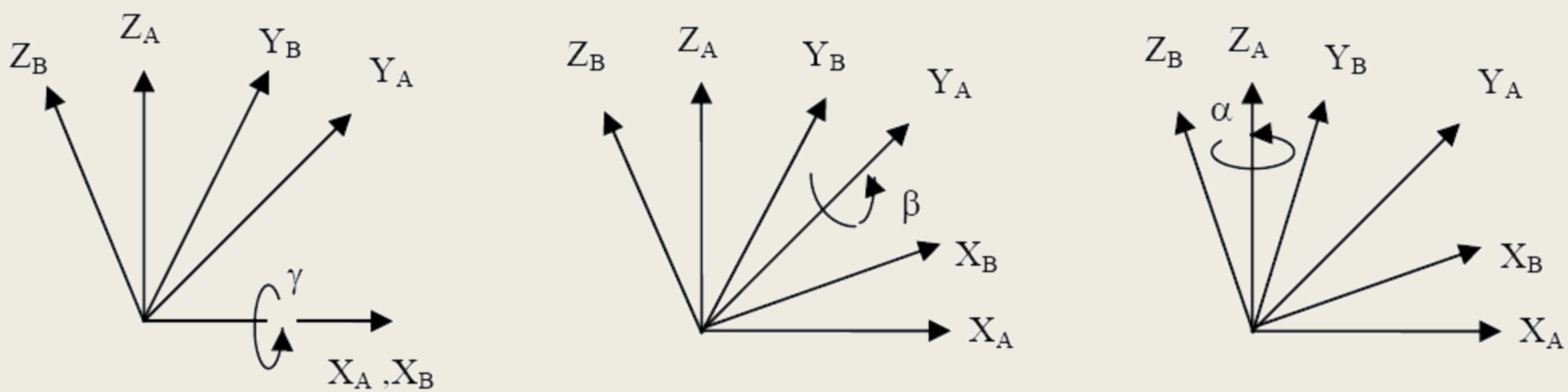
ZYX – Roll-Pitch-Yaw (RPY)



RPY – Vehicle Orientation



RPY – Rotation Matrix



$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R(Z,\alpha)\; R(Y,\beta)\; R(X,\gamma) = $$

$$ \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

RPY from Homogenous matrix

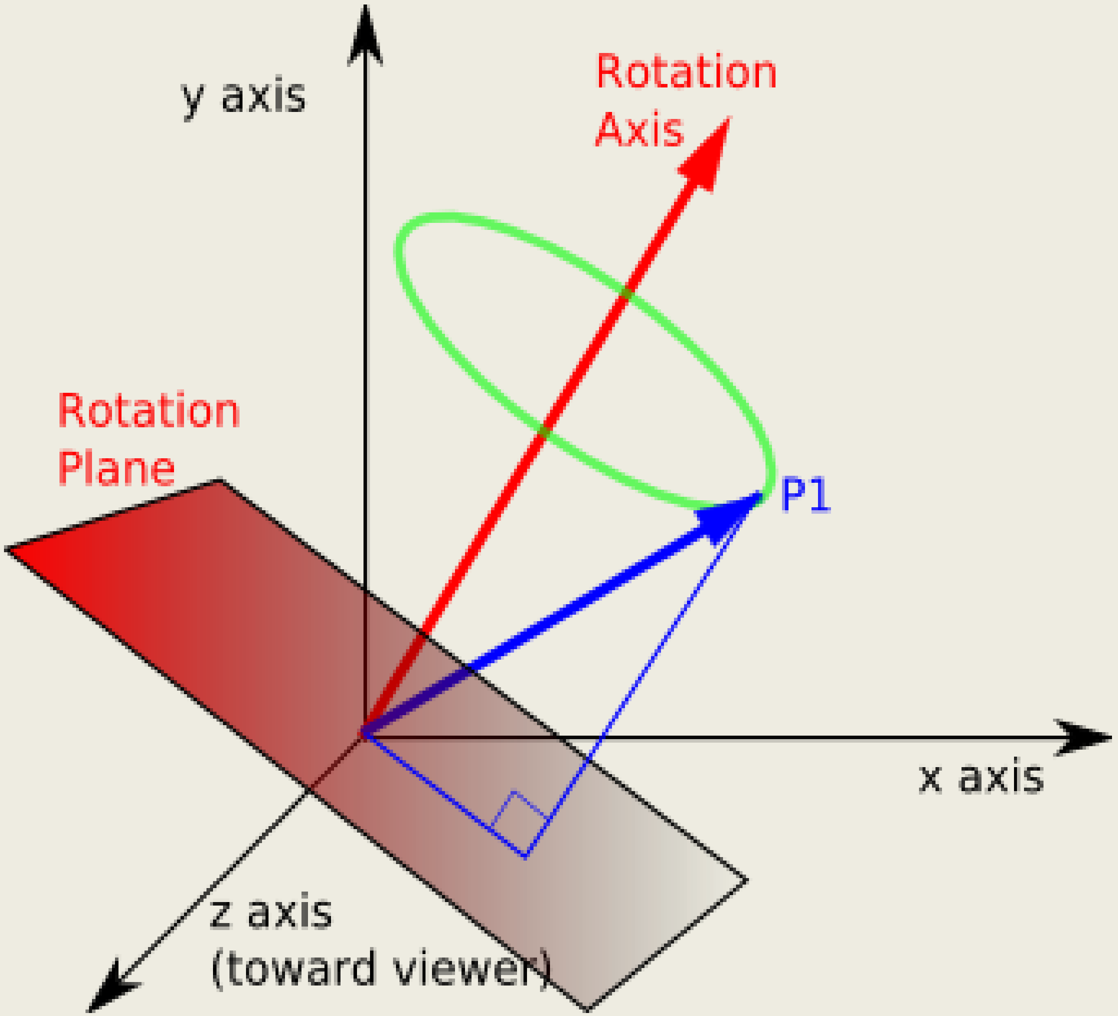
$$\beta = A\tan 2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$$

$$\alpha = A\tan 2(\frac{r_{21}}{c\beta}, \frac{r_{11}}{c\beta})$$

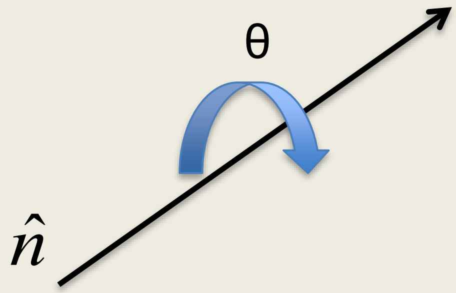
$$\gamma = A\tan 2(\frac{r_{32}}{c\beta}, \frac{r_{33}}{c\beta})$$

Singularity for the last two angles when β=90^o or 270^o

Quaternions – Basic Concept



Quaternions – Rotation as axis/angle



Any rotation is:

$$(q, \hat{n})$$

So it is safe to describe it as:

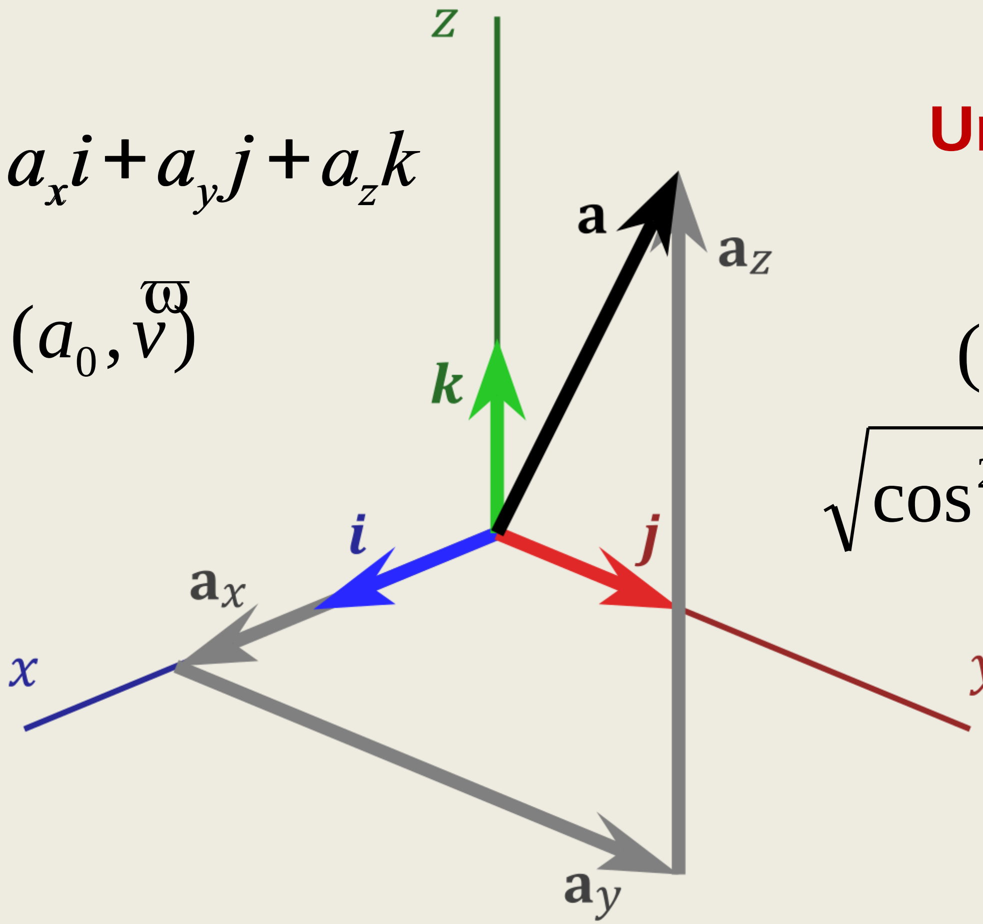
$$(q, x, y, z)$$

where x, y, z are coordinates of axis of rotation.

Quaternions – The Complex Number 3D space

$$a_0 + a_x i + a_y j + a_z k$$

$$(a_0, \vec{v})$$



Unit quaternions
(length = 1)

$$(\cos\theta, \sin\theta \cdot \vec{u})$$

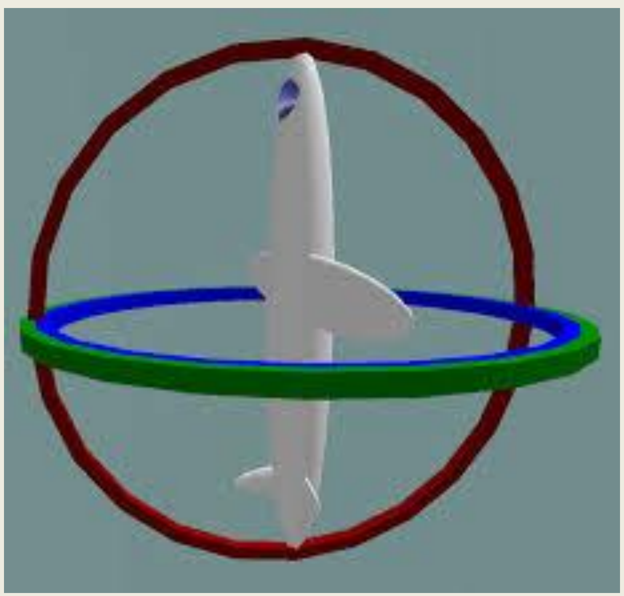
$$\sqrt{\cos^2\theta + \left(\sin\theta \cdot |\vec{v}|\right)^2} = 1$$

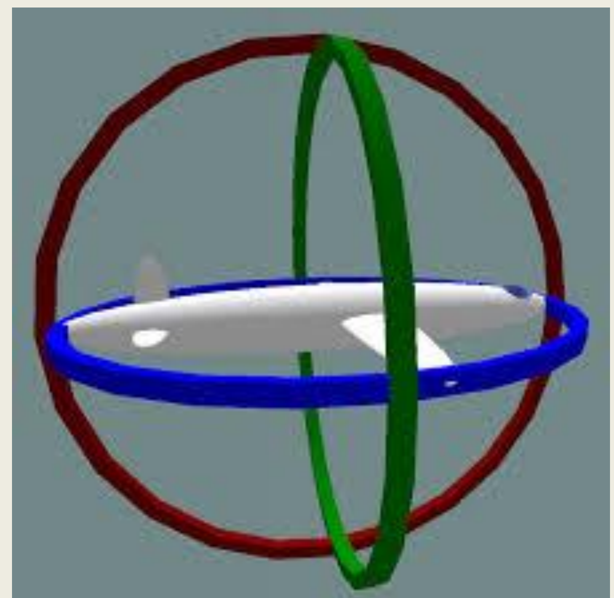
Quaternions – Advantages of Quaternions

Combination of Rotations (Simpler Computations)

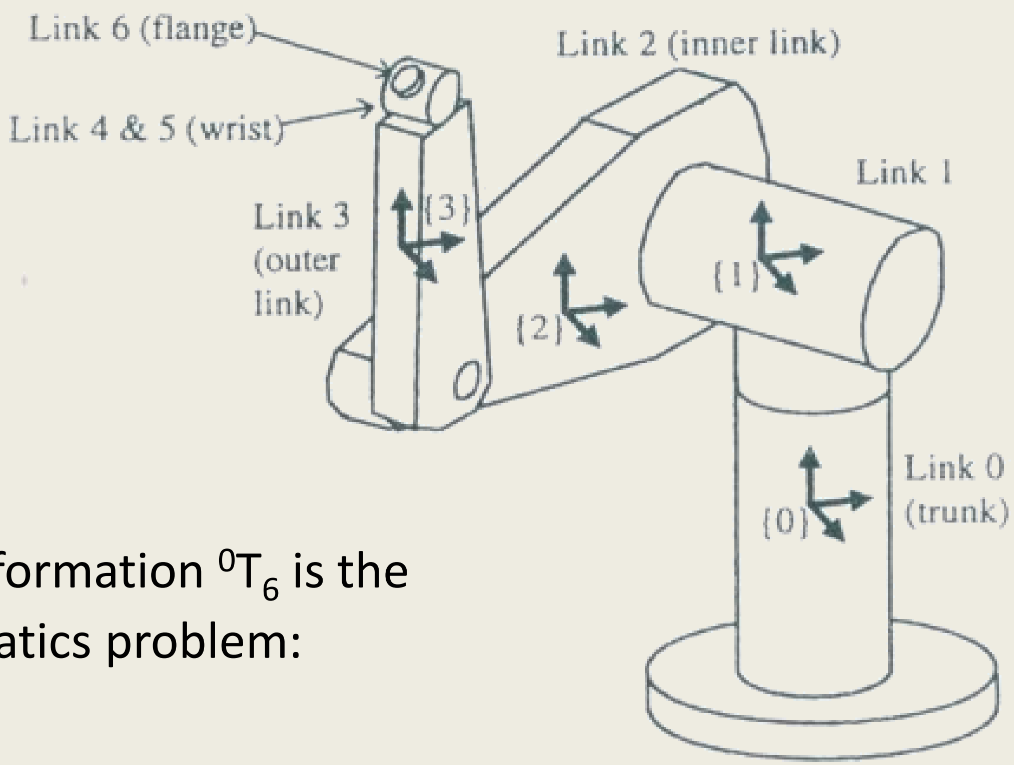
Rounding error robustness

No Gimbal Lock Issue





Forward kinematics – Composition of Homogeneous Transformations



For the Puma robot, the transformation ${}^{0}T_{6}$ is the solution to the forward kinematics problem:

$${}^{0}T_{6} = {}^{0}T_{1}\,{}^{1}T_{2}\,{}^{2}T_{3}\,{}^{3}T_{4}\,{}^{4}T_{5}\,{}^{5}T_{6}$$

Composition of homogeneous transforms

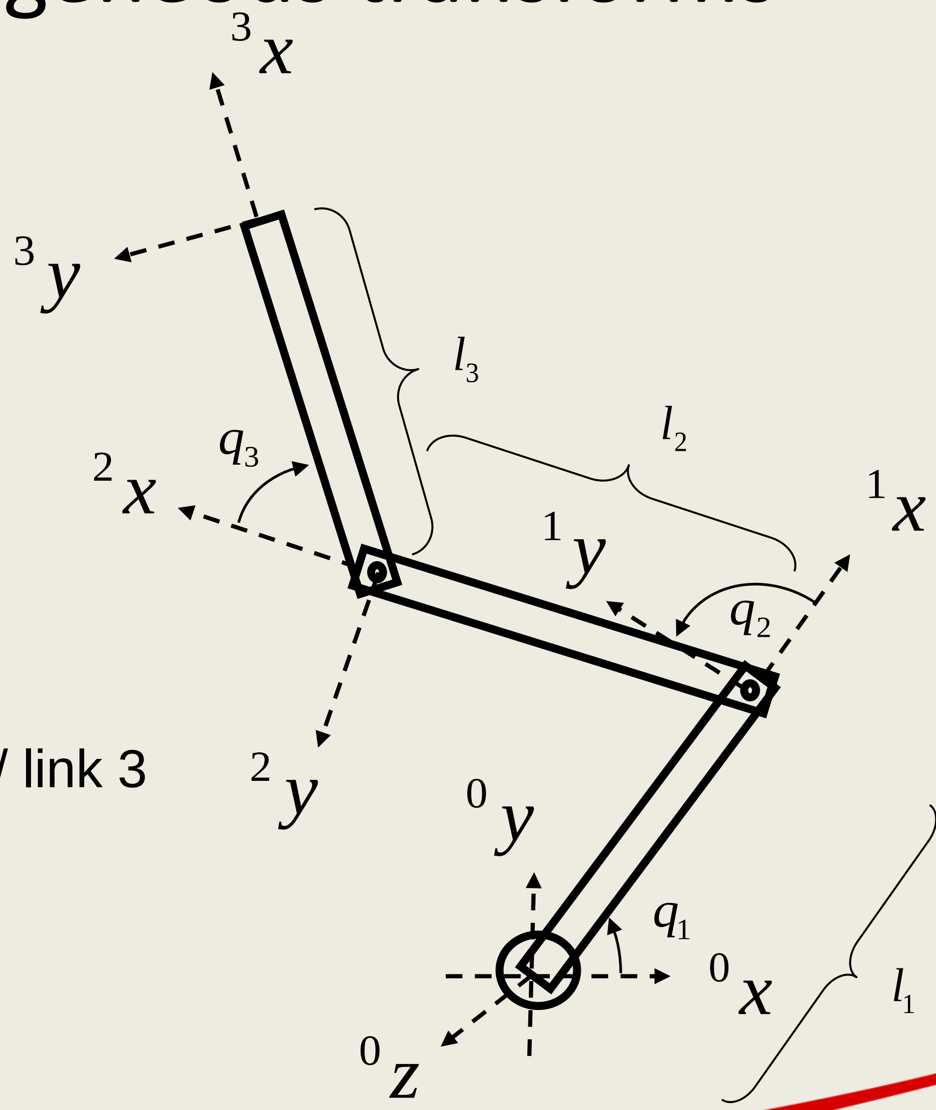
Base to *eff* transform

$$^0T_3 = {}^0T_1\,{}^1T_2\,{}^2T_3$$

Transform associated w/ link 3

Transform associated w/ link 2

Transform associated w/ link 1

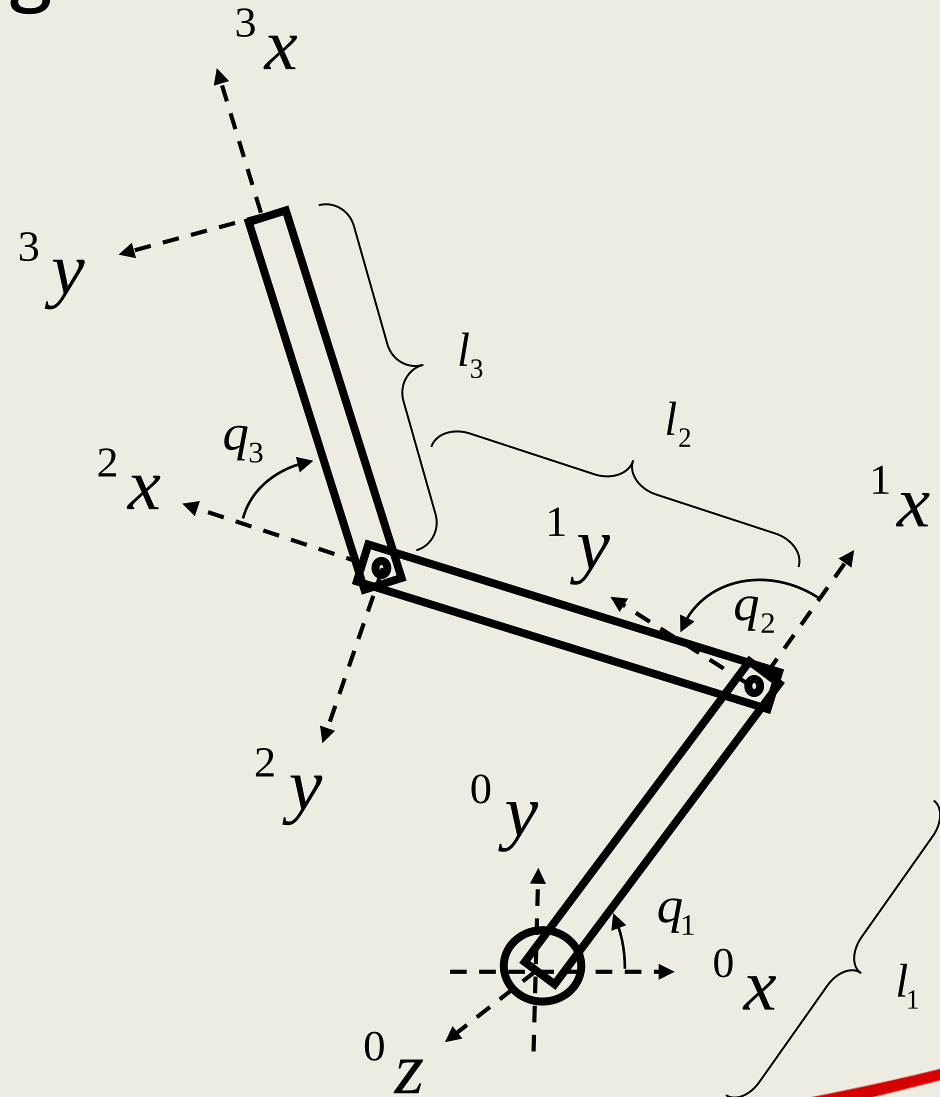


Composition of homogeneous transforms

$$^{0}T_3 = {}^{0}T_1\,{}^{1}T_2\,{}^{2}T_3$$

$$^{0}T_1 = \begin{pmatrix} c_1 & -s_1 & 0 & l_1c_1 \\ s_1 & c_1 & 0 & l_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

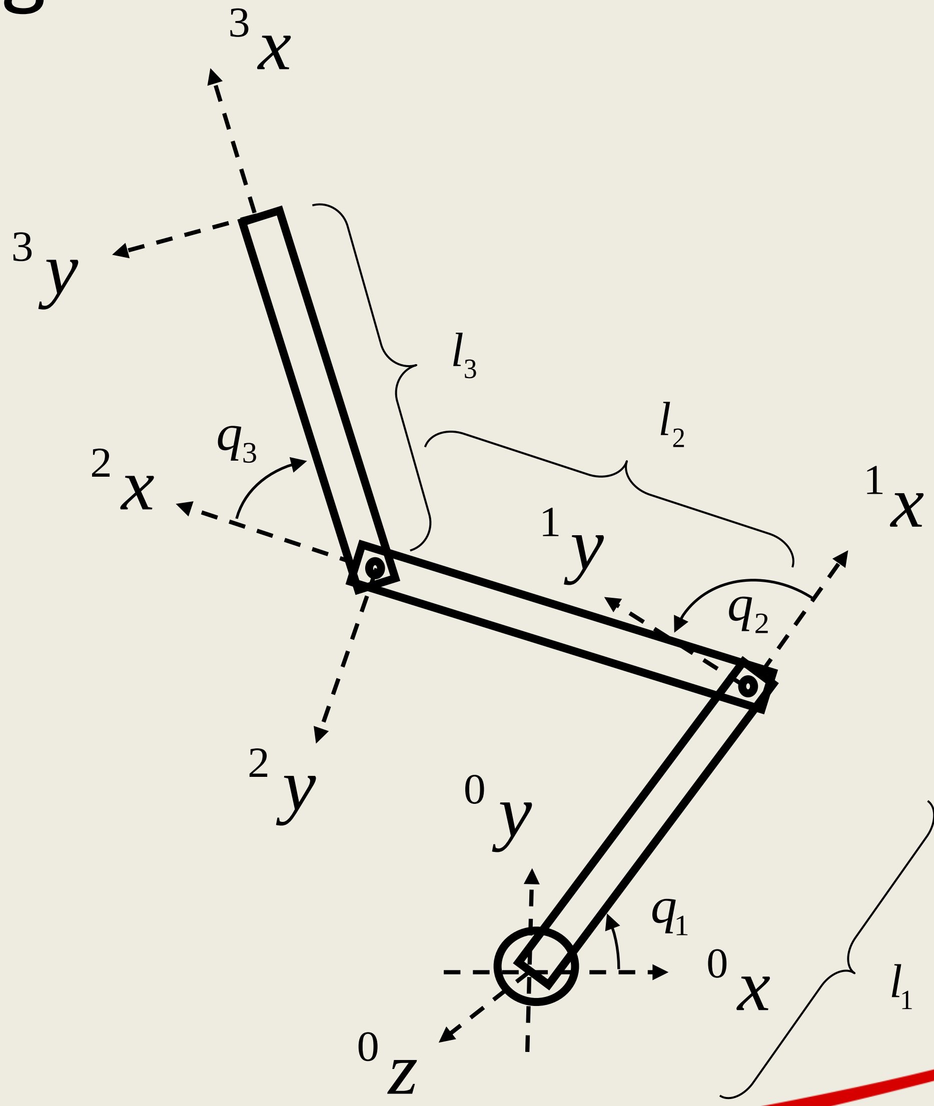
$$^{1}T_2 = \begin{pmatrix} c_2 & -s_2 & 0 & l_2c_2 \\ s_2 & c_2 & 0 & l_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Composition of homogeneous transforms

$$ {}^{0}T_{3} = {}^{0}T_{1}\, {}^{1}T_{2}\, {}^{2}T_{3} $$

$$ {}^{2}T_{3} = \begin{pmatrix} c_3 & -s_3 & 0 & l_3 c_3 \\ s_3 & c_3 & 0 & l_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$



Composition of homogeneous transforms

$$^{0}T_{3} = {}^{0}T_{1}\,{}^{1}T_{2}\,{}^{2}T_{3}$$

$$^{0}T_{3} = \begin{pmatrix} c_1 & -s_1 & 0 & l_1c_1 \\ s_1 & c_1 & 0 & l_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & -s_2 & 0 & l_2c_2 \\ s_2 & c_2 & 0 & l_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_3 & -s_3 & 0 & l_3c_3 \\ s_3 & c_3 & 0 & l_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^{0}T_{3} = \begin{pmatrix} c_{123} & -s_{123} & 0 & l_1c_1 + l_2c_{12} + l_3c_{123} \\ s_{123} & c_{123} & 0 & l_1s_1 + l_2s_{12} + l_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where: $\begin{cases} s_{ij} = \sin(\theta_i + \theta_j) \\ c_{ij} = \cos(\theta_i + \theta_j) \end{cases}$

…Remember those trigonometric identities…

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi)$$

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi)\, \mu \,\sin(\theta)\sin(\phi)$$

…and some more commonly used formulas…

$$\sin\left(\pi - \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$$

$$\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$$

S	A
T	C

$$\tan\left(-\frac{2\pi}{3}\right) = -\tan\left(\frac{2\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right)$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

Summary

Kinematics and Reference frames – Basis of the Analysis

Connecting Frames – Translation Vectors and Rotation Matrices

Unified Representation – Homogeneous Transformations – Compound Transformations

ROTATION REPRESENTATION