ROBOTIC FUNDAMENTALS (UMFM4X-15-M)

Dynamics (RNE)

UWE Bristol | University of the West of England

Previously on ROBOTIC FUNDAMENTALS

Introduction in Dynamics & Lagrange

Questions?

Today's Lecture

Recursive Newton-Euler (RNE)

Closed Form Newton-Euler

Computational Considerations and Conclusions

Approaches to Dynamic modeling

Newton Euler balance of forces and torques	**Lagrange** Energy-based approach
• Equations written for <u>each link</u> • Inverse dynamics in <u>real time</u> • Best suited for synthesis (implementation) of <u>model based control</u>	• Multi-body robot <u>seen as a whole</u> • Internal reaction forces between the links <u>are eliminated</u> • <u>Closed-form</u> equations are directly obtained • Best suited for study of <u>dynamic properties and analysis of control</u>

State – Space Equation

The equation of motion from both approaches can be described as:

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

Mass matrix

Gravity term

Centripetal and Coriolis matrix

Newton-Euler

THE PRINCIPLES

Newton – Euler

Forces/torques relate to velocities, accelerations

- Translation - Rigid body, mass *m*, centre of mass accelerating at a_c. Force *F* acting on its centre of mass causes this acceleration, given by Newton's Law:

$$F = ma_c$$

- Rotation - Rigid body rotating with angular velocity ω and with angular acceleration $d\omega/dt$. The moment *N* which must be acting on the body to cause this motion is given by Euler's Law:

$$N = \cancel{{}^{C}I\dot{\omega} + \omega \times {}^{C}I\omega} \qquad {}^{c}I\dot{\omega} + \omega \times {}^{c}I\omega$$

Principle of action and reaction forces/torques remain the same:

F/T applied by body i to body i+1 = - F/T applied by body i+1 to body i

Newton – Euler Forms

- The process of calculating joint torques from manipulator motion can be performed using the Recursive Newton-Euler method.

  ↳ efficient.

- Force and torque to be applied to a joint by an actuator as a function of all other forces and torques acting on that joint – Closed Form Equations.

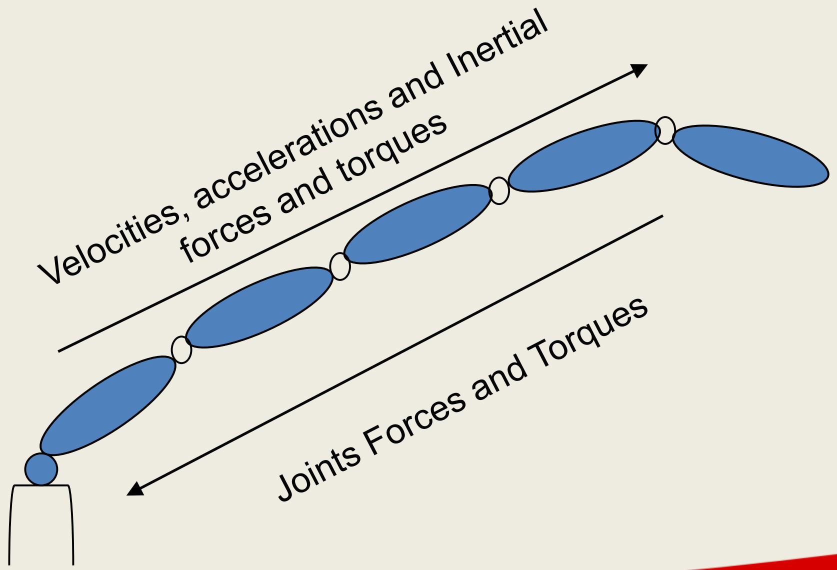
  Difficult to solve numerically, require a lot of computation time.

Newton-Euler

RECURSIVE APPROACH

Recursive method

- Enable real-time forward and inverse dynamics
- Recursive algorithm to compute from one link to another:
  - Forward: velocity and acceleration
  - Backward: forces and torques



Propagation from link to link

- As with locations, we <u>attach frames to the joints and links</u>

  Here using standard DH convention.

- When one link moves <u>all other links move</u>
- Motion of link *n* in respect to the base is <u>a sum of motions</u> of all links from the base to *n*
- Velocities and accelerations have to be <u>transferred</u> from frame to frame <u>using rotation transforms</u>

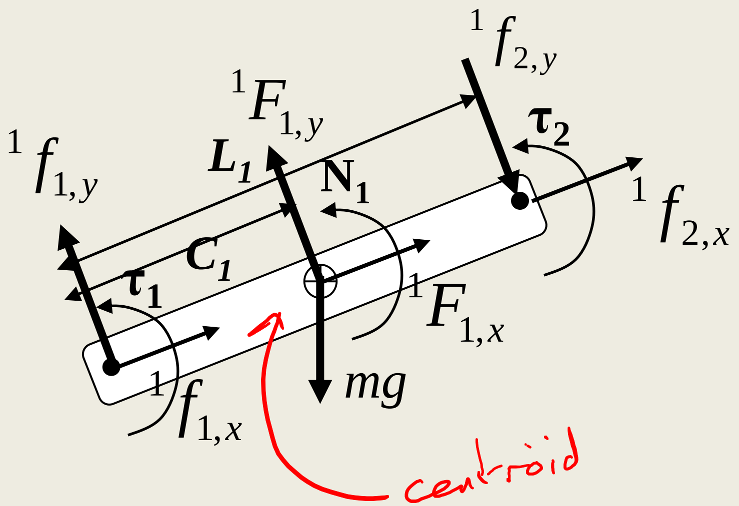
Summary of notation

All linear and angular velocities and accelerations are measured relative to a fixed frame, but can be expressed in any frame like any other vector. *In the planar case angular velocities/accelerations are scalar so this issue does not arise for angular motion*

- Linear accelerations/forces: where Q represents a linear quantity, ${}^{j}Q_{k}$ is that quantity for link k expressed in the frame $\{j\}$ coordinate system. A further subscript (x or y) indicates a particular scalar component of ${}^{j}Q_{k}$
- Angular velocities/accelerations/torques: where Q represents an angular quantity, Q_k is that quantity for link k.
- Links are defined by their dimensions (length L_i and centroid position c_i) and inertial properties (mass m_i and mass moment of inertia I_i)
- Matrices ${}^{j}R_{k}$ are rotation matrices as defined previously, except that only 2x2 matrices are required in the planar case

Link Balance Equations – Newton

↳ Balance of forces, torque etc.



Newton equation:

$$f_i - f_{i+1} + m_i g = m_i a_{Ci}$$

Absolute linear acceleration of centre C_i

τ_i Torque in joint *i*

N_i Inertial torque of link *i*

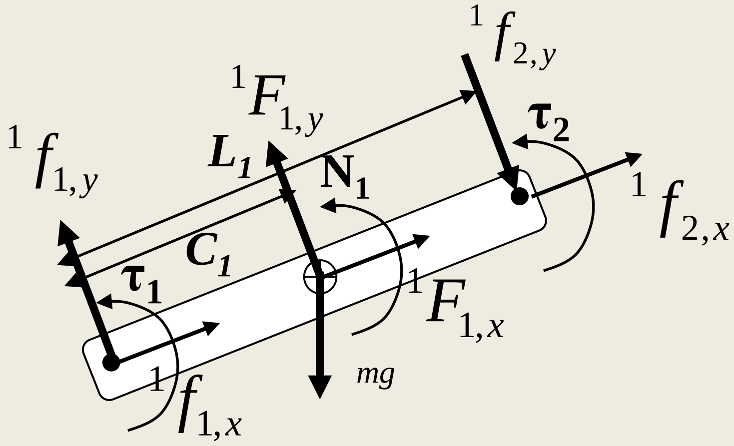
$^1f_{1,y}c_1$ Torque from coupling force in joint 1

$^1f_{2,y}(L_1 - c_1)$ Torque from coupling force in joint 2

$^1f_{1,y}$ Force acting on link 1

$^1f_{2,y}$ Force that link 2 acts on link 1

Link Balance Equations – Translational



Translation Forces:

$$^1f_{1,y} - {}^1f_{2,y} + mg - ma_{Ci} = 0$$

Inertia

τ_i Torque in joint *i*

N_i Inertial torque of link *i*

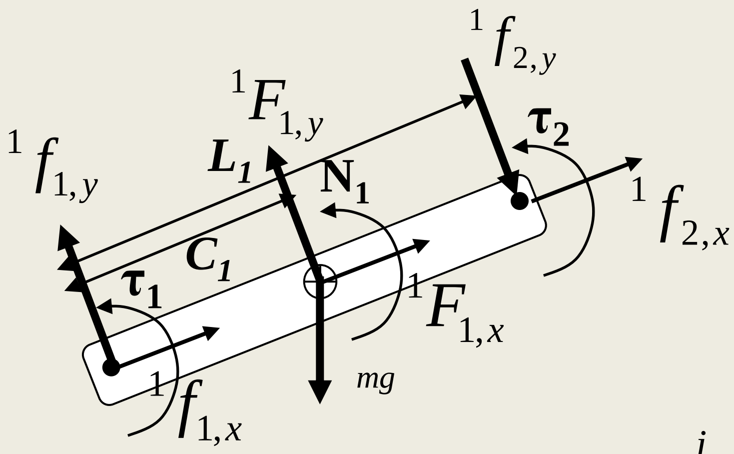
$^1f_{1,y}c_1$ Torque from coupling force in joint 1

$^1f_{2,y}(L_1 - c_1)$ Torque from coupling force in joint 2

$^1f_{1,y}$ Force acting on link 1

$^1f_{2,y}$ Force that link 2 acts on link 1

Link Balance Equations – Rotational



Rotation forces (torques):

$$\tau_1 = \tau_2 + c_1\,{}^1f_{1,y} + (L_1 - c_1)\,{}^1f_{2,y} + N_1$$

Propagation from link to link in general:

$$^i f_i = {}^iR_{i+1}\,{}^{i+1}f_{i+1}$$

$$^i\tau_i = {}^iR_{i+1}\,{}^{i+1}\tau_{i+1} + c_i \cdot {}^if_i + ({}^iP_{i+1} - c_i) \cdot {}^if_{i+1} + N_i$$

τ_i Torque in joint *i*

N_i Inertial torque of link *i*

${}^1f_{1,y}c_1$ Torque from coupling force in joint 1

${}^1f_{2,y}(L_1 - c_1)$ Torque from coupling force in joint 2

${}^1f_{1,y}$ Force acting on link 1

${}^1f_{2,y}$ Force that link 2 acts on link 1

Outward and Inward recursions

Outward recursions – For link i=1 to n, calculate:

1. Angular velocity of link (ω_i)
2. Angular acceleration of link ($d\omega_i/dt$)
3. Linear acceleration of link at frame origin (${}^i dv_i/dt$)
4. Linear acceleration of link at centroid (${}^i a_i$)
5. Resultant force acting on link at centroid (${}^i F_i$)
6. Resultant moment acting on link around centroid (N_i)

Inward recursions – For link i=n to 1, calculate:

1. Force exerted on link i by link i-1 (${}^i f_i$)
2. Torque exerted on link i by link i-1 (τ_i)

Outward Recursion

For an n-link planar jointed manipulator, assuming <u>each link's centroid lies on the link's x axis</u> (and <u>zero gravity</u>)

For link i=1-n, calculate:

1. Angular velocity of link
2. Angular acceleration of link
3. Linear acceleration of link at frame origin
4. Linear acceleration of link at centroid
5. Resultant force acting on link at centroid
6. Resultant moment acting on link around centroid

$$\omega_i = \omega_{i-1} + \dot{\theta}_i$$

$$\dot{\omega}_i = \dot{\omega}_{i-1} + \ddot{\theta}_i$$

!!!

Note how the rot. matrix is the other way to previously. ie ${}^{1}R_{0}$, ie zero in terms of 1.

$${}^{i}\dot{v}_{i-1} = {}^{i}R_{i-1}\,{}^{i-1}\dot{v}_{i-1}$$

$${}^{i}\dot{v}_i = {}^{i}\dot{v}_{i-1} + L_i \begin{bmatrix} -\omega_i^2 \\ \dot{\omega}_i \end{bmatrix}$$

$${}^{i}a_{Ci} = {}^{i}\dot{v}_{i-1} + c_i \begin{bmatrix} -\omega_i^2 \\ \dot{\omega}_i \end{bmatrix}$$

$$F_i = m_i\,{}^{i}a_{Ci}$$

$$N_i = I_i \dot{\omega}_i$$

Inward Recursion

For link i=n to 1, calculate:

1. Force exerted on link i by link i+1

$$ {}^{i}f_{i+1} = {}^{i}R_{i+1} \, {}^{i+1}f_{i+1} $$

$$ {}^{i}f_{i} = {}^{i}F_{i} + {}^{i}f_{i+1} $$

1. Torque exerted on link i by link i+1

$$ \tau_i = N_i + \tau_{i+1} + {}^{i}f_{i,y} c_i + {}^{i}f_{i+1,y} \left(L_i - c_i \right) $$

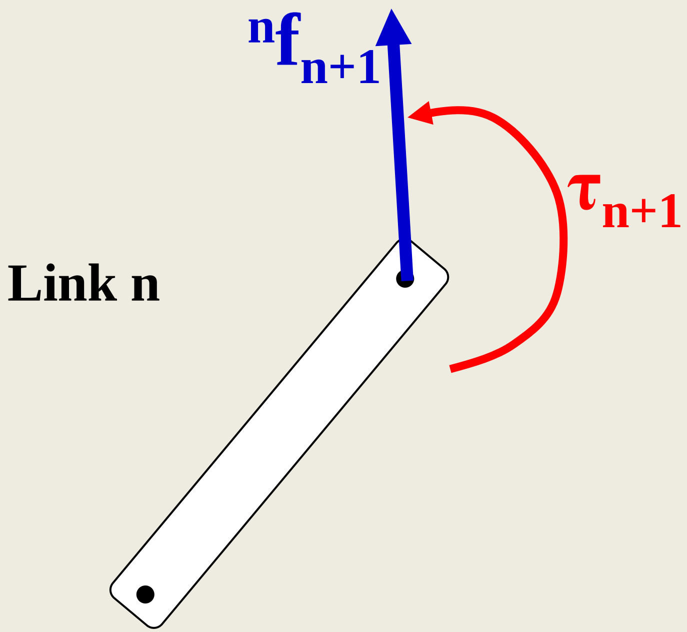
External forces

RNE can also be used to calculate joint torques where external forces act on the manipulator, including gravity

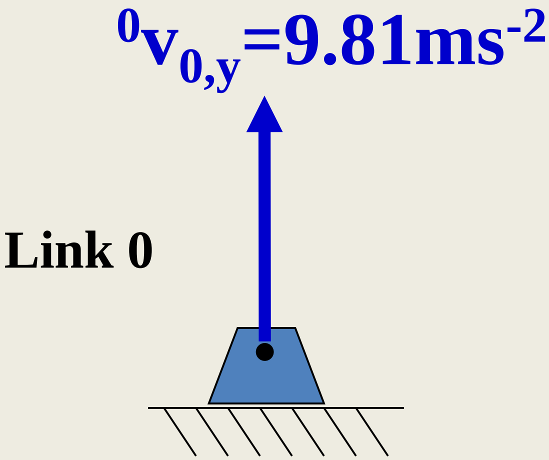
- **External forces** or moments exerted by the manipulator at/around a point or a link can simply be added to the force or moment equations during the inward recursions. A particularly common special case is end-effector force and torque, which are represented by ${}^{n}f_{n+1}$ and τ_{n+1} *${}^{i}f_{i+1}$ and τ_{i+1} would be defined as the force/ torque exerted by the manipulator, not on the manipulator*
- There is an easy way to include **gravity**: say that link 0 is accelerating vertically upwards at 1g. Thus assuming y_0 points vertically up, the following should be used:

$${}^{0}\dot{v}_{0} \qquad \cancel{{}^{0}\dot{v}_{0}} = \begin{bmatrix} 0 \\ 9.81 \end{bmatrix} m/s^{2}$$

External forces (2)

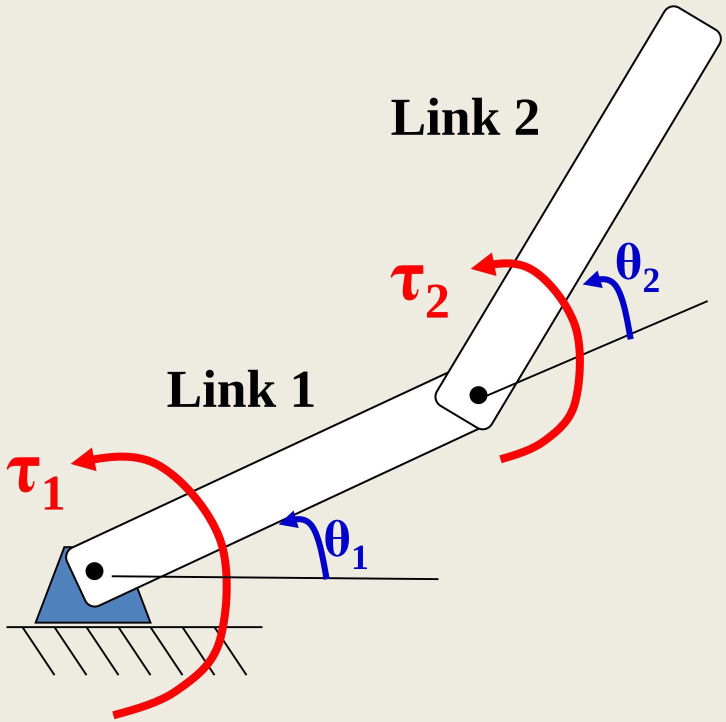


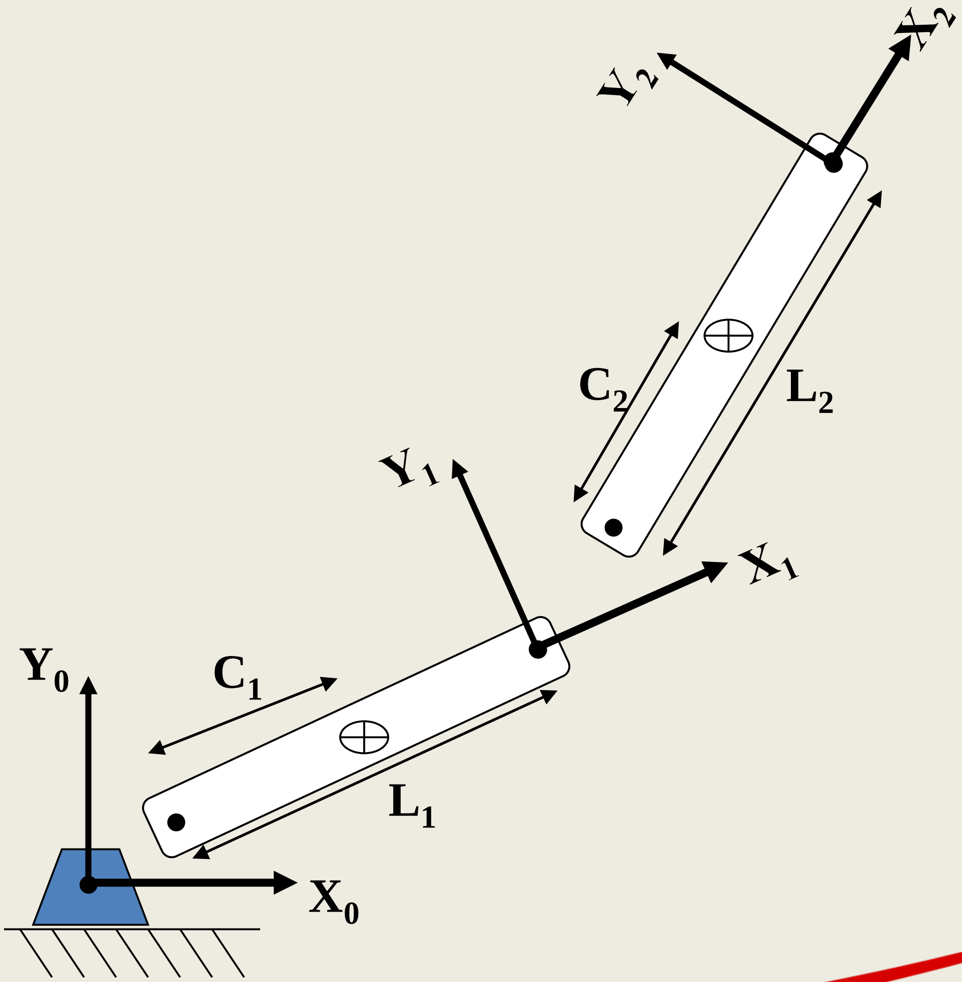
External Forces and Torques



Gravitational Forces

Inverse dynamics
Example: 2 link planar mechanism





We know angles, angular velocities and angular accelerations for θ_1 and θ_2.

We want to find torques τ_1 and τ_2

Inverse dynamics
Example: 2 link planar mechanism

Outward – Link1:

1. $\cancel{\omega_1 = \dot{\theta}_1}$ $\omega_1 = \dot{\theta}_1$

2. $\cancel{\dot{\omega}_1 = \ddot{\theta}_1}$ $\dot{\omega}_1 = \ddot{\theta}_1$

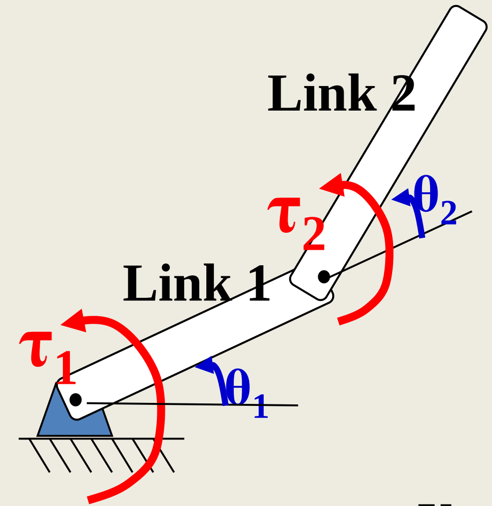
3. $${}^1\dot{V}_1 = \begin{bmatrix} -L_1\omega_1^2 \\ L_1\dot{\omega}_1 \end{bmatrix} \begin{matrix} \leftarrow x-component \\ \leftarrow y-component \end{matrix}$$

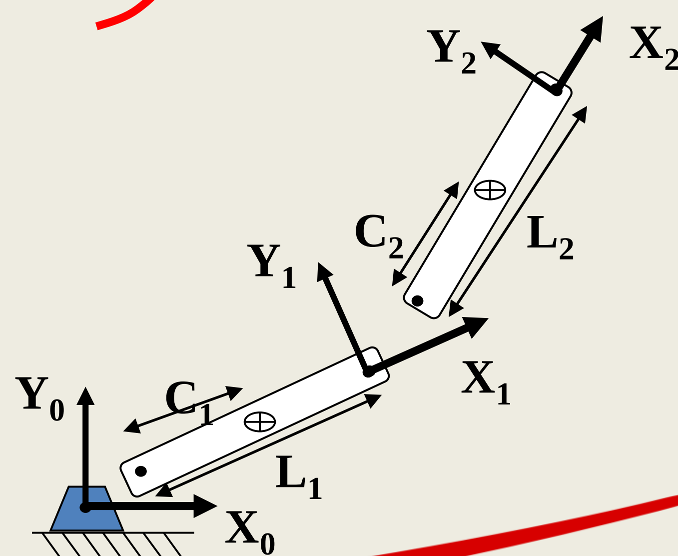
4. $${}^1a_1 = \begin{bmatrix} -c_1\omega_1^2 \\ c_1\dot{\omega}_1 \end{bmatrix}$$

5. $${}^1F_1 = m_1\,{}^1a_1, \quad i.e. \quad {}^1F_{1,x} = -m_1c_1\omega_1^2$$
$${}^1F_{1,y} = -m_1c_1\dot{\omega}_1$$

6. $N_1 = I_1\dot{\omega}_1 \quad (scalar)$

$\hookrightarrow$ Moment.





Inverse dynamics
Example: 2 link planar mechanism

Outward – Link2:

$$ROT(z,\theta)=\begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

!!!

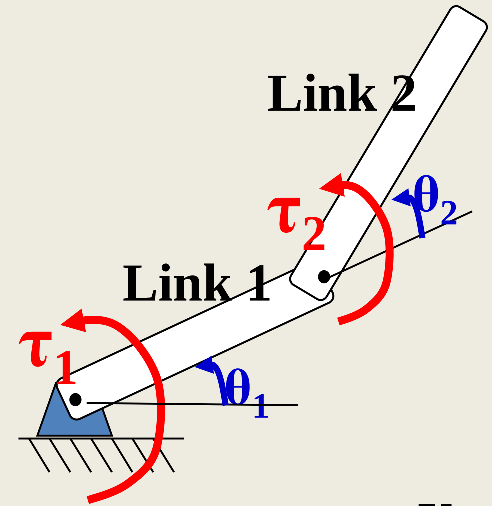
↳ note, only 2D.

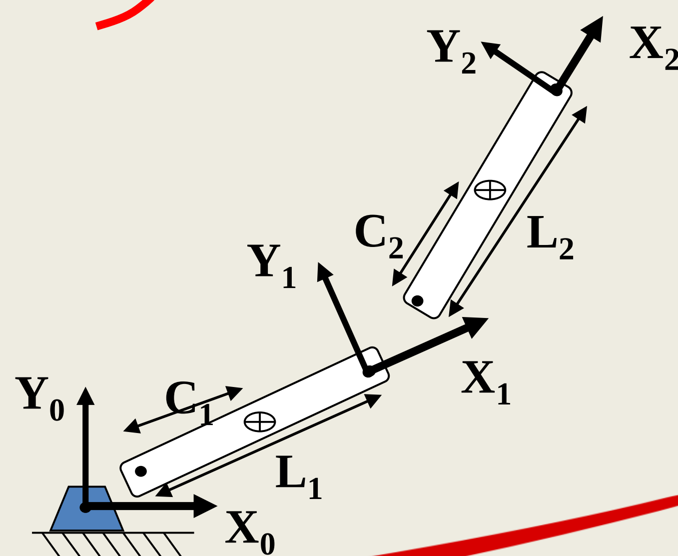
1. $\omega_2 = \omega_1 + \dot{\theta}_2$
2. $\dot{\omega}_2 = \dot{\omega}_1 + \ddot{\theta}_2$
3. ${}^2\dot{V}_1 = {}^2R_1\,{}^1\dot{V}_1$

$$where: \quad {}^2R_1 = \begin{bmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{bmatrix}$$

$${}^2\dot{V}_2 = {}^2\dot{V}_1 + \begin{bmatrix} -L_2\omega_2^2 \\ L_2\dot{\omega}_2 \end{bmatrix}$$

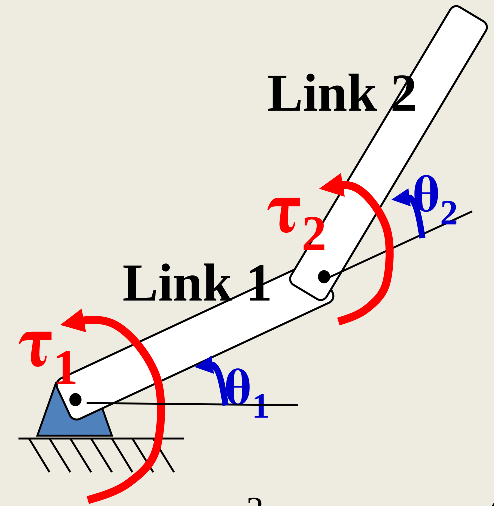
4. ${}^2a_2 = {}^2\dot{V}_1 + \begin{bmatrix} -c_2\omega_2^2 \\ c_2\dot{\omega}_2 \end{bmatrix}$
5. ${}^2F_2 = m_2\,{}^2a_2$
6. $N_2 = I_2\dot{\omega}_2$





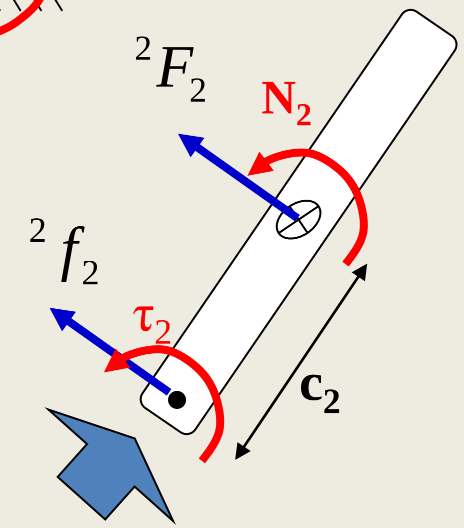
Inverse dynamics
Example: 2 link planar mechanism

Inward– Link2:



1. $Adding\ \ forces \quad {}^{2}f_{2} = {}^{2}F_{2}$
2. $Adding\ moments\ about\ centroid:$

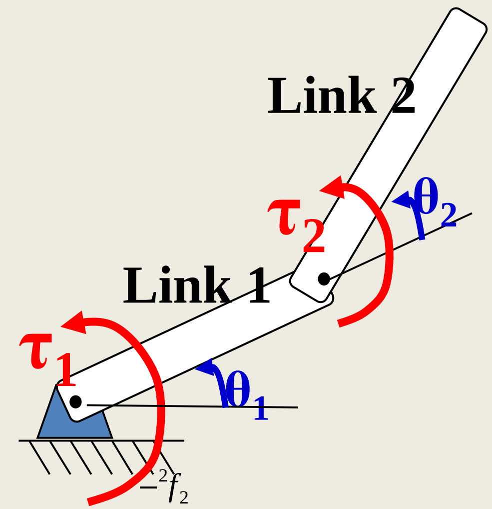
$$\tau_2 = N_2 + {}^{2}f_{2,y} c_2$$



Force and torque exerted on link 2 by link 1

Inverse dynamics
Example: 2 link planar mechanism

Inward– Link2:



1. $\quad {}^1f_2 = {}^1R_2\,{}^2f_2$

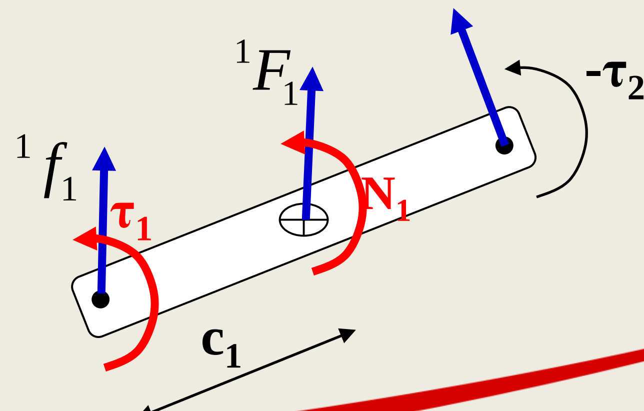
$$Adding\ \ forces: {}^1F_1 = {}^1f_1 - {}^1f_2$$

$${}^1f_1 = {}^1F_1 + {}^1f_2$$

2. $\quad Adding\ \ moments\ about\ centroid:$

$$N_1 = \tau_1 - \tau_2 - {}^1f_{1,y}c_1 - {}^1f_{2,y}\left(L_1 - c_1\right)$$

$$\tau_1 = N_1 + \tau_2 + {}^1f_{1,y}c_1 + {}^1f_{2,y}\left(L_1 - c_1\right)$$



Computational Considerations

- RNE calculations

  newton Euler

  increases linearly

  150n-48 multiplications

  131n-48 additions

- Lagrangian

  not linear.

  $32n^4+86n^3+171n^2+53n-128$ multiplications

  $25n^4+66n^3+129n^2+42n-96$ additions

- For n =6 (typical)

  852 vs 66394 multiplications

  738 vs 51456 additions

Both useful for different things.

Newton-Euler

CLOSED FORM

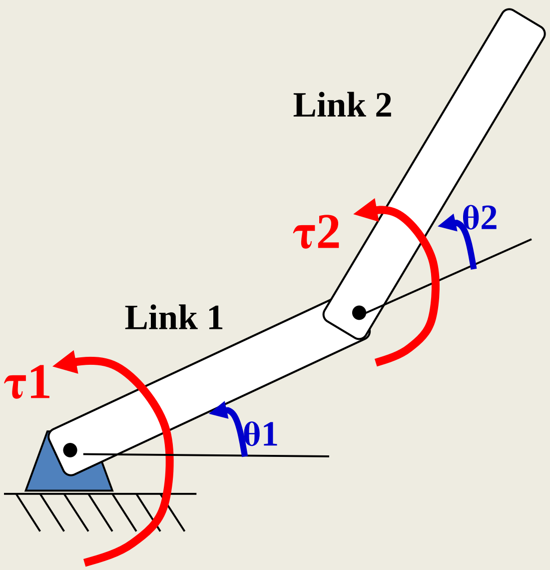
Comments on Newton Euler

- Recursive NE
  - substituting numeric values at each step
  - computational complexity of each step remains constant
  - grows in a linear fashion with the number of joints

- Closed form of NE
  - substituting expressions in a recursive way
  - closed-form dynamic model is obtained, which is identical to the one obtained using Euler-Lagrange (or any other) method

Closed Form Newton – Euler

- Explicit input-output relations
- Dynamic equations in terms of joints displacement and joints torques



NE eq. for Link 1:

$$ {}^{0}f_1 - {}^{1}f_2 + m_1 g - m_1 a_{C1} = 0 \quad (1) $$

$$ {}^{0}\tau_1 = {}^{1}\tau_2 + C_1 \times {}^{0}f_1 + (L_1 - C_1) \times {}^{1}f_2 - I_1 \dot{\omega}_1 \quad (2) $$

NE eq. for Link 2:

$$ {}^{1}f_2 + m_2 g - m_2 a_{C_2} = 0 \quad (3) $$

$$ {}^{1}\tau_2 = C_2 \times {}^{1}f_2 - I_2 \dot{\omega}_2 \quad (4) $$

Closed Form Newton – Euler (2)

- Eliminate coupling forces and separate them from the joint torques
- For the planar manipulator joint torques are equal to the coupling moments $\tau_2 = {}^1\tau_2$ $\tau_1 = {}^0\tau_1$

Replacing (3) in (4):

$$\tau_2 - C_2 \times \mathrm{m}_2 a_{C2} - C_2 \times \mathrm{m}_2 g + I_2 \dot{\omega}_2 = 0$$

and (1) in (2):

$$\tau_1 - \tau_2 - C_1 \times \mathrm{m}_1 a_{C1} - (L_1 - C_1) \times \mathrm{m}_2 a_{C2} + C_1 \times \mathrm{m}_1 g + (L_1 - C_1) \times \mathrm{m}_2 g - \mathrm{I}_1 \dot{\omega}_1 = 0$$

Closed Form Newton – Euler (3)

- Replacing with the different parameters:

$$\omega_1 = \dot{\theta}_1 \qquad \omega_2 = \dot{\theta}_1 + \dot{\theta}_2 \qquad {}^1\dot{\mathbf{v}}_1 = {}^1\mathbf{R}_0\, {}^0\dot{\mathbf{v}}_0 \qquad {}^0\dot{\mathbf{v}}_0 = \begin{bmatrix} 0 \\ \mathbf{g} \end{bmatrix}$$

$$\dot{\omega}_1 = \ddot{\theta}_1 \qquad \dot{\omega}_2 = \ddot{\theta}_1 + \ddot{\theta}_2 \qquad \mathbf{a}_{C1} = {}^1\dot{\mathbf{v}}_1 + \mathbf{C}_1 \begin{bmatrix} -\omega_1^2 \\ \dot{\omega}_1 \end{bmatrix} \qquad \mathbf{a}_{C2} = {}^2\dot{\mathbf{v}}_2 + \mathbf{C}_2 \begin{bmatrix} -\omega_2^2 \\ \dot{\omega}_2 \end{bmatrix}$$

$$\tau_2 - C_2 \times \mathrm{m}_2 a_{C2} - C_2 \times \mathrm{m}_2 g + I_2 \dot{\omega}_2 = 0$$

$$\tau_1 - \tau_2 - C_1 \times \mathrm{m}_1 a_{C1} - (L_1 - C_1) \times \mathrm{m}_2 a_{C2} + C_1 \times \mathrm{m}_1 g + (L_1 - C_1) \times \mathrm{m}_2 g - \mathrm{I}_1 \dot{\omega}_1 = 0$$

Closed Form Newton – Euler (3)

- Replacing with the different parameters:

$$\omega_1 = \dot{\theta}_1 \qquad \omega_2 = \dot{\theta}_1 + \dot{\theta}_2 \qquad {}^{1}\dot{\mathbf{v}}_1 = {}^{1}\mathbf{R}_0\,{}^{0}\dot{\mathbf{v}}_0 \qquad {}^{0}\dot{\mathbf{v}}_0 = \begin{bmatrix} 0 \\ \mathbf{g} \end{bmatrix}$$

$$\dot{\omega}_1 = \ddot{\theta}_1 \qquad \dot{\omega}_2 = \ddot{\theta}_1 + \ddot{\theta}_2 \qquad \mathbf{a}_{\mathbf{C}1} = {}^{1}\dot{\mathbf{v}}_1 + \mathbf{C}_1 \begin{bmatrix} -\omega_1^2 \\ \dot{\omega}_1 \end{bmatrix} \qquad \mathbf{a}_{\mathbf{C}2} = {}^{2}\dot{\mathbf{v}}_2 + \mathbf{C}_2 \begin{bmatrix} -\omega_2^2 \\ \dot{\omega}_2 \end{bmatrix}$$

- We get the Closed Form Equations:

$$\tau_2 = \mathrm{H}_{22}\,\ddot{\vartheta}_2 + \mathrm{H}_{12}\,\ddot{\vartheta}_1 + \mathrm{h}\,\dot{\vartheta}_1^2 + \mathrm{G}_2$$

$$\tau_1 = \mathrm{H}_{11}\,\ddot{\vartheta}_1 + \mathrm{H}_{12}\,\ddot{\vartheta}_2 - \mathrm{h}\,\dot{\vartheta}_2^2 - 2\mathrm{h}\,\dot{\vartheta}_1\dot{\vartheta}_2 + \mathrm{G}_1$$

Closed Form Newton – Euler (4)

$$t_2 = \mathrm{H}_{22}\ddot{J}_2 + \mathrm{H}_{12}\ddot{J}_1 + \mathrm{h}\dot{J}_1^2 + \mathrm{G}_2$$

$$\tau_1 = \mathrm{H}_{11}\ddot{\vartheta}_1 + \mathrm{H}_{12}\ddot{\vartheta}_2 - \mathrm{h}\dot{\vartheta}_2^2 - 2\mathrm{h}\dot{\vartheta}_1\dot{\vartheta}_2 + \mathrm{G}_1$$

where

$$\mathrm{H}_{11} = m_1c_1^2 + I_1 + m_2(l_1^2 + c_2^2 + 2l_1c_2\cos J_2) + I_2$$

$$H_{22} = m_2c_2^2 + I_2$$

$$H_{12} = m_2(l_2^2 + l_1c_2\cos J_2) + I_2$$

$$h = m_2l_1c_2\sin J_2$$

$$G_1 = m_1c_1g\cos J_1 + m_2g[c_2\cos(J_1 + J_2) + l_1\cos J_1]$$

$$G_2 = m_2c_2g\cos(J_1 + J_2)$$

Physical meaning of dynamic equations

$$\tau_1 = H_{11}\ddot{\vartheta}_1 + H_{12}\ddot{\vartheta}_2 - h\dot{\vartheta}_2^2 - 2h\dot{\vartheta}_1\dot{\vartheta}_2 + G_1$$

$$t_2 = H_{22}\ddot{J}_2 + \cancel{H_{12}}\ddot{J}_1 + h\dot{J}_1^2 + G_2$$

H_{21}

- **G$_i$** – effect of gravity, max. when links are fully extended along the x-axis
- **H$_{ii}$** – accounts for the inertia of both links as seen from the joint i.
- **H$_{ij}$** – accounts for the inertial effects of the j link's motion upon the i link
- **h term**– centrifugal force acting on the one joint due to rotation of the other joint
- **2h term** – Coriolis effect due to relative motion of the link 2 centroid that is also rotating around joint 1.

  Since the Coriolis force acts in parallel to the link 2 it does not create a moment about the second joint.

Conclusions on Dynamics

Dynamics is about the state of the manipulator

Two main approaches Lagrange + Newton Euler

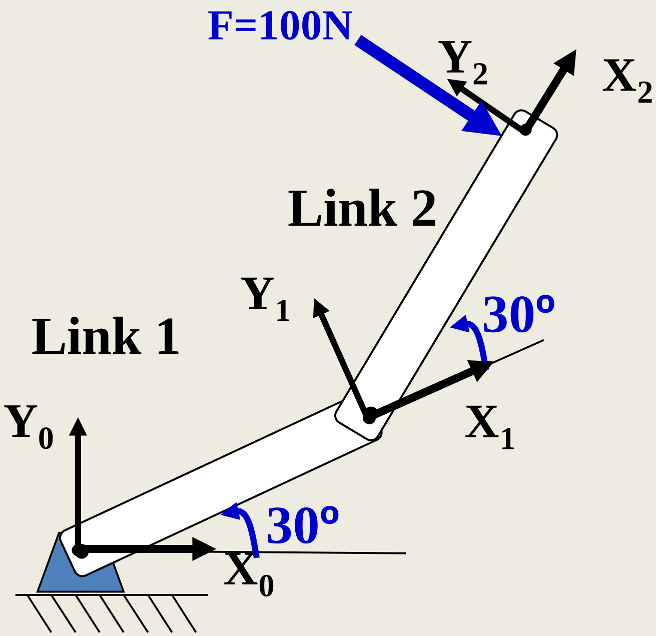
Recursive NE

- substituting numeric values at each step
- computational complexity of each step remains constant
- grows in a linear fashion with the number of joints

NUMERICAL EXAMPLE

Inverse dynamics

Numerical Example: 2 link mechanism with external force and gravity



Masses: m_1=20kg

m_2=10kg

Lengths: L_1=0.6m

L_2=0.4m

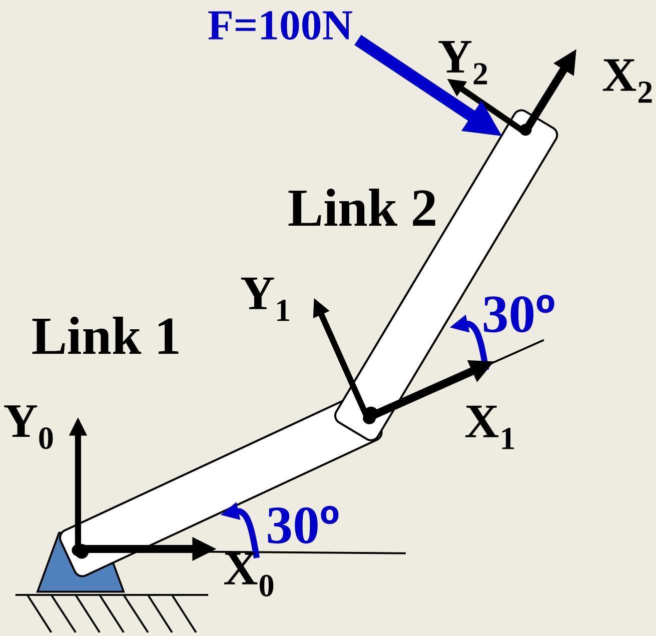
Centroid locations: c_1=0.3m

c_2=0.2m

Calculate the joint torques τ_1 and τ_2

Inverse dynamics

Numerical Example: 2 link mechanism with external force and gravity



$Outward\ Link1(i=1):$

1. $\omega_1 = 0$
2. $\dot{\omega}_1 = 0$
3. $^0\dot{V}_0 = \begin{bmatrix} 0 \\ 9.81 \end{bmatrix} m/s^2$

$$Rotation\ about\ z\ by\ -\theta : {}^1R_0 = \begin{bmatrix} C\theta & S\theta \\ -S\theta & C\theta \end{bmatrix} = \begin{bmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{bmatrix}$$

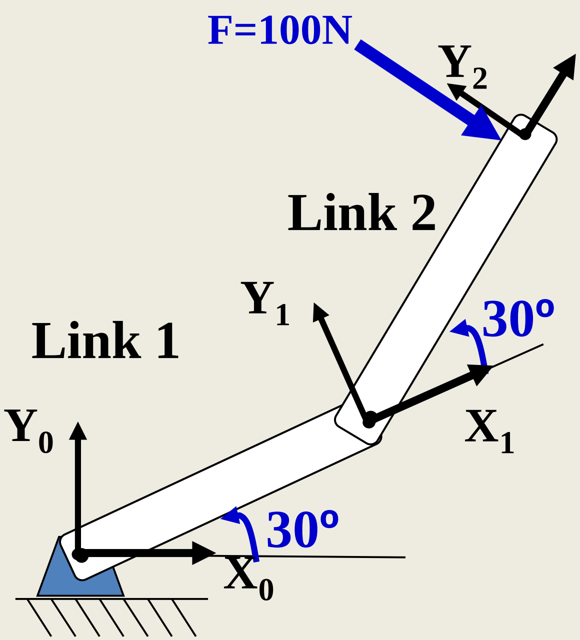
$$^1\dot{V}_0 = {}^1R_0\,{}^0\dot{V}_0 = \begin{bmatrix} 4.91 \\ 8.50 \end{bmatrix} m/s^2$$

$$^1\dot{V}_1 = {}^1\dot{V}_0 + \begin{bmatrix} -c_1\omega_1^2 \\ c_1\dot{\omega}_1 \end{bmatrix} = \begin{bmatrix} 4.91 \\ 8.50 \end{bmatrix} m/s^2$$

4. $^1a_1 = \begin{bmatrix} 4.91 \\ 8.50 \end{bmatrix} m/s^2$
5. $^1F_1 = m_1\,{}^1a_1 = 20 \times \begin{bmatrix} 4.91 \\ 8.50 \end{bmatrix} = \begin{bmatrix} 98.2 \\ 170 \end{bmatrix} N$
6. $N_1 = I_1\dot{\omega}_1 = 0$

Inverse dynamics

Numerical Example: 2 link mechanism with external force and gravity



Outward Link $2(i=2)$:

1. $\omega_2 = 0$

2. $\dot{\omega}_2 = 0$

3. $${}^2\dot{V}_1 = {}^2R_1\,{}^1\dot{V}_1 = \begin{bmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{bmatrix}\begin{bmatrix} 4.91 \\ 8.50 \end{bmatrix} = \begin{bmatrix} 8.50 \\ 4.91 \end{bmatrix} m/s^2$$

$${}^2\dot{V}_2 = {}^2\dot{V}_1 + \begin{bmatrix} -c_2\omega_2^2 \\ c_2\dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 8.50 \\ 4.91 \end{bmatrix} m/s^2$$

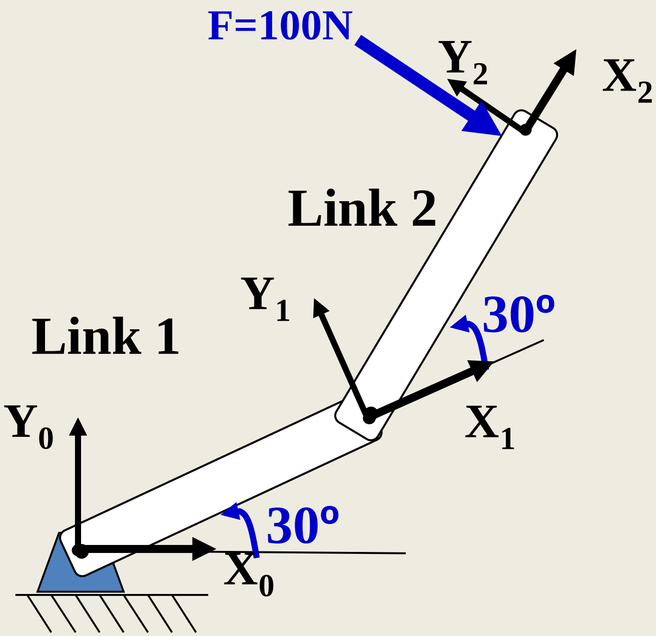
4. $${}^2a_2 = \begin{bmatrix} 8.50 \\ 4.91 \end{bmatrix} m/s^2$$

5. $${}^2F_2 = m_2\,{}^2a_2 = 10 \times \begin{bmatrix} 8.50 \\ 4.91 \end{bmatrix} = \begin{bmatrix} 85.0 \\ 49.1 \end{bmatrix} N$$

6. $N_2 = I_2\dot{\omega}_2 = 0$

Inverse dynamics

Numerical Example: 2 link mechanism with external force and gravity



$$Inward,\ Link\,2\ (i=2):$$

1. ${}^{2}f_3$ *is force being exerted on next link*

$$^{2}f_3 = \begin{bmatrix} 0 \\ 100 \end{bmatrix} \leftarrow Note\ sign$$

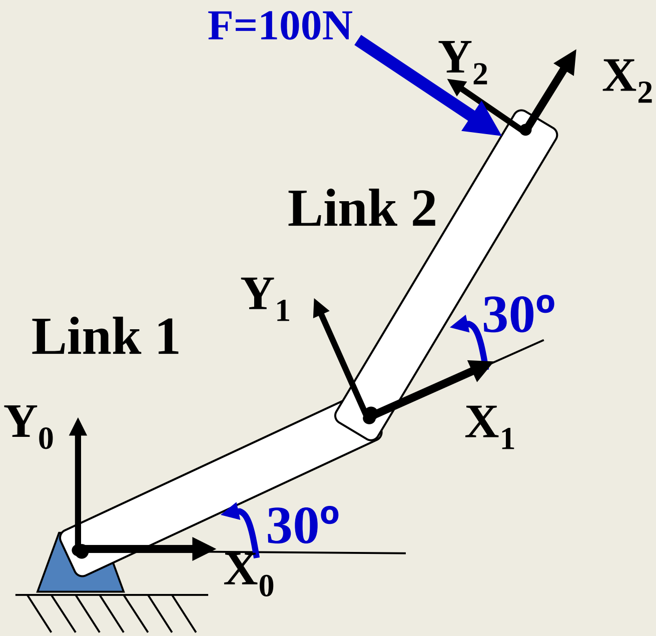
$$^{2}f_2 = \begin{bmatrix} 85.0 \\ 49.1 \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} = \begin{bmatrix} 85.0 \\ 149.1 \end{bmatrix} N$$

2. $\tau_2 = {}^{2}f_{2,y} c_2 + {}^{2}f_{3,y}\left(L_2 - c_2\right)$

$$= 149.1 \times 0.2 + 100 \times 0.2 = 49.8\,Nm$$

Inverse dynamics

Numerical Example: 2 link mechanism with external force and gravity



$$Inward\ \ Link\ 1\left(i=1\right):$$

$$1.\ {}^{1}f_{2}={}^{1}R_{2}\,{}^{2}f_{2}=\begin{bmatrix}0.866 & -0.5\\ 0.5 & 0.866\end{bmatrix}\begin{bmatrix}85.0\\ 149.1\end{bmatrix}=\begin{bmatrix}1.0\\ 172\end{bmatrix}N$$

$${}^{1}f_{1}={}^{1}F_{1}+{}^{1}f_{2}=\begin{bmatrix}98.2\\ 170\end{bmatrix}+\begin{bmatrix}1.0\\ 172\end{bmatrix}=\begin{bmatrix}99.2\\ 342\end{bmatrix}N$$

$$2.\ \tau_{1}=\tau_{2}+{}^{1}f_{1,y}c_{1}+{}^{1}f_{2,y}\left(L_{1}-c_{1}\right)$$

$$=49.8+342\times 0.3+172\times 0.3=204Nm$$