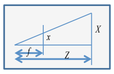
Epipolar Geometry - Maths

Rigid transformation between cameras:

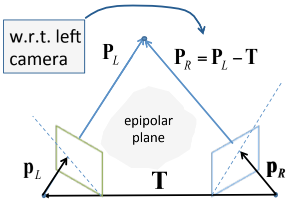
$$\mathbf{P}_L = R^T \mathbf{P}_R + \mathbf{T} \implies \mathbf{P}_R = R(\mathbf{P}_L - \mathbf{T})$$

Perspective projection:



$$\mathbf{P}_L = \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} \qquad \mathbf{p}_L = \begin{bmatrix} x_L \\ y_L \\ f \end{bmatrix} = \frac{f\mathbf{P}_L}{Z_L} \qquad \mathbf{p}_R = \begin{bmatrix} x_R \\ y_R \\ f \end{bmatrix} = \frac{f\mathbf{P}_R}{Z_R}$$

Epipolar Geometry - Maths



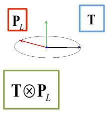
Vectors $\mathbf{P}_L$, $\mathbf{T}$ and $\mathbf{P}_L - \mathbf{T}$ all lie in epipolar plane

in plane

perpendicular to plane

$$\underbrace{(\mathbf{P}_L - \mathbf{T})^T}\underbrace{(\mathbf{T} \otimes \mathbf{P}_L)} = 0$$

$\otimes$ cross product :



Inner product = 0 for **perpendicular** vectors

Epipolar Geometry - Maths

$$(\mathbf{P}_L - \mathbf{T})^T (\mathbf{T} \otimes \mathbf{P}_L) = 0$$

$$(\mathbf{T} \otimes \mathbf{P}_L) = S\,\mathbf{P}_L$$

$$S = \begin{bmatrix} 0 & -T_Z & T_Y \\ T_Z & 0 & -T_X \\ -T_Y & T_X & 0 \end{bmatrix}$$

w.r.t. right camera

$$\mathbf{P}_R = R(\mathbf{P}_L - \mathbf{T})$$

$R^T = R^{-1}$ Rotation matrix

$$\Downarrow$$

$$R^T \mathbf{P}_R = (\mathbf{P}_L - \mathbf{T})$$

$$\mathbf{P}_R^T R = (\mathbf{P}_L - \mathbf{T})^T$$

$$\boxed{\mathbf{P}_R^T\, R\, S\, \mathbf{P}_L = 0}$$

The Essential Matrix

$$\mathbf{P}_R^T \underbrace{R\, S}\, \mathbf{P}_L = 0 \implies \mathbf{P}_R^T\, E\, \mathbf{P}_L = 0$$

$E = RS \implies$ the essential matrix

$$\mathbf{p}_L = \frac{f\mathbf{P}_L}{Z_L} \quad \mathbf{p}_R = \frac{f\mathbf{P}_R}{Z_R} \implies \frac{\cancel{Z_R}}{\cancel{f}} \mathbf{p}_R^T\, E\, \frac{\cancel{Z_L}}{\cancel{f}} \mathbf{p}_L = 0$$

$$\implies \mathbf{p}_R^T\, E\, \mathbf{p}_L = 0$$

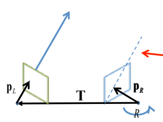
Epipolar Lines

$$\mathbf{p}_R^T\, E\, \mathbf{p}_L = 0$$

Let $\mathbf{u}_L = E\,\mathbf{p}_L = \begin{bmatrix} u_{L1} \\ u_{L2} \\ u_{L3} \end{bmatrix}$

$$\Downarrow$$

$$\mathbf{p}_R^T E\, \mathbf{p}_L = \mathbf{p}_R^T \mathbf{u}_L = \underbrace{x_R\, u_{L1} + y_R\, u_{L2} + f\, u_{L3} = 0}$$



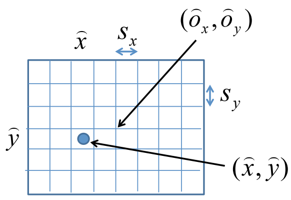
Equation of epipolar line in right image

Image Points and Pixels

- Pixel values represent light intensity within small region of image plane, e.g. of size $s_x \times s_y$
- Each pixel has:
  - row and column coordinates $(\hat{x}, \hat{y})$
- Image coordinates:

$$x = s_x(\hat{x} - \hat{o}_x)$$

$$y = s_y(\hat{y} - \hat{o}_y)$$



Fundamental Matrix

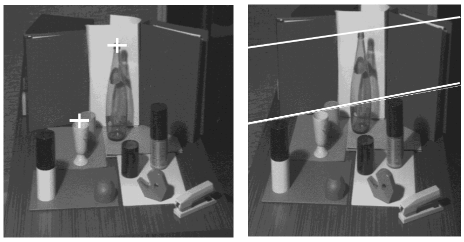
$$x = s_x(\hat{x} - \hat{o}_x) \quad y = s_y(\hat{y} - \hat{o}_y) \Rightarrow \mathbf{p}_L = \begin{bmatrix} x_L \\ y_L \\ f \end{bmatrix} = M_L \begin{bmatrix} \hat{x}_L \\ \hat{y}_L \\ f \end{bmatrix} = M_L \hat{\mathbf{p}}_L$$

$$\mathbf{p}_R^T E\, \mathbf{p}_L = 0 \Rightarrow \hat{\mathbf{p}}_R^T M_R^T E M_L\, \hat{\mathbf{p}}_L = 0$$

$$\Rightarrow \hat{\mathbf{p}}_R^T F \hat{\mathbf{p}}_L = 0 \qquad F = M_R^T E M_L$$

The fundamental matrix

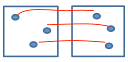
Epipolar Lines - Example



F from Correspondences

- Given set of correspondences, $i = 1 \ldots N$, we can also estimate the fundamental matrix :

$$\hat{\mathbf{p}}_R^T(i) F \hat{\mathbf{p}}_L(i) = 0 \qquad i = 1 \ldots N$$



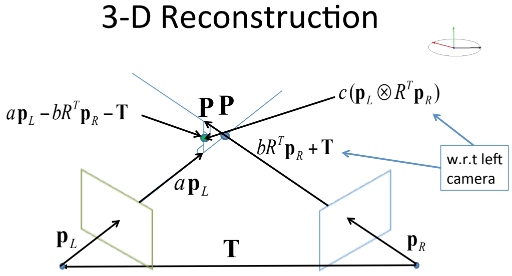
$$\Rightarrow A\,\mathbf{f} = 0$$

Nx9 matrix defined by correspondence vectors $\hat{\mathbf{p}}_R(i),\ \hat{\mathbf{p}}_L(i)$

Components of F

Solve for $\mathbf{f}$ using Singular Value Decomposition

3-D Reconstruction



find a,b,c s.t: $a\mathbf{p}_L - bR^T\mathbf{p}_R - \mathbf{T} - c(\mathbf{p}_L \otimes R^T\mathbf{p}_R) = 0$

3-D Reconstruction

find a,b,c s.t: $a\,\mathbf{p}_L - b\,R^T\mathbf{p}_R - \mathbf{T} - c\,(\mathbf{p}_L \otimes R^T\mathbf{p}_R) = 0$

Given **corresponding points**, we know : $\mathbf{p}_L, \mathbf{p}_R$

Given **calibrated views**, we know : $R, \mathbf{T}$

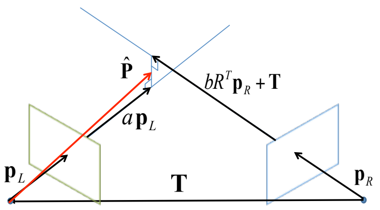
$$a \underbrace{\begin{bmatrix} \bullet \\ \mathbf{p}_L \\ \bullet \end{bmatrix}}_{3\text{x}1} - b \underbrace{\begin{bmatrix} R^T\mathbf{p}_R \end{bmatrix}}_{3\text{x}1} - c \underbrace{\begin{bmatrix} \mathbf{p}_L \otimes R^T\mathbf{p}_R \end{bmatrix}}_{3\text{x}1} = \underbrace{\begin{bmatrix} \mathbf{T} \end{bmatrix}}_{3\text{x}1}$$

3-D Reconstruction

$$a \underbrace{\begin{bmatrix} \bullet \\ \mathbf{p}_L \\ \bullet \end{bmatrix}}_{3\text{x}1} - b \underbrace{\begin{bmatrix} R^T\mathbf{p}_R \end{bmatrix}}_{3\text{x}1} - c \underbrace{\begin{bmatrix} \mathbf{p}_L \otimes R^T\mathbf{p}_R \end{bmatrix}}_{3\text{x}1} = \underbrace{\begin{bmatrix} \mathbf{T} \end{bmatrix}}_{3\text{x}1}$$

$$\Rightarrow H \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{T} \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = H^{-1}\,\mathbf{T}$$

3-D Reconstruction



$$\hat{\mathbf{P}} = (a\,\mathbf{p}_L + b\,R^T\mathbf{p}_R + \mathbf{T})/2$$