Basics of Matrices

In mathematics, a **matrix** is a rectangular array of numbers, symbols, or expressions, arranged in *rows* and *columns*.

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ is an (m, n) or $m \times n$ matrix.

The individual items in a matrix are called its *elements* or *entries*:

a_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq n$

It has m rows and n columns. An example of a matrix with 2 rows and 2 columns is:

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Transpose matrix

The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A^T, formed by turning rows into columns and vice versa:

$$(A^T)_{ij} = A_{ji}$$

Matrix Multiplication:

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}$

$AB = \begin{bmatrix} a_{11}\beta_{11} + a_{12}\beta_{21} & a_{11}\beta_{12} + a_{12}\beta_{22} \\ a_{21}\beta_{11} + a_{22}\beta_{21} & a_{21}\beta_{12} + a_{22}\beta_{22} \end{bmatrix}$

Multiplication is possible only if the number of columns of the first matrix is equal to the number of rows of the second matrix!

Identity matrix

The identity matrix I_n of size n is the $n \times m$ matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0, e.g.:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is called identity matrix because multiplication with it leaves a matrix unchanged: $A_{m \times n} I_n = A$.

Inverse of a matrix

The inverse of a matrix $A_{n \times n}$ is a matrix A^{-1} that satisfies the following property:

$$AA^{-1} = A^{-1}A = I_n$$

Determinant of a matrix

Let R be a square matrix nxn: $R = \left[\begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{bmatrix}\right]$

$$\det(R) = |R| = \sum_{i-1}^{n} (-1)^{i+j} r_{ij} M_{ij}$$

where M_{ij} is the minor of matrix R, i.e. the determinant of the matrix which we get after eliminating row I and column j from matrix R.

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Multiply the following matrices:

1. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix} =$

2. $\begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} =$

3. In the previous two exercises, is $AB = BA$?

4. $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 6 & 8 \\ 4 & 3 & 1 \end{bmatrix} =$

5. $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 9 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} =$

6*. $\begin{bmatrix} cos\alpha & sin\beta \\ cos\beta & sin\alpha \end{bmatrix} \begin{bmatrix} cos\beta & cos\alpha \\ -sin\alpha & sin\beta \end{bmatrix} =$

7. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix} =$

8. $\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \\ 1 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} =$

9. $det\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right)$ =

10. $det\left(\left[\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 9 \\ 0 & 0 & 1 \end{bmatrix}\right]\right)$

* $cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$
* $cos(\alpha - \beta) = cos\alpha cos\beta + sin\alpha sin\beta$
* $sin(\alpha \pm \beta) = sin\alpha cos\beta \pm sin\beta cos\alpha$

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Solutions

1. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 16 & 14 \end{bmatrix} = 2\begin{bmatrix} 4 & 4 \\ 8 & 7 \end{bmatrix}$

2. $\begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 17 \\ 8 & 10 \end{bmatrix}$

3. Multiplication of matrices is <u>not commutative</u>!
In general: $AB \neq BA$

4. $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 & 8 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 10 \\ 32 & 33 & 35 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 9 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 12 \\ 4 & 0 & 27 \\ 0 & 0 & 3 \end{bmatrix}$

6. If we write: $cos\alpha = c_\alpha$ and $= c_\beta$:

$$\begin{bmatrix} c_\alpha & s_\beta \\ c_\beta & s_\alpha \end{bmatrix}\begin{bmatrix} c_\beta & c_\alpha \\ -s_\alpha & s_\beta \end{bmatrix} = \begin{bmatrix} c_\alpha c_\beta - s_\beta s_\alpha & {c_\alpha}^2 + {s_\beta}^2 \\ {c_\beta}^2 - {s_\alpha}^2 & c_\alpha c_\beta + s_\beta s_\alpha \end{bmatrix} = \begin{bmatrix} c_{\alpha+\beta} & {c_\alpha}^2 + {s_\beta}^2 \\ {c_\beta}^2 - {s_\alpha}^2 & c_{\alpha-\beta} \end{bmatrix}$$

7. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 1 \\ -6 & -4 & -18 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 & 5 \\ 1 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$: multiplication is impossible.

9. $det\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) = -2$

10. $det\left(\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 9 \\ 0 & 0 & 1 \end{bmatrix}\right) = -6$

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