ROBOTIC FUNDAMENTALS (UMFM4X-15-M)

Velocity Kinematics

UWE Bristol
University of the West of England

Previously on ROBOTIC FUNDAMENTALS

Workspace of a robot – examples on how to calculate it

Inverse kinematics – redundant solutions due to the trigonometric functions

Iterative solutions slower and out of scope (but possibly the ONLY solution for some problems – Parallel Robots)

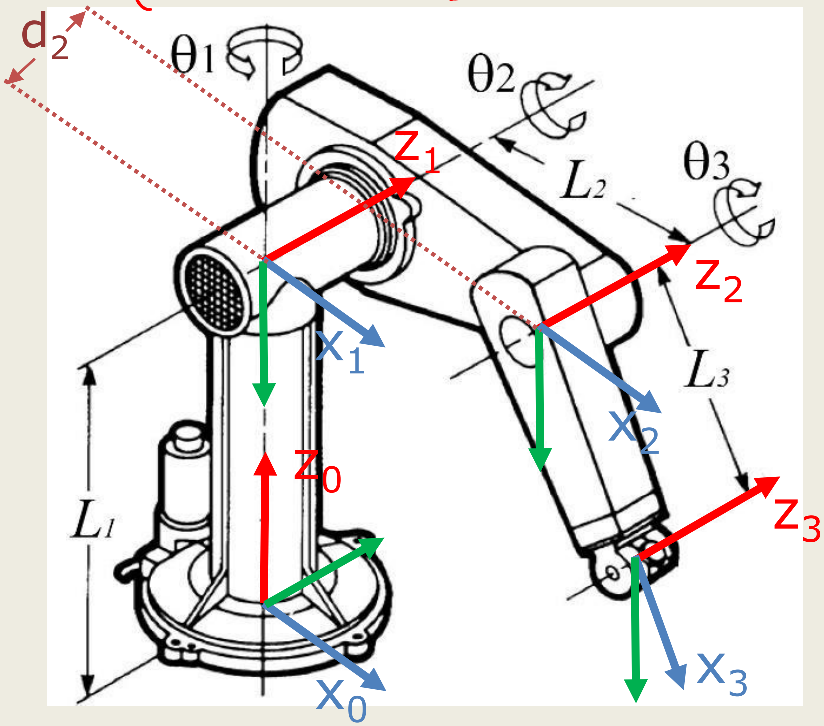
Closed form solutions (a.k.a. analytical) give information about the entire configuration of a manipulator

Test your skills on DH

Questions?

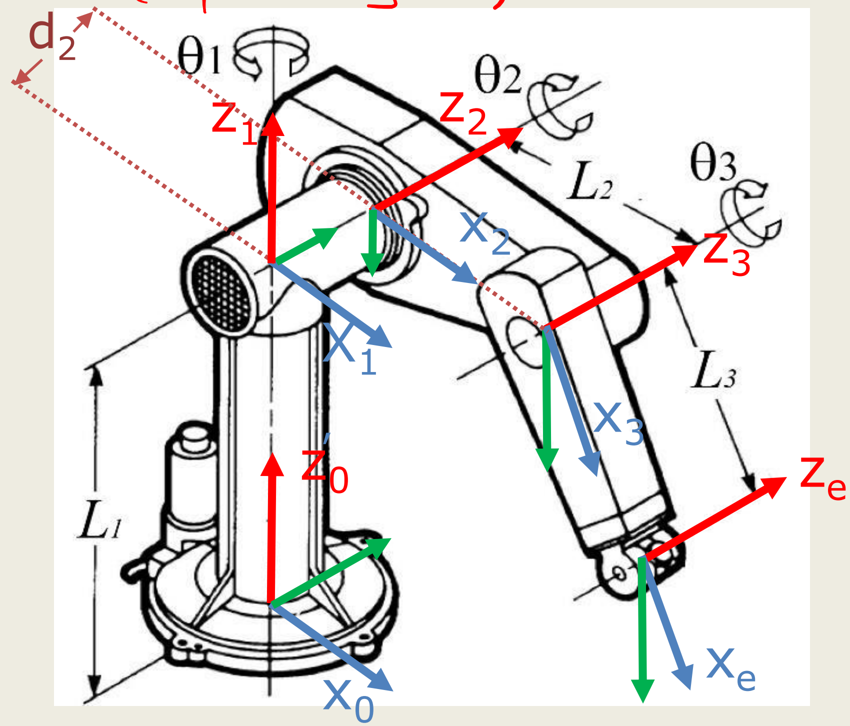
Test your skills on DH - exercises

Standard (distal) DH convention (first joint is base)
(X follows line from previous joint - eg x_2)



n	a_n	α_n	d_n	θ_n
1	0	-90º	L_1	θ_1
2	L_2	0	d_2	θ_2
3	L_3	0	0	θ_3

Modified (proximal) DH convention
(X points along link)



n	a_{n-1}	α_{n-1}	d_n	θ_n
1	0	0	L_1	θ_1
2	0	-90º	d_2	θ_2
3	L_2	0	0	θ_3
e	L_3	0	0	0

Test your skills on DH - exercises

1. When filling out the DH table, make sure all robot characteristics are included. For example, when using the proximal convention, you might have to place frame {0} and frame {1} at different origins.

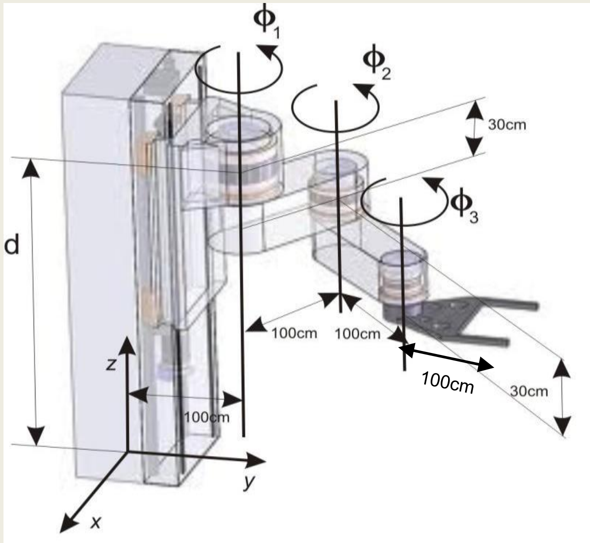
2. You should always have a frame {0}. In proximal, this is the base frame (before the first joint). In distal, this is the frame of the 1st joint.

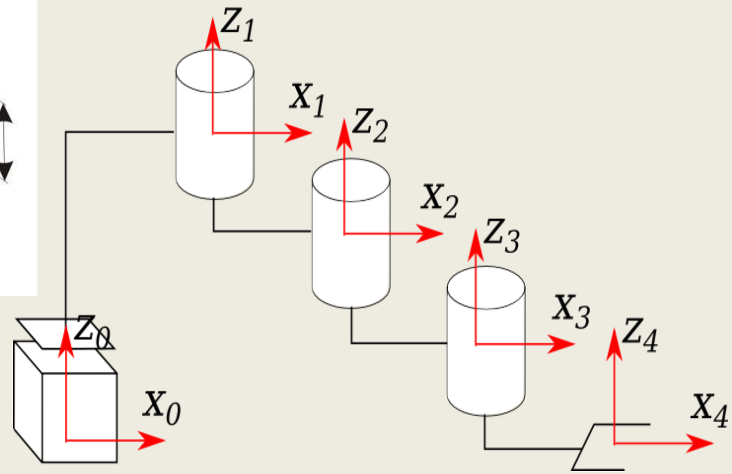
3. Do not forget indices! Do not simply write x, y, z, a, α, d, θ. Indicate what frame the belong to e.g. a_i or a_{i-1} and x_2/x_3 etc.

4. When showing the DH frames, it is advisable not to draw the robot in the 'home' position. In the 'home' position, x axes of consecutive joints align and therefore the true direction of the x axes will not be shown (e.g. in distal, the x-axis of a joint will follow the movement of the immediately previous link, while in proximal, it will follow the movement of the link following the joint.

5. Confusion about the two conventions: when using one convention, stick with it. E.g. do not place frames according to proximal and then make a distal DH table.

Test your skills on DH - exercises





Lengths are in m:

n	a_n	α_n	d_n	θ_n
1	1	0	d	0
2	1	0	-0.3	ϕ_1
3	1	0	-0.15	ϕ_2
4	1	0	-0.15	ϕ_3

Test your skills on DH - exercises

6. It is better to not start the indices of ‘θ’ from 0. Both proximal and distal DH tables start from i=1, e.g, $\theta_1 \ldots \theta_n$.

7. In some cases, the direction of ‘θ’ will be indicated in the drawing: place the z axis of the joint according to the right hand rule. This might be important, depending on application.

8. Do not forget that prismatic joints are joints! They will have a ‘d’ parameter which is variable. Generally, each row of a DH table represents one DOF and should usually contain only one variable (d or θ).

9. Pay attention to the definition of each parameter. E.g. ‘d’ is measured from x_{i-1} to x_i along the direction z_{i-1}(distal) or z_i(proximal). This means that, depending on the location of the two x axes and the direction that the z axis is pointing, it can be positive or negative

Today's Lecture

Velocities and Accelerations

The Jacobian

Differential Motion

Singular Matrix – Singularities

A glimpse in Dynamics

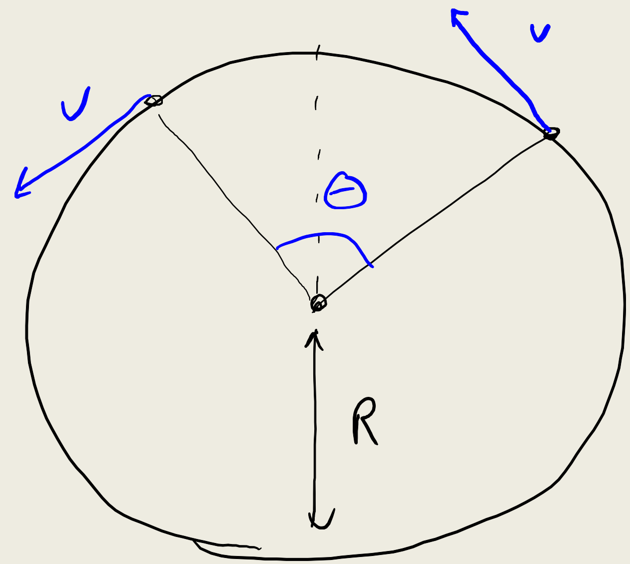
The Problem

How do we relate

end-effector linear and angular velocities

to

joint velocities?



angular displacement $= \theta$

ω = angular vel.

$$\omega = \frac{d\theta}{dt}$$

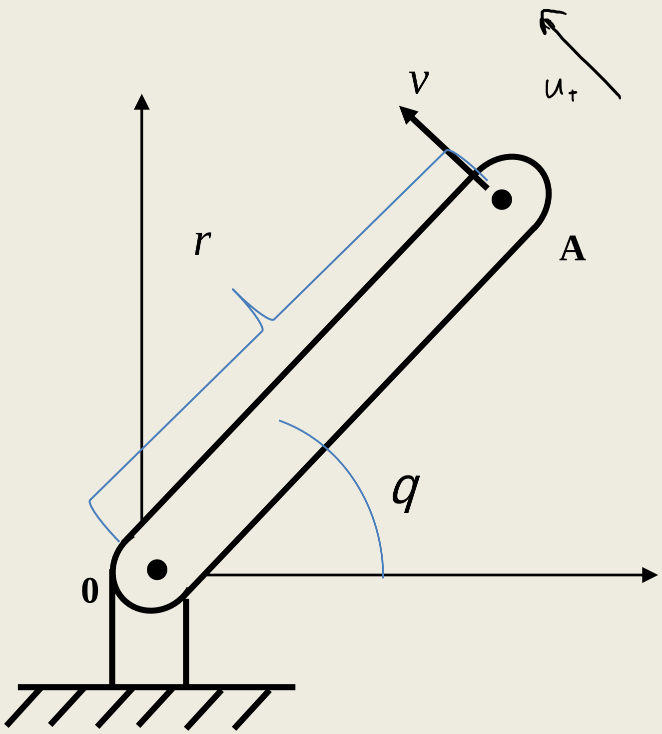
a = linear acc.

Velocity Kinematics

VELOCITIES AND ACCELERATIONS

Velocities – Linear and Angular

v = linear velocity.



$$\mathbf{v} = v \cdot \mathbf{u_t}$$

$$v = r \cdot \dot{\theta}$$

Angular velocity

$v = \omega r$

Fixed length link, pivoted at one end

Accelerations

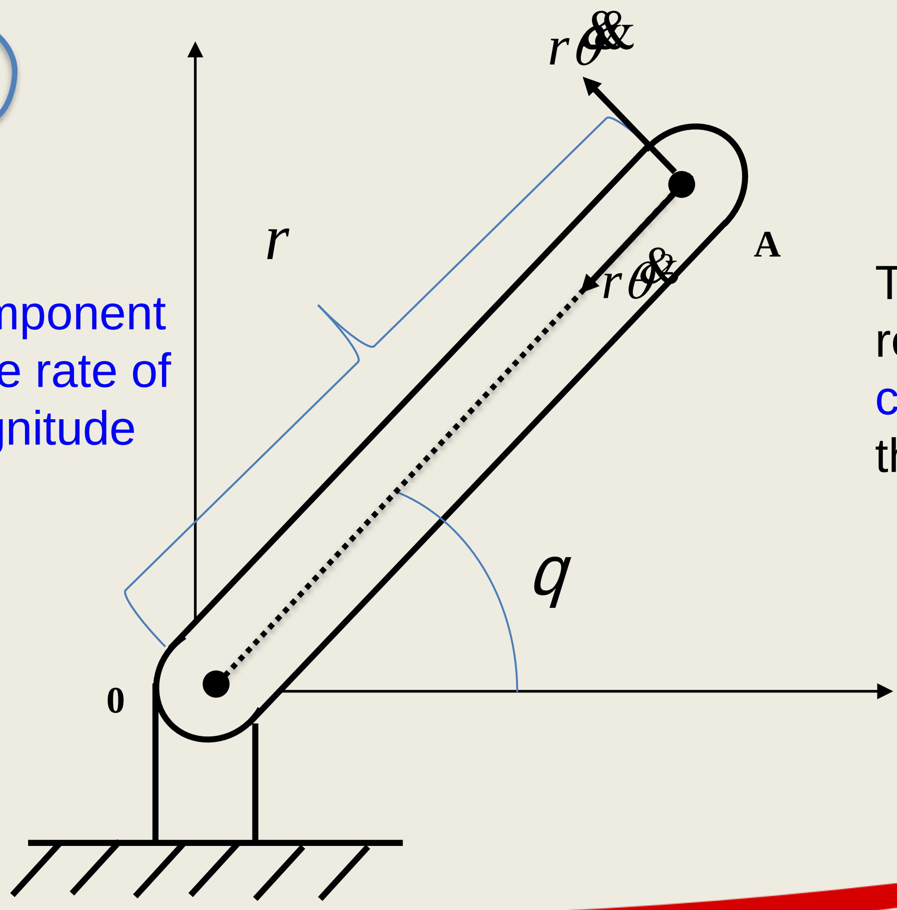
Acceleration is the time rate of change (derivative) of velocity

$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(v \cdot \mathbf{u_t}\right) = \underset{\dot{v}\cdot\underline{u}_t}{\overset{a_t\mathbf{u_t}}{\boxed{\cancel{\dot{v}\mathbf{u_t}}}}} + \underset{v\cdot\dot{\underline{u}}_t}{\overset{a_n\mathbf{u_n}}{\boxed{\cancel{v\dot{\mathbf{u}}_\mathbf{t}}}}}$$

Angular acceleration

$$a_t = r\ddot{\theta}$$

The tangential component represents the time rate of change in the magnitude of the velocity.



$$a_n = r\dot{\theta}^2$$

The normal component represents the time rate of change in the direction of the velocity.

Velocity propagation (link to link)

- Chain of links - at any instant a link has a linear and/or angular velocity component
- Propagation of velocities from the frame {O_0} to the end effector
- For most manipulators we will want to find the angular velocity of one frame due to the rotations of multiple frames. Consider two frames O_1, O_2 with rotation matrix $R_2^1(t)$

Their **angular velocities** are related as:

$$ {}^1\omega_2 = {}^1\omega_1 + \qquad {}^1_2R\ \dot{\theta}_2\ \vec{z}_2 $$

Angular velocities can be **added** once they are **projected into the same coordinate frame.**

Velocity propagation (link to link) (2)

- **Angular velocities** can be **added** once they are **projected into the same coordinate frame.**
- This can be extended to calculate the angular velocity for an *n*-link manipulator:
  - Suppose we have an *n*-link manipulator whose coordinate frames are related as follows: ${}^{0}_{n}R = {}^{0}_{1}R\,{}^{1}_{2}R \cdots {}^{n-1}_{n}R$
  - We can define the angular velocity of the tool frame (*n*) in the base frame (*0*):

$${}^{0}\omega_n = {}^{0}\omega_1 + {}^{0}_{1}R\,{}^{1}\omega_2 + {}^{0}_{2}R\,{}^{2}\omega_3 + {}^{0}_{3}R\,{}^{3}\omega_4 + \ldots + {}^{0}_{n-1}R\,{}^{n-1}\omega_n$$

Angular velocity of point attached to frame n (on link n) due to the rotation of frame {n-1} (also expressed in frame {n-1}

Linear velocities

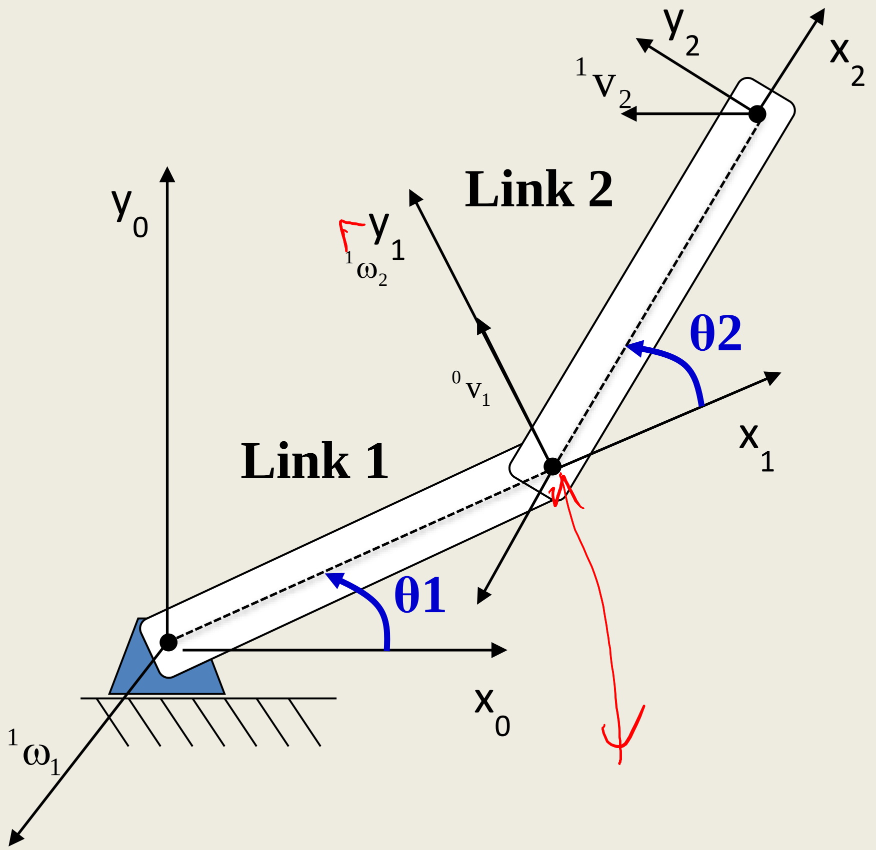
- The **linear velocity** of any point on a rigid body is the **sum** of the *linear velocity of the rigid body* and the velocity of the particle *due to rotation of the rigid body*.
- The linear velocity of the **end effector** can be due to the motion of revolute and/or prismatic joints.
- First, the position of a point *p* attached to a rigid body is:

$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG} $$

- To find the velocity, take the derivative as follows:

$$ {}^{A}\dot{p} = {}^{A}\omega_{B} \times {}^{A}_{B}R\,{}^{B}P + {}^{A}\dot{P}_{BORG} $$

Linear velocity of links



Frames are attached to each joint

Frame 1 does not translate with respect to 0 frame. Because **Link 1** rotates, *frame {1}* has a linear velocity with respect to the reference *frame {0}*

$$^{1}v_1 = {}^{1}\omega_1 \cdot L_1$$

Linear velocity of **Link 2** due to rotation of **Link 2** around reference *frame {1}*:

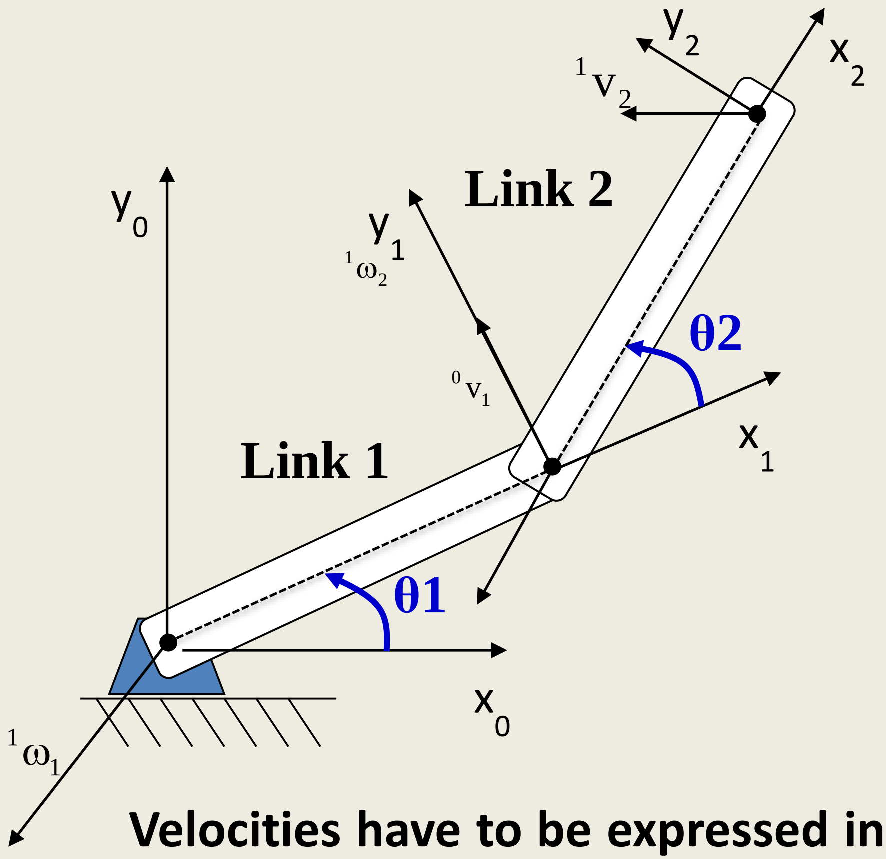
$$^{1}\omega_2 \cdot L_2$$

Angular velocity due to rotation of L2 expressed in frame {1}

$$^{1}v_2 = {}^{1}v_1 + {}^{1}\omega_2 \cdot {}^{1}L_2$$

THIS IS A SCALAR CALCULATION

Linear velocity of links



Velocities have to be expressed in relation to the same frame so we can add them up.

In general for revolute joints stands:

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1})$$

THIS IS A VECTOR CALCULATION

Acceleration – Angular

$$
{}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\dot{\omega}_{i} + {}^{i+1}_{i}R \cdot {}^{i}\omega_{i} \times \dot{\theta}_{i+1} + \ddot{\theta}_{i+1}
$$

Angular acceleration of joint *i* + Coriolis acceleration + acceleration of joint *i+1*

Acceleration – Linear

$$ {}^{i+1}a_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}a_i + {}^{i}\omega_i^2 \cdot {}^{i}P_{i+1} + {}^{i}\dot{\omega}_i \cdot {}^{i}P_{i+1}\right) $$

Linear acceleration of frame

Linear acceleration of *frame i +*
***+* normal and tangential components of Link *i*+1,**
in respect to *frame i*

$$ {}^{i+1}a_{Ci} = {}^{i}a_i + {}^{i}\omega_i^2 \cdot {}^{i}P_{Ci} + {}^{i}\dot{\omega}_i \cdot {}^{i}P_{Ci} $$

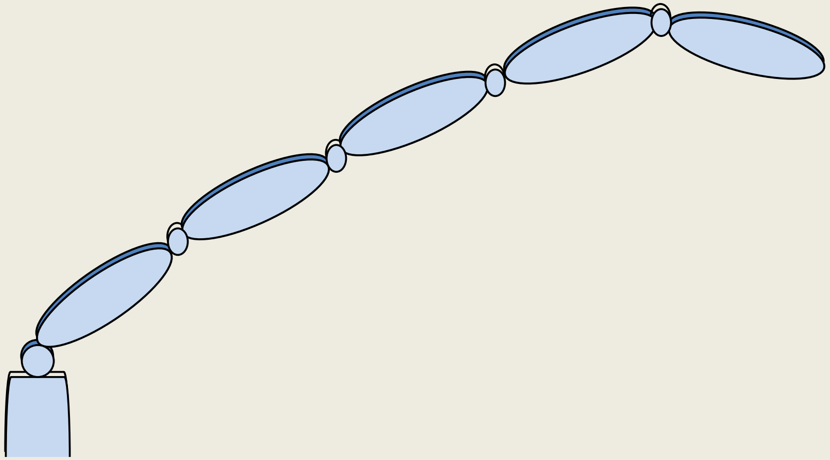
Linear acceleration of the <u>centre of mass</u>

Differential Motion and Jacobian

THE JACOBIAN

Differential motion

Forward kinematics	$q \rightarrow X$
Differential motion	$q + \delta q \rightarrow X + \delta X$
Which is a link between velocities:	$\dot{q} \rightarrow \dot{X}$



A relationship described by **the Jacobian matrix**

End effector vs joint velocities

$$\dot{X} = J(q)\dot{q}$$

End position velocity

Jacobian matrix

Joint velocity

The Jacobian

Instantaneous ***transformation*** *between* ***a vector in*** $\mathbf{R^n}$ *representing* *joint velocities* *to* ***a vector in*** $\mathbf{R^6}$ *representing the* *linear and angular velocities of the end-effector*

$$\dot{X} = J(q)\dot{q}$$

The Jacobian is mapping velocities from Joint space to Cartesian space.

Jacobian Solutions

Jacobian matrix is of **m**$\times$***n*** dimension

(**m** is the number of EE position and orientation parameters and **n** is the number of DOFs)

Represents the partial derivatives of each EE position and orientation parameter to each joint parameter

$$J_{ij} = \frac{\partial f_i}{\partial q_j}$$

Differential motion and the Jacobian (1)

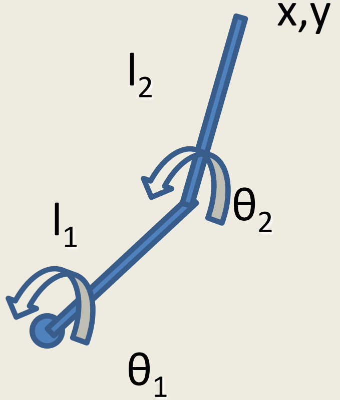
If the end-effector position and orientation are given by:

$$X = f(q)$$

then the Jacobian $J(q)$ is defined as:

$$\begin{aligned} \delta X_1 &= \frac{\partial f_1}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_1}{\partial q_n}\delta q_n \\ &\vdots \\ \delta X_m &= \frac{\partial f_m}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_m}{\partial q_n}\delta q_n \end{aligned} \Rightarrow \delta X = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} \delta q \Rightarrow \delta X = \boldsymbol{J}(\boldsymbol{q})\delta q$$

Example – 2-link Robot Jacobian



chain rule.

$$x = l_1 c_1 + l_2 c_{12}$$
$$y = l_1 s_1 + l_2 s_{12}$$

$$\frac{\delta x}{\partial \theta} = \frac{\partial}{\partial \theta}(l_1 c_1) + \frac{\partial}{\partial \theta}(l_2 c_{12})$$

$$\delta x = -(l_1 s_1 + l_2 s_{12})\delta\theta_1 - l_2 s_{12}\delta\theta_2$$
$$\delta y = (l_1 c_1 + l_2 c_{12})\delta\theta_1 + l_2 c_{12}\delta\theta_2$$

$$\delta X = \begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = \begin{bmatrix}-y & -l_2 s_{12}\\ x & l_2 c_{12}\end{bmatrix}\begin{bmatrix}\delta\theta_1\\ \delta\theta_2\end{bmatrix}$$

$$\dot{X} = \boldsymbol{J}(\boldsymbol{q})\dot{q} \Rightarrow \begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = \begin{bmatrix}\boldsymbol{-y} & \boldsymbol{-l_2 s_{12}}\\ \boldsymbol{x} & \boldsymbol{l_2 c_{12}}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

At any time instant, q has a certain value and $J(\theta)$ is a linear transformation.

<u>Jacobians are time-varying linear transformations</u>

The Inverse Jacobian

Inverse kinematics $\mathrm{X} \rightarrow q$

Differential form $\mathrm{X} + \delta X \rightarrow q + \delta q$

Which a link between velocities: $\dot{X} \rightarrow \dot{q}$

Which can be expressed by the

inverse of the Jacobian

$$\boldsymbol{J^{-1}}$$

Useful to find position of end effector from vel. of effector.

The Inverse Jacobian (2)

$$\dot{q} = J(q)^{-1}\dot{X}$$

For systems that do not **have exactly 6DOF**, we cannot directly invert the Jacobian

because J dimensions are $6 \times n$*, where* $n \neq 6$

Thus there is a solution to finding the joint velocities if X and *J* have the same rank

OR

Use the pseudo-inverse[1]

[1] R. Penrose, *A generalised inverse for Matrices*, 1954

Differential Motion and Jacobian

SINGULARITIES

Singular matrix

A matrix is singular if **its determinant is equal to zero.**

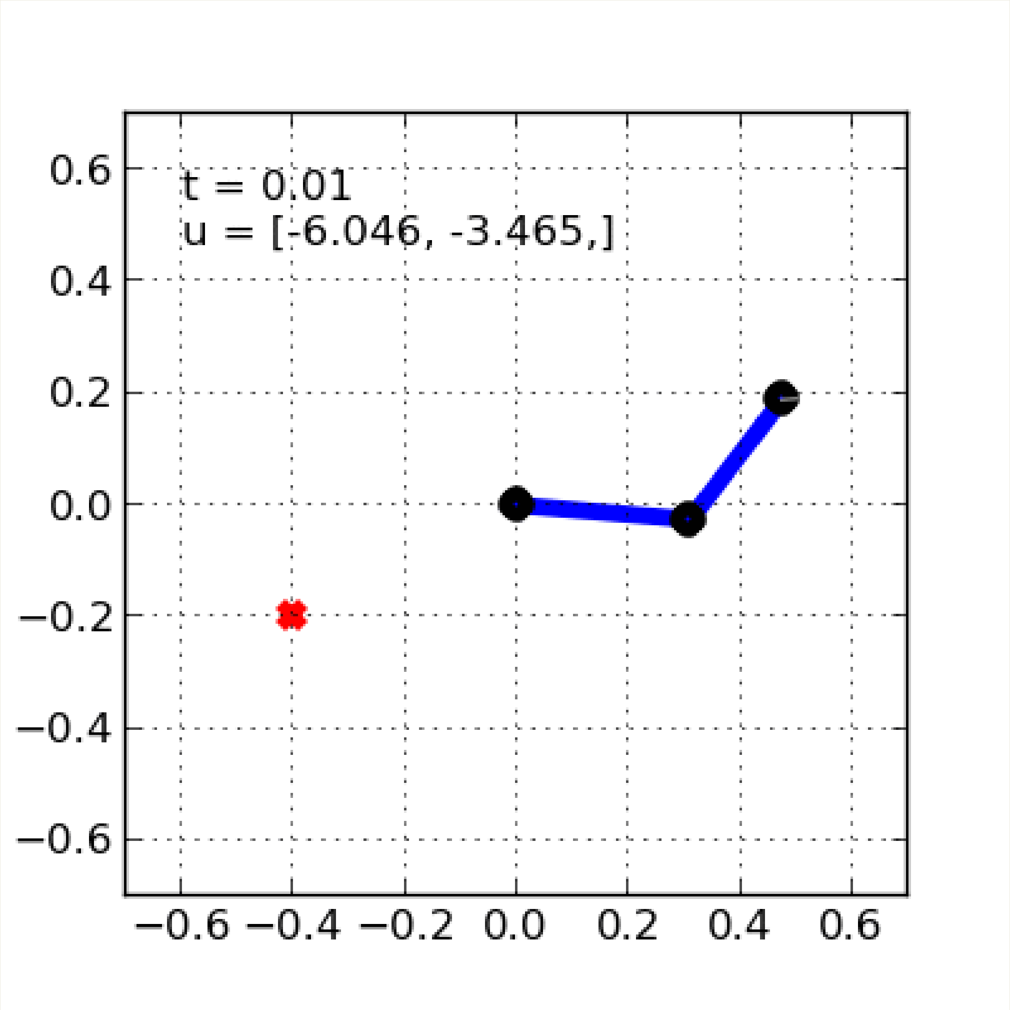
Singular matrix – Implications

A singular matrix **cannot be inverted.**

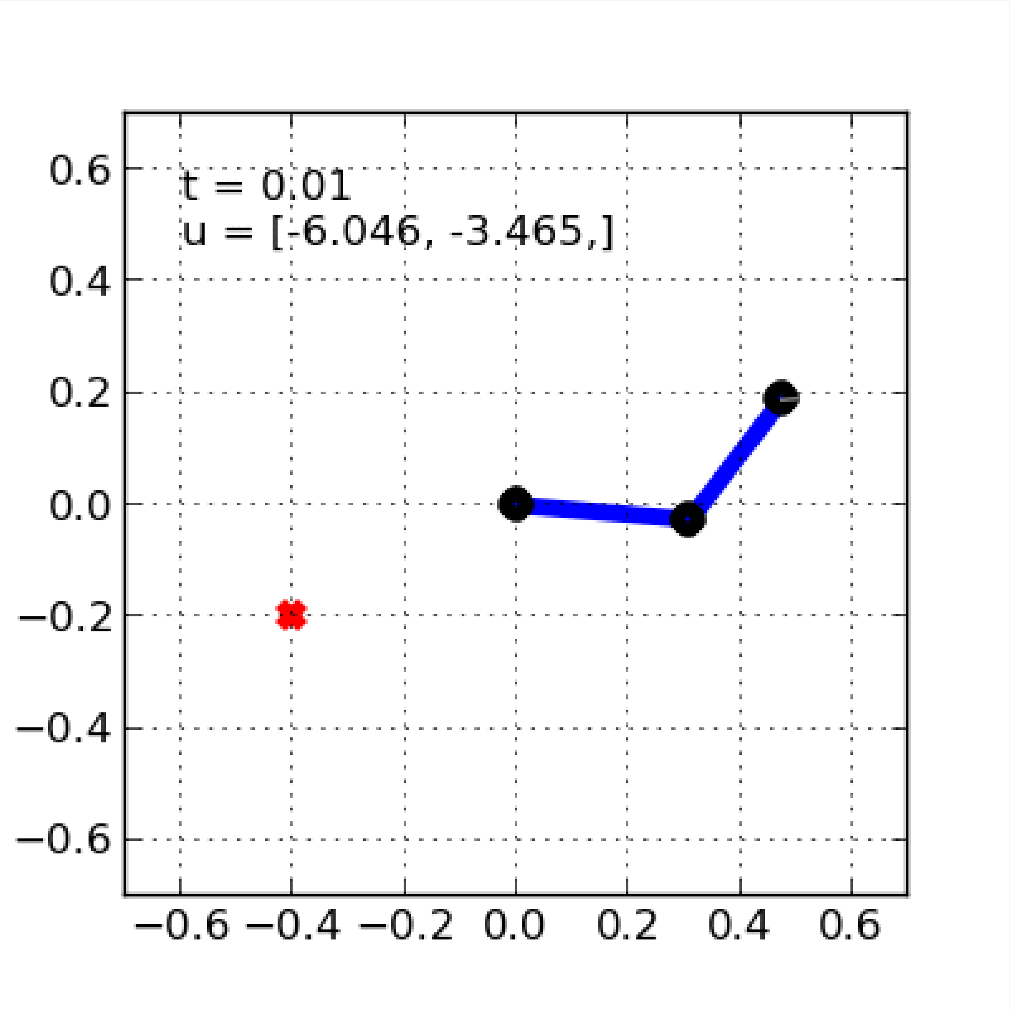
If you cannot invert a Jacobian you **cannot calculate joint** velocities.

<u>These are Singularities</u>

Singularity an example



Singularity an example - fixed



Singularities

- Singularities are points in the configurations space where **infinitesimal motion in a certain direction is not possible** and the manipulator loses one or more degrees of freedom
- When operating in a singular point **small end effector velocities** may correspond to large joint velocities
- Singularities are often found **on the extents of the workspace**
- Mathematically, singularities exist where the Jacobian inverse does not exist (**det(J)=0**).

Singularities

- **Boundary singularities** – manipulator <u>outstretched or retracted</u>. These singularities do not represent a true drawback, since they *can be avoided on condition that the manipulator is not driven to the boundaries of its reachable workspace.*

- **Internal singularities** – caused by a <u>lining up of two or more joint axes</u>. They are *more serious* and happen within the workspace under certain conditions.

Singularity analysis

The analysis is done by checking where the Jacobian is singular, i.e. check the Jacobian determinant.

$$\det(J)$$

Since J is a function of q this should give the values of q that will make the Jacobian zero.

These values of q are **Singularity points**

Singularity analysis – Example 2DOF

For a two link manipulator:

$$x = L_1 cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$y = L_1 sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

The Jacobian is:

$$J = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

Its determinant is:

$$\begin{aligned} det(J) &= (-L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2)) L_2 \cos(\theta_1 + \theta_2) + (L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)) L_2 \sin(\theta_1 + \theta_2) \\ &= -L_1 L_2 \sin\theta_1 \cos(\theta_1 + \theta_2) - L_2^2 \sin(\theta_1 + \theta_2)\cos(\theta_1 + \theta_2) + L_1 L_2 \cos\theta_1 \sin(\theta_1 + \theta_2) + L_2^2 \cos(\theta_1 + \theta_2)\sin(\theta_1 + \theta_2) \\ &= \ldots\ldots\ldots\ldots = L_1 L_2 \sin\theta_2 \end{aligned}$$

This determinant is equal to zero when θ_2=0 or 180^o. At these angles the manipulator is fully stretched or retracted.

Singularity analysis – Example 3DOF

For 3 DOF manipulator:

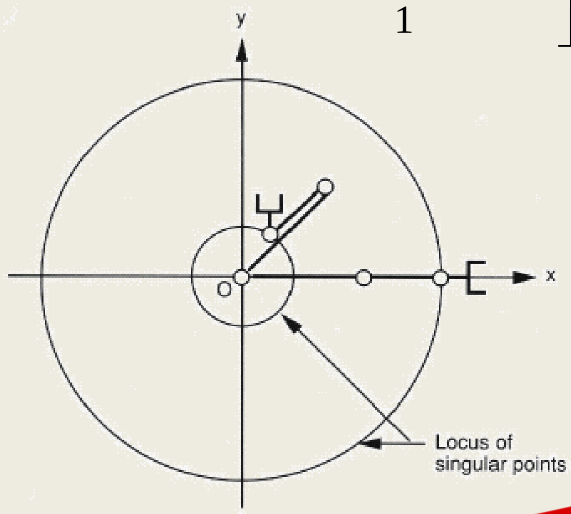
$$x = l_1 cos\theta_1 + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3)$$
$$\mathrm{y} = l_1 sin\theta_1 + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3)$$

$$\begin{bmatrix} l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) & l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) & l_3\cos(\theta_1+\theta_2+\theta_3) \\ l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) & l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) & l_3\sin(\theta_1+\theta_2+\theta_3) \\ 1 & 1 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} -(l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) & -(l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) & -l_3\sin(\theta_1+\theta_2+\theta_3) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) & l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) & l_3\cos(\theta_1+\theta_2+\theta_3) \\ 1 & 1 & 1 \end{bmatrix}$$ [2]

$$\det J = 0 \Rightarrow l_1 l_2 \sin\theta_2 = 0$$

This determinant is equal to zero when θ_2=0 or 180º. At these angles the manipulator is fully stretched or retracted. Manipulator can not move in radial direction, only tangential. In either case, it loses 1 DOF.



[2] Amiri, M., Fathy, M. and Bayat, M., 2010. Generalization of some determinantal identities for non-square matrices based on Radic's definition. *TWMS J. Pure Appl. Math*, *1*(2), pp.163-175.

Jacobian – Force/torque relationships

Similar to the relationship between the joint velocities and the end effector velocities, we can express the relationship between the **joint torques** and the **forces and moments at the end effector**

Important for dynamics and force control.

Jacobian – Force/torque relationships

Let the vector of forces and moments required at the end effector at a joint configuration q be represented as:

$$F_{ee} = \left[F_x \; F_y \; F_z \; n_x \; n_y \; n_z\right]$$

and the desired joint torques (for revolute or force for prismatic) are given by $\boldsymbol{\tau}$**.**

These is a relationship that relates $F \; \rightarrow \boldsymbol{\tau}$

Which can be derived using the transpose of the Jacobian at q:

$$\boldsymbol{\tau = J^T(q)F}$$

Be careful of the Jacobian Dimensions…

Conclusion

- Velocities, accelerations and forces propagate in serial robots
- Using the Jacobian to represent this
- Investigate Manipulator Singularities via the Jacobian Matrix
- Simple ”Inverse Dynamics” using the Jacobian.