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The coefficients of the characteristic polynomial in terms of the eigenvalues and the elements of an $n \times n$ matrix

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Abstract

The coefficients of the characteristic polynomial of an $n \times n$ matrix are derived in terms of the eigenvalues and in terms of the elements of the matrix. The connection between the two expressions allows the sum of the products of all sets of k eigenvalues to be calculated using cofactors of the matrix.

Keywords: Derivative of a determinant; Eigenvalue; Jacobian

1. Motivations

In order to derive general stability conditions for a first order three-dimensional discrete dynamic the coefficients of the characteristic polynomial of the Jacobian evaluated at equilibrium need to be expressed in terms of the three eigenvalues [1]. In order to extend this stability analysis technique to a first order n-dimensional discrete dynamic the coefficients of the characteristic polynomial of an $n \times n$ Jacobian evaluated at the equilibrium must first be expressed in terms of the eigenvalues and in terms of the elements of the Jacobian. Thus, given an $n \times n$ matrix $A = [a_{ij}]$, $a_{ij} \in R$, we wish to determine the relationship between the eigenvalues of A and the coefficients of the characteristic polynomial $C(x)$

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and the relationship between the coefficients of the characteristic polynomial and the real elements of the matrix.

2. The coefficients of the characteristic polynomial in terms of the eigenvalues of A

We can express the characteristic polynomial as

$$C(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_k)\cdots(x - \lambda_n)$$

where λ_i are the eigenvalues of the matrix A. Expanding the above form clearly shows that the coefficient for x^{n-1} is the negative of the trace of A and the magnitude of the constant term is the determinant of A:

$$C(x) = x^n - \operatorname{tr}(A)x^{n-1} + \cdots + (-1)^n \det(A).$$

Note that the signs of the coefficients alternate. We have, by convention, made the sign of the 1 coefficient of x^n positive:

$$C(x) = x^n - \operatorname{tr}(A)x^{n-1} + \cdots + (-1)^k c_k x^{n-k} + \cdots + (-1)^n \det(A).$$

Next we need to show by induction that c_k is the sum of all products of k eigenvalues. Thus $c_k = \sum_{\text{all sets of } k\lambda's} \lambda_i\lambda_j..\lambda_l$ and $c_1 = \textit{trace}$, $c_n = \det(A)$. Consider the $n = 3$ case:

$$\begin{aligned} C(x) &= (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) \\ &= x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)x - \lambda_1\lambda_2\lambda_3 \\ &= x^3 - c_1x^2 + c_2x - c_3 \\ &= x^3 - \operatorname{tr}(A)x^2 + c_2x - \det(A). \end{aligned}$$

Now assume that

$$\begin{aligned} C(x) &= (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_k)\cdots(x - \lambda_n) \\ C(x) &= x^n - c_1x^{n-1} + c_2x^{n-2} - \cdots + (-1)^k c_k x^{n-k} + \cdots (-1)^n c_n. \end{aligned}$$

Next we will consider a new expanded $\overline{C}(x) = (x - \overline{\lambda})C(x)$:

$$\begin{aligned} \overline{C}(x) &= (x - \overline{\lambda})[x^n - c_1x^{n-1} + c_2x^{n-2} - \cdots + (-1)^k c_k x^{n-k} + \cdots (-1)^n c_n] \\ &= x^{n+1} - c_1x^n + c_2x^{n-1} - \cdots + (-1)^k c_k x^{n-k+1} + \cdots (-1)^n c_n x - \overline{\lambda}x^n + \overline{\lambda}c_1x^{n-1} \\ &\quad - \overline{\lambda}c_2x^{n-2} - \cdots - (-1)^k\overline{\lambda}c_k x^{n-k} - \cdots (-1)^{n+1}\overline{\lambda}c_n. \end{aligned}$$

Now we collect on powers of x:

$$\overline{C}(x) = x^{n+1} - [\overline{\lambda} + c_1]x^n + [\overline{\lambda}c_1 + c_2]x^{n-1} - \cdots + (-1)^k[\overline{\lambda}c_{k-1} + c_k]x^{n-k+1} \cdots (-1)^{n+1}\overline{\lambda}c_k$$

and therefore, by induction, the coefficient of x^{n-k} can be seen to be the sum of all products of k eigenvalues. The sign of that coefficient is $(-1)^{n-k}$.

Thus the characteristic polynomial can be expressed as

$$\begin{aligned} C(x) = x^n &- \left[\sum_{\textit{sets} \text{ of } 1} \lambda\right] x^{n-1} + \left[\sum_{\textit{sets} \text{ of } 2} \lambda\lambda\right] x^{n-2} - \cdots + (-1)^k \left[\sum_{\textit{sets} \text{ of } k} \lambda\lambda..\lambda\right] x^{n-k} \\ &+ \cdots (-1)^n \left[\sum_{\textit{sets} \text{ of } n} \lambda\lambda\lambda..\lambda\right]. \end{aligned}$$

An alternative proof can be found in [2]. Next the characteristic polynomial will be expressed using the elements of the matrix A, $C(x) = (-1)^n \det[A - xI]$, with the sign factor, $(-1)^n$, used so that the coefficient of x^n is $+1$. The coefficients will now be generated by differentiating $C(x)$ as a determinant. The formula for the kth derivative of a general determinant will now be shown.

3. The *k*th derivative of a determinant

Theorem 1. *The kth derivative of a determinant of the $n \times n$ matrix $M = [m_{ij}]$ is given by the formula*

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\begin{vmatrix} m_{11}(t) & m_{12}(t) & \cdots & m_{1n}(t) \\ m_{21}(t) & m_{22}(t) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ m_{n1}(t) & m_{n2}(t) & \cdots & m_{nn}(t) \end{vmatrix}$$

$$= \sum_{\substack{k_i = 0 \\ \forall i,\, i = 1, 2, \ldots, p \\ k_1 + k_2 + \cdots + k_p = k}}^{k} \frac{k!}{k_1! k_2! \cdots k_p!} \begin{vmatrix} \frac{\mathrm{d}^{k_1} m_{11}(t)}{\mathrm{d}t^{k_1}} & \frac{\mathrm{d}^{k_1} m_{12}(t)}{\mathrm{d}t^{k_1}} & \cdots & \frac{\mathrm{d}^{k_1} m_{1n}(t)}{\mathrm{d}t^{k_1}} \\ \frac{\mathrm{d}^{k_2} m_{21}(t)}{\mathrm{d}t^{k_2}} & \frac{\mathrm{d}^{k_2} m_{22}(t)}{\mathrm{d}t^{k_2}} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\mathrm{d}^{k_p} m_{n1}(t)}{\mathrm{d}t^{k_p}} & \frac{\mathrm{d}^{k_p} m_{n2}(t)}{\mathrm{d}t^{k_p}} & \cdots & \frac{\mathrm{d}^{k_p} m_{nn}(t)}{\mathrm{d}t^{k_p}} \end{vmatrix}$$

where $m_{ij}(t)$ are real functions of $t \in R$.

Proof. Express the determinant as a sum of products:

$$\det(M) = \begin{vmatrix} m_{11}(t) & m_{12}(t) & \cdots & m_{1n}(t) \\ m_{21}(t) & m_{22}(t) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ m_{n1}(t) & m_{n2}(t) & \cdots & m_{nn}(t) \end{vmatrix} = \sum_{\pi = S_n} \mathrm{sgn}(\pi) \prod_{i=1}^{n} m_{i\pi_i}.$$

Take the kth derivative with respect to t using the generalized Leibniz rule to differentiate the products:

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\det(M) = \sum_{\pi = S_n} \mathrm{sgn}(\pi) \sum_{\substack{k_i = 0 \\ \forall i,\, i = 1, 2, \ldots, p \\ k_1 + k_2 + \cdots + k_p = k}}^{k} \frac{k!}{k_1! k_2! \cdots k_p!} \prod_{i=1}^{n} \frac{\mathrm{d}^{k_i} m_{i\pi_i}}{\mathrm{d}t^{k_i}}$$

$$= \sum_{\substack{k_i = 0 \\ \forall i,\, i = 1, 2, \ldots, p \\ k_1 + k_2 + \cdots + k_p = k}}^{k} \frac{k!}{k_1! k_2! \cdots k_p!} \sum_{\pi = S_n} \mathrm{sgn}(\pi) \prod_{i=1}^{n} \frac{\mathrm{d}^{k_i} m_{i\pi_i}}{\mathrm{d}t^{k_i}}$$

$$= \sum_{\substack{k_i=0 \\ \forall i,\, i=1,2,\ldots,p \\ k_1+k_2+\cdots+k_p=k}}^{k} \frac{k!}{k_1!k_2!\cdots k_p!} \begin{vmatrix} \frac{\mathrm{d}^{k_1} m_{11}(t)}{\mathrm{d}t^{k_1}} & \frac{\mathrm{d}^{k_1} m_{12}(t)}{\mathrm{d}t^{k_1}} & \cdots & \frac{\mathrm{d}^{k_1} m_{1n}(t)}{\mathrm{d}t^{k_1}} \\ \frac{\mathrm{d}^{k_2} m_{21}(t)}{\mathrm{d}t^{k_2}} & \frac{\mathrm{d}^{k_2} m_{22}(t)}{\mathrm{d}t^{k_2}} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\mathrm{d}^{k_p} m_{n1}(t)}{\mathrm{d}t^{k_p}} & \frac{\mathrm{d}^{k_p} m_{n2}(t)}{\mathrm{d}t^{k_p}} & \cdots & \frac{\mathrm{d}^{k_p} m_{nn}(t)}{\mathrm{d}t^{k_p}} \end{vmatrix}. \quad \square$$

4. The coefficients of the characteristic polynomial in terms of the elements of A

Next let us express the characteristic polynomial as

$$C(x) = (-1)^n \det[A - xI].$$

It is known that the coefficient of the x^{n-1} term in $C(x)$ is the negative of the trace of the matrix A and the constant term is the determinant of A. To calculate the coefficient of the x^k term, $\frac{1}{k!}\frac{\mathrm{d}^k C}{\mathrm{d}x^k}|_{x=0}$ will be constructed from elements of the matrix, A, using the above-derived derivative of a determinant formula. The elements of A are all real constants and all the determinants in the sum below when $k_i \neq 0$ or 1 are zero:

$$\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}(-1)^n \det[A - xI] = \frac{(-1)^n}{k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k} \begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{vmatrix}$$

$$\frac{1}{k!}\frac{\mathrm{d}^k C}{\mathrm{d}x^k}|_{x=0} = (-1)^n \sum_{\substack{i=1\ldots k \\ p_{ii}=-1,\, p_{ij}=p_{ji}=0 \\ else\ p_{mj}=a_{mj}}} \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & \cdots & p_{nn} \end{vmatrix}.$$

Thus the coefficient of the x^k term in $C(x)$ can be generated by summing the $\binom{n}{k}$ determinants of the matrices created by replacing k of the diagonal elements of matrix A with -1 and the remaining elements in those corresponding rows and columns with 0.

Example 1. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 1 & 0 & 5 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 4 & 1 & 2 \\ 1 & 3 & 4 & 5 & 2 \end{bmatrix}.$$

The coefficient of the x^2 term in the characteristic polynomial of A can be calculated by summing the $\binom{5}{2} = 10$ determinants:

$$\begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 4 & 5 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 5 & 0 & 1 & 2 \\ 0 & 3 & 0 & 5 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 4 \\ 0 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 3 & 4 & 0 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 2 & 3 & 4 & 0 \\ 0 & 5 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix}$$
$$+ \begin{vmatrix} 1 & 0 & 0 & 4 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 5 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 5 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 4 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 3 & 4 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 4 & 0 \\ 4 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 & 2 \end{vmatrix}$$
$$+ \begin{vmatrix} 1 & 2 & 0 & 4 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 4 & 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} = 273$$

and multiplying by $(-1)^5$. Thus the coefficient of x^2 is -273. We also know that the sum of all ten products of three eigenvalues is 273. This is of course valid regardless of the eigenvalues being real or complex or distinct.

5. Conclusions

Of course in the above example the characteristic polynomial can be calculated directly. The real strength of this concept occurs when each real element in the matrix in question is a complicated expression involving parameters, such as occurs in many mathematical models. This method can be used in the derivation of the stability conditions for a first order two-dimensional discrete dynamic [3] and for the derivation of the stability conditions for a first order three-dimensional discrete dynamic [1]. In an upcoming work the stability conditions for a first order four-dimensional discrete dynamic and stability conditions for a first order n-dimensional discrete dynamic will be derived using the results of this work.

References

[1] B.P. Brooks, Linear stability conditions for a first order 3-dimensional discrete dynamic, Applied Mathematics Letters 17 (4) (2004) 463–466.
[2] R.R. Silva, The trace formulas yield the inverse metric formula, Journal of Mathematical Physics 39 (1998) 6206–6213.
[3] J.M.T. Thompson, Non-Linear Dynamics and Chaos, John Wiley & Sons, Toronto, 1986.