# 1 Computation proofs

## 1.1 Multivariate case toy example

**Fact 1** *Let* $[D_w]_{j,j} = w_j \in \mathbb{C}^{k \times k}$ *and*

$$V_d = \begin{pmatrix} e^{i\pi\mu_1^{(1)}} & \dots & e^{i\pi\mu_1^{(k)}} \\ \vdots & \ddots & \vdots \\ e^{i\pi\mu_d^{(1)}} & \dots & e^{i\pi\mu_d^{(k)}} \end{pmatrix} \in \mathbb{C}^{d \times k}, \tag{1}$$

*where* $\mu_n^{(i)}$ *are i.i.d.* $\sim \mathcal{U}([-1,+1])$*. Furthermore, let* $F_{n_1,n_2,n_3} = f(s)\big|_{s=e_{n_1}+e_{n_2}+e_{n_3}}$*, for all* $n_1, n_2, n_3 \in [d]$*. Then,* $F$ *admits the tensor decomposition* $F = V_d \otimes V_d \otimes (V_d D_w)$*.*

**Proof** We wish to show that

$$f(e_1 + e_2 + e_3) = \sum_{j=1}^{k} w_j e^{i\pi(\mu_1^{(j)} + \mu_2^{(j)} + \mu_3^{(j)})}.$$

To do so, we first start by computing the matrix product $V_d D_w$. We now have

$$\begin{aligned} V_d D_w &= \begin{pmatrix} e^{i\pi\mu_1^{(1)}} & \dots & e^{i\pi\mu_1^{(k)}} \\ \vdots & \ddots & \vdots \\ e^{i\pi\mu_d^{(1)}} & \dots & e^{i\pi\mu_d^{(k)}} \end{pmatrix} \begin{pmatrix} w_1 & 0 & \dots & 0 \\ 0 & w_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_k \end{pmatrix} \\ &= \begin{pmatrix} w_1 e^{i\pi\mu_1^{(1)}} & w_2 e^{i\pi\mu_1^{(2)}} & \dots & w_k e^{i\pi\mu_1^{(k)}} \\ w_1 e^{i\pi\mu_2^{(1)}} & w_2 e^{i\pi\mu_2^{(2)}} & \dots & w_k e^{i\pi\mu_2^{(k)}} \\ \vdots & \vdots & \ddots & \vdots \\ w_1 e^{i\pi\mu_d^{(1)}} & w_2 e^{i\pi\mu_d^{(2)}} & \dots & w_k e^{i\pi\mu_d^{(k)}} \end{pmatrix}, \end{aligned}$$

so that

$$\begin{aligned} F_{n_1,n_2,n_3} &= \sum_{j=1}^{k} [V_d]_{n_1,j} [V_d]_{n_2,j} [V_d D_w]_{n_3,j} \\ &= \sum_{j=1}^{k} e^{i\pi\mu_{n_1}^{(j)}} e^{i\pi\mu_{n_2}^{(j)}} w_j e^{i\pi\mu_{n_3}^{(j)}} \\ &= \sum_{j=1}^{k} w_j e^{i\pi(\mu_{n_1}^{(j)} + \mu_{n_2}^{(j)} + \mu_{n_3}^{(j)})} = f(e_1, e_2, e_3), \end{aligned}$$

as required. ■

## 1.2 Tensor decomposition in the exact recovery case

**Fact 2** *Let* $\tilde{F}_{n_1,n_2,n_3} = \tilde{f}(s)\big|_{s=s^{(n_1)}+s^{(n_2)}+v^{(n_3)}}$, *for all* $n_1, n_2 \in [m']$ *and* $n_3 \in \{1,2\}$ *where* $m' = m+d+1$. *Furthermore, we have*

$$V_{S'} = \begin{pmatrix} V_S & V_d & 1,\dots,1 \end{pmatrix}^T \in \mathbb{C}^{m' \times k}, \tag{2}$$

*where*

$$V_S = \begin{pmatrix} e^{i\pi\langle \mu^{(1)}, s^{(1)}\rangle} & \dots & e^{i\pi\langle \mu^{(k)}, s^{(1)}\rangle} \\ \vdots & \ddots & \vdots \\ e^{i\pi\langle \mu^{(1)}, s^{(m)}\rangle} & \dots & e^{i\pi\langle \mu^{(k)}, s^{(m)}\rangle} \end{pmatrix} \in \mathbb{C}^{m \times k}, \tag{3}$$

*and* $V_d$ *is defined as in (1), that is,* $[V_d]_{r,c} = e^{i\pi\mu_r^{(c)}} \in \mathbb{C}^{d\times k}$ *for* $r \in [d]$ *and* $c \in [k]$. *Finally, we have*

$$V_2 = \begin{pmatrix} e^{i\pi\langle \mu^{(1)}, v^{(1)}\rangle} & \dots & e^{i\pi\langle \mu^{(k)}, v^{(1)}\rangle} \\ e^{i\pi\langle \mu^{(1)}, v^{(2)}\rangle} & \dots & e^{i\pi\langle \mu^{(k)}, v^{(2)}\rangle} \\ 1 & \dots & 1 \end{pmatrix} \in \mathbb{C}^{3 \times k}. \tag{4}$$

*Then, in the exact recovery case where* $\epsilon_z = 0$, $\tilde{F}$ *admits the tensor decomposition*

$$\tilde{F} = V_{S'} \otimes V_{S'} \otimes (V_2 D_w).$$

**Proof** Yet to come. ■