1 Computation proofs

1.1 Univariate Matrix-Pencil method

Fact 1 *Define the Vandermonde matrix $V_m \in \mathbb{C}^{m\times k}$ as $[V_m]_{r,c} = \left(e^{i\pi\mu^{(c)}}\right)^r$ for every $0 \leq r \leq m-1$ and $c \in [k]$. Since all the elements of V_m lie on the unit circle in the complex plane, then*

$$\lim_{m\to\infty} \operatorname{cond}_2(V_m) = 1.$$

1.2 Multivariate case toy example

Fact 2 *Let $[D_w]_{j,j} = w_j \in \mathbb{C}^{k\times k}$ and*

$$V_d = \begin{pmatrix} e^{i\pi\mu_1^{(1)}} & \dots & e^{i\pi\mu_1^{(k)}} \\ \vdots & \ddots & \vdots \\ e^{i\pi\mu_d^{(1)}} & \dots & e^{i\pi\mu_d^{(k)}} \end{pmatrix} \in \mathbb{C}^{d\times k}, \tag{1}$$

where $\mu_n^{(i)}$ are i.i.d. $\sim \mathcal{U}([-1,+1])$. Furthermore, let $F_{n_1,n_2,n_3} = f(s)\big|_{s=e_{n_1}+e_{n_2}+e_{n_3}}$, for all $n_1, n_2, n_3 \in [d]$. Then, F admits the tensor decomposition $F = V_d \otimes V_d \otimes (V_d D_w)$.

Proof We wish to show that

$$f(e_1+e_2+e_3) = \sum_{j=1}^{k} w_j e^{i\pi(\mu_1^{(j)}+\mu_2^{(j)}+\mu_3^{(j)})}.$$

To do so, we first start by computing the matrix product $V_d D_w$. We now have

$$\begin{aligned} V_d D_w &= \begin{pmatrix} e^{i\pi\mu_1^{(1)}} & \dots & e^{i\pi\mu_1^{(k)}} \\ \vdots & \ddots & \vdots \\ e^{i\pi\mu_d^{(1)}} & \dots & e^{i\pi\mu_d^{(k)}} \end{pmatrix} \begin{pmatrix} w_1 & 0 & \dots & 0 \\ 0 & w_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_k \end{pmatrix} \\ &= \begin{pmatrix} w_1 e^{i\pi\mu_1^{(1)}} & w_2 e^{i\pi\mu_1^{(2)}} & \dots & w_k e^{i\pi\mu_1^{(k)}} \\ w_1 e^{i\pi\mu_2^{(1)}} & w_2 e^{i\pi\mu_2^{(2)}} & \dots & w_k e^{i\pi\mu_2^{(k)}} \\ \vdots & \vdots & \ddots & \vdots \\ w_1 e^{i\pi\mu_d^{(1)}} & w_2 e^{i\pi\mu_d^{(2)}} & \dots & w_k e^{i\pi\mu_d^{(k)}} \end{pmatrix}, \end{aligned}$$

so that

$$\begin{aligned} F_{n_1,n_2,n_3} &= \sum_{j=1}^{k} [V_d]_{n_1,j}[V_d]_{n_2,j}[V_d D_w]_{n_3,j} \\ &= \sum_{j=1}^{k} e^{i\pi\mu_{n_1}^{(j)}} e^{i\pi\mu_{n_2}^{(j)}} w_j e^{i\pi\mu_{n_3}^{(j)}} \\ &= \sum_{j=1}^{k} w_j e^{i\pi(\mu_{n_1}^{(j)}+\mu_{n_2}^{(j)}+\mu_{n_3}^{(j)})} = f(e_1+e_2+e_3), \end{aligned}$$

as required. ∎

1.3 Tensor decomposition in the exact recovery case

Fact 3 *Let* $\tilde{F}_{n_1,n_2,n_3} = \tilde{f}(s)\big|_{s=s^{(n_1)}+s^{(n_2)}+v^{(n_3)}}$, *for all* $n_1, n_2 \in [m']$ *and* $n_3 \in \{1,2\}$ *where* $m' = m+d+1$. *Furthermore, we have*

$$V_{S'} = \begin{pmatrix} V_S, & V_d, & [1]^k \end{pmatrix}^T \in \mathbb{C}^{m' \times k}, \tag{2}$$

where

$$V_S = \begin{pmatrix} e^{i\pi\langle \mu^{(1)}, s^{(1)} \rangle} & \dots & e^{i\pi\langle \mu^{(k)}, s^{(1)} \rangle} \\ \vdots & \ddots & \vdots \\ e^{i\pi\langle \mu^{(1)}, s^{(m)} \rangle} & \dots & e^{i\pi\langle \mu^{(k)}, s^{(m)} \rangle} \end{pmatrix} \in \mathbb{C}^{m \times k}, \tag{3}$$

and V_d *is defined as in (1), that is,* $[V_d]_{r,c} = e^{i\pi\mu_r^{(c)}} \in \mathbb{C}^{d\times k}$ *for* $r \in [d]$ *and* $c \in [k]$. *Finally, we have*

$$V_2 = \begin{pmatrix} e^{i\pi\langle \mu^{(1)}, v^{(1)} \rangle} & \dots & e^{i\pi\langle \mu^{(k)}, v^{(1)} \rangle} \\ e^{i\pi\langle \mu^{(1)}, v^{(2)} \rangle} & \dots & e^{i\pi\langle \mu^{(k)}, v^{(2)} \rangle} \\ 1 & \dots & 1 \end{pmatrix} \in \mathbb{C}^{3 \times k}. \tag{4}$$

Then, in the exact recovery case where $\epsilon_z = 0$, $\tilde{F}$ *admits the tensor decomposition*

$$\tilde{F} = V_{S'} \otimes V_{S'} \otimes (V_2 D_w).$$

Proof Yet to come. ■