1 Conditional Probability Tables

1. For each of the following equations, select the *minimal set* of conditional independence assumptions necessary for the equation to be true. Note: By minimal, we just mean the set containing the smallest possible number of selected assumptions.

   (a) $\mathbf{P(A, C) = P(A \mid B)\, P(C)}$

   - ☐ $A \perp\!\!\!\perp B$
   - ☐ $A \perp\!\!\!\perp B \mid C$
   - ☐ $A \perp\!\!\!\perp C$
   - ☐ $A \perp\!\!\!\perp C \mid B$
   - ☐ $B \perp\!\!\!\perp C$
   - ☐ $B \perp\!\!\!\perp C \mid A$
   - ☐ No independence assumptions needed.

   (b) $\mathbf{P(A \mid B, C) = \frac{P(A)\ P(B|A)\ P(C|A)}{P(B|C)\ P(C)}}$

   - ☐ $A \perp\!\!\!\perp B$
   - ☐ $A \perp\!\!\!\perp B \mid C$
   - ☐ $A \perp\!\!\!\perp C$
   - ☐ $A \perp\!\!\!\perp C \mid B$
   - ☐ $B \perp\!\!\!\perp C$
   - ☐ $B \perp\!\!\!\perp C \mid A$
   - ☐ No independence assumptions needed.

   (c) $\mathbf{P(A, B) = \sum_c P(A \mid B, c)\, P(B \mid c)\, P(c)}$

   - ☐ $A \perp\!\!\!\perp B$
   - ☐ $A \perp\!\!\!\perp B \mid C$
   - ☐ $A \perp\!\!\!\perp C$
   - ☐ $A \perp\!\!\!\perp C \mid B$
   - ☐ $B \perp\!\!\!\perp C$
   - ☐ $B \perp\!\!\!\perp C \mid A$
   - ☐ No independence assumptions needed.

   (d) $\mathbf{P(A, B \mid C, D) = P(A \mid C, D)\, P(B \mid A, C, D)}$

   - ☐ $A \perp\!\!\!\perp B$
   - ☐ $A \perp\!\!\!\perp B \mid C$
   - ☐ $A \perp\!\!\!\perp B \mid D$
   - ☐ $C \perp\!\!\!\perp D$
   - ☐ $C \perp\!\!\!\perp D \mid A$
   - ☐ $C \perp\!\!\!\perp D \mid B$
   - ☐ No independence assumptions needed.

2. Mark **all** expressions that are equal to $\mathbf{P(A, B, C)}$, given that $\mathbf{A \perp\!\!\!\perp B}$.

   - ☐ $P(A \mid C)\ P(C \mid B)\ P(B)$
   - ☐ $P(A)\ P(B)\ P(C \mid A, B)$
   - ☐ $P(C)\ P(A \mid C)\ P(B \mid C)$
   - ☐ $P(A)\ P(C \mid A)\ P(B \mid C)$
   - ☐ $P(A)\ P(B \mid A)\ P(C \mid A, B)$
   - ☐ $P(A, C)\ P(B \mid A, C)$
   - ☐ None of the provided options.

3. For the following questions, you will be given a set of probability tables and a set of conditional independence assumptions. Given these tables and independence assumptions, write an expression for the requested probability tables. Keep in mind that your expressions cannot contain any probabilities other than the given probability tables. If it is not possible, mark "Not possible."

   (a) Using probability tables $\mathbf{P(A), P(A \mid C), P(B \mid A), P(C \mid A, B)}$ and no conditional independence assumptions, write an expression to calculate the table $\mathbf{P(B \mid A, C)}$.

   $\mathbf{P(B \mid A, C)} =$ ____________________________ ○ Not possible.

(b) Using probability tables $\mathbf{P}(\mathbf{A} \mid \mathbf{B}), \mathbf{P}(\mathbf{B}), \mathbf{P}(\mathbf{B} \mid \mathbf{A}, \mathbf{C}), \mathbf{P}(\mathbf{C} \mid \mathbf{A})$ and conditional independence assumption $\mathbf{A} \perp\!\!\!\perp \mathbf{B}$, write an expression to calculate the table $\mathbf{P}(\mathbf{C})$.

$\mathbf{P}(\mathbf{C}) =$ ________________________________ ○ Not possible.

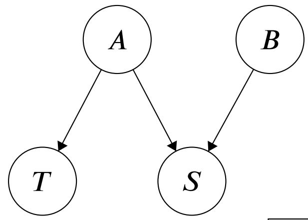
(c) Using probability tables $\mathbf{P}(\mathbf{A} \mid \mathbf{B}, \mathbf{C}), \mathbf{P}(\mathbf{B}), \mathbf{P}(\mathbf{B} \mid \mathbf{A}, \mathbf{C}), \mathbf{P}(\mathbf{C} \mid \mathbf{B}, \mathbf{A})$ and conditional independence assumption $\mathbf{A} \perp\!\!\!\perp \mathbf{B} \mid \mathbf{C}$, write an expression for $\mathbf{P}(\mathbf{A}, \mathbf{B}, \mathbf{C})$.

$\mathbf{P}(\mathbf{A}, \mathbf{B}, \mathbf{C}) =$ ________________________________ ○ Not possible.

2 Bayes' Net of Disease

Suppose that a patient can have a symptom (S) that can be caused by two different diseases (A and B). Disease A is much rarer, but there is a test T that tests for the presence of A. The Bayes' Net and corresponding conditional probability tables for this situation are shown below. For each part, you may leave your answer as a fraction.

$\mathbf{P}(A)$	
$+a$	0.1
$-a$	0.9



$\mathbf{P}(B)$	
$+b$	0.5
$-b$	0.5

$\mathbf{P}(T\|A)$		
$+a$	$+t$	1
$+a$	$-t$	0
$-a$	$+t$	0.2
$-a$	$-t$	0.8

$\mathbf{P}(S\|A,B)$			
$+a$	$+b$	$+s$	1
$+a$	$+b$	$-s$	0
$+a$	$-b$	$+s$	0.80
$+a$	$-b$	$-s$	0.20
$-a$	$+b$	$+s$	1
$-a$	$+b$	$-s$	0
$-a$	$-b$	$+s$	0
$-a$	$-b$	$-s$	1

1. Compute the following entry from the joint distribution:

   $\mathbf{P}(-a, -t, +b, +s) =$

2. What is the probability that a patient has disease A given that they have disease B?

   $\mathbf{P}(+a | +b) =$

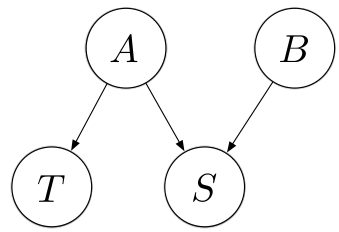
3. What is the probability that a patient has disease A given that they have symptom S and test T returns positive?

   $\mathbf{P}(+a | +t, +s) =$

4. Suppose that both diseases A and B become more likely as a person ages. Add any necessary variables and/or arcs to the Bayes' net to represent this change. For any variables you add, *briefly* (one sentence or less) state what they represent. Also, state one independence or conditional independence assertion that is **removed** due to your changes.

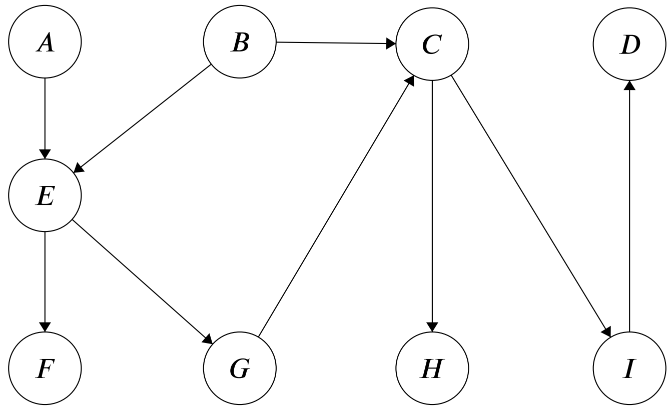
New variable(s) (and brief definition):

Removed conditional independence assertion:



3 Variable Elimination

The following questions use the Bayes' net below. All variables have binary domains:



1. Karthik wants to see the ocean, and so decides to compute the query $P(B, E, A, C, H)$. He wants you to help him run variable elimination to compute the answer, with the following elimination ordering: I, D, G, F. Complete the following description of the factors generated in this process:

   He initially has the following factors to start out with:

   $$P(A), P(B), P(C|B, G), P(D|I), P(E|A, B), P(F|E), P(G|E), P(H|C), P(I|C)$$

   When eliminating I we generate a new factor f_1 as follows:

   $$f_1(C, D) = \sum_i P(i|C)P(D|i)$$

   This leaves us with the factors:

   $$P(A), P(B), P(C|B, G), P(E|A, B), P(F|E), P(G|E), P(H|C), f_1(C, D)$$

   When eliminating D we generate a new factor f_2 as follows:

   This leaves us with the factors:

   When eliminating G we generate a new factor f_3 as follows:

   $$f_3(C, B, E) = \sum_g P(C|B, G = g)P(G = g|E)$$

This leaves us with the factors:

$$P(A), P(B), P(E|A, B), P(F|E), P(H|C), f_2(C), f_3(C, B, E)$$

When eliminating F we generate a new factor f_4 as follows:

$$f_4(E) = \sum_f P(F = f|E)$$

This leaves us with the following factors.

2. Among f_1, f_2, f_3, f_4, which is the largest factor generated, and how large is it? Assume all variables have binary domains and measure the size of each factor by the number of rows in the table that would represent the factor.

4 Simple HMM

Consider a Markov Model with a binary state X (i.e., X_t is either 0 or 1). The transition probabilities are given as follows:

X_t	X_{t+1}	$P(X_{t+1} \mid X_t)$
0	0	0.7
0	1	0.3
1	0	0.6
1	1	0.4

1. The prior belief distribution over the initial state X_0 is uniform, i.e., $P(X_0 = 0) = P(X_0 = 1) = 0.5$. After one timestep, what is the new belief distribution, $P(X_1)$?

X_1	$P(X_1)$
0	
1	

Now, we incorporate sensor readings. The sensor model is parameterized by a number $\beta \in [0, 1]$:

X_t	E_t	$P(E_t \mid X_t)$
0	0	$(1-\beta)$
0	1	β
1	0	β
1	1	$(1-\beta)$

2. At $t = 1$, we get the first sensor reading, $E_1 = 0$. Use your answer from part (1) to compute $P(X_1 = 0 \mid E_1 = 0)$. You may leave your answer in terms of β.

3. For what range of values of β will a sensor reading $E_1 = 0$ increase our belief that $X_1 = 0$? That is, what is the range of β for which $P(X_1 = 0 \mid E_1 = 0) > P(X_1 = 0)$?

4. Unfortunately, the sensor breaks after just one reading, and we receive no further sensor information. Compute $P(X_\infty \mid E_1 = 0)$, the stationary distribution *very many* timesteps from now.

X_∞	$P(X_\infty \mid E_1 = 0)$
0	
1	

5. How would your answer to part (4) change if we never received the sensor reading E_1, i.e. what is $P(X_\infty)$ given no sensor information?

X_∞	$P(X_\infty)$
0	
1	