Homework 4

Task 1 - Encrypt & MAC (15 points)

(a)

From the ciphertexts, we can infer that the plaintexts are the same.

Justification:
We can see that the last 256 bits of both ciphertexts C_1 and C_2 are the same. Given that the tag produced by the MAC algorithm is also 256 bits, we can infer that these common bits are the MAC tags.

Since the MACs are done on the plaintexts and both ciphertexts are generated using the same secret key, given that the MAC tags are equal, this implies that the actual plaintexts are the same.

Additionally, since both ciphertexts are encrypted using CTR with AES, the encryption algorithm will produce different ciphertexts even if the plaintexts are the same (assuming C_1 and C_2 used different initialization vector). Thus, the fact that the first 50 bytes of C_1 and C_2 are different doesn't contradict our inference that the plaintexts are the same.

(b)

E&M is a good scheme in terms of IND-CPA security.

Explanation:
Since HMAC with SHA-256 produces tags which are computationally indistinguishable from random by someone without the key, the tags themselves do not reveal any information about the plaintext. Since CTR mode with AES is IND-CPA secure, the ciphertext produced does not allow an attacker to determine which plaintext was encrypted.

Appending a MAC tag to the ciphertext is similar to appending a fixed pattern to an IND-CPA secure ciphertext, which we have proven does not impact the scheme's IND-CPA security in Homework 2 Task 4b). The tag acts similar to a fixed suffix that provides integrity without compromising confidentiality from the CTR encryption.

Therefore, the E&M is a good scheme in terms of IND-CPA security.

(c)

The main observation is that the E&M scheme produces identical tags for the same plaintexts since SHA-256 is deterministic. Let $\Pi = (\mathsf{Kg}, \mathsf{Enc}, \mathsf{Dec})$ be the encryption scheme E&M, We can construct an adversary against Π in the following way:

adversary $A^{O}()$:
$C'_1[0]C'_1[1]...C'_1[\ell] \mid\mid T_1 \leftarrow O.\mathsf{Encrypt}(0^{n\cdot\ell})$
$C'_2[0]C'_2[1]...C'_2[2\ell] \mid\mid T_2 \leftarrow O.\mathsf{Encrypt}(0^{n\cdot 2\ell})$
$C^* = C'_2[0]C'_2[1]...C'_2[\ell] \mid\mid T_1$
return C^*

Consider the INT-CTXT oracle. When running $\mathsf{A}^{\mathsf{INT\text{-}CTXT}[E\&M]}$, for the adversary to succeed, two things need to be satisfied: (1) $C^* \notin \mathcal{Q}$, and (2) $\mathsf{Dec}(K, C^*) \neq \perp$.

Firstly, the oracle's state is $\mathcal{Q} = \{C_1, C_2\}$, where $C_1 = C'_1[0]C'_1[1]...C'_1[\ell] \mid\mid T_1$ from $\mathsf{Enc}(K, 0^{n\cdot\ell})$, and $C_2 = C'_2[0]C'_2[1]...C'_2[2\ell] \mid\mid T_2$ from $\mathsf{Enc}(K, 0^{n\cdot 2\ell})$. Assuming the IV is different for each encryption, $C'_2[0] \neq C'_1[0]$ and because C^* is composed only half of the blocks from C_2 with the tag T_1 from C_1, we have that $C_1 \neq C_2 \neq C^*$, and thus, $C^* \notin \mathcal{Q}$.

Secondly, by the E&M construction, decrypting the counter-mode ciphertexts of C_1 and C^* will yield the same plaintext, because the first ℓ blocks of $0^{n\cdot 2\ell}$ are the same as $0^{n\cdot\ell}$, and CTR blocks can be decrypted independently. Since SHA-256 is deterministic, running the HMAC algorithm on the decrypted counter-mode ciphertext of C^* will produce the same tag as T_1. Therefore, $\mathsf{Dec}(K, C^*)$ returns $M \neq \perp$.
In conclusion,

$$\begin{aligned}\mathsf{Adv}^{\mathsf{int\text{-}ctxt}}_{\Pi}(A) &= \Pr[\mathsf{A}^{\mathsf{INT\text{-}CTXT}[\Pi]} \Rightarrow 1] \\ &= \Pr[C^* \notin \mathcal{Q} \wedge \mathsf{Dec}(K, C^*) \neq \perp] = 1,\end{aligned}$$

and since A returns in polynomial time, this means Π is not INT-CTXT secure.

Task 2 - Authenticated Encryption (15 points)

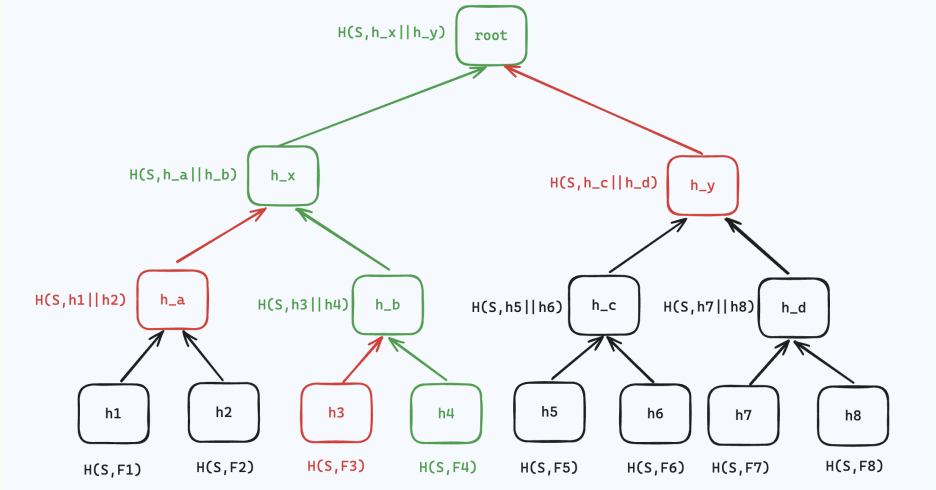
(a)

(b)

Task 3 - Set Commitments (10 points)

(a)

To verify the integrity of the file F_i with respect to com, the server needs to provide the client with: (1) The hash value of F_i, $H(S, F_i)$, (2) The sibling hash values for each level of the tree on the path from F_i to the root.
In the example below with $m = 8$, the client wants to retrieve and verify the integrity of F_4. The server sends client the file F_4, the com (the root), and the sibling hash values for each level of the tree (marked in red, $[h_3, h_a,$ and $h_y]$) on the path from F_4 to the root (marked in green).



The client can verify the integrity of F_i by doing the following:
1. Compute $H(S, F_i)$ to get the hash of the file.
2. At each level of the binary tree, the client computes the parent node hash by concatenating the hash of F_i and its sibling with the sibling's hash provided by the server, and then hashing the result with the seed S.
3. The client repeats this process up the tree, each time use the newly computed hash along with the next sibling hash provided by the server.
4. When the client reaches the root, they compare the computed root hash with the commitment com published by the server. If they match, then file F_i is valid.

The content sent by the server (excluding F_i itself) is $O(n \log m)$ bits long because each hash is n bits long and there are $\log m$ sibling hashes to send (for a balanced binary tree with m leaves, the path from any leaf to the root has $\log m$ levels [since m is a power of two], and we have one sibling on each level).
Because the hash function H is collision-resistant, it is computationally infeasible for an adversary to find two different inputs that hash to the same output. So changing any file F_i would result in a different hash at the leaf, and change would propagate up the tree result in a different root hash. Therefore, any change in the files would be detected when the client computes a root hash that does not match the commitment.

Task 4 - Number Theory & Groups (10 points)

(a)

By the definition, $\mathbb{Z}_{35}^* = \{a \in \mathbb{Z}_{35} : \gcd(a, 35) = 1\}$. Essentially, we are looking for all numbers between 0 and 34 that are coprime with 35. Since 35 is the product of two prime numbers 5 and 7, any number that is not a multiple of 5 or 7 is coprime with 35. Therefore, $\mathbb{Z}_{35}^* = \{1, 2, 3, 4, 6, 8, 9, 11, 12, 13, 16, 17, 18, 19, 22, 23, 24, 26, 27, 29, 31, 32, 33, 34\}$.

(b)

Since 37 is a prime number, $\mathbb{Z}_{37}^* = \{a : 1 \leq a \leq 36\}$. When $g = 2$, we have the following:

e =	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	1	2	4	8	16	32	27	17	34	31	25	13	26	15	30	23	9	18
e =	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
	36	35	33	29	21	5	10	20	3	6	12	24	11	22	7	14	28	19

So, we have: $\langle 2 \rangle = \{2^0 = 1, 2^1, ..., 2^{36-1}\} = \mathbb{Z}_{37}^*$.
Therefore, 2 is a generator of $\mathbb{Z}_{37}^*$.

(c)

To find $x \in \mathbb{Z}_{187}$ such that $125x \equiv 4 \pmod{187}$, we are essentially looking for $x, y \in \mathbb{Z}_{187}$ such that $125x + 187y = 4$. We can use the extended Euclidean algorithm to find x and y:

Euclidean algorithm	Backtrack
$187 = 125 \cdot 1 + 62$	$125 - 62 \cdot 2 = 1$
$125 = 62 \cdot 2 + 1$	$125 - (187 - 125) \cdot 2 = 1$

Here we get $125 - (187 - 125) \cdot 2 = 125 \cdot 3 + 187 \cdot (-2) = 1$. Multiplying both sides by 4, we get $125 \cdot 12 + 187 \cdot (-8) = 4$. Thus, we have $x = 12$ and $y = -8$.
Therefore, $x = 12$ is the solution to $125x \equiv 4 \pmod{187}$.