Quantitative Macroeconomics.
Homework 2.

Sebastian Zalas

in cooperation with Maciej Sztachera.

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Question 1.

In order to replicate results use Python codes `trans1C.py` and `trans1D.py`. Also `shooting.m` was utilized, but not necessary for replication.

(a).

Reformulate original problem:

$$\max_{\{k_{t+1},c_t\}_{t=0}^{\infty}} \{E\sum_{t=0}^{\infty}\beta^t \log(c_t)\}$$

$$\text{subject to: } c_t + k_{t+1} - (1-\delta)k_t - k_t^{1-\theta}(zh_t)^\theta = 0$$

Set up Lagrangian:

$$\mathscr{L} = \sum_{t=0}^{\infty}\beta^t \log(c_t) - \lambda_t(c_t + k_{t+1} - (1-\delta)k_t - k_t^{1-\theta}(zh_t)^\theta).$$

First order conditions:

$$\frac{\partial \mathscr{L}}{\partial k_{t+1}} : \lambda_t = \lambda_{t+1}((1-\delta) + (1-\theta)k_{t+1}^{-\theta}(zh_{t+1})^\theta)$$

$$\frac{\partial \mathscr{L}}{\partial c_t} : \beta^t\frac{1}{c_t} = \lambda_t$$

Substituting from second FOC in to first, I recieve Euler equation:

$$\frac{c_{t+1}}{c_t} = \beta((1-\delta) + (1-\theta)k_{t+1}^{-\theta}(zh_{t+1})^\theta)$$

Now, I shall derive steady state. For $k_t = k_{t+1}$ and $c_t = c_{t+1}$:

$$1 = \beta((1-\delta) + (1-\theta)k^{-\theta}(zh)^\theta)$$

$$c + k - (1-\delta)k - (1-\theta)k^{-\theta}(zh)^\theta = 0$$

From first of above equations I obtain k^*, from second c^* depending on k^*:

$$k^* = (\frac{1-\theta}{\beta^{-1} - 1 + \delta})^{1/\theta} zh$$

$$c^* = k^{*1-\theta}(zh)^\theta + \delta k^*.$$

Set investment-output ratio equal to 0.25:

$$\frac{i^*}{y^*} = \frac{\delta k^*}{k^{*1-\theta}(zh)^\theta} = \frac{1}{4} \Rightarrow \delta = \frac{1}{4}(\frac{zh}{k^*})^\theta$$

Set capital-output ratio equal to 4:

$$\frac{k^*}{y^*} = \frac{k^*}{k^{*1-\theta}(zh)^\theta} = 4 \Rightarrow k^* = 4^{\frac{1}{\theta}} zh$$

From investment-output and capital-output ratios, I obtain $\delta = \frac{1}{16}$.
Now substitute for k^* into capital-output ratio:

$$(\frac{1-\theta}{\beta^{-1} - 1 + \delta})^{1/\theta} zh = 4^{\frac{1}{\theta}} zh \Rightarrow \beta = \frac{4}{1 - \theta + 4(1-\delta)}$$

Moreover, z drops out from above equations, which means that **any** level of z will satisfy capital-output ratio equal to 4.
Final values of parameters: $\theta = 0.67$, $h_t = 0.31$, $z = 1$, $\delta = 0.0625$, $\beta = 0.98039$.
Steady state values: $k_a^* = 2.45448$, $c_a^* = 0.46021$, $y_a^* = 0.61362$.

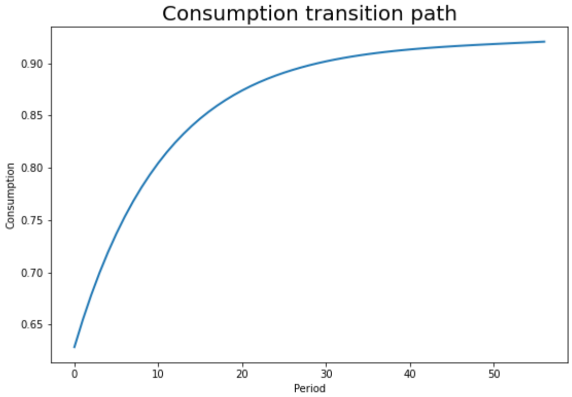
(b).

I redo computations using derivations and parameters from point (a), but now $z = 2$. New steady state values are as follows: $k_b^* = 4.90895$, $c_b^* = 0.92042$, $y_b^* = 1.2272$.

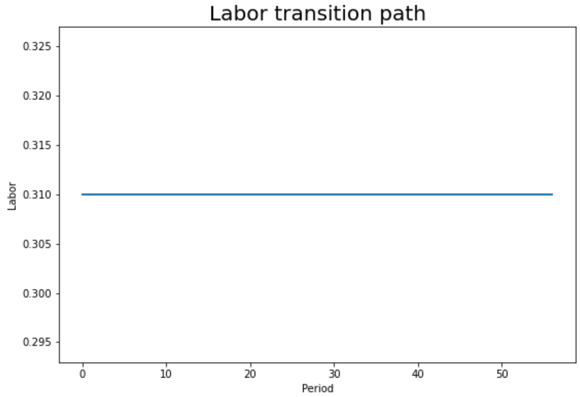
(c).

To compute transition path from steady state from (a) to steady state form (b), I have to find initial level of consumption, after doubling parameter z. I do this using shooting algorithm (to replicate result use `shooting.m`) and obtain $c_0 = 0.6286$. On Figure 1 there are requested transition paths for consumption, labor, savings and output.

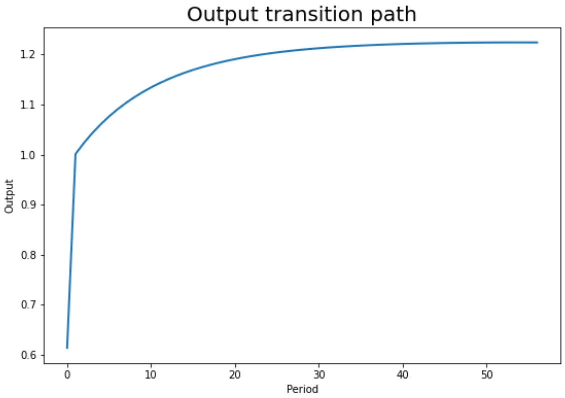
Figure 1: Transition paths for exercice 1.c.



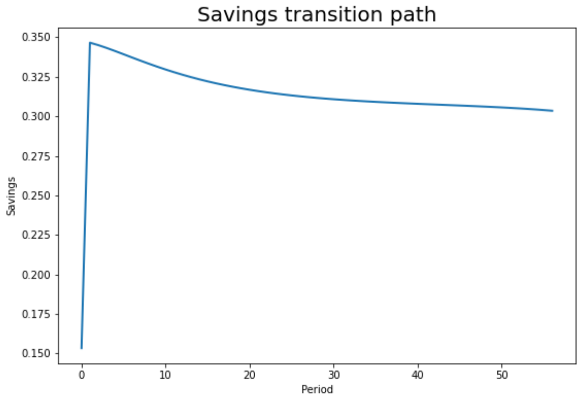
(a) Consumption's transition.



(b) Labor's transition.



(c) Output's transition.

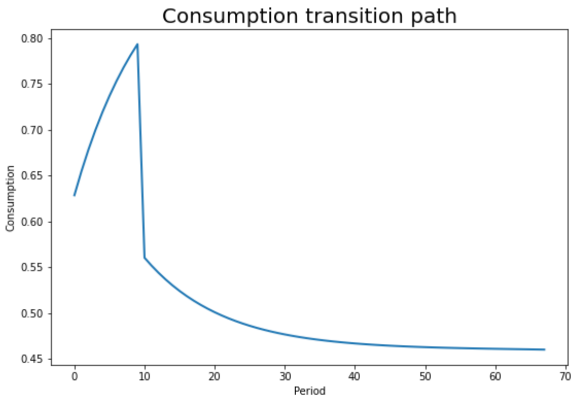


(d) Savings' trasition.

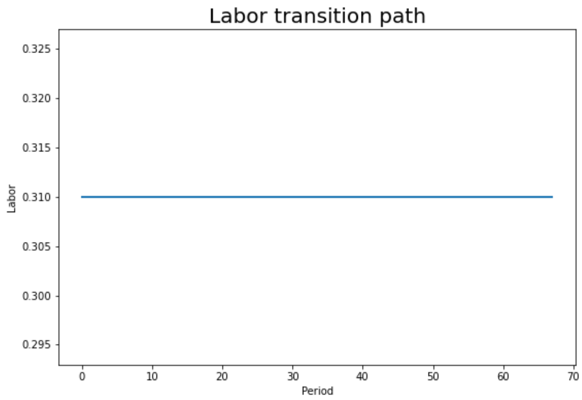
(d).

As an answer I present on Figure 2 transition paths for consumption, labor, savings and output.

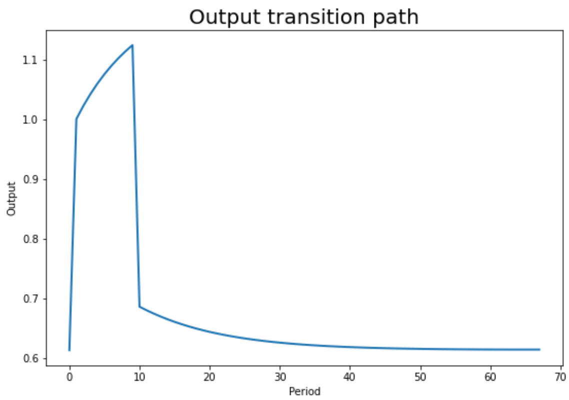
Figure 2: Transition paths for exercise 1.d.



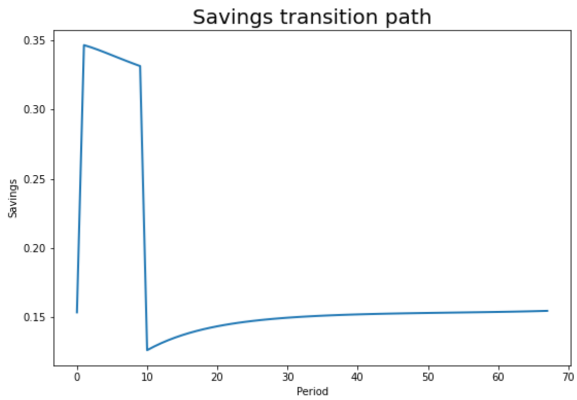
(a) Consumption's transition.



(b) Labor's transition.



(c) Output's transition.



(d) Savings' trasition.

Question 2.

In this section I present my anwser to Question 2, that concerned model presented during first lectures. It is organized as follows: in first subsection I present analytical part of solution and briefly discuss computational strategy which is implied by analytical part; in second subsection I refer to points (a) and (b) from homework commands; in third subsection I place requested plots. In order to replicate results use Python code `toymodelsolution.py`.

1. Solution of the model.

In order to solve the Toy model I define following problem:

$$\max_{H_f, H_{nf}} \{Y(H_f, H_{nf}) - \kappa_f H_f - \kappa_{nf} H_{nf} - \omega(1-\gamma) i H_f\}$$

$$\text{subject to: } H_f + H_{nf} \leq N$$

Since it is maximization problem with inequality constraint I apply Karusk-Kuhn-Tucker method. Firsty, set up Lagrangian:

$$\mathscr{L} = Y(H_f, H_{nf}) - \kappa_f H_f - \kappa_{nf} H_{nf} - \omega(1-\gamma) i H_f - \lambda(H_f + H_{nf} - N)$$

First order conditions:

$$\frac{\partial \mathscr{L}}{\partial H_f} : \frac{\partial Y}{\partial H_f} - \kappa_f - \omega(1-\gamma) i H_f - \lambda = 0$$

$$\frac{\partial \mathscr{L}}{\partial H_{nf}} : \frac{\partial Y}{\partial H_{nf}} - \kappa_{nf} - \lambda = 0$$

and:

$$\lambda(H_f + H_{nf} - N)$$

Now there are two cases: when constraint is active and inactive.

In case of active constraint, $\lambda \geq 0$ and constraint is binding so $H_f = N - H_{nf}$. Then two FOCs may be equalled,λ cancells out and it H_f may be substituted to obtain one equation with one unknown. Then `fsolve` function from Python is utilized to obtain solution. Of course one has to remember to check if $\lambda \geq 0$.

In case of inactive constraint $\lambda = 0$ so it drops from FOC equations. Then there are left two equations with two unknowns. Function *least_ squares* is utilized to solve such system of nonlinear equations, which also cares about bounds, so the solution is not out of domain.

These strategy is implemented in file `toymodelsolve.py`. This code solves two cases for each combination of β and $c(TW)$. Finally solution that gives greater value of objective function is chosen. Such strategy guarantees that both biding and nonbiding solutions are found. Solutions are stored in proper arrays, and then plotted as a heatmap.

2. Comments.

(a).

Plots of solutions for requested parametrization ($\rho = 1.1$ and $\omega = 20$) are located in subsection **Plots**, always in panel (a) of the figures. (to ensure tidiness and facilitate comparisons in next point). On the plots the darker the area, the higher value of variable of interest, conditional on parameters β and $c(TW)$.

Deaths are plotted on Figure 3a. The highest value of death rate occurs when $\beta = 1$ and $c(TW) = 0$, so when conditional infection rate is equal to 1 and productivity loss of teleworking is full. The lowest value of deaths rate occurs for opposite situation - no loss of productivity and infection rate equal to 0. Results here are compatible with intuition. The lower loss of productivity related to teleworking (which means higher $c(TW)$), the lower deaths rate since more people can work from home and there is less human contact. Also death rate diminishes with lower conditional infection rate.

Infections rate is plotted on Figure 4a. There is the same scheme as it was for deaths rate (Figure 3a): the lower loss of productivity the lower infections rate (as more people can work from home) and the lower conditional infections rate, the lower infections rate.

Welfare is plotted on Figure 5a. It seem from that graphs that welfare is independent from β and mainly relies on value of $c(TW)$. The lower loss of productivity, in other words, the closer to situation without pandemic, the higher is welfare. Loss of productivity causes loss of wealth, as less people can work optimally.

Output is plotted on Figure 6a. It reveals similar pattern ans previous graph of welfare. It is because these two are strongly related. Output is high when loss of productivity is low, such implication seems very intuitive.

On site labor supply is plotted on Figure 7a. It also shows logical patterns. When loss of productivity is low, and conditional infection rate is high, this encourages to work from home. However situation in which loss of productivity is high and (or) conditional infections rate is low enhances on site work. The same patterns are observed on Figure 9a which presents fraction of on site labor supply: when β is low and (or) $c(TW)$ is low it makes people to work on site and fraction increases; when parameters rise, people tend to telework, so fraction diminishes.

Telework labor supply is presented on Figure 8a. One can observe contrary tendences in comparison with on site labor supply. People tend to telework when loss of productivity is low, and conditional infection rate is high, so environment makes people to easily stay home. But when loss of productivity is significant and (or) conditional infections rate lowers, people are back in workplaces. These patterns are reasonable.

Finally, total labor supply is plotted on Figure 10a. In majority of area it is equal to one, it it means that there are very few situations in which lockdown occurs (and it is not very severe). Probably parametrization causes that tradeoff between deaths rate and quantity of output for planner in model is small.

Generally, model's behavoiur is very reasonable and consistent with basic intuition.

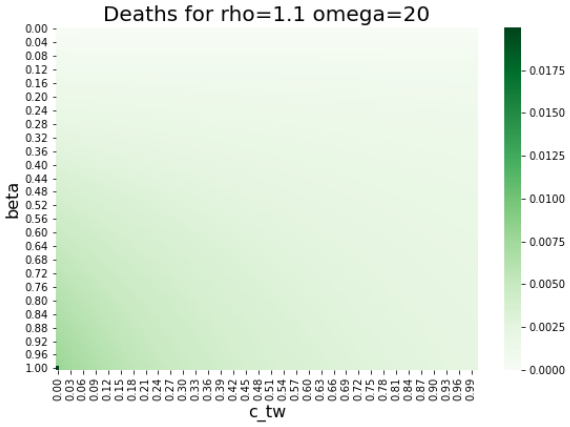
(b).

In point (a). results were based on parametrization with $\rho = 1.1$ and $\omega = 20$. I computed solutions for two alternative combinations of parameters. One consists of $\rho = 0.8$ and $\omega = 40$. Such values of parameters mean that substitution between working on site and telework is more difficult, and at the same time planner cares more about death rate than it was in case of point (a). Second specification consists of $\rho = 1.5$ and $\omega = 10$. In such situation planner even cares less about deaths in comparison with (a) and substitution between on site work and telework is even higher than in (a).
The latter combination of parameters should produce similar results to these presented in point (a) because planner do not cares much about pandemic (represented bu low ω and it is easy to shift people from on site work to telework (high substitution). Plots presented in panels (c) in section **Plots** presents results. All figures from (c) are similar to figures from (a). Thus this scenario is not very interesting, however shows that model behaves well, and do not produce counterintuitive outcomes.
Much more interesting results occurs in case of $\rho = 0.8$ and $\omega = 40$. Results that come from this specification are presented in panels (b) of figures located in section **Plots**. They reveal very different pattern than it occured in case of plots in panels (a) and (c). The most weird situation may be observed by comparing Figure 7b (Labor supply of on site work) and Figure 8b (Labor supply of telework). In previous specification these were reciprocals to some extent. Now both figures imply same pattern: there is high labor supply for low balues of telework productivity and for low values of β. It is probably caused by gross complementarity of factors of production in CES production function. It also shows how strong is tradeoff in for parameters $\rho = 0.8$ and $\omega = 40$. Some workers must work on site, but to ensure them conditions for that (planner strongly cares about deaths rate) rest of workers must work at home to diminish interactions and deaths rate. So they do telework even they are highy inefficient. Outcomes of that situation can be also observed at Figure 6b which is high for high loss of productivity from telework and values of β around half. Welfare reveals the same logic and is presented at Figure 5b. Deaths rate (Figure 3b) and infection rate (4b) reports highest values in the same region as output, so it means that high deaths and infection rates is the cost for that and tradeoff is strong.

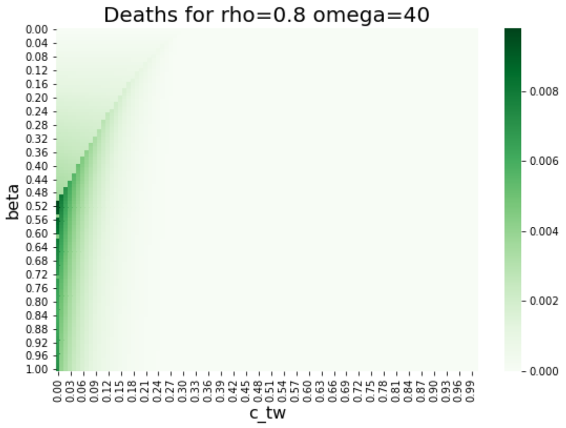
3. Plots.

Below I present plots for all combinations of β (conditional infection rate) and $c(TW)$ (productivity loss of teleworking) and for three specifications of ρ (elasticity of substitution between working on site and teleworking) and ω (weight that planner gives to deaths rate).

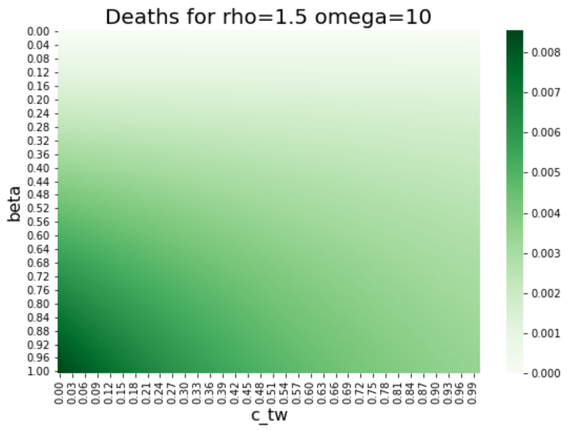
Figure 3: Deaths rate for different specifications.



(a)

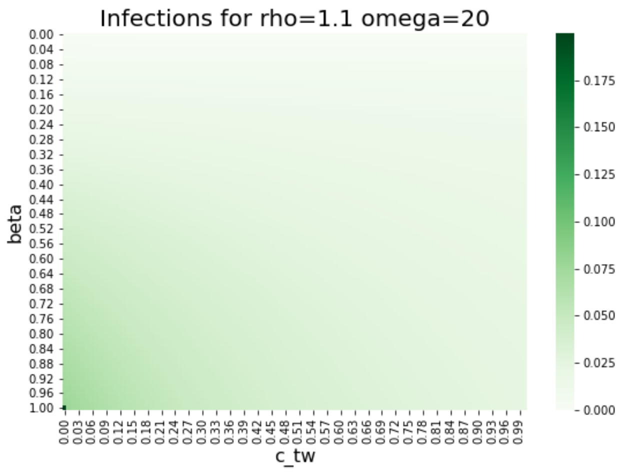


(b)

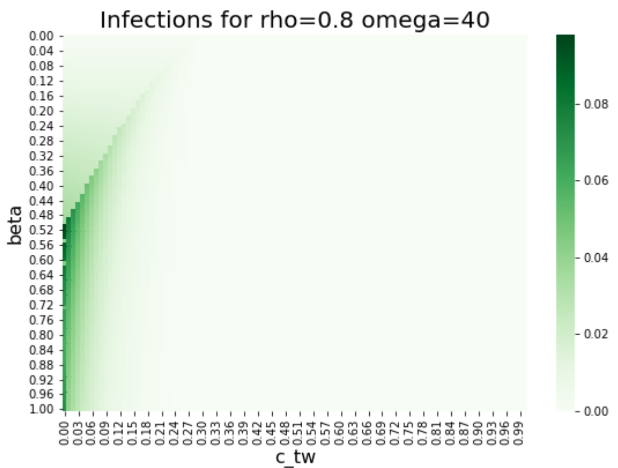


(c)

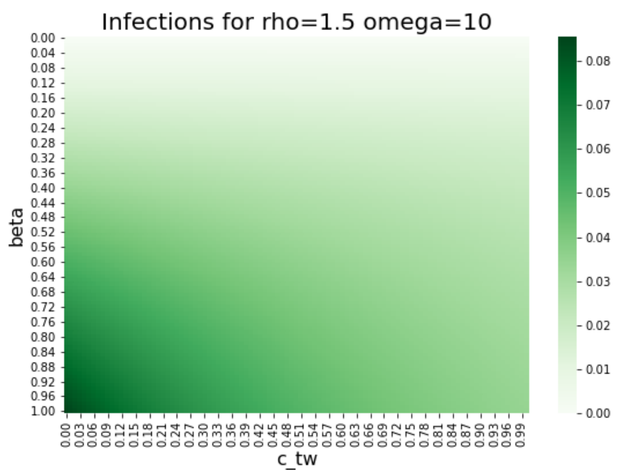
Figure 4: Infections rate for different specifications.



(a)

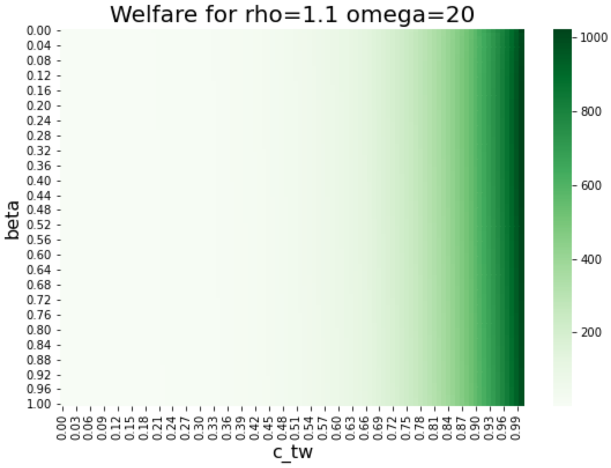


(b)

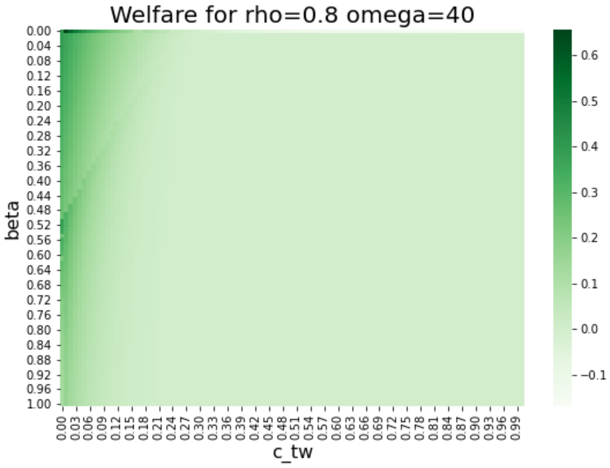


(c)

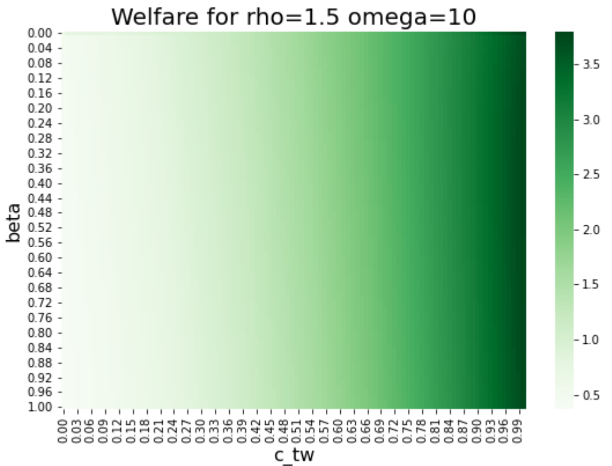
Figure 5: Welfare for different specifications.



(a)

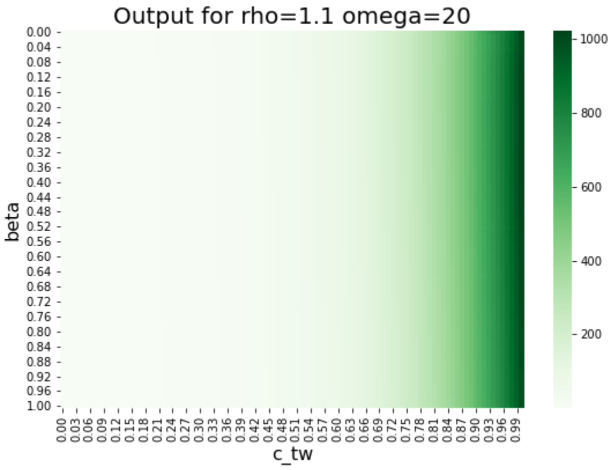


(b)

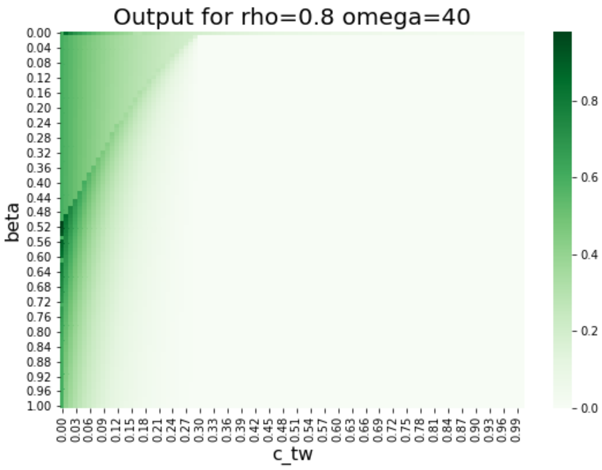


(c)

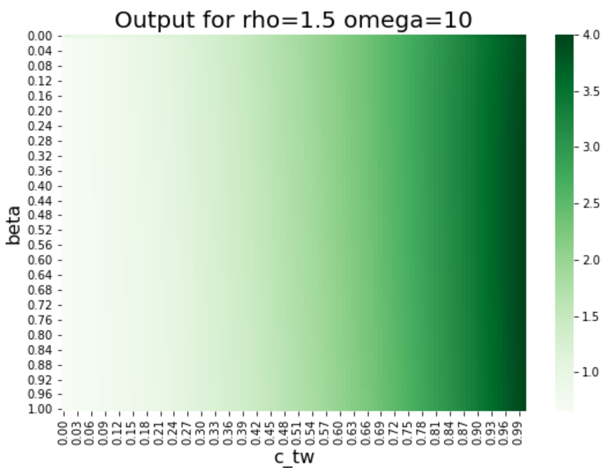
Figure 6: Output for different specifications.



(a)

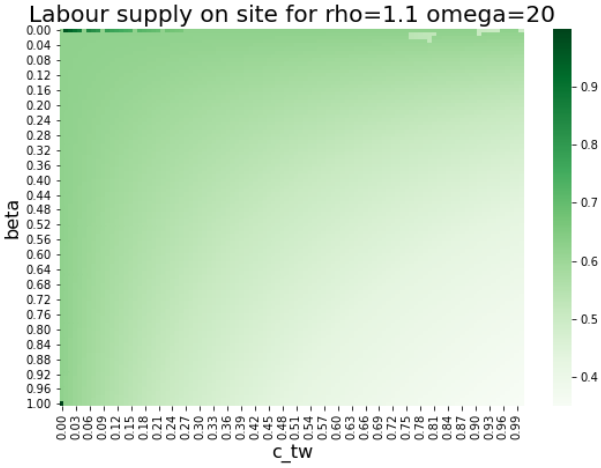


(b)

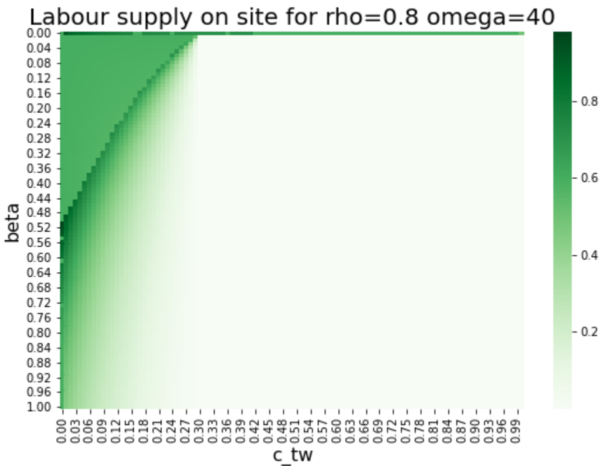


(c)

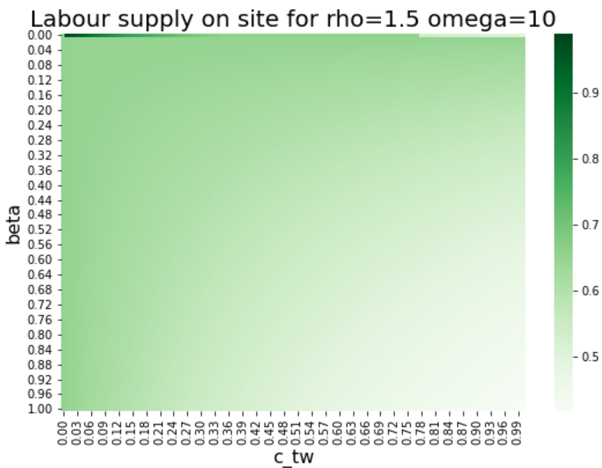
Figure 7: On site labor supply for different specifications.



(a)

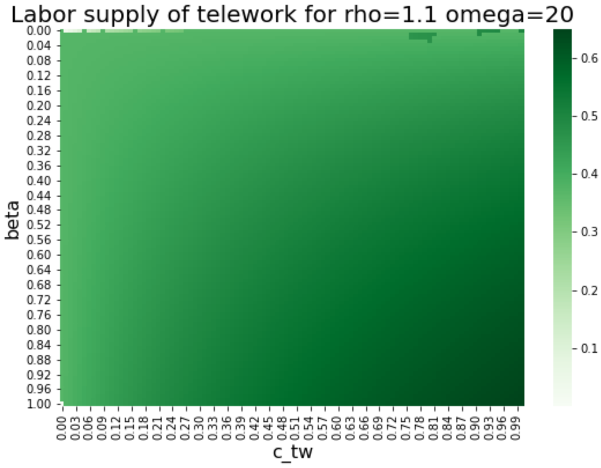


(b)

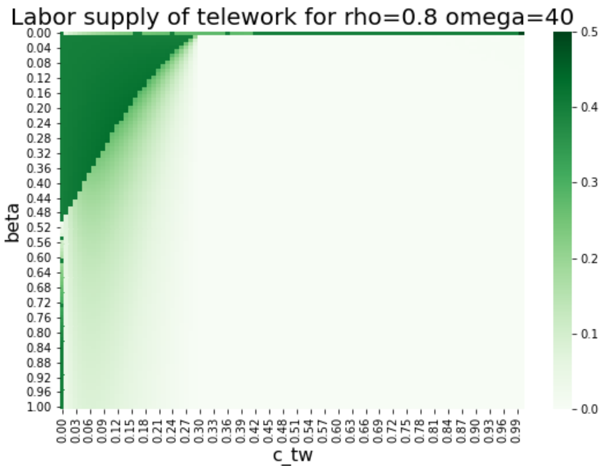


(c)

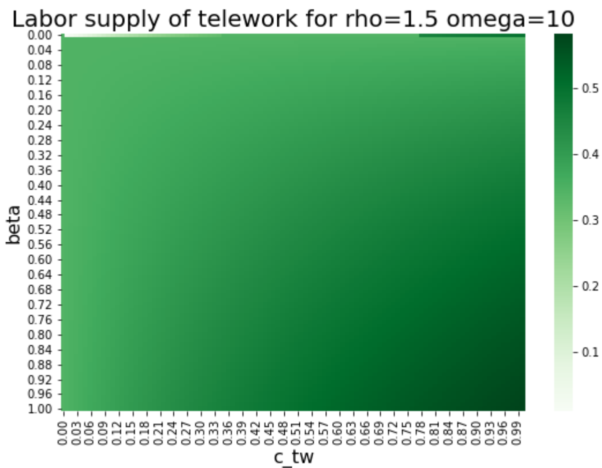
Figure 8: Teleworking labor supply for different specifications.



(a)

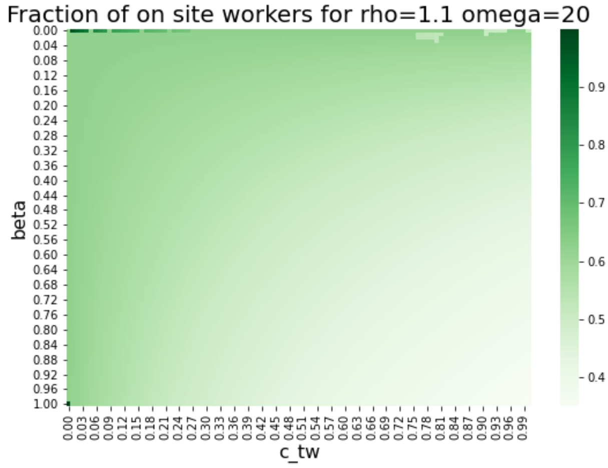


(b)

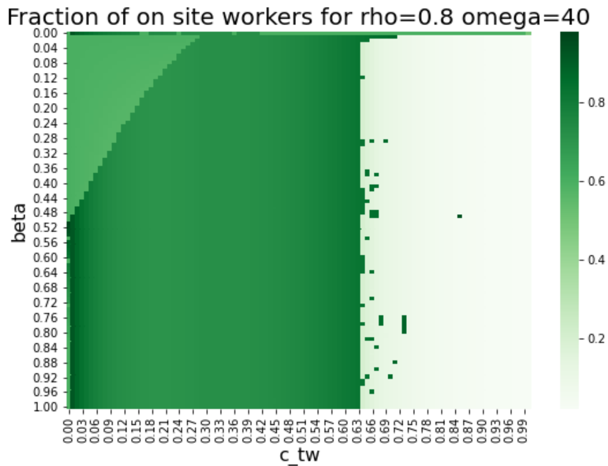


(c)

Figure 9: Fraction of on site labor supply for different specifications.



(a)



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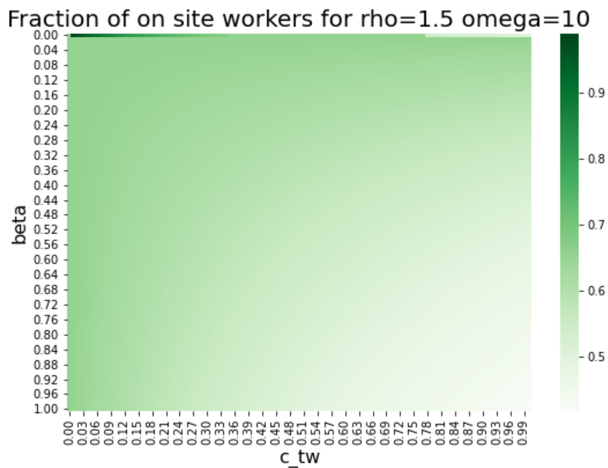
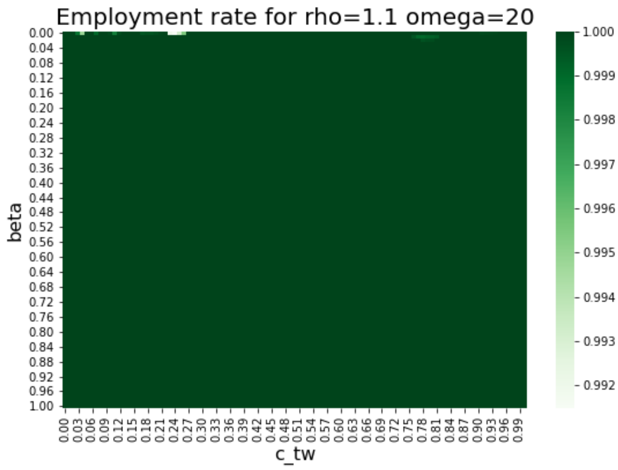
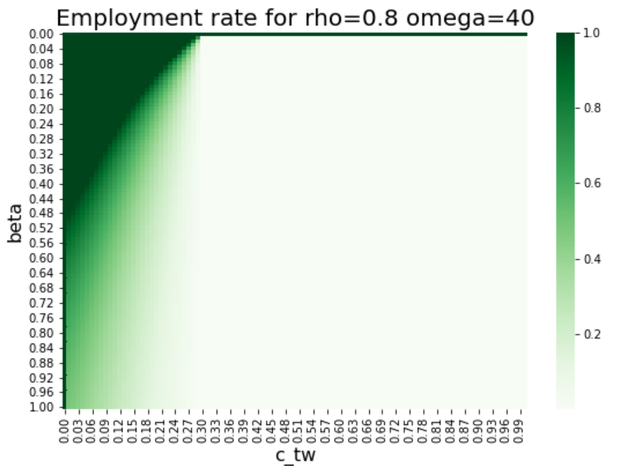


(c)

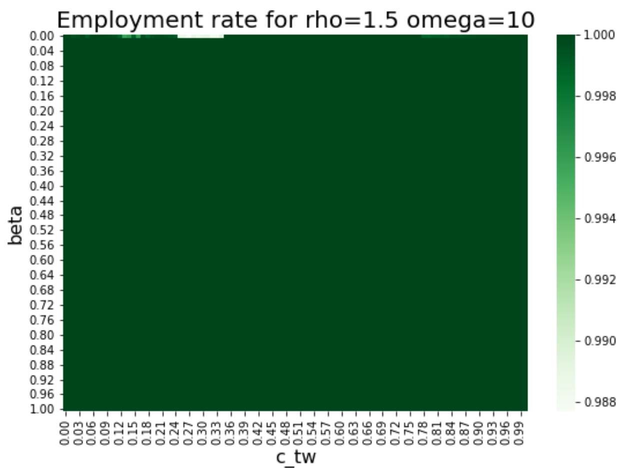
Figure 10: Total labor supply for different specifications.



(a)



(b)



(c)