MATHS 7027 Mathematical Foundations of Data Science

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Assignment 3 - Question 1

1(a)

The λ of $X \sim Pois(\lambda)$ is 5, which is the mathematical expectation of how many tickets are received per hour.

1(b)

The probability asked is $P(3 \leq X \leq 6)$, which equals $P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$.
According to Poisson distribution, we can get:

$$\begin{aligned}
P(3 \leq X \leq 6) &= P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) \\
&= \frac{e^{-5} \times 5^3}{3!} + \frac{e^{-5} \times 5^4}{4!} + \frac{e^{-5} \times 5^5}{5!} + \frac{e^{-5} \times 5^6}{6!} \\
&= \frac{e^{-5} \times 5^3}{3!} + \frac{e^{-5} \times 5^4}{4!} + \frac{e^{-5} \times 5^5}{5!} + \frac{e^{-5} \times 5^6}{6!} \\
&= \frac{125e^{-5}}{6} + \frac{625e^{-5}}{24} + \frac{3125e^{-5}}{120} + \frac{15625e^{-5}}{720} \\
&= \frac{13625}{144} e^{-5} \\
&\approx 0.6375
\end{aligned}$$

So the probability should be 0.6375.

1(c)

To find the minutes make the probability exceed $\frac{1}{4}$, we should first consider Y as the number of tickets recived in t minutes, and the new expectation should be $\lambda_t = \frac{t}{12}$.
We could get an expression to find t:

$$P(Y \geq 1) > \frac{1}{4} P(Y \geq 1) = 1 - P(Y = 0)$$

So we could get:

$$
\begin{aligned}
1 - P(Y = 0) &> \frac{1}{4} \\
1 - e^{-\lambda_t} &> \frac{1}{4} \\
e^{-\lambda_t} &< \frac{3}{4} \\
-\lambda_t &< \ln(\frac{3}{4}) \\
\lambda_t &> -\ln(\frac{3}{4})
\end{aligned}
$$

Since $\lambda_t = \frac{t}{12}$,

$$t > -12\ln(\frac{3}{4}) \approx 3.4522$$

So it would be approximately 3 minutes and 27 seconds to make the probability exceed $\frac{1}{4}$.

Assignment 3 - Question 2

2(a)

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n} P(X = x_i) \cdot x_i \\
&= 0.05 \times (-4) + 0.1 \times 2 + 0.15 \times -1 + 0.2 \times 0 + 0.5 \times 1 \\
&= -0.2 + -0.2 + -0.15 + 0 + 0.5 \\
&= -0.05
\end{aligned}
$$

2(b)

$$
\begin{aligned}
var(X) &= E[(X - E[X])^2] \\
&= \sum_{i=1}^{n} P(X = x_i) \cdot (x_i - E[X])^2 \\
&= 0.05 \times (-4 + 0.05)^2 + 0.1 \times (-2 + 0.05)^2 + 0.15 \times (-1 + 0.05)^2 + 0.2 \times (0 + 0.05)^2 + 0.5 \times (1 + 0.05 \\
&= 1.8475
\end{aligned}
$$

2(c)

$$
\begin{aligned}
var(X) &= E[X^2] - E[X]^2 \\
&= \sum_{i=1}^{n} P(X = x_i) \cdot {x_i}^2 - E[X]^2 \\
&= 0.05 \times 16 + 0.1 \times 4 + 0.15 \times 1 + 0.2 \times 0 + 0.5 \times 1 - 0.0025 \\
&= 1.8475
\end{aligned}
$$

Assignment 3 - Question 3

3(a)

Consider the event of the account is controlled by a bot is X_1, in the contrast the event of the account is controlled by a human is X_2. And consider the test is positive is Y.So according to the statement, we can learn:

$$P(Y|X_1) = 0.77P(Y|X_2) = 0.24P(X_1) = 0.68$$

We can calculate the probabilty of the account is controlled by a human is:

$$P(X_2) = 1 - P(X_1) = 0.32$$

As the total probability:

$$P(Y) = P(Y|X_1)P(X_1) + P(Y|X_2)P(X_2) = 0.77 \times 0.68 + 0.24 \times 0.32 = 0.6004$$

So the probability of a positive test result $P(Y)$ is 0.6004.

3(b)

Consider the probability of a negative result is $P(Y^*)$, which is:

$$P(Y^*) = 1 - P(Y) = 0.3996$$

The probability of an account is controlled by a human given that the test result is negative should be:

$$P(X_2|Y^*) = \frac{P(X_2)P(Y^*|X_2)}{P(Y^*)}$$

For the $P(Y^*|X_2)$:

$$P(Y^*|X_2) = 1 - P(Y|X_2) = 0.76$$

Now we can get:

$$\begin{aligned} P(X_2|Y^*) &= \frac{P(X_2)P(Y^*|X_2)}{P(Y^*)} \\ &= \frac{0.32 \times 0.76}{0.3996} \\ &\approx 0.6086 \end{aligned}$$

Assignment 3 - Question 4

$$FGH - H^TF^T = \begin{bmatrix} 0 & -1 & 1 \\ w & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 1 \\ 0 & x & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -2 & y \\ -1 & z \end{bmatrix} - \begin{bmatrix} 1 & -2 & -1 \\ 0 & y & z \end{bmatrix} \cdot \begin{bmatrix} 0 & w \\ -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \cdot 3 + (-1) \cdot 0 + (-1) \cdot 0 & 0 \cdot 0 + (-1) \cdot x + (-1) \cdot 0 & 0 \cdot 1 + (-1) \cdot 0 + 1 \cdot (-1) \\ w \cdot 3 + (-1) \cdot 0 + 0 \cdot 0 & w \cdot 0 + (-1) \cdot x + 0 \cdot 0 & w \cdot 1 + (-1) \cdot 0 + 0 \cdot (-1) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -2 & y \\ -1 & z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -x & -1 \\ 3w & -x & w \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -2 & y \\ -1 & z \end{bmatrix} - \begin{bmatrix} 1 & w+2 \\ -y+z & -y \end{bmatrix}$$

$$= \begin{bmatrix} 0 \cdot 1 + (-x) \cdot (-2) + (-1) \cdot (-1) & 0 \cdot 0 + (-x) \cdot y + (-1) \cdot z \\ 3w \cdot 1 + (-x) \cdot (-2) + w \cdot (-1) & 3w \cdot 0 + (-x) \cdot y + w \cdot z \end{bmatrix} - \begin{bmatrix} 1 & w+2 \\ -y+z & -y \end{bmatrix}$$

$$= \begin{bmatrix} 2x+1 & -xy-z \\ 2w+2x & wz-xy \end{bmatrix} - \begin{bmatrix} 1 & w+2 \\ -y+z & -y \end{bmatrix}$$

$$= \begin{bmatrix} 2x & -xy-z-w-2 \\ 2w+2x+y-z & wz-xy+y \end{bmatrix}$$