MATHS7027 Mathematical Foundations of Data Science Assignment 5

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Assignment 5 - Question 1

To find the eigenvalues and eigenspaces of A, we could consider an expression follow:

$$Ax = \lambda x, x \neq 0$$

According to the concept of characteristic equation, we could write:

$$\det(\lambda I - A) = 0$$

And we could get:

$$\begin{vmatrix} \lambda - 1 & 0 & 2 \\ 0 & \lambda - 3 & 0 \\ 2 & 0 & \lambda - 1 \end{vmatrix} = 0$$

Calculate the determinant:

$$\begin{aligned} \begin{vmatrix} \lambda - 1 & 0 & 2 \\ 0 & \lambda - 3 & 0 \\ 2 & 0 & \lambda - 1 \end{vmatrix} &= (\lambda - 1) \begin{vmatrix} \lambda - 3 & 0 \\ 0 & \lambda - 1 \end{vmatrix} - 0 + 2 \begin{vmatrix} 0 & \lambda - 3 \\ 2 & 0 \end{vmatrix} \\ &= (\lambda - 1)(\lambda - 3)(\lambda - 1) - 4(\lambda - 3) \\ &= (\lambda^2 - 2\lambda + 1 - 4)(\lambda - 3) \\ &= (\lambda^2 - 2\lambda - 3)(\lambda - 3) \\ &= (\lambda + 1)(\lambda - 3)(\lambda - 3) \end{aligned}$$

So we could find the eigenvalues of A is $\lambda = -1, 3, 3$
Next we solve the characteristic equations to find the corresponding eigenvectors:

Let $\lambda = -1$. Solve $(\lambda I - A)x = 0$ where $x = (x_1, x_2, x_3)$, and $\lambda = -1$.
Which is,

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{rrr|r} -2 & 0 & 2 & 0 \\ 0 & -4 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \quad \begin{array}{c} (R_3 = R_3 + R_1) \\ (R_1 = -\frac{1}{2}R_1) \\ (R_2 = -\frac{1}{4}R_2) \end{array}$$

$$\therefore x_1 = x_3 = t$$
$$x_2 = 0$$

The corresponding eigenvalue of $\lambda = -1$ is $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Let $\lambda = 3$. Solve $(\lambda I - A)x = 0$ where $x = (x_1, x_2, x_3)$, and $\lambda = 3$.
Which is,

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{rrr|r} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \quad \begin{array}{c} (R_3 = R_3 - R_1) \\ (R_1 = \frac{1}{2}R_1) \end{array}$$

$$\therefore x_1 = t$$
$$\Rightarrow x_3 = -t$$
$$x_2 = 0$$

The corresponding eigenvalue of $\lambda = 3$ is $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Assignment 5 - Question 2

2(a)

According to the concept of eigenvectors, we could get equations below

$$Bv_1 = \lambda_1 v_1$$
$$Bv_2 = \lambda_2 v_2$$

So we calculate Bv_1:

$$Bv_1 = \begin{bmatrix} 2 & a & -1 \\ 0 & 2 & b \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2-a \\ -2 \\ 0 \end{bmatrix}$$

Since $Bv_1 = \lambda_1 v_1$, so we have

$$\begin{bmatrix} 2-a \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ -\lambda_1 \\ 0 \end{bmatrix}$$

Then we calculate Bv_2

$$Bv_2 = \begin{bmatrix} 2 & a & -1 \\ 0 & 2 & b \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4-a \\ 2b-2 \\ 8 \end{bmatrix}$$

Since $Bv_2 = \lambda_2 v_2$, so we have

$$\begin{bmatrix} -4-a \\ 2b-2 \\ 8 \end{bmatrix} = \begin{bmatrix} -\lambda_2 \\ -\lambda_2 \\ 2\lambda_2 \end{bmatrix}$$

We could easily obtain that $\lambda_1 = 2, \lambda_2 = 4$, and substitute them into the equations,

$$\begin{aligned} 2-a &= 2 \\ -4-a &= -4 \Rightarrow a = 0, b = -1 \\ 2b-2 &= -4 \end{aligned}$$

So $a = 0, b = -1$

2(b)

The trace of B is the sum of its diagonal elements.
So we could get

$$\mathrm{tr}(B) = 2 + 2 + 3 = 7$$

Let λ_3 be the remaining eigenvalue, and the sum of the eigenvalues is equal to the trace.
So

$$\lambda_3 = 7 - 2 - 4 = 1$$

Assignment 5 - Question 3

3(a)

We could add the matrix I to the right of P, and then change it into the form$[I|P^{-1}]$:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Then we apply the row reduction to change the P to I:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 0 & 0 & 1 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \end{array}\right] \quad (\text{Swap } R_3 \text{ and } R_1)$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 & 0 & 0 \end{array}\right] \quad (R_2 = R_2 - R_1)$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 1 & -1 \\ 0 & -1 & -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \end{array}\right] \quad (R_3 = R_3 - \frac{1}{2}R_1)$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & -1 \end{array}\right] \quad (R_3 = R_3 + \frac{1}{2}R_2)$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right] \quad (R_3 = 2R_3)$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -4 & -1 & 3 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right] \quad (R_2 = R_2 - 2R_3)$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right] \quad (R_2 = \frac{1}{2}R_2)$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -2 & -1 & 3 \\ 0 & 1 & 0 & -2 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right] \quad (R_1 = R_1 - R_3)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 & -2 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right] \quad (R_1 = \frac{1}{2}R_1)$$

so P^{-1} should be

$$\begin{bmatrix} -1 & -\frac{1}{2} & \frac{3}{2} \\ -2 & -\frac{1}{2} & \frac{3}{2} \\ 2 & 1 & -2 \end{bmatrix}$$

3(b)

As C is diagonal, we could get:

$$C = PDP^{-1}$$

for any positive integer$n$, there is C^n:

$$C^n = (PDP^{-1})^n = PD^nP^{-1}$$

As $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix}$, so,

$$D^n = \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 0^n & 0 \\ 0 & 0 & c^n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^n \end{bmatrix}$$

So C^n should be

$$\begin{aligned} C^n = PD^nP^{-1} &= P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^n \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^n \end{bmatrix} \begin{bmatrix} -1 & -\frac{1}{2} & \frac{3}{2} \\ -2 & -\frac{1}{2} & \frac{3}{2} \\ 2 & 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 3c^n \\ 2 & 0 & c^n \end{bmatrix} \begin{bmatrix} -1 & -\frac{1}{2} & \frac{3}{2} \\ -2 & -\frac{1}{2} & \frac{3}{2} \\ 2 & 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -\frac{1}{2} & \frac{3}{2} \\ -2+6c^n & -1+3c^n & 3-6c^n \\ -2+2c^n & -1+c^n & 3-2c^n \end{bmatrix} \end{aligned}$$

Assignment 5 - Question 4

Consider the network comprising nodes A, B, C, and D, and represented by the matrix $M = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$.

4(a)

Enter M as an `array`.

In [ ]:
```
import numpy as np
M = np.array([
    [0, 0, 0, 0.5],
    [1, 0, 0.5, 0.5],
    [0, 0.5, 0, 0],
    [0, 0.5, 0.5, 0]
])
```

4(b)

Let $\boldsymbol{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \end{bmatrix}$. Use the PageRank algorithm to solve $M\boldsymbol{x} = \boldsymbol{x}$.

Hint: You may find Computer Exercise 5 helpful.

In [ ]:
```
results = np.linalg.eig(M)
eigenvector = results[1][:,0].real
eigenvector
```

Out[ ]:
```
array([0.26832816, 0.71554175, 0.35777088, 0.53665631])
```

4(c)

Hence, order the nodes A, B, C, and D from most important to least important.

In [ ]:
```
print('B,D,C,A')
```

```
B,D,C,A
```