MATHS7027 Mathematical Foundations of Data Science

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Assignment 4 - Question 1

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, to make $A^2 = \begin{bmatrix} 4 & -2 \\ 0 & 1 \end{bmatrix}$, we could write an expression below:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 0 & 1 \end{bmatrix}$$

And then we could get the relationships of a, b, c and d:

$$\begin{cases} a^2 + bc = 4 & (1) \\ ab + bd = -2 & (2) \\ ac + cd = 0 & (3) \\ bc + d^2 = 1 & (4) \end{cases}$$

from (3), we could learn that: $ac = -cd$, if $c \neq 0$, so $a = -d$.
So we can change (2) to $ab - ab = -2$ which is obviously incorrect.
Then $c = 0$, which means a and d could be any value based on (3).
But still with (1) and (4) we could obtain $a^2 = 4, d^2 = 1$, So a could be either 2 or -2 while d could be either 1 or -1. And calculate (2) with all the cases, we could get:

$$\begin{aligned} 2b + b = -2, a = 2, d = 1 \\ 2b - b = -2, a = 2, d = -1 \\ -2b + b = -2, a = -2, d = 1 \\ -2b - b = -2, a = -2, d = -1 \end{aligned}$$

Consider a vector of (a, d, b), which could be:

$$(2, 1, -\frac{2}{3}),\ (2, -1, -2),\ (-2, 1, 2)\ or\ (-2, -1, \frac{2}{3})$$

So the matrix A could be:

$$\begin{bmatrix} 2 & -\frac{2}{3} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix} or \begin{bmatrix} -2 & \frac{2}{3} \\ 0 & -1 \end{bmatrix}$$

Assignment 4 - Question 2

According to $W^{-1}(X + (YZ)^T)$ is defined, we could know W has the same row number to column number so it could be invertible. Assump that W is an $n \times n$ matrix.
Next we could learn that X has the same row number and column number to $(YZ)^T$, as far as we know that Y has 4 columns and Z has 5 columns, so Z should have 4 rows and 5 columns to make YZ is defined. Since X has 3 columns and could be added to YZ^T, so YZ^T should have 3 columns, which means Y has 3 rows and 4 columns, and YZ^T should be a 5×3 matrix.
So we can learn $X + (YZ)^T$ is a 5×3 matrix, and W should have the same columns to the rows of $X + (YZ)^T$ which means W is a 5×5 matrix.
To make conclusions:

$$\begin{aligned} W &: \ 5 \times 5 \\ X &: \ 5 \times 3 \\ Y &: \ 3 \times 4 \\ Z &: \ 4 \times 5 \end{aligned}$$

Assignment 4 - Question 3

3(a)
Consider $\delta = 4$, we could obtain that:

$$x_1 + 4x_2 = 10 \tag{5}$$
$$x_1 - 2x_2 = 4 \tag{6}$$

Doulbe the formula (6) and add it to (5) then we could get:

$$(1 + 2)x_1 = 18$$

which lead to $x_1 = 6$
then put it into (5), we could obtain$x_2 = 1$.
3(b)

To find the value of δ, we could change the system to the following form:

$$x_1 = -\delta x_2 + 10$$
$$x_1 = 2x_2 + \delta$$

If the coeffcient of both x_2 in the two equtions are equal, that means the lines of the two equtions are parallel which leads to the no solutions of this system. So the δ should be -2.

3(c)

Substract $x_2 = 3$ into the equtions,

$$x_1 + 3\delta = 10$$
$$x_1 - 6 = \delta$$

With $x_1 = \delta + 6$, the first eqution would be:

$$4\delta + 6 = 10$$

And we could easily get $x_1 = 7,\ \delta = 1$

Assignment 4 - Question 4

We could check the equation of each junction:

A:

$$x_1 = 10 + 7 + x_4$$

B:

$$10 + x_3 + 3 = x_2$$

C:

$$7 = x_3 + x_5$$

D:

$$2 + x_4 + x_5 = 3 + 8$$

Then we transform these equtions into $Ax = b$,

first:

$$x_1 - x_4 = 17$$
$$x_2 - x_3 = 13$$
$$x_3 + x_5 = 7$$
$$x_4 + x_5 = 9$$

so we can obtain A and b:

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 17 \\ 13 \\ 7 \\ 9 \end{bmatrix}$$

So the linear system should be like:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 17 \\ 13 \\ 7 \\ 9 \end{bmatrix}$$