SAMPLE Examination in the School of Computer and Mathematical Sciences

MATHS 7027	**SAMPLE EXAM** **Mathematical Foundations of Data Science**

Instructions:

- Attempt all questions.
- Start your answer to each question on a new page.
- Clearly label your answers by question and part.
- You must show working in your answers to every question. Full marks will not be awarded without supporting working.
- This is a closed book exam. You are only permitted access to the materials listed below.

Permitted Materials:

- A formula sheet, provided at the end of this exam paper.
- A scientific calculator or a basic/four function calculator. Calculators with graphics, remote communication, or CAS capabilities are NOT permitted.
- One hard copy of an English language translation dictionary.

Duration:

Writing Time: 150 minutes

This exam has 6 questions for a total of 60 marks.

Question:	1	2	3	4	5	6	Total
Marks	10	10	10	10	10	10	60

DO NOT OPEN THIS BOOKLET OR COMMENCE WRITING UNTIL INSTRUCTED TO DO SO. STOP WRITING IMMEDIATELY WHEN INSTRUCTED.

$\boxed{\text{3 marks}}$ 1. (a) Given $A = \{1, 4, \alpha, H\}$, $B = \{4, u, 2, -2\}$, and $C = \{u\}$, determine each of the following sets:

(i) $(A \cup B) \cap C$

Solution:
$(A \cup B) \cap C = \{u\}$

(ii) $B \cup C$

Solution:
$B \cup C = \{4, u, 2, -2\}$

(iii) $A \setminus B$

Solution:
$A \setminus B = \{1, \alpha, H\}$

$\boxed{\text{2 marks}}$ (b) Let $f(x) = \begin{cases} 0, & x < 0 \\ -1, & x \geq 0 \end{cases}$ and let $g(x) = \begin{cases} x+1, & x < 1 \\ x^2, & x \geq 1 \end{cases}$.
Evaluate the following:

(i) $f(g(0))$

Solution:
$f(g(0) = f(1) = -1$

(ii) $g(g(1/2))$

Solution:
$g(g(1/2)) = g(3/2) = 9/4$

$\boxed{\text{2 marks}}$ (c) Consider the function $h(x) = x^2$ for $x \in [-1, \infty)$. Explain why this function does not have an inverse, and give the largest subset of the domain on which the inverse may be defined.

Solution:
$h(x)$ is not 1-1, e.g., $h(-1/2) = h(1/2) = 1/4$.
Restricting the domain to $[0, \infty)$ makes $h(x)$ 1-1.

$\boxed{\text{3 marks}}$ (d) Show that

$$\sum_{i=2}^{n} (1-i)^2 = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$$

.

Solution:

$$\begin{aligned} \sum_{i=2}^{n} (1-i)^2 &= (-1)^2 + (-2)^i + \ldots + (1-n)^2 \\ &= 1^2 + 2^2 + \ldots + (n-1)^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1} i^2 \\
&= \frac{n-1}{6}(2(n-1)+1)((n-1)+1) \\
&= \frac{n-1}{6}(2n-1)n \\
&= \frac{n}{6}(2n^2-3n+1) \\
&= \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}
\end{aligned}
$$

Alternatively, expand $(1-i)^2$.

4 marks 2. (a) A binary sequence is an ordered sequence of ones and zeros.

(i) How many binary sequences are there of length 8?

Solution:
Each time, there are two choices (zero or one), and we choose 8 times $\Rightarrow 2^8 = 256$.

(ii) How many binary sequences of length 8 contain exactly 5 ones?

Solution:
Of the 8 total digits, we select 5 of them to be 1, which can be done in $\binom{8}{5} = \frac{8!}{5!3!} = 56$ ways.

(iii) What is the probability that a binary sequence of length 8 contains exactly 5 ones given that the first digit is a one.

Solution:
There are $2^7 = 128$ binary sequences of length 8 that start with a 1, so $P(\text{first is one}) = \frac{128}{256} = \frac{1}{2}$.
How many sequences start with a 1 and contain exactly 5 ones? 4 of the 7 remaining digits must be 1, and this can be done in $\binom{7}{4} = \frac{7!}{4!3!} = 35$ ways.
$P(5 \text{ ones}|\text{first is one}) = \frac{35}{128}$.

3 marks (b) Consider the following discrete probability distribution

x	1	2	3
$P(X = x)$	0.1	0.6	a

for some unknown a.

(i) What value of a makes X a valid discrete distribution?

Solution:
$1 = \sum_i P(X = x_i) = 0.1 + 0.6 + a \Rightarrow a = 0.3$.

(ii) What is the expected value of X?

Solution:

$$\begin{aligned} E[X] &= \sum_i x_i P(X = x_i) \\ &= 1 \times 0.1 + 2 \times 0.6 + 3 \times 0.3 \\ &= 2.2 \end{aligned}$$

(iii) What is the variance of X?

Solution:

$$E[X^2] = \sum_i x_i^2 P(X = x_i)$$

$$
\begin{aligned}
&= 1^2 \times 0.1 + 2^2 \times 0.6 + 3^2 \times 0.3 \\
&= 5.2 \\
\text{var}(x) &= E[X^2] - E[X]^2 \\
&= 5.2 - 2.2^2 \\
&= 0.36
\end{aligned}
$$

3 marks

(c) Central Adelaide comprises two electoral divisions, A and B. The population of division A is 107,000, and the population of division B is 105,000. At the 2019 Federal Election, the probability that a randomly selected voter voted for The Greens party was 0.16 in division A and 0.07 in division B.

(i) What is the probability that a randomly chosen voter in Central Adelaide is from division A?

Solution:
$P(A) = \frac{107{,}000}{107{,}000+105{,}000} = \frac{107}{212}$.

(ii) What is the probability that a randomly chosen voter in Central Adelaide voted for The Greens party? (Use the law of total probability to find the probability of selecting a Greens voter at random from central Adelaide.)

Solution:
$P(G) = P(G|A)P(A)+P(G|B)P(B) = 0.16\times\frac{107}{212}+0.07\times\frac{105}{212} \approx 0.1154$.

(iii) What is the probability that a randomly chosen voter in Central Adelaide is from division A given that they voted for The Greens party?

Solution:
$P(A|G) = \frac{P(G|A)P(A)}{P(G)} \approx \frac{0.16\times\frac{107}{212}}{0.1154} \approx 0.6996$.

5 marks 3. (a) Consider the following matrices

$$X = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -5 & 3 \end{bmatrix}$$

If possible, calculate each of the following. If not possible, explain why.

(i) $X + Y$

Solution:
Not possible. X is 2×2 and Y is 3×2.

(ii) XY

Solution:
Not possible. Cannot do $(2 \times \mathbf{2}) \times (\mathbf{3} \times 2)$.

(iii) YX

Solution:
$$YX = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ 0 & 0 \\ -38 & 19 \end{bmatrix}.$$

(iv) Y^T

Solution:
$$Y^T = \begin{bmatrix} 1 & 3 & -5 \\ -1 & 2 & 3 \end{bmatrix}$$

(v) $|X|$

Solution:
$|X| = 4 \times 3 - (-2 \times -6) = 12 - 12 = 0.$

3 marks (b) Suppose that Alice buys packets of chips, nuts, and dried fruit for a party. Each packet of chips costs \$3, each packet of nuts costs \$5, and each packet of dried fruit costs \$6.

In total, Alice spends \$74. She buys twice as many packets of chips as packets of nuts, and she buys one more packet of dried fruit than she buys packets of nuts.

Formulate a system of linear equations which could be used to determine the number of each type of snack that Alice buys.

You do NOT need to put the equations in matrix form or solve them, just give the equations with clear definitions for your variables.

Solution:
Let c be the number of packets of chips, n be the number of packets of

nuts, and f be the number of packets of dried fruit.
$3c + 5n + 6f = 74$
$c = 2n$, or $c - 2n = 0$
$f = n + 1$, or $-n + f = 1$

2 marks

(c) Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 \\ -1 & \alpha & \alpha \\ 1 & 5 & 0 \end{bmatrix}$$

Find the determinant of A, showing all working, and thus determine the value(s) of α for which A is invertible.

Solution:
Easiest using the top row, as it contains two zeros.
$|A| = 2 \times \begin{vmatrix} -1 & \alpha \\ 1 & 5 \end{vmatrix} = 2 \times (-5 - \alpha) = -10 - 2\alpha.$
A is invertible if $|A| \neq 0$, i.e., if $2\alpha \neq -10 \Rightarrow \alpha \neq -5$.

2 marks 4. (a) Suppose that a system of equations $A\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \boldsymbol{b}$ has a corresponding reduced row echelon form

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 4 \end{array}\right]$$

Find the set of solutions to this system of equations.

Solution:
Let $x_2 = s$, $x_4 = t$.
Then $x_1 - 2s + 3t = 0$ and $x_3 = 4$.
Thus, $[x_1, x_2, x_3, x_4] = [2s - 3t, s, 4, t]$, $s, t \in \mathbb{R}$.

6 marks (b) Consider the system of equations

$$\begin{aligned} x + 2y + 2z &= 1 \\ 2x + y - 2z &= 0 \\ 2x - 2y + z &= 1 \end{aligned}$$

(i) Write down the matrix A and vector $\boldsymbol{b}$ such that the above system of equations can be written in matrix form $A\boldsymbol{x} = \boldsymbol{b}$.

Solution:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(ii) After reducing the augmented system $[A|I]$ to row-echelon form, we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 6/9 & -3/9 & 0 \\ 0 & 0 & 1 & 2/9 & -2/9 & 1/9 \end{array}\right]$$

Perform further row operations to find A^{-1}. Please show details of all row operations you use.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 6/9 & -3/9 & 0 \\ 0 & 0 & 1 & 2/9 & -2/9 & 1/9 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 - 2R_3 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 5/9 & 4/9 & -2/9 \\ 0 & 1 & 0 & 2/9 & 1/9 & -2/9 \\ 0 & 0 & 1 & 2/9 & -2/9 & 1/9 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 2R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/9 & 2/9 & 2/9 \\ 0 & 1 & 0 & 2/9 & 1/9 & -2/9 \\ 0 & 0 & 1 & 2/9 & -2/9 & 1/9 \end{array}\right]$$

i.e., $A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

(iii) Hence, find the solution $\boldsymbol{x}$ to the equation in part (i).

Solution:

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \\ 1/3 \end{bmatrix}.$$

2 marks

(c) Suppose the set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly dependent. Prove that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is linearly dependent for any vector $\boldsymbol{v}_4$.

Solution:

As $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly dependent, $\exists c_1, c_2, c_3$ not all zero such that $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0}$.

Now, consider $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$. From above, we know $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + 0\boldsymbol{v}_4 = \mathbf{0}$ has non-trivial solutions, so the set must be linearly dependent.

4 marks

5. (a) Consider the matrix

$$A = \begin{bmatrix} -1 & 4 \\ 4 & 5 \end{bmatrix}$$

(i) Solve the characteristic equation to show that the eigenvalues of A are $\lambda = 7$ and $\lambda = -3$.

Solution:
$|\lambda I - A| = \begin{vmatrix} \lambda+1 & -4 \\ -4 & \lambda-5 \end{vmatrix} = (\lambda+1)(\lambda-5) - 16 = \lambda^2 - 4\lambda - 21 = (\lambda-7)(\lambda+3) = 0 \Rightarrow \lambda = 7, -3.$

(ii) Hence, find the eigenspaces of A.

Solution:
For $\lambda = 7$, $\lambda I - A = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow E_7 = \{t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid t \in \mathbb{R}\}$.
For $\lambda = -3$, $\lambda I - A = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow E_{-3} = \{t \begin{bmatrix} -2 \\ 1 \end{bmatrix} | t \in \mathbb{R}\}$.

2 marks

(b) Suppose that a 2×2 matrix has trace 1 and determinant $-4/9$. Determine its eigenvalues, showing all working.

Solution:
$\lambda_1 + \lambda_2 = 1, \quad \lambda_1\lambda_2 = -4/9.$
Then $\lambda_2 = -4/9\lambda_1$.
So $\lambda_1 - 4/9\lambda_1 = 1 \Rightarrow 9\lambda_1^2 - 9\lambda_1 - 4 = 0.$
Using the quadratic formula, $\lambda_1 = \frac{9 \pm \sqrt{81+144}}{18} = \ldots = 4/3$ or $-1/3$.
Thus, the eigenvalues are $4/3$, $-1/3$.

3 marks

(c) Suppose that a matrix B satisfies $B \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

(i) Give a matrix P which diagonalises B and the corresponding diagonal matrix D.

Solution:
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has eigenvalue 1. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has eigenvalue -1.
$P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$

(ii) Use this diagonalisation to prove that $B^{2k} = I$ for any positive integer k.

Solution:
$B = PDP^{-1}$, so $B^{2k} = PD^{2k}P^{-1}$.
$D^{2k} = \begin{bmatrix} 1^{2k} & 0 \\ 0 & (-1)^{2k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$
Then $B^{2k} = PIP^{-1} = PP^{-1} = I.$

1 mark

(d) Consider a principal component analysis of some dataset X. The dataset has covariance matrix

$$C = \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 3 \end{bmatrix}$$

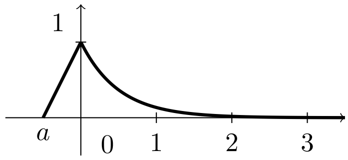
Show that the vector $\boldsymbol{v} = (-1, -1, 1)$ is a principal component of the dataset X.

Solution:

$$\begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-3-1 \\ 3-5-1 \\ 3-3+3 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix} = 3\boldsymbol{v}$$

$\Rightarrow$ $\boldsymbol{v}$ is an eigenvector of C, which is the definition of a principal component of X.

5 marks 6. (a) Consider the following graph of a probability density function. For x-values greater than zero, the function is defined by $f(x) = e^{-2x}$. For x-values less than zero, the function is a straight line that meets the x-axis at some point $a < 0$.



(i) Determine the value of a.

Solution: For $x > 0$, we can integrate
$\int_0^\infty e^{-2x}dx = \lim_{b\to\infty}\left[-\frac{e^{-2x}}{2}\right]_0^b = \lim_{b\to\infty}\left(\frac{1}{2} - \frac{e^{-2b}}{2}\right) = \frac{1}{2}$.
For $x < 0$, we thus require the area of the triangle to be $\frac{1}{2}$ in order for the total area to be 1.
Hence, $a = -1$.

(ii) Consider the continuous random variable X defined by this probability density function. What is the probability that X lies between 1 and 2?

Solution:

$$\begin{aligned} P(1 \le X \le 2) &= \int_1^2 e^{-2x}dx \\ &= \left[-\frac{e^{-2x}}{2}\right]_1^2 \\ &= \frac{e^{-4}}{2} + \frac{e^{-2}}{2} \\ &= \frac{1}{2}(e^{-2} - e^{-4}) \\ &\approx 0.0585 \end{aligned}$$

(iii) What is the probability that X is greater than 2?

Solution:

$$\begin{aligned} P(X > 2) &= \int_2^\infty e^{-2x}dx \\ &= \lim_{b\to\infty}\left[-\frac{e^{-2x}}{2}\right]_2^b \\ &= \lim_{b\to\infty}\left[-\frac{e^{-2b}}{2} + \frac{e^{-4}}{2}\right] \\ &= \frac{e^{-4}}{2} \end{aligned}$$

$$\approx 0.0092$$

3 marks

(b) Let $f(x) = x^{5/2}$.

(i) Find the degree 3 Taylor polynomial for $f(x)$ about the point $a = 1$, showing all working.

Solution:

$f(x) = x^{5/2}$ $\quad f(1) = 1$

$f'(x) = \frac{5}{2}x^{3/2}$ $\quad f'(1) = \frac{5}{2}$ $\quad$ 1]

$f''(x) = \frac{15}{4}x^{1/2}$ $\quad f''(1) = \frac{15}{4}$

$f'''(x) = \frac{15}{8}x^{-1/2}$ $\quad f'''(1) = \frac{15}{8}$

$P_3(x) = 1 + \frac{5}{2}(x-1) + \frac{15}{8}(x-1)^2 + \frac{15}{48}(x-1)^3.$

(ii) Use your Taylor polynomial to find an approximation for $2^{5/2}$.

Solution:

$$\begin{aligned} 2^{5/2} &\approx P_3(2) \\ &= 1 + \frac{5}{2} + \frac{15}{8} + \frac{15}{48} \\ &= \frac{48 + 120 + 90 + 15}{48} \\ &= \frac{273}{48} \\ &\approx 5.6875 \end{aligned}$$

2 marks

(c) Calculate the first two steps (i.e., m_1, m_2) of the gradient descent algorithm with an initial point $m_0 = 0$ and a learning rate of $\eta = 0.1$ for the function $f(x) = x^5 - x^3 + x^2 - x - 1$.

Solution:

$f'(x) = 5x^4 - 3x^2 + 2x - 1$

$h = -\eta f'(m_0) = -0.1f'(0) = 0.1 \Rightarrow m_1 = 0 + 0.1 = 0.1.$

Then $h = -0.1f'(m_1) = -0.1f'(0.1) = 0.08295 \Rightarrow m_2 = 0.18295.$

End of examination.