fourier transforms

Sinusoid $\Rightarrow$ $f(x) = \overset{\text{amplitude}}{A}\cdot\sin(\underbrace{2\pi u}_{\omega} x + \phi)$

- If you have periodic function, you can describe it with sum of sinusoids, lossless.

- Represents a signal $f(x)$ in terms of Amplitudes and phases of its Constituent of sinusoids.

$$f(x) \rightarrow \boxed{FT} \rightarrow F(x)$$

spatial representation — frequency representation

the other way is IFT

infinite sum of sine waves $\Rightarrow$ $A\sum_{k=1}^{\infty}\frac{1}{k}\cdot\sin(2\pi k x)$

discrete

FT $\rightarrow$ $F(k) = \frac{1}{N}\cdot\sum_{x=0}^{N-1} f(x)\cdot e^{-j2\pi kx/N}$

IFT $\rightarrow$ $f(x) = \sum_{x=0}^{N-1} F(k)\cdot e^{j2\pi kx/N}$

continuous

FT $\Rightarrow$ $F(k) = \int_{-\infty}^{\infty} f(x)\cdot e^{j2\pi kx}\,dx$

we represent frequencies both negative and positive

IFT $\Rightarrow$ $f(x) = \int_{-\infty}^{\infty} F(k)\cdot e^{j2\pi kx}\,du$

$x \Rightarrow$ space, $k \Rightarrow$ frequency, $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{j\theta} = \cos\theta + j\cdot\sin\theta$ $\rightarrow$ Euler's formula.

$$f(x) = \sum_{k=0}^{N-1} \underset{\text{scaling parameter}}{F(k)}\,\underbrace{\{\cos(2\pi kx) + j\cdot\sin(2\pi kx)\}}_{\text{wave components}}$$

DFT $\rightarrow$ discrete fourier transform

$$\boldsymbol{F} = \boldsymbol{W}\boldsymbol{f}$$

$$\begin{bmatrix} F(0)\\ F(1)\\ F(2)\\ F(3)\\ \vdots\\ F(N-1)\end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0\\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1}\\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2}\\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3}\\ \vdots & & & & \ddots & \vdots\\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1\end{bmatrix}\begin{bmatrix} f(0)\\ f(1)\\ f(2)\\ f(3)\\ \vdots\\ f(N-1)\end{bmatrix} \qquad W = e^{-j2\pi/N}$$

Frequency Domain Filtering

- $F\{g*h\} = F\{g\}\cdot F\{h\}$
- $F^{-1}\{gh\} = F^{-1}\{g\} * F^{-1}\{h\}$

Convolution for 1D continuous signals

Definition of linear shift-invariant filtering as convolution.

$$\underbrace{(f*g)(x)}_{\text{filtered signal}} = \int_{-\infty}^{\infty} \underbrace{f(y)}_{\text{filter}}\cdot \underbrace{g(x-y)}_{\text{input signal}}\cdot dy$$

- we can interpret and implement all kinds of linear shift-invariant filtering as multiplication in frequency domain.

Properties of Fourier Transform

Property	Spatial Domain	Frequency Domain
Linearity	$\alpha f_1(x) + \beta f_2(x)$	$\alpha F_1(u) + \beta F_2(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a\rvert}F\left(\frac{u}{a}\right)$
Shifting	$f(x-a)$	$e^{-i2\pi ua}F(u)$
Differentiation	$\frac{d^n}{dx^n}(f(x))$	$(i2\pi u)^n F(u)$