

# Discrete conformal maps and ideal hyperbolic polyhedra

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## Abstract

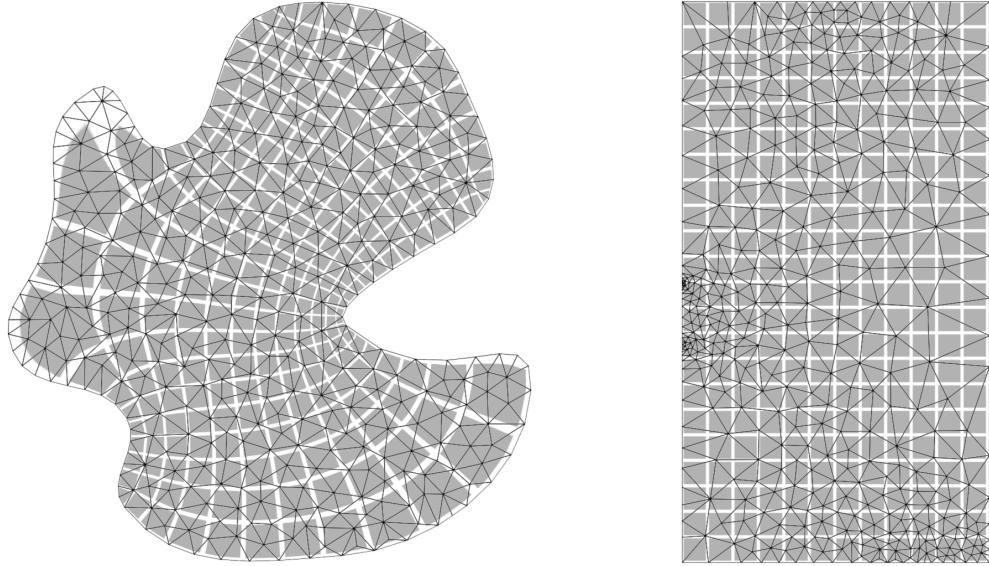
We establish a connection between two previously unrelated topics: a particular discrete version of conformal geometry for triangulated surfaces, and the geometry of ideal polyhedra in hyperbolic three-space. Two triangulated surfaces are considered discretely conformally equivalent if the edge lengths are related by scale factors associated with the vertices. This simple definition leads to a surprisingly rich theory featuring Möbius invariance, the definition of discrete conformal maps as circumcircle preserving piecewise projective maps, and two variational principles. We show how literally the same theory can be reinterpreted to addresses the problem of constructing an ideal hyperbolic polyhedron with prescribed intrinsic metric. This synthesis enables us to derive a companion theory of discrete conformal maps for hyperbolic triangulations. It also shows how the definitions of discrete conformality considered here are closely related to the established definition of discrete conformality in terms of circle packings.

## 1 Introduction

Recall that two Riemannian metrics  $g$  and  $\tilde{g}$  on a smooth manifold  $M$  are called conformally equivalent if

$$\tilde{g} = e^{2u} g \quad (1.1)$$

for a function  $u \in C^\infty(M)$ . In the discrete theory that we consider here, smooth manifolds are replaced with triangulated piecewise euclidean manifolds, and the discrete version of a conformal change of metric is to multiply all edge lengths with scale factors that are associated with the vertices (Definition 2.1.1). Apparently, this idea to model conformal transformations in a discrete setting by attaching scale factors to the vertices appeared first in the four-dimensional Lorentz-geometric context of the Regge calculus [32]. The Riemann-geometric version of this notion appeared in Luo's work on a discrete version of the Yamabe flow on surfaces [20]. He showed that this discrete Yamabe flow is the gradient flow of a locally convex function. Later, an explicit formula for this function was found ( $E_{T,\Theta,\lambda}$  defined in equation (3.3), with  $\Theta = 0$ ), and this lead to an efficient numerical method to compute discrete conformal maps, suitable for applications in computer graphics [34]. (Much of the basic theory of conformal equivalence and conformal maps in Section 2 and the first variational principle in Section 3 are already covered or at least touched upon in this earlier paper. Since everything else in the present paper relies on this material, we deemed it necessary to review it here, including more mathematical detail than is appropriate in a graphics paper.) The first variational principle described in Section 3.3 reduces the discrete conformal mapping problems (Section 3.1) to problems of convex optimization. Figures 1 and 2 show examples of discrete conformal maps that were obtained this way. Möbius transformations preserve the discrete conformal



*Figure 1.* Discrete conformal map to a rectangle.

class (Section 2.6), and this makes it possible to construct discrete conformal maps to regions bounded by circular polygons (Section 3.7), discrete analogs of the classical Riemann maps.

The first variational principle involves a function of the (logarithmic) scale factors  $u$ . The second variational principle (Section 3.6) involves a function of the triangle angles. Both functions are related by a singular version of Legendre duality. The corresponding variational principles of the classical smooth theory are discussed in Section 3.8.

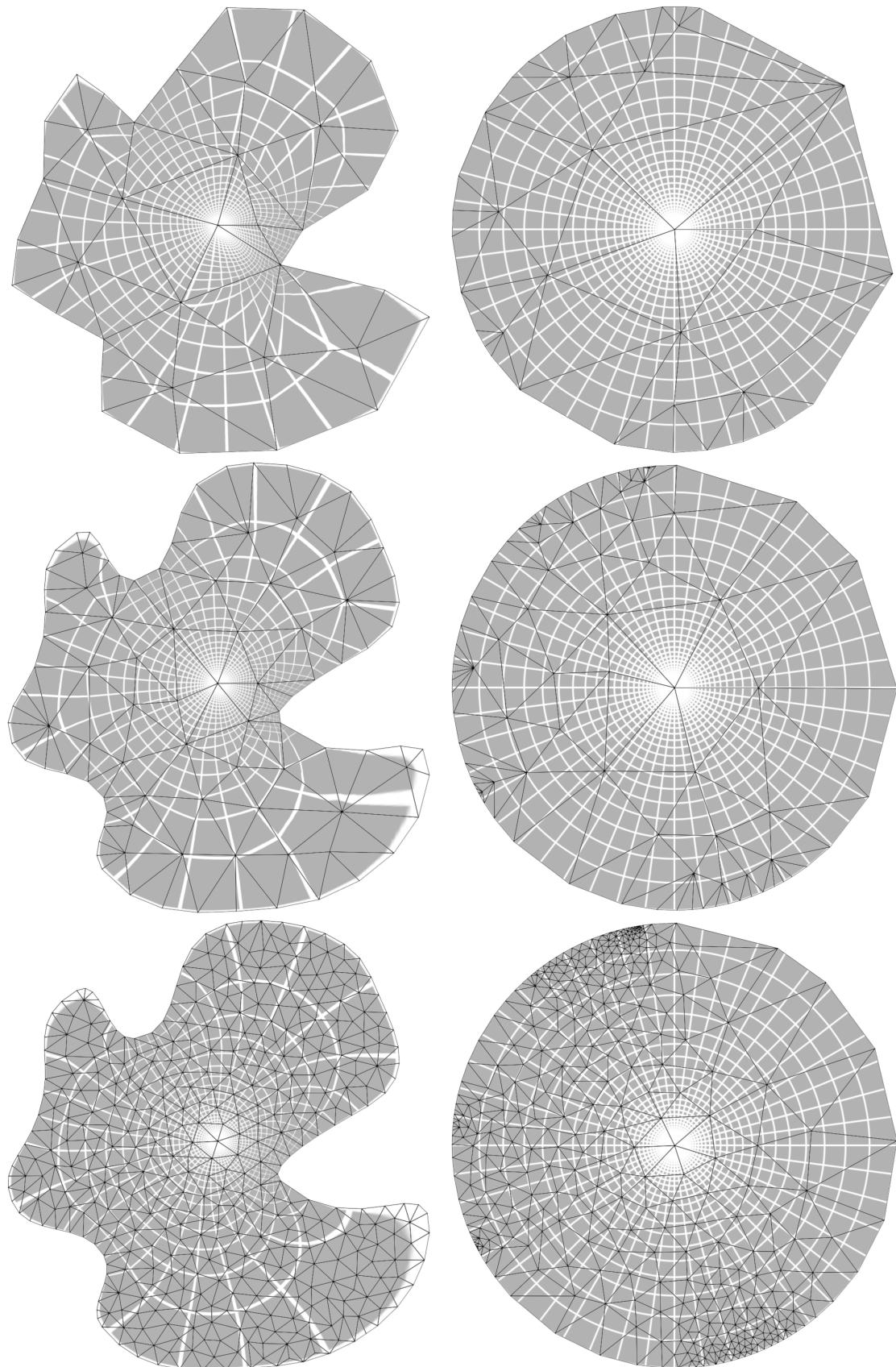
There are clear signs in Sections 2 and 3 that indicate a connection with hyperbolic geometry: the appearance of Milnor's Lobachevsky function  $J(x)$ , the fact that the second variational principle is almost the same as Rivin's variational principle for ideal hyperbolic polyhedra with prescribed dihedral angles [29], and the definition of discrete conformal maps in terms of circumcircle preserving piecewise projective functions (Section 2.7). This connection with two- and three-dimensional hyperbolic geometry is the topic of Section 4. Reversing a construction of Penner [27] [7], we equip a triangulated piecewise euclidean surface with a canonical hyperbolic metric with cusps. Discrete conformal maps are precisely the isometries with respect to this hyperbolic metric. The logarithmic edge lengths  $\lambda$  (Section 2.1) and the length-cross-ratios that characterize a discrete conformal class (Section 2.3) are Penner coordinates and shear coordinates, respectively, of the corresponding hyperbolic surface (Section 4.3). The problem of flattening a triangulation discretely conformally is equivalent to constructing an ideal hyperbolic polyhedron with prescribed intrinsic metric (Section 4.4). With this interpretation of discrete conformality in terms of three-dimensional hyperbolic geometry, the two variational principles of Section 3 are seen to derive from Schläfli's differential volume formula and Milnor's equation for the volume of an ideal tetrahedron (Section 4.5).

Once this connection between discrete conformality and hyperbolic polyhedra is established, it is straightforward to obtain a modified version of discrete conformality that pertains to triangulations composed of hyperbolic triangles instead of euclidean ones (Section 5). This is the theory of discrete conformal uniformization of triangulated higher genus surfaces

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*Figure 2.* Discrete Riemann maps.  
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over the hyperbolic plane that has been applied for the hyperbolization of euclidean ornaments [38]. (It is equally straightforward to obtain a corresponding theory for spherical triangulations, but the functions involved in the corresponding variational principles are not convex. We do not pursue this branch of the theory here.)

The connection with hyperbolic polyhedra entrains a connection between the discrete notion of conformality considered here and circle patterns, another discretization of the same concept. Thurston introduced patterns of circles as an elementary geometric visualization of hyperbolic polyhedra [36, Chapter 13]. He rediscovered Koebe’s circle packing theorem [17] and showed that it followed from Andreev’s work on hyperbolic polyhedra [1] [2]. Thurston’s conjecture that circle packings could be used to approximate the classical Riemann map, which was later proved by Rodin and Sullivan [31], set off a flurry of research that lead to a full-fledged theory of discrete analytic functions and conformal maps based on packings and patterns of circles [35]. (The circle packing version of Luo’s discrete Yamabe flow is the discrete Ricci flow of Chow and Luo [4].) The relationship between these two theories of discrete conformality is now clear: The circle packing theory deals with hyperbolic polyhedra with prescribed dihedral angles and the notion of discrete conformality considered here deals with hyperbolic polyhedra with prescribed metric. In Section 6.1 we discuss the relationship between the variational principles for discrete conformal maps (Section 3) and two variational principles for circle patterns. One is due to Rivin [29] (see also the recent survey article by Futer and Guéritaud [11], which provides a wealth of material that is otherwise difficult to find), and the other is again related to it by the same sort of singular Legendre duality.

In the last two sections, we discuss two extensions of discrete conformality that are motivated by the circle pattern connection discussed in Section 6.1. Instead of triangulations one can consider meshes composed of polygons that are inscribed in circles (Section 6.2), and we consider the problem to map multiply connected domains to domains bounded by polygons inscribed in circles, a discrete version of circle domains (Section 6.3).

Two important questions are not addressed in this paper. The first is the question of convergence. Of course we do believe that (under not too restrictive assumptions that have yet to be worked out) discrete conformal maps approximate conformal maps if the triangulation is fine enough. Figure 2 clearly suggests that a version of the Rodin–Sullivan theorem [31] also holds in this case. But all this has yet to be proved.

The other question concerns the solvability of the discrete conformal mapping problems of Section 3.1. A solution may not exist due to violated triangle inequalities. Fairly obvious necessary conditions and how they relate to properties of the function  $E_{T,\Theta,\lambda}$  appearing in the first variational principle are discussed in Section 3.5. In the numerous numerical experiments that we have made, we have observed that a solution exists if the necessary conditions are satisfied, no triangles are almost degenerate to begin with, and the triangulation is not too coarse. But to find necessary and sufficient conditions for solvability seems to be an intractable problem in this setting. After all, this would amount to giving necessary and sufficient conditions for the existence of a (not necessarily convex) ideal hyperbolic polyhedron with prescribed intrinsic metric and prescribed combinatorial type. The way out is to restrict oneself to convex polyhedra while widening the concept of discrete conformal map to allow for combinatorial changes (Section 4.1). Rivin proved that any hyperbolic metric with cusps on the sphere is realized by a unique ideal polyhedron [30]. This translates into an existence statement for discrete conformal maps. We will discuss the issue of solvability in greater detail in a future paper [33], in which we will present a constructive proof for Rivin’s realization theorem that is based on the methods presented here. (The approach is very similar to the recent construc-

tive proof of Alexandrov's classical polyhedral realization theorem [3].) Rivin considered the non-constructiveness of his proof its greatest drawback [30]. Although the theorem has in the meantime been greatly generalized [8] [9], all proofs are non-constructive.

## 2 Discrete conformal equivalence and discrete conformal maps

### 2.1 Discrete conformal equivalence

A *surface* is a connected 2-dimensional manifold, possibly with boundary. A *surface triangulation*, or *triangulation* for short, is a surface that is a CW complex whose faces (2-cells) are triangles which are glued edge-to-edge. We will denote the sets of vertices (0-cells), edges (1-cells), and faces of a triangulation  $T$  by  $V_T$ ,  $E_T$ , and  $T_T$ , and we will often drop the subscript  $T$  if the triangulation is clear from the context. We will also write  $A_T$  for the set of triangle angles, where angles means corners, or triangle-vertex incidences, not angle measures.

A *euclidean surface triangulation*, or *euclidean triangulation* for short, is a surface triangulation equipped with a metric so that  $T \setminus V_T$  is locally isometric to the euclidean plane, or half-plane if there is boundary, and the edges are geodesic segments. In other words, a euclidean surface triangulation is a surface consisting of euclidean triangles that are glued edge-to-edge. At the vertices, the metric may have cone-like singularities.

A euclidean triangulation is uniquely determined by a triangulation  $T$  and a function  $\ell : E_T \rightarrow \mathbb{R}_{>0}$  assigning a length to every edge in such a way that the triangle inequalities are satisfied for every triangle in  $T_T$ . We call such a positive function  $\ell$  on the edges that satisfies all triangle inequalities a *discrete metric* on  $T$ , and we denote the resulting euclidean triangulation by  $(T, \ell)$ .

In this paper, we will assume for simplicity that the triangulations are simplicial complexes. This means that a triangle may not be glued to itself at a vertex or along an edge, and the intersection of two triangles is either empty or it consists of one vertex or one edge. This restriction to simplicial complexes allows us to use simple notation: we will denote by  $ij$  the edge with vertices  $i$  and  $j$ , by  $ijk$  the triangle with vertices  $i$ ,  $j$ , and  $k$ , and by  ${}^i_{jk}$  the corner at vertex  $i$  in triangle  $ijk$ . If  $f, g, h$ , and  $\phi$  are functions on  $V$ ,  $E$ ,  $T$ , and  $A$ , respectively, we will write  $f_i$ ,  $g_{ij}$ ,  $h_{ijk}$ , and  $\phi_{jk}^i$  for  $f(i)$ ,  $g(ij)$ ,  $h(ijk)$ , and  $\phi({}^i_{jk})$ . But while this restriction to simplicial complexes is notationally very convenient, it is *a priori* uncalled for. There are a few exceptions, like Section 3.7 on mapping to the sphere and disk, but in general the domain of validity of the theory presented here extends beyond the simplicial case. (This will be essential in the sequel [33].)

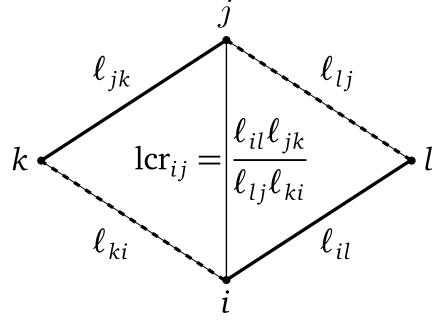
**2.1.1 Definition** (Luo [20]). Two combinatorially equivalent euclidean triangulations,  $(T, \ell)$  and  $(T, \tilde{\ell})$ , are *discretely conformally equivalent* if the discrete metrics  $\ell$  and  $\tilde{\ell}$  are related by

$$\tilde{\ell}_{ij} = e^{\frac{1}{2}(u_i + u_j)} \ell_{ij} \quad (2.1)$$

for some function  $u : V \rightarrow \mathbb{R}$ . This defines an equivalence relation on the set of discrete metrics on  $T$ , of which an equivalence classes is called a *discrete conformal class* of discrete metrics, or a *discrete conformal structure* on  $T$ .

Instead of the edge lengths  $\ell$  we will often use the logarithmic lengths

$$\lambda = 2 \log \ell. \quad (2.2)$$



**Figure 3.** The length-cross-ratio on edge  $ij$ . The lengths of the bold solid and bold dashed edges appear in the numerator and denominator, respectively.

(The reason for the factor of 2 will become apparent in Section 4.) In terms of these logarithmic lengths, relation (2.1) between  $\ell$  and  $\tilde{\ell}$  becomes linear:

$$\tilde{\lambda}_{ij} = \lambda_{ij} + u_i + u_j. \quad (2.3)$$

## 2.2 The two most simple cases

(1) If the triangulation  $T$  consists of a single triangle  $ijk$ , then any two euclidean triangulations  $(T, \ell)$  and  $(T, \tilde{\ell})$  are discretely conformally equivalent, because the three equations

$$\tilde{\ell}_{ij} = e^{\frac{1}{2}(u_i+u_j)}\ell_{ij}, \quad \tilde{\ell}_{jk} = e^{\frac{1}{2}(u_j+u_k)}\ell_{jk}, \quad \tilde{\ell}_{ki} = e^{\frac{1}{2}(u_k+u_i)}\ell_{ki}$$

always have a unique solution for  $u_i$ ,  $u_j$  and  $u_k$ :

$$e^{u_i} = \frac{\tilde{\ell}_{ij}\ell_{jk}\tilde{\ell}_{ki}}{\ell_{ij}\tilde{\ell}_{jk}\ell_{ki}}, \quad \dots$$

(2) Now let  $T$  be the triangulation consisting of two triangles  $ijk$  and  $ilj$  glued along edge  $ij$  as shown in Figure 3, and let  $\ell$  and  $\tilde{\ell}$  be two discrete metrics on  $T$ . What is the condition for  $(T, \ell)$  and  $(T, \tilde{\ell})$  to be discretely conformally equivalent? For each triangle considered separately, the corresponding equations (2.1) determine unique solutions for the values of  $u$  on its vertices. For each of the common vertices  $i$  and  $j$  one obtains two values and the necessary and sufficient condition for discrete conformal equivalence is that they are equal, which is equivalent to the condition

$$\frac{\ell_{il}\ell_{jk}}{\ell_{lj}\ell_{ki}} = \frac{\tilde{\ell}_{il}\tilde{\ell}_{jk}}{\tilde{\ell}_{lj}\tilde{\ell}_{ki}}.$$

## 2.3 Length-cross-ratios

The simple reasoning of Section 2.2 extends to the general case: Let  $T$  be any triangulation, and let  $\ell$  and  $\tilde{\ell}$  be two discrete metrics on  $T$ . For each triangle  $ijk$  of  $T$ , considered separately, equations (2.1) determine unique values for  $u$ . Thus, for each vertex  $i \in V$ , one obtains one value for  $u_i$  per adjacent triangle. These values are in general different. They agree for each vertex if and only if the discrete metrics  $\ell$  and  $\tilde{\ell}$  are discretely conformally equivalent. Since

the vertex links are connected, it suffices to consider values obtained from adjacent triangles. This leads to Proposition 2.3.2 below, where the condition for discrete conformal equivalence is given in terms of the so-called length-cross-ratios:

**2.3.1 Definition.** For each interior edge  $ij$  between triangles  $ijk$  and  $ilj$  as in Figure 3, define the *length-cross-ratio* induced by  $\ell$  to be

$$\text{lcr}_{ij} = \frac{\ell_{il}\ell_{jk}}{\ell_{lj}\ell_{ki}}. \quad (2.4)$$

This definition implicitly assumes that an orientation of the triangulated surface has been chosen. The other choice of orientation leads to reciprocal values for the length-cross-ratios. (For non-orientable surfaces, the length-cross-ratios are well defined on the interior edges of the oriented double cover.)

If the quadrilateral  $iljk$  is embedded in  $\mathbb{C}$ , then the length-cross-ratio  $\text{lcr}_{ij}$  is just the absolute value of the complex cross ratio of the vertex positions  $z_i, z_l, z_j, z_k$ ,

$$\text{cr}(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

Discretely conformally equivalent metrics  $\ell, \tilde{\ell}$  induce the same length-cross-ratios, because the scale factors  $e^{u/2}$  cancel. By the reasoning above, the converse is also true.

**2.3.2 Proposition.** Two euclidean triangulations  $(T, \ell)$  and  $(T, \tilde{\ell})$  are discretely conformally equivalent if and only if for each interior edge  $ij \in E_T$ , the induced length-cross-ratios are equal:  $\text{lcr}_{ij} = \widetilde{\text{lcr}}_{ij}$ .

## 2.4 The product of length-cross-ratios around a vertex

Let us denote the sets of interior edges and interior vertices by  $E_{int}$  and  $V_{int}$ , respectively. Which functions  $E_{int} \rightarrow \mathbb{R}_{>0}$  can arise as length-cross-ratios? A necessary condition is that the product of length-cross-ratios on the edges around an interior vertex is 1, because all lengths  $\ell$  cancel.

$$\text{For all } i \in V_{int} : \prod_{j:ij \in E} \text{lcr}_{ij} = 1. \quad (2.5)$$

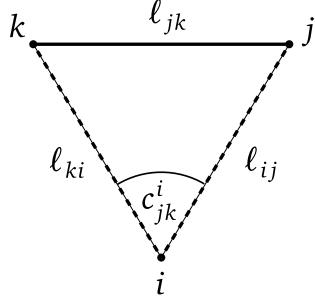
If we ignore the triangle inequalities, this condition is also sufficient:

**2.4.1 Proposition.** Let  $\text{lcr} : E_{int} \rightarrow \mathbb{R}_{>0}$  be any positive function on the set of interior edges. There exists a positive function  $\ell : E \rightarrow \mathbb{R}_{>0}$  on the set of edges satisfying (2.4) for every interior edge  $ij$ , if and only if condition (2.5) holds.

*Proof.* It remains to show that if  $\text{lcr} \in (\mathbb{R}_{>0})^{E_{int}}$  satisfies condition (2.5), then the system of equations (2.4) has a solution. In fact, we will explicitly construct such a solution. To this end, we introduce auxiliary parameters  $c$ , which are defined on the set of angles  $A$  of the triangulation: Given  $\ell \in (\mathbb{R}_{>0})^E$ , define  $c \in (\mathbb{R}_{>0})^A$  by

$$c_{jk}^i = \frac{\ell_{jk}}{\ell_{ij}\ell_{ki}}, \quad (2.6)$$

see Figure 4. In terms of these parameters, the length-cross-ratios induced by  $\ell$  are



**Figure 4.** The parameters  $c^i_{jk}$ , defined on the set of triangle angles  $A$ .

$$\text{lcr}_{ij} = \frac{c^i_{jk}}{c^i_{lj}}, \quad (2.7)$$

where  $l, j, k$  occur in the link of  $i$  in this cyclic order, as in Figure 3. (For a geometric interpretation of the parameters  $c^i_{jk}$  in terms of hyperbolic geometry, see Section 4.2.)

Now suppose  $\text{lcr} \in (\mathbb{R}_{>0})^{E_{\text{int}}}$  satisfies condition (2.5). Then it is easy to find a solution  $c \in (\mathbb{R}_{>0})^A$  of equations (2.7), because each equation involves only two values of  $c$  on consecutive angles at the same vertex. So one can freely choose one  $c$ -value per vertex and successively calculate the values on neighboring angles around the same vertex by multiplying (or dividing) with the values of  $\text{lcr}$  on the edges in between.

Next, solve equations (2.6) for  $\ell$ , where  $c$  is the solution to equations (2.7) just constructed. This is also easy: The length of an edge  $ij$  is determined by the values of  $c$  on the two adjacent angles on either side,

$$\ell_{ij} = (c^i_{jk} c^j_{ki})^{-\frac{1}{2}}.$$

(Check that the two  $c$ -values on the other side give the same value.) Thus we have constructed a function  $\ell \in (\mathbb{R}_{>0})^E$  satisfying equations (2.4) for the given function  $\text{lcr} \in (\mathbb{R}_{>0})^{E_{\text{int}}}$ .  $\square$

## 2.5 Some spaces and dimension counts

The set of discrete metrics on a triangulation  $T$  is the open subset of  $\mathbb{R}^E$  containing all  $\ell \in \mathbb{R}^E$  that satisfy  $\ell > 0$  and the strict triangle inequalities. Since it is not empty (always containing the constant function  $1_E$ ) and the inequalities are linear, it is the interior of an  $|E|$ -dimensional convex polytope. The discrete conformal structures on  $T$  are submanifolds which form a fibration of the polytope of discrete metrics. If  $\ell$  is a discrete metric, its conformal class is parametrized by  $u \in \mathbb{R}^V$  via equation (2.1). The space of discrete conformal classes has every right to be called the *discrete Teichmüller space* of the triangulation  $T$ . It is parameterized by the length-cross-ratios. But these are excessive parameters, for they are constrained by the product conditions (2.5), see Section 2.4.

The logarithmic lengths  $\lambda$  defined by equation (2.2) are coordinates on the space of discrete metrics. The space of discrete metrics corresponds to the parameter domain  $\Lambda \subset \mathbb{R}^E$  of all  $\lambda \in \mathbb{R}^E$  satisfying

$$\begin{aligned} -e^{\lambda_{ij}/2} + e^{\lambda_{jk}/2} + e^{\lambda_{ki}/2} &> 0 \\ e^{\lambda_{ij}/2} - e^{\lambda_{jk}/2} + e^{\lambda_{ki}/2} &> 0 \\ e^{\lambda_{ij}/2} + e^{\lambda_{jk}/2} - e^{\lambda_{ki}/2} &> 0 \end{aligned} \quad (2.8)$$

for all triangles  $ijk \in T$ . While the domain  $\Lambda$  is not a polytope any more, not even a convex set, the fibration by conformal classes has become simpler. For  $\lambda \in \mathbb{R}^E$  let  $U_\lambda$  be the affine subspace of all  $\tilde{\lambda}$  parameterized by  $u \in \mathbb{R}^E$  as in equation (2.3). If  $\lambda \in \Lambda$ , its discrete conformal class is  $U_\lambda \cap \Lambda$ . If  $\lambda \notin \Lambda$ , the intersection may be empty; if it is not empty, it is also a discrete conformal class. The fibration of the space of discrete metrics by conformal classes corresponds to a fibration of the domain  $\Lambda$  by the family  $\{U_\lambda\}_{\lambda \in \mathbb{R}^E}$  of parallel affine subspaces, and the discrete Teichmüller space corresponds to an open set in the factor space  $\mathbb{R}^E/U_0$ . Thus, we have:

**2.5.1 Proposition.** *The dimension of each discrete conformal class is  $|V_T|$ , and the dimension of the discrete Teichmüller space is  $|E_T| - |V_T|$ .*

**Example.** Consider the case where  $T$  is a triangulation of the closed orientable surface of genus  $g$ . Using  $2 - 2g = |T| - |E| + |V|$  and  $3|T| = 2|E|$ , one obtains  $|E| - |V| = 6g - 6 + 2|V|$ . This is also the dimension of  $\mathcal{T}_{g,|V|}$ , the Teichmüller space of a genus  $g$  Riemann surfaces with  $|V|$  punctures. It is no coincidence that the dimensions agree. The discrete conformal classes actually correspond to points in the Teichmüller space  $\mathcal{T}_{g,|V|}$  (see Section 4.1).

The subspaces  $U_\lambda$  were described above in parameter form. An equivalent description in equation form is also available. If  $\text{lcr} : \mathbb{R}^{E_{int}} \rightarrow \mathbb{R}_{>0}$  is the length-cross-ratio function induced by  $\ell = e^{\lambda/2}$ , then  $U_\lambda$  is the space of all  $\tilde{\lambda} \in \mathbb{R}^E$  satisfying

$$\tilde{\lambda}_{il} - \tilde{\lambda}_{lj} + \tilde{\lambda}_{jk} - \tilde{\lambda}_{ki} = 2 \log \text{lcr}_{ij} \quad (2.9)$$

for all  $ij \in E_{int}$ , where  $k$  and  $l$  are the opposite vertices as in Figure 3. The logarithmic length-cross-ratios  $\log \text{lcr}$  are coordinates for the discrete Teichmüller space; but they are excessive coordinates because they are constrained by the condition (2.5) that their sum around each interior vertex is zero. From this one obtains again  $|E| - |V|$  (#variables – #constraints) for the dimension of the discrete Teichmüller space.

## 2.6 Möbius invariance of discrete conformal structures

The group of Möbius transformations of  $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$  is the group generated by inversions in spheres. (Planes are considered spheres through  $\infty$ .) The group of Möbius transformations is also generated by the similarity transformations (which fix  $\infty$ ), and inversion in the unit sphere. Möbius transformations are conformal, and a famous theorem of Liouville says that for  $n > 2$ , any conformal map of a domain  $U \subset \mathbb{R}^n$  is the restriction of a Möbius transformation.

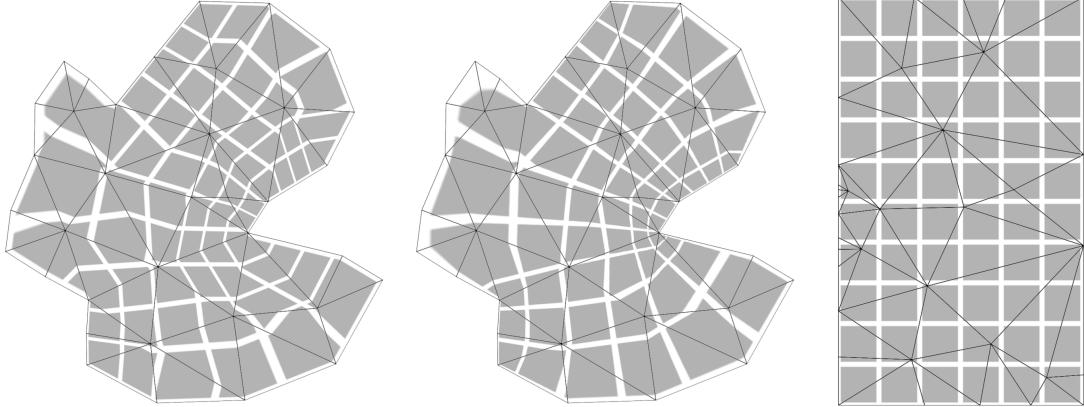
Let  $T$  be a triangulation and let  $\|\cdot\|$  denote the euclidean norm on  $\mathbb{R}^n$ ,  $n \geq 2$ . Suppose  $v : V_T \rightarrow \mathbb{R}^n$  maps the vertices of each triangle to three affinely independent points. Then  $v$  induces a discrete metric  $\ell_{ij} = \|v_i - v_j\|$ . Two maps  $v, \tilde{v} : V \rightarrow \mathbb{R}^n \subset \widehat{\mathbb{R}^n}$  are related by a Möbius transformation if there is a Möbius transformation  $T$  such that  $\tilde{v} = T \circ v$ .

**2.6.1 Proposition.** *If two maps  $V_T \rightarrow \mathbb{R}^n$  are related by a Möbius transformation, then the induced discrete metrics are discretely conformally equivalent.*

*Proof.* The claim is obvious if the relating Möbius transformation is a similarity transformation. For inversion in the unit sphere,  $x \mapsto \frac{1}{\|x\|^2} x$ , it follows from the identity

$$\left\| \frac{1}{\|p\|^2} p - \frac{1}{\|q\|^2} q \right\| = \frac{1}{\|p\| \|q\|} \|p - q\|.$$

□



**Figure 5.** A coarsely triangulated domain in the plane (middle) is mapped to a rectangle (right) by a discrete conformal map (see Definition 2.7.1). Instead of using circumcircle preserving piecewise projective interpolation, one can also interpolate linearly in each triangle. The result (left) looks noticeably “less smooth”.

**2.6.2 Remark.** For  $n = 2$  there is an obvious alternative argument involving the complex cross ratio. One can extend this argument to  $n > 2$ . The only difficulty is to define a complex cross-ratio for four points in  $\mathbb{R}^n$  if  $n > 2$ , such that it is invariant under Möbius transformations. Such a cross-ratio can be defined up to complex conjugation by identifying a 2-sphere through the four points conformally with the extended complex plane  $\hat{\mathbb{C}}$ . This involves several choices: a choice of 2-sphere if the four points are cocircular, a choice of orientation of the 2-sphere, and choice of orientation preserving conformal map to  $\hat{\mathbb{C}}$ . Only the choice of orientation makes a difference, the two choices leading to conjugate values for the cross-ratio. The length-cross-ratio is the absolute value of this complex cross-ratio, so the ambiguity with respect to complex conjugation does not matter.

## 2.7 Discrete conformal maps

So far we have only talked about discrete conformal equivalence. This section deals with the matching notion of discrete conformal maps (see Figure 5).

For any two euclidean triangles (with labeled vertices to indicate which vertices should be mapped to which), there is a unique projective map that maps one triangle onto the other and the circumcircle of one onto the circumcircle of the other (see Lemma 2.7.3 below). Let us call this map the *circumcircle preserving projective map* between the two triangles.

**2.7.1 Definition.** A *discrete conformal map* from one euclidean triangulation  $(T, \ell)$  to a combinatorially equivalent euclidean triangulation  $(\tilde{T}, \tilde{\ell})$  is a homomorphism whose restriction to every triangle is the circumcircle preserving projective map onto the corresponding image triangle.

Consider two combinatorially equivalent euclidean triangulations,  $(T, \ell)$  and  $(\tilde{T}, \tilde{\ell})$ . For each individual triangle of  $(T, \ell)$ , there is a circumcircle preserving projective map to the corresponding triangle of  $(\tilde{T}, \tilde{\ell})$ . But these maps do in general not fit together continuously across edges. However, they do fit together, forming a discrete conformal map, precisely if the euclidean triangulations are discretely conformally equivalent:

**2.7.2 Theorem.** *The following two statements are equivalent:*

- (i)  $(T, \ell)$  and  $(\tilde{T}, \tilde{\ell})$  are discretely conformally equivalent.
- (ii) There exists a discrete conformal map  $(T, \ell) \rightarrow (\tilde{T}, \tilde{\ell})$ .

The rest of this section is concerned with the proof of Theorem 2.7.2. It follows easily from Lemma 2.7.3 below, which provides an analytic description of the circumcircle preserving projective map between two individual triangles.

Consider two triangles  $\Delta$  and  $\tilde{\Delta}$  in the euclidean plane, and let  $(x_i, y_i)$  and  $(\tilde{x}_i, \tilde{y}_i)$ ,  $i \in \{1, 2, 3\}$ , be the coordinates of their vertices in a Cartesian coordinate system. Let  $\ell_{ij}$  and  $\tilde{\ell}_{ij}$  be the side lengths,

$$\ell_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2,$$

and similarly for  $\tilde{\ell}_{ij}$ . Consider the euclidean plane as embedded in the projective plane  $\mathbb{RP}^2$  and let  $v_i = (x_i, y_i, 1)$  and  $\tilde{v}_i = (\tilde{x}_i, \tilde{y}_i, 1)$  be the homogeneous coordinate vectors of the vertices, normalized so that the last coordinate is 1. Then the projective maps  $f : \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$  that map  $\Delta$  to  $\tilde{\Delta}$  correspond via  $f([v]) = [F(v)]$  to the linear maps  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of homogeneous coordinates that satisfy

$$F(v_i) = \mu_i \tilde{v}_i \quad (2.10)$$

for some “weights”  $\mu_i \in \mathbb{R} \setminus \{0\}$ .

**2.7.3 Lemma.** *The projective map  $f : [v] \mapsto [F(v)]$  maps the circumcircle of  $\Delta$  to the circumcircle of  $\tilde{\Delta}$  if and only if*

$$(\mu_1, \mu_2, \mu_3) = \mu(e^{-u_1}, e^{-u_2}, e^{-u_3}) \quad (2.11)$$

where  $u_1, u_2, u_3$  are the logarithmic scale factors satisfying the three equations (2.1) for a single triangle and  $\mu \in \mathbb{R} \setminus \{0\}$  is an arbitrary factor.

*Proof of Lemma 2.7.3.* The circumcircle of  $\Delta$  is

$$\{[v] \in \mathbb{RP}^2 \mid q(v) = 0\},$$

where  $q$  is the quadratic form

$$q(x, y, z) = x^2 + y^2 + 2axz + 2byz + cz^2$$

with  $a, b, c \in \mathbb{R}$  uniquely determined by the condition that

$$q(v_1) = q(v_2) = q(v_3) = 0. \quad (2.12)$$

In the same fashion, let the quadratic form describing the circumcircle of  $\tilde{\Delta}$  be

$$\tilde{q}(x, y, z) = x^2 + y^2 + 2\tilde{a}xz + 2\tilde{b}yz + \tilde{c}z^2$$

so that

$$\tilde{q}(\tilde{v}_1) = \tilde{q}(\tilde{v}_2) = \tilde{q}(\tilde{v}_3) = 0. \quad (2.13)$$

We will also denote by  $q$  and  $\tilde{q}$  the corresponding symmetric bilinear forms:

$$q(v) = q(v, v) \quad \text{and} \quad \tilde{q}(v) = \tilde{q}(v, v).$$

The projective map  $f$  maps circumcircle to circumcircle if and only if  $q$  and the pull-back  $F^*\tilde{q}$  are linearly dependent. That is, if and only if

$$\mu^2 q(v, w) = \tilde{q}(F(v), F(w))$$

for all  $v, w \in \mathbb{R}^3$  for some  $\mu \in \mathbb{R}$ . Since  $v_1, v_2, v_3$  is a basis of  $\mathbb{R}^3$  and because of equations (2.12) and (2.13), this is the case if and only if

$$\mu^2 q(v_i, v_j) = \mu_i \mu_j \tilde{q}(\tilde{v}_i, \tilde{v}_j) \quad (2.14)$$

for  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ . Now note that

$$\ell_{ij}^2 = q(v_i - v_j, v_i - v_j) = -2q(v_i, v_j),$$

and similarly  $\tilde{\ell}_{ij}^2 = -2\tilde{q}(\tilde{v}_i, \tilde{v}_j)$ . So condition (2.14) is equivalent to

$$\mu^2 \ell_{ij} = \mu_i \mu_j \tilde{\ell}_{ij}.$$

Solve equations (2.1) for  $u_i$  to obtain (2.11). This completes the proof of Lemma 2.7.3.  $\square$

To prove Theorem 2.7.2, consider two euclidean triangulations  $(T, \ell)$  and  $(T, \tilde{\ell})$ , and a pair of adjacent triangles  $ijk$  and  $jil$  of  $T$ . Embed the corresponding euclidean triangles of  $(T, \ell)$  simultaneously isometrically in the euclidean plane, and do the same for the corresponding two euclidean triangles of  $(T, \tilde{\ell})$ . Lemma 2.7.3 tells us what the circumcircle preserving projective maps are, and we might as well choose  $\mu = 1$  in both cases. These two maps fit together continuously along edge  $ij$  if and only if the values of  $\mu_i = e^{-u_i}$  and  $\mu_j = e^{-u_j}$  from one triangle are proportional to those of the other triangle. Since the value of  $e^{(u_i+u_j)/2} = \tilde{\ell}_{ij}/\ell_{ij}$  is the same for both triangles, this is the case if and only if the values of  $\mu_i$  and  $\mu_j$ , hence also those of  $u_i$  and  $u_j$ , coincide for both triangles. This holds for all interior edges if and only if  $(T, \ell)$  and  $(T, \tilde{\ell})$  are discretely conformally equivalent. This completes the proof of Theorem 2.7.2.

### 3 Two variational principles

#### 3.1 Discrete conformal mapping problems

Consider the following type of discrete conformal mapping problem:

##### 3.1.1 Problem. Given

- a surface triangulation  $T$ ,
- a discrete conformal class  $\mathcal{C}$  of discrete metrics on  $T$ ,
- a desired angle sum  $\Theta_i$  for each vertex  $i \in V_T$ ,

**find** a discrete metric  $\tilde{\ell}$  in the conformal class  $\mathcal{C}$  such that the euclidean triangulation  $(T, \tilde{\ell})$  has angle sum  $\Theta_i$  around each vertex  $i \in V_T$ .

In particular, if the given desired angle sum  $\Theta_i$  equals  $2\pi$  for every interior vertex  $i$ , then Problem 3.1.1 asks for a *flat* euclidean triangulation in the given conformal class which has prescribed angles at the boundary. A flat and simply connected euclidean triangulation can be developed in the plane by laying out one triangle after the other. Thus, Problem 3.1.1 comprises as a special case the following problem.

### 3.1.2 Problem. *Given*

- A euclidean triangulation  $(T, \ell)$  that is topologically a disc and
- a desired angle sum  $\Theta_i$  for each boundary vertex  $i$ ,

*find* a discretely conformally equivalent planar triangulation  $(T, \tilde{\ell})$  with the given angle sums at the boundary. (Self-overlap is allowed.)

We also consider a more general type of problem than Problem 3.1.1. Suppose the discrete conformal class  $\mathcal{C}$  is given in the form of a representative metric  $\ell \in (\mathbb{R}_{>0})^E$ . For some vertices  $i$  we may prescribe the (logarithmic) scale factor  $u_i$  instead of the angle sum  $\Theta_i$ :

### 3.1.3 Problem. *Given*

- a triangulation  $T$ ,
- a function  $\ell \in (\mathbb{R}_{>0})^E$  representing a conformal class,
- a partition  $V = V_0 \dot{\cup} V_1$  of the vertex set,
- a prescribed logarithmic scale factor  $u_i \in \mathbb{R}$  for each vertex  $i \in V_0$
- a prescribed angle sum  $\Theta_i$  for each vertex  $i \in V_1$

*find* logarithmic scale factors  $u_i \in \mathbb{R}$  for the remaining vertices  $i \in V_1$  so that  $\tilde{\ell}$  determined by equations (2.1) is a discrete metric and  $(T, \tilde{\ell})$  has the prescribed angle sum  $\Theta_i$  around each vertex  $i \in V_1$ .

For  $V_0 = \emptyset$ ,  $V_1 = V$ , this is just Problem 3.1.1. (If the conformal class  $\mathcal{C}$  is given in the form of length-cross-ratios  $\text{lcr} \in (\mathbb{R}_{>0})^{E_{\text{int}}}$ , one can obtain a representative  $\ell \in (\mathbb{R}_{>0})^E$  using the method described in the constructive proof of Proposition 2.4.1.) Note that any instance of Problem 3.1.3 can be reduced to the special case where  $u_i = 0$  is prescribed for  $i \in V_0$ : Simply apply first a discrete conformal change of metric (2.1) with the arbitrary prescribed  $u_i$  for  $i \in V_0$ .

In Section 3.7 we will see how one can also find discrete conformal maps to the sphere and to the disk (that is, solve the discrete Riemann mapping problem) if one can solve Problem 3.1.3.

**3.1.4 Remark.** An important special case of Problem 3.1.3 is the following: prescribe the angle sum  $\Theta_i = 2\pi$  for interior vertices, and  $u_i$  on the boundary. This is analogous to the following boundary value problem of the smooth theory: Given a smooth 2-manifold with boundary  $M$  equipped with a Riemannian metric  $g$ , find a conformally equivalent flat Riemannian metric  $e^{2u}g$  with prescribed  $u|_{\partial M}$ . Suppose we measure the relative distortion of a conformally equivalent Riemannian metric by the Dirichlet energy of  $u$ ,  $D(u) = \frac{1}{2} \int_M du \wedge *du$ . Then the conformally equivalent flat Riemannian metrics with least distortion are those with  $u|_{\partial M} = \text{const}$ . Thus, up to scale there is a unique least distortion solution obtained by setting  $u|_{\partial M} = 0$ . [34, Appendix E]

**3.1.5 Theorem.** *If Problem 3.1.3 (with  $V_0 \neq \emptyset$ ) has a solution, it is unique. If Problem 3.1.1 has a solution, it is unique up to scale. In any case, if a solution exists, it can be found by minimizing a convex function.*

*Proof.* This follows from Propositions 3.3.3, 3.3.4, and 3.3.5. □

## 3.2 Analytic description

Analytically, these discrete conformal mapping problems amount to solving systems of nonlinear equations. Suppose  $\ell \in \mathcal{C}$  is a representative of the given discrete conformal class, and

consider  $\tilde{\ell}$  as function of  $u$  defined by equations (2.1). Then solving the general Problem 3.1.3 amounts to solving for the unknown  $u_i$  ( $i \in V_1$ ) a system of angle-sum equations

$$\sum_{jk:ijk \in T} \tilde{\alpha}_{jk}^i = \Theta_i, \quad (3.1)$$

with one equation for each vertex  $i \in V_1$ . Here,  $\tilde{\alpha}_{jk}^i$  is the angle at  $i$  in triangle  $ijk$  of  $(T, \tilde{\ell})$ . The angles  $\tilde{\alpha}$  are nonlinear functions of the new lengths  $\tilde{\ell}$ . They can be obtained by invoking, for example, the cosine rule or the half-angle formula

$$\tan\left(\frac{\alpha_{jk}^i}{2}\right) = \sqrt{\frac{(-\ell_{ij} + \ell_{jk} + \ell_{ki})(\ell_{ij} + \ell_{jk} - \ell_{ki})}{(\ell_{ij} - \ell_{jk} + \ell_{ki})(\ell_{ij} + \ell_{jk} + \ell_{ki})}}. \quad (3.2)$$

(Tilde marks over  $\alpha$  and  $\ell$  have been omitted in this equation to avoid visual clutter.)

### 3.3 The first variational principle

The system of nonlinear equations described in the previous section turns out to be variational. Solutions of the conformal mapping problems correspond to the critical points of the function  $E_{T,\Theta,\lambda}$  defined as follows. The precise statement of this first variational principle is Proposition 3.3.3.

Let  $T$  be a surface triangulation,  $\Theta \in \mathbb{R}^V$ , and  $\lambda \in \mathbb{R}^E$ . For now (we will later extend the domain of definition to  $\mathbb{R}^V$ ) define the real valued function  $E_{T,\Theta,\lambda}(u)$  on the open subset of  $\mathbb{R}^V$  containing all  $u$  such that  $\tilde{\ell}$  determined by (2.1) is a discrete metric (that is, satisfies the triangle inequalities):

$$E_{T,\Theta,\lambda}(u) = \sum_{ijk \in T} \left( \tilde{\alpha}_{ij}^k \tilde{\lambda}_{ij} + \tilde{\alpha}_{jk}^i \tilde{\lambda}_{jk} + \tilde{\alpha}_{ki}^j \tilde{\lambda}_{ki} + 2\Lambda(\tilde{\alpha}_{ij}^k) + 2\Lambda(\tilde{\alpha}_{jk}^i) + 2\Lambda(\tilde{\alpha}_{ki}^j) \right. \\ \left. - \frac{\pi}{2} (\tilde{\lambda}_{ij} + \tilde{\lambda}_{jk} + \tilde{\lambda}_{ki}) \right) + \sum_{i \in V} \Theta_i u_i. \quad (3.3)$$

The first sum is taken over all triangles,  $\tilde{\alpha}_{jk}^i$  denotes the angle at vertex  $i$  in triangle  $ijk$  with side lengths  $\tilde{\ell} = e^{\tilde{\lambda}/2}$ ,  $\tilde{\lambda}$  is the function of  $u$  defined by equations (2.3), and  $\Lambda(x)$  is Milnor's Lobachevsky function,

$$\Lambda(x) = - \int_0^x \log |2 \sin(t)| dt. \quad (3.4)$$

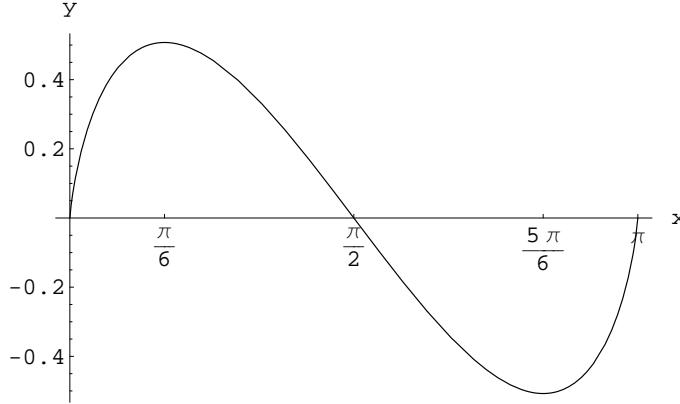
(Figure 6 shows a graph of this function.) The second sum is taken over all vertices. It is linear in  $u$ .

**3.3.1 Remark.** The notation  $\Lambda(x)$ , using a letter from the Cyrillic alphabet, and the name "Lobachevsky function" are due to Milnor [22] [23]. Lobachevsky used a slightly different function which is also known as the Lobachevsky function and often denoted  $L(x)$ . To distinguish these two functions, we call  $\Lambda(x)$  Milnor's Lobachevsky function. It is almost the same as Clausen's integral [19],  $\text{Cl}_2(x) = \frac{1}{2}\Lambda(2x)$ .

**3.3.2 Proposition** (First derivative). *The partial derivative of  $E_{T,\Theta,\lambda}$  with respect to  $u_i$  is*

$$\frac{\partial}{\partial u_i} E_{T,\Theta,\lambda} = \Theta_i - \sum_{jk:ijk \in T} \tilde{\alpha}_{jk}^i,$$

where the sum is taken over all angles around vertex  $i$ .



**Figure 6.** Graph of Milnor's Lobachevsky function,  $y = JI(x)$ . The function is  $\pi$ -periodic, odd, and smooth except at  $x \in \pi\mathbb{Z}$ , where its tangents are vertical.

*Proof.* This follows from equation (3.7) and Proposition 3.4.1.  $\square$

**3.3.3 Proposition** (First variational principle). *Let  $\mathcal{C}$  be a discrete conformal class with representative metric  $\ell = e^{\lambda/2}$ , and let  $\tilde{\ell} = e^{\tilde{\lambda}/2}$  where  $\tilde{\lambda}$  is the function of  $u$  defined by equations 2.3. Then*

- $\tilde{\ell}$  solves Problem 3.1.1 if and only if  $u$  is a critical point of  $E_{T,\Theta,\lambda}$ ,
- $\tilde{\ell}$  solves Problem 3.1.3 if and only if  $u$  is a critical point of  $E_{T,\Theta,\lambda}$  with fixed  $u_i$  for  $i \in V_0$ . (In this case, the values of  $\Theta$  for  $i \in V_0$  are irrelevant.)

*Proof.* This follows immediately from Proposition 3.3.2 because  $\frac{\partial}{\partial u_i} E_{T,\Theta,\lambda} = 0$  is equivalent to the angle sum condition (3.1).  $\square$

**3.3.4 Proposition** (Local convexity). *The function  $E_{T,\Theta,\lambda}$  is locally convex, that is, its second derivative  $\sum \frac{\partial^2 E_{T,\Theta,\lambda}}{\partial u_i \partial u_j} du_i du_j$  is a positive semidefinite quadratic form. The kernel is 1-dimensional and consists of the constants in  $\mathbb{R}^V$ .*

*Proof.* This follows from equation (3.7) and Proposition 3.4.3.  $\square$

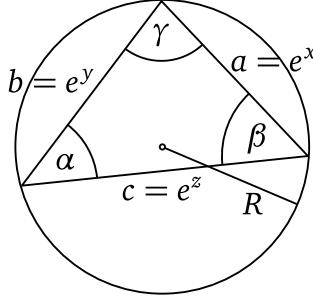
**3.3.5 Proposition** (Extension). *The function  $E_{T,\Theta,\lambda}$  can be extended to a convex continuously differentiable function on  $\mathbb{R}^V$ .*

*Proof.* This follows from equation (3.7) and Proposition 3.4.4.  $\square$

In fact, one has an explicit formula for the second derivative of  $E_{T,\Theta,\lambda}$ . This is helpful from the practical point of view, because it allows one to use more powerful algorithms to minimize  $E_{T,\Theta,\lambda}$  and thus solve the discrete conformal mapping problems. It is also interesting from the theoretical point of view, because the second derivative of  $E_{T,\Theta,\lambda}$  at  $u$  is the well known finite-element approximation of the Dirichlet energy (the cotan-formula) for a triangulation with edge lengths  $\tilde{\ell}$  [6] [28]:

**3.3.6 Proposition** (Second derivative). *The second derivative of  $E_{T,\Theta,\lambda}$  at  $u$  is*

$$\sum_{i,j \in V} \frac{\partial^2 E_{T,\Theta,\lambda}}{\partial u_i \partial u_j} du_i du_j = \frac{1}{2} \sum_{ij \in E} w_{ij}(u) (du_i - du_j)^2,$$



**Figure 7.** Triangle with sides  $a = e^x$ ,  $b = e^y$ ,  $c = e^z$  and angles  $\alpha$ ,  $\beta$ ,  $\gamma$ . The radius of the circumcircle is  $R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$ .

where  $w_{ij}(u) = \frac{1}{2}(\cot \tilde{\alpha}_{ij}^k + \cot \tilde{\alpha}_{ij}^l)$  if  $ij$  is an interior edge with opposite vertices  $k$  and  $l$  and  $w_{ij}(u) = \frac{1}{2}\cot \tilde{\alpha}_{ij}^k$  if  $ij$  is a boundary edge with opposite vertex  $k$ . This assumes all triangle inequalities are satisfied. If triangle inequalities are violated, the cotangent terms for the corresponding triangles have to be replaced with 0.

*Proof.* This follows from equation (3.7) and Proposition 3.4.2.  $\square$

### 3.4 A peculiar triangle function

Consider the function

$$f(x, y, z) = \alpha x + \beta y + \gamma z + \Pi(\alpha) + \Pi(\beta) + \Pi(\gamma), \quad (3.5)$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are the angles in a euclidean triangle with sides  $a = e^x$ ,  $b = e^y$ , and  $c = e^z$  as shown in Figure 7. Such a triangle exists if and only if the triangle inequalities are satisfied. So  $f$  is (for now) only defined on the set

$$\begin{aligned} \mathcal{A} = \{(x, y, z) \in \mathbb{R}^3 \mid & -e^x + e^y + e^z > 0, \\ & e^x - e^y + e^z > 0, \\ & e^x + e^y - e^z > 0\}. \end{aligned} \quad (3.6)$$

The function  $f(x, y, z)$  is the fundamental building block of  $E_{T, \Theta, \lambda}(u)$  since

$$E_{T, \Theta, \lambda}(u) = \sum_{ijk \in T} \left( 2f\left(\frac{\tilde{\lambda}_{ij}}{2}, \frac{\tilde{\lambda}_{jk}}{2}, \frac{\tilde{\lambda}_{ki}}{2}\right) - \frac{\pi}{2}(\tilde{\lambda}_{ij}, \tilde{\lambda}_{jk}, \tilde{\lambda}_{ki}) \right) + \sum_{i \in V} \Theta_i u_i, \quad (3.7)$$

and Propositions 3.3.2, 3.3.4, 3.3.5, and 3.3.6 follow from corresponding statements regarding  $f(x, y, z)$ .

**3.4.1 Proposition** (First derivative). *The partial derivatives of  $f$  are*

$$\frac{\partial f}{\partial x} = \alpha, \quad \frac{\partial f}{\partial y} = \beta, \quad \frac{\partial f}{\partial z} = \gamma.$$

*Proof.* Using  $\Pi'(x) = -\log|2\sin(x)|$  we obtain from (3.5) that

$$\frac{\partial f}{\partial x} = \alpha + (x - \log(2\sin\alpha)) \frac{\partial \alpha}{\partial x} + (y - \log(2\sin\beta)) \frac{\partial \beta}{\partial x} + (z - \log(2\sin\gamma)) \frac{\partial \gamma}{\partial x}.$$

Since

$$x - \log(2 \sin \alpha) = y - \log(2 \sin \beta) = z - \log(2 \sin \gamma) = \log R,$$

where  $R$  is the radius of the circumcircle, and since

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial x} + \frac{\partial \gamma}{\partial x} = 0$$

(because  $\alpha + \beta + \gamma = \pi$ ), this implies  $\frac{\partial f}{\partial x} = \alpha$ .  $\square$

**3.4.2 Proposition** (Second derivative). *The second derivative of  $f$  is*

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}^T \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \cot \alpha (dy - dz)^2 + \cot \beta (dz - dx)^2 + \cot \gamma (dx - dy)^2 \quad (3.8)$$

*Proof.* By Proposition 3.4.1,

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}^T \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = (d\alpha \ d\beta \ d\gamma),$$

so the left-hand side of equation (3.8) equals

$$d\alpha dx + d\beta dy + d\gamma dz.$$

We will show that

$$d\alpha = \cot \gamma (dx - dy) + \cot \beta (dx - dz). \quad (3.9)$$

This and the analogous equations for  $d\beta$  and  $d\gamma$  imply

$$d\alpha dx + d\beta dy + d\gamma dz = \cot \alpha (dy - dz)^2 + \cot \beta (dz - dx)^2 + \cot \gamma (dx - dy)^2,$$

and hence equation (3.8).

To derive equation (3.9), differentiate the cosine rule

$$2bc \cos \alpha = b^2 + c^2 - a^2$$

to get

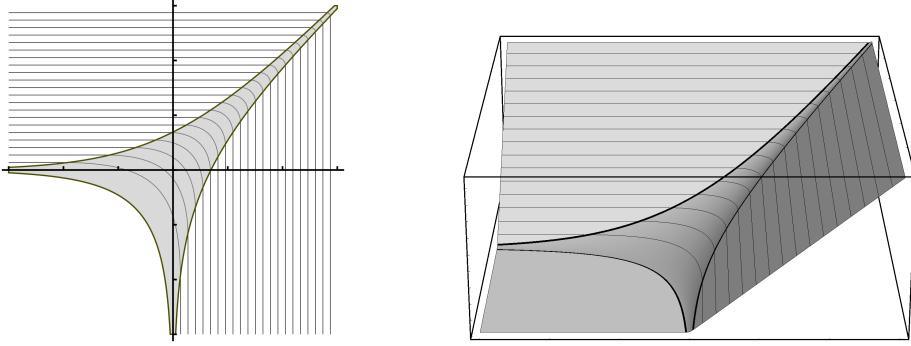
$$-2bc \sin \alpha d\alpha + 2bc \cos \alpha (dy + dz) = 2b^2 dy + 2c^2 dz - 2a^2 dx.$$

Apply the cosine rule three more times to get

$$\begin{aligned} 2bc \sin \alpha d\alpha &= (b^2 - c^2 + a^2)(dx - dy) + (-b^2 + c^2 + a^2)(dx - dz) \\ &= 2ab \cos \gamma (dx - dy) + 2ac \cos \beta (dx - dz). \end{aligned}$$

Divide through by  $2bc \sin \alpha$  and apply the sine rule to obtain equation (3.9).  $\square$

**3.4.3 Proposition** (Local convexity). *The function  $f$  is locally convex, that is, the second derivative (3.8) is a positive semidefinite quadratic form. Its kernel is one-dimensional and spanned by  $(1, 1, 1) \in \mathbb{R}^3$ .*



**Figure 8.** Left: Contour plot of  $(x, y) \mapsto f(x, y, 0)$ . The intersection of the domain  $\mathcal{A}$  (see equation (3.6)) with the  $xy$ -plane is shaded. Right: Graph of the same function, also showing contour lines.

*Proof.* Writing  $(dy - dz)$  as  $((dy - dx) - (dx - dz))$  we obtain

$$\begin{aligned} & \cot \alpha(dy - dz)^2 + \cot \beta(dz - dx)^2 + \cot \gamma(dx - dy)^2 \\ &= (\cot \alpha + \cot \beta)(dx - dz)^2 + (\cot \alpha + \cot \gamma)(dx - dy)^2 \\ &\quad - 2 \cot \alpha(dx - dy)(dx - dz) \end{aligned}$$

Thus, in terms of  $(dx - dz)$  and  $(dx - dy)$ , the matrix of this quadratic form is

$$M = \begin{pmatrix} \cot \alpha + \cot \beta & -\cot \alpha \\ -\cot \alpha & \cot \alpha + \cot \gamma \end{pmatrix}$$

Using  $\alpha + \beta + \gamma = \pi$ , we obtain

$$M = \frac{1}{\sin \alpha \sin \beta \sin \gamma} \begin{pmatrix} \sin \gamma & -\cos \alpha \sin \beta \sin \gamma \\ -\cos \alpha \sin \beta \sin \gamma & \sin \beta \end{pmatrix}$$

and  $\det M = 1$ . Since  $M_{11} > 0$  and  $\det M > 0$ ,  $M$  is positive definite. The claim about the second derivative of  $f$  follows.  $\square$

**3.4.4 Proposition** (Extension). *Extend the definition of  $f$  from  $\mathcal{A}$  to  $\mathbb{R}$  as follows. Define  $f(x, y, z)$  by equation (3.5) for all  $(x, y, z) \in \mathbb{R}^3$ , where for  $(x, y, z) \notin \mathcal{A}$  the angles  $\alpha, \beta$ , and  $\gamma$  are defined to be  $\pi$  for the angle opposite the side that is too long and 0 for the other two. The so extended function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  is continuously differentiable and convex.*

Figure 8 shows contour lines of the extended function  $f(x, y, z)$  in the plane  $z = 0$  and its graph.

*Proof.* The so-defined functions  $\alpha, \beta, \gamma$  are continuous on  $\mathbb{R}^3$ . This implies the continuity of  $f$  and, together with Proposition 3.4.1, the continuity of its first derivative. Since  $f$  is locally convex in  $\mathcal{A}$  (Proposition 3.4.3) and linear outside, it is convex.  $\square$

**3.4.5 Remark** (Amoebas and Ronkin functions). In fact,  $\mathcal{A}$  is an amoeba, the extended  $f$  is a Ronkin function, and the convexity of  $f$  that we have proved by elementary means follows also from a general theorem of Passare and Rullgård [26], which says that a Ronkin function

is convex. Amoebas were introduced by Gelfand, Kapranov and Zelevinsky [12]. The amoeba  $\mathcal{A}_p$  of a complex polynomial  $p(z_1, \dots, z_n)$  with  $n$  indeterminates is defined as the domain in  $\mathbb{R}^n$  that is the image of the set of zeros of  $p$  under the map  $(z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|)$ . So the domain  $\mathcal{A}$  defined by equation (3.6) is the amoeba of the linear polynomial  $z_1 + z_2 + z_3$ . The Ronkin function of a polynomial  $p$  is defined as the function  $N_p : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$N_p(x_1, \dots, x_n) = \frac{1}{(2\pi i)^n} \int_{S^1(e^{x_1}) \times \dots \times S^1(e^{x_n})} \log |p(z_1, \dots, z_n)| \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n},$$

where  $S^1(r)$  is the circle in  $\mathbb{C}$  around 0 with radius  $r$ . As it turns out,

$$f(x_1, x_2, x_3) = \pi N_{z_1, z_2, z_3}(x_1, x_2, x_3).$$

We will not spoil the reader's fun by presenting a proof here. The same Ronkin function also appears in the work of Kenyon, Okounkov, and Sheffield on the dimer model [15] [16] as the Legendre dual of a "surface tension" in a variational principle governing the limit shape of random surfaces. (See in particular Kenyon's survey article on dimers [14] and Mikhalkin's survey article on amoebas [21].) Whether or how this is related to the variational principles discussed in this paper is unclear.

### 3.5 Necessary conditions for the existence of a solution

In this section, we will discuss some rather obvious and rather mild necessary conditions for the solvability of the discrete mapping problems and how they relate to the behavior of the function  $E_{T, \Theta, \lambda}(u)$ . In short, the conditions are necessary for the problems to have a solution and sufficient to ensure that  $E_{T, \Theta, \lambda}(u)$  behaves "sanely", so that the following solvability alternative (see the corollary to Proposition 3.5.3) holds: Provided that we are able to find a minimizer of a convex function if it exists, then the variational principle allows us to either solve a discrete conformal mapping problem or to ascertain that it is not solvable.

**Condition A.**  $\sum_{i \in V} \Theta_i = \pi |T|$

If Problem 3.1.1 has a solution then clearly Condition A is satisfied (because the sum of angle sums around vertices equals the sum of angle sums in triangles). This is actually a discrete version of the Gauss-Bonnet formula. If we set  $K_i = 2\pi - \Theta_i$  for interior vertices and  $\kappa_i = \pi - \Theta_i$  for boundary vertices then Condition A is equivalent to

$$\sum_{i \in V_{int}} K_i + \sum_{i \in V_{bdy}} \kappa_i = 2\pi(|T| - |E| + |V|).$$

**3.5.1 Proposition.** *The function  $E_{T, \Theta, \lambda}(u)$  is scale-invariant, that is,*

$$E_{T, \Theta, \lambda}(u + h \mathbf{1}_V) = E_{T, \Theta, \lambda}(u),$$

*if and only if Condition A is satisfied.*

*Proof.* Because  $f(x + h, y + h, z + h) = f(x, y, z) + \pi h$ ,

$$E_{T, \Theta, \lambda}(u + h \mathbf{1}_V) = E_{T, \Theta, \lambda}(u) + h \left( \sum_{i \in V} \Theta_i - \pi |T| \right). \quad \square$$

The following stronger Conditions B and C are also obviously necessary for the existence of a solution of Problem 3.1.1. Moreover, if a solution to the general Problem 3.1.3 exists (where  $\Theta_i$  is given only for  $i \in V_1$ ), then positive  $\Theta$ -values can be assigned also to the vertices in  $V_0$  so that Conditions B and C are satisfied.

**Condition B.** There exists a system of angles  $\widehat{\alpha} > 0$ , such that

$$\widehat{\alpha}_{jk}^i + \widehat{\alpha}_{ki}^j + \widehat{\alpha}_{ij}^k = \pi \quad \text{for all } ijk \in T,$$

and

$$\sum_{jk:ijk \in T} \widehat{\alpha}_{jk}^i = \Theta_i \quad \text{for all } i \in V.$$

**Condition C.** If  $T_1$  is any subset of  $T$  and  $V_1 \subseteq V$  is the set of all vertices of the triangles in  $T_1$ , that is,

$$V_1 = \bigcup_{ijk \in T} \{i, j, k\},$$

then

$$\pi |T \setminus T_1| \geq \sum_{i \in V \setminus V_1} \Theta_i,$$

where equality holds if and only if  $T_1 = \emptyset$  or  $T_1 = T$ .

**3.5.2 Proposition.** *Conditions B and C are equivalent.*

*Proof.* The implication ‘Condition B  $\Rightarrow$  Condition C’ is easy to see. Regarding the converse implication, Colin de Verdière proves a similar statement using the feasible flow theorem [5, Section 7]. It is straightforward to adapt his proof for Proposition 3.5.2.  $\square$

**3.5.3 Proposition.** *If Condition B or C is satisfied (and hence both of them are), then*

$$E_{T,\Theta,\lambda}(u) \longrightarrow \infty \quad \text{if} \quad \max_{i \in V} u_i - \min_{i \in V} u_i \longrightarrow \infty.$$

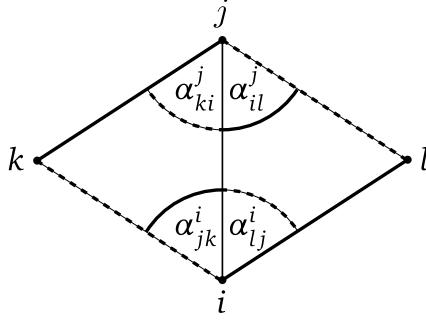
**3.5.4 Corollary** (Solvability alternative). *Suppose Condition B or C is satisfied (and hence both of them and Condition A are). Then  $E_{T,\Theta,\lambda}(u)$  (maybe with some variables  $u_i$  fixed) has a minimizer  $u_{\min}$ . Either  $u_{\min}$  is contained in the domain where all triangle inequalities are satisfied, then it is unique (up to a constant if no variables are fixed) and corresponds to the solution of the corresponding discrete conformal mapping problem, or it lies outside that domain, then the corresponding discrete conformal mapping problem does not have a solution.*

*Proof of Proposition 3.5.3.* Using the (constant) angles  $\widehat{\alpha}$  we can rewrite the sum over vertices on the right-hand side of equation (3.7) as a sum over triangles:

$$\sum_{i \in V} \Theta_i u_i = \sum_{ijk \in T} (\widehat{\alpha}_{jk}^i u_i + \widehat{\alpha}_{ki}^j u_j + \widehat{\alpha}_{ij}^k u_k).$$

Expressing  $u$  in terms of  $\tilde{\lambda}$  and  $\lambda$ , we obtain

$$\begin{aligned} \widehat{\alpha}_{jk}^i u_i + \widehat{\alpha}_{ki}^j u_j + \widehat{\alpha}_{ij}^k u_k &= \left(\frac{\pi}{2} - \widehat{\alpha}_{ij}^k\right)(\tilde{\lambda}_{ij} - \lambda_{ij}) + \left(\frac{\pi}{2} - \widehat{\alpha}_{jk}^i\right)(\tilde{\lambda}_{jk} - \lambda_{jk}) \\ &\quad + \left(\frac{\pi}{2} - \widehat{\alpha}_{ki}^j\right)(\tilde{\lambda}_{ki} - \lambda_{ki}), \end{aligned}$$



**Figure 9.** The sine theorem lets us express the length-cross-ratios (see Figure 3) in terms of angles.

so

$$E_{T,\Theta,\lambda}(u) = \sum_{ijk \in T} \left( 2f\left(\frac{\tilde{\lambda}_{ij}}{2}, \frac{\tilde{\lambda}_{jk}}{2}, \frac{\tilde{\lambda}_{ki}}{2}\right) - \tilde{\alpha}_{ij}^k \tilde{\lambda}_{ij} - \tilde{\alpha}_{jk}^i \tilde{\lambda}_{jk} - \tilde{\alpha}_{ki}^j \tilde{\lambda}_{ki} \right) + \text{const.},$$

where here and in the following “*const.*” stands for terms that do not depend on  $u$ . The estimate

$$f(x, y, z) \geq \pi \max\{x, y, z\}$$

yields

$$E_{T,\Theta,\lambda}(u) \geq \min\{\tilde{\alpha}_{ij}^k\} \sum_{ijk \in T} (\max\{\tilde{\lambda}_{ij}, \tilde{\lambda}_{jk}, \tilde{\lambda}_{ki}\} - \min\{\tilde{\lambda}_{ij}, \tilde{\lambda}_{jk}, \tilde{\lambda}_{ki}\}) + \text{const.}$$

Now if  $ijk \in T$ , then  $u_i - u_j = \tilde{\lambda}_{ki} - \tilde{\lambda}_{jk} - \lambda_{ki} + \lambda_{jk}$ , so

$$\begin{aligned} \max\{u_i, u_j, u_k\} - \min\{u_i, u_j, u_k\} &\leq \max\{\tilde{\lambda}_{ij}, \tilde{\lambda}_{jk}, \tilde{\lambda}_{ki}\} - \min\{\tilde{\lambda}_{ij}, \tilde{\lambda}_{jk}, \tilde{\lambda}_{ki}\} \\ &\quad + \text{const.}, \end{aligned}$$

and because the triangulated surface is connected this implies

$$E_{T,\Theta,\lambda}(u) \geq \min\{\tilde{\alpha}_{ij}^k\} (\max_{i \in V} u_i - \min_{i \in V} u_i) + \text{const.} \quad \square$$

### 3.6 The second variational principle

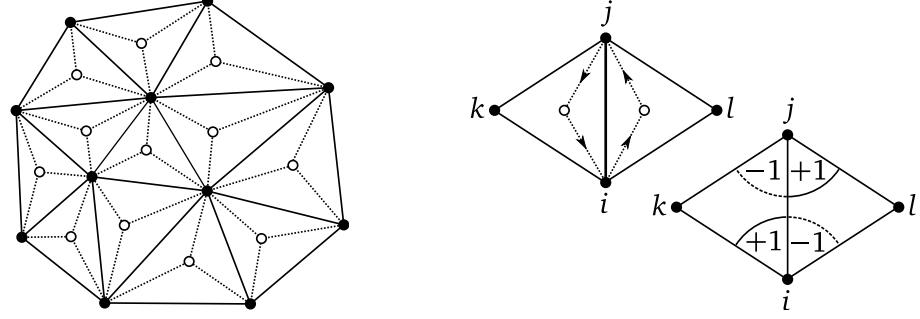
The second variational principle has angles as variables. It is based on the two elementary observations that, first, the sine theorem lets us express the length-cross-ratios in terms of angles,

$$\text{lcr}_{ij} = \frac{\sin(\alpha_{il}^j) \sin(\alpha_{jk}^i)}{\sin(\alpha_{lj}^i) \sin(\alpha_{ki}^j)} \quad (3.10)$$

(see Figure 9), and that, second, if we know the angles in a Euclidean triangulation, then we can (again using the sine theorem) reconstruct the lengths up to a global scale factor.

For a triangulated surface  $T$  and  $\lambda \in \mathbb{R}^E$ , define

$$\begin{aligned} S_{T,\lambda} : \mathbb{R}^A &\longrightarrow \mathbb{R}, \\ S_{T,\lambda}(\alpha) &= \sum_{ijk \in T} \left( V(\alpha_{ij}^k, \alpha_{jk}^i, \alpha_{ki}^j) + \alpha_{ij}^k \lambda_{ij} + \alpha_{jk}^i \lambda_{jk} + \alpha_{ki}^j \lambda_{ki} \right), \end{aligned} \quad (3.11)$$



**Figure 10.** Left: Triangulation  $T$  (solid) and the corresponding graph  $\Gamma$  (dotted). Right: The cycle of edges of  $\Gamma$  (dashed) that corresponds to an interior edge  $ij \in E_T$ , and the corresponding tangent vector  $v_{ij} \in \mathbb{R}^A$  to  $C_\Theta$ .

where

$$V(\alpha, \beta, \gamma) = \Pi(\alpha) + \Pi(\beta) + \Pi(\gamma). \quad (3.12)$$

**3.6.1 Proposition** (Rivin [29]). *The function  $V$  is strictly concave on the domain*

$$\{(\alpha, \beta, \gamma) \in (\mathbb{R}_{>0})^3 \mid \alpha + \beta + \gamma = \pi\}.$$

So  $S_{T,\lambda}$  is also strictly concave on the domain of positive angle assignments that sum to  $\pi$  in each triangle.

**3.6.2 Proposition** (Second variational principle I). *Let  $\mathcal{C}$  be a discrete conformal class on  $T$  with representative  $\ell = e^{\lambda/2} \in \mathbb{R}^E$ , let  $\Theta \in \mathbb{R}^V$ , and define the subset  $C \subseteq \mathbb{R}^A$  by*

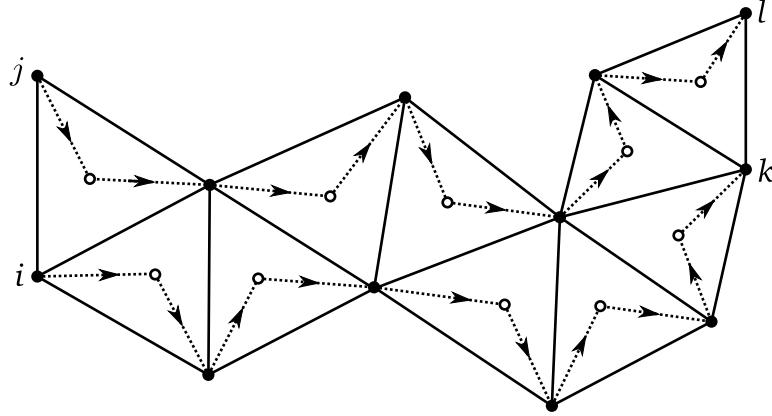
$$C_\Theta = \left\{ \alpha \in \mathbb{R}^A \mid \alpha > 0, \text{ for all } ijk \in T: \alpha_{jk}^i + \alpha_{ki}^j + \alpha_{ij}^k = \pi, \right. \\ \left. \text{for all } i \in V: \sum_{jk:ijk \in T} \alpha_{jk}^i = \Theta_i \right\}. \quad (3.13)$$

Then  $\tilde{\alpha} \in C_\Theta$  is the angle function of a solution  $\tilde{\ell} = e^{\tilde{\lambda}/2}$  of Problem 3.1.1 if and only if  $S_{T,\lambda}(\tilde{\alpha})$  is the maximum of the restriction  $S_{T,\lambda}|_C$ .

*Proof.* Consider the graph  $\Gamma$  that is obtained by choosing one point in each triangle of  $T$  and connecting it to the vertices of the triangle (see Figure 10, left). The vertex set  $V_\Gamma$  is in one-to-one correspondence with  $V_T \cup T_T$ , and the edge set  $E_\Gamma$  is in one-to-one correspondence with the set of angles  $A_T$ . The tangent space to  $C_\Theta \subseteq \mathbb{R}^A$ , which consists of those vectors in  $\mathbb{R}^A$  that sum to 0 in each triangle and around each vertex, is in one-to-one correspondence with the space of closed edge chains of  $\Gamma$  with coefficients in  $\mathbb{R}$ .

First, assume that  $\tilde{\alpha}$  is a critical point of  $S_{T,\lambda}|_C$ . Suppose  $ij \in E_T$  is an interior edge and consider the cycle of edges of  $\Gamma$  shown on the right in Figure 10. The corresponding tangent vector to  $C_\Theta$  in  $\mathbb{R}^A$  is

$$\frac{\partial}{\partial \alpha_{il}^j} - \frac{\partial}{\partial \alpha_{lj}^i} + \frac{\partial}{\partial \alpha_{jk}^i} - \frac{\partial}{\partial \alpha_{ki}^j}, \quad (3.14)$$



**Figure 11.** Edge-connected chain of triangles from edge  $ij$  to edge  $kl$  of  $T$ . The corresponding chain  $\gamma$  of  $\Gamma$ -edges (dotted) consists of the edges of  $\Gamma$  opposite the initial edge  $ij$ , the terminal edge  $kl$ , and the intermediate connecting edges of the triangle chain.

and

$$\left( \frac{\partial}{\partial \alpha_{il}^j} - \frac{\partial}{\partial \alpha_{lj}^i} + \frac{\partial}{\partial \alpha_{jk}^i} - \frac{\partial}{\partial \alpha_{ki}^j} \right) S_{T,\lambda}(\tilde{\alpha}) = -\log \left( \frac{\sin(\tilde{\alpha}_{il}^j) \sin(\tilde{\alpha}_{jk}^i)}{\sin(\tilde{\alpha}_{lj}^i) \sin(\tilde{\alpha}_{ki}^j)} \right) + \lambda_{il} - \lambda_{lj} + \lambda_{jk} - \lambda_{ki}.$$

Provided that  $\tilde{\alpha}$  is in fact the system of angles of a discrete metric  $\tilde{\ell} = e^{\tilde{\lambda}/2}$ , this implies that  $\tilde{\ell}$  and  $\ell$  are discretely conformally equivalent. It remains to show that  $\tilde{\alpha}$  is indeed the system of angles of a discrete metric. Construct such a metric as follows: Pick one edge  $ij \in E_T$  and choose an arbitrary value for  $\tilde{\lambda}_{ij}$ . To define  $\tilde{\lambda}_{lk}$  for any other edge  $lk \in E_T$ , connect it to  $ij$  by an edge-connected sequence of triangles as shown in Figure 11, let  $\gamma$  be the chain of edges of the graph  $\Gamma$  as indicated in the figure. Let  $w_\gamma$  be the corresponding vector in  $\mathbb{R}^A$  and define

$$\tilde{\lambda}_{kl} = \tilde{\lambda}_{ij} + dS_{T,\lambda} \Big|_{\tilde{\alpha}} (w_\gamma) + \lambda_{kl} - \lambda_{ij}.$$

The value of  $\tilde{\lambda}_{kl}$  obtained in this way is independent of the choice of triangle chain: Another triangle chain connecting  $ij$  to  $kl$  leads to an edge-chain  $\gamma'$  such that  $\gamma' - \gamma$  is a closed edge-chain so that  $w_{\gamma'} - w_\gamma \in \mathbb{R}^A$  is tangent to  $C_\Theta$ . Further,  $\tilde{\ell} = e^{\tilde{\lambda}/2}$  is a discrete metric with angles  $\tilde{\alpha}$ : If  $ij$  and  $kl$  belong to the same triangle (that is, if the triangle chain consists of only one triangle) then this follows from the sine rule. The general case follows by induction over the length of  $\gamma$ . So  $\tilde{\ell}$  is a solution of Problem 3.1.1.

The converse implication (solution of Problem 3.1.1 implies critical point) follows from the fact that the cycle space of  $\Gamma$  is spanned by the cycles corresponding to interior edges of  $T$  as shown in Figure 10 (right) together with the cycles in  $\Gamma$  corresponding to edge-connected triangle sequences as shown in Figure 11 but closed.  $\square$

Solutions to the more general Problem 3.1.3 (with  $u|_{V_0} = 0$ ) are also in one-to-one correspondence with critical points of  $S_{T,\lambda}$ . The only difference is that the angle sums are not constrained for vertices in  $V_0$ :

**3.6.3 Proposition** (Second variational principle II). *Let  $\mathcal{C}$  be a discrete conformal class on  $T$  with representative  $\ell = e^{\lambda/2} \in \mathbb{R}^E$ , let  $V = V_0 \dot{\cup} V_1$  be a partition of  $V$ , let  $\Theta \in \mathbb{R}^{V_1}$ , and define  $C'_\Theta \subseteq \mathbb{R}^A$  by*

$$C'_\Theta = \left\{ \alpha \in \mathbb{R}^A \mid \begin{array}{l} \alpha > 0, \text{ for all } ijk \in T: \alpha_{jk}^i + \alpha_{ki}^j + \alpha_{ij}^k = \pi, \\ \text{for all } i \in V_1: \sum_{jk:ijk \in T} \alpha_{jk}^i = \Theta_i \end{array} \right\}. \quad (3.15)$$

Then  $\tilde{\alpha} \in C'_\Theta$  is the angle function of a solution of Problem 3.1.3 with fixed  $u|_{V_0} = 0$  if and only if  $S_{T,\lambda}(\tilde{\alpha})$  is the maximum of the restriction  $S_{T,\lambda}|_{C'_\Theta}$ .

We omit the proof because no essential new ideas are necessary beyond those used in the proof of Proposition 3.6.2.

**3.6.4 Remark.** It also makes sense to consider critical points of  $S_{T,\lambda}$  under variations of the type shown in Figure 10 alone, disallowing variations corresponding homologically non-trivial cycles in  $\Gamma$ . These correspond to discretely conformally equivalent *similarity structures*, that is, “metrics” which may have global scaling holonomy.

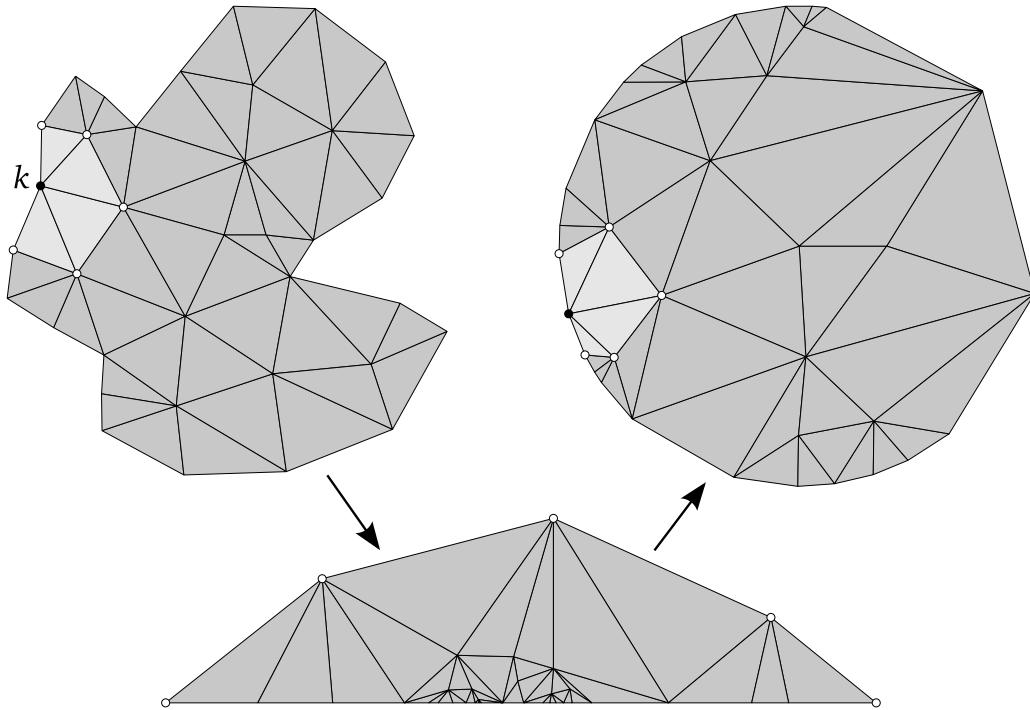
### 3.7 Mapping to the sphere and to the disk

If one can solve Problem 3.1.3, one can also find discrete conformal maps from euclidean triangulations that are topological spheres to polyhedra with vertices on the unit sphere in  $\mathbb{R}^3$ , and from euclidean triangulations that are topological disks to planar triangulations with boundary vertices on the unit circle. The method for mapping to the sphere presented here is a slight variation of a method that was previously described [34]. The old version requires an input triangulation that is immersed in some  $\mathbb{R}^n$  with straight edges. The variation presented here requires only a triangulated surface with discrete metric (but yields the same maps for immersed triangulations).

**Mapping to the sphere.** Suppose  $(T, \ell)$  is a euclidean triangulation that is topologically a sphere. Proceed as follows:

1. Choose a vertex  $k$  and apply a discrete conformal change of metric (2.1) so that afterwards all edges incident with  $k$  have the same length, say  $\tilde{\ell}_{ki} = 1$  for all neighbors  $i$  of  $k$ . For example, let  $e^{u_i/2} = \ell_{ki}^{-1}$  if  $i$  is a neighbor of  $k$  and 1 otherwise.
2. Let  $T'$  be  $T$  minus the open star of  $k$ . This is topologically a closed disk.
3. Solve Problem 3.1.3 for  $T'$  with prescribed  $\Theta_i = 2\pi$  for interior vertices  $i$  and prescribed  $u_i = 0$  for boundary vertices. (Suppose a solution exists.) The result is a planar triangulation.
4. Map the vertices of this planar triangulation to the unit sphere by stereographic projection. Add another vertex (the image of the removed vertex  $k$ ) on the sphere at the center of the stereographic projection. Build a geometric simplicial complex using these vertices and the combinatorics of  $T$ .

**3.7.1 Proposition.** *The result of this procedure is a polyhedron with vertices on the sphere that is discretely conformally equivalent to  $(T, \ell)$ . (It may not be convex. It is also possible that the planar triangulation obtained in step three overlaps with itself. In this case the star of  $k$  in the image polyhedron is not embedded.)*



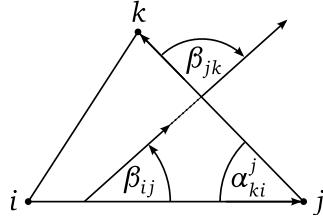
**Figure 12.** Mapping to the disk.

*Proof.* After Step 1, the length-cross-ratio for an edge  $ki$  incident with  $k$  is the quotient of the lengths of two consecutive edges  $mi, ij$  in the boundary of  $T'$ . This is not changed in Step 3 because  $u = 0$  on the boundary. Further, the length-cross-ratio for an edge  $ij$  opposite  $k$  as in Figure 3 is then the quotient  $\ell_{il}/\ell_{lj}$ . This is also not changed in Step 3 because  $u = 0$  on the boundary. Now imagine that before Step 4 you reinsert  $k$  at  $\infty$  in the plane, which you identify with the (extended) complex plane. Then the absolute values of the complex cross-ratios for all edges are the same as in  $(T, \ell)$ .  $\square$

**Mapping to the disk.** Suppose  $(T, \ell)$  is a euclidean triangulation that is topologically a closed disk. Proceed as follows (see Figure 12):

1. Choose a boundary vertex  $k$  and apply a discrete conformal change of metric (2.1) so that afterwards all edges incident with  $k$  have the same length, say  $\tilde{\ell}_{ki} = 1$  for all neighbors  $i$  of  $k$ . For example, let  $e^{u_i/2} = \ell_{ki}^{-1}$  if  $i$  is a neighbor of  $k$  and 1 otherwise.
2. Let  $T'$  be  $T$  minus the open star of  $k$ . Suppose this is topologically still a closed disk.
3. Solve Problem 3.1.3 for  $T'$ , with prescribed  $\Theta_i = 2\pi$  for interior vertices of  $T'$ ,  $\Theta_i = \pi$  for boundary vertices of  $T'$  that are not neighbors of  $k$  in  $T$ , and prescribed  $u_i = 0$  for the neighbors of  $k$  in  $T$ . (Suppose a solution exists.) The result is a planar triangulation. All boundary edges except the neighbors of  $k$  in  $T$  are contained in one straight line.
4. Apply a Möbius transformation to the vertices that maps this straight line to a circle and the other vertices inside this circle. Reinsert  $k$  at the image point of  $\infty$  under this Möbius transformation.

**3.7.2 Proposition.** *The result of this procedure is a planar triangulation that is discretely conformally equivalent to  $(T, \ell)$  and has a boundary polygon that is inscribed in a circle.*



**Figure 13.** Parallel transport of directions from edge to edge.

We omit the proof because no new ideas are needed.

**3.7.3 Remark.** Note that it is a necessary condition for the solvability of Problem 3.1.3 in step 3 that  $T$  has no ears (triangles with two edges on the boundary).

### 3.8 On the corresponding smooth theory

A natural question regarding the two variational principles for discrete conformal maps is: “What are the corresponding variational principles in the classical smooth theory of conformal maps?” In fact, even the question “What exactly are the corresponding smooth mapping problems?” deserves a comment. For the second variational principle it is not even obvious how the variables—triangle angles—translate to the smooth theory.

To answer this last question, consider a system  $\alpha \in \mathbb{R}^A$  of angles in a triangulation  $T$  that sums to  $\pi$  in each triangle. Even if this angle assignment does not allow a consistent assignment of edge lengths, it does allow a sensible definition of parallel transport of *directions* from edge to edge: The direction that makes an angle  $\beta_{ij}$  with the directed edge  $ij$  in triangle  $ijk$  makes an angle  $\beta_{jk} = \beta_{ij} + \alpha_{ki}^j - \pi$  with edge  $jk$  (see Figure 13). So angle assignments in triangulated surfaces correspond to connections in the direction bundle of a smooth surface.

Our discussion of the second variational principle will focus on the special case of flat connections in the direction bundle (this corresponds to angle sum  $2\pi$  around interior vertices) that also have trivial global holonomy. For any such connection there exists a parallel direction field and this is unique up to rotation by a constant angle. Conversely, any direction field is parallel for a unique flat connection with trivial global holonomy.

Before we enter our discussion of the smooth conformal mapping problems and variational principles, we will first review some background material about conformal metrics on Riemann surfaces from the point of view of direction fields. This is of course classic material, only the presentation is a bit unusual. Nevertheless, the smooth version of the second variational principle seems to be little known, although it is essentially just the theory of harmonic maps from a surface to  $S^1$ .

Let  $M$  be a smooth oriented surface, possibly with boundary, equipped with a Riemannian metric  $g$ . Let  $J : TM \rightarrow TM$  be the rotation by  $90^\circ$ , so that the area 2-form is  $\sigma = g(J\cdot, \cdot)$ , and let  $\nabla$  be the Levi–Civita connection of  $g$ . The tensor  $J$  is parallel, that is,  $\nabla_X(JZ) = J(\nabla_X Z)$ . The Riemann curvature tensor is

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where  $[\cdot, \cdot]$  is the Lie bracket for vector fields.

By a *direction field on  $M$  with respect to  $g$*  we mean simply a vector field  $Y$  with  $g(Y, Y) = 1$ , that is, a unit vector field with respect to  $g$ . Of course the existence of a direction field imposes

certain restrictions on the topology of  $M$ . In any case, direction fields exist locally, so purely local considerations remain valid for arbitrary  $M$ .

The Gauss curvature of  $g$  is

$$K = -g(R(Y, JY)Y, JY),$$

for any direction field  $Y$ . The curvature 2-form of  $g$  is

$$\Omega = K\sigma.$$

Define the *rotation 1-form*  $\rho_Y$  of a direction field  $Y$  with respect to  $g$  by

$$\rho_Y(X) = g(\nabla_X Y, JY).$$

**3.8.1 Proposition.** *For any direction field  $Y$ ,*

$$d\rho_Y = -\Omega.$$

*Proof.* The claim follows by a straightforward calculation:

$$\begin{aligned} d\rho_Y(Y, JY) &= Y \cdot g(\nabla_{JY} Y, JY) - (JY) \cdot g(\nabla_Y Y, JY) - g(\nabla_{[Y, JY]} Y, JY) \\ &= g(\nabla_Y \nabla_{JY} Y, JY) + \underbrace{g(\nabla_{JY} Y, \nabla_Y JY)}_{=0} \\ &\quad - g(\nabla_{JY} \nabla_Y Y, JY) - \underbrace{g(\nabla_Y \nabla_{JY} Y, JY)}_{=0} - g(\nabla_{[Y, JY]} Y, JY) \\ &= g(R(Y, JY)Y, JY) = -K. \end{aligned}$$

We have used that  $\nabla_V Y \perp Y$  and  $\nabla_V JY \perp JY$  for any vector field  $V$ , so that, because  $M$  is two-dimensional,  $g(\nabla_V Y, \nabla_W JY) = 0$  for any  $V, W$ .  $\square$

Now consider a conformal change of metric by equation (1.1), which is also characterized by the fact that the  $90^\circ$ -rotation with respect to  $\tilde{g}$  is the same  $J$ . The Levi–Civita connection  $\tilde{\nabla}$  of  $\tilde{g}$  is related to the Levi–Civita connection  $\nabla$  of  $g$  by

$$\tilde{\nabla}_X Z = \nabla_X Z + g(X, G)Z + g(Z, G)X - g(X, Z)G, \quad (3.16)$$

where  $G = \text{grad}_g u$ , that is,  $du = g(G, \cdot)$ . A direction field  $Y$  with respect to  $g$  naturally determines a direction field

$$\tilde{Y} = e^{-u} Y$$

with respect to  $\tilde{g}$ . Its rotation 1-form is

$$\tilde{\rho}_{\tilde{Y}}(X) = \tilde{g}(\tilde{\nabla}_X \tilde{Y}, J\tilde{Y}).$$

**3.8.2 Proposition.** *The rotation 1-forms  $\rho_Y, \tilde{\rho}_{\tilde{Y}}$  are related by*

$$\tilde{\rho}_{\tilde{Y}} = \rho_Y + *du,$$

where  $*$  denotes the Hodge star operator for  $g$ .

The Hodge star operator maps a 1-form  $\omega$  to the 1-form  $*\omega = -\omega(J\cdot)$ . It also maps a function (0-form)  $f$  to the 2-form  $*f = f\sigma$  and vice versa,  $**f = f$ . Note that on a 2-dimensional manifold, the Hodge star operator for 1-forms depends only on the conformal class of the metric.

*Proof of Proposition 3.8.2.* First,

$$\tilde{\nabla}_X \tilde{Y} = e^{-u} (-du(X)Y + \tilde{\nabla}_X Y),$$

so

$$\tilde{\rho}_{\tilde{Y}}(X) = g(\tilde{\nabla}_X Y, JY).$$

Using (3.16) one obtains

$$\tilde{\rho}_{\tilde{Y}}(X) = \rho_Y(X) + g(Y, G)g(X, JY) - g(X, Y)g(G, JY).$$

Finally, since  $J$  is skew,  $J^2 = -1$ , and  $(Y, JY)$  is an orthonormal frame,

$$\begin{aligned} g(Y, G)g(X, JY) - g(X, Y)g(G, JY) &= \\ &- g(G, Y)g(Y, JX) - g(G, JY)g(JY, JX) \\ &= -g(G, JX) = -du(JX) = *du(X), \end{aligned}$$

and this completes the proof.  $\square$

As a corollary of Propositions 3.8.1 and 3.8.2, we obtain the equation relating the curvature 2-forms of  $g$  and  $\tilde{g}$ ,

$$\tilde{\Omega} = \Omega - d * du, \quad (3.17)$$

and hence Liouville's equation for the curvatures,

$$e^{2u}\tilde{K} = K + \Delta u, \quad (3.18)$$

where  $\Delta$  is the Laplace–Beltrami operator with respect to  $g$ ,

$$\Delta f = -*d*df.$$

(We use the sign convention for the Laplace operator that renders it positive semidefinite.)

Which problems in the smooth theory are analogous to the discrete conformal mapping problems discussed in this paper? There are two fairly obvious candidates:

**3.8.3 Problem.** Given  $(M, g)$  and the function  $\tilde{K}$ , find a conformally equivalent Riemannian metric  $\tilde{g}$  with curvature  $\tilde{K}$ .

This amounts to solving Liouville's equation (3.18) for  $u$ .

**3.8.4 Problem.** Given  $(M, g)$  and the 2-form  $\tilde{\Omega}$ , find a conformally equivalent Riemannian metric  $\tilde{g}$  with curvature 2-form  $\tilde{\Omega}$ .

This amounts to solving equation (3.17), which is equivalent to Poisson's equation

$$\Delta u = f$$

with right-hand side  $f = * \tilde{\Omega} - K$ .

For both problems, one may prescribe either  $u$  on the boundary (Dirichlet conditions) or  $*du|_{\partial M}$  with  $\int_{\partial M} *du = -\int_M (\tilde{\Omega} - \Omega)$  (Neumann conditions).

Both Liouville's equation (3.18) and the Poisson equation (3.17) are variational, with the respective functionals

$$E_A(u) = \int_M \left( \frac{1}{2} du \wedge *du - \left( \frac{1}{2} e^{2u} \tilde{K} - uK \right) \sigma \right) \quad (3.19)$$

for Liouville's equation, where  $\tilde{K}$  and  $K$  are fixed given functions on  $M$ , and

$$E_B(u) = \int_M \left( \frac{1}{2} du \wedge *du - u(\tilde{\Omega} - \Omega) \right) \quad (3.20)$$

for equation (3.17), where  $\tilde{\Omega}$  and  $\Omega$  are fixed given 2-forms on  $M$ .

We would argue that the discrete conformal mapping problems in Section 3.1 are analogous to Problem 3.8.4 rather than to Problem 3.8.3, and that the function  $E_{T,\Theta,\lambda}$  of the first variational principle corresponds to  $E_B$  rather than  $E_A$ . Although we did denote the angle defect at vertex  $i$  by  $K_i$  (Section 3.5), prescribing the angle defect is analogous to prescribing the curvature 2-form rather than the curvature, the latter being an angle defect *per surface area*. Probably the clearest manifestation of this difference is in the scaling behavior: The angle defects at vertices and the curvature 2-form do not change when all lengths are scaled by a constant factor. The curvature on the other hand is inversely proportional to the square of this factor. Note, however, that for  $\tilde{K} = 0$ ,  $\tilde{\Omega} = 0$  there is no difference between Problems 3.8.3 and 3.8.4, and  $E_A = E_B$ .

The remaining question is: What is the smooth version of the second variational principle (Section 3.6)?

Assume for now that  $M$  is diffeomorphic to a closed disk and  $\tilde{\Omega} = 0$ , that is, we are looking for a conformally equivalent flat metric. We will discuss this case in detail, because it allows a more intuitive treatment using direction fields rather than connections or connection 1-forms. Afterwards, we will indicate how to treat the general case.

For a direction field  $Y$  define

$$S(Y) = \frac{1}{2} \int_M \rho_Y \wedge * \rho_Y.$$

**3.8.5 Proposition** (First variation). *Let  $Y_t$  be a variation of the direction field  $Y$  with*

$$\frac{d}{dt} \Big|_{t=0} Y_t = h J Y,$$

*where  $h \in C^\infty(M)$ . Then*

$$\frac{d}{dt} \Big|_{t=0} S(Y_t) = - \int_M h d * \rho_Y + \int_{\partial M} h * \rho.$$

*Proof.* This follows from  $\frac{d}{dt}\Big|_{t=0} \rho_{Y_t} = dh$ .  $\square$

**3.8.6 Corollary.** *A direction field  $Y$  is a critical point of  $S$  under variations that fix  $Y$  on the boundary  $\partial M$  if and only if*

$$d*\rho_Y = 0.$$

*It is also a critical point of  $S$  under arbitrary variations if and only if, additionally,*

$$*\rho_Y\Big|_{T\partial M} = 0. \quad (3.21)$$

Loosely speaking, the following proposition says that straightest direction fields with respect to  $g$  are parallel direction fields with respect to a conformally equivalent flat metric  $\tilde{g}$  with trivial global holonomy.

**3.8.7 Proposition** (Smooth version of the second variational principle). *Suppose the direction field  $Y$  is a critical point of  $S$  under variations that fix  $Y$  on the boundary. Define the function  $u$  up to an additive constant by*

$$du = *\rho_Y.$$

(This is possible since  $*\rho_Y$  is closed by the above corollary and we assumed that  $M$  was diffeomorphic to a disk.) Define  $\tilde{g}$  by the conformal change of metric (1.1). Then:

- (i) *The direction field  $\tilde{Y} = e^{-u}Y$  is parallel with respect to  $\tilde{g}$ , so  $\tilde{g}$  is flat.*
- (ii) *The geodesic curvature of the boundary  $\partial M$  with respect to  $\tilde{g}$  is*

$$\tilde{\kappa} = \kappa - \rho_Y(T),$$

where  $\kappa$  is the geodesic curvature with respect to  $g$  and  $T$  is the positively oriented unit tangent vector field to  $\partial M$ .

- (iii) *If  $Y$  is also a critical point of  $S$  under arbitrary variations, then  $u|_{\partial M}$  is constant.*

*Proof.* Since  $*du = -\rho_Y$ , the rotation form of  $\tilde{\rho}_{\tilde{Y}}$  vanishes by Proposition 3.8.2. This implies (i). The geodesic boundary curvatures are  $\kappa = \rho_T(T)$  and  $\tilde{\kappa} = \tilde{\rho}_{\tilde{T}}\tilde{T}$ . (Locally extend the direction field  $T$  inwards from the boundary.) Again by Proposition 3.8.2, this implies (ii). Finally, (iii) follows immediately from (3.21).  $\square$

In the general case,  $M$  is not restricted to be diffeomorphic to closed disk and one is looking for a conformally equivalent metric  $\tilde{g}$  with prescribed curvature 2-form  $\tilde{\Omega}$ . To treat this case variationally, consider the functional

$$S(\rho) = \frac{1}{2} \int_M \rho \wedge *\rho$$

on the affine space of 1-forms  $\rho$  satisfying  $d\rho = \tilde{\Omega} - \Omega$ . We leave the details to the reader, not because they are tedious but because they are interesting. The critical points correspond to conformally equivalent similarity structures, that is, “metrics” which may have global scaling holonomy. (Compare the remark at the end of Section 3.6.)

## 4 Ideal hyperbolic polyhedra

### 4.1 Hyperbolic structure on a euclidean triangulation

This section deals with the inverse of a construction of Penner [27] [7], which equips a hyperbolic manifold with cusps with a piecewise euclidean metric. Here, we construct a natural hyperbolic metric with cusps on any euclidean triangulation.

Consider a euclidean triangle with its circumcircle. If we interpret the interior of the circumcircle as a hyperbolic plane in the Klein model, then the euclidean triangle becomes an ideal hyperbolic triangle, that is, a hyperbolic triangle with vertices at infinity. This construction equips any euclidean triangle (minus its vertices) with a hyperbolic metric. If it is performed on all triangles of a euclidean triangulation  $(T, \ell)$ , then the hyperbolic metrics induced on the individual triangles fit together so  $T \setminus V$  is equipped with a hyperbolic metric with cusps at the vertices. Thus,  $T$  becomes an ideal triangulation of a hyperbolic surface with cusps.

**4.1.1 Theorem.** *Two euclidean triangulations  $(T, \ell)$  and  $(\tilde{T}, \tilde{\ell})$  with the same combinatorics are discretely conformally equivalent if and only if the hyperbolic metrics with cusps induced by the circumcircles are isometric. Discrete conformal maps are isometries with respect to the induced hyperbolic metrics.*

*Proof.* This follows immediately from Theorem 2.7.2 (Section 2.7), because the projective circumcircle preserving maps between triangles are precisely the hyperbolic isometries.  $\square$

**4.1.2 Remark.** Each discrete conformal structure on  $T$  corresponds therefore to a point in the classical Teichmüller space  $\mathcal{T}_{g,n}$  of a punctured surface. This explains the dimensional agreement observed in Section 2.5.

Theorem 4.1.1 also suggests a way to extend the concepts of discrete conformal equivalence and discrete conformal maps to triangulations which are not combinatorially equivalent:

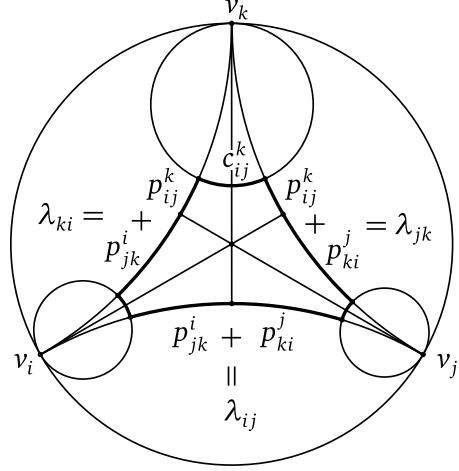
**4.1.3 Definition.** Two euclidean triangulations  $(T, \ell)$  and  $(\tilde{T}, \tilde{\ell})$ , which need not be combinatorially equivalent, are *discretely conformally equivalent* if they are isometric with respect to the induced hyperbolic metrics with cusps. The corresponding isometries are called *discrete conformal maps*.

This idea will be pursued further in the sequel [33]. Here, we will stick with Definition 2.1.1.

### 4.2 Decorated ideal triangles and tetrahedra

In this section we review some basic facts about ideal triangles and tetrahedra that will be needed in subsequent sections.

All ideal hyperbolic triangles are congruent with respect to the group of hyperbolic isometries. A *decorated ideal triangle* is an ideal hyperbolic triangle together with a choice of horocycles, one centered at each vertex (see Figure 14). We denote by  $\lambda_{ij}$  the signed distance between the horocycles at vertices  $i$  and  $j$  as measured along the edge  $ij$  and taken negatively if the horocycles intersect. Any triple of real numbers  $(\lambda_{ij}, \lambda_{jk}, \lambda_{ki}) \in \mathbb{R}^3$  corresponds to one and only one choice of horocycles. Figure 14 shows also the lines of symmetry of the ideal



**Figure 14.** Decorated ideal triangle in the Poincaré disk model.

triangle. (They are its heights as well.) We denote the signed distances from their base points to the horocycles by  $p_{ij}^k$  as shown. Clearly,

$$\lambda_{ij} = p_{jk}^i + p_{ki}^j, \quad \lambda_{jk} = p_{ki}^j + p_{ij}^k, \quad \lambda_{ki} = p_{ij}^k + p_{jk}^i,$$

so

$$\begin{aligned} p_{ij}^k &= \frac{1}{2}(-\lambda_{ij} + \lambda_{jk} + \lambda_{ki}), \\ p_{jk}^i &= \frac{1}{2}(\lambda_{ij} - \lambda_{jk} + \lambda_{ki}), \\ p_{ki}^j &= \frac{1}{2}(\lambda_{ij} + \lambda_{jk} - \lambda_{ki}). \end{aligned} \tag{4.1}$$

**4.2.1 Lemma** (Penner [27]). *The length  $c_{ij}^k$  of the part of the horocycle centered at  $v_k$  that is the ideal triangle as shown in Figure 14 is*

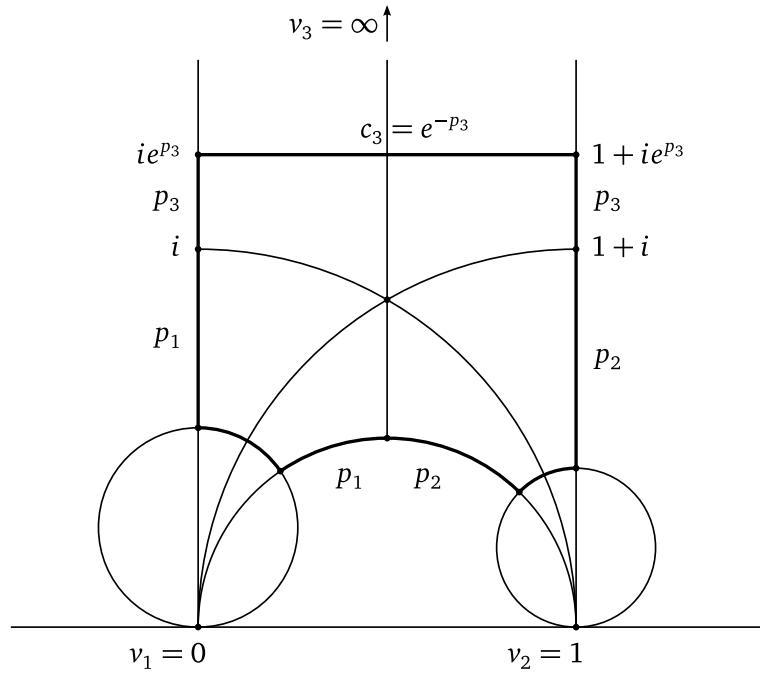
$$c_{ij}^k = e^{\frac{1}{2}(\lambda_{ij} - \lambda_{jk} - \lambda_{ki})}.$$

*Proof.* See Figure 15, which shows the ideal triangle in the half-plane model. Recall that in the half-plane model, the hyperbolic plane is represented by  $\{z \in \mathbb{C} \mid \text{Im } z > 0\}$  with metric  $ds = \frac{1}{\text{Im } z} |dz|$ .  $\square$

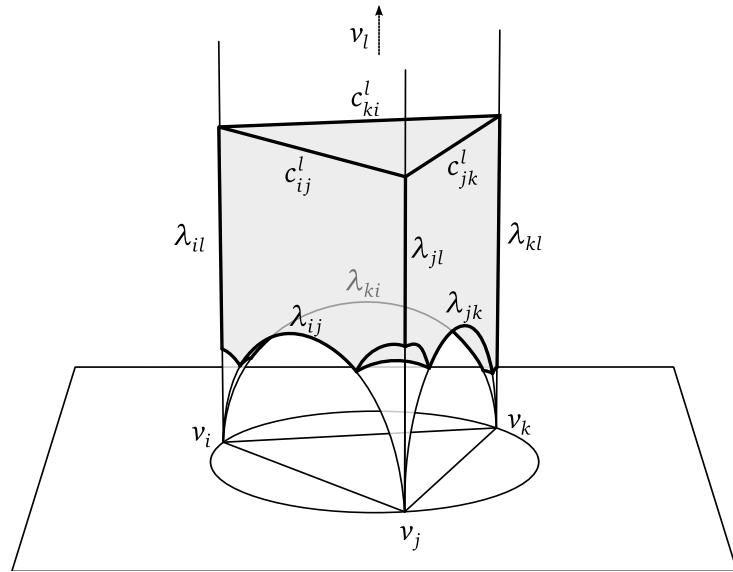
**4.2.2 Remark.** Together with Proposition 4.3.2 of the next section, this provides a geometric interpretation for the auxiliary parameters  $c_{jk}^i$  introduced in Section 2.4.

All ideal tetrahedra are not isometric. There is a complex 1-parameter family of them, the parameter being the complex cross-ratio of the vertices in infinite boundary of hyperbolic 3-space. A *decorated ideal tetrahedron* is an ideal hyperbolic tetrahedron together with a choice of horospheres centered at the vertices. Figure 16 shows a decorated ideal tetrahedron, truncated at its horospheres, in the half-space model. Again, we denote the signed distances between the horospheres by  $\lambda_{ij}$ .

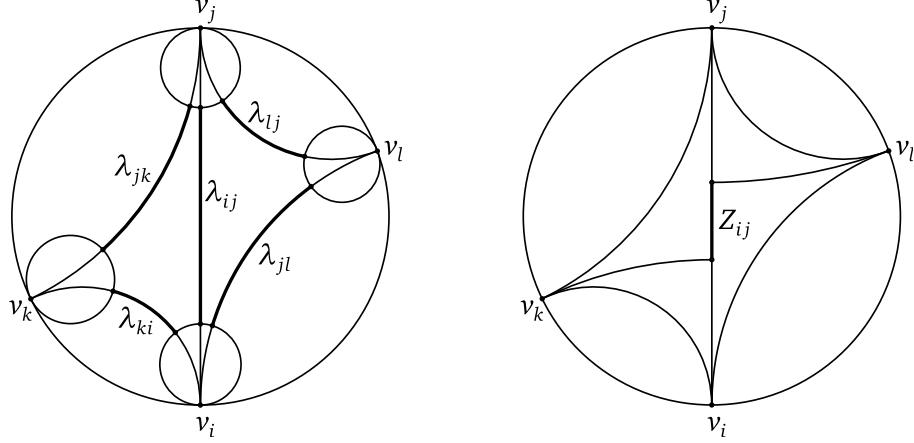
The intrinsic geometry of a horosphere in hyperbolic space is euclidean. So the intersection of the tetrahedron with the horosphere centered at, say,  $v_l$  is a euclidean triangle with side lengths  $c_{ij}^l$ ,  $c_{jk}^l$ ,  $c_{ki}^l$  determined by Lemma 4.2.1. One easily deduces the following lemma.



**Figure 15.** Decorated ideal triangle in the half-plane model.



**Figure 16.** Decorated ideal tetrahedron in the half-space model.



**Figure 17.** Penner coordinates (left) and shear coordinates (right).

**4.2.3 Lemma.** Six real numbers  $\lambda_{ij}, \lambda_{jk}, \lambda_{ki}, \lambda_{il}, \lambda_{jl}, \lambda_{kl}$  are the signed distances between horospheres of a decorated ideal tetrahedron as shown in Figure 16 (which is then unique) if and only if  $c_{ij}^l, c_{jk}^l, c_{ki}^l$  determined by Lemma 4.2.1 satisfy the triangle inequalities.

So the six parameters  $\lambda$  determine the congruence class of the ideal tetrahedron (2 real parameters) and the choice of horospheres (4 parameters).

Note that the angles of the euclidean triangles in which the tetrahedron intersects the four horospheres are the dihedral angles of the tetrahedron. This implies that the dihedral angles sum to  $\pi$  at each vertex, and further, that the dihedral angles at opposite edges are equal. The space of ideal tetrahedra is therefore parametrized by three dihedral angles  $\alpha_{ij} = \alpha_{kl}$ ,  $\alpha_{jk} = \alpha_{il}$ ,  $\alpha_{ki} = \alpha_{jl}$  satisfying  $\alpha_{il} + \alpha_{jl} + \alpha_{kl} = \pi$ .

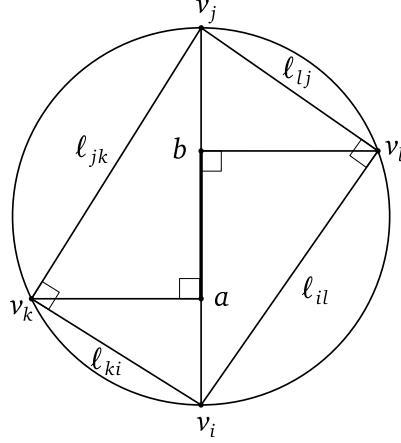
### 4.3 Penner coordinates and shear coordinates

In Section 4.1, we equipped a euclidean triangulation  $(T, \ell)$  with a hyperbolic cusp metric that turns it into an ideal hyperbolic triangulation. In this section, we will identify the logarithmic edge lengths  $\lambda$  (see equation (2.2)) with the Penner coordinates [27] and the logarithmic length-cross-ratios log lcr (see equation (2.4)) with the shear coordinates [10] [37] for this ideal triangulation. (The handbook [25] is a good reference for the pertinent aspects of Teichmüller theory.)

*Warning.* Our notation differs from Penner's in a potentially confusing way. His "lambda-lengths" are  $\sqrt{2}e^{\lambda/2} = \sqrt{2}\ell$  in our notation. Our  $\lambda$ s are the signed hyperbolic distances between horocycles.

Since the sides of an ideal hyperbolic triangle are complete geodesics, there is a one-parameter family of ways to glue two sides together. Penner coordinates and shear coordinates can be seen as two ways to describe how ideal triangles are glued together along their edges to form a hyperbolic surface with cusps.

Suppose  $T$  is a triangulated surface and  $\lambda \in \mathbb{R}^E$ . For each triangle  $ijk \in T$ , take the decorated ideal triangle with horocycle distances  $\lambda_{ij}, \lambda_{jk}, \lambda_{ki}$ , and glue them so that the horocycles fit together (see Figure 17, left). The result is a hyperbolic surface with cusps at the vertices, together with a particular choice of horocycles centered at the cusps. In this way, the Penner coordinates  $\lambda$  parametrize the *decorated Teichmüller* space, that is, the space of



**Figure 18.** Shear and length-cross-ratio (see proof of Proposition 4.3.2).

hyperbolic cusp metrics on a punctured surface (modulo isotopy) with horocycles centered at the cusps.

The shear coordinates represent another way to prescribe how ideal triangles are glued, for which no choice of horospheres is necessary. The shear coordinate  $Z$  on an interior edge of an ideal triangulation is the signed distance of the base points of the heights from the opposite vertices (see Figure 17, right). The following relation between Penner coordinates and shear coordinates is well known.

**4.3.1 Lemma.** *If  $\lambda \in \mathbb{R}^E$  are the Penner coordinates for an ideal triangulation with a particular choice of horocycles, then the shear coordinates  $Z \in \mathbb{R}^{E_{int}}$  are*

$$Z_{ij} = \frac{1}{2}(\lambda_{il} - \lambda_{lj} + \lambda_{jk} - \lambda_{ki}),$$

where  $k$  and  $l$  are the vertices opposite edge  $ij$  as in Figure 17.

*Proof.* The claim follows from  $Z_{ij} = p_{ki}^j - p_{il}^j$  and equations (4.1).  $\square$

**4.3.2 Proposition.** *Let  $(T, \ell)$  be a euclidean triangulation. The shear coordinates  $Z \in \mathbb{R}^{E_{int}}$  for the corresponding ideal triangulation (see Section 4.1) are*

$$Z_{ij} = \log \text{lcr}_{ij} \tag{4.2}$$

(see equation (2.4)). Thus, for a suitable choice of horocycles, the Penner coordinates  $\lambda \in \mathbb{R}^E$  are given by equation (2.2).

*Proof.* Consider an interior edge  $ij \in E$  between triangles  $ijk$  and  $jil$ . Without loss of generality, we may assume that the triangles have a common circumcircle. For otherwise we can change  $\ell$  discretely conformally so that this holds, and this changes neither  $\text{lcr}_{ij}$  nor the hyperbolic cusp metric on  $T$ . We may further assume that  $ij$  is a diameter of the common circumcircle. For otherwise we may apply a projective transformation that maps the circle onto itself so that this holds. This is an isometry with respect to the hyperbolic metric of the Klein model, and it is a discrete conformal map of the quadrilateral formed by the two triangles. We arrive at the situation shown in Figure 18 in the Klein model. The hyperbolic heights are also

the euclidean heights, and in the hyperbolic metric of the Klein model, the distance between their base points  $a$  and  $b$  is

$$Z_{ij} = \frac{1}{2} \log \frac{|av_j| |bv_i|}{|av_i| |bv_j|},$$

where  $|xy|$  denotes the euclidean distance between  $x$  and  $y$ . Since by elementary euclidean geometry

$$\frac{|av_j|}{|av_i|} = \frac{\ell_{jk}^2}{\ell_{ki}^2} \quad \text{and} \quad \frac{|bv_i|}{|bv_j|} = \frac{\ell_{lj}^2}{\ell_{il}^2},$$

this implies equation (4.2). Now Lemma 4.3.1 implies the statement about Penner coordinates.  $\square$

#### 4.4 Ideal hyperbolic polyhedra with prescribed intrinsic metric

The discrete conformal mapping problems described in Section 3.1 are equivalent to problems involving the polyhedral realization of surfaces with hyperbolic cusp metrics, like the following.

**4.4.1 Problem.** *Given* an ideal triangulation  $T$  of a punctured sphere equipped with a hyperbolic metric with cusps, *find* an isometric embedding of  $T$  as ideal hyperbolic polyhedron in  $H^3$ . The polyhedron is not required to be convex, but it is required that the edges of the polyhedron are edges of  $T$ .

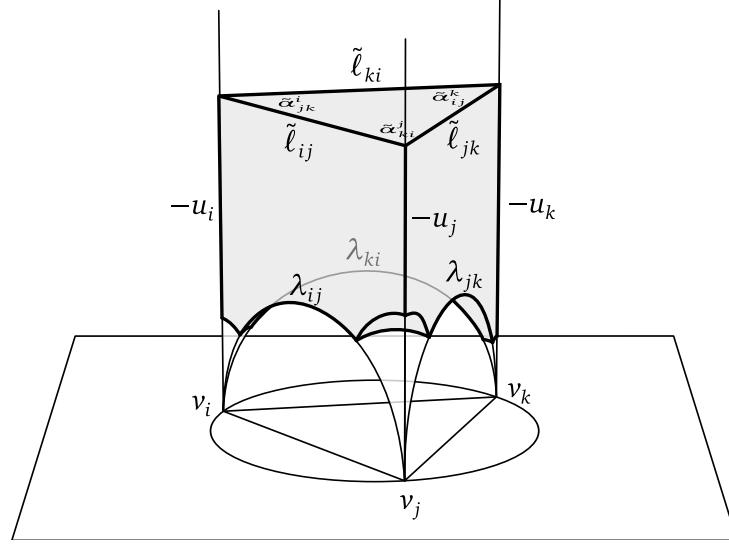
**4.4.2 Theorem.** *For any vertex  $l$  of  $T$ , Problem 4.4.1 has at most one solution that is star-shaped with respect to  $l$ .*

Problem 4.4.1 is equivalent to (a special case of) Problem 3.1.3, so Theorem 4.4.2 follows from Theorem 3.1.5. Indeed, to solve Problem 4.4.1 one may proceed as follows. Let  $\lambda \in \mathbb{R}^3$  be the Penner coordinates for the ideal triangulation  $T$ , and let  $\ell = e^{\lambda/2}$ . Choose a vertex  $l$  of  $T$  and let the triangulation  $T'$  be  $T$  minus the open star of  $l$ . Solve Problem 3.1.3 for  $T'$ , prescribing  $\Theta_i = 2\pi$  if  $i$  is an interior vertex and  $u_i = -\lambda_{il}$  if  $i$  is a boundary vertices. Suppose a solution  $u \in \mathbb{R}^{V'}$  exists. This leads to a flat triangulation  $(T', \tilde{\ell})$ . Suppose further that  $(T', \tilde{\ell})$  does not overlap with itself when developed in the plane. For each triangle  $ijk$  of  $T'$ , construct the decorated ideal tetrahedron (see Section 4.2) with horosphere-distances  $\lambda_{ij}$ ,  $\lambda_{jk}$ ,  $\lambda_{ki}$ , and  $-u_i$ ,  $-u_j$ ,  $-u_k$  as shown in Figure 19. They exist by Lemma 4.2.3, because by Lemma 4.2.1, the intersection of the ideal tetrahedron with the horosphere centered at the vertex opposite  $ijk$  is the euclidean triangle with side lengths  $\tilde{\ell}_{ij}$ ,  $\tilde{\ell}_{jk}$ ,  $\tilde{\ell}_{ki}$  as shown in the figure. Hence, all these ideal tetrahedra fit together to form a solution of Problem 4.4.1 that is star-shaped with respect to  $l$ .

Conversely, any solution of Problem 4.4.1 that is star-shaped with respect to  $l$  yields a solution without self-overlap of the corresponding instance of Problem 3.1.3.

**4.4.3 Remark.** Note the similarity with the procedure for mapping to a sphere described in Section 3.7.

Numerous variations of Problem 4.4.1 can be treated in similar fashion. We mention only the following.



**Figure 19.** Discretely conformally flattening a euclidean triangulation is equivalent to constructing an ideal polyhedron with prescribed hyperbolic cusp metric.

**4.4.4 Problem.** *Given* an ideal triangulation  $T$  of a punctured torus equipped with a hyperbolic metric with cusps, *find* an isometric embedding of the universal cover of  $T$  as an ideal polyhedron in  $H^3$  that is symmetric with respect to an action of the fundamental group of  $T$  by parabolic isometries. The polyhedron is not required to be convex, but it is required that the polyhedron is star-shaped with respect to the ideal fixed point of the parabolic isometries and that the edges of the polyhedron are edges of  $T$ .

**4.4.5 Theorem.** *If Problem 4.4.4 has a solution, it is unique.*

## 4.5 The variational principles and hyperbolic volume

The connection with hyperbolic polyhedra elucidates the nature and origin of the variational principles for discrete conformal maps. In this section, we will indicate how to derive the first variational principle for discrete conformal maps (Proposition 3.3.3) from Milnor's equation for the volume of an ideal tetrahedron and Schläfli's formula.

Milnor [22] [23] showed that the volume of an ideal tetrahedron with dihedral angles  $\alpha, \beta, \gamma$  is  $V(\alpha, \beta, \gamma)$  as defined by equation (3.12). Schläfli's differential volume formula (more precisely, Milnor's generalization which allows for ideal vertices [24]) says that its derivative is

$$dV = -\frac{1}{2} \sum_{ij} \lambda_{ij} d\alpha_{ij}, \quad (4.3)$$

where the sum is taken over the six edges  $ij$ ,  $\lambda_{ij}$  is the signed distance between horospheres centered at the vertices  $i$  and  $j$ , and  $\alpha_{ij}$  is the interior dihedral angle. (The choice of horospheres does not matter because the angle sum at a vertex is constant.) Define

$$\widehat{V}(\lambda_{12}, \lambda_{23}, \lambda_{31}, \lambda_{14}, \lambda_{24}, \lambda_{34}) = \frac{1}{2} \sum_{ij} \alpha_{ij} \lambda_{ij} + V(\alpha_{14}, \alpha_{24}, \alpha_{34}), \quad (4.4)$$

where the dihedral angles  $\alpha_{12} = \alpha_{34}$ ,  $\alpha_{23} = \alpha_{14}$ ,  $\alpha_{31} = \alpha_{24}$  of the decorated tetrahedron are considered as functions of the  $\lambda_{ij}$ . (They are the angles in a euclidean triangle with side lengths  $e^{(\lambda_{12}-\lambda_{14}-\lambda_{24})/2}$ ,  $e^{(\lambda_{23}-\lambda_{24}-\lambda_{34})/2}$ ,  $e^{(\lambda_{31}-\lambda_{34}-\lambda_{14})/2}$ , see Section 4.2.) Then, by equation (4.3),

$$d\hat{V} = \frac{1}{2} \sum \alpha_{ij} d\lambda_{ij}. \quad (4.5)$$

This implies Proposition 3.3.2 on the partial derivatives of  $E_{T,\Theta,\lambda}$ , and therefore Proposition 3.3.3 (the first variational principle), because using (4.4) (and (2.3)) we can rewrite equation (3.3) as

$$\begin{aligned} E_{T,\Theta,\lambda}(u) &= \sum_{ijk \in T} 2\hat{V}(\lambda_{ij}, \lambda_{jk}, \lambda_{ki}, -u_i, -u_j, -u_k) \\ &\quad - \sum_{ij \in E} \Phi_{ij} \lambda_{ij} + \sum_{i \in V} \Theta_i u_i, \end{aligned} \quad (4.6)$$

where

$$\Phi_{ij} = \begin{cases} \pi, & \text{if } ij \text{ is an interior edge,} \\ \frac{\pi}{2}, & \text{if } ij \text{ is a boundary edge.} \end{cases} \quad (4.7)$$

(See also Figure 19.)

## 5 Hyperbolic triangulations

### 5.1 Discrete conformal equivalence of hyperbolic triangulations

In Section 4.5 we derived the first variational principle for discrete conformal maps from Milnor's equation for the volume of an ideal tetrahedron and Schläfli's formula. A straightforward modification of this derivation leads to a companion theory of discrete conformality for *hyperbolic* triangulations. This makes it possible, for example, to construct discretely conformal uniformizations of higher genus surfaces as shown in Figure 20.

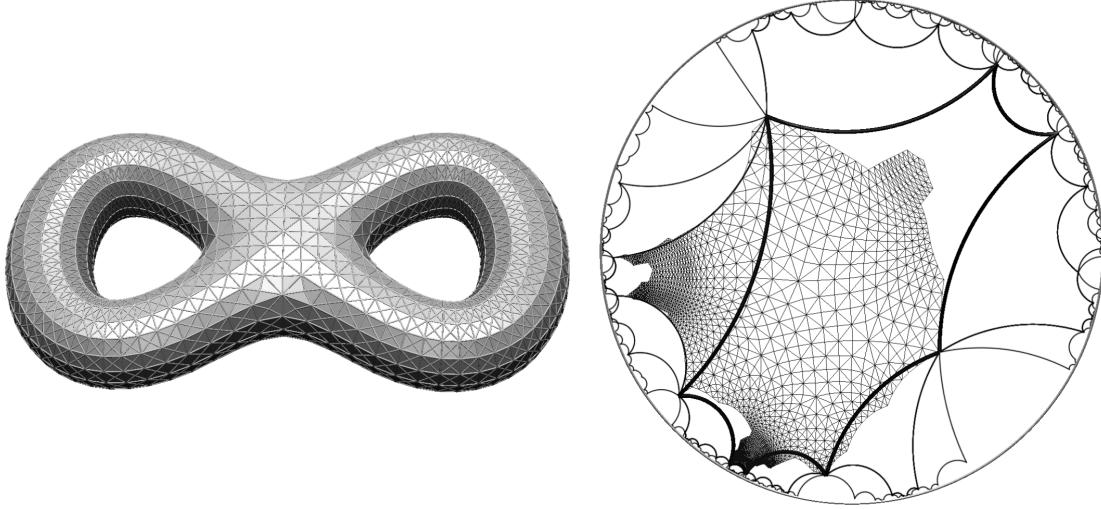
We will present the basic theory in this section, and show how to derive it by hyperbolic volume considerations in the next. For the discretely conformal uniformization of a euclidean triangulation over the hyperbolic plane, as shown in Figure 20, we need to define what it means for a euclidean triangulation to be discretely conformally equivalent to a hyperbolic triangulation. This is the topic of Section 5.3.

Suppose  $T$  is a surface triangulation and  $\ell \in \mathbb{R}_{>0}^E$  is a discrete metric, that is, a real valued function on the set of edges that satisfies all triangle inequalities. Then there is not only a euclidean triangulation  $(T, \ell)$  with these edge lengths. One can equally construct hyperbolic triangles  $ijk$  with hyperbolic side lengths  $\ell_{ij}, \ell_{jk}, \ell_{ki}$  and glue them together. The result is a *hyperbolic triangulation* which we denote by  $(T, \ell)_h$ .

**5.1.1 Definition.** Two combinatorially equivalent hyperbolic triangulations,  $(T, \ell)_h$  and  $(T, \tilde{\ell})_h$ , are *discretely conformally equivalent* if the discrete metrics  $\ell$  and  $\tilde{\ell}$  are related by

$$\sinh \frac{\tilde{\ell}_{ij}}{2} = e^{\frac{1}{2}(u_i+u_j)} \sinh \frac{\ell_{ij}}{2} \quad (5.1)$$

for some function  $u : V \rightarrow \mathbb{R}$ .



**Figure 20.** Discretely conformal uniformization of a genus-two surface.

Thus, in the hyperbolic version of the theory, equation (2.2) is replaced by

$$\lambda = 2 \log \sinh \frac{\ell}{2}, \quad (5.2)$$

so that in terms of  $\lambda$  and  $\tilde{\lambda}$ , the relation (5.1) is again (2.3). The role of  $E_{T,\Theta,\lambda}(u)$  is played by the function

$$E_{T,\Theta,\lambda}^h(u) = \sum_{ijk \in T} 2\widehat{V}_h(\lambda_{ij}, \lambda_{jk}, \lambda_{ki}, -u_i, -u_j, -u_k) + \sum_{i \in V} \Theta_i u_i, \quad (5.3)$$

where

$$\begin{aligned} 2\widehat{V}_h(\lambda_{12}, \lambda_{23}, \lambda_{31}, \lambda_1, \lambda_2, \lambda_3) = & \alpha_1 \lambda_1 + \alpha_2 \lambda_2 + \alpha_3 \lambda_3 + \alpha_{12} \lambda_{12} + \alpha_{23} \lambda_{23} + \alpha_{31} \lambda_{31} \\ & + \Pi(\alpha_1) + \Pi(\alpha_2) + \Pi(\alpha_3) + \Pi(\alpha_{12}) + \Pi(\alpha_{23}) + \Pi(\alpha_{31}) \\ & + \Pi\left(\frac{1}{2}(\pi - \alpha_1 - \alpha_2 - \alpha_3)\right), \end{aligned} \quad (5.4)$$

and  $\alpha_1, \alpha_2, \alpha_3$  are the angles in a hyperbolic triangle with side lengths

$$\begin{aligned} \tilde{\ell}_{23} &= 2 \operatorname{arsinh}\left(e^{\frac{1}{2}(\lambda_{23}-\lambda_2-\lambda_3)}\right), \\ \tilde{\ell}_{31} &= 2 \operatorname{arsinh}\left(e^{\frac{1}{2}(\lambda_{31}-\lambda_3-\lambda_1)}\right), \\ \tilde{\ell}_{12} &= 2 \operatorname{arsinh}\left(e^{\frac{1}{2}(\lambda_{12}-\lambda_1-\lambda_2)}\right), \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} \alpha_{23} &= \frac{1}{2}(\pi + \alpha_1 - \alpha_2 - \alpha_3), \\ \alpha_{31} &= \frac{1}{2}(\pi - \alpha_1 + \alpha_2 - \alpha_3), \\ \alpha_{12} &= \frac{1}{2}(\pi - \alpha_1 - \alpha_2 + \alpha_3). \end{aligned} \quad (5.6)$$

Thus,  $\widehat{V}_h$  is defined only on the domain where  $\tilde{\ell}_{12}, \tilde{\ell}_{23}, \tilde{\ell}_{31}$  satisfy the triangle inequalities. However, exactly as in the case of  $E_{T,\Theta,\lambda}(u)$ , we can extend the domain of definition of  $E_{T,\Theta,\lambda}^h(u)$  to the whole of  $\mathbb{R}^V$ :

**5.1.2 Proposition.** *Extend the domain of definition of  $E_{T,\Theta,\lambda}^h(u)$  to  $\mathbb{R}^V$  by declaring the angles in “broken” triangles to be 0, 0,  $\pi$ , respectively. The resulting function is continuously differentiable on  $\mathbb{R}^V$ .*

*Proof.* See Section 5.2. □

**5.1.3 Remark.** To compute the angles  $\alpha, \beta, \gamma$  in a hyperbolic triangle with side lengths  $a, b, c$ , one can use, for example, the hyperbolic cosine rule or the hyperbolic half-angle formula

$$\tan\left(\frac{\alpha}{2}\right) = \sqrt{\frac{\sinh((a-b+c)/2)\sinh((a+b-c)/2)}{\sinh((-a+b+c)/2)\sinh((a+b+c)/2)}}.$$

**5.1.4 Proposition.** *Let  $\ell \in \mathbb{R}^E$ , let  $\lambda$  be defined by equation (5.2), and suppose  $u \in \mathbb{R}^V$  is in the domain where  $\tilde{\ell}$  defined by equation (5.1) satisfies all triangle inequalities. Then the partial derivative of  $E_{T,\Theta,\lambda}^h$  with respect to  $u_i$  is*

$$\frac{\partial}{\partial u_i} E_{T,\Theta,\lambda}^h = \Theta_i - \sum_{jk:ijk \in T} \tilde{\alpha}_{jk}^i,$$

where  $\tilde{\alpha}$  are the angles in the hyperbolic triangulation  $(T, \tilde{\ell})_h$ , and the sum is taken over all angles around vertex  $i$ .

*Proof.* See Section 5.2. □

**5.1.5 Proposition.** *The function  $E_{T,\Theta,\lambda}^h(u)$  is convex on  $\mathbb{R}^V$  and locally strictly convex in the domain where  $\tilde{\ell}$  defined by equation (5.1) satisfies all triangle inequalities.*

*Proof.* See Section 5.2. □

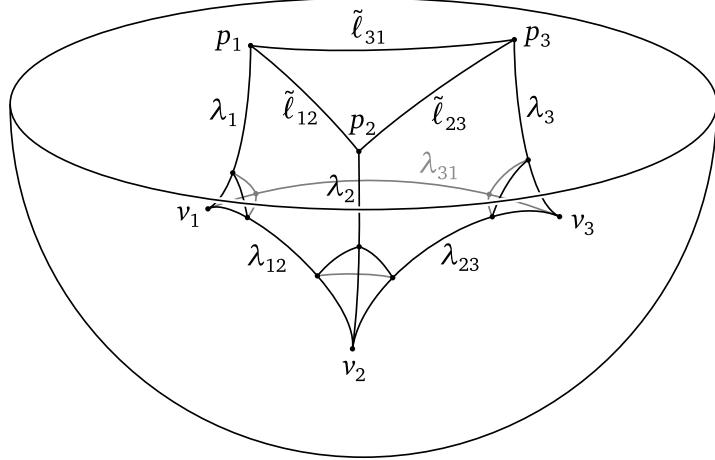
Consider the discrete conformal mapping problems for hyperbolic triangulations that are analogous to those for euclidean triangulations described in Section 3.1. Propositions 5.1.4 and 5.1.7 imply the following hyperbolic version of Theorem 3.1.5.

**5.1.6 Theorem.** *If the discrete mapping problems for hyperbolic triangulations have a solution, it is unique and can be found by minimizing  $E_{T,\Theta,\lambda}^h(u)$ .*

In contrast to the euclidean case, there is never any ambiguity of scale. The following relatively simple explicit formula for the second derivative facilitates the numerical minimization of  $E_{T,\Theta,\lambda}^h$ .

**5.1.7 Proposition.** *The second derivative of  $E_{T,\Theta,\lambda}^h$  at  $u$  is*

$$\begin{aligned} \sum_{i,j \in V} \frac{\partial^2 E_{T,\Theta,\lambda}^h}{\partial u_i \partial u_j} du_i du_j &= \\ \frac{1}{2} \sum_{ij \in E} w_{ij}(u) \left( (du_i - du_j)^2 + \tanh^2\left(\frac{\tilde{\ell}_{ij}}{2}\right) (du_i + du_j)^2 \right), \quad (5.7) \end{aligned}$$



**Figure 21.** The polyhedral building block (shown in the Poincaré ball model) used to derive the theory of discrete conformal equivalence of hyperbolic triangulations.

with  $\tilde{\ell}$  defined by equation (5.1) and

$$w_{ij}(u) = \frac{1}{2} \left( \cot\left(\frac{1}{2}(\pi - \tilde{\alpha}_{jk}^i - \tilde{\alpha}_{ki}^j + \tilde{\alpha}_{ij}^k)\right) + \cot\left(\frac{1}{2}(\pi - \tilde{\alpha}_{il}^j - \tilde{\alpha}_{lj}^i + \tilde{\alpha}_{ji}^l)\right) \right)$$

for interior edges  $ij$  with opposite vertices  $k$  and  $l$  if  $\tilde{\ell}$  satisfies the triangle equalities for  $ijk$  and  $jil$  (so that the corresponding angles  $\tilde{\alpha}$  are positive and smaller than  $\pi$ ). If  $ij$  is a boundary edge, there is only one cotangent term. For “broken” triangles, replace the three corresponding cotangent terms with 0.

We omit the proof, which consists of a lengthy but elementary calculation.

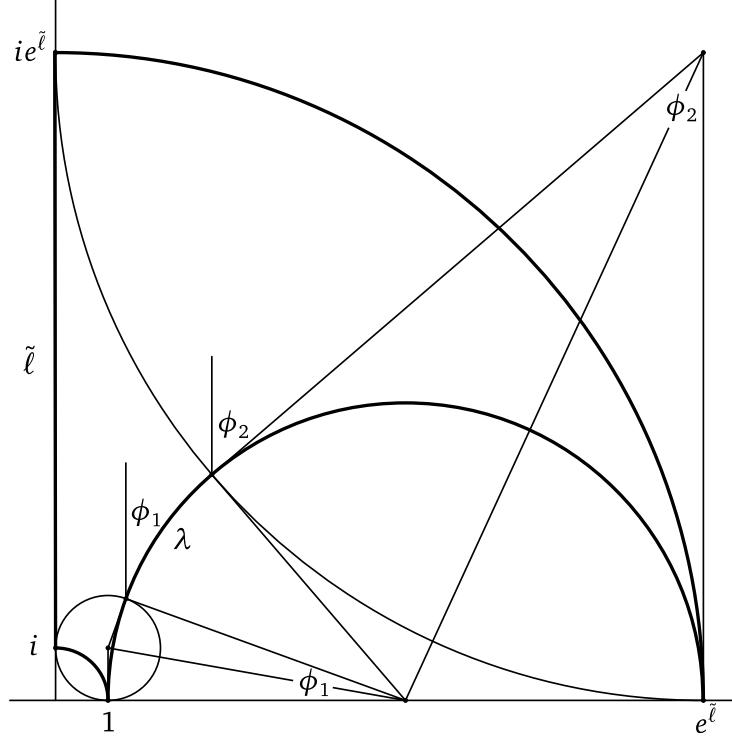
## 5.2 Derivation by volume considerations

The theory of discrete conformal equivalence for hyperbolic triangulations is based on volume considerations for the type of polyhedron shown in Figure 21. From the vertices  $p_1, p_2, p_3$  of a triangle in hyperbolic 3-space, three rays run orthogonally to the plane of the triangle until they intersect the infinite boundary in the ideal points  $v_1, v_2, v_3$ . The convex hull of these six points is a prism with three ideal vertices and right dihedral angles at the base triangle  $p_1p_2p_3$ . Let the dihedral angles at the three rays from  $p_1, p_2, p_3$  be  $\alpha_1, \alpha_2, \alpha_3$ . Since the dihedral angles sum to  $\pi$  at the ideal vertices, the dihedral angles  $\alpha_{12}, \alpha_{23}, \alpha_{31}$  at edges  $v_1v_2, v_2v_3, v_3v_1$  satisfy equations (5.6). Let  $\tilde{\ell}_{ij}$  be the lengths of the finite edges, and let  $\lambda_i$  and  $\lambda_{ij}$  be the lengths of the infinite edges truncated at some horospheres centered at the ideal vertices  $v_i$ , as shown in Figure 21.

**5.2.1 Lemma.** *The (truncated) edge lengths of the prism shown in Figure 21 are related by equations (5.5).*

*Proof.* We consider the case when  $\lambda_1 = \lambda_2 = \lambda_3 = 0$  (that is, when the truncating horospheres touch the base plane in  $p_1, p_2, p_3$ ), from which the general case follows easily. Figure 22 shows one of the side quadrilaterals of the prism in the half-plane model. We will show that

$$\lambda = 2 \log \sinh\left(\frac{\tilde{\ell}}{2}\right), \quad (5.8)$$



**Figure 22.** Proof of Lemma 5.2.1.

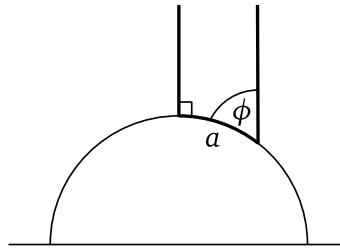
which proves this special case. We have

$$\phi_1 = 2 \operatorname{arccot} \left( \frac{e^{\tilde{\ell}} - 1}{2} \right), \quad \phi_2 = 2 \operatorname{arccot} \left( \frac{2e^{\tilde{\ell}}}{e^{\tilde{\ell}} - 1} \right),$$

The equation for the “angle of parallelity” (see Figure 23) implies that

$$\lambda = \log \cot(\phi_1/2) - \log \cot(\phi_2/2),$$

and hence equation (5.8).  $\square$



**Figure 23.** The angle  $\phi$  and side length  $a$  in a right angled hyperbolic triangle with an ideal vertex satisfy the equation  $a = \log \cot(\phi/2)$ .

The volume of the polyhedron shown in Figure 21 is

$$V_h(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{2} \left( \text{J}(\alpha_1) + \text{J}(\alpha_2) + \text{J}(\alpha_3) + \text{J}(\alpha_{12}) + \text{J}(\alpha_{23}) + \text{J}(\alpha_{31}) + \text{J}\left(\frac{1}{2}(\pi - \alpha_1 - \alpha_2 - \alpha_3)\right) \right). \quad (5.9)$$

This was shown by Leibon [18], who also showed that the volume function  $V_h$  is strictly concave on its domain of definition,

$$\{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \mid \alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0, \alpha_1 + \alpha_2 + \alpha_3 < \pi\}.$$

By Schläfli's formula,

$$dV_h = -\frac{1}{2}(\lambda_1 d\alpha_1 + \lambda_2 d\alpha_2 + \lambda_3 d\alpha_3 + \lambda_{12} d\alpha_{12} + \lambda_{23} d\alpha_{23} + \lambda_{31} d\alpha_{31}).$$

(The choice of horospheres does not matter because the angle sum at the ideal vertices is constant. Also note that the lengths  $\tilde{\ell}_{ij}$  of the finite edges do not appear in the equation because their dihedral angles are constant.)

The function  $\widehat{V}_h$  defined by equation (5.4) is

$$\begin{aligned} \widehat{V}_h(\lambda_{12}, \lambda_{23}, \lambda_{31}, \lambda_1, \lambda_2, \lambda_3) = \\ \frac{1}{2}(\alpha_1 \lambda_1 + \alpha_2 \lambda_2 + \alpha_3 \lambda_3 + \alpha_{12} \lambda_{12} + \alpha_{23} \lambda_{23} + \alpha_{31} \lambda_{31}) + V_h(\alpha_1, \alpha_2, \alpha_3), \end{aligned}$$

so that

$$d\widehat{V}_h = \frac{1}{2}(\alpha_1 d\lambda_1 + \alpha_2 d\lambda_2 + \alpha_3 d\lambda_3 + \alpha_{12} d\lambda_{12} + \alpha_{23} d\lambda_{23} + \alpha_{31} d\lambda_{31}).$$

From this one obtains Proposition 5.1.4 on the partial derivatives of  $E_{T,\Theta,\lambda}^h$ . By extending  $\widehat{V}_h$  using essentially the same argument as in the proof of Proposition 3.3.5, one obtains Proposition 5.1.2 on the extension of  $E_{T,\Theta,\lambda}^h$ . To prove the convexity of  $E_{T,\Theta,\lambda}^h$  (Proposition 5.1.5), note that the function

$$\widehat{V}_h(\lambda_{12}, \lambda_{23}, \lambda_{31}, \lambda_1, \lambda_2, \lambda_3) - \frac{\pi}{4}(\lambda_{12} + \lambda_{23} + \lambda_{31})$$

really only depends on the three parameters

$$\begin{aligned} x_1 &= \frac{1}{4}(+\lambda_{12} - \lambda_{23} + \lambda_{31} - 2\lambda_1) = \frac{\partial V_h}{\partial \alpha_1}, \\ x_2 &= \frac{1}{4}(+\lambda_{12} + \lambda_{23} - \lambda_{31} - 2\lambda_2) = \frac{\partial V_h}{\partial \alpha_2}, \\ x_3 &= \frac{1}{4}(-\lambda_{12} + \lambda_{23} + \lambda_{31} - 2\lambda_3) = \frac{\partial V_h}{\partial \alpha_3}, \end{aligned}$$

As function of these parameters, it is minus the Legendre transform of the strictly concave function  $V_h$ :

$$\widehat{V}_h - \frac{\pi}{4}(\lambda_{12} + \lambda_{23} + \lambda_{31}) = -\alpha_1 x_1 - \alpha_2 x_2 - \alpha_3 x_3 + V_h.$$

Hence,  $\widehat{V}_h$  is a locally strictly convex function of  $x_1, x_2, x_3$ , and hence also of  $\lambda_1, \lambda_2, \lambda_3$ , if  $\lambda_{12}, \lambda_{23}, \lambda_{31}$  are considered constant. The  $C^1$  extension of  $\widehat{V}_h$  is linear outside the domain where the triangle inequalities are satisfied, hence still convex.

**5.2.2 Remark.** In the same way, one can derive a theory of discrete conformal equivalence for *spherical* triangulations. In this case, the polyhedral building block is a tetrahedron with one finite and three ideal vertices. However, the functions involved in the corresponding variational principles are not convex. So in this case, the variational principles do not immediately lead to a uniqueness theorem, nor to a computational method for discrete conformal maps.

### 5.3 When are a euclidean and a hyperbolic triangulation discretely conformally equivalent?

We propose the following definition:

**5.3.1 Definition.** A euclidean triangulation  $(T, \ell)$  and a hyperbolic triangulation  $(T, \tilde{\ell})_h$  are *discretely conformally equivalent* if  $\ell$  and  $\tilde{\ell}$  are related by

$$\sinh \frac{\tilde{\ell}_{ij}}{2} = e^{\frac{1}{2}(u_i + u_j)} \ell_{ij} \quad (5.10)$$

for some function  $u \in \mathbb{R}^V$ .

This is based on the following interpretation of equations (5.2) and (2.3). Consider the hyperboloid model of the hyperbolic plane,

$$H^2 = \{x \in \mathbb{R}^3 \mid \langle x, x \rangle = -1, x_3 > 0\},$$

where  $\langle \cdot, \cdot \rangle$  denotes the indefinite scalar product

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3,$$

and the hyperbolic distance  $d_h(x, y)$  between two points  $x, y \in H^2$  satisfies

$$\cosh d_h(x, y) = -\langle x, y \rangle.$$

This implies

$$\|x - y\|_h = 2 \sinh \left( \frac{1}{2} d_h(x, y) \right),$$

where  $\|v\|_h = \sqrt{\langle v, v \rangle}$ .

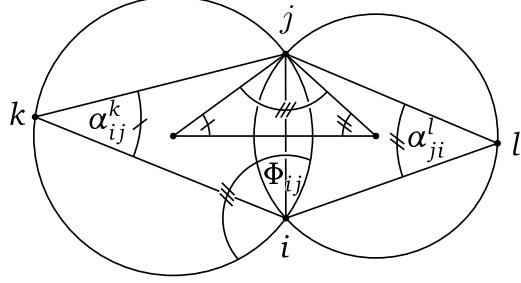
To every triangle hyperbolic triangle in  $H^2$  with sides of length  $\ell_{12}, \ell_{23}, \ell_{31}$ , there corresponds a *secant triangle* whose sides are the straight line segments connecting the vertices in  $\mathbb{R}^3$ . Their lengths, as measured by  $\|\cdot\|$ , are therefore  $2 \sinh(\ell_{ij}/2)$ .

**5.3.2 Remark.** Let  $p_1, p_2, p_3 \in H^2 \subset \mathbb{R}^3$  be the vertices of a hyperbolic triangle. Then the following statements are equivalent:

- (i) The restriction of the semi-Riemannian metric  $\langle \cdot, \cdot \rangle$  on  $\mathbb{R}^3$  to the affine plane in spanned by  $p_1, p_2, p_3$  is positive definite and therefore induces a euclidean metric on that plane.
- (ii) The side lengths of the secant triangle satisfy the triangle inequalities.
- (iii) The circumcircle of the hyperbolic triangle is a proper circle. (The circumcircle of a hyperbolic triangle is either a proper circle or a horocycle or a curve of constant distance from a geodesic.)

The analogous statement for ideal triangles is Penner's Lemma 2.2 [27].

Now let  $(T, \ell)_h$  and  $(T, \tilde{\ell})_h$  be a two hyperbolic triangulations. The edge lengths of the secant triangles of  $(T, \ell)_h$  are  $2e^{\lambda/2}$  with  $\lambda$  defined by equation (5.2), and similarly for  $(T, \tilde{\ell})_h$ . Now  $(T, \ell)_h$  and  $(T, \tilde{\ell})_h$  are discretely conformally equivalent if and only if  $\ell' = e^{\lambda/2}$  and  $\tilde{\ell}' = e^{\tilde{\lambda}/2}$  are related by equation (2.1), that is, related like discrete metrics of discretely conformally equivalent euclidean triangulations.



**Figure 24.** Circumcircle intersection angles  $\Phi_{ij} = \alpha_{ij}^k + \alpha_{ji}^l$ . For a boundary edge  $ij \in E_{\text{bdy}}$ , define  $\Phi_{ij} = \alpha_{ij}^k$ .

## 6 Circle patterns, circular meshes, and circle domains

### 6.1 Two variational principles for circle patterns

While the discrete conformal mapping problems essentially ask for ideal hyperbolic polyhedra with prescribed metric, the circle pattern problem below asks for an ideal polyhedron with prescribed dihedral angles. Rivin's variational principle for this type of problem is very similar to our second variational principle for discrete conformal maps. The function is essentially the same, only the constraints placed on the angle assignments are stronger. Also, the first variational principle needs only a slight modification to become a variational principle for circle patterns.

**6.1.1 Problem** (Circle pattern problem). *Given* a surface triangulation  $T$  and a function  $\Phi \in (0, \pi]^E$ , *find* a discrete metric  $\tilde{\ell}$  so that the euclidean triangulation  $(T, \tilde{\ell})$  has circumcircle intersection angles  $\Phi_{ij}$  as shown in Figure 24.

**6.1.2 Proposition** (Rivin [29]). *Angles  $\tilde{\alpha} \in \mathbb{R}^A$  are the angles of a euclidean triangulation  $(T, \tilde{\ell})$  that solves Problem 6.1.1 if and only if  $S_{T,\lambda}(\tilde{\alpha})$  is the maximum of  $S_{T,\lambda}$  on the set all  $\alpha \in \mathbb{R}^A$  that satisfy*

- (i)  $\alpha > 0$ ,
- (ii)  $\alpha_{ij}^k + \alpha_{jk}^i + \alpha_{ki}^j = \pi$  for all triangles  $ijk \in T$ ,
- (iii)  $\alpha_{ij}^k + \alpha_{jk}^l = \Phi_{ij}$  for all interior edges  $ij \in E_{\text{int}}$ ,
- (iv)  $\alpha_{ij}^k = \Phi_{ij}$  for all boundary edges  $ij \in E_{\text{bdy}}$ .

Due to condition (iii), the choice of the parameter  $\lambda \in \mathbb{R}^E$  of  $S_{T,\lambda}$  does not matter because

$$S_{T,\lambda}(\alpha) = S_{T,0}(\alpha) + \sum_{ij \in E} \theta_{ij} \lambda_{ij},$$

So in connection with circle patterns, it makes sense to consider only

$$S_{T,0}(\alpha) = \sum \mathcal{J}(\alpha_{ij}^k).$$

Now consider the first variational principle for discrete conformal maps. For  $\Phi \in \mathbb{R}^E$ ,  $\Theta \in \mathbb{R}^V$  define

$$\begin{aligned} \mathcal{E}_{T,\Phi,\Theta} : \mathbb{R}^E \times \mathbb{R}^V &\longrightarrow \mathbb{R}, \\ \mathcal{E}_{T,\Phi,\Theta}(\lambda, u) &= \sum_{ijk \in T} 2\hat{V}(\lambda_{ij}, \lambda_{jk}, \lambda_{ki}, -u_i, -u_j, -u_k) - \sum_{ij \in E} \Phi_{ij} \lambda_{ij} + \sum_{i \in V} \Theta_i u_i. \end{aligned} \quad (6.1)$$

Compare equation (4.6). If  $\Phi$  is defined by equation (4.7), then

$$\mathcal{E}_{T,\Phi,\Theta}(\lambda, u) = E_{T,\Theta,\lambda}(u).$$

So if we fix  $\lambda$  and vary  $u$ , then we obtain the first variational principle for discrete conformal maps. If, on the other hand, we fix  $u$  and vary  $\lambda$ , then we obtain a variational principle for circle patterns (see Proposition 6.1.3 below). Interpret the circumcircles and triangle sides as hyperbolic planes in the half-plane model. Then, using equation (4.5) and the fact that opposite dihedral angles in an ideal tetrahedron are equal, one gets for an interior edge  $ij \in E_{int}$  that

$$\frac{\partial}{\partial \lambda_{ij}} \mathcal{E}_{T,\Phi,\Theta} = \tilde{\alpha}_{ij}^k + \tilde{\alpha}_{ji}^l - \Phi_{ij},$$

where  $\tilde{\alpha}$  are the angles in the euclidean triangulation  $(T, \tilde{\ell})$  with  $\tilde{\ell}$  determined by equations (2.1) and (2.2). (Here we assume that  $\tilde{\ell}$  satisfies the triangle inequalities. Otherwise the angles are 0 or  $\pi$  as stipulated in Proposition 3.4.4.) In the same way, one gets for a boundary edge  $ij$

$$\frac{\partial}{\partial \lambda_{ij}} \mathcal{E}_{T,\Phi,\Theta} = \tilde{\alpha}_{ij}^k - \Phi_{ij}.$$

This implies the following variational principle for circle patterns. (Note that  $\mathcal{E}_{T,\Phi,\Theta}(\lambda, u)$  is also convex if we fix  $u$  and consider  $\lambda$  as variables.)

**6.1.3 Proposition.** *The function  $\tilde{\ell} \in \mathbb{R}_{>0}^E$  defined in terms of  $\lambda$  and  $u$  by equations (2.1) and (2.2) is a solution of Problem 6.1.1 if and only if  $\tilde{\ell}$  satisfies all triangle inequalities and  $\mathcal{E}_{T,\Theta,\lambda}(\lambda, u)$  is the minimum of the function  $\lambda \mapsto \mathcal{E}_{T,\Theta,\lambda}(\lambda, u)$ . That is,  $u$  is arbitrary but constant. (Without loss of generality one could fix  $u = 0$ .)*

## 6.2 Discrete conformal equivalence for circular polyhedral surfaces

In this section, we generalize the notion of discrete conformal equivalence from surfaces composed of triangles to surfaces composed of polygons inscribed in circles. The variational principle described below is like a mixture of the first variational principle for discrete conformal maps of Proposition 3.3.3 and the variational principle for circle patterns of Proposition 6.1.3.

An (abstract) *polyhedral surface* is a surface that is a CW complex. A *euclidean polyhedral surface* is a polyhedral surface obtained by gluing euclidean polygons edge-to-edge. If all of the polygons have a circumscribed circle, we speak of a (*euclidean*) *circular polyhedral surface*. A circular polyhedral surface is determined by the polyhedral surface  $P$  and the function  $\ell \in (\mathbb{R}_{>0})^{E_P}$  that assigns to each edge its length. Conversely, a function  $\ell \in (\mathbb{R}_{>0})^{E_P}$  defines a circular polyhedral surface if and only if it satisfies the “polygonal inequalities”: In each polygon, the length of any edge is smaller than the sum of lengths of the other edges. If  $\ell$  satisfies these conditions, we denote the resulting circular polyhedral surface by  $(P, \ell)$ .

**6.2.1 Definition.** Two circular polyhedral surfaces,  $(P, \ell)$  and  $(P, \tilde{\ell})$ , are *discretely conformally equivalent* if  $\ell$  and  $\tilde{\ell}$  are related by equation (2.1) for some function  $u \in \mathbb{R}^{V_P}$ .

To solve the discrete conformal mapping problems for circular polyhedral surfaces that are analogous to those described in Section 3.1, proceed as follows: First triangulate the non-triangular faces of the given circular polyhedral surface  $(P, \ell)$  to obtain a euclidean triangulation  $(T, \tilde{\ell})$  (where  $\tilde{\ell} : E_T \rightarrow \mathbb{R}_{>0}$ ,  $\tilde{\ell}|_{E_P} = \ell$ ). Then define  $\Phi$  by equations (4.7) and minimize

$\mathcal{E}_{T,\Phi,\Theta}(\lambda, u)$ , where  $\lambda_{ij} = 2 \log \hat{\ell}_{ij}$  is held fixed if  $ij \in E_P$  and considered a variable if  $E_T \setminus E_P$ , and the  $u_i$  are variables or fixed depending on the mapping problem, as in the case of triangulations. If  $\tilde{\ell}$  determined by equations (2.1) and (2.2) for the minimizing  $(\lambda, u)$  satisfies the triangle inequalities, it is a solution of the mapping problem.

Note that the values  $\hat{\ell}_{ij}$  for edges  $ij \in E_T \setminus E_P$  do not enter because the corresponding  $\lambda_{ij}$  are variables.

### 6.3 Discrete circle domains

A domain in the Riemann sphere  $\widehat{\mathbb{C}}$  is called a *circle domain* if every boundary component is either a point or a circle. Koebe conjectured that every domain in  $\mathbb{C}$  is conformally equivalent to a circle domain. For a simply connected domain, this is just the Riemann mapping theorem. Koebe himself proved the conjecture for finitely connected domains, and after various generalizations by several other people, He and Schramm gave a proof for domains with at most countably many boundary components [13]. (Their proof is based on circle packings.)

The method for mapping to the sphere described in Section 3.7 works (*mutatis mutandis*) also for the circular polyhedral surfaces discussed in the previous section. This allows us to map euclidean triangulations to “discrete circle domains”, that is, domains in the plane that are bounded by circular polygons.

Suppose  $(T, \ell)$  is a euclidean triangulation that is topologically a disc with holes. To map  $(T, \ell)$  to a discrete circle domain, simply fill the holes by attaching a face to each boundary polygon and map the resulting circular polyhedral surface to the sphere.

Note that for a topological disk with 0 holes, we recover in a different guise the procedure for mapping to a disk that was described in Section 3.7.

## Acknowledgments

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Figure 20 was made by Stefan Sechelmann, using software written by himself in *Java* together with *jReality*, a visualization and sonification library focused on mathematical and scientific content.

The other figures of discrete conformal maps were made with the 3D content creation suite *Blender* and *Python* scripts written by the third author. These rely on other pieces of software to do the real work: the *GNU Scientific Library*, which provides an implementation of Clausen’s integral, the Python software *CVXOPT* for convex optimization by Joachim Dahl and Lieven Vandenberghe, and the program *Triangle*, Jonathan Shewchuk’s two-dimensional quality mesh generator and Delaunay triangulator.

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