Discrete differential geometry of surfaces. Variational principles, algorithms, and implementation

Stefan Sechelmann

April 25, 2012

Contents

List of Figures

1 Introduction

2 Discrete Uniformization

2.1 Discrete conformal equivalence

Definition 1. *Two Euclidean triangulations T and $\tilde{T}$ are* discretely conformally equivalent *if there is a map $u : V \to \mathbb{R}$ such that for any edge ij it is*

$$l_{ij} = e^{u_i+u_j}\tilde{l}_{ij}$$

where l_{ij} is the length of the edge ij.

Definition 2. *A* discrete flat metric *is a map $l : E \to \mathbb{R}_+$ such that triangle inequalities are satisfied and angle sums around each inner vertex are equal to 2π.*

2.2 Variational principles for discrete metrics in $\mathbb{E}^2$, $\mathbb{H}^2$, and $\mathbb{S}^2$

Construction of discrete flat metrics. A discrete Euclidean flat metric is the minimizer of a convex functional.

$$\lambda_{ij} := 2\log l_{ij} \tag{1}$$

$$\tilde{\lambda}_{ij} := \lambda_{ij} + u_i + u_j \tag{2}$$

$$f_{Euc}(u_i, u_j, u_k) := \alpha_i\tilde{\lambda}_{jk} + \alpha_j\tilde{\lambda}_{ki} + \alpha_k\tilde{\lambda}_{ij} + 2\left(\text{Л}(\alpha_i) + \text{Л}(\alpha_j) + \text{Л}(\alpha_k)\right) \tag{3}$$

Definition 3.

$$E_{Euc}(u) := \sum_{ijk\in F}\left(f_{Euc}(u_i, u_j, u_k) - \frac{\pi}{2}\left(\tilde{\lambda}_{jk} + \tilde{\lambda}_{ki} + \tilde{\lambda}_{ij}\right)\right) + \sum_{i\in V}\Theta_i u_i \tag{4}$$

This definition and the derivatives can be found in [BPS10]

For the hyperbolic case λ and $\tilde{\lambda}$ are defined as before. Further define

$$\beta_i := \frac{1}{2}\left(\pi + \alpha_i - \alpha_j - \alpha_k\right) \tag{5}$$

$$\beta_j := \frac{1}{2}\left(\pi - \alpha_i + \alpha_j - \alpha_k\right) \tag{6}$$

$$\beta_k := \frac{1}{2}\left(\pi - \alpha_i - \alpha_j + \alpha_k\right) \tag{7}$$

$$f_{Hyp}(u_i, u_j, u_k) := \beta_i\tilde{\lambda}_{jk} + \beta_j\tilde{\lambda}_{ki} + \beta_k\tilde{\lambda}_{ij} \tag{8}$$

$$+\text{Л}(\alpha_i) + \text{Л}(\alpha_j) + \text{Л}(\alpha_k) + \text{Л}(\beta_i) + \text{Л}(\beta_j) + \text{Л}(\beta_k) \tag{9}$$

$$+\text{Л}\left(\frac{1}{2}(\pi - \alpha_i - \alpha_j - \alpha_k)\right) \tag{10}$$

Definition 4.

$$E_{Hyp}(u) := \sum_{ijk\in F}\left(f_{Hyp}(u_i, u_j, u_k) - \frac{\pi}{2}\left(\tilde{\lambda}_{jk} + \tilde{\lambda}_{ki} + \tilde{\lambda}_{ij}\right)\right) + \sum_{i\in V}\Theta_i u_i \tag{11}$$

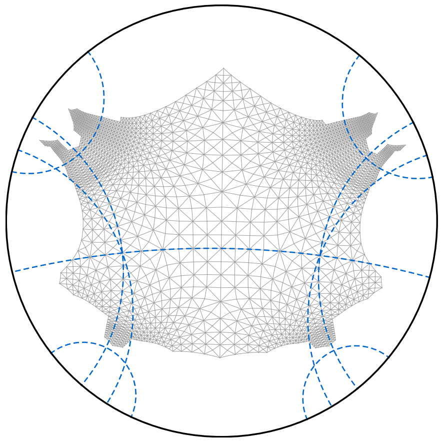

Figure 1: Hyperbolic flat metric on a genus 2 surface and the axes of the associated hyperbolic motions.

2.3 Realization

3 Uniformization of surfaces of higher genus

Triangulated surfaces of genus $g \geq 2$ without boundary can be equipped with a discretely conformally equivalent flat hyperbolic metric [BPS10]. By flat hyperbolic metric we mean that the edge length are hyperbolic and for any vertex the angle sum is 2π. To realize this metric in the hyperbolic plane e.g. in the Poicaré disk model one has to introduce cuts along a basis of the homotopy. This creates a simply connected domain in $\mathbb{H}^2$. Matching cut paths are realated by a hyperbolic motion i.e. the Möbius transformations that leave the unit disk invariant (Figure 1).

3.1 The cut-graph and fuchsian groups

Want so say here: the number of transformations generated by the mapping of corresponding edges equals the number of path segments in the homotopy-cut-graph. They generate a fuchsian group with #vertices relations

Proposition 1.

3.2 Minimal presentation

4 Canonical fundamental domains of fuchsian groups

4.1 Separated handles

4.2 Opposite sides identified

5 Uniformization of tori

5.1 Elliptic Functions

5.2 The modul space

5.3 Numerical convergence analysis

6 Uniformization of hyperelliptic surfaces

6.1 Construction

Any hyperelliptic Riemann surface can be expressed as an algebraic curve of the form

$$\mu^2 = \prod_{i=1}^{n} (\lambda - \lambda_i)^2 \qquad n \geq 3, \quad \lambda_i \neq \lambda_j \forall i \neq j.$$

Here λ_i are the branch points of the doubly covered Riemann sphere.

6.2 Weierstrass points on hyperelliptic surfaces

A hyperelliptic surface comes together with a holomorphic involution h called the hyperelliptic involution. The branch points are fixed points under this transformation. For a hyperelliptic algebraic curve it is $h(\mu, \lambda) = (-\mu, \lambda)$

6.3 Canonical domains

7 Discrete quasi-isothermic parametrizations

The notion of quasi-conformal parameterizations

7.1 Discrete quasi-isothermic parameterizations

7.2 Boundary value problem

7.3 Variational principle for S-isothermic surfaces

7.4 A discrete ellipsoid and its dual surface

8 References

[BPS10] Alexander I. Bobenko, Ulrich Pinkall, and Boris Springborn. Discrete conformal maps and ideal hyperbolic polyhedra. Preprint; `http://arxiv.org/abs/1005.2698`, 2010.