# Discrete differential geometry of surfaces. Variational principles, algorithms, and implementation

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# 2 Discrete Uniformization

## 2.1 Discrete conformal equivalence

**Definition 1.** *Two Euclidean triangulations $T$ and $\tilde{T}$ are* discretely conformally equivalent *if there is a map $u : V \to \mathbb{R}$ such that for any edge $ij$ it is*

$$l_{ij} = e^{u_i+u_j}\tilde{l}_{ij}$$

*where $l_{ij}$ is the length of the edge $ij$.*

**Definition 2.** *A* discrete flat Euclidean metric *is a map $l : E \to \mathbb{R}_+$ such that triangle inequalities are satisfied and angle sums around each inner vertex are equal to $2\pi$.*

## 2.2 Variational principles for discrete metrics in $\mathbb{E}^2$, $\mathbb{H}^2$, and $\mathbb{S}^2$

Construction of discrete flat metrics. A discrete Euclidean flat metric is the minimizer of a convex functional.

$$\lambda_{ij} := 2\log l_{ij} \tag{1}$$

$$\tilde{\lambda}_{ij} := \lambda_{ij} + u_i + u_j \tag{2}$$

$$f_{Euc}(u_i, u_j, u_k) := \alpha_i\tilde{\lambda}_{jk} + \alpha_j\tilde{\lambda}_{ki} + \alpha_k\tilde{\lambda}_{ij} + 2\left(Л(\alpha_i) + Л(\alpha_j) + Л(\alpha_k)\right) \tag{3}$$

**Definition 3.**

$$E_{Euc}(u) := \sum_{ijk\in F}\left(f_{Euc}(u_i, u_j, u_k) - \frac{\pi}{2}\left(\tilde{\lambda}_{jk} + \tilde{\lambda}_{ki} + \tilde{\lambda}_{ij}\right)\right) + \sum_{i\in V}\Theta_i u_i \tag{4}$$

This definition and the derivatives can be found in [BPS10]

For the hyperbolic case $\lambda$ and $\tilde{\lambda}$ are defined as before. Further define

$$\beta_i := \frac{1}{2}(\pi + \alpha_i - \alpha_j - \alpha_k) \tag{5}$$

$$\beta_j := \frac{1}{2}(\pi - \alpha_i + \alpha_j - \alpha_k) \tag{6}$$

$$\beta_k := \frac{1}{2}(\pi - \alpha_i - \alpha_j + \alpha_k) \tag{7}$$

$$f_{Hyp}(u_i, u_j, u_k) := \beta_i\tilde{\lambda}_{jk} + \beta_j\tilde{\lambda}_{ki} + \beta_k\tilde{\lambda}_{ij} \tag{8}$$

$$+Л(\alpha_i) + Л(\alpha_j) + Л(\alpha_k) + Л(\beta_i) + Л(\beta_j) + Л(\beta_k) \tag{9}$$

$$+Л\left(\frac{1}{2}(\pi - \alpha_i - \alpha_j - \alpha_k)\right) \tag{10}$$

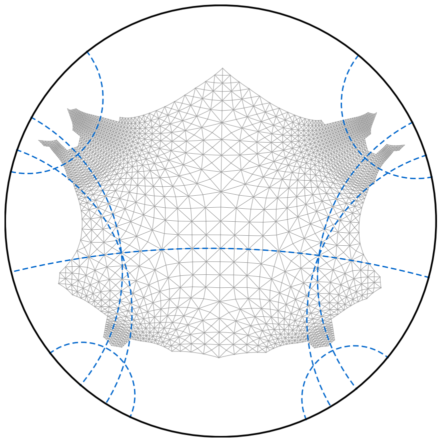

Figure 1: Hyperbolic flat metric on a genus 2 surface and the axes of the associated hyperbolic motions.

**Definition 4.**

$$E_{Hyp}(u) \quad := \quad \sum_{ijk \in F} \left( f_{Hyp}(u_i, u_j, u_k) - \frac{\pi}{2} \left( \tilde{\lambda}_{jk} + \tilde{\lambda}_{ki} + \tilde{\lambda}_{ij} \right) \right) + \sum_{i \in V} \Theta_i u_i \tag{11}$$

### 2.3 Realization

## 3 Uniformization of surfaces of higher genus

Triangulated surfaces of genus $g \geq 2$ without boundary can be equipped with a discretely conformally equivalent flat hyperbolic metric [BPS10]. By flat hyperbolic metric we mean that the edge length are hyperbolic and for any vertex the angle sum is $2\pi$. To realize this metric in the hyperbolic plane e.g. in the Poicaré disk model one has to introduce cuts along a basis of the homotopy. This creates a simply connected domain in $\mathbb{H}^2$. Matching cut paths are realated by a hyperbolic motion i.e. the Möbius transformations that leave the unit disk invariant (Figure 1).

### 3.1 The cut-graph and fuchsian groups

*Want so say here: the number of transformations generated by the mapping of corresponding edges equals the number of path segments in the homotopy-cut-graph. They generate a fuchsian group with #vertices relations*

**Proposition 1.**

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### 6.1 Construction

Any hyperelliptic Riemann surface can be expressed as an algebraic curve of the form

$$\mu^2 = \prod_{i=1}^{n} (\lambda - \lambda_i)^2 \qquad n \geq 3, \quad \lambda_i \neq \lambda_j \forall i \neq j.$$

Here $\lambda_i$ are the branch points of the doubly covered Riemann sphere.

### 6.2 Weierstrass points on hyperelliptic surfaces

A hyperelliptic surface comes together with a holomorphic involution $h$ called the hyperelliptic involution. The branch points are fixed points under this transformation. For a hyperelliptic algebraic curve it is $h(\mu, \lambda) = (-\mu, \lambda)$

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[BPS10] Alexander I. Bobenko, Ulrich Pinkall, and Boris Springborn. Discrete conformal maps and ideal hyperbolic polyhedra. Preprint; http://arxiv.org/abs/1005.2698, 2010.