Variational Methods for Discrete Surface Parameterization. Applications and Implementation.

vorgelegt von
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1.1 Branched coverings of $\hat{\mathbb{C}}$

In this section we discuss Riemann surfaces that arise as branched coverings of $\hat{\mathbb{C}}$. A Riemann surface that can be represented as a double cover of $\hat{\mathbb{C}}$ is called elliptic for $g = 1$ and hyperelliptic for $g > 1$ [2, p. 235].

Let $\lambda_1, \ldots, \lambda_{2g+2} \in \hat{\mathbb{C}}$, $\lambda_i \neq \lambda_j \forall i \neq j$.. The algebraic curve

$$C = \{(z, \mu) \in \mathbb{C}^2 \mid \mu^2 = \prod_{i=1}^{2g+2} (z - \lambda_i)\} \tag{1.1}$$

is a one dimensional complex manifold. A branched double cover of $\hat{\mathbb{C}}$ is the projection $\pi : C \to \mathbb{C}$, $C \ni (z, \mu) \mapsto z$. A discrete branched cover of $\hat{\mathbb{C}}$ is a triangulation of $\pi(C)$ with vertices at λ_i.

1.1.1 Elliptic curves

1.1.2 The moduli space

1.1.3 Numerical convergence analysis

1.1.4 Construction of hyperelliptic surfaces

Any hyperelliptic Riemann surface can be expressed as an algebraic curve of the form

$$w^2 = \prod_{i=1}^{2g+2} (z - \lambda_i) \qquad g \geq 1, \quad \lambda_i \neq \lambda_j \forall i \neq j.$$

Here λ_i are the branch points of the doubly covered Riemann sphere.

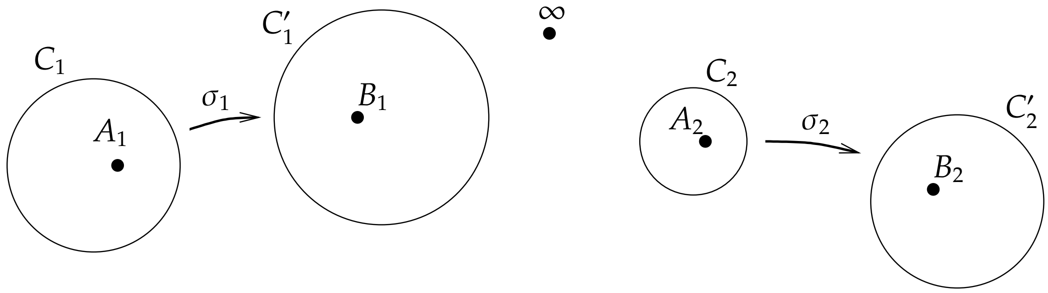

Figure 1.1: Schottky group generating a Riemann surface of genus 2. The point at infinity is not contained in any of the circles. The parameters A and B lie inside the cirlces by definition.

1.1.5 Weierstrass points on hyperelliptic surfaces

A hyperelliptic surface comes together with a holomorphic involution h called the hyperelliptic involution. The branch points are fixed points under this transformation. For a hyperelliptic algebraic curve it is $h(\mu, \lambda) = (-\mu, \lambda)$

1.1.6 Canonical domains

1.1.7 The Wente torus

1.1.8 Lawsons surface

1.2 Schottky to Fuchsian Uniformization

Let $C_1, C_1' \ldots, C_g, C_g'$ be disjoint circles in $\hat{\mathbb{C}}$. A classical Schottky group G is a Kleinian group with generators $\sigma_1, \ldots, \sigma_g$ satisfying

$$\frac{\sigma_i z - B_i}{\sigma_i z - A_i} = \mu_i \frac{z - B_i}{z - A_i}, \qquad 0 < |\mu_i| < 1,$$

where σ_i maps the exterior of C_i onto the interior of C_i'. The points A_i and B_i lie inside the circles C_i and B_i respectively [1]. Figure 1.1 illustrates this construction. The quotient space $\hat{\mathbb{C}}/G$ is a Riemann surface of genus g. The group G is called the uniformizing Schottky group

Given a Riemann surface of genus g and a cassical Schottky uniformization, we calculate a corresponding Fuchsian uniformization using discrete conformal equivalence of spherical and hyperbolic triangle meshes. A fundamental domain of the Schottky uniformization is the Riemann sphere without the interior of the circles.

1.2.1 Discretization

We start with a classical Schottky group G. Let $\sigma_1, \ldots, \sigma_g$ be the generators of G. We choose circles around the fixed points of the

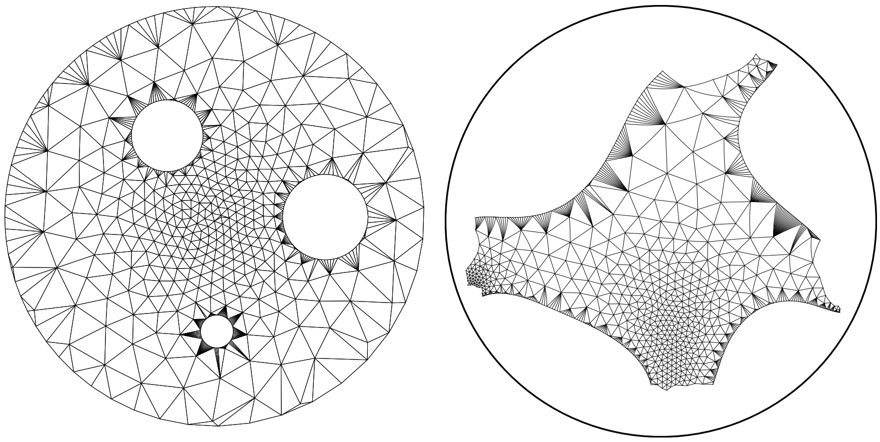

Figure 1.2: Fuchsian uniformization of a Riemann surface (right) given by Schottky data (left).

The triangulation of a fundamental domain has to respect the mappings between the circles. Vertices on identified circles have to be identified as well. It is however not clear a priori what the conformal structure of a triagulated fundamental domain is. The edge length on identified circles differ since there is a Möbius transformation between them.

The idea to define the conformal structure is to use length cross-ratios instead of edge lengths. Both definitions are equivalent and define the conformal structure of a discrete Riemann surface. So given length cross ratios on edges we can calculate representative edge lengths in the discrete conformal class of the surface and, more obvious, vice versa. On boundary edges of the fundamental domain the length cross ratios agree since length-cross ratios are Möbius invariant.

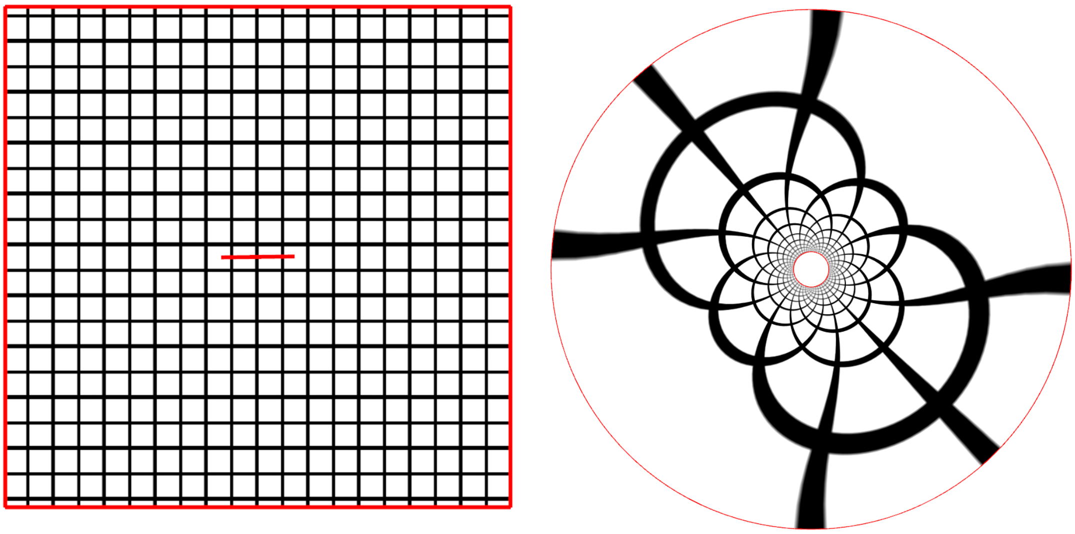

Figure 1.3: Square with symmetric slit to the circle

1.2.2 Images of isometric circles

1.2.3 Hyperelliptic data

1.3 Conformal maps to $\hat{\mathbb{C}}$

1.3.1 Selection of Branch Data

1.3.2 Examples

1.4 Surfaces with boundary

1.4.1 Variation of edge length

1.4.2 Examples

1.5 Conformal maps of planar domains

1.5.1 Boundary conditions

1.5.2 Comparison with examples of the Schwarz-Christoffel community

Part II

Variational Methods for Discrete Surface Parameterization

Part III

Software Packages

Bibliography

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Acknowledgements