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A unique representation of polyhedral types. Centering via Möbius transformations

Boris A. Springborn[⋆]

Technische Universität Berlin, Institut für Mathematik, Sekr. MA 8-5, Strasse des 17. Juni 136, 10623 Berlin, Germany (e-mail: `springb@math.tu-berlin.de`)

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Abstract. For $n \geq 3$ distinct points in the d-dimensional unit sphere $S^d \subset \mathbb{R}^{d+1}$, there exists a Möbius transformation such that the barycenter of the transformed points is the origin. This Möbius transformation is unique up to post-composition by a rotation. We prove this lemma and apply it to prove the uniqueness part of a representation theorem for 3-dimensional polytopes as claimed by Ziegler (1995): For each polyhedral type there is a unique representative (up to isometry) with edges tangent to the unit sphere such that the origin is the barycenter of the points where the edges touch the sphere.

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In today's language, Steinitz' *fundamental theorem of convex types* [14], [15] (for a modern treatment see [7], [17]) states that the combinatorial types of convex 3-dimensional polyhedra correspond to the strongly regular cell decompositions of the 2-sphere. (A cell complex is *regular* if the closed cells are attached without identifications on the boundary. A regular cell complex is *strongly regular* if the intersection of two closed cells is a closed cell or empty.)

Grünbaum and Shephard [8] posed the question whether for every combinatorial type there is a polyhedron with edges tangent to a sphere. This question has been answered affirmatively:

Theorem 1 (Koebe [10], Andreev [1], [2], Thurston [16], Brightwell and Scheinerman [5], Schramm [12])**.** *For every combinatorial type of convex* 3*-dimensional polyhedra, there is a representative with edges tangent to the unit sphere. This representative is unique up to projective transformations which fix the sphere and do not make the polyhedron intersect the plane at infinity.*

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A proof which makes use of a variational principle was given by A. Bobenko and the author [4], [13].

The purpose of this article is to prove Theorem 2 below, which singles out a unique representative for each convex type. (The proof given here is also contained in the author's doctoral dissertation [13].) The claim of Theorem 2 is not new (see Ziegler [17], p. 118, and the second edition of Grünbaum's classic [7], p. 296a) but this proof seems to be.

Theorem 2. *For every combinatorial type of convex 3-dimensional polyhedra there is a unique polyhedron (up to isometry) with edges tangent to the unit sphere $S^2 \subset \mathbb{R}^3$, such that the origin $0 \in \mathbb{R}^3$ is the barycenter of the points where the edges touch the sphere.*

Theorem 2 follows from Theorem 1 and Lemma 1 below (with $d = 2$). Indeed, the projective transformations of $\mathbb{R}\mathrm{P}^{d+1}$ that fix S^d correspond to the Möbius transformations of S^d. Lemma 1 is also of interest in its own right.

Lemma 1. *Let $v_1, \dots, v_n$ be $n \geq 3$ distinct points in the d-dimensional unit sphere $S^d \subset \mathbb{R}^{d+1}$. There exits a Möbius transformation T of S^d, such that*

$$\sum_{j=1}^{n} T v_j = 0.$$

If $\widetilde{T}$ is another such Möbius transformation, then $\widetilde{T} = RT$, where R is an isometry of S^d.

Our proof of Lemma 1 is based on the fundamental relationship between projective, hyperbolic, and Möbius geometry. The equation

$$-x_0^2 + x_1^2 + x_2^2 + \dots + x_{d+1}^2 = 0$$

represents the d-dimensional sphere S^d as a quadric in $(d+1)$-dimensional projective space $\mathbb{R}\mathrm{P}^{d+1}$. The group of projective transformations of $\mathbb{R}\mathrm{P}^{d+1}$ which fix S^d is $O(d+1, 1)/\{\pm 1\}$, where the orthogonal group $O(d+1, 1) \subset GL(d+2)$ acts linearly on the homogeneous coordinates. At the same time, $O(d+1, 1)/\{\pm 1\}$ acts faithfully as the Möbius group on S^d, and as the isometry group of $(d+1)$-dimensional hyperbolic space H^{d+1}, which is identified with the open ball bounded by S^d (the Klein model of hyperbolic space). For a detailed account of this classical material see, for example, Hertrich-Jeromin [9] and Kulkarni, Pinkall [11].

A similar interplay of geometries leads Bern and Eppstein, to another choice of a unique representative for each polyhedral type. Given n spheres in S^d, Bern and Eppstein apply that Möbius transformation which makes the smallest sphere as large as possible [3]. It is not difficult to see that this Möbius transformation is unique up to post-composition with a rotation if $n \geq 3$. Since edge-tangent polyhedra correspond to circle packings, this leads to another choice of unique representative for each polyhedral type [6].

For symmetric polyhedral types (more precisely, for those polyhedral types with a symmetry group of orientation preserving isomorphisms which is not just a cyclic group) the unique representative of Bern and Eppstein coincides with ours.

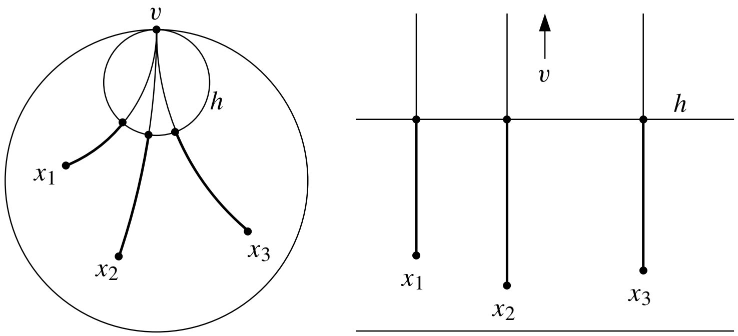

Fig. 1. The 'distance' to an infinite point v is measured by cutting off at some horosphere through v. *Left:* Poincaré ball model. *Right:* half-space model

Proof of Lemma 1. The Möbius transformations of S^d correspond to isometries of the hyperbolic space H^{d+1}, of which S^d is the infinite boundary.

For $n \geq 3$ points $v_1, \dots, v_n \in S^d$, we are going show that there is a unique point $x \in H^{d+1}$ such that the sum of the 'distances' to $v_1, \dots, v_n$ is minimal. Of course, the distance to a point in the infinite boundary is infinite. The right quantity to use instead is the distance to a horosphere through the infinite point (see the figure). □

Definition. For a horosphere h in H^{d+1}, define

$$\delta_h : H^{d+1} \to \mathbb{R},$$

$$\delta_h(x) = \begin{cases} -\operatorname{dist}(x, h) & \text{if } x \text{ is inside } h, \\ 0 & \text{if } x \in h, \\ \operatorname{dist}(x, h) & \text{if } x \text{ is outside } h, \end{cases}$$

where $\operatorname{dist}(x, h)$ is the distance from the point x to the horosphere h.

Suppose v is the infinite point of the horosphere h. Then the shortest path from x to h lies on the geodesic connecting x and v. If h' is another horosphere through v, then $\delta_h - \delta_{h'}$ is constant. If $g : \mathbb{R} \to H^{d+1}$ is an arc-length parametrized geodesic, then $\delta_h \circ g$ is a strictly convex function, unless v is an infinite endpoint of the geodesic g. In that case, $\delta_h \circ g(s) = \pm(s - s_0)$. These claims are straightforward to prove using the Poincaré half-space model, where hyperbolic space is identified with the upper half space:

$$H^{d+1} = \{(x_0, \dots, x_d) \in \mathbb{R}^{d+1} \mid x_0 > 0\},$$

and the metric is

$$ds^2 = \frac{1}{x_0^2}\left(dx_0^2 + dx_1^2 + \cdots + dx_d^2\right)$$

Also, one finds that, as $x \in H^{d+1}$ approaches the infinite boundary,

$$\lim_{x \to \infty} \sum_{j=1}^{n} \delta_{h_j}(x) = \infty,$$

where h_j are horospheres through different infinite points and $n \geq 3$. Thus, the following definition of the *point of minimal distance sum* is proper.

Lemma (and Definition) 2. *Let $v_1, \dots, v_n$ be n points in the infinite boundary of H^{d+1}, where $n \geq 3$. Choose horospheres $h_1, \dots, h_n$ through $v_1, \dots, v_n$, respectively. Then there is a unique point $x \in H^{d+1}$ for which $\sum_{j=1}^{n} \delta_{h_j}(x)$ is minimal. This point x does not depend on the choice of horospheres. It is the* point of minimal distance sum *from the infinite points $v_1, \dots, v_n$.*

In the Poincaré ball model, hyperbolic space is identified with the unit ball as in the Klein model, but the metric is $ds^2 = \frac{4}{(1-\sum x_j^2)^2} \sum dx_j^2$. (Since the Klein model and the Poincaré ball model agree on the infinite boundary and in the center of the sphere, one might as well use the Klein model in the following lemma.)

Lemma 3. *Let $v_1, \dots, v_n$ be $n \geq 3$ different points in the infinite boundary of H^{d+1}. In the Poincaré ball model, $v_j \in S^d \subset \mathbb{R}^{d+1}$. The origin is the point of minimal distance sum, if and only if $\sum v_j = 0$.*

Proof. If h_j is a horosphere through v_j, then the gradient of δ_{h_j} at the origin is the unit vector $-\frac{1}{2} v_j$. □

Lemma 1 is now almost immediate. Let x be the point of minimal distance sum from the $v_1, \dots, v_n$ in the Poincaré ball model. There is a hyperbolic isometry T which moves x into the origin. If $\widetilde{T}$ is another hyperbolic isometry which moves x into the origin, then $\widetilde{T} = RT$, with R is an orthogonal transformation of $\mathbb{R}^{d+1}$.

This concludes the proof of Lemma 1.

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