Quasiisothermic Mesh Layout

Abstract. *The quality of a quad-mesh depends on the shape of the individual quadrilaterals. The ideal shape from an architectural point of view is the planar square or rectangles with fixed aspect ratio. A parameterization that divides a surface into such shapes is called isothermic, i.e., angle-preserving and curvature-aligned. Such a parameterization exists only for the special class of isothermic surfaces. We extend this notion and introduce quasiisothermic parameterizations for arbitrary triangulated surfaces.*

We describe an easy-to-implement algorithm that creates quasiisothermic meshes. Interestingly many surfaces appearing in architecture are close to isothermic surfaces, namely those coming from form finding methods and physical simulation. For those surfaces our method works particularly well and gives a high quality and robust mesh layout. We show how to optimize such meshes further to obtain disk packing representations. The quadrilaterals of these meshes are planar and possess touching incircles.

1 Introduction

A key problem in architectural geometry is to convert surfaces created by form finding methods, physical simulation, or manual modeling to quadrilateral meshes, which are preferred for glass-steel structures. There are many possible quad-meshes that approximate a given shape and we study those that consist of principle-curvature-aligned conformal squares (see Fig. 2). Not all surface shapes can be approximated by such meshes. A smooth analog of a surface with this property is called an isothermic surface. These surfaces admit conformal curvature line parameterizations, i.e., angle-preserving parameterizations aligned with the principle curvature directions. Their discrete counterpart are so-called S-isothermic meshes. These meshes have the additional property that neighboring quadrilaterals possess touching incircles (see Fig. 8).

The class of isothermic surfaces comprises, for example, constant mean curvature surfaces. Roofs that act shell-like turn out to have almost constant mean curvature. These are the kinds of surfaces that initiated our study of conformal curvature line parameterizations in the architectural context. Both conformality and alignment with curvature lines are favorable properties for meshes.

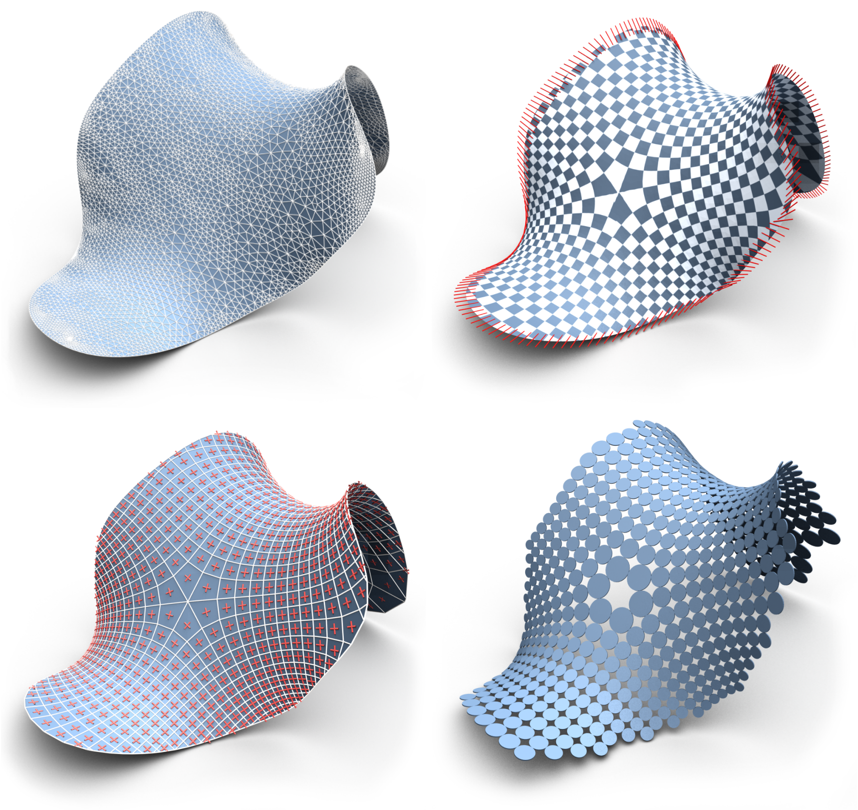

Figure 1: The algorithmic steps of this paper: For a triangulated surface we calculate texture coordinates by solving a boundary value problem for principle curvature directions on boundary edges (checker board texture and red directions). The edges of the corresponding quad mesh align with the curvature directions (red crosses). The mesh is then optimized towards planar quads with touching incircles.

The contributions of this paper are:
Definition of quasiisothermic parameterizations: We propose a definition of quasiisothermic parameterizations of triangle meshes. It is based on angles between curvature directions and edges of a triangle mesh. We define the quasiisothermic modulus that measures how isothermic a parameterization is. If this modulus is zero we obtain discrete isothermic parameterizations in the sense of our definition.
Algorithm for quasiisothermic parameterizations: We give an algorithm that creates quasiisothermic parameterizations based on discrete conformal maps of triangle meshes to the plane. This approach is build on top of the conformal mapping technique of [Springborn et al. 2008]. We inherit the speed and superior projective mapping properties of their parameterizations.

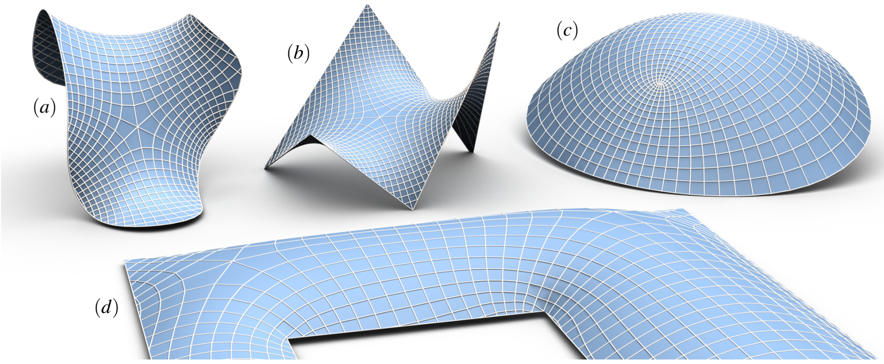

Figure 2: The surface examples of this paper. All have been parameterized and remeshed. (a) The TEASER surface is the minimizer of a spring energy with a smooth fixed boundary curve. (b) A MINIMAL surface with polygonal boundary curve. (c) DOME: Part of a NURBS surface exhibiting positive curvature and two curvature field singularities. (d) ROOF structure with planar boundary curve and regions of positive and negative curvature.

Variational principle for circle packing quad-meshes: The obtained parameterizations are used for remeshing and we optimize quadrilaterals to have touching incircles by minimizing a novel energy. These S-isothermic meshes have been studied in discrete differential geometry and possess some remarkable properties, e.g., minimal S-isothermic surfaces may be deformed isometrically retaining the same Gauß map.

The rest of the paper is organized as follows: Section 2 gives an overview of existing parameterization schemes and their relation to our approach. We also give reference to the related mathematical literature in discrete differential geometry. In Section 3 we define quasiisothermic parameterizations and a corresponding quality measure. In Section 4 we describe an algorithm to obtain quasiisothermic parameterizations with small modulus. We describe the connection to discrete conformal maps and discuss how we deal with singularities. A variational principle to generate S-isothermic meshes is presented in Section 5. At the end of the section we show the effect of our optimization on several examples from different classes of surfaces. In the final Section 6 we sum up the results and propose extensions and enhancements subject to further research.

2 Related Work

There has been considerable work on conformal parameterizations as well as on curvature line parameterizations related to our quasiisothermic scheme. We can only give a selection of previous work here. For a general background on mesh parameterization we refer to the surveys by [Floater and Hormann 2005] and [Sheffer et al. 2006].

Our algorithmic approach is based on the discrete conformal equivalence of triangle meshes introduced in [Springborn et al. 2008] (see [Bobenko et al. 2010] for the mathematical background). The convex functional optimized in Springborn *et al.* constructs a conformally equivalent flat mesh for specified boundary conditions and singularities. Our work is related to [Sheffer and de Sturler 2001]. They aim for conformal parameterizations and express this by the additional constraint, that triangle angles have to be close to the original ones on the surface. For discrete isothermic parameterizations the definitions coincide.

Parameterizations aligning with lines of principle curvature were constructed by [Alliez et al. 2003]. Their method involves the integration of curvature vector fields and does not include an optimization towards conformality. Global parameterizations following arbitrary frame fields (including in particular principle curvature fields) are constructed in [Kälberer et al. 2007]. They use discrete Hodge decomposition and harmonic vector fields to obtain a globally consistent parameterization. Their QuadCover algorithm can deal with surfaces of arbitrary genus and treats singularities using a suitable branched cover. Both algorithms cover arbitrary triangulated surfaces and implementations are highly complex.

The use of variational principles to enforce desired properties such as planarity of quadrilateral faces has been successfully used in architectural geometry. [Liu et al. 2006] propose an algorithm to optimize a quadrilateral mesh to become planar and even conical. [Pottmann et al. 2008] use functionals to approximate freeform surfaces with single curved panels. The energy minimized in Section 5 is a combination of a new functional with an energy recently described by [Schiftner et al. 2009]. They construct circle packing triangle meshes that approximate a given surface by minimizing a combination of energies. Discrete S-isothermic minimal surfaces are defined in terms of their Gauß map in [Bobenko et al. 2006]. This Gauß map is a Koebe polyhedron with edges tangent to a sphere. These Koebe polyhedra also occur in the study of edge offset meshes by [Pottmann et al. 2007], which again use a variational approach to obtain support structures. Another parametrization technique creating quad-dominant meshes guided by conjugate parameter directions is given by [Zadravec et al. 2010]. Their algorithm includes a level set approach to circumvent the integration of a vector field.

The notion of discrete S-isothermic meshes was introduced in the mathematical context by [Bobenko and Pinkall 1999] as a special class of quad meshes. The mathematical theory of these meshes has since then been an active field of research in discrete differential geometry. A good overview of the recent development and literature can be found in the book [Bobenko and Suris 2008].

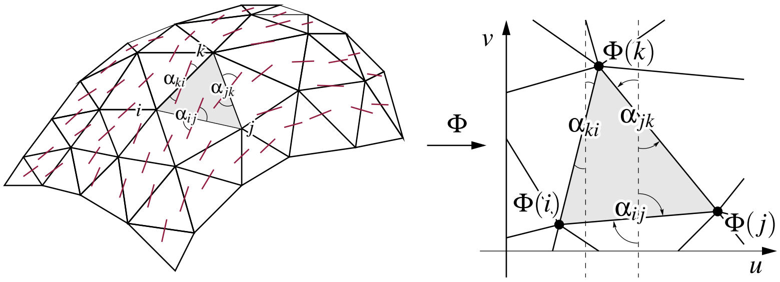

Figure 3: A discrete isothermic parameterization. Angles between triangle edges and a curvature direction family are preserved by the map.

3 Discrete Quasiisothermic Parameterization

In this section we introduce the notion of quasiisothermic parameterizations and the corresponding quality measure.

3.1 Discrete Parameterizations

Let $M = (V, E, F)$ be a triangle mesh. The elements of V are the vertices of the mesh denoted by simple indices $i \in V$. Edges are denoted by double indices $ij \in E$, and faces are denoted $ijk \in F$. A triangulated surface is a map $S : V \to \mathbb{R}^3$, $i \mapsto (x_i, y_i, z_i)$. We call a map $\Phi : V \to \mathbb{R}^2$, $i \mapsto (u_i, v_i)$ a *discrete parameterization* of the surface S. We only consider orientable surfaces and parameterizations that preserve the orientation of the triangles with respect to the canonical orientation of $\mathbb{R}^2$.

The next definition connects arbitrary parameterizations with certain directions tangent to the surface S, e.g., curvature directions. Such a direction is encoded as an angle per edge.

Definition 3.1 *Let $\alpha : E \to]-\frac{\pi}{2}, \frac{\pi}{2}]$, $ij \mapsto \alpha_{ij}$ be a map that assigns an angle to each edge. A discrete parameterization $\Phi : V \to \mathbb{R}^2$, $i \mapsto (u_i, v_i)$ is called a* discrete parameterization with α *if*

$$\tan \alpha_{ij} = \frac{u_i - u_j}{v_i - v_j} \tag{1}$$

for all edges of the mesh.

In other words, in a parameterization with α the image of an edge $ij \in E$ under the map Φ encloses the prescribed angle α_{ij} with the v-axis of the parameter space. One could equally use the u-axis here.

3.2 Quasiisothermic Parameterizations

Our main example of a parameterization with an angle function α comes from α defined by the curvature directions of a surface $S : V \to \mathbb{R}^3$ (see Fig. 3). For a triangulated surface curvature directions and magnitudes can be calculated and assigned to edges. This is usually done by averaging curvature information over neighborhoods of points on the surface [Cohen-Steiner and Morvan 2003]. A discrete parameterization with angle function α stemming from the curvature direction field is then called a *discrete isothermic parameterization*. Indeed, the latter is just a curvature line parameterization. In our case the curvature directions are mapped to the coordinate directions in the (u,v)-plane. The map can be treated as conformal: Angles between edges and curvature directions are preserved.

Generic surfaces do not allow for isothermic parameters (those admitting isothermic parameterizations are called isothermic surfaces). Therefore we do not expect a parameterization with given α to exist in general. To be able to deal with arbitrary surfaces we introduce the notion of discrete quasiisothermic parameterization. The idea is to obtain a parameterization with angles $\tilde{\alpha}$ as close to the curvature directions α as possible. Let

$$Q^{\alpha}(ij) = |\alpha(ij) - \tilde{\alpha}(ij)| \tag{2}$$

where $\tilde{\alpha}(ij)$ is angle the between the v-axis and the edge ij in parameter space.

Definition 3.2 *We call a discrete parameterization* Φ *with angle function* α quasi-isothermic *with modulus* $Q \in \mathbb{R}_+$ *if*

$$Q^{\alpha}(ij) \leq Q \tag{3}$$

for all edges $ij \in E$.

The motivation for this measure of quasiisothermicity is the following: If Q is small the directions of principle curvature on the surface are almost tangent to the parameter lines of the parameterization. For a modulus of zero we have an in a sense angle preserving map where edges enclose the same angles with the coordinate axes in parameter space as with curvature directions on the surface. We will now create parameterizations that have small Q.

4 Minimization of the Functional

In this section the surface M is a triangulated surface with one boundary component. We will now construct a function Φ that has zero approximation error Q^{α} at boundary edges and is a discrete conformal map in the sense of [Springborn et al. 2008] in the interior of the surface. We argue under which circumstances this leads to nice behaviour in the interior. We start by briefly introducing discrete conformal maps and the boundary conditions we need for our purposes.

4.1 Discrete Conformal Maps

We recall the definition by Springborn *et al.* of discrete conformal maps via conformal equivalence of triangle meshes. It is stated in terms of lengths of the surface edges and corresponding parameter edges in the (u,v)-plane.

Definition 4.1 *A discrete parameterization* Φ *is* conformal *if there exists a function* $\mu : V \to \mathbb{R}$, $i \mapsto \mu_i$ *such that the following condition for the edge lengths* l_{ij} *on the surface and* $\tilde{l}_{ij} = \|\Phi(i) - \Phi(j)\|$ *in parameter space holds*

$$\tilde{l}_{ij} = \mu_i \mu_j l_{ij}. \tag{4}$$

For a given triangle mesh there is a unique solution μ that retains the boundary edge lengths. Another unique solution can be obtained by fixing the angles between consecutive boundary edges. These angles have to be chosen consistently obeying the Gauß-Bonnet relation (see Sec. 4.2).

The function μ for a given triangle mesh can be found as the minimizer of a convex functional. Thus its computation is efficient. The resulting parameterization is created by a breadth-first layout that enumerates all triangles and assigns texture coordinates. In addition to boundary angles one can ask for solutions that contain special interior vertices where the sum of triangle angles is not equal to 2π (see Fig. 1). These so-called cone points will be inserted at singularities of the parameterization.

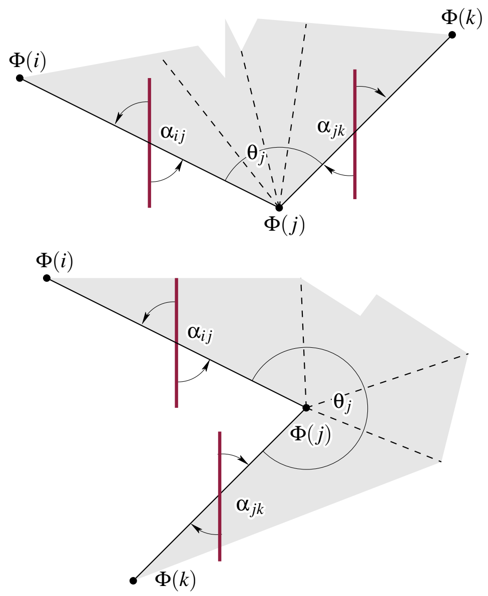

Figure 4: Curvature boundary conditions: In parameter space the interior angle θ_j at the vertex $\Phi(j)$ has to be chosen such that curvature directions given by angles α_{ij} and α_{jk} align with the v-axis. This choice is unique up to addition of $k\pi$. The above pictures show two possible layouts in the parameter plane depending on the interior angle sum on the surface.

4.2 Curvature Boundary Conditions

Let $\alpha : E \to]-\frac{\pi}{2}, \frac{\pi}{2}]$ be an angle function derived from numerical curvature directions and the given surface orientation. Let $ij \in E$ and $jk \in E$ be two consecutive boundary edges with common vertex j and let $\tilde{\theta}_j$ be the sum of interior triangle angles on the surface at this vertex. The direction of the edge $\Phi(ij)$ (resp. $\Phi(jk)$) is determined by the angle α_{ij} (resp. α_{jk}), since the curvature directions encoded by the α's should align with the v-axis. The orientation of the surface (resp. the boundary edges) defines the angle θ_j between the edges $\Phi(ij)$ and $\Phi(jk)$ up to addition of $k\pi$ (see Fig. 4). To make the angle unique we require that the difference between the original surface angle $\tilde{\theta}_j$ and the angle in the parameter plane θ_j is as small as possible, i.e., choose $k > 0$ as small as possible such that

$$|\underbrace{\angle(\Phi(ij), \Phi(jk)) + k\pi}_{\theta_j} - \tilde{\theta}_j| \leq \pi/2.$$

See Figure 4 for an illustration of the alignment in the parameter plane. The angles θ_j at boundary vertices serve as boundary conditions for the discrete conformal parameterization. A conformal parameterization with these boundary conditions will then have perfect alignment of curvature directions at the boundary.

4.3 Singularities

There exists an analog of the smooth Gauß-Bonnet theorem for discrete surfaces that relates the Gaussian and the boundary curvature to the Euler characteristic. During parameterization we construct a metric that is flat everywhere except for cone singularities where positive or negative curvatures are introduced. For the purpose of curvature line parameterizations we can only have cone points with discrete curvatures of π, 0, or $-k\pi$ at singularities of the curvature direction field. The Gaussian curvature κ_i at interior vertices $i \in V_I$ is the angle defect, i.e., $\kappa_i = 2\pi - \theta_i$, where θ_i is the sum of the angles at the vertex i. For a boundary vertex $j \in V_B$ the corresponding geodesic curvature is defined by $\kappa_j^g = \pi - \theta_j$. So if we split the vertex set $V = V_B \cup V_I$ into boundary vertices V_B and interior vertices V_I the discrete Gauß-Bonnet theorem becomes:

$$\sum_{i \in V_I} \kappa_i + \sum_{j \in V_B} \kappa_j^g = 2\pi\chi, \tag{5}$$

where χ is the Euler characteristic of the surface ($\chi = 1$ for disks). Since all curvature directions at boundary edges in parameter space become parallel, the boundary curvature adds up to a multiple of π. If this sum happens to differ from 2π, there must be singularities in the curvature field and we have to compensate the deficit at interior vertices to satisfy Equation (5). In Figure 5 the boundary curvature sum of the domain is 4π. So by inspection of the curvature field in the interor of the surface we picked two singularities each of curvature $-\pi$ to satisfy the Gauß-Bonnet equation. They correspond to cone points with angle 3π in the parameterization.

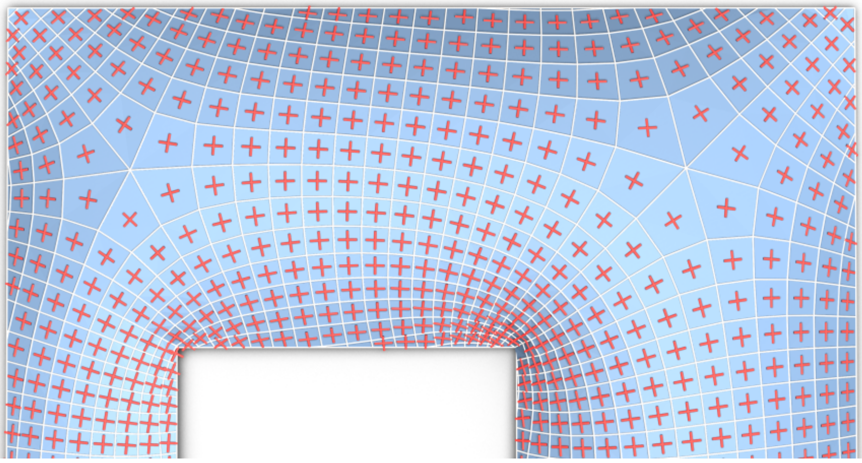

Figure 5: Parameterization of the ROOF model. The discrete curvature lines approximate curvature directions with high quality. See Section 4.5 for a discussion.

4.4 Implementation

The algorithm to compute a quasiisothermic parameterization with vanishing modulus at the boundary of the domain is the following:

1. Generate curvature directions and compute α at the boundary
2. Calculate boundary angles θ and pick singularities
3. Compute conformal parameterization with given θs
4. Perform remeshing
5. Remove cone point cuts

To estimate principle curvature fields we use the method of Cohen-Steiner and Morvan [Cohen-Steiner and Morvan 2003], where the curvature tensor is averaged over a disk of a given radius centered at edge midpoints. Together with a fixed orientation of the surface this defines the angle function α.

We deduce the angles θ for the boundary vertices as described in Section 4.2. These angles are the boundary curvatures we plug into the algorithm of Springborn *et al.* to obtain a conformal parameterization. If necessary, we pick singularities for the curvature field at vertices and prescribe corresponding cone angles by inspection of the curvature direction field on the interior of the surface. A consistent singularity choice can easily be checked using Equation (5). By construction we can only process curvature fields with isolated singularities.

We layout the new edges in the parameter plane such that an arbitrary boundary edge $\Phi(ij)$ intersects the v-axis in the desired angle α_{ij}. By construction the intersection angles coincide with the prescribed α's for all boundary edges. The domain of parameterization can contain singularities, which are modeled as cone points with prescribed curvature. Therefore we have to cut along paths from the cone points to the boundary of the mesh. The layout overlaps if singularities with negative curvature are used. To create seamless parameter lines we use the rectification approach described in [Springborn et al. 2008].

Finally, we create a new mesh based on a regular (u,v)-grid in $\mathbb{R}^2$. The remeshing process is carried out as a subdivision step followed by some cleanup and regluing: We use the projective interpolation in the texture domain to increase the quality of the result. Previously cut paths from singularities to the boundary are sewed up to obtain the final remesh.

4.5 Examples and Quality

With the quasiisothermic modulus Q^α on edges introduced in Equation (2) we are now able to measure the quality of our parameterizations.

There are two kinds of examples to consider: The first class of meshes stems from smooth surfaces that admit conformal curvature line parameterizations, i.e., triangulations approximating isothermic surfaces. The second class consists of arbitrary non-isothermic surfaces. For almost isothermic surfaces we expect our parameterization to reconstruct the isothermic coordinates up to numerical precision and hence Q^α to be reasonably small. For non-isothermic surfaces we achieve the correct directions on the boundary but lack accuracy in the interior. In Table 1 we summarize the numerical results obtained from the surfaces of Figure 2.

Isothermic surfaces. The class of smooth isothermic surfaces contains surfaces of constant mean curvature, surfaces of revolution, and conic sections. We use the MINIMAL example as an instance of an isothermic surface with mean curvature zero (Figure 2b). As expected this surface exhibits the highest curvature line quality of all tested meshes. The error however cannot vanish completely since the surface's curvature field contains singularities. In the vicinity of these points the numerical curvature directions contain significant amounts of noise.

Non-isothermic surfaces. Non-isothermic surfaces are surfaces that do not admit a parameterization with conformal curvature lines. We investigate the properties of surfaces (a), (c), and (d) displayed in Figure 2.

The TEASER surface was created as a minimizer of a spring functional fixing the boundary and modelling interior edges as springs of rest length zero. It is not far away from a minimal surface with the same boundary. The curvature line pattern however differs substantially as it contains singularities whereas the minimal surface with this boundary curve does not. The quality of the curvature line pattern is also very high. The mean angle error of the numerical directions is 3.2 degrees. Note that the deviation Q_σ from the mean value is also very low. For this surface

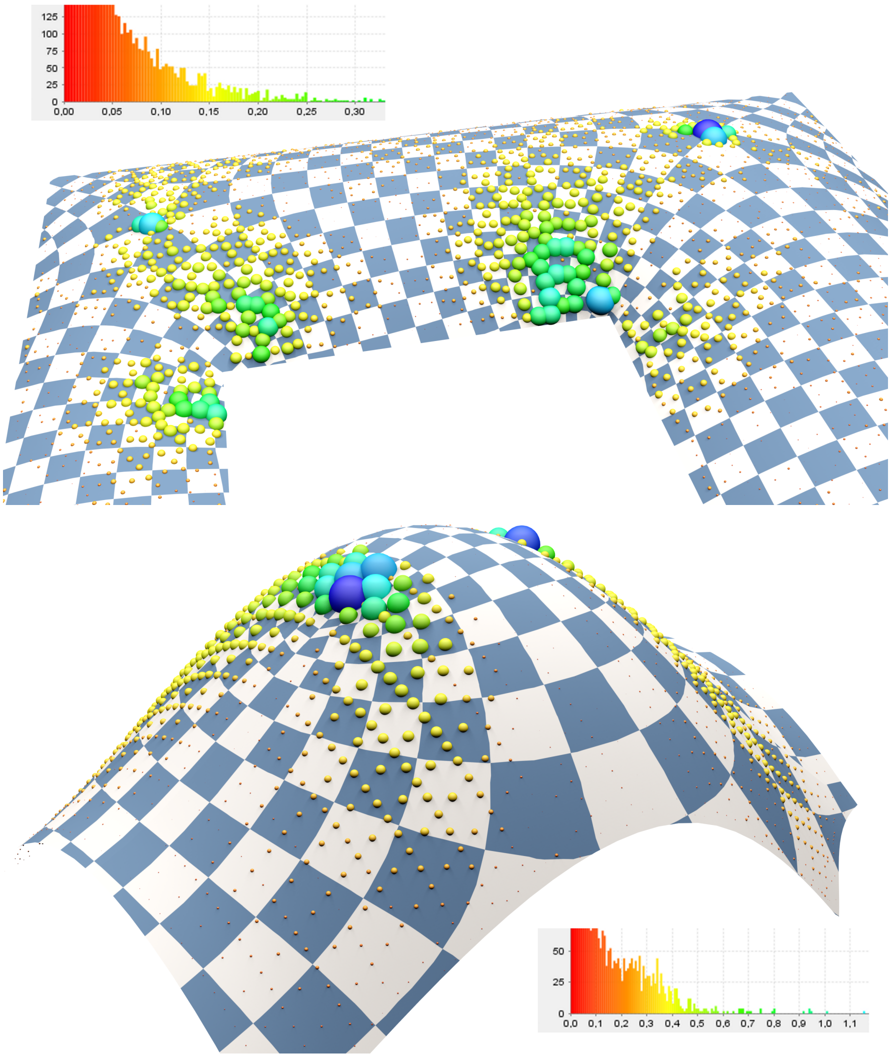

Figure 6: The quality of the parameterization is measured in radians per edge of the underlying triangulation. The checkerboard texture indicates the parameter lines of the map. Small (red and yellow) beads represent good curvature direction quality, big beads (green and blue) represent high deviation. The color of the histogram corresponds to the color of the beads. Note that the mean error of the ROOF surface (top) is half the error of the DOME. See also Table 1 for detailed quality measures of the other surfaces.

	$\#E$	$\#\partial E$	Q^{α}_{mean}	$Q^{\alpha}_{\max}$	Q^{α}_{σ}
Minimal	6260	450	0.036	0.603	0.033
Teaser	17550	1000	0.057	1.20	0.066
Roof	3766	470	0.051	0.610	0.059
Dome	1900	350	0.133	1.52	0.157

Table 1: The curvature line approximation quality of the examples. $\#\partial E$ are the number of boundary edges. Q^{α}_{σ} is the standard deviation of Q^{α}.

the coordinates generated by our algorithm are a globally good approximation to conformal curvature lines.

A quality plot of the ROOF surface (Figures 2d and 5) is shown in Figure 6. Surprisingly, the quality of the curvature lines is as high as in the TEASER or the MINIMAL case. This suggests that a slight variation of the surface yields an isothermic surface. See also Figure 5 for a visual impression of the quality of the curvature lines.

The DOME model (Figure 2c) is created from a NURBS surface. The quality plot (Figure 6) reveals areas of high angle deviation especially around the singularities. Other areas, in particular those near the boundary, are of high curvature line quality. The distance to the nearest isothermic surface is expected to be larger than in the previous examples. More evidence for this is given in Section 5.

Discussion. Our parameterization scheme works well for surfaces that are not too far away from surfaces that possess isothermic coordinates. In the case of surfaces stemming from minimal or constant mean curvature surfaces we get almost perfect approximation quality of curvature lines. These are surfaces that are particularly interesting when designing beam layouts for roof structures that where form-found. For other surfaces the parameterization is conformal and the parameter line pattern captures the combinatorics of the curvature line pattern while approximating the curvature line geometry. There are of course surfaces for which our method is not applicable. If the boundary is too short compared to the overall size of the surface we cannot expect the solution to follow curvature lines as the distance to the boundary increases.

5 Discrete S-isothermic Surfaces

Starting with a quasiisothermically parameterized mesh with low modulus we now aim to create discrete S-isothermic surfaces that stay in the vicinity of the input surface. S-isothermic surfaces were introduced by [Bobenko and Pinkall 1999]: A quadrilateral mesh is *S-isothermic* if (i) all the quadrilaterals are planar, (ii) all faces have incircles, and (iii) the incircles of adjacent quadrilaterals touch. Figure 8 displays S-isothermic surfaces derived from our parametrizations shown in Figure 2.

5.1 Variational Principle

In this section we introduce an energy whose minimizers are S-isothermic surfaces. We denote quadilaterals by $ijkm \in F$ where the indices are in cyclic order. The S-isothermic energy E_S consists of three parts:

$$E_S := \lambda_1 E_{\text{planar}} + \lambda_2 E_{\text{incircle}} + \lambda_3 E_{\text{touch}} \tag{6}$$

The planarity energy E_{planar} penalizes non-planar quadrilateral faces. For each quad it can be defined either by the distance of the diagonals (an idea attributed to Peter Schröder in [Pottmann et al. 2008]) or the volume of the tetrahedron spanned by the four vertices. We give the formula for the former here.

$$E_{\text{planar}} = \sum_{ijkm \in F} \frac{\left\langle \Delta_{ji}, \Delta_{mj} \times \Delta_{ki} \right\rangle^2}{\|\Delta_{mj} \times \Delta_{ki}\|^2} \tag{7}$$

Here Δ_{ij} is the vector pointing from vertex i to j.

For E_{incircle} we use the energy defined by [Schiftner et al. 2009] based on the fact that the sum of opposite edge lengths must be equal for a planar quad to possess an incircle.

$$E_{\text{incircle}} = \sum_{ijkm \in F} \left(l_{ij} + l_{km} - l_{jk} - l_{mi} \right)^2 \tag{8}$$

The energy E_{touch} is a new energy that enforces touching incircles if faces are planar and possess incircles. It is defined per edge, see Figure 7 for the exact labeling of the angles at one edge. For an interior edge $ij \in E$ we define

$$E_{\text{touch}}(ij) = \left(\cot \frac{\beta_l^j}{2} \cot \frac{\beta_r^i}{2} - \cot \frac{\beta_r^j}{2} \cot \frac{\beta_l^i}{2} \right)^2 . \tag{9}$$

On boundary edges the energy is zero. All energies can be formulated in terms of the vertex coordinates and the derivatives can be calculated explicitly.

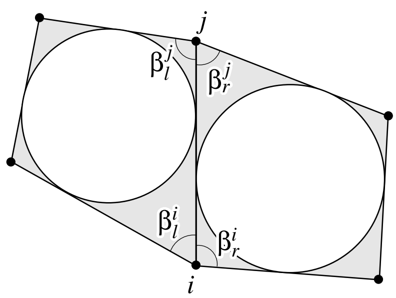

Figure 7: Labels for the touching-circles-functional at an edge. The circles touch if the ratio $\cot\left(\beta^i/2\right) / \cot\left(\beta^j/2\right)$ is equal on both sides of the edge.

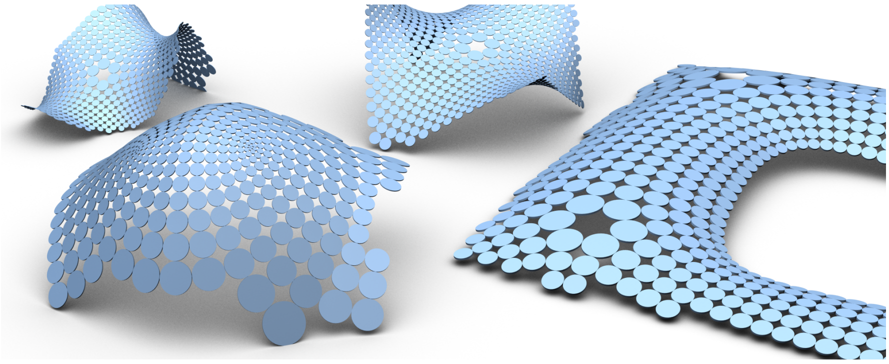

Figure 8: S-isothermic meshes created from the models presented in Figure 2. The inner quadrilaterals are optimized towards touching incircles. A series of touching circles in a row can be interpreted as discrete curvature line.

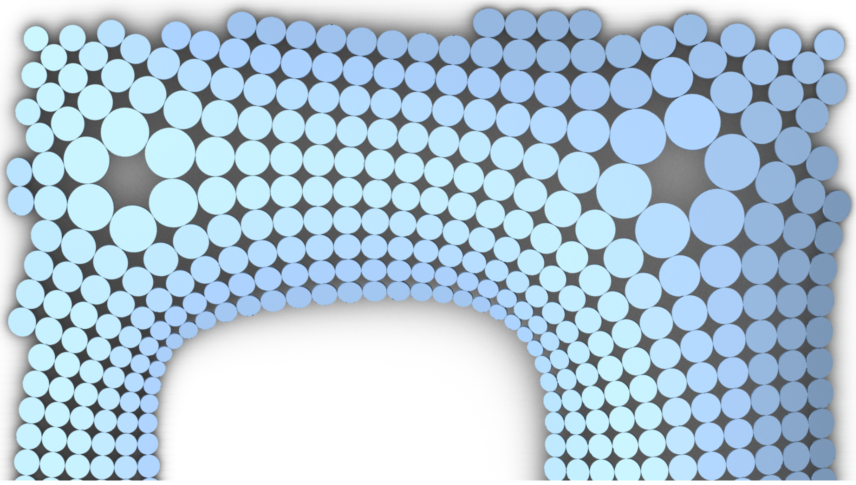

Figure 9: The S-isothermic circle packing on the ROOF model in detail.

Since all energies are in general non-convex we need a good initial guess to find meaningful minimizers of E_S. S-isothermic minimal surfaces converge to isothermic parameterizations of smooth minimal surfaces [Bobenko et al. 2006]. For general S-isothermic surfaces this is an open conjecture. The parameterizations obtained in Section 3 are good candidates to start from with the optimization of the functional. We use the non-linear optimization package PETSc/TAO [Balay et al. 2011, Benson et al. 2007] and its java binding [Sommer 2010] to find minimizers of E_S. Figure 10 shows convergence plots of E_S for the four models that were discussed in the previous section.

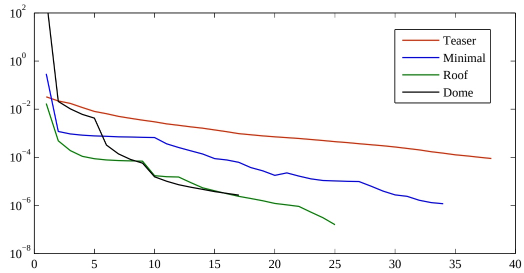

Figure 10: Convergence behavior of E_S during optimization. We use the meshes displayed in Figure 2 as initial guesses for the minimization. The convergence of the Teaser geometry is slower due to the high complexity.

As seen in the quality analysis of Section 4.5, the TEASER, the MINIMAL, and the ROOF models are quasiisothermic surfaces with low modulus. For these models the corresponding S-isothermic surface is also close to the input surface. Figure 10 shows the energy during the optimization. Here the three close-to-isothermic meshes start with a lower energy than the DOME model. After the DOME has passed some iterations it exhibits convergence properties similar to the other models. As this surface converges against a discrete S-isothermic surface, we observe a considerable change in shape during the first iterations especially around the singularities.

6 Conclusions and Future Research

The main contributions of this article are, on the one hand, the definition of quasiisothermic parameterizations together with a new algorithm to compute parameterizations of surfaces that optimizes the corresponding quality measure. On the other hand, we have defined a new energy for meshes with touching incircles.

We see the main advantage of the algorithm presented in Section 3 in its simplicity and its applicability to shell-like roof structures which arise in architectural models. Since these models often have almost constant mean curvature and thus allow for an almost isothermic parameterization, our algorithm performs particularly well on these examples.

The new energy described in Section 5 is closely related to the article of [Schiftner et al. 2009] dealing with circle packing meshes. They explicitly do not treat quadrilateral meshes since they are aware of the shape restrictions and focus on triangle meshes instead. The shape restriction lies in the core of isothermic surfaces but did not influence our results dramatically for the surfaces in our focus. How to approximate arbitrary surfaces by isothermic surfaces is unknown and will be subject to future research. The results of Section 5 suggest that this might be possible using related methods.

Our new functional generates quad circle packing meshes in the sense of Schiftner *et al.*. For surfaces arising in architectural context (in particular for shell-like roofs) we are able to construct aesthetically pleasing quad meshes supporting a circle packing.

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