The Uniformization Theorem for Circle Packings

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ABSTRACT: Circle packings were proposed as a means for discrete approximation of conformal mappings by William Thurston in 1985, based on his reinterpretation of a result of E. M. Andreev. However, they also appear to provide a setting for a discrete analogue of large parts of classical function theory, particularly its more geometric features. In earlier work, the authors proved the Discrete Schwarz-Pick Lemma for circle packings. Here we establish a notion of spherical, parabolic, and hyperbolic "type" for circle packing configurations, discrete versions of the Uniformization Theorem and universal covering surfaces, a generalization of Andreev's Theorem to arbitrary compact Riemann surfaces, and a preliminary definition of discrete analytic functions. The emphasis throughout is on the parallels to classical complex variables, our contention being that circle packings share the geometric rigidity which in the continuous setting characterizes analytic functions.

1. Introduction and Summary. Circle packings are configurations of circles having mutually disjoint interiors, prescribed patterns of tangency, and triangular interstices. William Thurston initiated their study with his interpretation of a result of E. M. Andreev: *Any triangulation of the Riemann sphere may be realized as the pattern underlying a circle packing, and the resulting circle configuration is unique up to inversions and automorphisms.* (See Figure 1(a).) Applying this, Thurston went on to conjecture [T2] that one could approximate classical analytic conformal maps by simplicial maps obtained from circle packings, a conjecture verified in the case of hexagonal patterns by B. Rodin and D. Sullivan [RS].

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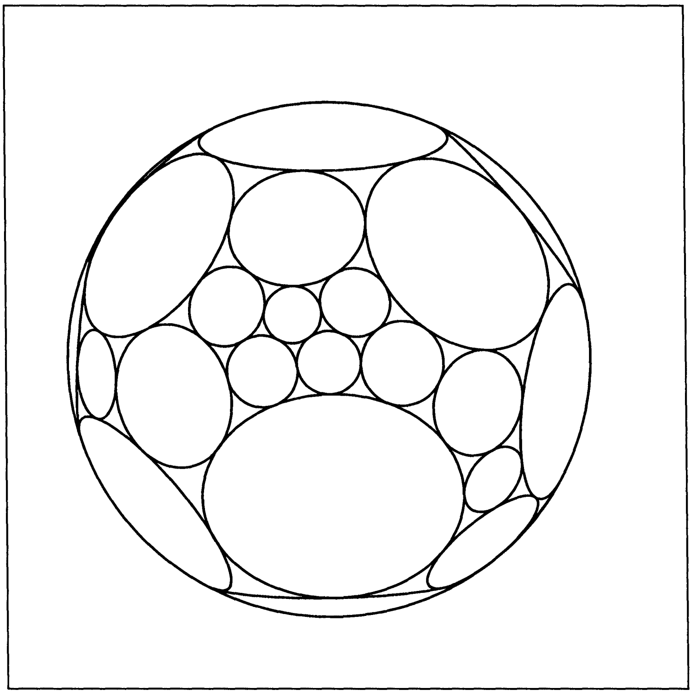
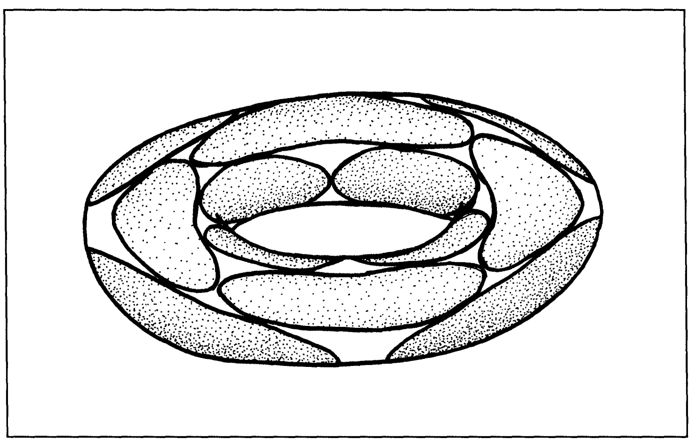


(A)



(B)

FIGURE 1. CIRCLE PACKINGS IN THE SPHERE AND TORUS

However, circle packings provide not only a means of approximating analytic functions, but also an opportunity to develop discrete analogues. The authors began to investigate this more closely in [BSt1], establishing the Discrete Schwarz-Pick Lemma for circle packings. This lemma has been the key to vast parts of classical analysis, and here we exploit our discrete version to develop discrete parallels. In particular, we establish a notion of spherical, parabolic, and hyperbolic "type" for circle packing configurations, discrete versions of the Uniformization Theorem and universal covering surfaces, and a generalization of Andreev's Theorem to arbitrary compact Riemann surfaces. This last topic is addressed in Thurston's notes [Chp. 13, T1]; however, our methods are independent and more in the vein of classical analysis. Indeed, our emphasis throughout is precisely on the parallels to classical complex variables, adding weight to our contention in [BSt1] that circle packings share the geometric rigidity which in the continuous setting characterizes analytic functions.

The combinatorics of circle configurations will be encoded by certain simplicial 2-complexes K, termed CP-complexes; the vertices represent circles, edges indicate tangent circles, and faces correspond to triangular interstices of triples of circles. Subject to formalities to come later, we say that K has spherical, parabolic, or hyperbolic "type" if it can be realized as a circle packing filling the sphere, the plane, or the hyperbolic plane, respectively. It is not at all obvious that a complex will have a type nor, if it does, that the type is unique. However, for appropriate complexes K, we prove:

(a) K can be realized as a circle packing,
(b) K has a unique type, and moreover
(c) there is a certain extremal packing which realizes K and is unique up to automorphisms of the underlying space.

Disregarding, for the moment, some of our technical terms, we state our main result in a form familiar to complex analysts:

Discrete Uniformization Theorem

Every simply connected CP–complex can be realized as a maximal circle packing of one and only one of the sphere, the plane, or the unit disc. The maximal circle packing fills the underlying space and is unique up to automorphisms.

In referring to the standard domains, the sphere, the plane, and the unit disc, we will use the letter $\mathcal{D}$ to denote any one of these. In a thoroughly classical way, we use the Discrete Uniformization Theorem and covering theory to handle non-simply connected complexes: If K is not simply connected, it has a simply connected complex $\widetilde{K}$ as a complete, simplicial universal covering, whose type it inherits. $\widetilde{K}$ then has a circle packing in $\mathcal{D}$, and the group Γ of deck transformations of the (simplicial) covering projection $\pi : \widetilde{K} \longrightarrow K$ induces a group of automorphisms of $\mathcal{D}$ under which that circle packing is invariant. The original complex K is homeomorphic to the Riemann surface $\mathcal{D}/\Gamma$, and the invariant circle packing projects to a maximal circle packing, *in situ*, on K. Among the consequences, we prove a generalization of Andreev's theorem:

> *Any triangulation of a compact oriented 2-manifold S determines a conformal structure and metric on S so that K may be realized as the complex of a circle packing of S; the conformal structure is unique and the circle packing is unique up to automorphisms.*

With the emphasis here on the connections between combinatorics and geometry, we should point out that in the classical setting, Riemann surfaces have conformal structures and complete Riemannian metrics of constant curvature. We will pursue our discrete parallels in very concrete terms; however, we might paraphrase the analogy at what is perhaps its most fundamental level:

> *A Riemann surface S has a* **conformal structure** *which determines an* **infinitesimal metric** *of constant curvature, while a complex K has a* **combinatorial structure** *which determines a* **discrete metric** *of constant curvature.*

Thus, a conformal structure reflects infinitesimal combinatorics; a conformal metric, infinitesimal radii. Though we concentrate on geometric structures rather than on mappings, the parallels with analytic functions show through repeatedly. In addition to the Discrete Uniformization theorem and universal covers, the interested reader will recognize classical type conditions, the Little Picard Theorem, Liouville's Theorem, covering maps, the Carathéodory Kernel Theorem, and more. It is only near the end of the paper that "ambient" circle packings and the concept of discrete analytic functions as those preserving circle packings can be introduced. In the opinion of the authors, this lays the foundation for a much richer analogy with classical analysis, but development of the ideas awaits further study.

Here is a brief outline of the paper's organization: In Section 2 we recall some of the classical theory of Riemann surfaces and their metrics, including surfaces with geodesic borders; we formally define circle packings and introduce CP-complexes. In Section 3, we prove basic lemmas about circle patterns and state known results, including Andreev's Theorem, the Discrete Schwarz-Pick Lemma, and Sullivan's uniqueness result for circle packings in the plane. In Section 4 we prove the existence of the cannonical "maximal" circle packings in spherical, parabolic, and hyperbolic settings, establishing the Discrete Uniformization Theorem and defining "type". We study the properties of maximal circle packings in the three geometries in Section 5, placing particular emphasis on extremal and uniqueness questions, and ending with the generalization of Andreev's original result. Section 6 gives examples and proves various type conditions. Section 7 introduces more general "ambient" circle packings, and we suggest how discrete analytic functions might be defined. The final section gathers remarks and questions and ends with brief comments on Thurston's original topic, the approximation of classical analytic functions. Throughout the paper, we provide numerous examples, not only for the intuition this can provide, but also for the pure beauty of the patterns themselves. Our thanks to Phil Bowers for interesting conversations on this material.

2. Riemann Surfaces, Circle Configurations, and Complexes. In this section we discuss Riemann surfaces and their metrics, the details of the patterns of "circle packings", and the complexes which encode these patterns. (See [AhS] and [Ca] for general background.)

Riemann Surfaces. The classical geometries are the spherical, euclidean, and hyperbolic. Serving as models will be the Riemann sphere $\widehat{\mathbf{C}}$, the complex plane $\mathbf{C}$, and the unit disc Δ. The notation $\mathcal{D}$ will denote one of these three standard domains, as the situation requires. Observe that as topological spaces the three domains are nested, that circles in one are circles in any larger one, and that their automorphism groups are groups of Möbius transformations and hence preserve circles. An entity in $\mathcal{D}$ is said to be "essentially" unique if it is unique up to automorphisms. The metrics we use will be conformal and of constant curvature; a metric is complete if geodesics may be extended without limit. The sphere, the plane, and the disc each support an essentially unique complete conformal metric of constant curvature: the spherical metric of curvature 1 on $\widehat{\mathbf{C}}$; the euclidean or flat metric of curvature 0 on $\mathbf{C}$; and the hyperbolic or Poincarè metric of curvature -1 on Δ.

If S is an open simply connected Riemann surface, then it is conformally equivalent to precisely one of $\widehat{\mathbf{C}}$, $\mathbf{C}$, and Δ, and is said to be of spherical, parabolic, or hyperbolic type, respectively. If S is not simply connected, then it has a universal covering surface, $\widetilde{S}$, which is conformally equivalent to one of the standard domains $\mathcal{D}$. The surface S inherits its type from $\widetilde{S}$. As it happens, only the sphere itself has spherical type; the plane, the once-punctured plane, and all tori of genus 1 have parabolic type; while the remaining Riemann surfaces (including all bordered Riemann surfaces) have hyperbolic type. The covering projection $\pi : \widetilde{S} \longrightarrow S$ is analytic. The covering group of S (the deck transformations) is the group of automorphisms γ of $\widetilde{S}$ satisfying $\pi \circ \gamma \equiv \pi$, and S is conformally equivalent to the Riemann surface $\widetilde{S}/\Gamma$ obtained by identification. There is a complete conformal metric on S inherited from that on $\widetilde{S}$ and having the same curvature; this will in all cases be denoted ρ.

We will also encounter bordered Riemann surfaces S, which in the discrete setting will correspond with complexes having boundaries. (Note that we also include as bordered those surfaces with ∂S a mixture of border and ideal boundary components.) S is of hyperbolic type, and its usual universal covering map would map boundary arcs of Δ to its border components. However, we want the borders to be covered by geodesics, so we introduce special covering maps and terminology. Putting aside some elementary cases, the covering is obtained as follows: Double S across its border to obtain a hyperbolic surface $\widehat{S}$; let $\pi : \Delta \longrightarrow \widehat{S}$ be a universal covering map, Γ its covering group of Möbius transformations, and U a component of $\pi^{-1}(S)$. Reflection of S across its border is an anticonformal involution j of $\widehat{S}$. It is not difficult to verify that each component of $\pi^{-1}(\partial S) \cap \Delta$ is the fixed set of an anticonformal automorphism of Δ arising as an appropriate lift of j. In particular, U is a simply connected subset of Δ with geodesic borders. The analytic covering projection $\pi : U \longrightarrow S$ is invariant under the subgroup of Γ which leaves U invariant; this subgroup serves as the covering group. Under π, S inherits from the Poincarè metric on U a conformal metric ρ in which its borders are complete geodesics. It is convenient to treat this all as a part of the classical uniformization theorem—thus we include these sets U among those 'standard domains' for which we have introduced the notation $\mathcal{D}$, and we refer to Δ as the universal covering space, though we should actually refer to $U \subseteq \Delta$. (Note that these domains U are obtained from Δ by removing some collection of mutually disjoint open halfplanes; see Figure 3(b) for an example.)

Definition. *Let S be an open or bordered Riemann surface. The complete conformal metric ρ obtained from its universal cover will be termed the "intrinsic" metric on S. Unless stated otherwise, ρ will be the metric used to define circles in the surface S.*

Note that ρ is unique only up to scaling if S is parabolic, and only up to automorphisms if S is the sphere; nonetheless, we assume in these cases that a fixed choice for ρ has been made. In the hyperbolic case, ρ is unique. For open surfaces, ρ is complete in the usual sense of having complete geodesics. If S is bordered, which can happen only in the hyperbolic setting, we still say that ρ is complete, though geodesics may end at frontier points. Note that border components are complete geodesics. One final detail in the case of the hyperbolic plane: horocycles in Δ are euclidean circles internally tangent to $\partial\Delta$ and will be regarded as circles in the hyperbolic metric having their centers at the point of tangency and having infinite hyperbolic radius. In the case of one of the bordered domains U above, the intersection of U with a horocycle whose center is a point of ∂U will still qualify as a "circle" in the metric ρ. (E.g., superimpose Figures 3(a) and (b).)

Circle Packings. The notion of a circle packing is intuitively very simple. Figure 1(a) illustrates a circle packing in the sphere, Figure 2 shows two in the plane, and Figure 3(a) (see p. 1391) shows one in the hyperbolic plane. We hate to saddle these pretty pictures with heavy formalism, but we will need definitions and terminology which avoid various pathologies and ambiguities, particularly in the setting of general Riemann surfaces.

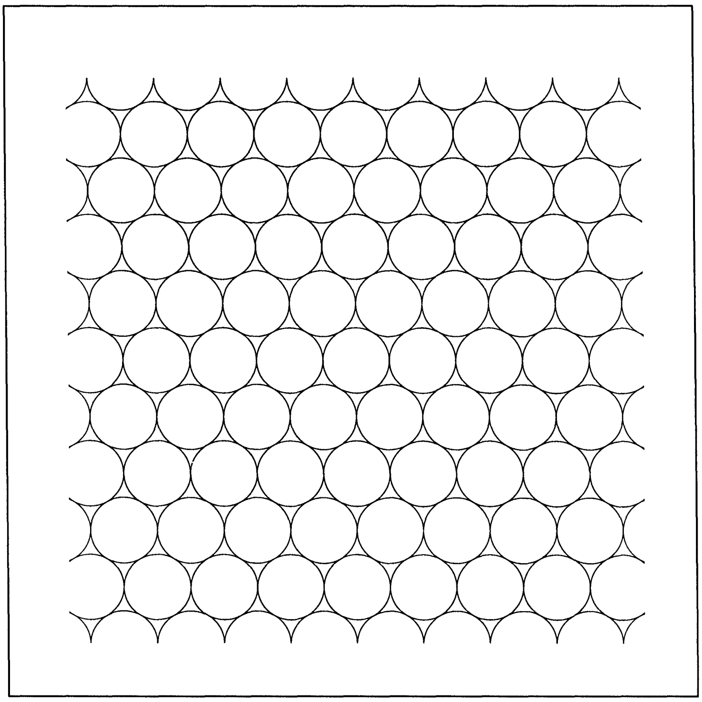


FIGURE 2(A). CIRCLE PACKINGS IN $\mathbf{C}$.

To introduce the ideas, we start in the familiar setting of the plane: Let P be a collection of circles having mutually disjoint interiors. The "carrier" of P is the simplicial 2-complex formed by taking the centers of circles as vertices, the line segments connecting (the centers of) tangent circles as edges, and the triangles formed by triples of mutually tangent circles as faces (when the interstice of such triple contains no other circles). The abstract complex K we obtain is termed the "complex for P" and the carrier of P may be thought of as an embedding of K in the plane. The main requirement for P to be a circle packing will be that the carrier be connected and that it equal the closure of its faces (the definition

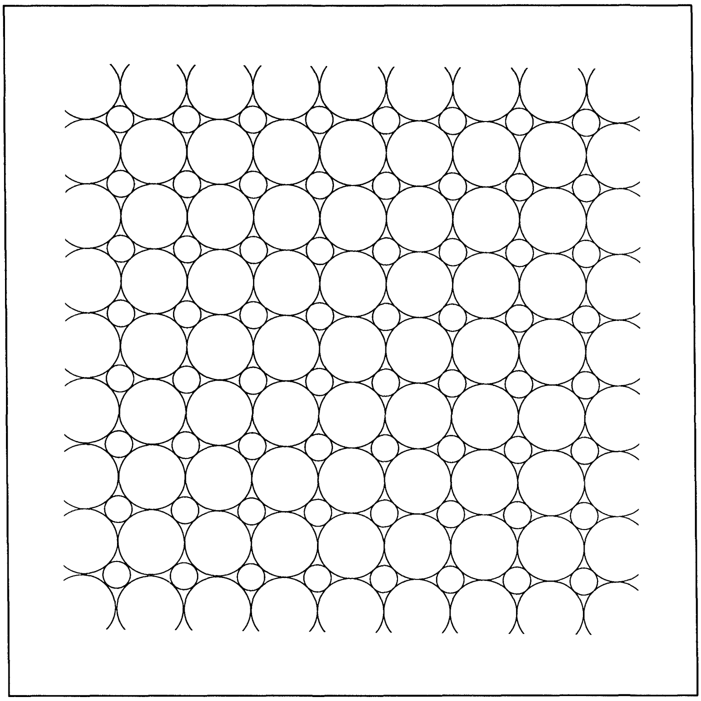


FIGURE 2(B). CIRCLE PACKINGS IN **C**.

will come shortly). Thus, we will be avoiding, among other things, disconnected patterns, dangling circles, and interstices which are not triangular. The more general setting for circle packings is a Riemann surface S (with its intrinsic metric ρ). Given a collection P of circles having mutually disjoint interiors, the complex K is obtained as above, using centers and geodesics associated with ρ. However, in the general setting, some additional assumptions are needed:

Assumptions. *The following assumptions are made regarding circle configurations on Riemann surfaces:*

1. *Each circle has a simply connected interior.*
2. *Circles with disjoint interiors are tangent (i.e., intersect) at no more than one point.*
3. *No circle is tangent to itself.*
4. *Triangular faces of the carrier formed by triples of mutually tangent circles are simply connected.*

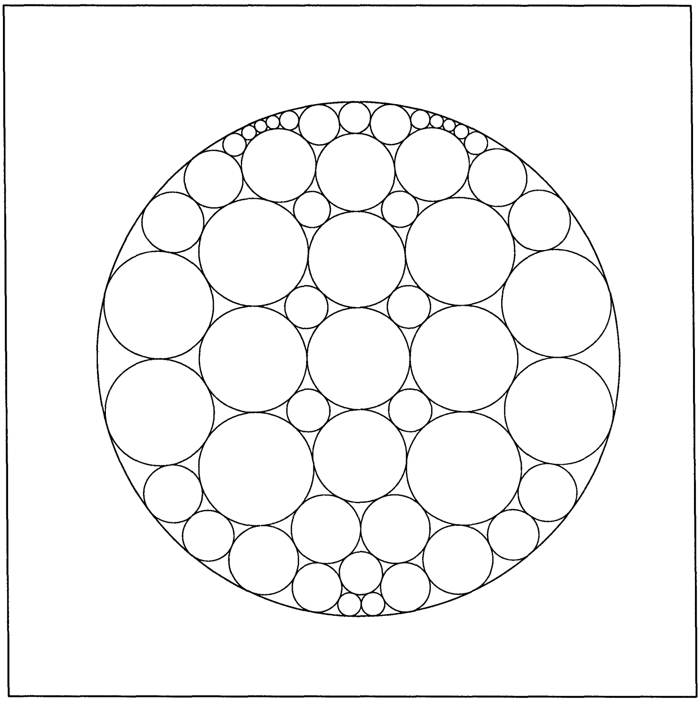


FIGURE 3(A). AN ANDREEV PACKING AND ITS HYPERBOLIC CARRIER.

There are some additional conditions to be placed on P; these are best phrased as conditions on the abstract complex K.

Definition. *An abstract simplicial 2-complex K is termed a "CP–complex" if the following conditions hold:*

(1) *K is (simplicially equivalent to) a triangulation of a connected, oriented 2-manifold,*
(2) *K has bounded degree,*
(3) *K has nonempty connected interior, and*
(4) *every boundary vertex of K has an interior vertex as neighbor.*

We will generally refer to CP-complexes simply as complexes in the sequel, unless special emphasis on these conditions is desired. Condition (1) means that K is an oriented topological 2-manifold, that no more than 2 faces share a given edge, that every edge belongs to at least one face, that every interior vertex belongs to at least three faces, that two faces which intersect share an edge, and that ∂K, if nonempty, consists of simple, oriented edge chains. Condition

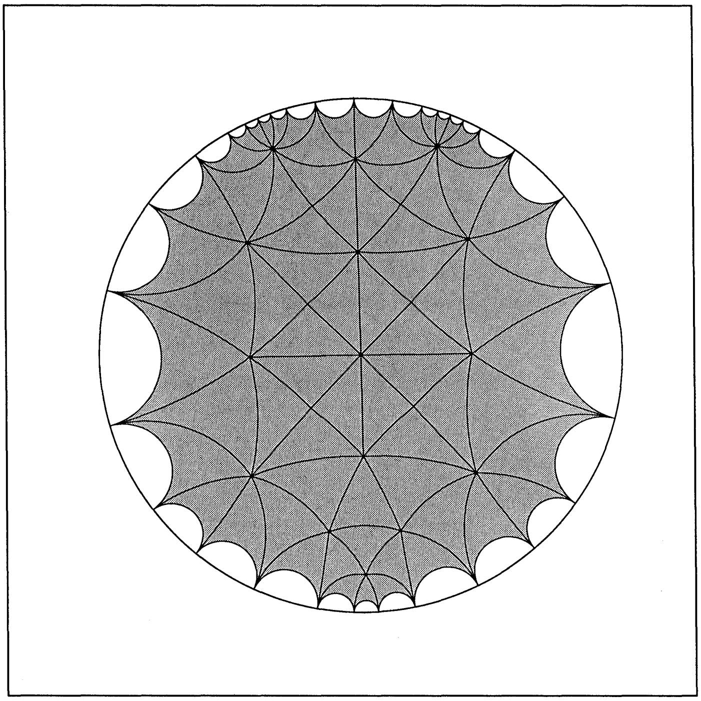


FIGURE 3(B). AN ANDREEV PACKING AND ITS HYPERBOLIC CARRIER.

(2) means that there exists a finite maximum d (the degree of the complex) on the number of neighbors of any one vertex; only in the concluding section do we pose questions concerning complexes of unbounded degree. We note that conditions (3) and (4) are included for convenience—they allow us to avoid some minor pathologies. Condition (3) means that any two interior vertices may be connected by a chain of edges having only interior vertices as endpoints.

Definition. *In a Riemann surface S, a configuration P of circles having mutually disjoint interiors is termed a "circle packing in S" if its complex K is a* CP*-complex. Moreover, P is said to "fill" S if its carrier is all of S.*

Among the consequences of this definition for the circle configurations: each pair of tangent circles belongs to a mutually tangent triple; no circle is tangent to more than a finite number of others; there are no dangling circles or disconnected pieces in the pattern; and we include in the carrier of the packing only the triangular (i.e., bounded by three circles) interstices of the pattern. A circle packing in Δ is also a circle packing in $\mathbf{C}$, and one in $\mathbf{C}$ is also one in $\widehat{\mathbf{C}}$.

However, the choice of settings carries with it the choice of metric for the radius and location for the center. If P is a circle packing which fills S, then the embedding of its complex K as the carrier of P is a homeomorphism of K onto S. In particular, a circle packing filling $\widehat{\mathbf{C}}$ will necessarily be finite, while those filling $\mathbf{C}$ or Δ will necessarily be infinite.

In our work, we will generally be starting with a CP-complex K and then finding a circle packing which has K as its complex.

Definition. *Given a CP–complex K, a circle packing P with complex K in a Riemann surface S is called a "realization" of K. We say S "supports" a circle packing for K.*

The complexes used in [BSt1], which will occur frequently in the sequel, might be thought of as disc-like: A CP-complex K is called a "DL-complex" if it is simplicially equivalent to a finite triangulation of a closed disc. A couple of final points: A complex is termed "compact" if it is finite and has no boundary vertices. Complexes with boundary will be treated as bordered surfaces, with each boundary edge being a border component—that is, the boundary vertices themselves are not considered to be part of the border.

Barycentric coordinates. Any complex we consider, whether abstract or geometric, is assumed to have a fixed system of barycentric coordinates; maps between complexes are always taken as barycentric simplicial maps. It is only in the euclidean plane that we will have occasion to consider specific point mappings between embedded complexes; for these we use the standard euclidean barycentric coordinates. As a point of information, barycentric coordinates are available in the spherical and hyperbolic setting, though there are difficulties with faces having vertices at infinity in the hyperbolic case.

3. Basic Results and Lemmas. We are now in position to state some basic results and important geometric lemmas, beginning with the result of Andreev [A1] as interpreted by Thurston:

Theorem 1. *Let K be a triangulation of the sphere. Then $\widehat{\mathbf{C}}$ supports a circle packing for K. The corresponding circle configuration P fills $\widehat{\mathbf{C}}$ and is unique up to automorphisms.*

Note that our CP-complexes are oriented; otherwise, uniqueness would be up to automorphisms *and* inversions. A circle packing of the sphere is illustrated in Figure 1(a). For Figure 3(a), that packing has been modified by an automorphism so that one of its circles has the northern hemisphere as interior, and then projected to Δ stereographically. Following Thurston, this suggests how the theorem for the sphere may be used to establish what we will call Andreev's Theorem (see [Theorem 2, BSt1]).

Theorem 2 (Andreev's Theorem). *Let K be a DL–complex. There is a circle packing P in Δ with complex K and having all boundary circles internally tangent to $\partial\Delta$. The configuration is unique up to automorphisms of Δ.*

Though the configuration P is unique only up to automorphisms, we refer to it as the "Andreev packing" for K and write P_a. Its hyperbolic radii are unique, with boundary circles having infinite hyperbolic radius (i.e., horocycles). Figure 3(b) (p. 1392) illustrates the carrier of an Andreev packing with its complete geodesic boundary. The extremal properties of Andreev packings are the subject of the Discrete Schwarz-Pick Lemma (DSPL) proven in [BSt1]. We state a portion of that result in the form we will need.

Discrete Schwarz-Pick Lemma. *Let K be a DL–complex with Andreev packing P_a and assume P is a circle packing in Δ with complex K.*

(a) *Each circle in P has hyperbolic radius less than or equal to that of the corresponding circle in P_a.*
(b) *The hyperbolic distance between (the centers of) two circles of P is less than or equal to that between the corresponding circles in P_a.*
(c) *Each face in the (hyperbolic) carrier of P has hyperbolic area less than or equal to that of the corresponding face in the carrier of P_a.*

Moreover, equality in (a) *for any interior vertex, finite equality in* (b) *for any distance, or equality in* (c) *for any face implies that P and P_a are automorphic images of one another.*

The following uniqueness result for circle packings of the plane is stated in [RS]. Recall that the complex here has bounded degree in accordance with the conventions we stated for CP-complexes.

Theorem 3 (Sullivan Uniqueness). *Let K be a CP–complex which may be realized as a circle packing P filling the plane. Then P is unique up to automorphisms of $\mathbf{C}$.*

In the next three lemmas we collect several results on the local geometry of circle packings. We stick with estimates which will be needed in the sequel, and no attempt is made to give best constants. We start by looking at an "n-flower" P, that is, a circle packing consisting of a single interior circle precisely sorrounded by a chain of $n \geq 3$ tangent circles, called "petals". Its geometry—radii, angles, etc.—depend on whether it lies in the sphere, the plane, or the hyperbolic plane.

Lemma 1. *Let P be an n-flower with central circle v. There exists a constant $\mathcal{C}_1 > 0$, depending only on n, so that the following hold:*

- *If P lies in $\mathbf{C}$ and R is the euclidean radius of v, then*
  (a) *the euclidean radius r of any petal satisfies $\frac{r}{R} \geq \mathcal{C}_1$, and*
  (b) *the circle of euclidean radius $R(1+\mathcal{C}_1)$ concentric with v lies in the euclidean carrier of P.*
- *If P lies in the sphere, R is the euclidean radius of v, and one of the petals is the exterior of the unit disc (so v is a horocycle), then*
  (c) *the euclidean radius r of any other petal satisfies $\frac{r}{R} \geq \mathcal{C}_1$.*
- *If P lies in Δ and R is the hyperbolic radius of v, then*
  (d) *the hyperbolic radius r of any petal satisfies $\frac{r}{R} \geq \mathcal{C}_1$.*

The various results here reflect the rigid nature of flowers—the petals must reach exactly around the center circle, each petal tangent to the next and the last precisely tangent to the first. Parts (a), (c), and (d) come from the Ring Lemma of [RS]; part (b) follows easily from (a).

Lemma 2. *Let K be a DL–complex of degree d, P_a its Andreev packing. There exist constants $\mathcal{C}_2 > 0$ and $\mathcal{C}_3 > 0$, depending only on d, so that the following hold:*

(a) *Any interior circle of P_a which is tangent to a boundary circle has hyperbolic radius R satisfying $R \geq \mathcal{C}_2$.*
(b) *For any interior circle of P_a, the concentric disc of hyperbolic radius $\mathcal{C}_2$ lies in the hyperboic carrier of P_a.*
(c) *For any interior circle of P_a centered at the origin, the concentric disc of euclidean radius $\mathcal{C}_3$ lies in the euclidean carrier of P_a.*

Proof.

(a) Let $w \in K$ be a boundary vertex, and let $w', v_1, v_2, \cdots, v_n, w''$ be the positively ordered list of neighboring vertices; w' and w'' are boundary vertices, while $v_1, \cdots, v_n$ are interior, with $n \leq d-2$. In the configuration P_a, let c, c', and c'' be the horocycles for w, w', and w'', respectively; applying a Möbius transformation to P_a, we may assume that c goes through the origin and the point 1 and that c' and c'' are symmetric with respect to the x-axis. A computation shows that the disc D of hyperbolic radius $a = \ln\left(\frac{5}{4}\right)$ centered at the origin is disjoint from c' and c''. (The extreme situation in which c' and c'' are tangent gives this radius.)
The vertex v of interest is one of the v_j's and has radius R. By repeated application of Lemma 1(d), the radius s of any v_i satisfies $s \leq (\mathcal{C}_1)^{2-d}R$, where $\mathcal{C}_1$ depends only on d. In particular, letting $b = 2(d-2)(\mathcal{C}_1)^{2-d}$, the diameter of the chain of circles for $v_1, \cdots, v_n$ is no greater than bR. Since

these circles stretch from c' to c'', one of them intersects the real axis. We conclude that $R + bR \geq \frac{a}{2} \Rightarrow R \geq= \frac{a}{2+2b} > 0$. This proves (a).

(b) For Part (b), let Ω denote the hyperbolic carrier of P_a. This is hyperbolically convex, and we have already shown that those interior circles neighboring the boundary have radius greater than $\mathcal{C}_2$. The union of faces having one or two boundary vertices therefore forms a uniformly thick annular region separating the rest of the interior circles from $\partial\Omega$. Part (b) then follows with some elementary geometric reasoning from (a).

(c) For Part (c) we switch to euclidean geometry. Let v be the interior vertex whose circle is centered at the origin (this can be arranged for any interior vertex by applying a Möbius transformation to P_a), and write Ω_e for the euclidean carrier of P_a. Let $\mathcal{C}$ be the euclidean radius of the circle at the origin of hyperbolic radius $\mathcal{C}_2$. The border of Ω_e consists of line segments connecting centers of neighboring horocycles. The horocycles don't contain the origin (an interior center) so their centers lie outside the circle of radius $\frac{1}{2}$. The points of tangency of neighboring horocycles lie in $\partial\Omega \cap \partial\Omega_e$, and by (b) are euclidean distance at least $\mathcal{C}$ from the origin. Elementary geometry implies a positive lower bound $\mathcal{C}_3$ on the distance from the origin to the complement of Ω_e. □

Our final preliminary concerns the quasiconformal nature of simplicial maps between embedded euclidean carriers. An appropriate reference for the theory of quasiconformal mappings is [LV]. We say that two circles are comparable with constant $\mathcal{C}$ if their radii r and R satisfy $\mathcal{C}^{-1}R < r < \mathcal{C}R$.

Lemma 3. *Let K be a CP–complex and let P_1 and P_2 be two circle packings for K in $\mathbf{C}$ with (euclidean) carriers Ω_1 and Ω_2. Assume further that there is a constant $\mathcal{C} > 1$ so that neighboring circles of P_j are comparable with constant $\mathcal{C}$, $j = 1,2$. If $f : \Omega_1 \longrightarrow \Omega_2$ is the simplicial mapping defined using euclidean barycentric coordinates, then f is κ-quasiconformal for a constant $1 \leq \kappa < \infty$ depending only on $\mathcal{C}$.*

The use of quasiconformal mappings was introduced in [RS] in precisely this setting. We may paraphrase their observations: The simplicial mapping f is piecewise affine. The comparability constant $\mathcal{C}$ for neighboring circles gives a uniform lower bound for the angles of the faces of Ω_1 and Ω_2, and hence an upper bound $1 \leq \kappa = \kappa(\mathcal{C}) < \infty$ on the quasiconformal dilatation of f within each face. The edges of triangles comprise a set of area zero, so the mapping has dilatation at most κ throughout Ω_1.

4. Maximal Circle Packings. In this section, we start with a CP-complex K and construct a cannonical realization termed its "maximal circle packing"; the "type" of K is determined by the supporting geometry. The arguments should be compared to those in the classical Carathéodory convergence theorem and covering theory. The section ends with several examples.

We start with simply connected complexes, dispensing with the finite complexes first: If K is finite and simply connected, then it is either a topological sphere (compact) or a DL-complex. In the former case its maximal circle packing $\mathcal{P}$ is the circle packing P of $\widehat{\mathbf{C}}$ given by Theorem 1; while in the latter case, it is the Andreev packing P_a in Δ given by Theorem 2.

Next, assume that K is simply connected but infinite; let d denote its degree. Let $\mathcal{K} = \{K_\alpha\}$ denote the directed net of DL-subcomplexes of K, partially ordered by inclusion. The countability of K and the bound on its degree imply that these exhaust K and that there exist cofinal sequences from the net. For each index α, let P_α denote the Andreev packing in Δ for K_α. Fix a vertex v and let r_α denote the hyperbolic radius of the corresponding circle in P_α, assuming $v \in K_\alpha$. If $K_\alpha, K_\beta \in \mathcal{K}$, $v \in K_\alpha \leq K_\beta$, then by the DSPL, $r_\beta \leq r_\alpha$. This monotonicity implies the directed limit $r = \lim_\alpha r_\alpha \geq 0$ exists. We conclude that there is a nonnegative "limiting radius" associated with each vertex of K.

Fix a second vertex w, with corresponding hyperbolic radii s_α and limiting radius $s = \lim_\alpha s_\alpha$. If both v and w are interior vertices, there is a chain γ of edges connecting v and w, passing through interior vertices only, and having, say, m edges. Suppose all the vertices of γ are interior vertices for some K_α; repeated use of the Ring Lemma (Lemma 1(d)) implies

$$s_\alpha \geq (\mathcal{C}_1)^m \cdot r_\alpha. \tag{$*$}$$

In the limit we have $s \geq (\mathcal{C}_1)^m r$. We conclude that one of two situations arises:

(I) All interior vertices have zero limiting radius.
(II) All interior vertices have positive limiting radius.

We will handle these two situations separately, in each case obtaining a limiting circle packing by applying a diagonalization argument of Rodin and Sullivan [RS], showing it fills the underlying space, and proving it is independent of the choices made during construction. For starters, because the directed limits of radii exist, it suffices to work with a fixed, totally ordered cofinal sequence $\{K_n\}_{n=1}^\infty$ of $\mathcal{K}$. Designate an interior vertex v of K_1 and, as before, let $r = \lim_\alpha r_\alpha = \lim_n r_n$ be its limiting radius. It is clear from $(*)$ that the case which arises is independent of the choice of v.

▷ Case I:

The limiting radius $r = 0$: First observe that K has no boundary; for if w were a boundary vertex, then by Lemma 2(a), a designated interior neighbor would have hyperbolic radius bounded below by $\mathcal{C}_2$ in each P_n, contradicting the condition for Case I. Thus, K is an infinite complex with only interior vertices.

Apply Möbius transformations to the configurations P_n so that each has the circle for v centered at the origin. Let R_n be the euclidean radius of this circle and note that $r_n \to 0$ implies $R_n \to 0$. Perform a scaling of each configuration P_n by the euclidean factor $1/R_n$. This leads to a sequence of circle packings P_n' in $\mathbf{C}$ with the unit circle centered at the origin as the circle for v. Arguing as we did for $(*)$, we see that for each vertex w of K, the euclidean radii of the corresponding circles of P_n' (for all large n) will be bounded above and bounded away from zero. Clearly, this also implies that the centers of those circles will be restricted to some compact subset of the plane. With a standard diagonalization argument, we may choose a subsequence P_{n_j}' so that every vertex for K has a limiting position for its circle's center and a positive limiting value for its euclidean radius. It is elementary to verify that the resulting infinite configuration of circles, call it $\mathcal{P}$, constitutes a circle packing and that its complex is K. Therefore, we realize K as a circle packing $\mathcal{P}$ in $\mathbf{C}$, and we define this to be the maximal circle packing for K.

The proof that $\mathcal{P}$ fills $\mathbf{C}$ requires the use of the Carathéodory kernel theorem for quasiconformal mappings. The argument is rather curious: Having constructed $\mathcal{P}$, its carrier provides a euclidean structure on the sub-packings of $\mathcal{P}$ associated with the K_n's. We now construct $\mathcal{P}$ once more, but with the metric structure available, we can exploit properties of quasiconformal mappings.

Denote by Ω the (euclidean) carrier of $\mathcal{P}$. Let P_n'' be the circle packing in $\mathcal{P}$ with complex K_n and recall that P_n' is the associated Andreev packing, scaled by $1/R_n$. Write Ω_n'' and Ω_n' for their carriers. Define the family of simplicial mappings $f_n : \Omega_n'' \longrightarrow \Omega_n'$ and note the following: (i). By Lemma 1(a) and (c) followed by Lemma 3, each f_n is κ-quasiconformal, with κ depending only on the degree d. (ii). Each f_n fixes the origin. (iii). The carriers Ω_n'' are nested and exhaust Ω. (iv). Write Δ_t for the disc centered at the origin of euclidean radius $t > 0$. By Lemma 2(c), for each n we have

$$\Delta_{t_n} \subset \Omega_n' = f_n(\Omega_n''),$$

where $t_n = \mathcal{C}_3/R_n$ and $\mathcal{C}_3$ depends only on d.

Consider the f_{n_j} associated with the subsequence P_{n_j}' which was originally used to define $\mathcal{P}$: focus on a point $\zeta \in \mathbf{C}$ which is the center of a circle of $\mathcal{P}$— say associated with vertex $w \in K$. For sufficiently large n_j, $f_{n_j}(\zeta)$ is defined and is the center of the circle for w in P_{n_j}'; these centers converge to the center of the circle for w in $\mathcal{P}$, which is ζ. That is, f_{n_j} converges on the vertices of Ω to the identity map! Since each f_{n_j} is piecewise affine, it is immediate that f_{n_j} converges to the identity map on all of Ω. Now apply the kernel theorem [Theorem 5.4, Chp. II, LV] for quasiconformal mappings: It tells us that the image of the limit function is the *kernel* of the sequence of image sets Ω'_{n_j}. In our case, the image of the limit function is Ω. Since $R_n \to 0$, (iv) above implies that each disc $\Delta_t, t > 0$, lies in Ω'_{n_j} for all but finitely many n_j. Therefore the kernel of the Ω'_{n_j} is the entire plane, and $\Omega = \mathbf{C}$.

Finally, since the carrier of $\mathcal{P}$ is the whole plane, we can appeal to Theorem 3 to conclude that $\mathcal{P}$ is essentially unique. Thus our construction choices do not affect the outcome.

▷ CASE II:

The limiting radius $r > 0$: Again, normalize the configurations P_n so that each has the circle for v centered at the origin. The circles for v decrease with n to a limit circle with positive hyperbolic radius, hence positive euclidean radius; the Ring Lemma gives bounds on the the euclidean radii associated with each vertex of K, and the euclidean centers all lie in Δ. The diagonalization process can proceed as in Case I to give a subsequence P_{n_j} with a limiting circle packing $\mathcal{P}$ for K, this time in Δ; we define this to be the maximal circle packing for K. (Note that any boundary vertex of K will lead to a horocycle in $\mathcal{P}$, since it corresponds to a horocycle in the Andreev packings P_n, for sufficiently large n. The carrier of $\mathcal{P}$, as we have defined it, includes the point of tangency with the unit circle, which is a point at infinity as far as the hyperbolic plane is concerned. We will nevertheless refer to the carrier as lying in the hyperbolic plane.)

The essential uniqueness of $\mathcal{P}$ follows because it is determined by its (non-zero) hyperbolic radii (this is simple trigonometry, see [Theorem 1, BSt1]), and these are independent of the cofinal sequence $\{K_n\}_{n=1}^{\infty}$ used in the construction. The proof that it fills Δ again relies on the kernel theorem. First, let Ω_h be the hyperbolic carrier of $\mathcal{P}$ and recall what "fill" means: If K has no border, then we must show $\Omega_h = \Delta$; if K has border edges, then each corresponds to a complete geodesic in $\partial\Omega_h$ connecting the centers of horocycles, and we must show that every point of $\partial\Omega_h \cap \Delta$ lies on such a geodesic. Now, consider a point $z \in \partial\Omega_h \cap \Delta$: if it is not on a geodesic border, then it must be a limit point of hyperbolic centers of interior circles of $\mathcal{P}$. To show Ω_h fills Δ, therefore, it will suffice to prove this:

Claim. *There exists a constant $C_4 > 0$ with the property that any interior circle of $\mathcal{P}$ has a concentric disc of hyperbolic radius C_4 lying in Ω_h.*

Without loss of generality we may assume the interior circle at issue is centered at the origin and that it is associated with our designated vertex v. Now we can move over to euclidean geometry. Let Ω be the euclidean carrier and proceed exactly as in case I, only stick with the Andreev packings P_{n_j} rather than the scaled ones P'_{n_j}. By Lemma 2(c), the euclidean carrier Ω_{n_j} of

P_{n_j} contains the euclidean disc Δ_{C_3}, for every n_j. As before, the kernel theorem implies Δ_{C_3} lies in Ω. Knowing that the origin lies in an interior circle and that boundary circles of $\mathcal{P}$ (if there are any) are horocycles, it is not difficult to see that the euclidean carrier Ω lies in Ω_h. We conclude that $\Delta_{C_3} \subset \Omega_h$. The claim is proven, with C_4 equal to the hyperbolic radius of Δ_{C_3}. This completes case II.

Definition. *Given a simply connected CP–complex K, the circle packing $\mathcal{P}$ constructed above is called the "maximal circle packing" for K and is unique up to automorphisms of the underlying space $\mathcal{D}$. K is said to be of spherical, parabolic, or hyperbolic "type" as $\mathcal{D}$ is the sphere, the plane, or the hyperbolic plane, respectively.*

At this point we have established the Discrete Uniformization Theorem stated in the introduction. As in the classical setting, with the simply connected case in hand, one proceeds via covering theory: Every complex K (which is a manifold) has a simply connected complex $\widetilde{K}$ as its universal cover; thus there is a locally one-to-one simplicial projection $\pi : \widetilde{K} \longrightarrow K$. $\widetilde{K}$ is unique up to simplicial equivalence, and we may impose barycentric coordinates so that π is the barycentric mapping. The group of deck or covering transformations will be denoted Λ; thus each $\lambda \in \Lambda$ is a simplicial automorphism of $\widetilde{K}$ satisfying $\pi \circ \lambda \equiv \pi$. It is elementary to verify that if K is a CP-complex, then $\widetilde{K}$ is also a CP-complex. $\widetilde{K}$ is finite iff K is simply connected and finite, and $\widetilde{K}$ has a boundary iff K has a boundary. A complex inherits its type from that of its universal covering complex.

Definition. *The CP–complex K is said to be of spherical, parabolic, or hyperbolic type, respectively, if $\widetilde{K}$ is of spherical, parabolic, or hyperbolic type.*

We may dispense with the spherical situation now: the sphere covers only itself, so K is of spherical type iff it is a topological sphere. The variety occurs in the other types; however, we will see that most complexes are hyperbolic. The reader will recognize here the direct analogue in the simplicial setting for the theory associated with analytic covering surfaces—indeed, a very faithful analogue.

Let K be a complex of parabolic or hyperbolic type. Denote by $\widetilde{P}$ a (specific) maximal circle packing of $\widetilde{K}$ in $\mathcal{D}$ ($\mathbf{C}$ or Δ), and let $\sigma : \widetilde{K} \longrightarrow \widetilde{P}$ be the map which identifies each vertex v of $\widetilde{K}$ with the corresponding circle $C(v)$ of $\widetilde{P}$. We say that $\widetilde{P}$ is invariant under an automorphism γ of $\mathcal{D}$ in case that C is a circle of $\widetilde{P}$ iff $\Gamma(C)$ is a circle of $\widetilde{P}$. The next lemma states that the group Λ of deck transformations of $\widetilde{K}$ lifts to a group of automorphisms of $\mathcal{D}$.

Lemma 4. *Given the CP–complex K, there exists a unique group Γ of automorphisms of $\mathcal{D}$ and a group isomorphism $\psi : \Lambda \longrightarrow \Gamma$ such that:*

(a) *$\widetilde{P}$ is invariant under Γ, and*

(b) *$\sigma \circ \lambda \equiv \psi(\lambda) \circ \sigma : \widetilde{K} \longrightarrow \widetilde{P}$, i.e., the following diagram commutes:*

$$\begin{tikzcd} \widetilde{P} \arrow[r, "\psi(\lambda)"] & \widetilde{P} \\ \widetilde{K} \arrow[u, "\sigma"] \arrow[r, "\lambda"'] & \widetilde{K} \arrow[u, "\sigma"'] \end{tikzcd}$$

Proof. Note that the elements of Γ play dual roles as automorphisms of $\mathcal{D}$ and self-maps of $\widetilde{P}$ as a collection of circles. The connection is via uniqueness results. Assume first that K is of parabolic type, so $\mathcal{D}$ is the plane. Let g be the map $\sigma \circ \lambda \circ \sigma^{-1} : \widetilde{P} \longrightarrow \widetilde{P}$. Since $\lambda(\widetilde{K}) = \widetilde{K}$, $\widetilde{P}$ and $g(\widetilde{P})$ are two circle packings in $\mathbf{C}$ having complex $\widetilde{K}$. By the uniqueness result of Sullivan, the identification g is that induced by an automorphism of $\mathbf{C}$. Denoting this by $\psi(\lambda)$, one routinely verifies the conclusions of the lemma.

We do not know whether something like Sullivan's uniqueness result holds in general for circle packings in the disc (more about this later), but we can rely instead on the cannonical nature of $\widetilde{P}$. In particular, fix $\lambda \in \Lambda$ and let v and $w = \lambda(v)$ be vertices of $\widetilde{K}$. Let $\{\widetilde{K}_n\} \subset \mathcal{K}$ be a cofinal sequence of DL-subcomplexes of $\widetilde{K}$. Then $\{\lambda(\widetilde{K}_n)\}$ is also cofinal. The "limit radii" for v and w may be obtained using either sequence. But note that the radius associated with v in the Andreev packing for $\widetilde{K}_n$ is the same as that for $w = \lambda(v)$ in the Andreev packing for $\lambda(\widetilde{K}_n)$ by the uniqueness of Andreev packings, Theorem 2. We conclude that the limit radii for v and w are the same. As noted earlier, the radii determine the circle packing up to an automorphism of Δ. The proof concludes as in the parabolic case. □

Theorem 4. *Given a CP–complex K, there exists a Riemann surface S and a circle packing P for K which fills S. The surface S is the sphere, parabolic, or hyperbolic as K is of spherical, parabolic, or hyperbolic type, respectively. If K has a boundary, S is a bordered Riemann surface with complete geodesic borders.*

Proof. If K is of spherical type, then it is equivalent to a triangulation of the sphere, so $S = \widehat{\mathbf{C}}$ and P is the circle packing in Andreev's original setting (Theorem 1).

If K is of parabolic and hyperbolic type, we have the maximal circle packing $\widetilde{P}$ and the group of automorphisms Γ of $\mathcal{D}$ ($\mathbf{C}$ or Δ) as above. Because $\widetilde{P}$ is invariant under Γ, Γ must act totally discontinuously on $\mathcal{D}$. Also, Γ has no nontrivial elements of finite order because Λ, as a group of deck transformations, has none. Using the invariance of $\widetilde{P}$ again, it follows that Γ has no nontrivial elements with fixed points. Therefore, we may define S to be the Riemann surface $\mathcal{D}/\Gamma$, with the smooth analytic projection $p : \mathcal{D} \longrightarrow S$.

If $\mathcal{D}$ is the plane, then $\widetilde{K}$ is homeomorphic to the carrier of $\widetilde{P}$, which is all of $\mathbf{C}$, and Γ is isomorphic to Λ, so K is homeomorphic to S. The euclidean metric on $\mathbf{C}$ induces under p the metric ρ of zero curvature on S. The image of each circle in $\widetilde{P}$ under p is automatically a circle in this metric on S. It is evident that the circle configuration $P = p(\widetilde{P})$ is a circle packing in S with complex K, as desired. Its carrier is all of S since the carrier of $\widetilde{P}$ is all of $\mathbf{C}$.

If $\mathcal{D} = \Delta$, the argument is precisely the same, only the metric ρ has constant curvature -1 and is obtained from the Poincarè metric on Δ. If K has a boundary, then the carrier $\widetilde{\Omega}$ of $\widetilde{K}$ in Δ is a bordered simply connected domain in the sense of Section 2. In particular, the border segments are the complete geodesics between centers (on the unit circle) of neighboring boundary horocycles of $\widetilde{P}$. Under p, these project to give the geodesic borders of S. □

In many cases, we will find that the Riemann surface S constructed in this proof is the unique one satisfying the conditions of Theorem 4, or perhaps is unique under some additional extremal assumption; more on this in the next section. Be that as it may, the construction gives a cannonical choice for S (up to conformal equivalence) and for P. We now subsume our previous definition:

Definition. *If K is a CP–complex, the circle packing obtained in the proof of Theorem 4 above is termed the "maximal circle packing" for K and will henceforth be denoted $\mathcal{P}_K$. The Riemann surface supporting $\mathcal{P}_K$ will be denoted S_K. The maximal circle packing $\widetilde{P}$ for $\widetilde{K}$ will be termed the "universal covering packing" for K.*

Figures 1(a), 2, and 3(a) above illustrate maximal circle packings in $\widehat{\mathbf{C}}$, $\mathbf{C}$, and Δ, respectively. We conclude with several additional examples.

▹ EXAMPLE 1:

"Constant degree" complexes are those in which every vertex has the same degree. In [BSt2], the authors show that simply connected complexes of constant degree 5, 4, 3, or 2 are of spherical type, with maximal circle packings associated with the regular icosahedron, octahedron, tetrahedron, and (planar) equilateral triangle, respectively. The degree 6 case is the regular hexagonal packing of the plane (see Figure 2(a)). Constant degree 7 and higher are of hyperbolic type.

These show in a striking way that combinatorics affect the geometry. The degree 7 case is illustrated in Figure 4.

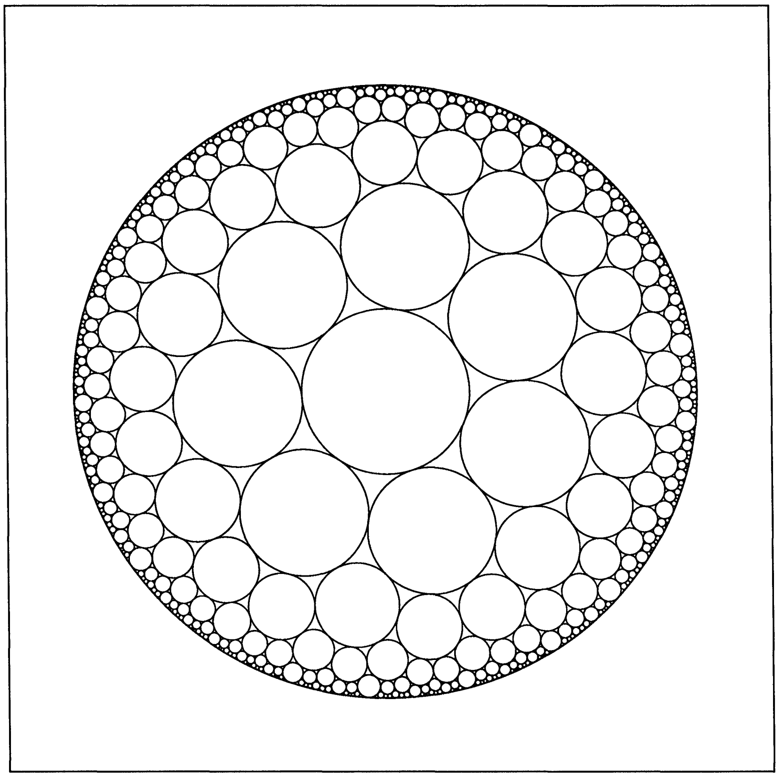


FIGURE 4. A CONSTANT DEGREE 7 CIRCLE PACKING.

▷ EXAMPLE 2:

Figure 5 illustrates a maximal circle packing having vertices of degree 6 and 12. It was obtained as follows: Construct a regular hyperbolic hexagon, each of whose interior angles measures $\frac{\pi}{6}$. It is known from the theory of Fuchsian groups that reflections in the sides of this hexagon and its copies will tesselate the hyperbolic plane—the circles go along to form the packing. (See [BSt2].) Similar constructions give several interesting configurations. We will have occassion to refer back to the $\{6,12\}$ and to the analogous $\{8,16\}$ circle packings later.

▷ EXAMPLE 3:

Circle packings of tori are illustrated in Figure 1(b) (p. 1384) and Figure 6.

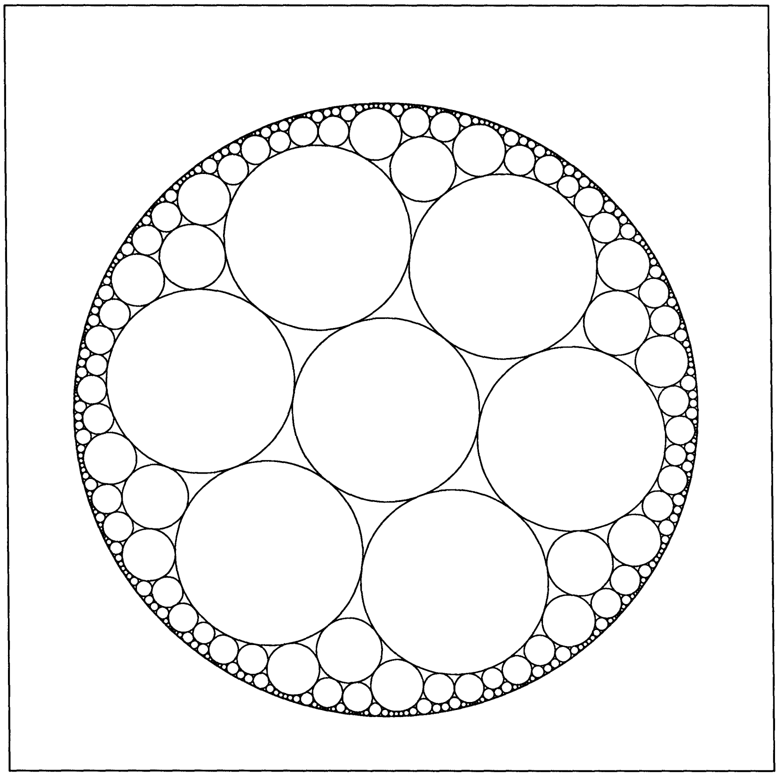


FIGURE 5. A DEGREE 6, 12 CIRCLE PACKING.

The universal covering packing is, in each case, that of Figure 2(a); however, the underlying complexes, the covering groups in $\mathbf{C}$, and the moduli of the tori are different.

▷ EXAMPLE 4:

Figures 7(a) and (b) (pp. 1404–05) suggest infinite complexes K and K' without boundary which are topological annuli. Since K has constant degree 6, its universal covering packing is that of Figure 2(a) (p. 1387), and we shall see that S_K must be the punctured plane. On the other hand, K' has constant degree 7, it universal covering packing is that of Figure 4 (p. 1403), so $S_{K'}$ must be a nondegenerate annulus. Thus, combinatorics can distinguish (conformally) between a puncture and an actual hole, as well as between the plane and the disc.

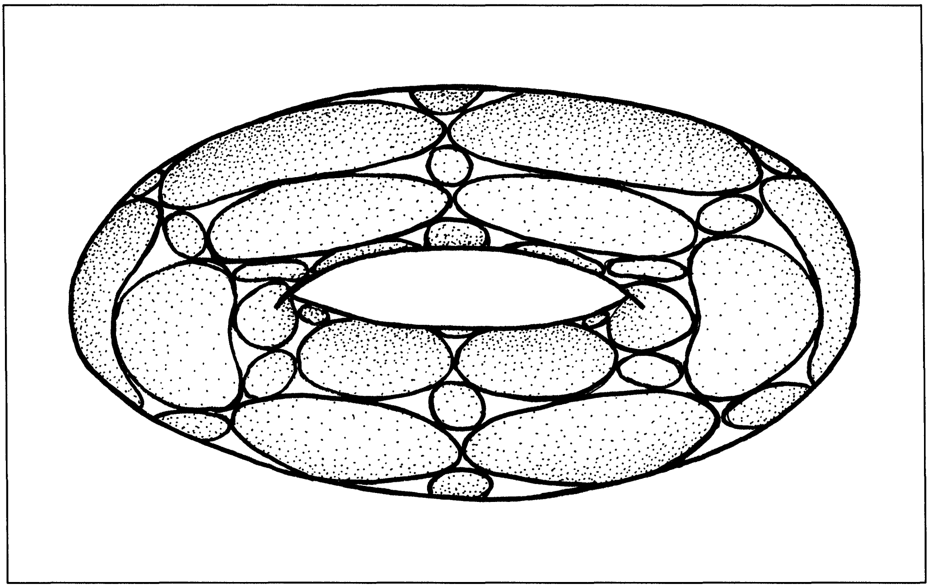


FIGURE 6. MAXIMAL CIRCLE PACKING OF A TORUS.

5. Properties of Maximal Circle Packings. We gather together properties of maximal circle packings in each of the geometries. Particular attention is paid to uniqueness: If S is a Riemann surface and P is a circle packing which fills S and has complex K, is S conformally equivalent to S_K? If so, is P the image of $\mathcal{P}_K$ under an automorphism of S_K? The answers are affirmative in the spherical and parabolic cases and in at least certain hyperbolic cases. We do settle the issue for compact complexes, so we close the section with a generalization of Andreev's original theorem, along with an example.

Spherical type. We have previously observed that K is of spherical type iff it is compact and simply connected. Since K is compact, there can be no realization in the plane or disc. The realization $\mathcal{P}_K$ in $\widehat{\mathbb{C}}$ is essentially unique by Theorem 1. The spherical radii themselves are not unique, since they change in complicated ways under automorphisms of $\widehat{\mathbb{C}}$.

Parabolic type. Let's begin by proving that a parabolic K cannot be realized as a circle packing in the hyperbolic plane:

Proposition 1. *A CP–complex K of parabolic type cannot be realized as a circle packing in Δ.*

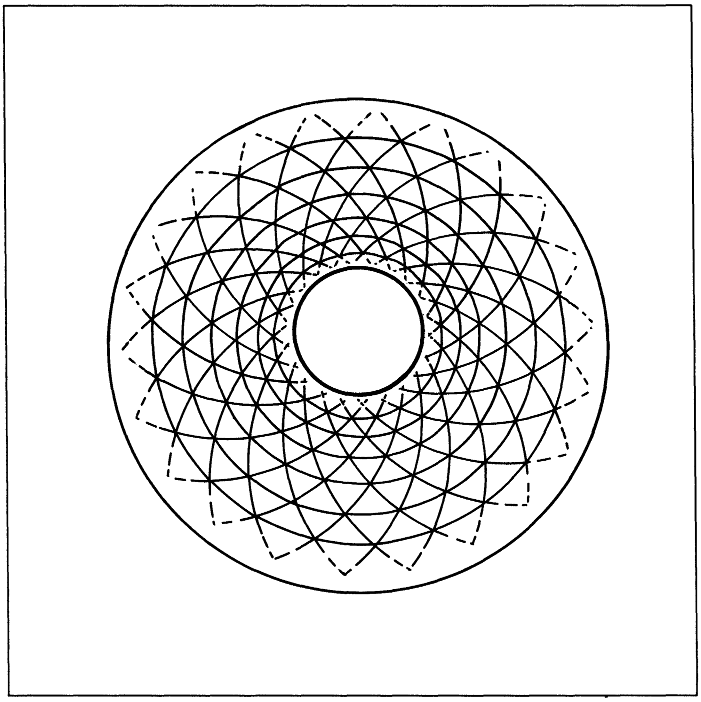


FIGURE 7(A). ANNULAR COMPLEXES.

Proof. Assume P is such a realization. Let Ω be its (euclidean) carrier, $\widetilde{\Omega}$ the carrier of the universal covering packing $\widetilde{P}$. The simplicial projection $\pi : \widetilde{K} \longrightarrow K$ induces a simplicial mapping $f : \widetilde{\Omega} \longrightarrow \Omega$ defined using euclidean barycentric coordinates. This is κ-quasiconformal for some $\kappa > 1$ by Lemma 3. The contradiction arises because $\widetilde{P}$ fills $\mathbf{C}$, so f is a κ-quasiconformal mapping from $\mathbf{C}$ into Δ. No such mappings exist. $\square$

In fact, this same proof implies that K cannot be realized as a circle packing in any hyperbolic subdomain of $\mathbf{C}$, that is, any subdomain having two or more finite boundary points.

Next, we observe that the list of complexes of parabolic type parallels that in the classical setting. Indeed, the group Λ of deck transformations of K is isomorphic to the group Γ of automorphisms of $\mathbf{C}$, by Lemma 4. Since Γ is properly discontinuous, it must be abelian, so only three situations arise:

(1) Λ is trivial: In this case, K is simply connected, so $K = \widetilde{K}$ and S_K is the plane.

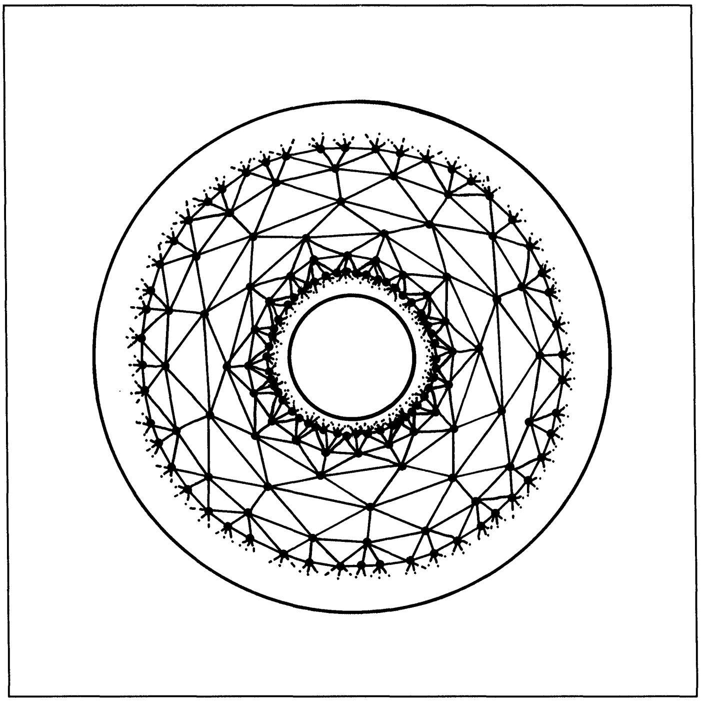


FIGURE 7(B). ANNULAR COMPLEXES.

(2) Λ is cyclic: In this case, K is topologically an annulus. Being fixed-point free, Γ must be generated by a translation, so S_K is the punctured plane.
(3) Λ is generated by two commuting elements: In this case, K is a torus. Γ is generated by two commuting translations, so S_K is a torus with modulus determined by the generators.

Lastly, the question of uniqueness of S_K and $\mathcal{P}_K$. Suppose S is a Riemann surface and P is a circle packing in S with complex K. We first show that S is conformally equivalent to S_K. Let S' be the universal covering surface of S, $p : S' \longrightarrow S$ the covering projection, and Γ' the covering group. We claim that the circle packing P lifts under p to a circle packing P' in S': This depends first on our standing assumption that circles on Riemann surfaces have simply connected interiors, and second on the fact that we are assuming the intrinsic metrics in S' and S, and these are related by the covering projection p'. Of course, $\widetilde{K}$ is the complex for P', so by the proposition above, $S' = \mathbf{C}$ and P' fills $\mathbf{C}$. Let $g : S_K \longrightarrow S$ be the simplicial homeomorphism from the carrier of $\mathcal{P}_K$ to the carrier of P. It is easily seen that g lifts to a simplicial homeomorphism G of the carrier of $\widetilde{P}$ to the carrier of P' with $G^{-1}\Gamma' G = \Gamma$. By the Sullivan uniqueness theorem, G is an automorphism of $\mathbf{C}$, so we conclude that g is conformal and hence S_K and S are conformally equivalent. Furthermore, in case $S_K = S$ we see that g is an automorphism of S_K. That means that, within S_K, the circle packing $\mathcal{P}_K$ is unique up to automorphisms. We have proven

Theorem 5. *Let K be a CP–complex of parabolic type. Then S_K is the unique Riemann surface in which K may be realized as a circle packing. The corresponding circle configuration fills S_K and is unique up to automorphisms of S_K.*

Parabolic surfaces S_K have uncountably many automorphisms. What, then, can be said about radii of a realization P of K? The euclidean metric ρ on S_K is determined only up to a scalar multiple, so the same holds for the radii. Suppose a particular ρ is fixed. If S_K is a torus, then automorphisms are isometries, so the radii of P are uniquely determined. For the plane and punctured plane, only the ratios of radii are determined.

Hyperbolic type. The remaining complexes are hyperbolic. Unlike the other types, we may have a variety of circle packings with the same complex—uniqueness questions are thus tougher, and, with no known analogue of Sullivan's uniqueness theorem, not completely resolved. Some positive results are obtained using the following extremal result, which also justifies use of the term "maximal":

Lemma 5. *Let K be a simply connected CP–complex of hyperbolic type, $\mathcal{P}_K$ its maximal circle packing with carrier $\Omega \subseteq \Delta$.*

(a) *Suppose P is another circle packing for K in Δ.*
   (i) *Each circle in P has hyperbolic radius less than or equal to that of the corresponding circle in $\mathcal{P}_K$.*
   (ii) *The hyperbolic distance between (the centers of) two circles of P is less than or equal to that between the corresponding circles in $\mathcal{P}_K$.*
   (iii) *Each face in the hyperbolic carrier of P has hyperbolic area less than or equal to that of the corresponding face in the carrier of $\mathcal{P}_K$.*

Moreover, equality in (i) *for any interior vertex, finite equality in* (ii) *for any distance, or equality in* (iii) *for any face implies that P and $\mathcal{P}_K$ are automorphic images of one another.*

(b) *If K has no boundary vertices, then $\Omega = \Delta$. Otherwise, $\Delta \backslash \Omega$ is a union of mutually disjoint open halfplanes determined by the geodesics connecting centers of contiguous horocycles of $\mathcal{P}_K$ associated with boundary vertices of K.*

Proof.

(a) The inequalities are immediate from the DSPL and our construction of $\mathcal{P}_K$ via nested exhaustion. Regarding equality, suppose that w is an interior vertex with at least one neighbor whose circle in P is strictly smaller than in $\mathcal{P}_K$. The remaining circles in the flower of w are no larger in P than in $\mathcal{P}_K$, so a monotonicity result in [Lemma 3, BSt1] implies that the circle for w is strictly smaller in P than in $\mathcal{P}_K$. Because K has connected interior, therefore, a smaller circle for any interior vertex implies smaller circles for all—that is, equality for one implies equality for all. Equality of interior radii implies that P and $\mathcal{P}_K$ are automorphic images of one another. Equalities in (ii) and (iii) will imply equality in (i) by monotonicity results in [Lemma 2, 4, BSt1]. This proves (a).

(b) This part was proven during the construction of maximal circle packings in the hyperbolic plane. □

To set the stage for the uniqueness question, one should be aware of the variety of circle packings realizing a simply connected complex K of hyperbolic type. First, any euclidean contraction of $\mathcal{P}_K$ gives an essentially distinct circle packing $P \subseteq \Delta$. If K is a DL-complex, a bewildering variety of "packings" (many of them circle packings) are provided by [Theorem 3, BSt1]. The reader should note, however, that these are circle packings in an "ambient" metric—one inherited by the carrier from the ambient space (in this case, the Poincarè metric on Δ)—whereas we are using the intrinsic metric ρ on the carrier. Nonetheless, they make one pause: the possibility of more subtle alternatives to $\mathcal{P}_K$ will be discussed in Section 8.

Let us start with DL-complexes: Suppose P is a circle packing for K which fills S. In its intrinsic metric ρ, S is to be a simply connected Riemann surface with complete geodesic boundary, so it is conformally equivalent to the interior of an ideal polygon in the hyperbolic plane (one with vertices on $\partial\Delta$). In particular, the boundary vertices correspond with circles of infinite radius, so P is an automorphic image of $\mathcal{P}_K$ by the uniqueness of Andreev's Theorem. Consequently, S is conformally equivalent to S_K.

Let K be a complex of hyperbolic type, with maximal circle packing $\mathcal{P}_K$ in the Riemann surface S_K and $\widetilde{P}$ the universal covering packing in Δ. Suppose P is another circle packing for K whose carrier is a Riemann surface S homeomorphic to K. By Proposition 1, the universal covering surface of S is Δ. As in the parabolic case discussed earlier, P lifts to a circle packing P' for $\widetilde{K}$ under the universal covering projection $p' : \Delta \longrightarrow S$. Let Γ' be the corresponding covering group. The embeddings of K in S_K and S yield a simplicial homeomorphism $g : S_K \longrightarrow S$ which lifts to a simplicial homeomorphism G of the carrier of $\widetilde{P}$ to the carrier of P' with $G^{-1}\Gamma' G = \Gamma$. We obtain uniqueness as before if we can show that G is an automorphism of Δ. The best we can do at this time is

Theorem 6. *Let K be a CP–complex of hyperbolic type and suppose that S_K is an open Riemann surface with finite area. Then S_K is the unique Riemann surface which is filled by a circle packing realizing K. The corresponding circle configuration is unique up to automorphisms of S_K.*

Proof. An open Riemann surface has finite area (with respect to ρ) if and only if it is a compact surface of genus g with some finite number N of punctures. In particular, since S_K is hyperbolic, it can only be the sphere with 3 or more punctures, a torus with 1 or more punctures, or a compact surface with $g \geq 2$ and $N \geq 0$ punctures. The area $A = (4g + 2N - 4)\pi$ is a topological invariant, so the homeomorphic surface S has the same area. However, consider fundamental regions $\widetilde{F}$ and F' of the coverings $\widetilde{P} \mapsto \mathcal{P}_K$ and $P' \mapsto P$, respectively. These both have area A. But by Lemma 5(a(iii)), this equality implies that $\widetilde{P}$ and P' are automorphic images of one another, and the map G above is an automorphism. □

There are additional hypotheses which imply uniqueness. For instance, we have noted the uniqueness for DL-complexes; and more generally, if S_K is bordered and of finite area, then one can use a doubling argument to reduce to the open case above. However, the question of uniqueness in full generality is more subtle and comments are left to Section 8.

Regarding the actual circle configurations, the hyperbolic setting gives much more rigidity than the others. The metric ρ is unique, so the circles of $\mathcal{P}_K$ have uniquely determined radii. All automorphisms of S_K are isometries, but the generic surface will have no or only finitely many automorphisms, thus, at most finitely many ways to realize K.

Compact Surfaces. Bringing together preceeding results, we get the complete picture for compact complexes and generalize Theorem 1.

Theorem 7. *Let K be a triangulation of a compact 2-manifold of genus g. There exists a unique conformal structure on K so that the resulting Riemann surface S supports a circle packing for K. The corresponding (oriented) circle configuration fills S and is unique up to automorphisms of S.*

The spherical case, $g = 0$, is Theorem 1. The new feature for higher genus is the determination of a conformal structure; unlike the sphere, K could be endowed with uncountably many distinct conformal structures, but only one supports a circle packing for K. Circle packings of the sphere and torus have been illustrated in Figure 1. Here's an example of higher genus:

▷ EXAMPLE 5:

In Figure 8, we illustrate a complex of genus 2 and its maximal circle packing *in situ*, i.e., in an embedding of S in $\mathbf{R}^3$. Note the different combinatorics of the two handles of the complex. We have tried to indicate that the combinatoric asymmetry will generally force differences in the geometries of the two handles (their moduli). In particular, lack of symmetry will typically prevent there being any automorphisms of S, and the circle configuration will be unique.

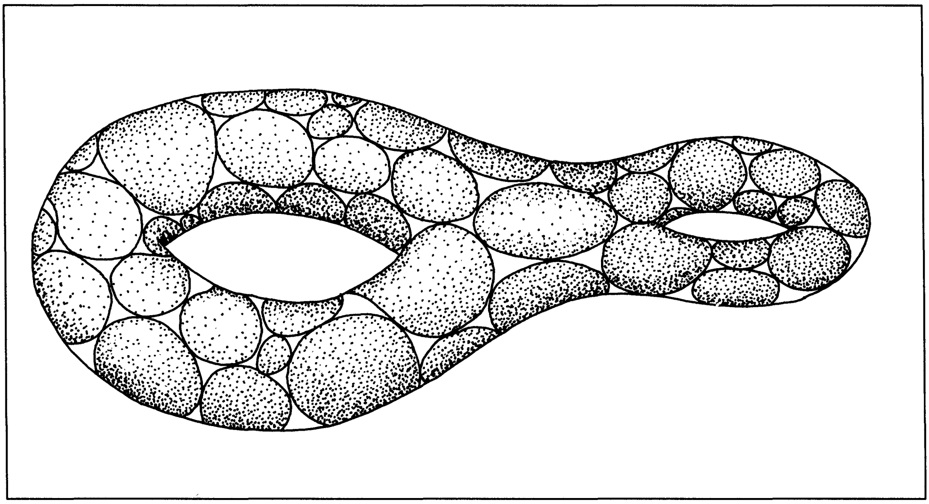


FIGURE 8. A MAXIMAL CIRCLE PACKING IN GENUS 2

6. The Type Problem. It is a classic problem in complex analysis to determine conditions which give the "type" of a Riemann surface—that is, to determine whether the universal covering surface will be the plane or the disc. The same question is now applicable to complexes. We start by gathering together some topological type conditions which have already been observed.

Proposition 2. *Let K be a CP-complex. Then:*

(a) *K is of spherical type $\iff$ it is a topological sphere $\iff$ it is a triangulation of the sphere $\iff$ it is compact and simply connected.*
(b) *K is of parabolic type if it is a torus; that is, compact and genus 1.*
(c) *K is of hyperbolic type if it has boundary vertices.*
(d) *K is of hyperbolic type if its fundamental group is nonabelian: in particular, if it has genus $g \geq 2$ or if it is m-connected, $m \geq 3$.*

Next, we indicate some purely combinatoric type conditions which arise in [BSt2]. The constant degree complexes of Example 1 above are the prototypes, but recall that d here refers to the maximum degree among all vertices of K.

Proposition 3. *Let K be a CP–complex of degree d with no boundary. Then:*

(a) *K is of spherical type (and finite) if $d \leq 5$.*
(b) *K is of parabolic type if it is infinite and $d \leq 6$.*
(c) *K is of hyperbolic type if every vertex has degree ≥ 7.*

The universal covering complex $\widetilde{K}$ shares the degree condition on K, so we may assume K is simply connected. The proofs in this case are given in [BSt2] and basically involve the rate at which radii must grow or decay as one moves along a chain of vertices of the complex. Note Example 7 in the next section, which suggests how exquisitely tight is the link between combinatorics and type.

More general type conditions follow from combinatoric versions of the length/area principle. We give two examples. The first is noted by Rodin and Sullivan [p. 357, RS] and follows from their Length-Area Lemma.

Propostion 4. *Let K be an infinite simply connected CP–complex without boundary. Fix a vertex v, and let n_k denote the number of vertices in the kth generation from v, $k = 1, 2, \cdots$. If $\sum n_k^{-1}$ diverges, then K is of parabolic type.*

The second also involves an infinite simply connected complex K without boundary. Let $\{K_j\}$ be an exhaustion by DL-subcomplexes, each lying in the interior of the next; let ℓ_j denoting the number of edges in the boundary curve of K_j. If ℓ_j grows sufficiently slowly, then K will be of parabolic type. Specifically, define $\sigma_j \in (0,1)$ by $\sigma_j = \frac{1}{1+C_j}$, where C_j is the constant $\mathcal{C}_1$ associated with ℓ_j-flowers by Lemma 1(b).

Proposition 5. *Let K and $\{\sigma_n\}$ be as above. If the infinite product $\Pi_n \sigma_n = 0$, then K is of parabolic type.*

Proof. Suppose v is a fixed vertex interior to all K_n. Let P_n be the Andreev packing for K_n, normalized so that the circle for v is at the origin; let r_n be the euclidean radius of that circle; and let Q_{n-1} be the configuration of circles from P_n associated with the complex $K_{n-1} \subseteq K_n$.

Now P_n has a chain of ℓ_n horocycles forming its boundary. Reflecting these in the unit circle gives a ℓ_n-flower with center circle Δ. By Lemma 1(b), the carrier of this flower contains the disc $\{|z| < 1 + C_n\}$. Reflecting back into the unit disc and observing that these horocycles are disjoint from Q_{n-1}, we see that $Q_{n-1} \subseteq \{|z| < \sigma_n\}$. In particular, the scaling of Q_{n-1} by the euclidean factor $1/\sigma_n$ gives a circle packing for K_{n-1} in Δ. The circle for v at the origin in this has euclidean radius r_n/σ_n. Since we are comparing circles centered at the origin, the hyperbolic inequality of DSPL implies the euclidean inequality $r_{n-1} \geq r_n/\sigma_n \Rightarrow r_n \leq \sigma_n r_{n-1}$. By induction, $r_n \leq \Pi_{j=1}^{n} \sigma_j r_1$, $n = 1, 2, \cdots$. The condition that the infinite product vanish implies $\lim_n r_n = 0$, hence K is of parabolic type (Case I, Section 4). $\square$

These type conditions suggest that our notion of "hyperbolic" is very much in the spirit of the classical notion. Moreover, it seems to be compatible with the broader use of the term in various settings involving comparisons of lengths and areas. It is perhaps an oversimplification, but a hyperbolic situation is one in which small areas can have long boundaries—there is an abundance of room near infinity. Examples: In the study of random walks, the length/area comparison is connected with recurrence vs. transience (Dodziuk [D]); in hyperbolic groups, with the rate of growth of words (Gromov [G]); in 3-manifolds, with the proliferation of copies of the fundamental domain (Thurston [T1]). A striking example of length/area methods is the recent work of Cannon [C] on combinatoric "conformal structures".

7. Discrete Analytic Functions. In the concluding section of [BSt1], we suggested a notion of "discrete analytic function" in the hyperbolic setting and stated analogues for the classical Schwarz and Pick lemmas. It seems that a more comprehensive notion is now possible. With our contention that circle packings determine a type of discrete conformal structure, "analytic" functions should somehow preserve circle packings. It is not our intention to go into this deeply here, but rather to suggest some preliminary ideas.

The first need is for more general circle configurations, so let's consider circles defined in what we term "ambient" metrics—until now, we have used only the intrinsic metric ρ, so we have ignored many configurations which might have occurred to the reader (not to mention the more abstract "packings" studied in [BSt1]). Let's start with several examples.

▷ EXAMPLE 6:

First, consider the infinite circle packing of Figure 9(a), which is one of a two-parameter family we term "Doyle spirals", as they were motivated by an idea of Peter Doyle. (These are investigated in greater detail in [BSt3].) It can be shown that this circle packing fills the punctured plane. However, these are circles in the "ambient" euclidean metric inherited from the plane, and not in the "intrinsic" euclidean metric projected from the plane (as universal covering). For comparison, Figure 9(b) shows the circle packing with the same complex, but now in the intrinsic metric.

▷ EXAMPLE 7:

An ambient packing and some Doyle spirals show how sensitive the type of a complex can be to its combinatorics: Choose two disjoint 6-flowers in the infinite hexagonal packing of Figure 2(a). The center circle of each may be replaced by an infinite spiral, as in Figure 10 (see p. 1416), where the petals of the original flower are shaded. The resulting ambient circle packing can be shown to fill the twice punctured plane and consequently determines a hyperbolic structure. Note

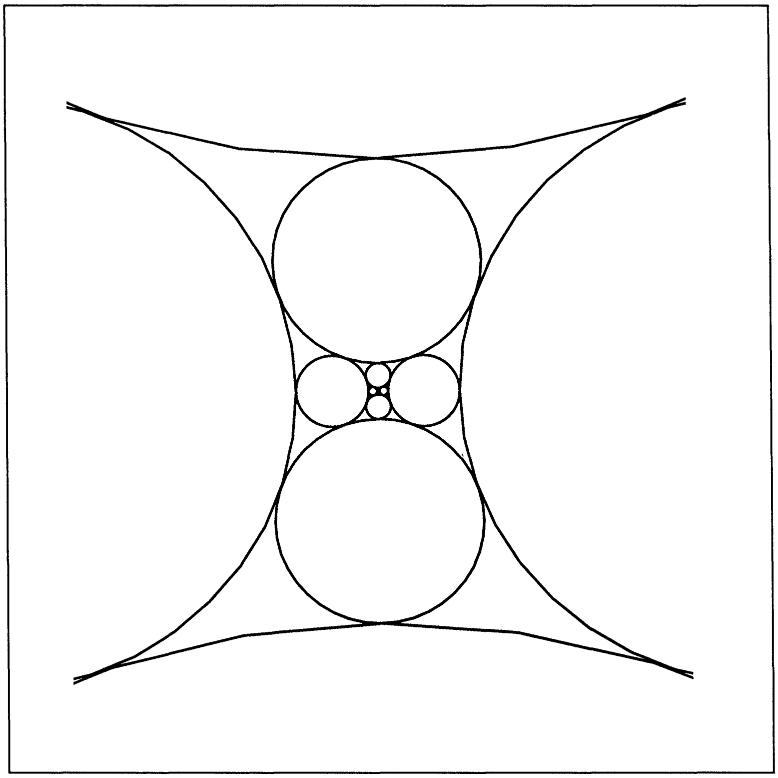


FIGURE 9(A). A DOYLE SPIRAL IN AMBIENT AND INTRINSIC METRICS.

that the degree is 6 at every circle, save for the 12 petals of our two flowers—those 12 have degree 7 and make all the difference in the type.

▷ EXAMPLE 8:

Figure 11(a) (see p. 1418) illustrates a circle packing of an annulus in Δ. Here we have a choice of ambient metric, either euclidean or hyperbolic. The combinatorially equivalent intrinsic packing is suggested by Figure 11(b) (see p. 1419). The supporting annuli will generally have different moduli; it is the conformal structure of the annulus underlying (b) which is picked out by Theorem 4.

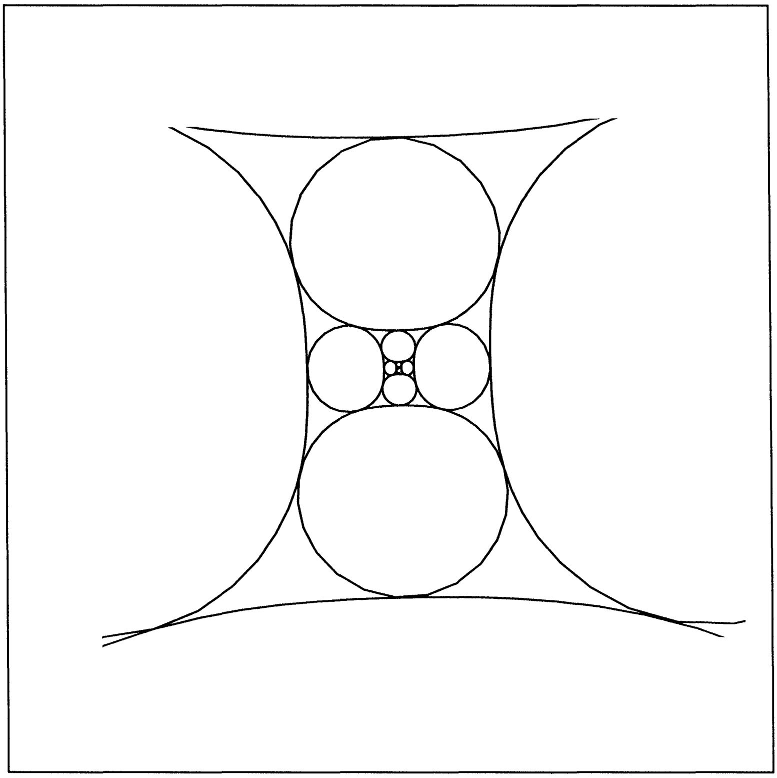


FIGURE 9(B). A DOYLE SPIRAL IN AMBIENT AND INTRINSIC METRICS.

▷ EXAMPLE 9:

Figure 12(a) (see p. 1420) illustrates yet another possibility (see [Figure 7, BSt1]). The configuration P winds around and overlaps itself. The overlap seems to violate our requirement for circles having mutually disjoint interiors; however, we may resolve the overlap by considering the circles as lying on the two-sheeted Riemann surface S suggested by Figure 12(b) (see p. 1421). The natural projection of S to Δ induces a metric λ of constant curvature -1 on S, and P qualifies as a circle packing in S using the ambient metric λ. The complex for the circle packing is embedded in S in the usual way as the carrier of P, but note that its projection to Δ would be termed an "immersion", since it is no longer one-to-one.

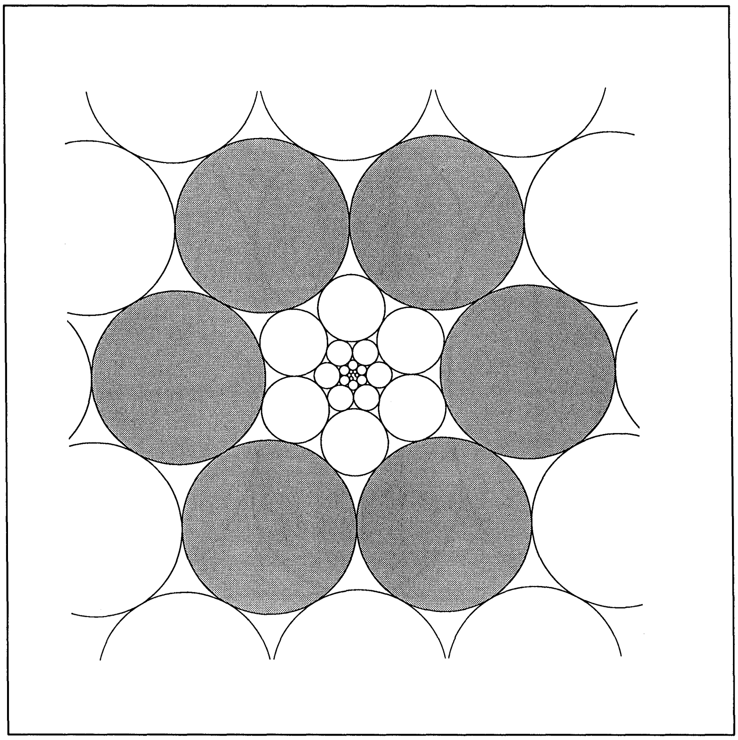


FIGURE 10. MODIFIED FLOWERS.

The use of ambient metrics, overlapping patterns, and so forth allows for very flexible circle configurations, ones which compare to the earlier patterns as multi-sheeted Riemann surfaces compare to single-sheeted ones—even branch points may be included in a very natural way. Discrete analytic functions, then, should somehow preserve circle packings. We propose a definition involving simplicial maps between complexes, along with the embeddings or immersions of those complexes as carriers of circle packings.

Let K be a simply connected CP-complex with maximal circle packing $\mathcal{P}_K \subset \mathcal{D}$ and with embedding $\varphi : K \longrightarrow \mathcal{D}$. Let K' be another CP-complex realized as a circle packing $P \subset \mathcal{A}$ with immersion $\psi : K' \longrightarrow \mathcal{A}$.

Definition. *Suppose $f : K \longrightarrow K'$ is a simplicial map. The function F : $\mathcal{D} \longrightarrow \mathcal{A}$ defined by $F \equiv \psi \circ f \circ \varphi^{-1}$ will be called a "discrete analytic function" on K.*

$$
\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "\varphi"'] & K' \arrow[d, "\psi"] \\
\mathcal{D} \arrow[r, "F"'] & \mathcal{A}
\end{tikzcd}
$$

We are not being very precise about the embeddings and immersions here. Of main importance is that the center of the circle of $\mathcal{P}_K$ corresponding to a vertex $v \in K$ is mapped to the center of the circle of P corresponding to $f(v) \in K'$. Generally, the rest of the mapping can be defined using appropriate barycentric coordinates. One other point: recall that φ is only determined up to an automorphism of $\mathcal{D}$, so a normalization is needed; for our purposes here, however, we may safely ignore this slight ambiguity.

If $K = K'$ were a DL-complex and $P \subset \Delta$, then F would be one of the "discrete" analytic functions of [BSt1]. However, we now have much more variety. For instance, "discrete entire" functions are those defined on circle packings of the plane; "discrete rational" functions, those defined on complexes of spherical type. The range might well be a compact Riemann surface rather than a domain in the plane or the hyperbolic plane.

If K were the universal covering complex of K', then F is evidently a discrete universal covering map. Thus, the map of the regular hexagonal packing (Figure 2(a)) to the Doyle spiral of Figure 9(a) is a discrete entire function which is, like the classical exponential function, a covering map of the punctured plane. Figure 13 (p. 1422) illustrates the discrete universal covering map of the disc to an annulus.

Many features of classical covering theory will go through in the category of discrete analytic functions. Suppose, for instance, that K is a universal covering complex of K_1, $\mathcal{P}_K$ is supported by $\mathcal{D}$, and Γ is the associated covering group of automorphisms of $\mathcal{D}$. Then any Γ-automorphic discrete analytic function on K defines a discrete analytic function on K_1. By such devices, one can in a very straightforward manner define discrete analytic functions between Riemann surfaces.

▷ EXAMPLE 10:

The tesselation of the hyperbolic plane by the embedded complex in Example 2 comes from a "triangle group" of Möbius transformations. If we form the identification space, we get a branched covering of the Riemann sphere, which can be interpreted as the covering of the circle packing of Figure 14 (p. 1423), with branch points of multiplicity 2 at the north and south poles and multiplicity 4 at the centers of the three circles about the equator.

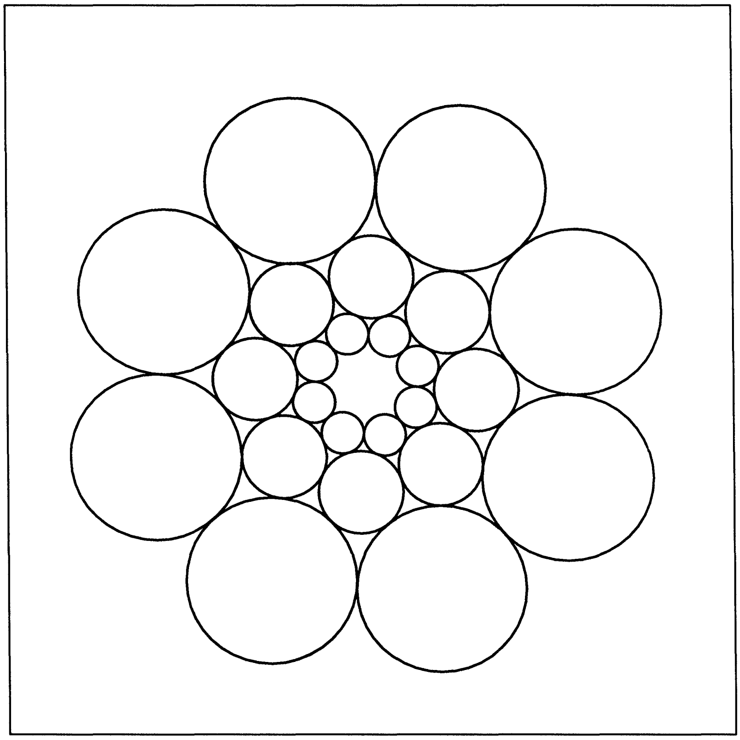


FIGURE 11(A). AMBIENT VS INTRINSIC CIRCLE PACKINGS OF AN ANNULUS.

Certain classical results are available in discrete form: For discrete analytic self-maps of Δ, Lemma 5(a) implies the analogue (in the case of infinite, as well as finite, complexes) of the Schwarz-Pick lemma. Proposition 1 converts to a statement of the small Picard theorem and Liouville's theorem for discrete entire functions. By compactness, discrete rational functions will be n-valent mappings of the sphere to itself, which (if not automorphisms) will require branch points in the image circle packing P. A fuller understanding of these ideas and their implications awaits further study.

8. Remarks and Questions. Circle packing is a new topic, and it holds out the potential for many interesting and perhaps useful results. We close with several comments and questions—many others have probably occurred to the reader along the way.

Uniqueness. Perhaps the most intriguing open question concerns the uniqueness of circle packings in Δ. Let K be a simply connected complex of

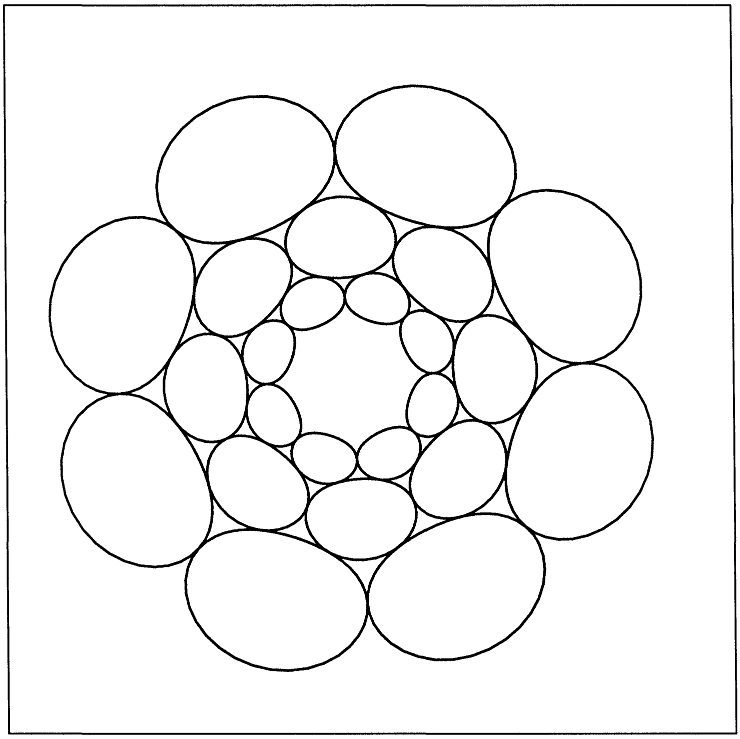


FIGURE 11(B). AMBIENT VS INTRINSIC CIRCLE PACKINGS OF AN ANNULUS.

hyperbolic type and without boundary. The maximal circle packing $\mathcal{P}_K$ fills the disc.

Question. *Is $\mathcal{P}_K$ the unique circle packing with complex K which fills the hyperbolic plane?*

An affirmative answer would give the infinite analogue of uniqueness of Andreev packings in Theorem 2. More to the point, however, it would be the hyperbolic analogue of Sullivan's uniqueness result in the plane. This would (along with a similar result which should follow for complexes with boundary) provide the uniqueness statement one anticipates in Theorem 4. As it is, we basically get uniqueness only with the help of some side conditions, such as in Theorem 6 where there is a Fuchsian group under which the circle packing is invariant. Geometric arguments in [BSt2] prove uniqueness for constant degree complexes.

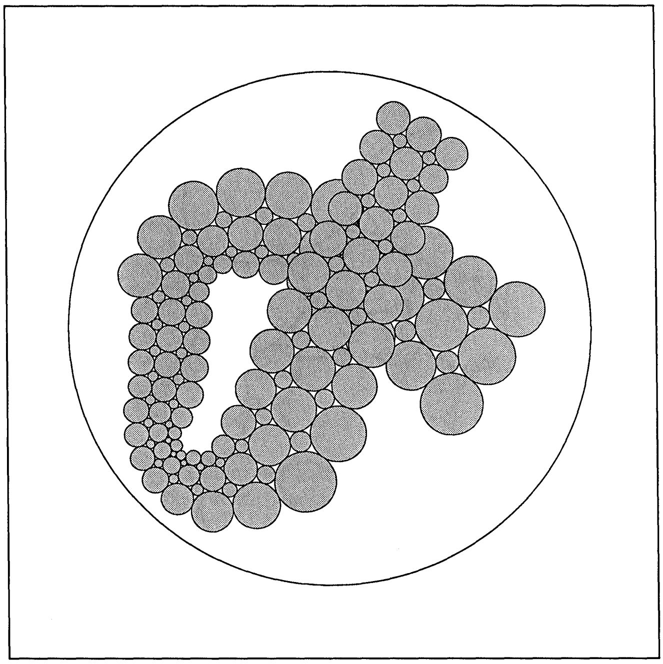


FIGURE 12(A). A CIRCLE PACKING ON A SURFACE.

Since Sullivan's result relies on Mostow rigidity, this question may be quite deep. We do have some affirmative signs, however. Foremost among these is the extremal property of $\mathcal{P}_K$ given by Lemma 5. If P is an alternate circle packing, we may define the euclidean barycentric map f from the carrier of $\mathcal{P}_K$ to that of P. If P fills Δ, this is a quasiconformal homeomorphism of Δ (see the proof of Proposition 1). By [Lemma 3(d), BSt1], the set of vertices of $\mathcal{P}_K$ is uniformly dense in the hyperbolic plane, and by Lemma 5(a(ii)), f is a hyperbolic contraction on this set. Can f fail to be a Möbius transformation?

Countability of structures. There are only countably many compact CP-complexes K, and hence only countably many compact Riemann surfaces S_K.

Question. *What distinguishes these "packable" surfaces?*

Of the uncountably many compact surfaces of genus $g > 0$, for example, only countably many support any circle packing whatsoever. Can one estimate moduli directly from the combinatorics of K? Can one find a packable complex with

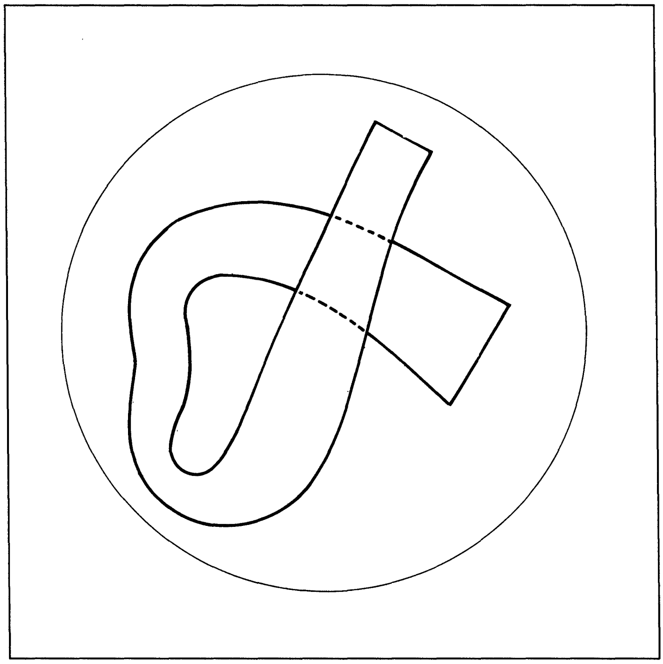


FIGURE 12(B). A CIRCLE PACKING ON A SURFACE.

moduli close to any preassigned values? Can a given compact Riemann surface be packable in (essentially) distinct ways?

Consider the regular hexagonal lattice of Figure 2(a). A sublattice with modulus τ determines a packable torus with modulus τ. There are infinitely many other sublattices with this same modulus, hence infinitely many combinatorially distinct circle packings of that torus. Also, the moduli of the sublattices are dense in the moduli space of the torus. Thus, one can approximate any torus with a packable one, though it may be at the expense of larger complexes. What is the situation for genus greater than one?

Countability when one discretizes something is not surprising, but the number theory and combinatoric connections here seem potentially very rich. An indication of such connections is given by the fascinating study of Brooks [Br]. He provides a continuous parameter t associated via continued fractions with a process of packing the quadrilateral interstice formed by four circles. For rational values of t, the process concludes with a circle packing (with our usual triangular interstices) after a finite number of steps; an additional countable set of parameters correspond to infinite packings of bounded degree; while the generic value of t leads to an infinite packing of unbounded degree. The parameter is related

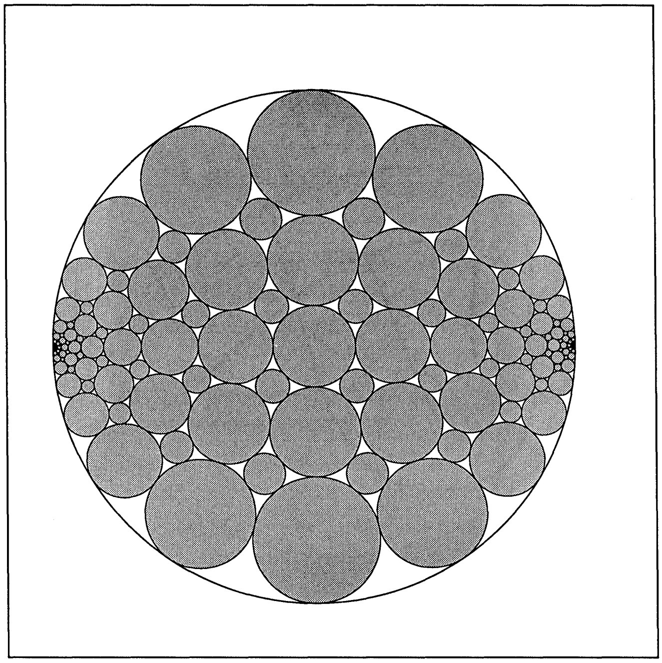


FIGURE 13. UNIVERSAL COVERING PACKING OF AN ANNULUS.

to the modulus of the original interstice: the rigidity of packing conditions forces the link t between the combinatorics and the geometry of these situations.

Other Complexes. Certain of the conditions on CP-complexes could be removed without serious damage to results (though not without a cost in added details), while others may be essential.

The triangulation K in Theorem 7, for example, can be replaced by a more general partition, say a cell-decomposition, as long as the universal cover $\widetilde{K}$ is a triangulation (CP-complex). After all, in nonplanar Riemann surfaces, circles may be tangent to themselves, pairs of circles with disjoint interiors may be tangent at more than one point, and so forth. We have avoided these cases simply for clarity and convenience. As an extreme illustration, consider the circle packing of Δ obtained as in Example 2, but having degrees 8 and 16. This projects to the "circle packing" of a Riemann surface of genus 2 suggested in Figure 15—it has only 2 circles!

In contrast to this, our restriction to complexes of bounded degree may be essential. If K fails to have bounded degree (though each individual vertex has bounded degree, of course), we still obtain a maximal circle packing $\mathcal{P}_K$ by the

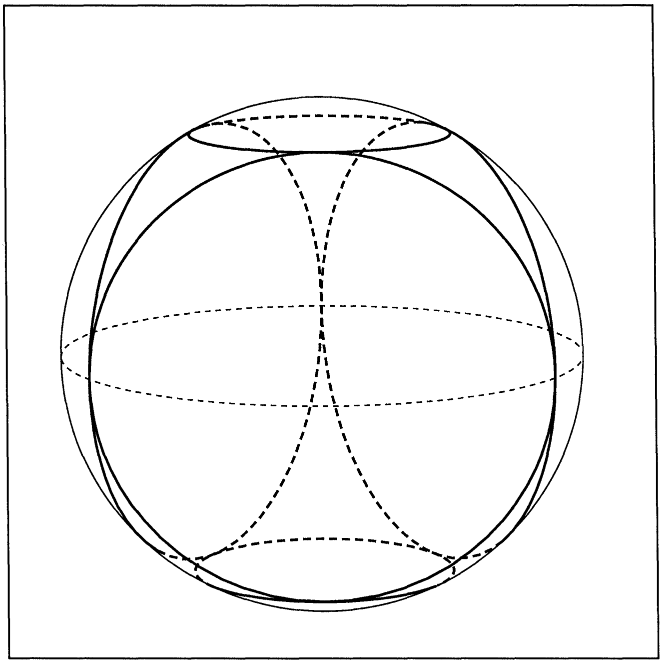


FIGURE 14. A BRANCHED CIRCLE PACKING OF THE SPHERE.

methods of Section 4. However, does it fill the plane or the disc? Could it have a realization in both? What about uniqueness? These questions seem quite difficult to handle at this time, but our guess is that severe pathologies might be possible.

Approximations. We have discussed several discrete analogues of continuous analytic objects. We close by suggesting that these may also serve as approximations. For instance, the discrete analytic functions as defined earlier should approximate classical analytic functions *à la* Thurston's original conjecture [T1].

To frame the issue precisely, consider the following situation: Let G be a bounded and finitely connected domain in the plane. Let P_j be a sequence of increasingly fine ambient circle packings in G whose carriers exhaust G and whose complexes K_j are homeomorphic to G and of uniformly bounded degree. For each j, let $\widetilde{K}_j$ be the universal covering complex and let $F_j : \Delta \longrightarrow G$ denote the discrete analytic function mapping $\mathcal{P}_{\widetilde{K}_j}$ to P_j. Apply a normalization so that F_j maps the origin to some fixed point $z_0 \in G$, and a positive value to some

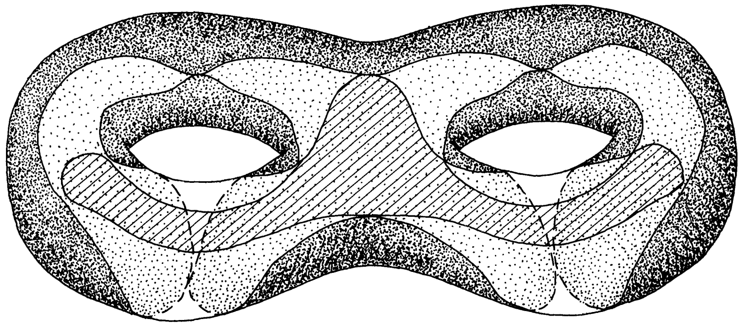


FIGURE 15. A TWO-CIRCLE PACKING OF A 2-TORUS

fixed point $z_1 \in G$. Based on techniques from this paper and [St], we conjecture that *F_j converges uniformly on compact subsets of Δ to the analytic universal covering projection $F : \Delta \longrightarrow G$.* This generalizes Thurston's conjecture in that it allows for non-simply connected, non-hexagonal, and infinite combinatorics. It seems, then, that whether one wants approximations or analogies to the classical theory of analytic functions, circle packings hold great promise.

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