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COMBINATORIAL YAMABE FLOW ON HYPERBOLIC SURFACES WITH BOUNDARY

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This paper studies the combinatorial Yamabe flow on hyperbolic surfaces with boundary. It is proved by applying a variational principle that the length of boundary components is uniquely determined by the combinatorial conformal factor. The combinatorial Yamabe flow is a gradient flow of a concave function. The long-time behavior of the flow and the geometric meaning is investigated.

Keywords: Combinatorial conformal factor; combinatorial Yamabe flow; variational principle; derivative cosine law.

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1. Introduction

1.1. *Piecewise flat metrics*

In trying to develop the analogous piecewise linear conformal geometry, Luo studied the combinatorial Yamabe problem for piecewise flat metrics on triangulated surfaces [16]. We summarize a part of this work in the following. Suppose Σ is a connected closed surface with a triangulation T so that V, E, F are sets of all vertices, edges and triangles of T. We identify a vertex of T with an index, i.e. $V = \{1, 2, \ldots, n\}$, where n is the number of vertices. We will use a Greek letter to denote an edge.

A piecewise flat metric on (Σ, T) is identified with a vector indexed by the set of edges E. More precisely, it is an assignment to each edge a positive number such that the three numbers assigned to the three edges of a triangle satisfy the triangle inequality. Equipped with a piecewise flat metric, each triangle of T can be realized as a Euclidean triangle and Σ can be realized as a Euclidean polyhedral surface.

Let us fix a piecewise flat metric on (Σ, T) as $l^0 \in \mathbb{R}^{|E|}_{>0}$. The assignment to the edge α is denoted by l^0_α. A *combinatorial conformal factor* on (Σ, T) is a vector $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ which assigns each vertex $i \in V$ a number w_i. (In [16], the notation u_i is used, where $u_i = e^{w_i}$.) From a combinatorial conformal factor, we obtained a new vector $l \in \mathbb{R}^{|E|}_{>0}$ as follows:

$$l_\alpha = e^{w_i + w_j} l^0_\alpha \tag{1}$$

for each edge $\alpha \in E$, where i and j are vertices incident to the edge α. Note that i and j can be the same vertex.

Let $\mathcal{W}_E$ be the space of combinatorial conformal factors w such that each vector l corresponding to a vector $w \in \mathcal{W}_E$ is indeed a piecewise flat metric. In other words, the triangle inequality holds for each triangle under the assignment l. Obviously, $\mathcal{W}_E$ depends on the initial metric l^0.

For a vector $w \in \mathcal{W}_E$, the vector l corresponding to w is a piecewise flat metric. Each triangle of T is realized as a Euclidean triangle under the metric l. At a vertex i, the curvature K_i of the metric l is defined as 2π minus the sum of all inner angles at the vertex i.

Under the rescaling, the curvature at each vertex is invariant. We only need to consider $\mathcal{W}_E \cap P$ where P is the hyperplane defined by $\sum_{i=1}^n w_i = 0$. By Gauss–Bonnet theorem, the possible curvature $(K_1, K_2, \ldots, K_n)$ is contained in the hyperplane Q which is defined by $\sum_{i=1}^n K_i = 2\pi\chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic of the surface Σ.

This produce a map

$$\begin{aligned} \psi_E : \mathcal{W}_E \cap P \quad &\to Q, \\ (w_1, w_2, \ldots, w_n) &\mapsto (K_1, K_2, \ldots, K_n) \end{aligned}$$

sending a combinatorial conformal factor to the curvature.

Theorem 1 (Luo). *The map ψ_E is a local diffeomorphism.*

The theorem is proved by applying a variational principle. A local convex energy function is constructed using the derivative cosine law, and ψ_E turns out to be the gradient of the energy function.

Motivated by establishing a discrete Uniformization Theorem, Luo introduced the combinatorial Yamabe flow

$$\begin{cases} \dfrac{dw_i(t)}{dt} &= -K_i(t), \\ w_i(0) &= 0. \end{cases} \tag{2}$$

Corollary 2 (Luo). *The combinatorial Yamabe flow (2) is the negative gradient flow of a locally convex function in terms of w, and $\sum_{i=1}^n K_i^2(t)$ is decreasing in time t.*

1.2. *Related work*

Motivated by the application in computer graphics, Springborn, Schröder and Pinkall [22] considered this combinatorial conformal change of piecewise flat metrics (1). They found an explicit formula of the energy function. Glickenstein

[8, 9] studied the combinatorial Yamabe flow on three-dimensional piecewise flat manifolds relating to the ball packing metric of Cooper and Rivin [7]. Recently Glickenstein [10] set the theory of combinatorial Yamabe flow of piecewise flat metric in a broader context including the theory of circle packing on surfaces. This combinatorial conformal change of metrics has appeared in physic literature [20] and numerical analysis literature [14, 18]. We were informed by Luo in 2009 that Springborn considered the combinatorial conformal change of hyperbolic metric on a triangulated closed surface. He introduced the combinatorial conformal change as

$$\sinh \frac{l_\alpha}{2} = e^{w_i+w_j} \sinh \frac{l^0_\alpha}{2}, \tag{3}$$

where i and j are vertices incident to the edge α. See [1, Formula (5.1)].

1.3. *Hyperbolic metrics*

In this paper we study the combinatorial Yamabe flow on hyperbolic surfaces with geodesic boundary. Let Σ be a connected compact surface with n boundary components. The set of boundary components is $B = \{1, 2, \dots, n\}$ where a boundary component is identified with an index.

A colored hexagon is a hexagon with three non-pairwise adjacent edges labeled by red and the opposite edges labeled by black. Take a finite disjoint union of colored hexagons and identify all red edges in pairs by homeomorphisms. The quotient is a compact surface with non-empty boundary together with an *ideal triangulation*. The faces in the ideal triangulation are quotients of the hexagons. The quotients of red edges are called the edges of the ideal triangulation while the quotients of black edges are called the boundary arcs.

It is well known that each connected compact surface Σ of non-empty boundary and negative Euler characteristic admits an ideal triangulation.

Let T be an ideal triangulation of Σ. Denote by E the set of edges of T, by F the set of faces of T. We will use a Greek letter to denote an edge. A hyperbolic metric on (Σ, T) is identified with a vector indexed by the set of edges E. More precisely, it is an assignment to each edge a positive number. It is well known that [4], for any three positive numbers, there exists a hyperbolic right-angled hexagon the length of whose three non-pairwise adjacent edges are the three numbers. Furthermore, the hexagon is unique up to isometry. Therefore, for a vector in $\mathbb{R}^{|E|}_{>0}$, each face of F can be realized as a unique hyperbolic right-angled hexagon and the surface Σ can be realized as a hyperbolic surface with geodesic boundary.

Let us fix a hyperbolic metric on (Σ, T) as $l^0 \in \mathbb{R}^{|E|}_{>0}$. The assignment to the edge α is denoted by l^0_α. A *combinatorial conformal factor* on (Σ, T) is a vector $w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$ which assigns each boundary component $i \in B$ a number w_i. From a combinatorial conformal factor, we obtained a new assignment $l \in \mathbb{R}^{|E|}$ as follows:

$$\cosh \frac{l_\alpha}{2} = e^{w_i+w_j} \cosh \frac{l^0_\alpha}{2} \tag{4}$$

for any edge $\alpha \in E$ where i and j are boundary components incident to the edge α. Note that i and j can be the same boundary component. This definition (4) is an analog of Springborn's definition of combinatorial conformal change of hyperbolic metrics on triangulated closed surfaces (3).

Denote by $\mathcal{W}$ the set of combinatorial conformal factors such that the corresponding assignment is positive, i.e. $l \in \mathbb{R}^{|E|}_{>0}$. Therefore, for $w \in \mathcal{W}$, we obtained a new hyperbolic metric on (Σ, T). Each face of T is a hyperbolic right-angled hexagon. The surface Σ is realized as a hyperbolic surface with geodesic boundary. Denote by B_i the length of the boundary component i.

This produces a map

$$\psi : \mathcal{W} \to \mathbb{R}^n,$$
$$(w_1, w_2, \ldots, w_n) \mapsto (B_1, B_2, \ldots, B_n)$$

sending a combinatorial conformal factor to the length of boundary components.

Theorem 3. *The map ψ is a diffeomorphism.*

This is a result of global rigidity while Theorem 1 is a result of local rigidity.

We also consider the combinatorial Yamabe flow in this situation

$$\begin{cases} \dfrac{dw_i(t)}{dt} = B_i(t), \\ w_i(0) = 0. \end{cases} \tag{5}$$

Corollary 4. *The combinatorial Yamabe flow* (5) *is the gradient flow of a concave function in terms of w, and $\sum_{i=1}^n B_i^2(t)$ is decreasing in time t.*

We investigate the long-time behavior of the flow.

Theorem 5. *The combinatorial Yamabe flow* (5) *has a solution for $t \in [0, \infty)$. Along the flow* (5), *any initial hyperbolic surface with geodesic boundary converges to a complete hyperbolic surface with cusps.*

1.4. *Variational principle*

The approach of variational principle of studying polyhedral surfaces was introduced by Colin de Verdiére [6] in his proof of Andreev–Thurston's circle packing theorem. Since then, many works about variational principles on polyhedral surfaces have appeared. For example, see [2, 3, 5, 11–13, 15, 17, 19, 21] and others.

1.5. *Organization of the paper*

Theorem 3 is proved in Sec. 2. Corollary 4 and Theorem 5 are proved in Sec. 3.

2. Diffeomorphism

2.1. *Hexagon*

Let $f \in F$ be a face of the ideal triangulation T of Σ which is a quotient of a hexagon H. There are four cases.

Case 1. The three red edges of H are identified as three different edges of T: α, β, γ. The three black edges of H become boundary arcs contained in three different boundary components: i, j, k. We say that f is of type of (3,3).

Case 2. The three red edges of H are identified as three different edges of T: α, β, γ. The three black edges of H become boundary arcs contained in two different boundary components: i, j. We say that f is of type of (3,2).

Case 3. The three red edges of H are identified as three different edges of T: α, β, γ. The three black edges of H become boundary arcs contained in one boundary component: i. We say that f is of type of (3,1).

Case 4. The three red edges of H are identified as two different edges of T: α, β. The three black edges of H become boundary arcs contained in two different boundary components: i, j. We say that f is of type of (2,2).

Once we deal with a face of T, we need to discuss these four cases (Fig. 1).

2.2. *Space of combinatorial conformal factors*

Let $l^0 \in \mathbb{R}^{|E|}_{>0}$ be a fixed hyperbolic metric on (Σ, T). We investigate the space of combinatorial conformal factors such that the corresponding new assignment is a hyperbolic metric. For a face $f \in T$, we denote by $\mathcal{W}^f$ the space of conformal factors incident to f such that the corresponding new assignment is a hyperbolic metric on f. The following is more precise description.

For a face $f \in F$ of type (3,3), $\mathcal{W}^f$ is the space of vectors (w_i, w_j, w_k) such that l_α, l_β and l_γ are positive.

For a face $f \in F$ of type (3,2), $\mathcal{W}^f$ is the space of vectors (w_i, w_j) such that l_α, l_β and l_γ are positive.

For a face $f \in F$ of type (3,1), $\mathcal{W}^f$ is the space of vectors (w_i) such that l_α, l_β and l_γ are positive.

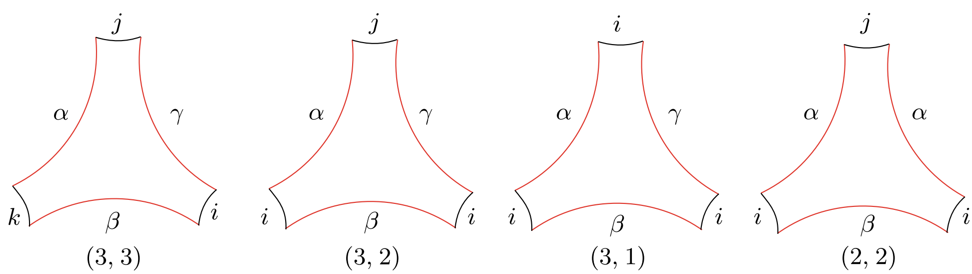

Fig. 1. (Color online)

For a face $f \in F$ of type (2,2), $\mathcal{W}^f$ is the space of vectors (w_i, w_j) such that l_α, l_β are positive.

Lemma 6. *$\mathcal{W}^f$ is a convex polytope.*

Proof. If f is of type (3,3), by definition (4) $\cosh \frac{l_\gamma}{2} = e^{w_i + w_j} \cosh \frac{l_\gamma^0}{2}$. The only requirement is $l_\gamma > 0$. Hence

$$w_i + w_j > -\ln \cosh \frac{l_\gamma^0}{2}.$$

Similar inequalities hold for $w_j + w_k$ and $w_k + w_i$. Therefore $\mathcal{W}^f$ is the intersection of three half space.

The similar argument holds for the three other types of a face f. □

Corollary 7. *The space $\mathcal{W}$ is a convex polytope.*

Proof. $\mathcal{W} = \cap_{f \in F} \mathcal{W}^f$. □

2.3. *Energy function*

Let us focus on one face $f \in F$. We will construct an energy function $\mathcal{E}_f$ for each face f. We need to discuss the four cases.

Case 1. If f is of type (3,3), when $(w_i, w_j, w_k) \in \mathcal{W}^f$, from (4), we obtain a hyperbolic right-angled hexagon whose non-pairwise adjacent edges have length l_α, l_β and l_γ. Denote by $\theta_f^\alpha, \theta_f^\beta$ and θ_f^γ the length of the boundary arcs of f opposite to the edges α, β and γ, respectively. Note that the boundary arc of length θ_f^α (or θ_f^β or θ_f^γ) is an arc in the boundary component i (or j or k). Now $\theta_f^\alpha, \theta_f^\beta$ and θ_f^γ are functions of w_i, w_j, w_k.

Lemma 8. *The Jacobian matrix of functions $\theta_f^\alpha, \theta_f^\beta, \theta_f^\gamma$ in terms of w_i, w_j, w_k is symmetric.*

Proof. The cosine law for hyperbolic right-angled hexagon induces the derivative cosine law:

$$\begin{pmatrix} d\theta_f^\alpha \\ d\theta_f^\beta \\ d\theta_f^\gamma \end{pmatrix} = \frac{-1}{\sinh \theta_f^\gamma \sinh l_\beta \sinh l_\alpha} \begin{pmatrix} \sinh l_\alpha & 0 & 0 \\ 0 & \sinh l_\beta & 0 \\ 0 & 0 & \sinh l_\gamma \end{pmatrix}$$

$$\times \begin{pmatrix} -1 & \cosh \theta_f^\gamma & \cosh \theta_f^\beta \\ \cosh \theta_f^\gamma & -1 & \cosh \theta_f^\alpha \\ \cosh \theta_f^\beta & \cosh \theta_f^\alpha & -1 \end{pmatrix} \begin{pmatrix} dl_\alpha \\ dl_\beta \\ dl_\gamma \end{pmatrix}.$$

By differentiating the two sides of Eq. (4), $\cosh \frac{l_\gamma}{2} = e^{w_i+w_j} \cosh \frac{l_\gamma^0}{2}$, we obtain

$$dl_\gamma = \frac{2\sinh l_\gamma}{\cosh l_\gamma - 1}(dw_i + dw_j).$$

Similar formulas hold for dl_α and dl_β. Then we have

$$\begin{pmatrix} d\theta_f^\alpha \\ d\theta_f^\beta \\ d\theta_f^\gamma \end{pmatrix} = \frac{-2}{\sinh\theta_f^\gamma \sinh l_\beta \sinh l_\alpha} \begin{pmatrix} \sinh l_\alpha & 0 & 0 \\ 0 & \sinh l_\beta & 0 \\ 0 & 0 & \sinh l_\gamma \end{pmatrix}$$

$$\times \begin{pmatrix} -1 & \cosh\theta_f^\gamma & \cosh\theta_f^\beta \\ \cosh\theta_f^\gamma & -1 & \cosh\theta_f^\alpha \\ \cosh\theta_f^\beta & \cosh\theta_f^\alpha & -1 \end{pmatrix} \begin{pmatrix} \frac{\sinh l_\alpha}{\cosh l_\alpha - 1} & 0 & 0 \\ 0 & \frac{\sinh l_\beta}{\cosh l_\beta - 1} & 0 \\ 0 & 0 & \frac{\sinh l_\gamma}{\cosh l_\gamma - 1} \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} dw_i \\ dw_j \\ dw_k \end{pmatrix}.$$

For simplicity of the notations, the above formula is written as

$$\begin{pmatrix} d\theta_f^\alpha \\ d\theta_f^\beta \\ d\theta_f^\gamma \end{pmatrix} = \frac{-2}{\sinh\theta_f^\gamma \sinh l_\beta \sinh l_\alpha} M \begin{pmatrix} dw_i \\ dw_j \\ dw_k \end{pmatrix}, \tag{6}$$

where M is a product of four matrixes. To prove the lemma, it is enough to show that the matrix M is symmetric.

Represent $\cosh\theta_f^\alpha, \cosh\theta_f^\beta, \cosh\theta_f^\gamma$ as functions of $\cosh l_\alpha, \cosh l_\beta, \cosh l_\gamma$ using the cosine law. For simplicity of the notations, let $a := \cosh l_\alpha, b := \cosh l_\beta, c := \cosh l_\gamma$, Then we have

$$M = \begin{pmatrix} \frac{c+ab}{b-1} + \frac{b+ac}{c-1} & \frac{a+b-c+1}{c-1} & \frac{a+c-b+1}{b-1} \\ \frac{a+b-c+1}{c-1} & \frac{c+ab}{a-1} + \frac{a+bc}{c-1} & \frac{b+c-a+1}{a-1} \\ \frac{a+c-b+1}{b-1} & \frac{b+c-a+1}{a-1} & \frac{b+ac}{a-1} + \frac{a+bc}{b-1} \end{pmatrix}. \tag{7}$$

□

Lemma 9. *The Jacobian matrix of functions $\theta_f^\alpha, \theta_f^\beta, \theta_f^\gamma$ in terms of w_i, w_j, w_k is negative definite.*

Proof. We only need to show that the matrix M is positive definite. Let M_{rs} be the entry of M at rth row and sth column. First, $M_{11} > 0$. Second,

$$M_{11} - M_{12} = \frac{c+ab}{b-1} + a + 1 > 0,$$

$$M_{11} + M_{12} = \frac{c+ab}{b-1} + \frac{2b+a+1+ac-c}{c-1} > 0,$$

$$M_{22} - M_{21} = \frac{c+ab}{a-1} + b + 1 > 0,$$

$$M_{22} + M_{21} = \frac{c+ab}{a-1} + \frac{2a+b+1+bc-c}{c-1} > 0.$$

Then $M_{11}M_{22} > |M_{12}||M_{21}| \geq M_{12}M_{21}$. Third,

$$\det M = \sinh l_\alpha \sinh l_\beta \sinh l_\gamma \cdot (\sinh\theta_f^\alpha \sinh\theta_f^\beta \sinh l_\gamma)^2 \cdot \frac{\sinh l_\alpha}{\cosh l_\alpha - 1} \frac{\sinh l_\beta}{\cosh l_\beta - 1} \frac{\sinh l_\gamma}{\cosh l_\gamma - 1} \cdot 2 > 0.$$

□

Corollary 10. *The differential* 1*-form* $\theta_f^\alpha dw_i + \theta_f^\beta dw_j + \theta_f^\gamma dw_k$ *is closed on* $\mathcal{W}^f$*. For any* $c \in \mathcal{W}^f$, *the integral*

$$\mathcal{E}_f(w_i, w_j, w_k) = \int_c^{(w_i,w_j,w_k)} (\theta_f^\alpha dw_i + \theta_f^\beta dw_j + \theta_f^\gamma dw_k)$$

is a strictly concave function on $\mathcal{W}^f$ *satisfying*

$$\frac{\partial \mathcal{E}_f}{\partial w_i} = \theta_f^\alpha, \quad \frac{\partial \mathcal{E}_f}{\partial w_j} = \theta_f^\beta, \quad \frac{\partial \mathcal{E}_f}{\partial w_k} = \theta_f^\gamma.$$

Proof. The differential 1-form is closed due to Lemma 8. Since $\mathcal{W}^f$ is connected and simply connected due to Lemma 6, then the function $\mathcal{E}(w_i, w_j, w_k)$ is well defined, i.e. independent the path of integration. By Lemma 9, the Hessian matrix of $\mathcal{E}(w_i, w_j, w_k)$ is negative definite. □

Case 2. If f is of type (3,2), when $(w_i, w_j) \in \mathcal{W}^f$, from (4), we obtain a hyperbolic right-angled hexagon whose non-pairwise adjacent edges have length l_α, l_β and l_γ. Denote by $\theta_f^\alpha, \theta_f^\beta$ and θ_f^γ the length of the boundary arcs of f opposite to the edges α, β and γ, respectively. Note that the boundary arc of length θ_f^α and the boundary arc of length θ_f^γ are arcs the boundary component i. The boundary arc of length θ_f^β is an arc in the boundary component j. Now $\theta_f^\alpha, \theta_f^\beta$ and θ_f^γ are functions of w_i, w_j.

Corollary 11. *The differential* 1*-form* $(\theta_f^\alpha + \theta_f^\gamma)dw_i + \theta_f^\beta dw_j$ *is closed on* $\mathcal{W}^f$*. For any* $c \in \mathcal{W}^f$, *the integral*

$$\mathcal{E}_f(w_i, w_j) = \int_c^{(w_i,w_j)} ((\theta_f^\alpha + \theta_f^\gamma)dw_i + \theta_f^\beta dw_j)$$

is a strictly concave function on $\mathcal{W}^f$ *satisfying*

$$\frac{\partial \mathcal{E}_f}{\partial w_i} = \theta_f^\alpha + \theta_f^\gamma, \quad \frac{\partial \mathcal{E}_f}{\partial w_j} = \theta_f^\beta.$$

Proof. To prove this corollary, we need to verify that the Jacobian matrix of functions $\theta_f^\alpha + \theta_f^\gamma, \theta_f^\beta$ in terms of w_i, w_j is symmetric and negative definite. In fact, we can use the Jacobian matrix (6) in Case 1 with the condition $w_k = w_i$, i.e.

$$\begin{pmatrix} d\theta_f^\alpha \\ d\theta_f^\beta \\ d\theta_f^\gamma \end{pmatrix} = s \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} dw_i \\ dw_j \\ dw_i \end{pmatrix},$$

where $s = \frac{-2}{\sinh \theta_f^\gamma \sinh l_\beta \sinh l_\alpha} < 0$ and the symmetric matrix $M = (M_{ij})$ is given by (7). Then we have

$$\begin{pmatrix} d\theta_f^\alpha + d\theta_f^\gamma \\ d\theta_f^\beta \end{pmatrix} = s \begin{pmatrix} M_{11} + M_{13} + M_{31} + M_{33} & M_{12} + M_{32} \\ M_{21} + M_{23} & M_{22} \end{pmatrix} \begin{pmatrix} dw_i \\ dw_j \end{pmatrix},$$

Since $M = (M_{ij})$ is symmetric, $M_{12} + M_{32} = M_{21} + M_{23}$. Therefore the Jacobian matrix of functions $\theta_f^\alpha + \theta_f^\gamma, \theta_f^\beta$ in terms of w_i, w_j is symmetric.

First $M_{11} + M_{13} + M_{31} + M_{33} > 0$, since

$$M_{11} + M_{13} > \frac{c + ab}{b - 1} + \frac{a + c - b + 1}{b - 1} > \frac{ab - b}{b - 1} = a,$$

$$M_{31} + M_{33} > \frac{a + c - b + 1}{b - 1} + \frac{a + bc}{b - 1} > \frac{-b + bc}{b - 1} = c.$$

Second,

$$\begin{aligned}
&M_{11} + M_{13} + M_{31} + M_{33} \\
&\quad = \frac{c + ab}{b - 1} + \frac{b + ac}{c - 1} + \frac{a + c - b + 1}{b - 1} + \frac{a + c - b + 1}{b - 1} + \frac{b + ac}{a - 1} + \frac{a + bc}{b - 1} \\
&\quad = \frac{3c + 3a + 2 + ab - b + bc - b}{b - 1} + \frac{b + ac}{c - 1} + \frac{b + ac}{a - 1} \\
&\quad > \frac{b + ac}{c - 1} + \frac{b + ac}{a - 1} \\
&\quad > \left| \frac{a + b - c + 1}{c - 1} \right| + \left| \frac{b + c - a + 1}{a - 1} \right| \\
&\quad = |M_{21}| + |M_{23}|,
\end{aligned}$$

and

$$M_{22} = \frac{c + ab}{a - 1} + \frac{a + bc}{c - 1} > \left| \frac{a + b - c + 1}{c - 1} \right| + \left| \frac{b + c - a + 1}{a - 1} \right| = |M_{12}| + |M_{32}|.$$

Thus

$$\begin{aligned}(M_{11}+M_{13}+M_{31}+M_{33})M_{22} &> (|M_{21}|+|M_{23}|)(|M_{12}|+|M_{32}|)\\ &\geq (M_{21}+M_{23})(M_{12}+M_{32}).\end{aligned}$$

Hence the Jacobian matrix is negative definite. □

Case 3. If f is of type (3,1), when $w_i \in \mathcal{W}^f$, from (4), we obtain a hyperbolic right-angled hexagon whose non-pairwise adjacent edges have length l_α, l_β and l_γ. Denote by $\theta_f^\alpha, \theta_f^\beta$ and θ_f^γ the length of the boundary arcs of f opposite to the edges α, β and γ, respectively. Note that the boundary arcs of lengths θ_f^α, θ_f^β, θ_f^γ are arcs in the boundary component i. Now $\theta_f^\alpha, \theta_f^\beta$ and θ_f^γ are functions of w_i.

Corollary 12. *For any $c \in \mathcal{W}^f$, the integral*

$$\mathcal{E}_f(w_i) = \int_c^{w_i} (\theta_f^\alpha + \theta_f^\beta + \theta_f^\gamma) dw_i$$

is a strictly concave function on $\mathcal{W}^f$ satisfying

$$\frac{\partial \mathcal{E}_f}{\partial w_i} = \theta_f^\alpha + \theta_f^\beta + \theta_f^\gamma.$$

Proof. We have

$$\begin{pmatrix} d\theta_f^\alpha \\ d\theta_f^\beta \\ d\theta_f^\gamma \end{pmatrix} = s \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} dw_i \\ dw_i \\ dw_i \end{pmatrix}.$$

Thus $\frac{\partial^2 \mathcal{E}_f}{\partial w_i \partial w_i} = s(M_{11}+M_{12}+M_{13}+M_{21}+M_{22}+M_{23}+M_{31}+M_{32}+M_{33}) < 0$ since $M_{11}+M_{12}+M_{13} > 0, M_{21}+M_{22}+M_{23} > 0, M_{31}+M_{32}+M_{33} > 0$. □

Case 4. If f is of type (2,2), when $(w_i, w_j) \in \mathcal{W}^f$, from (4), we obtain a hyperbolic right-angled hexagon whose non-pairwise adjacent edges have length l_α, l_α and l_β. By the sine law of a hyperbolic right-angled hexagon, the two boundary arcs opposite to the edge α have the same length which is denoted by θ_f^α. Denote by θ_f^β the length of the boundary arcs of f opposite to the edges β. Note that the two boundary arcs of length θ_f^α are arcs in the boundary component i and the boundary arc of length θ_f^β is an arc in the boundary component j. Now $\theta_f^\alpha, \theta_f^\beta$ are functions of w_i, w_j.

Corollary 13. *The differential 1-form $2\theta_f^\alpha dw_i + \theta_f^\beta dw_j$ is closed on $\mathcal{W}^f$. For any $c \in \mathcal{W}^f$, the integral*

$$\mathcal{E}_f(w_i, w_j) = \int_c^{(w_i,w_j)} (2\theta_f^\alpha dw_i + \theta_f^\beta dw_j)$$

is a strictly concave function on $\mathcal{W}^f$ satisfying

$$\frac{\partial \mathcal{E}_f}{\partial w_i} = 2\theta_f^\alpha, \quad \frac{\partial \mathcal{E}_f}{\partial w_j} = \theta_f^\beta.$$

Proof. The proof is similar to the proof of Case 2. We only need to add the condition $l_\alpha = l_\gamma$ and $\theta_f^\alpha = \theta_f^\gamma$. □

2.4. *Diffeomorphism*

In this subsection we prove Theorem 3. The following two lemmas are needed.

The first one is well known in analysis.

Lemma 14. *Suppose X is an open convex set in $\mathbb{R}^N$ and $f : X \to \mathbb{R}$ is a smooth function. If the Hessian matrix of f is positive definite for all $x \in X$, then the gradient $\nabla f : X \to \mathbb{R}^N$ is a smooth embedding.*

Lemma 15. *For a family of combinatorial conformal factor $w^{(m)} \in \mathcal{W}$, if $\lim_{m\to\infty} w_t^{(m)} = \infty$ for some $t \in \{1, 2, \ldots, n\}$, then $\lim_{m\to\infty} B_t^{(m)} = 0$ and the convergence is independent of the values of $\lim_{m\to\infty} w_i^{(m)}$ for $i \neq t$.*

Proof. Let f be a face with a boundary arc in the boundary component t. Assume the other two boundary arcs of f are in the boundary components r and s, respectively. Note that we do not assume that r, s, t are different due to the four possible types of f. Denote the edges of f be λ, μ, ν which is opposite to the boundary components r, s, t, respectively. Note that two of the edges may be the same. By Definition (4),

$$\begin{aligned}
\cosh l_\lambda^{(m)} &= e^{2w_s^{(m)}+2w_t^{(m)}} c_1 - 1,\\
\cosh l_\mu^{(m)} &= e^{2w_t^{(m)}+2w_r^{(m)}} c_2 - 1,\\
\cosh l_\nu^{(m)} &= e^{2w_r^{(m)}+2w_s^{(m)}} c_3 - 1,
\end{aligned}$$

where $c_1 = 2\cosh^2 \frac{l_\lambda^0}{2}, c_2 = 2\cosh^2 \frac{l_\mu^0}{2}, c_3 = 2\cosh^2 \frac{l_\nu^0}{2}$. Let θ_f^ν be the length of the boundary arc of f contained in the boundary component t. Then

$$\begin{aligned}
&\lim_{m\to\infty} \cosh(\theta_f^\nu)^{(m)}\\
&\quad= \lim_{m\to\infty} \frac{\cosh l_\nu^{(m)} + \cosh l_\lambda^{(m)} \cosh l_\mu^{(m)}}{\sinh l_\lambda^{(m)} \sinh l_\mu^{(m)}}\\
&\quad= \lim_{m\to\infty} \frac{e^{2w_r^{(m)}+2w_s^{(m)}} c_3 - 1 + (e^{2w_s^{(m)}+2w_t^{(m)}} c_1 - 1)(e^{2w_t^{(m)}+2w_r^{(m)}} c_2 - 1)}{e^{w_s^{(m)}+w_t^{(m)}} \sqrt{e^{2w_s^{(m)}+2w_t^{(m)}} c_1^2 - 2c_1} \cdot e^{w_t^{(m)}+w_r^{(m)}} \sqrt{e^{2w_t^{(m)}+2w_r^{(m)}} c_2^2 - 2c_2}}
\end{aligned}$$

$$= \lim_{m\to\infty} \frac{c_3 - e^{-2w_r^{(m)}-2w_s^{(m)}}}{e^{4w_t^{(m)}}\sqrt{(c_1^2 - 2c_1e^{-2w_s^{(m)}-2w_t^{(m)}})(c_2^2 - 2c_2e^{-2w_r^{(m)}-2w_t^{(m)}})}}$$

$$+ \lim_{m\to\infty} \frac{(c_1 - e^{-2w_s^{(m)}-2w_t^{(m)}})(c_2 - e^{-2w_r^{(m)}-2w_t^{(m)}})}{\sqrt{(c_1^2 - 2c_1e^{-2w_s^{(m)}-2w_t^{(m)}})(c_2^2 - 2c_2e^{-2w_r^{(m)}-2w_t^{(m)}})}}$$

$$= 0 + 1.$$

Hence $\lim_{m\to\infty}(\theta_f^\nu)^{(m)} = 0$ independent the values of $\lim_{m\to\infty} w_i^{(m)}$ for $i \neq t$. Thus $\lim_{m\to\infty} B_t^{(m)} = \lim_{m\to\infty} \sum_{f\in F}(\theta_f^\nu)^{(m)} = 0$. □

Proof of Theorem 3. Let $l^0 \in \mathbb{R}_{>0}^{|E|}$ be a fixed hyperbolic metric on (Σ, T). For any combinatorial conformal factor $w \in \mathcal{W}$, we obtain a new hyperbolic metric $l \in \mathbb{R}_{>0}^{|E|}$. By Corollaries 10–13, for each face $f \in F$, there is a function $\mathcal{E}_f$. Define a function $\overline{\mathcal{E}} : \mathcal{W} \to \mathbb{R}$ by

$$\overline{\mathcal{E}}(w_1, w_2, \ldots, w_n) = \sum_{f\in F} \mathcal{E}_f,$$

where the sum is over all faces in F. By Corollaryies 10–13, $\overline{\mathcal{E}}$ is strictly concave on $\mathcal{W}$ and

$$\frac{\partial\overline{\mathcal{E}}}{\partial w_i} = \sum_{f\in F} \frac{\partial\mathcal{E}_f}{\partial w_i} = B_i. \tag{8}$$

The last equality is obtained as follows. For a face $f \in F$, if none of the boundary arcs of f is in the boundary component i, then $\frac{\partial\mathcal{E}_f}{\partial w_i} = 0$. Otherwise, $\frac{\partial\mathcal{E}_f}{\partial w_i}$ is equal to the sum of the lengths of boundary arcs of f in the boundary component i. When f runs over all faces, we get the sum of the lengths of all boundary arcs in the boundary component i. The sum is exactly B_i the length of the boundary component i.

That means the gradient of $\overline{\mathcal{E}}$ is exactly the map ψ sending a combinatorial conformal factor w to the corresponding length of boundary components. Thus ψ is a smooth embedding due to Lemma 14.

To show that ψ is a diffeomorphism, we will prove that $\psi(\mathcal{W})$ is both open and closed in $\mathbb{R}_{>0}^n$.

Since ψ is a smooth embedding, $\psi(\mathcal{W})$ is open in $\mathbb{R}_{>0}^n$.

To show that $\psi(\mathcal{W})$ is closed in $\mathbb{R}_{>0}^n$, take a sequence of combinatorial conformal factor $w^{(m)}$ in $\mathcal{W}$ such that $\lim_{m\to\infty}(B_1^{(m)}, B_2^{(m)}, \ldots, B_n^{(m)}) \in \mathbb{R}_{>0}^n$. To prove the closeness, it is sufficient to show that there is a subsequence of $w^{(m)}$ whose limit is in $\mathcal{W}$.

Suppose otherwise, there is a subsequence, still denoted by $w^{(m)}$, so that its limit is on the boundary of $\mathcal{W}$. The first possibility is that there is some $t \in \{1, 2, \ldots, n\}$ such that $\lim_{m\to\infty} w_t^{(m)} = \infty$. By Lemma 15, $\lim_{m\to\infty} B_t^{(m)} = 0$. This contradicts the assumption that $\lim_{m\to\infty}(B_1^{(m)}, B_2^{(m)}, \ldots, B_n^{(m)}) \in \mathbb{R}_{>0}^n$.

The second possibility is that, for a face f of any type with edges λ, μ, ν and opposite boundary components r, s, t, we have $\lim_{m\to\infty}(w_r^{(m)} + w_s^{(m)}) = -\ln\cosh\frac{l_\nu^0}{2}$. That means $\lim_{m\to\infty} l_\nu(m) = 0$. Denote by θ_f^λ the length of the boundary arc of f in the boundary component of r. We have

$$\begin{aligned}\lim_{m\to\infty}(\theta_f^\lambda)^{(m)} &= \lim_{m\to\infty}\frac{\cosh l_\lambda^{(m)} + \cosh l_\mu^{(m)}\cosh l_\nu^{(m)}}{\sinh l_\mu^{(m)}\sinh l_\nu^{(m)}}\\ &\geq \lim_{m\to\infty}\frac{\cosh l_\mu^{(m)}\cosh l_\nu^{(m)}}{\sinh l_\mu^{(m)}\sinh l_\nu^{(m)}}\\ &\geq \lim_{m\to\infty}\frac{\cosh l_\nu^{(m)}}{\sinh l_\nu^{(m)}} = \infty.\end{aligned}$$

Therefore $\lim_{m\to\infty}(\theta_f^\lambda)^{(m)} = \infty$ and $\lim_{m\to\infty} B_r^{(m)} = \infty$. This contradicts the assumption that $\lim_{m\to\infty}(B_1^{(m)}, B_2^{(m)}, \ldots, B_n^{(m)}) \in \mathbb{R}^n_{>0}$. □

3. Flow

Proof of Corollary 4. The combinatorial Yamabe flow (5) is the gradient flow of the concave function $\overline{\mathcal{E}}(w_1, w_2, \ldots, w_n)$ due to Eq. (8).

For a face f and a boundary component i, denote by η_f^i be the sum of lengths of boundary arcs of f in the boundary component i.

Since

$$\frac{dB_i(t)}{dt} = \sum_{f\in F}\frac{d\eta_f^i(t)}{dt} = \sum_{f\in F}\sum_{j=1}^n \frac{d\eta_f^i}{dw_j}\frac{dw_j}{dt} = \sum_{f\in F}\sum_{j=1}^n\frac{d\eta_f^i}{dw_j}B_j(t),$$

we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{i=1}^n B_i^2(t) &= \sum_{i=1}^n B_i(t)\frac{dB_i(t)}{dt}\\ &= \sum_{i=1}^n B_i\left(\sum_{f\in F}\sum_{j=1}^n\frac{d\eta_f^i}{dw_j}B_j\right)\\ &= \sum_{f\in F}\sum_{i=1}^n\sum_{j=1}^n\frac{d\eta_f^i}{dw_j}B_iB_j.\end{aligned}$$

If f is of type (3,3) as in Fig. 1, then

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{d\eta_f^i}{dw_j}B_iB_j &= \frac{d\theta_f^\alpha}{dw_i}B_i^2+\frac{d\theta_f^\beta}{dw_j}B_j^2+\frac{d\theta_f^\gamma}{dw_k}B_k^2+\left(\frac{d\theta_f^\gamma}{dw_j}+\frac{d\theta_f^\beta}{dw_k}\right)B_jB_k\\ &\quad+\left(\frac{d\theta_f^\alpha}{dw_k}+\frac{d\theta_f^\gamma}{dw_i}\right)B_kB_i+\left(\frac{d\theta_f^\beta}{dw_i}+\frac{d\theta_f^\alpha}{dw_j}\right)B_iB_j\\ &= (B_i,B_j,B_k)\frac{\partial(\theta_f^\alpha,\theta_f^\beta,\theta_f^\gamma)}{\partial(w_i,w_j,w_k)}(B_i,B_j,B_k)^{\mathrm{T}}<0,\end{aligned}$$

Since by Lemma 9, the Jacobian matrix $\frac{\partial(\theta_f^\alpha,\theta_f^\beta,\theta_f^\gamma)}{\partial(w_i,w_j,w_k)}$ is negative definite.

If f is of type (3,2), (3,1) or (2,2), then by the proof of Corollary 11, Corollary 12 and Corollary 13, we have $\sum_{i=1}^n\sum_{j=1}^n\frac{d\eta_f^i}{dw_j}B_iB_j<0$.

Hence $\frac{1}{2}\frac{d}{dt}\sum_{i=1}^n B_i^2(t)<0$. Therefore $\sum_{i=1}^n B_i^2(t)$ is decreasing in t. □

Proof of Theorem 5. Since $B_i(t)>0$, then $w_i(t)>w_i(0)=0$.

For any $L<\infty$, we claim that $\lim_{t\to L}w_i(t)<\infty$.

Otherwise, if $\lim_{t\to L}w_i(t)=\infty$ for some $L<\infty$, then by Lemma 15, we see $\lim_{t\to L}B_i(t)=0$. Therefore, for any $\epsilon>0$, there exists some $\delta>0$ such that when $t\in(L-\epsilon,L)$, the inequalities $0<B_i(t)<\epsilon$ holds. Hence, by the flow (5), $0<\frac{dw_i(t)}{dt}<\epsilon$ holds. Thus $w_i(0)<w_i(t)<\epsilon t<\epsilon L$. This contradicts to the assumption that $\lim_{t\to L}w_i(t)=\infty$.

Hence the solution of the flow (5) exists for all time $t\in[0,\infty)$.

To obtain the geometric picture, we claim that $\lim_{t\to\infty}B_i(t)=0$ for each $1\le i\le n$. There are two cases to consider.

First, if $\lim_{t\to\infty}w_i(t)=\infty$, by Lemma 15, $\lim_{t\to\infty}B_i(t)=0$ holds.

Second, if $\lim_{t\to\infty}w_i(t)<\infty$, we claim $\lim_{t\to\infty}B_i(t)=0$ still holds. Otherwise, $\lim_{t\to\infty}B_i(t)=a>0$. Then, for any $\epsilon\in(0,a)$, there exists some $P>0$ such that when $t>P$, the inequality $B_i(t)>a-\epsilon$ holds. Hence, by the flow (5), $\frac{dw_i}{dt}>a-\epsilon$ holds. Therefore $w_i(t)>(a-\epsilon)t$. This contradicts to the assumption $\lim_{t\to\infty}w_i(t)<\infty$.

Let f be a face of any type with edges λ,μ,ν whose opposite boundary components are r,s,i. Then

$$\begin{aligned}\lim_{t\to\infty}l_\lambda(t) &= \lim_{t\to\infty}\frac{\cosh\theta_f^\lambda(t)+\cosh\theta_f^\mu(t)\cosh\theta_f^\nu(t)}{\sinh\theta_f^\mu(t)\sinh\theta_f^\nu(t)}\\ &\ge \lim_{t\to\infty}\frac{\cosh\theta_f^\mu(t)\cosh\theta_f^\nu(t)}{\sinh\theta_f^\mu(t)\sinh\theta_f^\nu(t)}\\ &\ge \lim_{t\to\infty}\frac{\cosh\theta_f^\nu(t)}{\sinh\theta_f^\nu(t)}=\infty,\end{aligned}$$

since $\lim_{t\to\infty}B_i(t)=0$ and, in consequence, $\lim_{t\to\infty}\theta_f^\nu(t)=0$.

Therefore each hyperbolic right-angled hexagon converges to a hyperbolic ideal triangle. □

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References

[1] A. I. Bobenko, U. Pinkall and B. A. Springborn, Discrete conformal maps and ideal hyperbolic polyhedra, preprint (2010); arXiv:1005.2698.

[2] A. I. Bobenko and B. A. Springborn, Variational principles for circle patterns and Koebe's theorem, *Trans. Amer. Math. Soc.* **356**(2) (2004) 659–689.

[3] W. Brägger, Kreispackungen und Triangulierungen, *Enseign. Math.* (2) **38**(3–4) (1992) 201–217 (in German).

[4] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces* (Birkhäuser Boston, Boston, MA, 1992).

[5] B. Chow and F. Luo, Combinatorial Ricci flows on surfaces, *J. Differential Geom.* **63** (2003) 97–129.

[6] Y. Colin de Verdière, Un principe variationnel pour les empilements de cercles, *Invent. Math.* **104**(3) (1991) 655–669 (in French).

[7] D. Cooper and I. Rivin, Combinatorial scalar curvature and rigidity of ball packings, *Math. Res. Lett.* **3**(1) (1996) 51–60.

[8] D. Glickenstein, A combinatorial Yamabe flow in three dimensions, *Topology* **44**(4) (2005) 791–808.

[9] ———, A maximum principle for combinatorial Yamabe flow, *Topology* **44**(4) (2005) 809–825.

[10] ———, Discrete conformal variations and scalar curvature on piecewise flat two and three dimensional manifolds, *J. Differential Geom.* **87**(2) (2011) 201–238.

[11] R. Guo, A note on circle patterns on surfaces, *Geom. Dedicata* **125** (2007) 175–190.

[12] ———, Local rigidity of inversive distance circle packing, *Trans. Amer. Math. Soc.* **363** (2011) 4757–4776.

[13] R. Guo and F. Luo, Rigidity of polyhedral surfaces, II, *Geom. Topol.* **13** (2009) 1265–1312.

[14] T. Kerkhoven, Piecewise linear Petrov–Galerkin error estimates for the box method, *SIAM J. Numer. Anal.* **33**(5) (1996) 1864–1884.

[15] G. Leibon, Characterizing the Delaunay decompositions of compact hyperbolic surface, *Geom. Topol.* **6** (2002) 361–391

[16] F. Luo, Combinatorial Yamabe flow on surfaces, *Commun. Contemp. Math.* **6**(5) (2004) 765–780.

[17] ———, Rigidity of polyhedral surfaces, to appear in *J. Differential Geom.* (2006).

[18] M. Putti and C. Cordes, Finite element approximation of the diffusion operator on tetrahedra, *SIAM J. Sci. Comput.* **19**(4) (1998) 1154–1168.

[19] I. Rivin, Euclidean structures on simplicial surfaces and hyperbolic volume, *Ann. of Math. (2)* **139**(3) (1994) 553–580.

[20] M. Roček and R. M. Williams, The quantization of Regge calculus, *Z. Phys. C* **21**(4) (1984) 371–381.
[21] B. A. Springborn, A variational principle for weighted Delaunay triangulations and hyperideal polyhedra, *J. Differential Geom.* **78**(2) (2008) 333–367.
[22] B. Springborn, P. Schröder and U. Pinkall, Conformal equivalence of triangle meshes, *ACM Trans. Graph.* **27**(3) (2008) 1–11.