Surface panelization using periodic conformal maps

Abstract. *We present a new method to obtain periodic conformal parameterizations of surfaces with cylinder topology and describe applications to architectural design and rationalization of surfaces. The method is based on discrete conformal maps from the surface mesh to a cylinder or cone of revolution. It comes with a number of degrees of freedom on the boundary that can be used to obtain a variety of interesting panelizations. We illustrate different choices of parameters for* NURBS *surface designs. Further, we describe how our parameterization can be used to get a periodic boundary aligned hex-mesh on a doubly-curved surface. We optimize this initial mesh to consist of a limited number of planar regular hexagons that panel the surface.*

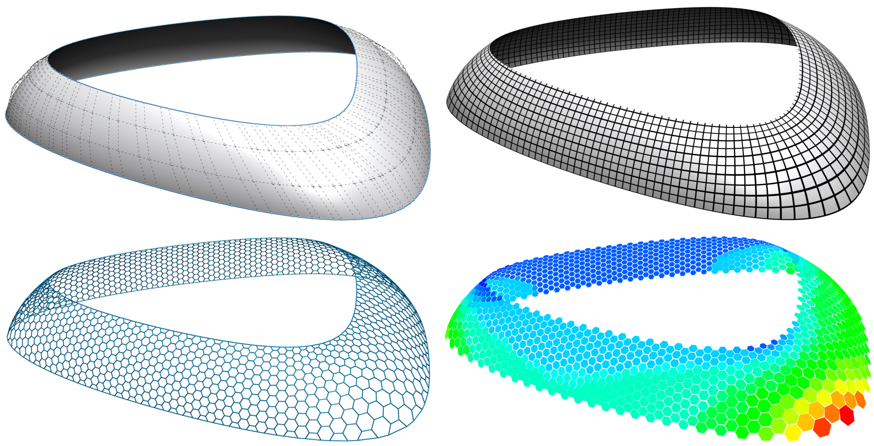


Figure 1: For a cylindrical NURBS surface (top-left) we create a seamless periodic conformal parameterization (top-right). A new mesh (bottom-left) is then rationalized and the panels are optimized for quantized regular hexagons (bottom-right).

1 Introduction

The panelization of surfaces remains a challenge in architectural design. CAD software such as Rhino, delivers powerful NURBS surface modelling to the designer. Their ease of use have made them a de facto standard for the design of freeform (and other) shapes in architectural design especially building envelopes and facades.

The scale of buildings introduces challenges to surface-based modelling strategies: The large scale of building elements demands that they are divided into smaller elements. The cost of material and labor, standardized production lines, green building concepts, availability and redundancy during construction periods demand for a high degree of similarity of these elements. Yet, the inherent UV subdivision of NURBS surfaces offers limited control over the layout, shape and configuration of the panels. While strategies for the controlled and careful creation of freeform surfaces have been presented (see [Glymph et al. 2004]) and realized, the tiling of true freeform surfaces through alternative algorithms is still a challenge.

The quality of a surface panelization solution can be defined in various ways. From an aesthetic point of view the shape of individual elements is important. Further there are global conditions such as alignment with surface boundaries and smooth transition of element shape. From the standpoint of fabrication, the elements should be repetitive. The contributions of this paper are:

- **Periodic discrete conformal parameterization**: We present an algorithm that maps a triangle mesh with cylinder topology conformally to a cylinder or a cone of revolution. This allows us to obtain seamless patterns on surfaces. It is a generalized version of the discrete conformal parameterization scheme described by [Springborn et al. 2008].
- **Regular elements**: We show how the periodic discrete parameterization can be used to construct a panelization of given periodic surface into a small number of repeating regular elements.
- **Applications to architectural surfaces**: We present a case study that initiated the development and where the described methods have successfully been applied in the architectural context.

The article is organized as follows: Section 2 describes various parameterization schemes available to architectural geometers. We briefly discuss the shortcomings of current methods for our task. Section 3 introduces the concept of periodic discrete conformal parameterizations. See Figure 2 for a schematic view of the proposed method. In Section 4 we present a case study that lead to the development of this work. Section 5 deals with the optimization of panels to meet requirements arising during the case study. We give implementational details in Section 6 and close with an outlook on further research.

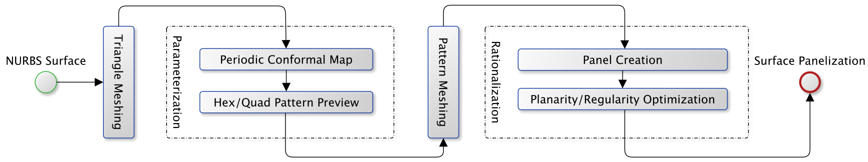


Figure 2: Flow diagram of the algorithms described in this paper. From a NURBS surface a triangle mesh is created. The parameterization part described in Section 3. A pattern-mesh is created using this parameterization. The creation and optimization of panels is described in Section 5.

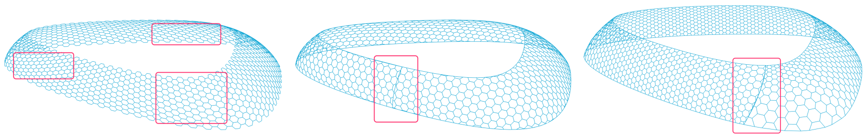


Figure 3: State of the art unroll methods can create patterns on a closed surface. The `ApplyCrv` command of Rhinoceros produces boundary alinged periodic patterns but introduces unacceptable non-isotropic stretch (left). The `SquishBack` method creates sufficiently regular elements but does not respect the periodicity of the surface (middle). Non-periodic conformal maps align with the boundary of a cut surface. Along the cut the map is not continuous (right).

2 Related Work

In this section we will review different methods used to unroll/parametrize surfaces that are accessible in the architectural design process. All available tools lack at least one of the key features of the proposed method:

- *Conformality* of the mapping avoids non-uniformly stretched panels
- *Periodicity* of the parameterization is needed to layout panels seamlessly on the surface

Rhinoceros CreateUVCrv/ApplyCrv. A NURBS surface is naturally equipped with a parameterization, i.e., a UV mapping from a rectangle domain to the surface. For the surface of Figure 1 such a map can be used to project a pattern from this rectangle to the surface (`ApplyCrv` in Rhinoceros). The pattern can be constructed periodically as it is defined in the UV domain of the surface. However, in general this method does not produce satisfactory results in terms of quality of elements for complex freeform surfaces. The UV parameterization is not conformal and thus introduces non-isotropic stretch and shear preventing the elements to be regular on the surface, see Figure 3, left. This limitation exists even for developable surfaces.

Rhinoceros Squish/SquishBack. The `Squish/SquishBack` command of Rhino maps a surface to the plane minimizing the amount of stretch. While this is geometrically not a conformal map it produces acceptable patterns on the surface. It is however not capable of calculating periodic maps to the surface. Thus it is not applicable in our situation, see Figure 3, middle.

PanelingTools for Rhino. The paneling tools of Rhinoceros use the UV parameterization of the underlying surface to populate grid-points over the surface. With the help of such a grid, panels are placed onto the surface, see [McNeel]. The shapes of the panels depend heavily on the NURBS-parameterization. An example is shown in Figure 3, left.

Hexagonal tilings. In the architectural context, hexagonal panelizations have been studied by [Zimmer et al. 2013, Troche 2008] and [Schiftner et al. 2009]. These aproaches however do not include regularity or special boundary alignment as introduced in the current work. In [Schiftner et al. 2009] the result of the panelization depends on the choice of an initial triangle mesh that is optimized towards touching incircles. This allows for a torsion free support structure of a non planar hex-mesh. Hexagonal tilings for triangulated surfaces also have been studied by [Nieser et al. 2012]. They focus on regularity but not on boundary conditions.

Mesh parameterizations. There are a vast number of parameterization schemes for meshes. To elaborate on all methods is beyond the scope of this section and we describe only the most relevant results here. General purpose parameterization methods for triangle meshes produce high quality quad or hex meshes for unstructured input data, see for instance [Bommes et al. 2009, Alexa et al. 2000, Springborn et al. 2008]. They have been used with success in the architectural context, e.g., by [Bo et al. 2011] and [Sechelmann et al. 2013].

The basis of our method are conformally equivalent triangle meshes as described by [Springborn et al. 2008]. The straight forward method to map a surface with this approach is to cut it open and map it to a rectangle domain. This method yields boundary aligned conformal maps that however do not match along the introduced cut, see Figure 3, right. How to generalize this method to overcome this limitation is the content of the following section.

3 Periodic conformal parametrization

In this section we describe our algorithm for the creation of periodic conformal maps for cylindrical meshes/surfaces, i.e., surfaces with the same topology as a cylinder. First we will review the discrete conformal maps of [Springborn et al. 2008]. Then we show how it can be generalized to yield periodic maps to cylinders or cones.

A smooth *conformal map* between two surfaces is a map that preserves angles. Intuitively, one can think of a conformal map as a map that preserves the shape but not the scale of small figures. For conformal surface parameterization, one looks for conformal maps from the plane to a surface and vice-versa. These can be used

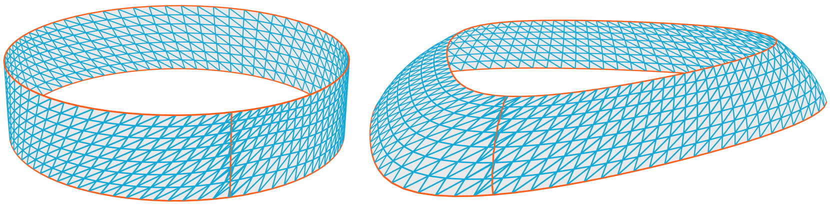


Figure 4: A discrete periodic map from a cylinder to a triangulated surface. On the cylinder all edges of the triangulation are geodesic arcs. If the cylinder is cut at the vertical orange path, then it can be unrolled to the plane creating a rectangular domain.

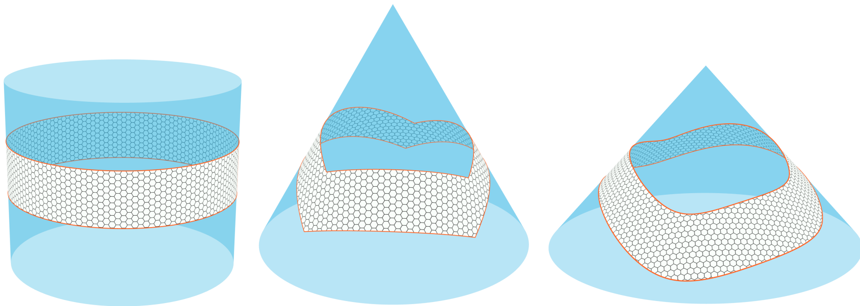


Figure 5: Periodic domains of parameterization of the surfaces shown in Figure 1. Left: Map to a cylinder with geodesic boundary curves. Middle: Map to a cone of revolution with hex-pattern-adapted angle. The domain is a polygon with quantized angles. Right: Isometric boundary on a cone with hex-pattern-adapted angle. Panelizations created with the help of these maps are shown in Figure 6.

to map different patterns onto surfaces in a way that only isotropic stretch/uniform scaling is applied to the pattern elements. The method described in [Springborn et al. 2008] is a triangle mesh based discretization of conformal maps. For a given triangle mesh and prescribed angle sums at the vertices, i.e., for each vertex v of the surface – interior or boundary – we may prescribe an angle θ_v that corresponds to the angle sum of adjacent triangles in the target mesh. Starting from an input mesh and target angles θ_v the method calculates new edge lengths for the triangles of the target mesh such that the angle sums at the target vertices are as prescribed. This goal is achieved by minimizing a convex functional. The prescribed angles have to satisfy a Gauss-Bonnet type condition, i.e., the angles at interior vertices have to match the angles at the boundary vertices. We will state the condition for the special cases treated later in the article, see Equation 1.

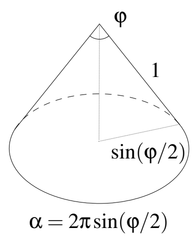
For the parameterization problem, we want to construct a map from a surface to the plane. To get a planar target mesh, the target angles have to be set to 2π for all interior vertices, i.e., the angles of the triangles adjacent to every interior vertex sum up to 2π. Thus the computed target triangles can be laid out in the plane. At the boundaries there is still a certain degree of freedom, which allows to map the surface to different shapes, e.g., a rectangle or a more general polygon with prescribed angles. An alternative choice of boundary conditions yields a target mesh whose boundary edges have the same lengths as the original mesh. Then the control over the boundary angles is no longer possible.

This method for the parameterization of triangle meshes can be generalized to triangle meshes with cylinder topology, see Figure 4. Instead of constructing a discrete conformal map from the surface to the plane, we construct a map to a cylinder or cone, whose image is isometric to a polygonal region in the plane, see Figure 5. This works with an approach very similar to the previous one. We start with the definition of a periodic parameterization.

Definition 1 *Let $M = (V,E,F)$ be a cylindrical triangle mesh with vertices V, edges E, and triangles F. Let $D \subset C$ be a region on a cone/cylinder of revolution. A continuous bijection $\Phi : D \to M$ is called a* discrete periodic parameterization. *D is called the* domain of parameterization.

In the latter we always assume that the preimages of edges of M are geodesic arcs on the cone/cylinder C. For panelization of periodic surfaces we need to make sure, that different patterns match around the cone or cylinder. This yields certain restrictions on the cone that serves as domain of parameterization.

Definition 2 *Let C be a cone with aperture φ and $\Phi : C \supset D \to M$ be a discrete periodic parameterization of a triangle mesh M with domain D. The map Φ is called* triangle adapted *if the cone angle α is a multiple of $\frac{\pi}{3}$ and* quad-pattern adapted *if it is a multiple of $\frac{\pi}{2}$.*



This definition ensures that either a quad-, triangle-, or hex-pattern fits seamlessly onto the surface after parameterization.

3.1 Periodic boundary conditions

If we want to construct periodic conformal maps we are allowed to specify angle sums θ_v at boundary vertices. The condition for the sums of boundary angles differs from the plane case in the following way: The curvature at a boundary vertex v is given by $\kappa_v = \pi - \theta_v$, where θ_v is the angle sum of the adjacent triangles in the target mesh. For the two boundary loops $(v_1,\dots,v_n)$ and $(w_1,\dots,w_m)$ we have:

$$\sum_{i=1}^{n} \kappa_{v_i} + \sum_{j=1}^{m} \kappa_{w_j} = 0. \tag{1}$$

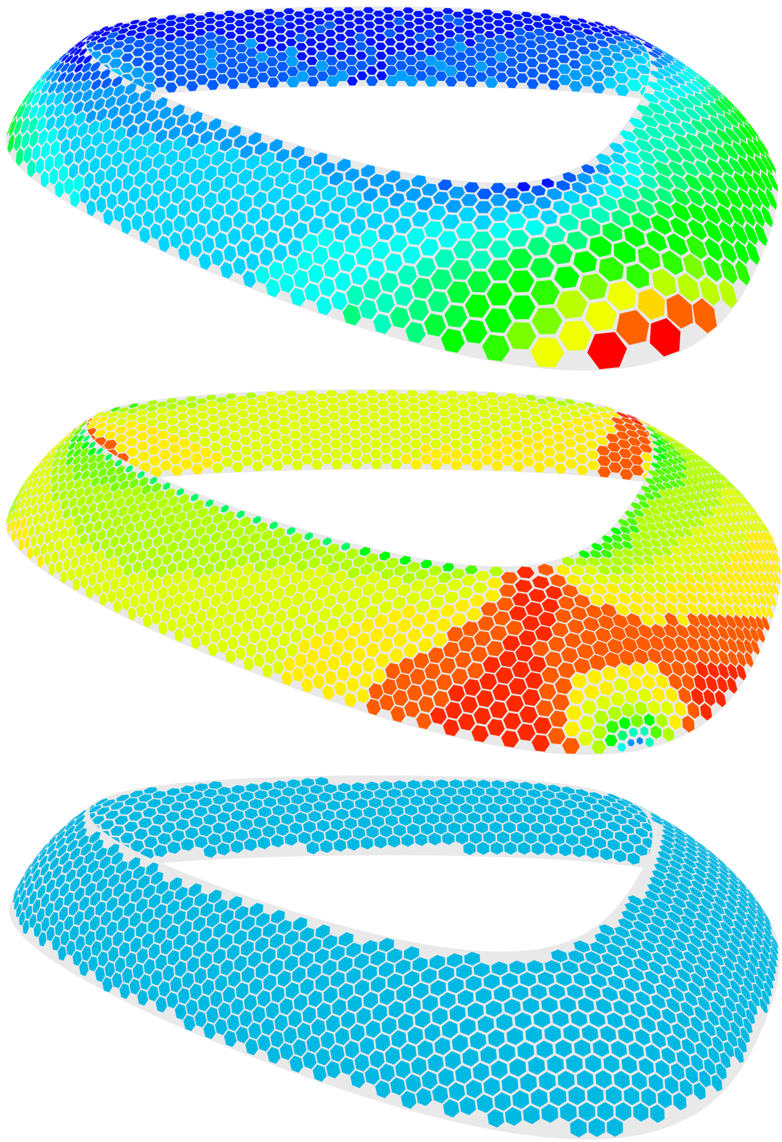


Figure 6: Quantized periodic hexagonal panelizations. Boundary conditions affect the amount of stretch in the interior of the surface. Top: Hexagonal pattern aligns with the boundary, a strong condition that produces large deviation of edge lengths. Middle: Map to a pattern-adapted polygon on a cone of revolution. The pattern contains exceptional points at the boundary. The stretch is minimized while at the same time the pattern alignes with the boundary. Bottom: Conformal map with the least stretch in the interior, pattern can be optimized to consist of congruent hexagons alone. In all images, panels with the same color are congruent. The corresponding domains of parameterization are shown in Figure 5.

This condition makes sure that the two boundary curves "bend" the same amount and can hence be wrapped around a cone. We will now show how boundary conditions can be used to construct periodic patterns on the studied models. We start with a discrete conformal map of the doubly-curved model from Figure 1 to a standard cylinder.

Straight cylinder. The simplest way to generate a map to the cylinder is to set the target angles for all boundary vertices to π. Hence the curvatures at the boundary vertices are zero and the two boundary loops are mapped to "straight" curves. In this case both angle sums of Equation 1 vanish and the target mesh can be wrapped around a cylinder, see Figure 5, left. The new edge lengths computed with the variational principle correspond to the lengths on a cylinder. This cylinder can be unrolled in the plane preserving angles and lengths. So the two boundary polygons are mapped to straight lines in the plane. These two straight lines have to be parallel and of equal lengths. If the lengths of the boundary curves in the original model differ a lot, then a map to a cylinder induces a lot of conformal stretch. This stretch can be reduced by specifying special boundary conditions for a parameterization on a cone of revolution.

Cone of revolution. As long as Equation (1) is satisfied we obtain a map to a general cone of revolution. In our case, we require that the periodic parameterization is adapted to the target pattern. This means that the two sums of Equation (1) need to be (the same) multiples of $\frac{\pi}{3}$ (triangle or hex) or multiples of $\frac{\pi}{2}$ (quad). We present two methods to achieve this requirement: a uniform distribution and a concentration of curvature.

If the boundary of the mesh should to be aligned with the pattern, then boundary angles need to be quantized, i.e., multiples of $\frac{\pi}{3}$ or $\frac{\pi}{2}$ need to be chosen as target angles. In Figure 5, middle, three vertices of the top and bottom boundary curve were manually assigned to $\frac{4}{3}\pi$ and $\frac{2}{3}\pi$, respectively. All other boundary angles are set to π, i.e., straight. Such a map can be used as a starting point to obtain a tesselation with equal hexagons as described in Section 5.

It is known that a discrete confomal map that does not change the lengths of the boundary edges exhibits the least stretch in the interior of the surface. To obtain such a parameterization we first construct a periodic conformal mapping onto an arbitrary cone such that the lengths of the boundary edges are not changed. The resulting angle sums at boundary vertices of the target mesh determine the cone angle of the map. The cone angle of a pattern adapted periodic parameterization is the closest multiple of the desired quantization. We distribute the difference to the closest quantized angle uniformly to the individual boundary vertices and recompute the map with these angle conditions. The obtained map is periodic and exhibits the lowest stretch of all periodic conformal maps (see Figure 5, right).

Design and structural opportunities. It is also possible to use special boundary conditions to support structural purposes or design requirements. If one aims for a

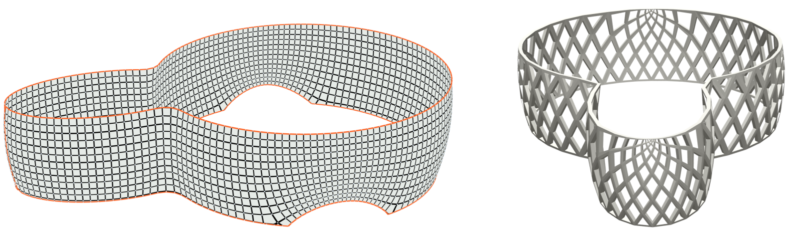


Figure 7: A periodic conformal map onto a cylinder with special vertices creates the opportunity to incorporate entrances (left) or concentration of support structure (right).

panelization with boundary aligned patterns, then the target boundary angles must be quantized.

To include entrances in a facade it is possible to incorporate special boundary conditions. An example with special boundary vertices with domain angles $\frac{4}{3}\pi$ and $\frac{\pi}{3}$ is shown in Figure 7, left. In the remeshed surface, the lower boundary curve bends inside at the vertices with angle $\frac{4}{3}\pi$ around the vertex with angle $\frac{\pi}{3}$. This incorporates a natural entrance into the facade.

Another effect of such angle conditions is a densification of the pattern at the vertices with small angles. Such a concentration of elements can be used to enforce structural properties of a geometry. An example of a diagrid generated with such boundary conditions is shown in Figure 7, right.

4 Case Study: Hexagonal surface panelization

Here we present the findings of a first case study, where the method of discrete conformal mappings was utilized for a real-world project. In this case study, a facade design, important questions concerning panel layout, similarity and therewith constructability had to be addressed at the early design stage.

Discrete conformal maps were used, as they allow the designer to explore alternative surface textures and surface panelizations with great design flexibility. This distinguishes the method from more constrained modelling techniques [Glymph et al. 2004]. Through the method of conformal mapping, opposed to naive UV mapping, the density of the surface panelization varies across the entire surface, yet the shape of elements does not. This can be used for structural purposes, such as diagrid layouts, or design driven, such as window distribution. The optimization of the surface panelization towards multiple criteria such as edge length and planarity was consequential.

For a commissioned competition entry we tested and developed the method of periodic conformal mappings. The project, which served as a case study, was highly constrained, as the architects were asked to propose an alternative facade design for an existing design proposal of a multifunctional exhibition center in China, see Figure 8. The massing was fixed, but there were 2 alternative massing options (1

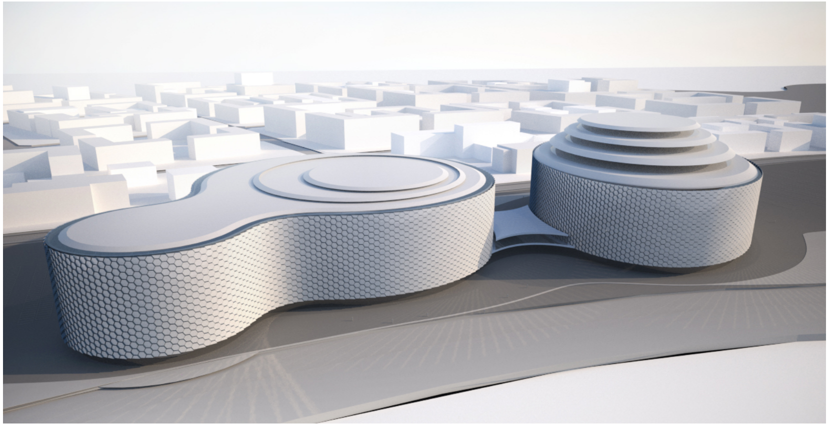


Figure 8: Rendering of Case Study - A project where the method of conformal mapping was utilized for the facade design.

single curved and 1 doubly-curved envelope) to be explored. Also, the client wanted a hexagonal tiling on the facade but only had a very limited budget of approximately 200 Euro / sqm for the entire facade including sub structure in mind.

These limitations, in combination with the very short timeframe of 2 weeks for the entire redevelopment of the facade including a feasibility study drove the development of the conformal mapping method. Especially, since existing solutions such as UV mapping led to unsatisfactory results producing anisotropic stretch and shear of some regions in the master surfaces. Some specific questions that had to be addressed for each massing option were:

- How many (different) panels would we need?
- Can we clad the entire surface with planar tiles?
- Can we equalize the edge lengths of each hexagon?
- Can we control the orientation of the panels?
- Can we achieve a regular pattern with a homogeneous visual appearance?

In the end, all the above questions were answered/solved.

The first step of development focused on achieving periodicity across the surface and alignment with the boundary. While the issue of periodicity directly addressed the last question, it is strongly related to the others as they could be achieved by successive optimization steps.

Already during the design phase a fully periodic tiling was achieved. In a following step the panels were planarized, grouped by dimension and their edge lengths were equalized. Finally, a control for the panel orientation based on the tangents of the NURBS master surface was implemented. This hexagonal pattern served as a base for the facade engineering team. Due to the high cost demands, a simple component system that served as a sub-structure for each panel was developed, see Figure 9.

Figure 9: The data for the component-like construction of each panel was derived from the mesh.

Unfortunately the given massing options for the building were not very challenging in terms of geometry. One massing option was a simple extrusion and the other had very little distortion. After the successful submission of the project, we decided to continue the development and test the method of discrete conformal mappings on more extreme base geometries.

During these tests a Grasshopper plug-in for VARYLAB, see [Sechelmann and Rörig 2013] has been developed and refined. We focused on tiling surfaces with a large distortion/stretch and double curvature. The main aim was to tile these surfaces without distorsion. This led to focus on the boundary conditions. The designer is now able to choose between an aligned mapping where tile-pattern aligns with the underlying surface boundary. The trade-off being, that panels need to vary in sizes. Or one chooses a "homogeneous tiling", where all the tiles are the same, but do not align with the boundary. VARYLAB's numerous optimization algorithms can be applied and combined with either of the two approaches, see Figure 6. During the development, we realized that through singularities and special boundary conditions, one is able to control the density and distribution of the pattern on the surface and along its boundaries, see Figure 7.

We also started to look into how patterns can be applied across multiple surface patches, such that the pattern aligns at the crease where the patches meet. This would necessitate to develop methods for more general mappings and the applications of multiple boundary conditions; A field that is yet to be developed.

The collaboration between architect and math department proved to be very satisfactory for both parties: The architects did provide specific questions related to real world projects whilst the mathematicians were able to translate these questions into mathematical formulas and provided meaningful results that could not have

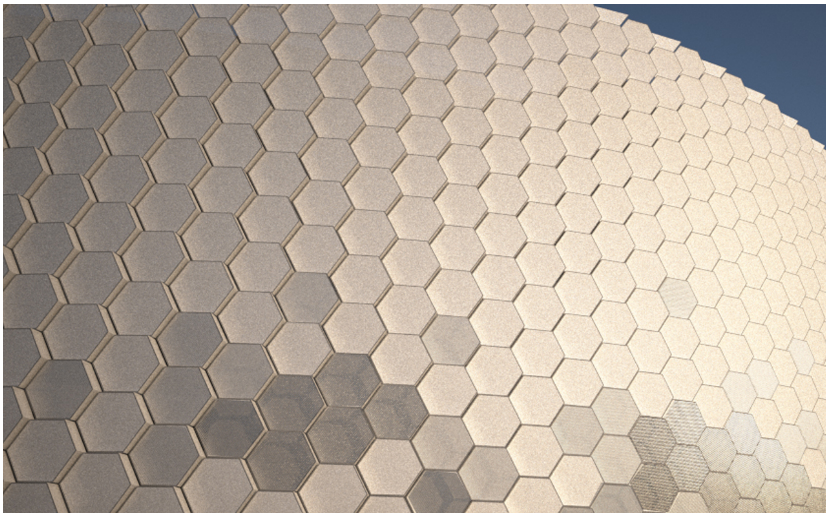


Figure 10: Close-up rendering of facade.

been achieved alternatively. The common design framework of Rhino and the basic knowledge of NURBS geometry and modelling techniques proved to be of essential importance for the successful collaboration between the teams.

5 Rationalization: Hexagon optimization

Starting from the conformal parameterization we optimize the obtained hex-mesh to have identical regular hexagons. We use a global optimization approach and define energies to achieve *planarity*, *regularity*, and *equality*.

Planarity. The planarity function is a simple adaptation of the usual energy used to planarize quad-meshes: A quadrilateral $\{A,B,C,D\}$ is planar if the volume of the tetrahedron $\{A,B,C,D\}$ is zero. So if we require the volume of all tetrahedra spanned by the vertices a polygon to be zero we obtain a planar polygon. The planarity energy E_{pl} can easily be expressed in terms of determinants.

Regularity. A regular planar polygon is characterized by having equal edge lengths and equal angles at all vertices. As for planarity we define an energy that is minimized in case of regular polygons. The interior angle at a vertex of a regular p-gon is $\frac{p-2}{p}\pi$. So an energy E_{reg} that is minimized for a regular p-gon with vertices

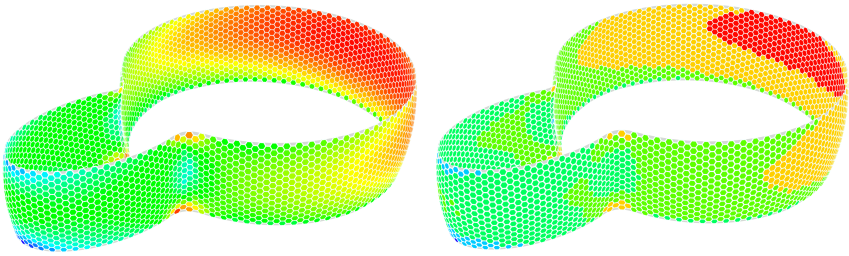


Figure 11: Panelization of a doubly curved design alternative of the case study shown in Figure 8. Left: Unquantized panelization. Right: Quantization to 11 panel sizes

$\{v_1,\ldots,v_p\}$ and corresponding angles $\{\alpha_1,\ldots,\alpha_p\}$ is

$$E_{reg}(P) = \lambda_P \, E_\alpha(P) + \mu_P \, E_\ell(P) \qquad \text{with}$$
$$E_\alpha(P) = \sum_{i=1}^{p}(\alpha_i - \frac{p-2}{p}\pi)^2 \qquad E_\ell(P) = \sum_{(v_i,v_{i+1})}(\|v_i - v_{i+1}\| - \ell_P)^2,$$

where ℓ_P is the desired target edge length for the polygon and λ_P and μ_P weights for the different energies. In a first step, the target length can be chosen to be the average edge length of the polygon or the shortest edge length among the edges to avoid overlap. Note that, the normalization of the angles already implies planarity of the polygons. Nevertheless, we consider the planarity energy since it increases the rate of convergence.

Starting from a cylindrical or conical periodic conformal parameterization we construct a hex-mesh that may or may not be aligned with the boundary. As a consequence of the conformality of the parameterization the angles of the hexagons are almost $\frac{2\pi}{3}$. In the following we do not work with a water tight mesh any more but split the surface into individual hexagonal panels. We optimize the edge lengths of the hexagons to be constant per face using E_ℓ. To avoid overlap we choose the length of the shortest edge of each face as target length. We add the planarity and angle regularity functionals E_{pl} and E_α to the optimization and obtain planar and regular hexagons. Each of the hexagons has its own constant edge length. Finally, we can rationalize the panelization further, by choosing a discrete set of edge lengths as target lengths for the polygons in the edge length functional E_ℓ. Due to the symmetry of the edge length functional for regular hexagons edge length optimization will not destroy the planarity and regularity of the hexagons. So it is possible to adjust the edge lengths using E_ℓ only. This quantization process is illustrated in Figure 11.

The range of lengths obtained depends on the initial hex-mesh constructed on the chosen target geometry. The effect of the different periodic conformal parametrizations on the quantization is shown in Figures 5 and 6.

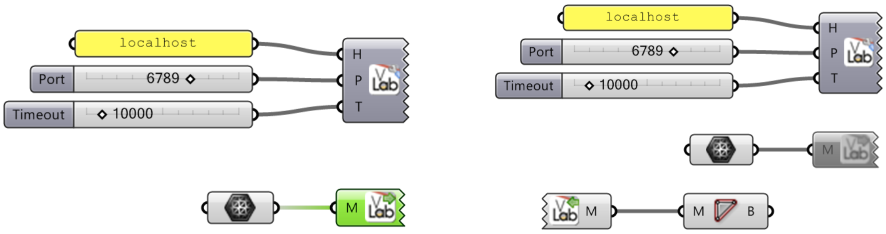


Figure 12: Grasshopper networks connecting Rhino and VaryLab. In the first step we send mesh data to VaryLab running on the same machine at `localhost:6789` (left). In the second step we collect the result from this VaryLab instance and create polygons from the mesh's faces (right).

6 Implementation

We use the software package VaryLab, [Sechelmann and Rörig 2013], in combination with Rhino's Grasshopper to calculate discrete conformal maps to the cylinder or cone as shown in Figure 5. Figure 12 shows the Grasshopper network used to connect Rhino and VaryLab. VaryLab uses the optimization package TAO/PETSc, see [Benson et al. 2007, Balay et al. 2011, Sommer 2010], to perform energy minimization.

7 Conclusion

We presented a method for homogeneous periodic panelization of NURBS surface geometry of cylinder type. A natural future development is the design of suitable support structures possibly with torsion free nodes. This can be derived from the panel layout by intersecting the panel planes. Furthermore the quantization of edge lengths of regular hexagonal panels is not yet fully explored. A denser distribution of quantized edge lengths in regions with great edge length variance can possibly improve the layout and number of different panels.

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