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An Introduction to Riemann Surfaces

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To Raghavan Narasimhan

Preface

A Riemann surface X is a connected 1-dimensional complex manifold, that is, a connected Hausdorff space that is locally homeomorphic to open subsets of $\mathbb{C}$ with complex analytic coordinate transformations (this also makes X a real 2-dimensional smooth manifold). Although a connected second countable real 1-dimensional smooth manifold is simply a line or a circle (up to diffeomorphism), the real 2-dimensional character of Riemann surfaces makes for a much more interesting topological characterization and a still more interesting complex analytic characterization.

A noncompact (i.e., an open) Riemann surface satisfies analogues of the classical theorems of complex analysis, for example the Mittag-Leffler theorem, the Weierstrass theorem, and the Runge approximation theorem (this development began only in the 1940s with the work of, for example, Behnke and Stein). On the other hand, by the maximum principle, a compact Riemann surface admits only constant holomorphic functions. However, compact Riemann surfaces do admit a great many meromorphic functions. This property leads to powerful theorems, in particular, the crucial Riemann–Roch theorem.

The theory of Riemann surfaces occupies a unique position in modern mathematics, lying at the intersection of analysis, algebra, geometry, and topology. Most earlier books on this subject have tended to focus on its algebraic-geometric and number-theoretic aspects, rather than its analytic aspects. This book takes the point of view that Riemann surface theory lies at the root of much of modern analysis, and it exploits this happy circumstance by using it as a way to introduce some fundamental ideas of analysis, as well as to illustrate some of the interactions of analysis with geometry and topology. The analytic methods applied in this book to the study of one complex variable are also useful in the study of several complex variables. Moreover, they contain the essence of techniques used generally in the study of partial differential equations and in the application of analytic tools to problems in geometry. Thus a careful reader will be rewarded not only with a good command of the classical theory of Riemann surfaces, but also with an introduction to these important modern techniques. While much of the book is intended for students at the second-year graduate level, Chaps. 1 and 2 and Sect. 5.2 (along with the required background material) could serve as the basis for the complex analysis component of a year-long first-year graduate-level course on real and complex analysis. A successful student in such a course would be well prepared for further study in analysis and geometry.

The analytic approach in this book is based on the solution of the inhomogeneous Cauchy–Riemann equation with L^2 estimates (or the L^2 $\bar{\partial}$*-method*) in a holomorphic line bundle with positive curvature. This powerful technique from several complex variables (see, for example, Andreotti–Vesentini [AnV], Hörmander [Hö], Skoda [Sk1], [Sk2], [Sk3], [Sk4], [Sk5], and Demailly [De1]) takes an especially nice form on a Riemann surface. Moreover, the 1-variable version serves as a gentle introduction to, and demonstration of, this important technique. For example, one may sometimes, with care, check signs in the formulas for higher dimensions by considering the dimension-one case. On the other hand, the higher-dimensional analogues of the main theorems considered in this book do require additional hypotheses.

Two central features of this book are a simple construction of a strictly subharmonic exhaustion function (which is a modified version of the construction in [De2]) and a simple construction of a positive-curvature Hermitian metric in the holomorphic line bundle associated to a nontrivial effective divisor. The simplicity of these constructions and the power of the L^2 $\bar{\partial}$-method make this approach to Riemann surfaces very efficient. The recent book of Varolin [V] also uses L^2 methods. However, although there is some overlap in the choice of topics, the proofs themselves are quite different. For a different treatment of this material that uses the solution of the inhomogeneous Cauchy–Riemann equation, but not L^2 methods, the reader may refer to, for example, [GueNs].

This book also contains (in Chaps. 5 and 6) proofs of some fundamental facts concerning the holomorphic, smooth, and topological structure of a Riemann surface, such as the Koebe uniformization theorem, the biholomorphic classification of Riemann surfaces, the embedding theorems, the integrability of almost complex structures, Schönflies' theorem (and the Jordan curve theorem), and the existence of a smooth structure on a second countable surface. The approach in this book to the above facts differs from the usual approaches (see, for example, [Wey], [Sp], or [AhS]) in that it mostly relies on the L^2 $\bar{\partial}$-method (in place of harmonic functions and forms) and on explicit holomorphic attachment of disks and annuli (in place of triangulations). The above facts, along with the facts concerning compact Riemann surfaces considered in Chap. 4, constitute some of the background required for the study of Teichmüller theory (as considered in, for example, [Hu]).

Riemann first introduced Riemann surfaces partly as a way of understanding multiple-valued holomorphic functions (see, for example, [Wey] or [Sp]), and Riemann surface theory has since grown into a vast area of study. Weyl's book [Wey] was the first book on the subject, and some elements of the point of view in [Wey] (and in the similar book [Sp]) are present in this book. For different approaches to the study of open Riemann surfaces, the reader may refer to, for example, [AhS] or [For]. For further study of compact Riemann surfaces (the main focus of most books on Riemann surfaces), the reader may refer to, for example, [FarK], [For], or [Ns4].

Much of the background material required for study of this book is provided in detail in Part III (Chaps. 7–11). It is strongly recommended that, rather than first studying all of Part III, the reader instead focus on Parts I and II, and consult the appropriate sections in Part III only as needed (tables providing the interdependence of the sections appear after table of contents). In fact, the background material has been placed in the last part of the book instead of the first in order to make such consultation more convenient. The main prerequisite for the book is some knowledge of point-set topology (as in, for example, [Mu]) and elementary measure theory (as in, for example, Chap. 1 of [Rud1]), although parts of these subjects are reviewed in Sects. 7.1 and 9.1. On the other hand, smooth manifolds (as in Chap. 9 and in, for example, [Mat], [Ns3], and [Wa]) and Hilbert spaces (as in Chap. 7 and in, for example, [Fol] and [Rud1]) are essential objects in this book, so it would be to the reader's advantage to have had some previous experience with these objects. It may surprise the reader to learn that previous experience with complex analysis in the plane (as in, for example, [Ns5]), while helpful, is not necessary. In fact, as indicated earlier, Chaps. 1 and 2 and Sect. 5.2 (along with the required background material and the corresponding exercises) together actually provide the material for a fairly complete course on complex analysis on domains in the plane along with the analogous study of complex analysis on Riemann surfaces (other suggested course outlines appear after the tables listing the interdependence of the sections).

Exercises are included for most of the sections, and some of the theory is developed in the exercises.

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Contents

Interdependence of Sections

The main prerequisites for each section appear in the tables below. If, for Sect. x, facts proved in Sect. y are needed only for a nonessential result, or only one or two small facts from Sect. y are needed, then Sect. y is omitted from the list of required sections for Sect. x. If Sect. x requires Sect. y and Sect. y requires Sect. z, then, in most cases, Sect. z is omitted from the list of required sections for Sect. x (for example, Sect. 2.3 requires Sects. 2.1 and 2.2, which in turn require Chap. 1 and Sect. 9.1). Italicized section numbers correspond to Part III sections, that is, sections containing background material.

Part I	Require(s)
1.1–1.6	*7.1–7.4*
2.1–2.2	Chap. 1, *9.1*
2.3	2.1, 2.2
2.4	2.1, 2.2, *8.1*, *9.2*, *9.4*
2.5	2.4, *8.2*, *9.5–9.7*
2.6	2.5, *7.5*
2.7	2.5, *9.8*
2.8, 2.9	2.6, 2.7
2.10	2.9 *or* 3.9
2.11	2.10
2.12	2.10
2.13	*9.1–9.3*
2.14	2.8, 2.11, 2.13
2.15	2.12, 2.14
2.16	2.12, 2.14
3.1–3.10	*8.3*, 2.5, 2.7, 2.8
3.11, 3.12	2.14, 3.10

Part II	Require(s)
4.1	3.10
4.2	2.14 (2.11 is *not* required), 4.1
4.3	4.2
4.4–4.9	4.2
5.1–5.5	*10.1–10.5*, 2.3, 2.16
5.6–5.9	5.5
5.10, 5.11	*9.9*, *10.1–10.6*
5.12–5.14	5.5, 5.10, 5.11
5.15	5.10, 5.11, *10.7*
5.16	5.13, 5.15 (Chap. 4 for Corollary 5.16.4)
5.17	5.2–5.4, 5.15
5.18	5.17
5.19	Chap. 4
5.20	5.1
5.21–5.22	Chap. 4, 5.20, *10.6*, *10.7*
6.1–6.6	2.9, *9.11*, *Chap. 11*, *7.6*
6.7–6.9	5.15, 5.17, *9.10*
6.10	5.10, 5.11, *10.1–10.7*
6.11	6.7–6.10

Suggested Course Outlines

A typical one-semester course would begin with careful study of Sects. 1.1 and 1.2, a quick reading of Sects. 1.3–1.6, careful study of Sects. 2.1 and 2.2, some (possibly brief) consideration of Sect. 2.3, and careful study of Sects. 2.4 and 2.5. The following are some of the natural directions in which the course could then proceed:

- *Holomorphic and meromorphic functions on an open Riemann surface*: Sections 2.6–2.16 and, possibly, 5.2.
- *Holomorphic line bundles on an open Riemann surface*: Sections 2.7, 2.8, 3.1–3.10, 2.11, 2.13, 2.14, 3.11, and 3.12.
- *Uniformization*: Sections 2.6–2.14, 2.16, and 5.1–5.9.
- *Compact Riemann surfaces*: Sections 2.7, 2.8, 3.1–3.10, 2.13, 2.14 (here, one needs the results of Sect. 2.14 only for an open subset of a compact Riemann surface, which is automatically second countable, so Radó's theorem from Sect. 2.11 is not required for this case), 4.1–4.9, and 5.19.
- *Integrability of almost complex structures on a surface*: Sections 2.6–2.9 and 6.1–6.6.

Suggested Topics for Further Study Beyond a one-semester course, the reader may wish to consider, for example:

- *The characterization of a Riemann surface in terms of holomorphic attachment of tubes*: Sections 5.10–5.14.
- *Further topics concerning compact Riemann surfaces*: Sections 5.10–5.13, 5.15, 5.16, 5.19–5.22.
- *The embedding theorem for an open Riemann surface*: Sections 5.17 and 5.18.
- *The existence of smooth structures on second countable topological surfaces*: Sections 5.17 and 6.7–6.11.

Part I
Analysis on Riemann Surfaces

Chapter 1
Complex Analysis in $\mathbb{C}$

In this chapter, following Hörmander [Hö], we consider elementary definitions and facts concerning complex analysis in $\mathbb{C}$ from the point of view of local solutions of the inhomogeneous Cauchy–Riemann equation $\partial u/\partial\bar{z} = v$. The *global* solution of the analogous inhomogeneous Cauchy–Riemann equation on a Riemann surface (see Chaps. 2 and 3) will allow us to obtain analogues of some of the central theorems of complex analysis in the plane (for example, the Riemann mapping theorem, the Mittag-Leffler theorem, and the Weierstrass theorem) for open Riemann surfaces, as well as some of the central theorems of the theory of compact Riemann surfaces (for example, the Riemann–Roch theorem). A reader who is familiar with complex analysis in the plane may wish to read Sects. 1.1 and 1.2 carefully, but only skim Sects. 1.3–1.6.

1.1 Holomorphic Functions

Identifying the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ under $(x, y) \mapsto z = x + iy = x + \sqrt{-1}y$ for $(x, y) \in \mathbb{R}^2$ (we will sometimes denote i by $\sqrt{-1}$), we may define the first-order constant-coefficient linear differential operators

$$\frac{\partial}{\partial z} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The operators $\partial/\partial z$ and $\partial/\partial\bar{z}$ obey the standard sum, product, and quotient rules. Moreover, if $\gamma\colon (a, b) \to \mathbb{C}$ is a (real) differentiable function on an interval (a, b) and u is a $\mathcal{C}^1$ function on a neighborhood of the image, then

$$\frac{d}{dt}u(\gamma(t)) = \frac{\partial u}{\partial z}(\gamma(t))\frac{d\gamma}{dt} + \frac{\partial u}{\partial \bar{z}}(\gamma(t))\overline{\frac{d\gamma}{dt}}.$$

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Observe that if f is a $\mathcal{C}^1$ function on an open set in $\mathbb{C}$, and $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, then

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \Longleftrightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We refer to the above as the *homogeneous Cauchy–Riemann equation(s)*. We use a similar notation if elements of $\mathbb{C}$ are represented by other names (for example, $\zeta = s + it$).

Definition 1.1.1 Let Ω be an open subset of $\mathbb{C}$. A function $f \in \mathcal{C}^1(\Omega)$ is called *holomorphic* (or *analytic* or *complex analytic*) on Ω if $\partial f/\partial \bar{z} \equiv 0$ on Ω. We denote the set of all holomorphic functions on Ω by $\mathcal{O}(\Omega)$, and for each function $f \in \mathcal{O}(\Omega)$, we define

$$f' = \frac{d}{dz}[f(z)] \equiv \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

We also say that f is a *primitive* for the function f'. A holomorphic function on $\mathbb{C}$ is also called an *entire* function.

The proof of the following is left to the reader (see Exercise 1.1.1):

Theorem 1.1.2 *Any sum, product, quotient, or composition of holomorphic functions is holomorphic on its domain. Moreover, if f and g are holomorphic functions, then*

$$(f+g)' = f' + g', \qquad (fg)' = f'g + fg',$$
$$(f/g)' = \frac{f'g - fg'}{g^2}, \quad \textit{and} \quad (f \circ g)' = (f' \circ g) \cdot g'.$$

Consequently, every rational function of z is holomorphic on its domain, and for every $n \in \mathbb{Z}$, $d(z^n)/dz = nz^{n-1}$.

An important tool in this book (and in the general study of analysis on manifolds) is the machinery of differential forms. The required definitions and facts appear in Chap. 9. For now, we take an informal approach to differential forms on open subsets of $\mathbb{C}$, which will suffice for this chapter.

We have the real differentials dx and dy on the complex plane $\mathbb{C}$ (identified with $\mathbb{R}^2$) and the complex differentials

$$dz = dx + i\,dy \quad \text{and} \quad d\bar{z} = dx - i\,dy.$$

Any complex 1-*form* α may be written

$$\alpha = P\,dx + Q\,dy = a\,dz + b\,d\bar{z},$$

for some (unique) pair of complex-valued functions P and Q and for $a = \frac{1}{2}(P - iQ)$ and $b = \frac{1}{2}(P + iQ)$. If α is continuous (i.e., if the coefficients P and Q are continuous) on a neighborhood of the image of a $\mathcal{C}^1$ path $\gamma = (u + iv)\colon [r,s] \to \mathbb{C}$, with $u = \operatorname{Re}\gamma$ and $v = \operatorname{Im}\gamma$, then

$$\begin{aligned}\int_\gamma \alpha &= \int_r^s \left(P(\gamma(t))\frac{du}{dt} + Q(\gamma(t))\frac{dv}{dt} \right) dt \\ &= \int_r^s \left(a(\gamma(t))\frac{d\gamma}{dt} + b(\gamma(t))\overline{\frac{d\gamma}{dt}} \right) dt.\end{aligned}$$

The *exterior product* of dx and dy is denoted by $dx \wedge dy = -dy \wedge dx$ (and $dx \wedge dx = dy \wedge dy = 0$). Any complex 2-*form* β may be written

$$\beta = f\,dx \wedge dy = f \cdot (i/2)\,dz \wedge d\bar{z},$$

for some (unique) function f. If f is integrable on a Lebesgue measurable set $E \subset \mathbb{C}$, then

$$\int_E f\,d\lambda = \int_E f\,dx \wedge dy = \int_E f \cdot \frac{i}{2}\,dz \wedge d\bar{z},$$

where λ denotes *Lebesgue measure* (see Sect. 7.1).

If f is a $\mathcal{C}^1$ function on an open set in $\mathbb{C}$, then the *differential* (or *exterior derivative*) of f satisfies

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar{z}}\,d\bar{z} = \partial f + \bar{\partial} f,$$

where

$$\partial f \equiv \frac{\partial f}{\partial z}\,dz \quad \text{and} \quad \bar{\partial} f \equiv \frac{\partial f}{\partial \bar{z}}\,d\bar{z}.$$

In particular, f is holomorphic if and only if $\bar{\partial} f = 0$ (equivalently, $df = h\,dz$ for some function h). If $\alpha = P\,dx + Q\,dy$ is a 1-form of class $\mathcal{C}^1$ (i.e., the coefficients P and Q are of class $\mathcal{C}^1$), then the *exterior derivative* of α is given by

$$d\alpha = dP \wedge dx + dQ \wedge dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$

Equivalently, for $\alpha = a\,dz + b\,d\bar{z}$ (with $a = \frac{1}{2}(P - iQ)$ and $b = \frac{1}{2}(P + iQ)$),

$$d\alpha = da \wedge dz + db \wedge d\bar{z} = \left(\frac{\partial b}{\partial z} - \frac{\partial a}{\partial \bar{z}} \right) dz \wedge d\bar{z} = \partial\alpha + \bar{\partial}\alpha,$$

where

$$\partial\alpha = (\partial b) \wedge d\bar{z} = \frac{\partial b}{\partial z}\,dz \wedge d\bar{z} \quad \text{and} \quad \bar{\partial}\alpha = (\bar{\partial} a) \wedge dz = -\frac{\partial a}{\partial \bar{z}}\,dz \wedge d\bar{z}.$$

According to Stokes' theorem, if $\Omega \Subset \mathbb{C}$ is a smooth domain and α is a $\mathcal{C}^1$ 1-form on a neighborhood of $\overline{\Omega}$, then

$$\int_{\partial\Omega} \alpha = \int_{\Omega} d\alpha,$$

where the left-hand side is given by the sum of the integrals of α over the finitely many boundary curves of Ω, each parametrized so that Ω lies to the left (i.e., each boundary curve is positively oriented relative to Ω).

Exercises for Sect. 1.1

1.1.1. Prove Theorem 1.1.2.

1.2 Local Solutions of the Cauchy–Riemann Equation

For $0 < r < R < \infty$ and $z_0 \in \mathbb{C}$, we set

$$\begin{aligned}
\Delta(z_0; R) &\equiv \{z \in \mathbb{C} \mid |z - z_0| < R\}, \\
\Delta(z_0; r, R) &\equiv \{z \in \mathbb{C} \mid r < |z - z_0| < R\}, \\
\Delta^*(z_0; R) &\equiv \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\} = \Delta(z_0; 0, R), \\
\mathbb{C}^* &\equiv \mathbb{C} \setminus \{0\}.
\end{aligned}$$

Lemma 1.2.1 *Let Ω be a smooth relatively compact domain in $\mathbb{C}$, and let f be a $\mathcal{C}^1$ function on a neighborhood of $\overline{\Omega}$.*

(a) Cauchy integral formula. *For each point $z \in \Omega$, we have*

$$f(z) = \frac{1}{2\pi i}\left[\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{\Omega} \frac{\partial f/\partial\bar{\zeta}}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}\right].$$

(b) Cauchy's theorem. *We have*

$$\int_{\partial\Omega} f(\zeta)\, d\zeta + \int_{\Omega} \frac{\partial f}{\partial\bar{\zeta}}\, d\zeta \wedge d\bar{\zeta} = 0.$$

Remarks 1. In particular, for f holomorphic on a neighborhood of $\overline{\Omega}$, we get (the more standard versions)

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \forall z \in \Omega \quad \text{and} \quad \int_{\partial\Omega} f(\zeta)\, d\zeta = 0.$$

2. The formula in part (a) is also called the *Cauchy–Pompeiu integral formula* or the $\bar{\partial}$-*Cauchy integral formula*.

Proof of Lemma 1.2.1 Observe that the function $\zeta \mapsto (\zeta - z)^{-1}$ is locally integrable in $\mathbb{C}$, so the integrals in (a) are defined. For $z \in \Omega$ and $0 < r < \operatorname{dist}(z, \partial\Omega)$, Stokes' theorem, together with the product rule and the holomorphicity of $\zeta \mapsto (\zeta - z)^{-1}$, gives

$$\begin{aligned}\int_{\Omega\setminus\overline{\Delta(z;r)}} \frac{\partial f/\partial\bar{\zeta}}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} &= \int_{\Omega\setminus\overline{\Delta(z;r)}} d\left[-\frac{f(\zeta)}{\zeta - z}\, d\zeta\right] \\ &= -\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{\partial\Delta(z;r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= -\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_0^{2\pi} f(z + re^{i\theta}) i\, d\theta.\end{aligned}$$

Letting $r \to 0^+$, we get (a).

Part (b) follows from Stokes' theorem (alternatively, one may fix a point $z \in \Omega$ and apply (a) to the function $\zeta \mapsto f(\zeta) \cdot (\zeta - z)$). □

Lemma 1.2.2 (Local solution of the inhomogeneous Cauchy–Riemann equation) *Let D be a disk in $\mathbb{C}$, let $\alpha \in \mathcal{C}^0(\partial D)$, let $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, let β be a $\mathcal{C}^k$ function on a neighborhood of $\overline{D}$, and, for each point $z \in D$, let*

$$f(z) = \frac{1}{2\pi i}\left[\int_{\partial D} \frac{\alpha(\zeta)}{\zeta - z}\, d\zeta + \int_D \frac{\beta(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}\right].$$

Then $f \in \mathcal{C}^k(D)$ and $\partial f/\partial\bar{z} = \beta$ on D. In particular, f is holomorphic on $D \setminus \operatorname{supp}\beta$.

Proof Setting $D = \Delta(z_0; R)$, we may fix a number r with $0 < r < R$, we may fix a $\mathcal{C}^k$ function β_1 with compact support in D such that $\beta_1 = \beta$ on a neighborhood of $\overline{\Delta(z_0; r)}$, and we may set $\beta_2 \equiv \beta - \beta_1$. Then $f = g + h$, where for each point $z \in D$,

$$g(z) \equiv \frac{1}{2\pi}\int_0^{2\pi} \frac{\alpha(z_0 + Re^{i\theta})}{z_0 + Re^{i\theta} - z} \cdot Re^{i\theta} d\theta - \frac{1}{\pi}\int_D \frac{\beta_2(\zeta)}{\zeta - z}\, d\lambda(\zeta)$$

and

$$h(z) \equiv -\frac{1}{\pi}\int_D \frac{\beta_1(\zeta)}{\zeta - z}\, d\lambda(\zeta) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\beta_1(\zeta)}{\zeta - z}\, d\lambda(\zeta).$$

The functions $(z, \theta) \mapsto \alpha(z_0 + Re^{i\theta})/(z_0 + Re^{i\theta} - z)$ and $(z, \zeta) \mapsto \beta_2(\zeta)/(\zeta - z)$, and their derivatives of arbitrary order in x and y ($z = x + iy$), are continuous and bounded on $\Delta(z_0; r) \times [0, 2\pi]$ and $\Delta(z_0; r) \times D$, respectively. Thus differentiation past the integral (Proposition 7.2.5) implies that g is of class $\mathcal{C}^\infty$ on $\Delta(z_0; r)$ and

$$\frac{\partial g}{\partial\bar{z}}(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial\bar{z}}\left[\frac{\alpha(z_0 + Re^{i\theta})}{z_0 + Re^{i\theta} - z}\right] \cdot Re^{i\theta} d\theta - \frac{1}{\pi}\int_D \frac{\partial}{\partial\bar{z}}\left[\frac{\beta_2(\zeta)}{\zeta - z}\right] d\lambda(\zeta) = 0.$$

Thus g is holomorphic and of class $\mathcal{C}^\infty$ on $\Delta(z_0; r)$.

Similarly, setting $R' \equiv R + r + |z_0|$, we get, for each point $z \in \Delta(z_0; r)$,

$$\begin{aligned}h(z) &= -\frac{1}{\pi}\int_{\mathbb{C}} \beta_1(\zeta + z)\cdot\zeta^{-1}\,d\lambda(\zeta) = -\frac{1}{\pi}\int_{\Delta(z_0 - z; R)} \beta_1(\zeta + z)\cdot\zeta^{-1}\,d\lambda(\zeta)\\ &= -\frac{1}{\pi}\int_{\Delta(z_0; R')} \beta_1(\zeta + z)\cdot\zeta^{-1}\,d\lambda(\zeta);\end{aligned}$$

and hence, since the function $\zeta \mapsto 1/\zeta$ is integrable on $\Delta(z_0; R')$, h is of class $\mathcal{C}^k$ on $\Delta(z_0; r)$. Moreover, setting $\gamma \equiv \partial\beta_1/\partial\bar{z}$, we get, for each point $z \in \Delta(z_0; r)$,

$$\begin{aligned}\frac{\partial h}{\partial \bar{z}}(z) &= -\frac{1}{\pi}\int_{\Delta(z_0; R')} \frac{\partial}{\partial \bar{z}}[\beta_1(\zeta + z)]\cdot\zeta^{-1}\,d\lambda(\zeta) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\gamma(\zeta + z)}{\zeta}\,d\lambda(\zeta)\\ &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\gamma(\zeta)}{\zeta - z}\,d\lambda(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial\beta_1/\partial\bar{\zeta}}{\zeta - z}\,d\zeta \wedge d\bar{\zeta}.\end{aligned}$$

Since $\operatorname{supp}\beta_1 \subset D$, the Cauchy integral formula (Lemma 1.2.1) now gives $\partial h/\partial\bar{z} = \beta_1 = \beta$ on $\Delta(z_0; r)$, and since the choice of $r \in (0, R)$ was arbitrary, the claim follows. □

Theorem 1.2.3 *If $\Omega \subset \mathbb{C}$ is an open set and $f \in \mathcal{O}(\Omega)$, then $f \in \mathcal{C}^\infty(\Omega)$ and $f' \in \mathcal{O}(\Omega)$.*

Proof For any disk $D \Subset \Omega$, the Cauchy integral formula (Lemma 1.2.1) and the local solution of the Cauchy–Riemann equation (Lemma 1.2.2) together imply that f is of class $\mathcal{C}^\infty$ on D. It follows that $f' \in \mathcal{O}(\Omega)$, because

$$\frac{\partial f'}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar{z}}\right) = 0.$$ □

Theorem 1.2.4 *For every open set $\Omega \subset \mathbb{C}$, every compact set $K \subset \Omega$, and every linear differential operator A with coefficients in $L^\infty_{\mathrm{loc}}(\Omega)$, there is a constant $C = C(\Omega, K, A)$ such that*

$$\|Af\|_{L^\infty(K)} \leq C\|f\|_{L^p(\Omega)} \quad \forall p \in [1, \infty],\ f \in \mathcal{O}(\Omega).$$

Proof Since A has bounded coefficients in a neighborhood of K, we may assume without loss of generality that $A = \partial^{|\alpha|}/\partial x^{\alpha_1}\partial y^{\alpha_2}$ for some multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and applying Hölder's inequality in a relatively compact neighborhood of K in Ω, we see that we may also assume that $p = 1$. Given a point $z_0 \in K$, we may choose $r, R_1, R_2 \in \mathbb{R}$ with

$$0 < r < R_1 < R_2 < \operatorname{dist}(z_0, \mathbb{C} \setminus \Omega),$$

and we may choose a function $\eta \in \mathcal{D}(\Delta(z_0; R_2))$ with $\eta \equiv 1$ on $\Delta(z_0; R_1)$. For each $f \in \mathcal{O}(\Omega)$ and each point $z \in \Delta(z_0; R_1)$, the Cauchy integral formula gives

$$f(z) = \eta(z) f(z) = -\frac{1}{\pi} \int_{\Delta(z_0; R_2) \setminus \Delta(z_0; R_1)} f(\zeta) \frac{\partial \eta}{\partial \bar{\zeta}}(\zeta)(\zeta - z)^{-1}\, d\lambda(\zeta).$$

The function $(z, \zeta) \mapsto f(\zeta) \cdot (\partial\eta/\partial\bar{\zeta})(\zeta) \cdot (\zeta - z)^{-1}$, and its derivatives of arbitrary order in x and y (for $z = x + iy$), are bounded on $\Delta(z_0; r) \times [\Delta(z_0; R_2) \setminus \Delta(z_0; R_1)]$. Differentiation past the integral now gives, for $z \in \Delta(z_0; r)$,

$$[Af](z) = -\frac{1}{\pi} \int_{\Delta(z_0; R_2) \setminus \Delta(z_0; R_1)} f(\zeta) \frac{\partial \eta}{\partial \bar{\zeta}}(\zeta) A[(\zeta - \cdot)^{-1}](z)\, d\lambda(\zeta).$$

Hence

$$\sup_{\Delta(z_0; r)} |Af| \le C_{z_0} \|f\|_{L^1(\Delta(z_0; R_2) \setminus \Delta(z_0; R_1))} \le C_{z_0} \|f\|_{L^1(\Omega)}$$

for some constant C_{z_0} independent of the choice of f. Covering K by finitely many such disks $\Delta(z_0; r)$, we get the desired constant C. □

Corollary 1.2.5 *For every open set $\Omega \subset \mathbb{C}$ and every nonempty compact set $K \subset \Omega$, there is a constant $C = C(\Omega, K)$ such that $|f(w) - f(z)| \le C\|f\|_{L^p(\Omega)}|w - z|$ for all $w, z \in K$, $p \in [1, \infty]$, and $f \in \mathcal{O}(\Omega)$ (cf. Definition* 7.2.3).

Proof Given a disk $D \Subset \Omega$, Theorem 1.2.4 provides a (real) constant B such that $\sup_D |\partial f/\partial z| \le B\|f\|_{L^p(\Omega)}$ for every $f \in \mathcal{O}(\Omega)$ and $p \in [1, \infty]$. Hence, for every pair of points $w, z \in D$, for every function $f \in \mathcal{O}(\Omega)$, and for every number $p \in [1, \infty]$, we have

$$\begin{aligned} |f(w) - f(z)| &= \left| \int_0^1 \frac{d}{dt}[f(z + t(w - z))]\, dt \right| \\ &= \left| \int_0^1 \frac{\partial f}{\partial z}(z + t(w - z)) \cdot (w - z)\, dt \right| \\ &\le B\|f\|_{L^p(\Omega)}|w - z|. \end{aligned}$$

Covering K by finitely many such disks and fixing a Lebesgue number $\delta > 0$ for the covering, we get a constant B_0 such that $|f(w) - f(z)| \le B_0\|f\|_{L^p(\Omega)}|w - z|$ for all $w, z \in K$ with $|w - z| < \delta$, all $f \in \mathcal{O}(\Omega)$, and all $p \in [1, \infty]$. There is also a constant B_1 such that $\max_K |f| \le B_1\|f\|_{L^p(\Omega)}$ for all $f \in \mathcal{O}(\Omega)$ and all $p \in [1, \infty]$. Thus the constant $C \equiv \max(B_0, 2B_1/\delta)$ has the required property. □

Corollary 1.2.6 *If $\{f_n\}$ is a sequence of holomorphic functions converging uniformly on compact subsets of an open set $\Omega \subset \mathbb{C}$ to a function f, then $f \in \mathcal{O}(\Omega)$, and for each $k \in \mathbb{Z}_{\ge 0}$, the sequence of kth derivative functions $\{f_n^{(k)}\}$ converges uniformly to $f^{(k)}$ on compact subsets of Ω.*

Proof For each disk $D \Subset \Omega$ and each point $z \in D$, the Cauchy integral formula gives

$$f(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial D} \frac{f_n(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Thus Lemma 1.2.2 implies that f is holomorphic on D. That $f_n^{(k)} \to f^{(k)}$ uniformly on each compact set $K \subset \Omega$ follows easily from Theorem 1.2.4. □

Corollary 1.2.7 (Montel's theorem) *If $\{f_n\}$ is a sequence of holomorphic functions that is uniformly bounded on each compact subset of an open set $\Omega \subset \mathbb{C}$, then some subsequence $\left\{f_{n_k}\right\}$ converges uniformly on compact subsets of Ω to a function $f \in \mathcal{O}(\Omega)$.*

Proof For any compact set $K \subset \Omega$, Corollary 1.2.5 implies that there is a constant $C > 0$ such that $|f_n(w) - f_n(z)| \leq C|w - z|$ for all $w, z \in K$ and for every $n \in \mathbb{Z}_{>0}$. Ascoli's theorem (together with Cantor's diagonal process) now provides a subsequence converging uniformly on compact subsets of Ω, and Corollary 1.2.6 implies that the limit is holomorphic. □

According to Definition 7.4.2, for L^1_{loc} functions u and v on an open set $\Omega \subset \mathbb{C}$, we have

$$\left(\frac{\partial u}{\partial \bar{z}}\right)_{\text{distr}} \equiv \left(\frac{\partial}{\partial \bar{z}}\right)_{\text{distr}} u = v$$

if and only if

$$\int_\Omega u\overline{\left(-\frac{\partial \varphi}{\partial z}\right)}\,d\lambda = \int_\Omega v\bar{\varphi}\,d\lambda \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Theorem 1.2.8 (Regularity theorem) *If Ω is an open subset of $\mathbb{C}$, $u \in L^1_{\text{loc}}(\Omega)$, $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, $\beta \in \mathcal{C}^k(\Omega)$, and $(\partial u/\partial \bar{z})_{\text{distr}} = \beta$, then $u \in \mathcal{C}^k(\Omega)$ (i.e., there is a unique $\mathcal{C}^k$ function that is equal to u almost everywhere in Ω). In particular, if $(\partial u/\partial \bar{z})_{\text{distr}} = 0$, then $u \in \mathcal{O}(\Omega)$ (i.e., there is a unique holomorphic function that is equal to u almost everywhere in Ω).*

Proof It suffices to show that u is of class $\mathcal{C}^k$ on every disk $D \Subset \Omega$. Thus, by subtracting the $\mathcal{C}^k$ solution v of $\partial v/\partial \bar{z} = \beta$ in D provided by Lemma 1.2.2, we may assume without loss of generality that $\beta \equiv 0$.

To show that u is holomorphic on D, we fix a disk D' with $D \Subset D' \Subset \Omega$ and a nonnegative $\mathcal{C}^\infty$ function κ with compact support in the unit disk $\Delta(0; 1)$ in $\mathbb{C}$ such that $\int_{\mathbb{C}} \kappa\,d\lambda = 1$. By Lemma 7.3.1 and Lemma 7.4.4, for each sufficiently large $n \in \mathbb{Z}_{>0}$, the function u_n given by

$$u_n(z) \equiv \int_\Omega u(\zeta)n^2\kappa(n(z - \zeta))\,d\lambda(\zeta) \quad \forall z \in D'$$

is of class $\mathcal{C}^\infty$ and $\partial u_n/\partial\bar{z}=0$; that is, $u_n\in\mathcal{O}(D')$. Moreover, $\|u_n-u\|_{L^1(D')}\to 0$ as $n\to\infty$. Hence Theorem 1.2.4 implies that the sequence $\{u_n|_{\overline{D}}\}$ is Cauchy in $(\mathcal{C}^0(\overline{D}),\|\cdot\|_{L^\infty(\overline{D})})$ and therefore uniformly convergent. Corollary 1.2.6 now implies that u is holomorphic on D and therefore that $u\in\mathcal{C}^k(D)$. $\square$

Theorem 1.2.9 (Mean value property) *If $f\in\mathcal{O}(\Delta(z_0;R))$, then for every $r\in(0,R)$,*

$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi}f\big(z_0+re^{i\theta}\big)\,d\theta=\frac{1}{\pi r^2}\int_{\Delta(z_0;r)}f(\zeta)\,d\lambda(\zeta).$$

Proof Lemma 1.2.1 gives the first equality. For the second, we integrate to get

$$\begin{aligned}\frac{r^2}{2}f(z_0)=\int_0^r f(z_0)s\,ds&=\frac{1}{2\pi}\int_0^r\int_0^{2\pi}f\big(z_0+se^{i\theta}\big)s\,d\theta\,ds\\&=\frac{1}{2\pi}\int_{\Delta(z_0;r)}f(\zeta)\,d\lambda(\zeta).\end{aligned}$$

$\square$

Theorem 1.2.10 (Riemann's extension theorem) *Let $\Omega\subset\mathbb{C}$ be an open set, let $z_0\in\Omega$, and let f be a function that is holomorphic on $\Omega\setminus\{z_0\}$. If*

$$\lim_{z\to z_0}(z-z_0)f(z)=0\quad or\quad f\in L^2(\Omega),$$

then there exists a (unique) function $\tilde{f}\in\mathcal{O}(\Omega)$ such that $\tilde{f}=f$ on $\Omega\setminus\{z_0\}$.

Proof Let $a(z)=z-z_0$ for each $z\in\mathbb{C}$ and fix $R>0$ with $D\equiv\Delta(z_0;R)\Subset\Omega$. Assuming first that $a(z)f(z)\to 0$ as $z\to z_0$, we get $f\in L^1_{\mathrm{loc}}(\Omega)$. If $\varphi\in\mathcal{D}(D)$, then for every $r\in(0,R)$, Stokes' theorem gives

$$\int_{D\setminus\overline{\Delta(z_0;r)}}f\frac{\partial\varphi}{\partial\bar{z}}\,dz\wedge d\bar{z}=-\int_{D\setminus\overline{\Delta(z_0;r)}}d[f\varphi\,dz]=\int_{\partial\Delta(z_0;r)}f(z)\varphi(z)\,dz.$$

Letting $r\to 0^+$, we get

$$\int_D f\frac{\partial\varphi}{\partial\bar{z}}\,d\lambda=0.$$

Thus $(\partial f/\partial\bar{z})_{\mathrm{distr}}=0$ in Ω, and the regularity theorem (Theorem 1.2.8) implies that $f\in\mathcal{O}(\Omega)$.

For $f\in L^2(\Omega)$ and for $z\in\Delta^*(z_0;R/2)$, the mean value property (Theorem 1.2.9) and the Schwarz (or Hölder) inequality give, for $r=|z-z_0|/2$,

$$\begin{aligned}|a(z)f(z)|&=\left|\frac{1}{\pi r^2}\int_{\Delta(z;r)}a(\zeta)f(\zeta)\,d\lambda(\zeta)\right|\\&\le\frac{1}{\pi r^2}\|a\|_{L^2(\Delta(z;r))}\cdot\|f\|_{L^2(\Delta(z;r))}\end{aligned}$$

$$\begin{aligned} &\le \frac{1}{\pi r^2}\|a\|_{L^2(\Delta(z_0;3r))}\cdot\|f\|_{L^2(\Delta(z_0;3r))} \\ &= \frac{9}{\sqrt{2\pi}}\|f\|_{L^2(\Delta(z_0;3r))}\to 0 \quad \text{as } r\to 0^+. \end{aligned}$$

It follows that $a(z)f(z)\to 0$ as $z\to z_0$, and hence that $f\in\mathcal{O}(\Omega)$. $\square$

Exercises for Sect. 1.2

1.2.1. Prove the following general version of Lemma 1.2.2. Let μ be a positive measure in $\mathbb{C}$ that is defined on the collection of Lebesgue measurable subsets of $\mathbb{C}$ and that satisfies $\mu(\mathbb{C}\setminus K)=0$ for some compact set $K\subset\mathbb{C}$, let $f\in L^1(\mathbb{C},\mu)$, and let

$$u(z) = -\frac{1}{\pi}\int_{\mathbb{C}}\frac{f(\zeta)}{\zeta - z}\,d\mu(\zeta)$$

for every point $z\in\mathbb{C}$ at which the above integral is defined. Then we have the following:

(i) The function u is defined and holomorphic on $\mathbb{C}\setminus K$.
(ii) If, for some open set Ω, we have $f\restriction_\Omega\in L^\infty(\Omega)$ and $d\mu = d\lambda$ in Ω, then u is defined and continuous on Ω and $(\partial u/\partial\bar{z})_{\text{distr}} = f$ on Ω.
(iii) If, for some open set Ω, we have $f\restriction_\Omega\in\mathcal{C}^k(\Omega)$ with $k\in\mathbb{Z}_{>0}\cup\{\infty\}$ and $d\mu = d\lambda$ in Ω, then u is defined and of class $\mathcal{C}^k$ on Ω and $\partial u/\partial\bar{z} = f$ on Ω.

1.3 Power Series Representation

In this section, we verify that holomorphic functions are precisely those functions that can be expressed locally as sums of power series, and we consider some important consequences.

Theorem 1.3.1 (Abel) *For any power series $\sum_{n=0}^{\infty}c_n(z-z_0)^n$, either there is an $R\in(0,\infty]$ such that the series converges uniformly and absolutely on compact subsets of $D\equiv\{z\in\mathbb{C}\mid |z-z_0|<R\}$ to a holomorphic function f and the series diverges for $|z-z_0|>R$, or the series converges only when $z=z_0$, in which case we set $R=0$. Furthermore, the power series*

$$\sum_{n=1}^{\infty} n\cdot c_n(z-z_0)^{n-1} \quad \textit{and} \quad \sum_{n=0}^{\infty}\frac{c_n}{n+1}(z-z_0)^{n+1}$$

have this same radius of convergence R; and if $R>0$, then on the above set D, the sum of the first is equal to f', and the sum of the second has derivative f.

Proof Let $R=\sup\{|z-z_0|\mid \text{the series converges at } z\}$. Then, clearly, the series diverges for $|z-z_0|>R$. If $0<r<R$, then the series converges at some point $z_1\in\mathbb{C}$

with $r < t = |z_1 - z_0| < R$, and hence in particular, the sequence $\{|c_n(z_1 - z_0)^n|\}$ is bounded above. For each $z \in \Delta(z_0; r)$ and each $n \in \mathbb{Z}_{\geq 0}$, we have $|c_n(z - z_0)^n| \leq |c_n(z_1 - z_0)^n|(r/t)^n$. Therefore, by the Weierstrass M-test, the series converges uniformly and absolutely on $\Delta(z_0; r)$. Corollary 1.2.6 implies that the sum $f(z)$ is holomorphic on D and that $f'(z) = \sum_{n=1}^{\infty} n \cdot c_n(z - z_0)^{n-1}$. In particular, this series representing $f'(z)$ converges for $|z - z_0| < R$. The proofs of the remaining claims are left to the reader (see Exercise 1.3.1). □

Conversely, we have the following:

Theorem 1.3.2 *If $R \in (0, \infty]$, $z_0 \in \mathbb{C}$, and f is a holomorphic function on the set $D = \{z \in \mathbb{C} \mid |z - z_0| < R\}$, then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \quad \forall z \in D.$$

Proof If $0 < r < s < R$, then $f(z) = \frac{1}{2\pi i}\int_{\partial\Delta(z_0;s)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$ for every point z in $\Delta(z_0; r)$. On the other hand, for all $z \in \Delta(z_0; r)$ and $\zeta \in \partial\Delta(z_0; s)$, we have $|(z - z_0)/(\zeta - z_0)| < r/s < 1$ and hence

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)/(\zeta - z_0)}{1 - [(z - z_0)/(\zeta - z_0)]} = \sum_{n=0}^{\infty} f(\zeta)(z - z_0)^n(\zeta - z_0)^{-n-1},$$

and for each $n \in \mathbb{Z}_{\geq 0}$,

$$|f(\zeta)(z - z_0)^n(\zeta - z_0)^{-n-1}| \leq \max_{\partial\Delta(z_0;s)} |f| \cdot r^{-1} \cdot (r/s)^{n+1}.$$

Thus the Weierstrass M-test gives uniform convergence in ζ and, integrating term by term, we get $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ for each $z \in \Delta(z_0; r)$, where

$$c_n = \frac{1}{2\pi i}\int_{\partial\Delta(z_0;s)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \quad \forall n \in \mathbb{Z}_{\geq 0}.$$

For $n \in \mathbb{Z}_{\geq 0}$, according to Theorem 1.3.1, differentiation of the above power series term by term n times gives $f^{(n)}(z)$. Evaluating at z_0, we get $f^{(n)}(z_0) = c_n \cdot n!$, as claimed. □

Corollary 1.3.3 *Let Ω be a domain (i.e., a connected open set) in $\mathbb{C}$, and let f be a nonconstant holomorphic function on Ω. Then:*

(a) *For every point $z_0 \in \Omega$, $f^{(m)}(z_0) \neq 0$ for some $m \in \mathbb{Z}_{\geq 0}$.*
(b) *For every point $z_0 \in \Omega$, there are a unique $m \in \mathbb{Z}_{\geq 0}$ and a unique function $g \in \mathcal{O}(\Omega)$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for every point $z \in \Omega$.*
(c) Identity theorem. *The set $f^{-1}(0)$ has no limit points in Ω.*

(d) *For* $u = \operatorname{Re} f$ *and* $v = \operatorname{Im} f$*, the sets* $u^{-1}(0)$ *and* $v^{-1}(0)$ *are nowhere dense in* Ω.

Remark Suppose f is a holomorphic function on an open set $\Omega \subset \mathbb{C}$ and $z_0 \in \mathbb{C}$. It follows from part (b) that if f is not identically zero on some connected neighborhood of z_0 in Ω, then there exist a unique nonnegative integer m and a function $g \in \mathcal{O}(\Omega)$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for every point $z \in \Omega$. The nonnegative integer m is called the *order of* f *at* z_0 and is denoted by $\operatorname{ord}_{z_0} f$. For $m = \operatorname{ord}_{z_0} f > 0$, z_0 is called a *zero of order* m. For $m = 1$, we also say that f has a *simple zero at* p. If f vanishes on some neighborhood of z_0, then we say that f has *order* $\operatorname{ord}_{z_0} f = \infty$ at z_0.

Proof of Corollary 1.3.3 The set $A \equiv \bigcap_{n=0}^{\infty} [f^{(n)}]^{-1}(0)$ is closed in Ω, and by Theorem 1.3.2, A is also open. Part (a) now follows.

For (b), observe that by (a), the set $\{n \in \mathbb{Z}_{\geq 0} \mid f^{(n)}(z_0) \neq 0\}$ is nonempty. Taking m to be the minimum of this set and applying Theorem 1.3.2, we get the claim.

For the proof of (c), suppose $z_0 \in \Omega$. By part (b), there are a nonvanishing holomorphic function g on a neighborhood U of z_0 in Ω and an integer $m \in \mathbb{Z}_{\geq 0}$ such that $f(z) = (z - z_0)^m g(z)$ for each $z \in U$. Hence $f^{-1}(0) \cap U \subset \{z_0\}$ and the claim follows.

Finally, for the proof of (d), observe that if $u \equiv 0$ on a disk $D \subset \Omega$, then on D, we have $\overline{\partial v / \partial z} = \partial v / \partial \bar{z} = i \partial u / \partial \bar{z} \equiv 0$, and hence $dv \equiv 0$ and v is constant on D. But by the identity theorem (part (c) above), this is impossible. Thus $u^{-1}(0)$ is nowhere dense. Since v is the real part of the holomorphic function $-if$, it follows that $v^{-1}(0)$ is also nowhere dense. □

Theorem 1.3.4 (Maximum principle) *If the modulus* $|f|$ *of a holomorphic function* f *on a domain* $\Omega \subset \mathbb{C}$ *attains a local maximum at some point* $z_0 \in \Omega$*, then* f *is constant on* Ω.

Proof Let $u = \operatorname{Re} f$ and let $D = \Delta(z_0; R) \Subset \Omega$ be a disk on which $|f| \leq |f(z_0)|$. By Corollary 1.3.3, it suffices to show that u is constant on D. For this, we may assume without loss of generality that $f(z_0) \neq 0$ (otherwise, we have $f \equiv 0$ on a neighborhood) and hence that $f(z_0) = 1$ (divide by $f(z_0)$). The mean value property (Theorem 1.2.9) gives

$$1 = \operatorname{Re}\left[\frac{1}{\pi R^2} \int_D f(\zeta)\, d\lambda(\zeta)\right] = \frac{1}{\pi R^2} \int_D u(\zeta)\, d\lambda(\zeta) \leq \frac{1}{\pi R^2} \int_D d\lambda(\zeta) = 1.$$

Hence the nonnegative function $1 - u$ integrates to 0, and therefore $u \equiv 1$ on D. □

Theorem 1.3.5 (Open mapping theorem) *If* f *is a nonconstant holomorphic function on a domain* $\Omega \subset \mathbb{C}$*, then the image* $U \equiv f(\Omega)$ *is open.*

Proof Suppose $w_0 = f(z_0) \in \partial U$ for some $z_0 \in \Omega$. By the identity theorem (Corollary 1.3.3), there is a disk $D \Subset \Omega$ containing z_0 such that w_0 is not in the compact set $K \equiv f(\partial D)$. Choosing a point $w_1 \in \mathbb{C} \setminus U$ with $|w_1 - w_0| < \operatorname{dist}(w_1, K)$, we get a function $g \equiv (f - w_1)^{-1} \in \mathcal{O}(\Omega)$ whose maximum modulus on $\overline{D}$ is *not* attained at any point in ∂D. This contradicts the maximum principle (Theorem 1.3.4), so U must be open. □

For a holomorphic function on a punctured disk, a punctured plane, or an annulus, we have the *Laurent series representation* provided by the following theorem, the proof of which is left to the reader (see Exercise 1.3.2):

Theorem 1.3.6 (Laurent series representation) *Let $z_0 \in \mathbb{C}$.*

(a) *Let $\{c_n\}_{n\in\mathbb{Z}}$ be a collection of complex constants, and let r and R be the radii of convergence for the power series $\sum_{n=1}^{\infty} c_{-n}\zeta^n$ and $\sum_{n=0}^{\infty} c_n\zeta^n$, respectively. If $1/r < R$, then the Laurent series $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ is absolutely summable (i.e., $\sum_{n\in\mathbb{Z}} |c_n(z-z_0)^n| < \infty$) and uniformly convergent (i.e., the series $\sum_{n=1}^{\infty} c_{-n}(z-z_0)^{-n}$ and $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ are uniformly convergent) on compact subsets of the set $A \equiv \{z \in \mathbb{C} \mid 1/r < |z-z_0| < R\}$, and the sum*

$$\begin{aligned} z \mapsto f(z) &\equiv \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n \equiv \sum_{n\in\mathbb{Z}} c_n(z-z_0)^n \\ &= \sum_{n=1}^{\infty} c_{-n}(z-z_0)^{-n} + \sum_{n=0}^{\infty} c_n(z-z_0)^n \end{aligned}$$

is a holomorphic function on A. Moreover,

(i) *For any constant $t \in (1/r, R)$,*

$$c_n = \frac{1}{2\pi i}\int_{\partial\Delta(z_0;t)} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta \quad \forall n \in \mathbb{Z};$$

(ii) *The power series $\sum_{n=1}^{\infty}(-n)c_{-n}\zeta^{n+1}$ and $\sum_{n=1}^{\infty} nc_n\zeta^{n-1}$ have radii of convergence r and R, respectively, and*

$$f'(z) = \sum_{n=-\infty}^{\infty} nc_n(z-z_0)^{n-1} \quad \forall z \in A;$$

and

(iii) *The power series $\sum_{n=2}^{\infty} c_{-n}\zeta^{n-1}/(-n+1)$ and $\sum_{n=0}^{\infty} c_n\zeta^{n+1}/(n+1)$ have radii of convergence r and R, respectively, and if $c_{-1} = 0$, then the holomorphic function*

$$z \mapsto F(z) \equiv \sum_{n=-\infty}^{\infty} \frac{c_n}{n+1}(z-z_0)^{n+1}$$

(the coefficient $c_n/(n+1)$ is taken to be 0 for $n=-1$) satisfies $F' = f$ on A.

(b) *Let s and S be constants with $0 \le s < S \le \infty$; let f be a holomorphic function on the set $B \equiv \{z \in \mathbb{C} \mid s < |z - z_0| < S\}$; let $t \in (s, S)$; let*

$$c_n \equiv \frac{1}{2\pi i} \int_{\partial \Delta(z_0;t)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \quad \forall n \in \mathbb{Z};$$

and let r and R be the radii of convergence of the power series $\sum_{n=1}^{\infty} c_{-n}\zeta^n$ and $\sum_{n=0}^{\infty} c_n \zeta^n$, respectively. Then $1/r \le s < S \le R$ and

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z - z_0)^n \quad \forall z \in B.$$

Exercises for Sect. 1.3

1.3.1 Complete the proof of Theorem 1.3.1.
1.3.2 Prove Theorem 1.3.6.

1.4 Complex Differentiability

Complex differentiability provides a third characterization of holomorphic functions.

Definition 1.4.1 A function f defined on a neighborhood of a point $z_0 \in \mathbb{C}$ is *complex differentiable at* z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We say that f is *complex differentiable* on an open set $\Omega \subset \mathbb{C}$ if f is complex differentiable at each point in Ω.

If f is a function that is holomorphic on a neighborhood of a point $z_0 \in \mathbb{C}$, then applying part (b) of Corollary 1.3.3 to the holomorphic function $f - f(z_0)$, we get a holomorphic function h on a neighborhood of z_0 such that $f(z) - f(z_0) = (z - z_0)h(z)$ for all z. It follows that f is complex differentiable at z_0. Conversely, if a function f is complex differentiable at a point $z_0 \in \mathbb{C}$, then f is continuous at z_0, and computing the limit in Definition 1.4.1 separately along the x and y directions, we get

$$\frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0) = \frac{\partial f}{\partial z}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}}(z_0) = 0.$$

Consequently, if a function f is of class $\mathcal{C}^1$ and complex differentiable on an open set, then f is holomorphic with derivative given by the limit appearing in Definition 1.4.1. Thus we have the following:

Lemma 1.4.2 *Let f be a $\mathcal{C}^1$ function on an open set $\Omega \subset \mathbb{C}$. Then $f \in \mathcal{O}(\Omega)$ if and only if f is complex differentiable on Ω. Moreover, if this is the case, then*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \forall z \in \Omega.$$

It is not evident that a complex differentiable function on an open set is of class $\mathcal{C}^1$, so we cannot conclude from the above that such a function is holomorphic. Nevertheless, this is the case (see Exercise 1.4.2). In fact, by a theorem of Looman and Menchoff, any continuous function that satisfies the Cauchy–Riemann equations on an open set is holomorphic (see [Ns5]). On the other hand, in this book, we will need only the $\mathcal{C}^1$ case addressed in Lemma 1.4.2.

Exercises for Sect. 1.4

1.4.1 Prove *Goursat's theorem*: If $S \equiv (a, b) \times (c, d) \subset \mathbb{C}$ is a bounded open rectangular region and f is a function that is complex differentiable on a neighborhood of $\overline{S}$, then $\int_{\partial S} f(z)\, dz = 0$ (cf. Lemma 1.2.1).

Outline of a proof. One must show that $M_0 \equiv |\int_{\partial S} f(z)\, dz| = 0$. Let $S_0 = S$. Breaking up S into four congruent rectangular regions, one sees that for one of the rectangles S_1, $M_1 \equiv |\int_{\partial S_1} f(z)\, dz| \geq \frac{M_0}{4}$. Proceeding inductively, one gets a decreasing sequence of open rectangular regions $\{S_\nu\}$ such that for each $\nu \in \mathbb{Z}_{>0}$, S_ν has side lengths $2^{-\nu}(b - a)$ and $2^{-\nu}(d - c)$ and

$$M_\nu \equiv \left| \int_{\partial S_\nu} f(z)\, dz \right| \geq 4^{-\nu} M_0.$$

There exists a point z_0 in the intersection of the closures, and the function

$$g(z) \equiv \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} - \frac{\partial f}{\partial z}(z_0) & \text{if } z \in \overline{S} \setminus \{z_0\}, \\ 0 & \text{if } z = z_0, \end{cases}$$

is continuous. For a constant $r > 0$, one gets $M_0 \leq \sup_{S_\nu} |g| \cdot r$. Letting $\nu \to \infty$, one gets the claim.

1.4.2 Let f be a complex differentiable function on an open rectangular region $S \subset \mathbb{C}$, let $z_0 = x_0 + i y_0 \in S$ with $x_0, y_0 \in \mathbb{R}$, and let $F \colon S \to \mathbb{C}$ be the function given by

$$z = x + iy \mapsto \int_{x_0}^{x} f(t + i y_0)\, dt + i \int_{y_0}^{y} f(x + it)\, dt;$$

that is, $F(z)$ is obtained by integration of $f(z)dz$ along the horizontal line segment from z_0 to $x + i y_0$ followed by integration along the vertical line segment from $x + i y_0$ to z.

(a) Show that $\partial F / \partial y = i f$ on S. Applying Goursat's theorem (see Exercise 1.4.1), express $F(z)$ in terms of similar integrals that yield $\partial F / \partial x = f$ on S. Conclude from this that F is holomorphic and $F' = f$.

(b) Using part (a), prove that a function on an open set is holomorphic if and only if it is complex differentiable.

1.4.3 Show that complex differentiability provides a more direct proof that the sum of a power series is holomorphic than that appearing in the proof of Theorem 1.3.1. More precisely, suppose that $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ is a power series with radius of convergence $R \in (0, \infty]$, and $f(z)$ is the sum of this series for each point $z \in D \equiv \{z \in \mathbb{C} \mid |z - z_0| < R\}$. The first part of the proof of Abel's theorem on power series (Theorem 1.3.1) implies that the series converges absolutely on D and uniformly on any disk of radius $< R$ centered at z_0.

(a) Prove directly, without invoking Abel's theorem, that the power series

$$\sum_{n=1}^{\infty} nc_n(z-z_0)^{n-1}$$

also converges for $z \in D$ (and therefore that this power series has radius of convergence at least R).

(b) Let $g(z)$ be the sum of the series in part (a) for each point $z \in D$. Prove directly from Definition 1.4.1 (without invoking Corollary 1.2.6) that f is complex differentiable on D with $f' = g$. Using the fact that the limit of a uniformly convergent sequence of continuous functions is continuous, show that f is of class $\mathcal{C}^1$, and hence that f is holomorphic.

1.5 The Holomorphic Inverse Function Theorem in $\mathbb{C}$

A holomorphic function f mapping an open set Ω bijectively onto an open set U is called a *biholomorphism* (or a *biholomorphic* mapping) if the inverse function f^{-1} is holomorphic. We also say that f *maps* Ω *biholomorphically onto* U and that Ω and U are *biholomorphic*. Observe that the chain rule implies that f' is nonvanishing and $[f^{-1}]' = 1/(f' \circ f^{-1})$. A function that maps a neighborhood of each point biholomorphically onto an open set is called a *local biholomorphism* (or a *locally biholomorphic* mapping). The following fact is fundamental in the study of Riemann surfaces:

Theorem 1.5.1 (Holomorphic inverse function theorem in $\mathbb{C}$) *Let f be a holomorphic function on an open set $\Omega \subset \mathbb{C}$.*

(a) *If $z_0 \in \Omega$ and $f'(z_0) \neq 0$, then f maps some neighborhood of z_0 biholomorphically onto a neighborhood of $w_0 \equiv f(z_0)$.*
(b) *If f is one-to-one, then f maps Ω biholomorphically onto an open set.*

The main step of the proof is the following inequality:

Lemma 1.5.2 *Let D be a disk in $\mathbb{C}$, let f be a holomorphic function on D, and let C_1 and C_2 be positive constants with $|f'| \geq C_1$ and $|f''| \leq C_2$ on D. Then*

$$|f(z_2) - f(z_1)| \geq C_1|z_2 - z_1| - \frac{C_2}{2}|z_2 - z_1|^2 \quad \forall z_1, z_2 \in D.$$

In particular, if $\operatorname{diam} D < 2C_1/C_2$, *then f is injective.*

Proof For every pair of points $z_1, z_2 \in D$, we have

$$\begin{aligned}
&f(z_2) - f(z_1) - f'(z_1)(z_2 - z_1) \\
&\quad = \int_0^1 \frac{d}{dt}\big[f(z_1 + t(z_2 - z_1))\big]\,dt - f'(z_1)(z_2 - z_1) \\
&\quad = (z_2 - z_1)\int_0^1 \big[f'(z_1 + t(z_2 - z_1)) - f'(z_1)\big]\,dt \\
&\quad = (z_2 - z_1)\int_0^1 \int_0^1 \frac{d}{ds}\big[f'(z_1 + st(z_2 - z_1))\big]\,ds\,dt \\
&\quad = (z_2 - z_1)^2 \int_0^1 \int_0^1 t \cdot f''(z_1 + st(z_2 - z_1))\,ds\,dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
|f(z_2) - f(z_1)| \geq{}& |f'(z_1)| \cdot |z_2 - z_1| \\
&- |z_2 - z_1|^2 \int_0^1 \int_0^1 t \cdot |f''(z_1 + st(z_2 - z_1))|\,ds\,dt,
\end{aligned}$$

and the claim follows. □

Proof of Theorem 1.5.1 For the proof of part (a), observe that we may choose a constant $R > 0$ so small that $D \equiv \Delta(z_0; R) \subset \Omega$, $C_1 \equiv \inf_D |f'| > 0$, and $C_2 \equiv \sup_D |f''| < C_1/R$. Lemma 1.5.2 then implies that $f \restriction_D$ is injective and hence, by the open mapping theorem (Theorem 1.3.5), f maps D homeomorphically onto a neighborhood U of w_0 in $\mathbb{C}$ with inverse function $h \equiv [f\restriction_D]^{-1} \colon U \to D$. Furthermore, for $w_1 \in U$ and $w \in U \setminus \{w_1\}$ sufficiently close to w_1, we have

$$\frac{h(w) - h(w_1)}{w - w_1} = \frac{h(w) - h(w_1)}{f(h(w)) - f(h(w_1))} \to 1/f'(h(w_1)) \quad \text{as } w \to w_1;$$

so h is a complex differentiable function. Moreover, h has continuous partial derivatives, since

$$\frac{\partial h}{\partial x} = -i\frac{\partial h}{\partial y} = \frac{\partial h}{\partial z} = \frac{1}{f' \circ h};$$

and hence by Lemma 1.4.2, $h \in \mathcal{O}(U)$.

For the proof of (b), observe that by the open mapping theorem (Theorem 1.3.5), f maps Ω homeomorphically onto an open set Θ. Moreover, by the identity theorem (Corollary 1.3.3), the zero set $Z \equiv (f')^{-1}(0)$ of the holomorphic function f' has no limit points in Ω, and hence its image $Y \equiv f(Z)$ has no limit points in Θ. Therefore, by part (a), f maps $\Omega \setminus Z$ biholomorphically onto $\Theta \setminus f(Z)$, and Riemann's extension theorem (Theorem 1.2.10) implies that f^{-1} is holomorphic on Θ. □

Exercises for Sect. 1.5

1.5.1 Give a proof of the holomorphic inverse function theorem (Theorem 1.5.1) that uses the $\mathcal{C}^\infty$ inverse function theorem (Theorem 9.9.1).

1.5.2 A subset S of $\mathbb{R}^n$ is *convex* if $(1-t)x + ty = x + t(y-x) \in S$ for all points $x, y \in S$ and every number $t \in [0,1]$ (in other words, S contains every line segment with endpoints in S). A real-valued function φ on a convex set $S \subset \mathbb{R}^n$ is *convex* if $\varphi((1-t)x+ty) \leq (1-t)\cdot\varphi(x)+t\cdot\varphi(y)$ for all points $x, y \in S$ and every number $t \in [0,1]$ (in other words, the graph of the restriction of the function to a line segment in S lies below the line segment between the endpoints on the graph).

(a) Prove that if φ is a convex function on a convex set $S \subset \mathbb{R}^n$, then for every $a \in \mathbb{R}$, $\{x \in S \mid \varphi(x) < a\}$ is a convex set.

(b) For a real-valued $\mathcal{C}^2$ function φ on an open set $\Omega \subset \mathbb{R}^n$, the *Hessian* at a point $p \in \Omega$ is the bilinear function $(\mathrm{Hess}(\varphi))_p(\cdot,\cdot)$ on $\mathbb{R}^n$ given by

$$(\mathrm{Hess}(\varphi))_p(u,v) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(p) u_i v_j$$

for all $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in \mathbb{R}^n$. Prove that if Ω is convex and

$$(\mathrm{Hess}(\varphi))_p(u,u) \geq 0 \quad \forall u \in \mathbb{R}^n,\ p \in \Omega,$$

then φ is convex.

Hint. First consider the case $n = 1$.

(c) Let f be a holomorphic function on a neighborhood of 0 with $f(0) = 0$ and $f'(0) \neq 0$. Prove that for every sufficiently small $\epsilon > 0$, f maps the disk $\Delta(0;\epsilon)$ biholomorphically onto a *convex* neighborhood of 0.

Hint. After applying Theorem 1.5.1 to obtain a local holomorphic inverse g in a neighborhood of 0, show that the function $|g|^2$ is convex near 0.

1.5.3 Let Ω be a convex domain in $\mathbb{C}$ (see Exercise 1.5.2), $f \in \mathcal{O}(\Omega)$, and let C_1 and C_2 be positive constants with $|f'| \geq C_1$ and $|f''| \leq C_2$ on Ω.

(a) Prove that

$$|f(z_2) - f(z_1)| \geq C_1|z_2 - z_1| - \frac{C_2}{2}|z_2 - z_1|^2 \quad \forall z_1, z_2 \in \Omega.$$

(b) Prove that if $\operatorname{diam}\Omega < 2C_1/C_2$, then f maps Ω biholomorphically onto a domain in $\mathbb{C}$.

1.6 Examples of Holomorphic Functions

In this section, we consider some examples of holomorphic functions.

Example 1.6.1 (Rational functions) As noted in Theorem 1.1.2, every rational function

$$z \mapsto \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}$$

is holomorphic (on its domain).

Example 1.6.2 (Exponential, logarithmic, and power functions) The *exponential function* on $\mathbb{C}$ is defined by

$$z = x + iy \mapsto \exp(z) \equiv e^x \cos y + i e^x \sin y = e^x \cdot e^{iy}$$

(see Example 7.2.8). One may verify directly that the exponential function is nonvanishing and holomorphic with derivative function $z \mapsto \exp z$ (see Exercise 1.6.1) and that $\overline{\exp(z)} = \exp(\bar{z})$ for each $z \in \mathbb{C}$. Fixing $w \in \mathbb{C}$, one may also verify directly that $\exp(w + x) = \exp w \cdot \exp x$ for $x \in \mathbb{R}$, and the identity theorem (Theorem 1.3.3) applied to the holomorphic function $z \mapsto \exp(w + z)$ then implies that $\exp(w+z) = \exp w \cdot \exp z$ for all $w, z \in \mathbb{C}$. The power series representation centered at 0 is given by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \forall z \in \mathbb{C}.$$

We also denote the exponential function by $z \mapsto e^z$.

The holomorphic map $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is surjective, and by the holomorphic inverse function theorem (Theorem 1.5.1), the map is locally biholomorphic. Thus the local inverses determine a *multiple-valued* holomorphic function, which is called the *logarithmic function* on $\mathbb{C}^*$. This multiple-valued function is denoted by $z \mapsto \log z$, provided there is no danger that it will be confused with the real-valued (and single-valued) logarithmic function $x \mapsto \log x$ on $(0, \infty)$. The derivative of the logarithmic function is the (single-valued) holomorphic function $z \mapsto 1/\exp(\log z) = 1/z$. A local holomorphic inverse of the exponential function is a single-valued holomorphic *branch* of the logarithmic function. The difference of any two single-valued holomorphic branches on a domain is a holomorphic function with values in the discrete set $2\pi i\mathbb{Z}$, and hence this difference must be constant.

We may describe the logarithmic function more explicitly as follows. Recall that polar coordinates in $\mathbb{R}^2$ are given by $\mathcal{C}^\infty$ local inverses of the surjective local diffeomorphism $(0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ given by $(r, \theta) \mapsto re^{i\theta}$ (see Examples 7.2.8, 9.2.8, and 9.7.20). Thus the local inverses provide a multiple-valued $\mathcal{C}^\infty$ function $z = re^{i\theta} \mapsto \theta$, which is denoted by $z \mapsto \arg z$ and is called the *argument function*. For any $\mathcal{C}^\infty$ local inverse $\Phi = (\rho, \alpha)\colon \Omega \to \Phi(\Omega) \subset (0, \infty) \times \mathbb{R}$ of the map $(r, \theta) \mapsto re^{i\theta}$ on a domain $\Omega \subset \mathbb{C}^*$ (in particular, $\rho\colon z \mapsto |z|$), the composition of the exponential function with the $\mathcal{C}^\infty$ function $z \mapsto L(z) = \log|z| + i\alpha(z)$ is the identity. Thus L is a holomorphic branch of the logarithmic function. Conversely, given a holomorphic branch $L = \tau + i\alpha$ of the logarithmic function on a domain $\Omega \subset \mathbb{C}^*$ with $\tau = \operatorname{Re} L$ and $\alpha = \operatorname{Im} L$, we have $z = e^{\tau(z)} \cdot e^{i\alpha(z)}$ for each $z \in \Omega$. It follows that $\tau : z \mapsto \log|z|$ and that (e^τ, α) is a $\mathcal{C}^\infty$ local inverse of the map $(r, \theta) \mapsto re^{i\theta}$. Thus the logarithmic function is given by

$$z \mapsto \log|z| + i \arg z.$$

In particular, the real-valued logarithmic function on $(0, \infty)$ is the restriction of any branch $L = \tau + i\alpha$ of the logarithmic function with $\alpha \equiv 0$ on $(0, \infty)$.

For any $\lambda > 0$ with $\lambda \neq 1$, the *exponential function with base* λ is the holomorphic function given by $z \mapsto \lambda^z \equiv e^{z \log \lambda}$ for $z \in \mathbb{C}$, and the *logarithmic function with base* λ is the multiple-valued holomorphic function given by $z \mapsto \log_\lambda z \equiv \frac{\log z}{\log \lambda}$ for $z \in \mathbb{C}^*$ (i.e., given by the local holomorphic inverses of the exponential function with base λ).

For any $\zeta \in \mathbb{C}^*$, the corresponding *power function* is the multiple-valued holomorphic function given by $z \mapsto z^\zeta \equiv \exp(\zeta \log z)$ for $z \in \mathbb{C}^*$.

Example 1.6.3 (Trigonometric functions) The *trigonometric functions* are defined, at each suitable z, by

$$\sin z \equiv -\frac{i}{2}(e^{iz} - e^{-iz}), \qquad \cos z \equiv \frac{1}{2}(e^{iz} + e^{-iz}),$$
$$\tan z \equiv \frac{\sin z}{\cos z}, \qquad \cot z \equiv \frac{\cos z}{\sin z},$$
$$\sec z \equiv \frac{1}{\cos z}, \qquad \csc z \equiv \frac{1}{\sin z}.$$

Exercises for Sect. 1.6

1.6.1 Verify that the exponential function $z \mapsto \exp z$ is nonvanishing and holomorphic with derivative function $z \mapsto \exp z$.

1.6.2 Using the identity $\exp(w + z) = \exp w \cdot \exp z$, prove that for all $w, z \in \mathbb{C}$,

$$\sin(w + z) = \sin w \cos z + \cos w \sin z,$$
$$\cos(w + z) = \cos w \cos z - \sin w \sin z.$$

1.6.3 Prove directly that the power series $\sum_{n=0}^\infty \frac{z^n}{n!}$ converges for every $z \in \mathbb{C}$, and conclude from Theorem 1.3.1 that the sum determines an entire function E with derivative $E' = E$ (thus Theorem 1.3.1 leads to a different approach to the exponential function).

1.6.4 *Holomorphic functions of several complex variables.* A complex-valued $\mathcal{C}^1$ function f on an open subset Ω of $\mathbb{C}^n$ is called *holomorphic* if f is holomorphic in each variable. That is, for each $j = 1, \dots, n$, and each choice of

$z_1, \dots, \widehat{z_j}, \dots, z_n \in \mathbb{C}$, the function $\zeta \mapsto f(z_1, \dots, z_{j-1}, \zeta, z_{j+1}, \dots, z_n)$ is a holomorphic function on the open set $\{\zeta \in \mathbb{C} \mid (z_1, \dots, z_{j-1}, \zeta, z_{j+1}, \dots, z_n) \in \Omega\}$ (with the obvious adjustments for $j = 1$ or n). A $\mathcal{C}^1$ map $F = (f_1, \dots, f_m) \colon \Omega \to \mathbb{C}^m$ is called a *holomorphic map* if f_i is a holomorphic function for each $i = 1, \dots, m$. For each $j = 1, \dots, n$, we set

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

(a) Prove that a $\mathcal{C}^1$ function f on an open set $\Omega \subset \mathbb{C}^n$ is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}_j} \equiv 0 \quad \forall j = 1, \dots, n.$$

(b) Prove that any composition of holomorphic mappings (of open subsets of complex Euclidean spaces) is holomorphic.

Chapter 2
Riemann Surfaces and the L^2 $\bar{\partial}$-Method for Scalar-Valued Forms

In this chapter, we consider some elementary properties of Riemann surfaces, as well as a fundamental technique called the L^2 $\bar{\partial}$-method, Radó's theorem on second countability of Riemann surfaces, and analogues of the Mittag-Leffler theorem and the Runge approximation theorem for open Riemann surfaces. Viewing holomorphic functions as solutions of the homogeneous Cauchy–Riemann equation $\partial f/\partial \bar{z} = 0$ in $\mathbb{C}$ allows one to very efficiently obtain their basic properties (see Chap. 1). The intrinsic form of the homogeneous Cauchy–Riemann equation on a Riemann surface is given by $\bar{\partial} f = 0$ (see Sect. 2.5). In order to obtain holomorphic functions (and holomorphic 1-forms) on a Riemann surface (even on an open subset of $\mathbb{C}$), it is useful to consider the *inhomogeneous* Cauchy–Riemann equation $\bar{\partial}\alpha = \beta$. One well-known approach to solving this differential equation (as well as differential equations in many other contexts) is to consider weak solutions in L^2. This is the approach taken in this book. In order to do so, we must develop suitable versions of an L^2 space of differential forms (see Sect. 2.6) and an (intrinsic) distributional $\bar{\partial}$ operator (see Sect. 2.7). The relatively simple approaches to the above appearing in this book are, in part, special to Riemann surfaces; but they do contain important elements of the higher-dimensional versions (see, for example, [Hö] or [De3] for the higher-dimensional versions).

In this chapter, we consider only scalar-valued differential forms. In Chap. 3, we will consider the analogue for forms with values in a holomorphic line bundle. The solution in line bundles is more efficient in some ways, and it also generalizes more readily to higher-dimensional complex manifolds.

In Sects. 2.1–2.9, we consider the definition and basic properties of a Riemann surface, the L^2 spaces of differential forms, and the fundamental theorem regarding the solution of the (inhomogeneous) Cauchy–Riemann equation for scalar-valued differential forms. In the remaining sections, we apply the above to obtain some important facts, namely, the existence of meromorphic 1-forms and functions, Radó's theorem on second countability of Riemann surfaces, the Mittag-Leffler theorem, and the Runge approximation theorem (see [R] for a historical perspective).

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2.1 Definitions and Examples

In this section, we consider the definition of a Riemann surface and some examples.

Definition 2.1.1 Let X be a Hausdorff space.

(a) A homeomorphism $\Phi : U \to U'$ of an open set $U \subset X$ onto an open set $U' \subset \mathbb{C}$ is called a *local complex chart of dimension* 1 (or a 1-*dimensional local complex chart*) in X. We also denote this local complex chart by (U, Φ, U').

(b) Two 1-dimensional local complex charts (U_1, Φ_1, U_1') and (U_2, Φ_2, U_2') in X are *holomorphically compatible* if the *coordinate transformations*

$$\Phi_1 \circ \Phi_2^{-1} \colon \Phi_2(U_1 \cap U_2) \to \Phi_1(U_1 \cap U_2)$$

and

$$\left(\Phi_1 \circ \Phi_2^{-1}\right)^{-1} = \Phi_2 \circ \Phi_1^{-1} \colon \Phi_1(U_1 \cap U_2) \to \Phi_2(U_1 \cap U_2)$$

are holomorphic; that is, each is a biholomorphism (see Fig. 2.1).

(c) A family of holomorphically compatible 1-dimensional local complex charts $\mathcal{A} = \{(U_i, \Phi_i, U_i')\}_{i \in I}$ that covers X (i.e., that satisfies $X = \bigcup_{i \in I} U_i$) is called a *holomorphic atlas of dimension* 1 on X.

(d) Two 1-dimensional holomorphic atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on X are *holomorphically equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a holomorphic atlas (this is an equivalence relation).

(e) An equivalence class $\mathcal{R}$ of holomorphic atlases on X is called a *holomorphic structure* (or a *complex analytic structure*) of dimension 1 on X, and the pair $(X, \mathcal{R})$ (which is usually denoted simply by X) is called a *complex manifold of dimension* 1 (or a *complex* 1-*manifold* or a *complex analytic manifold of dimension* 1). If, in addition, X is connected, then the pair $(X, \mathcal{R})$ (which again is usually denoted simply by X) is called a *Riemann surface*.

(f) A 1-dimensional local complex chart (U, Φ, U') in a holomorphic atlas in the holomorphic structure on a complex 1-manifold X is called a *local holomorphic chart in* X. Setting $z = \Phi$, we call z a *local holomorphic coordinate*, and for each point $p \in U$, we call (U, z) a *local holomorphic coordinate neighborhood of* p.

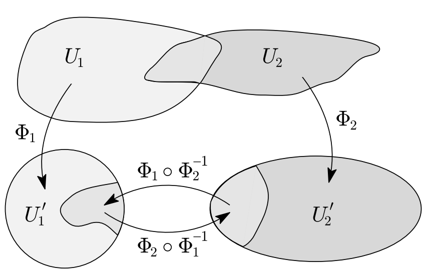

Fig. 2.1 Holomorphic coordinate transformations

Remarks 1. Higher-dimensional complex manifolds are also considered in this book, but only in some of the exercises (see, for example, Exercise 2.2.6). For this reason, all local complex charts, all local holomorphic charts, and all holomorphic atlases should be assumed to be of dimension 1 unless otherwise indicated.

2. A holomorphic structure on X determines a unique underlying real 2-dimensional $\mathcal{C}^\infty$ (in fact, real analytic) structure, with $\mathcal{C}^\infty$ atlas consisting of local $\mathcal{C}^\infty$ coordinates given by $(x, y) = (\operatorname{Re} z, \operatorname{Im} z)$ for any local holomorphic coordinate z (see Chap. 9). A map from (to) an open subset of X from (respectively, to) an open subset of a manifold or complex 1-manifold is said to be *of class* $\mathcal{C}^k$ if the map is of class $\mathcal{C}^k$ with respect to the underlying $\mathcal{C}^k$ structures.

3. In many treatments, manifolds are assumed to be second countable (often without comment). In fact, according to a theorem of Radó that will be proved in Sect. 2.11, every Riemann surface is automatically second countable.

4. A local holomorphic chart (U, Φ, V) and a local holomorphic coordinate neighborhood $(U, z = \Phi)$ are really the same object, but one generally uses the former terminology when emphasizing the mapping properties, and the latter when emphasizing the coordinate properties.

5. We will call a local holomorphic coordinate neighborhood $(U, z = \Phi)$ in which $\Phi(U)$ is a disk a *holomorphic coordinate disk* (or simply a *coordinate disk*). We will use the analogous terminology in similar contexts (for example, a *holomorphic coordinate annulus* will refer to a local holomorphic coordinate neighborhood with image an annulus). Similarly, given a set $S \subset U$ and a property of $\Phi(U)$, we will often say that S has this property *in* (or *with respect to*) the local coordinate neighborhood. For example, if $\Phi(S)$ is a closed rectangle in $\mathbb{R}^2$, then we will say that *S is a closed rectangle in* (U, z). We will also use terminology analogous to the above in the context of topological and $\mathcal{C}^\infty$ manifolds.

6. It turns out that the theory of compact Riemann surfaces and that of noncompact Riemann surfaces differ. A noncompact Riemann surface is also called an *open* Riemann surface.

7. For convenience, most of the main theorems, as well as some of the elementary facts, in this book are stated as applying to Riemann surfaces. However, analogues, with the appropriate modifications, also hold for complex 1-manifolds (see Example 2.1.4 below).

We now consider some examples. The verifications of some details are left to the reader.

Example 2.1.2 (The complex plane) The *complex plane* $\mathbb{C}$ together with the holomorphic structure determined by the holomorphic atlas $\{(\mathbb{C}, z \mapsto z, \mathbb{C})\}$ is a Riemann surface.

Example 2.1.3 An open set Ω in a complex 1-manifold X has a natural induced holomorphic structure in which each triple of the form $(U \cap \Omega, \Phi\restriction_{U\cap\Omega}, \Phi(U \cap \Omega))$, where (U, Φ, U') is a local holomorphic chart in X, is a local holomorphic chart in Ω (see Exercise 2.1.1). Unless otherwise indicated, we take this to be the holomorphic structure on such a given open subset Ω.

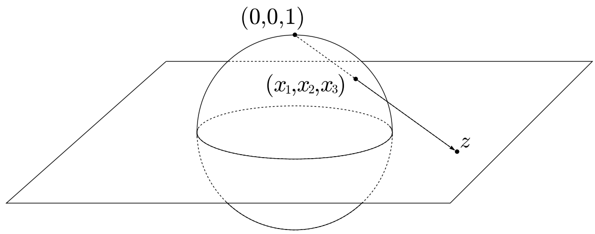

Fig. 2.2 The Riemann sphere

Example 2.1.4 Let $X \equiv \bigsqcup_{\alpha \in A} X_\alpha$ be a disjoint union of complex 1-manifolds $\{X_\alpha\}_{\alpha \in A}$, and let $\iota_\alpha : X_\alpha \hookrightarrow X$ be the inclusion map for each index $\alpha \in A$. Then X has a unique (natural) holomorphic structure in which each triple of the form $(\iota_\alpha(U), \Phi \circ \iota_\alpha^{-1}, U')$, where (U, Φ, U') is a local holomorphic chart in X_α for some $\alpha \in A$, is a local holomorphic chart in X (see Exercise 2.1.2). Unless otherwise indicated, we take this to be the holomorphic structure on such a given disjoint union. In particular, every complex 1-manifold is equal to a disjoint union of Riemann surfaces, that is, of its connected components. Consequently, most statements about Riemann surfaces may be easily modified to give an analogous statement about complex 1-manifolds.

Example 2.1.5 (The Riemann sphere) The *Riemann sphere* (or *extended complex plane*) is the one-point compactification $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ (see Definition 9.1.11) together with the holomorphic structure determined by the holomorphic atlas

$$\{(\mathbb{C}, z \mapsto z, \mathbb{C}), (\mathbb{C}^* \cup \{\infty\}, z \mapsto 1/z, \mathbb{C})\}$$

(where $1/\infty = 0$). The Riemann sphere is diffeomorphic to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ (see Examples 9.2.3) under the *stereographic projection* $\mathbb{S}^2 \to \mathbb{P}^1$ (see Fig. 2.2 and Exercise 2.1.3), i.e., the mapping $(x_1, x_2, x_3) \mapsto \frac{x_1}{1-x_3} + i\,\frac{x_2}{1-x_3}$ $((0,0,1) \mapsto \infty)$.

Example 2.1.6 (Complex tori) A *lattice* in $\mathbb{C}$ is a subgroup of the form $\Gamma = \mathbb{Z}\xi_1 + \mathbb{Z}\xi_2$, where $\xi_1, \xi_2 \in \mathbb{C}$ are complex numbers that are linearly independent over $\mathbb{R}$. We may associate to Γ an equivalence relation $\sim$ given by

$$z \sim w \quad \Longleftrightarrow \quad z - w \in \Gamma$$

(in other words, the equivalence class of each element $z \in \mathbb{C}$ is $z + \Gamma$). Let us denote the corresponding quotient space and quotient mapping by $\Upsilon : \mathbb{C} \to X$. That is, X is the set of equivalence classes for $\sim$, Υ is the mapping $z \mapsto z + \Gamma$, and X is given the quotient topology (i.e., $U \subset X$ is open if and only if its inverse image $\Upsilon^{-1}(U)$ is open).

Observe that Υ is an open mapping. For if $U \subset \mathbb{C}$ is open, then $\Upsilon^{-1}(\Upsilon(U))$ is the union of the open sets $\{U + \xi\}_{\xi \in \Gamma}$. For each point $z = u_1\xi_1 + u_2\xi_2 \in \mathbb{C}$ with

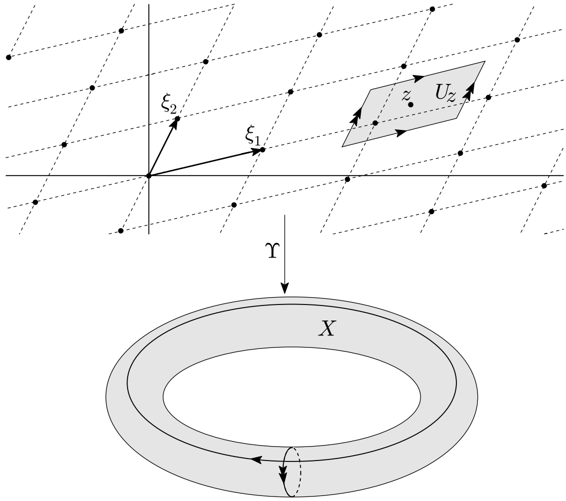

Fig. 2.3 A complex torus

$u_1, u_2 \in \mathbb{R}$, the set

$$U_z \equiv \left\{ z + t_1\xi_1 + t_2\xi_2 \mid -\frac{1}{2} < t_1, t_2 < \frac{1}{2} \right\}$$

is the image of the open square $(u_1 - 1/2, u_1 + 1/2) \times (u_2 - 1/2, u_2 + 1/2) \subset \mathbb{R}^2$ under the real linear isomorphism $\mathbb{R}^2 \to \mathbb{C}$ given by $(t_1, t_2) \mapsto t_1\xi_1 + t_2\xi_2$, and hence U_z is a relatively compact neighborhood of z in $\mathbb{C}$ (see Fig. 2.3). It is also easy to check that

$$U_z \cap (U_z + (\Gamma \setminus \{0\})) = \emptyset \quad \text{and} \quad \Upsilon(\overline{U}_z) = X.$$

In particular, the openness of Υ, together with the first equality, implies that Υ maps U_z homeomorphically onto $\Upsilon(U_z)$; and the second equality implies that X is compact. Furthermore, X is Hausdorff. For it is clear that each of the sets $\Upsilon(U_z)$ is Hausdorff, while if $w \in \partial U_z$, then the neighborhoods

$$V \equiv \left\{ z + t_1\xi_1 + t_2\xi_2 \mid -\frac{1}{4} < t_1, t_2 < \frac{1}{4} \right\}$$

and

$$W \equiv \left\{ w + t_1\xi_1 + t_2\xi_2 \mid -\frac{1}{4} < t_1, t_2 < \frac{1}{4} \right\}$$

of z and w, respectively, have disjoint images in X.

If $U' \subset \mathbb{C}$ is an open set with $U' \cap (U' + \xi) = \emptyset$ for each $\xi \in \Gamma \setminus \{0\}$, then Υ maps U' homeomorphically onto an open set $U \subset X$. Thus we get a local complex chart $(U, (\Upsilon\restriction_{U'})^{-1}, U')$ in X. The collection of such local complex charts determines a holomorphic structure on X. For if $(V, (\Upsilon\restriction_{V'})^{-1}, V')$ is another such local complex chart in X and $\Psi \equiv (\Upsilon\restriction_{V'})^{-1} \circ \Upsilon\restriction_{(\Upsilon\restriction_{U'})^{-1}(U\cap V)}$ is the associated coordinate transformation, then $z \mapsto \Psi(z) - z$ is a continuous function on $U' \cap (V' + \Gamma) = (\Upsilon\restriction_{U'})^{-1}(U \cap V)$ with values in the discrete set Γ. Hence this function must be locally constant, and it follows that Ψ is holomorphic. Thus X is a compact Riemann surface.

The Riemann surface X is called a *complex torus* because topologically, X is the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. In fact, we have a commutative diagram of $\mathcal{C}^\infty$ maps

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\alpha"] \arrow[d, "\Upsilon_0"'] & \mathbb{C} \arrow[d, "\Upsilon"] \\ \mathbb{T}^2 \arrow[r, "\beta"] & X \end{tikzcd}$$

where α is the real linear isomorphism (and therefore diffeomorphism) given by $(t_1, t_2) \mapsto t_1\xi_1 + t_2\xi_2$, Υ_0 is the $\mathcal{C}^\infty$ mapping onto the real torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ given by $(t_1, t_2) \mapsto (e^{2\pi i t_1}, e^{2\pi i t_2})$, and the induced mapping β is a well-defined diffeomorphism. For it is easy to verify that β is a well-defined bijection. Moreover, given a point $t = (t_1, t_2) \in \mathbb{R}^2$, Υ_0 maps the neighborhood $V \equiv (t_1 - 1/2, t_1 + 1/2) \times (t_2 - 1/2, t_2 + 1/2)$ diffeomorphically onto a neighborhood of $\Upsilon_0(t)$ (with inverse given by the product of $1/2\pi$ and a $\mathcal{C}^\infty$ argument function in each coordinate). Since locally, β is a composition of the $\mathcal{C}^\infty$ map Υ, the $\mathcal{C}^\infty$ map α, and a local $\mathcal{C}^\infty$ inverse of Υ_0, β must be a $\mathcal{C}^\infty$ map. Similarly, β^{-1} is also a $\mathcal{C}^\infty$ map, and therefore β is a diffeomorphism.

Exercises for Sect. 2.1

2.1.1 Verify that in any open subset of a complex 1-manifold, the local complex charts described in Example 2.1.3 form a holomorphic atlas.

2.1.2 Verify that in any disjoint union of complex 1-manifolds, the local complex charts described in Example 2.1.4 form a holomorphic atlas.

2.1.3 Verify that the Riemann sphere (Example 2.1.5) is a Riemann surface, and verify that the stereographic projection (see Fig. 2.2) is a diffeomorphism of the unit sphere $\mathbb{S}^2$ onto $\mathbb{P}^1$.

2.2 Holomorphic Functions and Mappings

Given a holomorphic structure, one gets the corresponding holomorphic functions and mappings.

Definition 2.2.1 Let X and Y be complex 1-manifolds.

(a) A *holomorphic function* on an open set $\Omega \subset X$ is a complex-valued function f on Ω such that $f \circ \Phi^{-1} \in \mathcal{O}(\Phi(\Omega \cap U))$ for every local holomorphic chart (U, Φ, U') in X. We denote the set of holomorphic functions on Ω by $\mathcal{O}(\Omega)$.

(b) A continuous mapping $\Psi\colon X \to Y$ is *holomorphic* if the function $\Phi \circ \Psi$ belongs to $\mathcal{O}(\Psi^{-1}(U))$ for every local holomorphic chart (U, Φ, U') in Y.

(c) A bijective holomorphic mapping $\Psi: X \to Y$ with holomorphic inverse is called a *biholomorphism* (or a *biholomorphic* mapping). If such a biholomorphism exists, then we say that X and Y are *biholomorphically equivalent* (or simply *biholomorphic*). For $Y = X$, we also call Ψ an *automorphism* of X.
(d) A holomorphic mapping $\Psi: X \to Y$ is a *local biholomorphism* (or a *locally biholomorphic* mapping) if Ψ maps a neighborhood of each point in X biholomorphically onto an open subset of Y.

Remarks 1. A holomorphic function on an open subset Ω of a complex 1-manifold X is precisely a holomorphic mapping of Ω into $\mathbb{C}$.

2. A function f on an open subset Ω of a complex 1-manifold X is holomorphic if and only if for each point $p \in \Omega$, *there exists* a local holomorphic chart (U, Φ, U') in X such that $p \in U$ and $f \circ \Phi^{-1} \in \mathcal{O}(\Phi(\Omega \cap U))$. Similarly, a continuous mapping of complex 1-manifolds $\Psi: X \to Y$ is holomorphic if and only if for each point $p \in X$, *there exists* a local holomorphic chart (U, Φ, U') in Y such that $\Psi(p) \in U$ and $\Phi \circ \Psi \in \mathcal{O}(\Psi^{-1}(U))$ (see Exercise 2.2.1). Furthermore, a holomorphic mapping of complex 1-manifolds is of class $\mathcal{C}^\infty$ with respect to the underlying $\mathcal{C}^\infty$ structures (in fact, the mapping is real analytic with respect to the underlying real analytic structures).

3. Biholomorphic equivalence is an equivalence relation (see Exercise 2.2.2), and we usually identify any two biholomorphic complex 1-manifolds.

4. Any sum or product of holomorphic functions is holomorphic, and any composition of holomorphic mappings is holomorphic (see Exercise 2.2.3). In particular, the set of holomorphic functions on an open set is an algebra.

5. In Example 2.1.6, the quotient mapping $\Upsilon: \mathbb{C} \to X$ of $\mathbb{C}$ to the complex torus X is locally biholomorphic, because the local holomorphic charts are given by local inverses of Υ. Although all complex tori are diffeomorphic, they are not all biholomorphic (see Exercise 5.9.1).

Many of the elementary theorems for holomorphic functions on domains in the plane immediately give analogues for holomorphic mappings on Riemann surfaces. For example, we have the following:

Theorem 2.2.2 *Let X and Y be Riemann surfaces.*

(a) Identity theorem. *If $\Phi: X \to Y$ is a nonconstant holomorphic mapping, then the fiber $\Phi^{-1}(p)$ over each point $p \in Y$ is discrete in X (i.e., $\Phi^{-1}(p)$ has no limit points in X). Moreover, if $\Psi: X \to Y$ is a holomorphic map that is not identically equal to Φ, then $\{x \in X \mid \Phi(x) = \Psi(x)\}$ is discrete in X.*
(b) Riemann's extension theorem. *A continuous mapping $\Phi: X \to Y$ that is holomorphic on the complement of a discrete subset of X is holomorphic.*
(c) Maximum principle. *If $f \in \mathcal{O}(X)$ and $|f|$ attains a local maximum at some point $p \in X$, then f is constant. In particular, if X is compact, then every holomorphic function on X is constant.*
(d) Open mapping theorem. *If $\Phi: X \to Y$ is a nonconstant holomorphic mapping, then the image $\Phi(X)$ is open.*

Proof The proof of the first part of (a) is provided here, and the proofs of the second part of (a) and of (b)–(d) are left to the reader (see Exercise 2.2.4). Suppose $\Phi\colon X \to Y$ is a holomorphic mapping for which the fiber $F \equiv \Phi^{-1}(p)$ over some point $p \in Y$ has a limit point $q \in X$. We may fix local holomorphic charts (U, Ψ, U') in X and (V, Λ, V') in Y such that $q \in U$, U is connected, and $\Phi(U) \subset V$. It follows that for the holomorphic function $f \equiv \Lambda \circ \Phi \circ \Psi^{-1}\colon U' \to \mathbb{C}$, the fiber $f^{-1}(\Lambda(p)) = \Psi(F \cap U)$ has a limit point $\Psi(q)$ in U', and hence $f \equiv \Lambda(p)$ on U' by the identity theorem in the plane (see Corollary 1.3.3). Thus $\Phi\restriction_U = \Lambda^{-1} \circ f \circ \Psi \equiv p$, and therefore $U \subset F$. Hence the interior $\overset{\circ}{F}$ of F is nonempty, and by the above argument, $\partial(\overset{\circ}{F}) = \emptyset$. It follows that $F = X$, and hence that Φ is constant. $\square$

Lemma 2.2.3 (Local representation of holomorphic mappings) *Let p be a point in a Riemann surface X.*

(a) *If f is a nonconstant holomorphic function on X, then there exist a positive integer m and a local holomorphic coordinate neighborhood (U, z) of p such that $z(p) = 0$ and $f = f(p) + z^m$ on U. Moreover, $m = m(f, p)$ is unique, and for any local holomorphic coordinate neighborhood (W, w) of p, the function $(w - w(p))^{-m}(f - f(p)) \in \mathcal{O}(W \setminus \{p\})$ extends to a holomorphic function on W that does not vanish at p.*

(b) *If $\Psi\colon X \to Y$ is a nonconstant holomorphic mapping of X to a Riemann surface Y, then for some $m \in \mathbb{Z}_{>0}$ and for every local holomorphic coordinate neighborhood (V, ζ) of $\Psi(p)$ in Y, there is a local holomorphic coordinate neighborhood (U, z) of p in X such that $z(p) = 0$ and $\zeta(\Psi) = \zeta(\Psi(p)) + z^m$ on $U \cap \Psi^{-1}(V)$. Moreover, the integer $m = m(\Psi, p)$ is unique, and for any local holomorphic coordinate neighborhood (W, w) of p, the holomorphic function $(w - w(p))^{-m}(\zeta(\Psi) - \zeta(\Psi(p)))$ on $W \cap \Psi^{-1}(V) \setminus \{p\}$ extends to a holomorphic function on $W \cap \Psi^{-1}(V)$ that does not vanish at p.*

Proof For the proof of (a), we may fix a local holomorphic chart $(U_0, \Phi = w, V_0)$ with $p \in U_0$. Corollary 1.3.3 then provides a unique integer $m \in \mathbb{Z}_{>0}$ and a unique function $g \in \mathcal{O}(U_0)$ such that $g(p) \neq 0$ and $f - f(p) = (w - w(p))^m \cdot g$ on U_0. Choosing U_0 to be sufficiently small, we also get a holomorphic branch L of the logarithmic function on a neighborhood of $g(U_0)$ (that is, $e^{L(\zeta)} = \zeta$), and hence we have the holomorphic mth root function $e^{L/m}$. The holomorphic function $z \equiv (w - w(p)) \cdot e^{L(g)/m}$ then satisfies $z(p) = 0$ and $f = f(p) + z^m$ on U_0. Moreover,

$$\left(z \circ \Phi^{-1}\right)'(w(p)) = e^{L(g(p))/m} \neq 0,$$

so the holomorphic inverse function theorem (Theorem 1.5.1) implies that the function z maps some neighborhood U of p in U_0 biholomorphically onto a neighborhood of 0 in $\mathbb{C}$. Finally, we have uniqueness of m. For if $f - f(p) = \zeta^n$ on Q for some $n \in \mathbb{Z}_{>0}$ and some local holomorphic chart $(Q, \Psi = \zeta, Q')$ with $p \in Q$, then by Corollary 1.3.3, since $(z \circ \Psi^{-1})'(0) \neq 0$, there is a holomorphic function h on a neighborhood of 0 in $\mathbb{C}$ with $h(0) \neq 0$ and $z(\Psi^{-1}(\xi)) = \xi h(\xi)$ for each $\xi \in Q'$ near 0. Hence, for each ξ near 0,

$$\xi^n = f\big(\Psi^{-1}(\xi)\big) - f(p) = \big[z\big(\Psi^{-1}(\xi)\big)\big]^m = \xi^m (h(\xi))^m.$$

The uniqueness part of Corollary 1.3.3 implies that $n = m$ (and $h^m \equiv 1$). Thus (a) is proved.

The proof of (b) is left to the reader (see Exercise 2.2.5). □

Definition 2.2.4 Let Ω be an open subset of a complex 1-manifold X, and let $p \in \Omega$.

(a) If f is a holomorphic function on Ω and m is a positive integer such that $f = z^m g$ on a neighborhood p for some local holomorphic coordinate z with $z(p) = 0$ and some nonvanishing holomorphic function g (i.e., $f(p) = 0$ and m is the integer provided by part (a) of Lemma 2.2.3), then we say that f has a *zero of order m at p*. For $m = 1$, we also say that f has a *simple zero at p*.
(b) If $\Psi\colon \Omega \to Y$ is a holomorphic mapping into a complex 1-manifold Y and m is a positive integer such that $\zeta(\Psi) - \zeta(\Psi(p))$ has a zero of order m at p for some local holomorphic coordinate ζ on a neighborhood of $\Psi(p)$ (i.e., m is the integer provided by part (b) of Lemma 2.2.3), then we say that Ψ *has multiplicity m at p*, and we write $\operatorname{mult}_p \Psi = m$.

Definition 2.2.5 A *meromorphic function* on an open subset Ω of a complex 1-manifold X is a function $f\colon \Omega \setminus P \to \mathbb{C}$ on the complement $\Omega \setminus P$ of a discrete subset P of Ω such that for each point $p \in P$, $|f(x)| \to \infty$ as $x \to p$. Each point $p \in P$ is called a *pole* of f. We denote the set of meromorphic functions on Ω by $\mathcal{M}(\Omega)$.

Proposition 2.2.6 *For any holomorphic function f on the complement $X \setminus P$ of a discrete subset P of a complex 1-manifold X, the following are equivalent*:

(i) *The function f is a meromorphic function on X with pole set P.*
(ii) *There exists a holomorphic mapping $h\colon X \to \mathbb{P}^1$ such that $h = f$ on $X \setminus P$ and $P = h^{-1}(\infty)$.*
(iii) *For each point $p \in P$ and for every local holomorphic coordinate neighborhood (U, z) of p in X, there exist a nonvanishing holomorphic function g on a neighborhood V of p in U and a positive integer ν_p such that $f = (z - z(p))^{-\nu_p} g$ on $V \setminus \{p\}$.*
(iv) *For each point $p \in P$, there exist a local holomorphic coordinate neighborhood (U, z) of p in X, a nonvanishing holomorphic function g on a neighborhood V of p in U, and a positive integer ν_p such that $f = (z - z(p))^{-\nu_p} g$ on $V \setminus \{p\}$.*
(v) *For each point $p \in P$, there exist a local holomorphic coordinate neighborhood (U, z) of p in X and a positive integer ν_p such that $z(p) = 0$ and $f = z^{-\nu_p}$ on $U \setminus \{p\}$.*

Moreover, in the above, for each point $p \in P$, the integer ν_p in (iii)–(v) *is equal to the multiplicity at p of the holomorphic mapping h in* (ii).

Proof Clearly, any one of the conditions (ii)–(v) implies (i). Conversely, if f is meromorphic with pole set P, then f extends to a unique continuous mapping $h\colon \Omega \to \mathbb{P}^1$ with $h^{-1}(\infty) = P$, and Riemann's extension theorem (Theorem 2.2.2) implies that h is a holomorphic mapping; that is, (ii) holds. If $p \in P$ and $\nu_p = \mathrm{mult}_p h$, then by the local representation of holomorphic mappings (Lemma 2.2.3), for every local holomorphic coordinate neighborhood (U, z) of p, there is a nonvanishing holomorphic function u on a neighborhood V of p in U such that $h^{-1}(\infty) \cap V = \{p\}$ and $u = (z - z(p))^{-\nu_p} \cdot (1/h)$ on $V \setminus \{p\}$. Setting $g \equiv 1/u$, we get (iii), and (iv) follows. Moreover, we may choose (U, z) so that $z(p) = 0$ and $1/h = z^{\nu_p}$ on U, so we get (v) as well. □

Definition 2.2.7 We identify any meromorphic function f on an open subset Ω of a complex 1-manifold X with the associated holomorphic mapping $\Omega \to \mathbb{P}^1$. If $p \in f^{-1}(\infty)$ is a pole of f at which this mapping has multiplicity m, then we say that f has a *pole of order m at p*. We say that f has a *zero of order m at $q \in \Omega$* if $q \in f^{-1}(0)$ and the holomorphic function $f\restriction_{\Omega \setminus f^{-1}(\infty)}$ has a zero of order m at q (note that this is consistent with the above identification). For any point $p \in \Omega$, the *order of f at p* is given by

$$\mathrm{ord}_p f \equiv \begin{cases} 0 & \text{if } p \notin f^{-1}(\{0, \infty\}), \\ m & \text{if } f \text{ has a zero of order } m \text{ at } p, \\ -m & \text{if } f \text{ has a pole of order } m \text{ at } p, \\ \infty & \text{if } f \equiv 0 \text{ on a neighborhood of } p. \end{cases}$$

We say that f has a *simple zero* (*simple pole*) *at* p if $\mathrm{ord}_p f = 1$ (respectively, $\mathrm{ord}_p f = -1$).

Remarks 1. If f and g are meromorphic functions on a connected neighborhood of a point p in a complex 1-manifold X and neither f nor g is identically zero, then in terms of some local holomorphic coordinate z on a neighborhood of p, we have $f = z^m f_0$ and $g = z^n g_0$ for some pair of integers m and n and some pair of nonvanishing holomorphic functions f_0 and g_0 on a neighborhood of p. It follows that each of the functions $f + g$, fg, and f/g is holomorphic on a neighborhood of p or has a removable singularity or a pole at p. Consequently, any sum, product, or quotient (provided the denominator does not vanish identically on any open set) of meromorphic functions is a meromorphic function, provided we holomorphically extend the resulting function over any removable singularities. In other words, for any connected open subset Ω of a complex 1-manifold, $\mathcal{M}(\Omega)$ is a field.

2. Suppose X is a complex 1-manifold, (U, z) is a local holomorphic coordinate neighborhood of a point $p \in X$, $f \in \mathcal{O}(X \setminus \{p\})$, and $f = \sum_{n=-\infty}^{\infty} c_n (z - z(p))^n$ is the corresponding Laurent series representation of f about p provided by Theorem 1.3.6. Then f has a pole of order $m \in \mathbb{Z}_{>0}$ at p if and only if $c_n = 0$ for all $n < -m$ and $c_{-m} \neq 0$.

Exercises for Sect. 2.2

2.2.1 Prove that a function f on an open subset Ω of a complex 1-manifold X is holomorphic if and only if for each point $p \in \Omega$, *there exists* a local holomorphic chart (U, Φ, U') in X such that $p \in U$ and $f \circ \Phi^{-1} \in \mathcal{O}(\Phi(\Omega \cap U))$. Prove also that a continuous mapping of complex 1-manifolds $\Psi \colon X \to Y$ is holomorphic if and only if for each point $p \in X$, *there exists* a local holomorphic chart (U, Φ, U') in Y such that $\Psi(p) \in U$ and $\Phi \circ \Psi \in \mathcal{O}(\Psi^{-1}(U))$.

2.2.2 Prove that biholomorphic equivalence of complex 1-manifolds is an equivalence relation.

2.2.3 Prove that any sum or product of holomorphic functions on a complex 1-manifold is holomorphic, and any composition of holomorphic mappings of complex 1-manifolds is holomorphic.

2.2.4 Prove the second statement in part (a) of Theorem 2.2.2 (i.e., if $\Phi, \Psi \colon X \to Y$ are distinct holomorphic mappings of Riemann surfaces, then $\{x \in X \mid \Phi(x) = \Psi(x)\}$ is discrete in X). Also prove parts (b)–(d) of Theorem 2.2.2.

2.2.5 Prove part (b) Lemma 2.2.3.

2.2.6 *Complex manifolds.* A *holomorphic atlas of dimension n* on a Hausdorff space X is an atlas in which each element is a homeomorphism $\Phi \colon U \to U'$ of an open set $U \subset X$ onto an open set $U' \subset \mathbb{C}^n$ and the coordinate transformations are holomorphic mappings (see Exercise 1.6.4 for the definition). Two holomorphic atlases of dimension n are *holomorphically equivalent* if their union is a holomorphic atlas. A *complex manifold of dimension n* is a Hausdorff space X together with an equivalence class of n-dimensional holomorphic atlases (i.e., an n-dimensional *holomorphic structure*). The elements of any atlas in the holomorphic structure are called *local holomorphic charts*. A continuous mapping of complex manifolds is *holomorphic* if its representation in local holomorphic charts is holomorphic. Prove that the Cartesian product of a pair of Riemann surfaces has a natural 2-dimensional holomorphic structure.

2.2.7 Prove that no two of the Riemann surfaces $\mathbb{C}$, $\mathbb{P}^1$, and $\Delta \equiv \Delta(0; 1)$ are biholomorphic.

2.2.8 Prove the following:

(a) *Liouville's theorem.* Every bounded entire function is constant.

Hint. Apply Theorem 1.2.10 near ∞ in $\mathbb{P}^1 \supset \mathbb{C}$.

(b) *The fundamental theorem of algebra* (*Gauss*). Every nonconstant complex polynomial has a zero.

Hint. Given a nonvanishing complex polynomial function g, consider the holomorphic function $1/g$.

2.3 Holomorphic Attachment

One may produce infinitely many examples of Riemann surfaces by holomorphic attachment. In fact, one goal of this book is a proof (appearing in Chap. 5) that *every* Riemann surface may be obtained by holomorphic attachment of tubes to a domain in the Riemann sphere $\mathbb{P}^1$. Holomorphic attachment is not required until Chap. 5, so a reader who prefers to skip this section and move on to other topics in this chapter may do so without fear of missing a required concept. It will be convenient to have holomorphic attachment formulated as follows:

Proposition 2.3.1 (Holomorphic attachment) *Let X and X' be complex 1-manifolds, and let $\Psi\colon G' \to G$ be a biholomorphism of an open set $G' \subset X'$ onto an open set $G \subset X$ such that $\Psi^{-1}(K \cap G)$ is closed in X' for every compact set $K \subset X$ (i.e., $\Psi(x) \to \infty$ in the one-point compactification of X as $x \to p \in \partial G'$) and $\Psi(K' \cap G')$ is closed in X for every compact set $K' \subset X'$ (i.e., $\Psi^{-1}(x) \to \infty$ in the one-point compactification of X' as $x \to p \in \partial G$). Let $\iota\colon X \hookrightarrow X \sqcup X'$ and $\iota'\colon X' \hookrightarrow X \sqcup X'$ be the inclusion mappings of X and X' into the disjoint union $X \sqcup X'$, let $\sim$ be the equivalence relation in the disjoint union $X \sqcup X'$ determined by $\iota'(x) \sim \iota(\Psi(x))$ for every point $x \in G'$, and let*

$$\Pi\colon X \sqcup X' \to Y = X \cup_\Psi X' \equiv (X \sqcup X')/\sim$$

be the associated quotient space. Then there is a unique 1-dimensional holomorphic structure on Y with respect to which $\Pi \circ \iota$ and $\Pi \circ \iota'$ are biholomorphisms of X and X', respectively, onto open subsets of Y (equivalently, there is a unique holomorphic structure on Y with respect to which Π is a local biholomorphism).

The proof is left to the reader (see Exercise 2.3.1).

Definition 2.3.2 For $X \supset G$, $X' \supset G'$, and $\Psi\colon G' \to G$ as in Proposition 2.3.1, the complex 1-manifold $X \cup_\Psi X'$ is called the *holomorphic attachment* of X and X' along Ψ.

Remarks 1. It is customary and convenient to identify X and X' with their (disjoint) images in $X \sqcup X'$. However, although it is customary to leave out any explicit mention of the inclusion maps ι and ι', we will often mention the inclusion maps when considering specific mappings of the above spaces. This will allow us to avoid any danger of confusion (for example, there is danger of confusion whenever $X = X'$).

2. The natural identification of $X \sqcup X'$ with $X' \sqcup X$ gives a natural identification of $X \cup_\Psi X'$ with $X' \cup_{\Psi^{-1}} X$.

In applications, usually one first removes a set and then holomorphically attaches a new set of a desired type. For example, several key arguments in Chap. 5 will require that we replace sets with caps (i.e., disks) or tubes (i.e., annuli). For now, we consider the following:

Example 2.3.3 (Holomorphic attachment of a tube) Let Y be a complex 1-manifold; let $R_0, R_1 > 1$; let $\{(D_\nu, \Phi_\nu, \Delta(0; R_\nu))\}_{\nu\in\{0,1\}}$ be disjoint local holomorphic charts in Y; let $A'_\nu \equiv \Phi_\nu^{-1}(\Delta(0; 1, R_\nu)) \subset D_\nu$ for $\nu = 0, 1$; let $T \equiv \Delta(0; 1/R_0, R_1)$; and

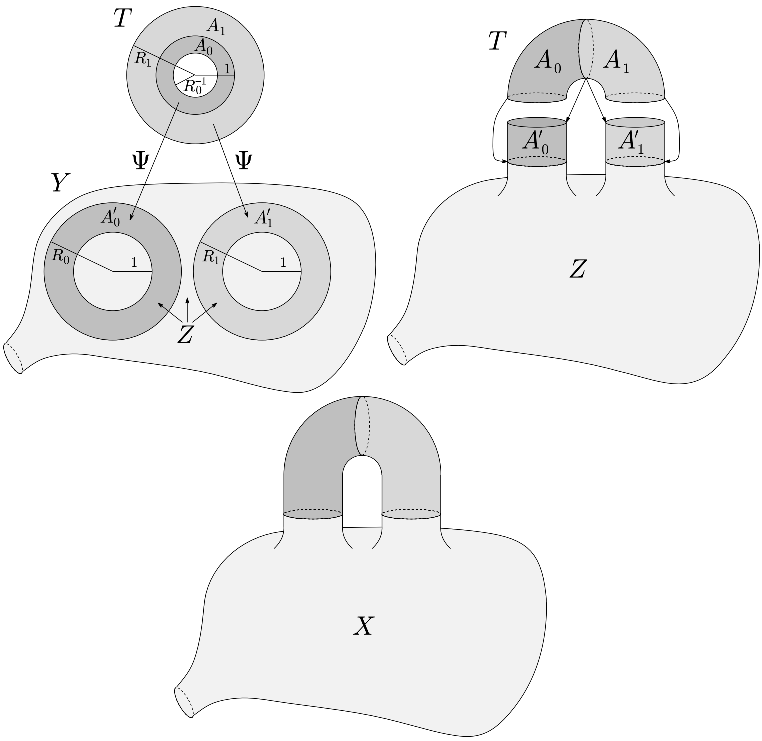

Fig. 2.4 Holomorphic attachment of a tube

let $A_0 \equiv \Delta(0; 1/R_0, 1) \subset T$ and $A_1 \equiv \Delta(0; 1, R_1) \subset T$. Thus we get a biholomorphism $\Psi\colon A_0 \cup A_1 \to A_0' \cup A_1'$ given by

$$\Psi(z) = \begin{cases} \Phi_0^{-1}(1/z) \in A_0' & \text{if } z \in A_0, \\ \Phi_1^{-1}(z) \in A_1' & \text{if } z \in A_1. \end{cases}$$

For $Z \equiv Y \setminus [\Phi_0^{-1}(\overline{\Delta(0; 1)}) \cup \Phi_1^{-1}(\overline{\Delta(0; 1)})] \supset A_0' \cup A_1'$, $\Psi^{-1}(K' \cap (A_0' \cap A_1'))$ is closed in T for every compact set $K' \subset Z$, and $\Psi(K \cap (A_0 \cup A_1))$ is closed in Z for every compact set $K \subset T$. We may therefore form the holomorphic attachment $X \equiv Z \cup_\Psi T = Z \sqcup T/{\sim}$, where for $p \in A_0'$ and $z \in A_0$, the images p_0 of p and z_0 of z in $Z \sqcup T$ under the inclusions $A_0', A_0 \hookrightarrow Z \sqcup T$ satisfy $p_0 \sim z_0$ if and only if $z \cdot \Phi_0(p) = 1$, and for $p \in A_1'$ and $z \in A_1$, the respective images p_0 and z_0 satisfy $p_0 \sim z_0$ if and only if $z = \Phi_1(p)$ (see Fig. 2.4). In other words, X is a complex 1-manifold obtained by removing the unit disks in each of the coordinate disks D_0 and D_1 and gluing in a tube (i.e., an annulus) T (equivalently, the boundaries of the unit disks are glued together). We call X the *complex* 1*-manifold obtained by holomorphic attachment of the tube T to Y at the coordinate disks* $\{(D_\nu, \Phi_\nu, \Delta(0; R_\nu))\}_{\nu\in\{0,1\}}$ (or simply *at* $\{D_\nu\}_{\nu\in\{0,1\}}$). This is a slight abuse of language, since actually we first removed the set $\Phi_0^{-1}(\overline{\Delta(0; 1)}) \cup \Phi_1^{-1}(\overline{\Delta(0; 1)})$ before performing the attachment.

Observe that if Y is connected, then X is connected; and if Y is compact, then X is compact. If X is compact, then Y is compact. However, X may be connected even if Y is not connected (see Exercise 2.3.2).

Fixing R_ν^* with $1 < R_\nu^* \leq R_\nu$ for $\nu = 0, 1$, we may form the holomorphic attachment $X^* = Z \sqcup T^*/\sim$ of the tube $T^* \equiv \Delta(0; 1/R_0^*, R_1^*) \subset T$ to Y at the coordinate disks $\{(D_\nu^* \equiv \Phi_\nu^{-1}(\Delta(0; R_\nu^*)), \Phi_\nu^* \equiv \Phi_\nu \restriction_{D_\nu^*}, \Delta(0; R_\nu^*))\}_{\nu \in 0,1}$. The natural inclusion $Z \sqcup T^* \subset Z \sqcup T$ then descends to a natural biholomorphism $X^* \xrightarrow{\cong} X$ (see Exercise 2.3.4), so we may identify X^* with X. In particular, we may always choose $R_0, R_1 \in (1, \infty)$ to be equal and arbitrarily close to 1.

By holomorphically attaching tubes, one obtains examples of complex 1-manifolds with complicated topology. In fact, one of the main goals of Chap. 5 will be to show that *every* Riemann surface may be obtained by holomorphic attachment of tubes at elements of a locally finite family of disjoint coordinate disks in a domain in $\mathbb{P}^1$ (see Theorem 5.13.1 and Theorem 5.14.1).

Exercises for Sect. 2.3

2.3.1 Prove Proposition 2.3.1.

2.3.2 Let X be a complex 1-manifold obtained by holomorphically attaching a tube to a complex 1-manifold Y as in Example 2.3.3. Prove that X is connected if Y is connected. Give an example that shows that X may be connected even if Y is not.

2.3.3 Write out a specific example of holomorphic attachment of a tube to the Riemann sphere (observe that the resulting Riemann surface appears to have the topological type of a torus).

2.3.4 In the notation of Example 2.3.3, verify that the natural inclusion of $Z \sqcup T^*$ into $Z \sqcup T$ descends to a biholomorphism of X^* onto X.

2.3.5 Let X and X' be complex 1-manifolds, let $\Psi\colon G' \to G$ be a biholomorphism of an open set $G' \subset X'$ onto an open set $G \subset X$, let $\iota\colon X \hookrightarrow X \sqcup X'$ and $\iota'\colon X' \hookrightarrow X \sqcup X'$ be the inclusion mappings of X and X' into the disjoint union $X \sqcup X'$, let $\sim$ be the equivalence relation in the disjoint union $X \sqcup X'$ determined by $\iota'(x) \sim \iota(\Psi(x))$ for every point $x \in G'$, and let $Y \equiv (X \sqcup X')/\sim$ be the associated quotient space. Prove that if there exists a compact set $K \subset X$ for which $\Psi^{-1}(K \cap G)$ is *not* closed in X', then Y is *not* Hausdorff.

2.4 Holomorphic Tangent Bundle

A complex 1-manifold X has an underlying real 2-dimensional $\mathcal{C}^\infty$ structure, and therefore a tangent bundle and a complexified tangent bundle

$$\Pi_{TX}\colon TX \to X \quad \text{and} \quad \Pi_{(TX)_{\mathbb{C}}}\colon (TX)_{\mathbb{C}} \to X,$$

respectively (see Sect. 9.4). We recall that for $p \in X$, a tangent vector $v \in (T_pX)_{\mathbb{C}}$ is a linear derivation on the germs of $\mathcal{C}^\infty$ functions at p; that is, for every pair of $\mathcal{C}^\infty$ functions f, g on a neighborhood of p, we have $v(fg) = v(f) \cdot g(p) + f(p) \cdot v(g) \in \mathbb{C}$. Given a local holomorphic coordinate neighborhood $(U, z = x + iy)$, the vector fields $\partial/\partial x$ and $\partial/\partial y$ form a basis for the tangent space at each point in U. Furthermore, the corresponding complex vector fields $\partial/\partial z$ and $\partial/\partial \bar{z}$ as in Chap. 1 form a different complex basis for the complexified tangent space at each point, and their spans yield a natural decomposition into a sum of 1-dimensional vector spaces. Moreover, a $\mathcal{C}^1$ function g on U is holomorphic if and only if $\partial g/\partial \bar{z} = \overline{\partial \bar{g}/\partial z} = 0$. Similarly, the differentials dz and $d\bar{z}$ form a dual basis that yields a decomposition of the complexified cotangent space at each point. The precise definitions and verifications appear below.

Definition 2.4.1 Let X be a complex 1-manifold.

(a) For each point $p \in X$, a tangent vector $v \in (T_pX)_{\mathbb{C}}$ is of *type* $(1,0)$ (of *type* $(0,1)$) if $d\bar{f}(v) = v(\bar{f}) = 0$ (respectively, $df(v) = v(f) = 0$) for every holomorphic function f on a neighborhood of p. The subspace $(T_pX)^{1,0}$ of $(T_pX)_{\mathbb{C}}$ formed by the tangent vectors at p of type $(1,0)$ is called the *holomorphic tangent space* (or $(1,0)$-*tangent space*) at p. The subspace $(T_pX)^{0,1} \subset (T_pX)_{\mathbb{C}}$ formed by the tangent vectors at p of type $(0,1)$ is called the $(0,1)$-*tangent space* at p. The spaces

$$\Pi_{(TX)^{1,0}} = \Pi_{(TX)_{\mathbb{C}}}\restriction_{(TX)^{1,0}} : (TX)^{1,0} \equiv \bigcup_{p \in X} (T_pX)^{1,0} \to X$$

and

$$\Pi_{(TX)^{0,1}} = \Pi_{(TX)_{\mathbb{C}}}\restriction_{(TX)^{0,1}} : (TX)^{0,1} \equiv \bigcup_{p \in X} (T_pX)^{0,1} \to X$$

are called the *holomorphic tangent bundle* (or $(1,0)$-*tangent bundle*) and $(0,1)$-*tangent bundle*, respectively.

(b) For each $p \in X$, an element $\alpha \in (T_p^*X)_{\mathbb{C}}$ is of *type* $(1,0)$ (of *type* $(0,1)$) if $\alpha(v) = 0$ for every tangent vector $v \in (T_pX)^{0,1}$ (respectively, $v \in (T_pX)^{1,0}$). The subspace $(T_p^*X)^{1,0} \subset (T_p^*X)_{\mathbb{C}}$ formed by the elements of type $(1,0)$ is called the *holomorphic cotangent space* (or $(1,0)$-*cotangent space*) at p. The subspace $(T_p^*X)^{0,1} \subset (T_p^*X)_{\mathbb{C}}$ formed by the elements of type $(0,1)$ is called the $(0,1)$-*cotangent space* at p. The spaces

$$\Pi_{(T^*X)^{1,0}} = \Pi_{(T^*X)_{\mathbb{C}}}\restriction_{(T^*X)^{1,0}} : (T^*X)^{1,0} \equiv \bigcup_{p \in X} (T_p^*X)^{1,0} \to X$$

and

$$\Pi_{(T^*X)^{0,1}} = \Pi_{(T^*X)_{\mathbb{C}}}\restriction_{(T^*X)^{0,1}} : (T^*X)^{0,1} \equiv \bigcup_{p \in X} (T_p^*X)^{0,1} \to X$$

are called the *holomorphic cotangent bundle* (or $(1,0)$-*cotangent bundle*) and $(0,1)$-*cotangent bundle*, respectively. The holomorphic cotangent bundle

$(T^*X)^{1,0}$ is also called the *canonical line bundle* (or *canonical bundle*) and is also denoted by $\Pi_{K_X} : K_X \to X$.

(c) Given a local holomorphic coordinate neighborhood $(U, \Phi = z = x + iy)$ in X (with $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$), we call the vector fields

$$\frac{\partial}{\partial z} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \overline{\partial/\partial z}$$

the *partial derivative operators with respect to z and $\bar{z}$*, respectively. In other words, denoting the standard complex coordinate on $\mathbb{C}$ by w, we have, for any suitable function f,

$$\frac{\partial f}{\partial z} = \frac{\partial(f \circ \Phi^{-1})}{\partial w} \circ \Phi \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{\partial(f \circ \Phi^{-1})}{\partial \bar{w}} \circ \Phi.$$

Proposition 2.4.2 *Let X be a complex 1-manifold.*

(a) *For each point $p \in X$, we have direct sum decompositions*

$$(T_pX)_{\mathbb{C}} = (T_pX)^{1,0} \oplus (T_pX)^{0,1} \quad \textit{and} \quad (T_p^*X)_{\mathbb{C}} = (T_p^*X)^{1,0} \oplus (T_p^*X)^{0,1};$$

and we have isomorphisms

$$(T_p^*X)^{1,0} \xrightarrow{\cong} ((T_pX)^{1,0})^* \quad \textit{and} \quad (T_p^*X)^{0,1} \xrightarrow{\cong} ((T_pX)^{0,1})^*$$

given by $\alpha \mapsto \alpha\restriction_{(T_pX)^{1,0}}$ and $\alpha \mapsto \alpha\restriction_{(T_pX)^{0,1}}$, respectively.

(b) *Conjugation gives $\overline{(TX)^{1,0}} = (TX)^{0,1}$ and $\overline{(T^*X)^{1,0}} = (T^*X)^{0,1}$.*

(c) *For every local holomorphic coordinate neighborhood $(U, z = x + iy)$ of a point $p \in X$, we have*

$$(T_pX)^{1,0} = \mathbb{C} \cdot (\partial/\partial z)_p, \qquad (T_pX)^{0,1} = \mathbb{C} \cdot (\partial/\partial \bar{z})_p,$$
$$(T_p^*X)^{1,0} = \mathbb{C} \cdot (dz)_p, \qquad (T_p^*X)^{0,1} = \mathbb{C} \cdot (d\bar{z})_p,$$

and

$$dz((\partial/\partial z)_p) = 1, \qquad d\bar{z}((\partial/\partial \bar{z})_p) = 1,$$
$$dz((\partial/\partial \bar{z})_p) = 0, \qquad d\bar{z}((\partial/\partial z)_p) = 0.$$

Moreover, for every complex number $\zeta = a + ib$ with $a, b \in \mathbb{R}$, we have

$$a\left(\frac{\partial}{\partial x}\right)_p + b\left(\frac{\partial}{\partial y}\right)_p = \zeta\left(\frac{\partial}{\partial z}\right)_p + \bar{\zeta}\left(\frac{\partial}{\partial \bar{z}}\right)_p$$

and

$$a(dx)_p + b(dy)_p = \frac{\bar{\zeta}}{2}(dz)_p + \frac{\zeta}{2}(d\bar{z})_p.$$

(d) *A $\mathcal{C}^1$ function f on an open set $\Omega \subset X$ is holomorphic if and only if for each point $p \in \Omega$, $\partial f/\partial \bar{z} \equiv 0$ on $U \cap \Omega$ for some (equivalently, for every) local holomorphic coordinate neighborhood (U, z) of p. If $\Psi \colon X \to Y$ is a $\mathcal{C}^1$ mapping of X into a complex 1-manifold Y and $(r, s) \in \{(1,0), (0,1)\}$, then the following are equivalent:*
  (i) *Ψ is holomorphic.*
  (ii) $\Psi_*(TX)^{r,s} \subset (TY)^{r,s}$.
  (iii) $\Psi^*(T^*Y)^{r,s} \subset (T^*X)^{r,s}$.

Proof Let $(U, \Phi = z, U')$ be a local holomorphic chart in X and let $p \in U$. A function $f \in \mathcal{C}^1(U)$ is holomorphic if and only if $f \circ \Phi^{-1} \in \mathcal{O}(U')$. Let w denote the standard complex coordinate on $\mathbb{C}$. Since

$$\frac{\partial f}{\partial \bar{z}} \circ \Phi^{-1} = \frac{\partial (f \circ \Phi^{-1})}{\partial \bar{w}},$$

we see that f is holomorphic if and only if $\overline{\partial \bar{f}/\partial z} = \partial f/\partial \bar{z} \equiv 0$, and hence that $(\partial/\partial z)_p \in (T_pX)^{1,0} \setminus \{0\}$ and $(\partial/\partial \bar{z})_p \in (T_pX)^{0,1} \setminus \{0\}$. Moreover,

$$\left(\frac{\partial}{\partial x}\right)_p = \left(\frac{\partial}{\partial z}\right)_p + \left(\frac{\partial}{\partial \bar{z}}\right)_p \quad \text{and} \quad \left(\frac{\partial}{\partial y}\right)_p = i\left(\frac{\partial}{\partial z}\right)_p - i\left(\frac{\partial}{\partial \bar{z}}\right)_p.$$

Therefore, since $\dim_{\mathbb{C}}(T_pX)_{\mathbb{C}} = 2$, we have the direct sum decomposition

$$(T_pX)_{\mathbb{C}} = \mathbb{C} \cdot \left(\frac{\partial}{\partial z}\right)_p \oplus \mathbb{C} \cdot \left(\frac{\partial}{\partial \bar{z}}\right)_p = (T_pX)^{1,0} \oplus (T_pX)^{0,1}.$$

We also have

$$dz\left(\frac{\partial}{\partial z}\right) = (dx + i\,dy)\left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}\right) = \frac{1}{2} - \frac{i}{2} \cdot 0 + i \cdot \frac{1}{2} \cdot 0 - i \cdot \frac{i}{2} \cdot 1 = 1.$$

A similar computation gives $dz(\partial/\partial \bar{z}) = 0$, and it follows that

$$d\bar{z}(\partial/\partial \bar{z}) = \overline{dz(\partial/\partial z)} = 1 \quad \text{and} \quad d\bar{z}(\partial/\partial z) = \overline{dz(\partial/\partial \bar{z})} = 0.$$

Thus $(dz)_p \in (T_p^*X)^{1,0}$ and $(d\bar{z})_p \in (T_p^*X)^{0,1}$ form the basis of $(T_p^*X)_{\mathbb{C}}$ which is dual to the basis $(\partial/\partial z)_p, (\partial/\partial \bar{z})_p$. Parts (a)–(c) now follow easily.

If Ψ is a $\mathcal{C}^1$ mapping of X into a complex 1-manifold Y and (V, ζ) is a local holomorphic coordinate neighborhood of $\Psi(p)$ in Y, then

$$\begin{aligned} \frac{\partial(\zeta \circ \Psi)}{\partial \bar{z}} &= d(\zeta \circ \Psi)(\partial/\partial \bar{z}) = d\zeta(\Psi_*(\partial/\partial \bar{z})) = (\Psi^* d\zeta)(\partial/\partial \bar{z}) \\ &= \overline{d(\bar{\zeta} \circ \Psi)(\partial/\partial z)} = \overline{d\bar{\zeta}(\Psi_*(\partial/\partial z))} = \overline{(\Psi^* d\bar{\zeta})(\partial/\partial z)}. \end{aligned}$$

Part (c) now gives the equivalence of (i)–(iii) in (d). □

Remarks 1. It follows from the above proposition that if $(U, z = x + iy)$ is a local holomorphic coordinate neighborhood in a complex 1-manifold X, $p \in U$, $v \in (T_pX)_{\mathbb{C}}$, $\alpha \in (T_p^*X)_{\mathbb{C}}$, and f is a $\mathcal{C}^1$ function on a neighborhood of p, then

$$v = dx(v)\left(\frac{\partial}{\partial x}\right)_p + dy(v)\left(\frac{\partial}{\partial y}\right)_p = dz(v)\left(\frac{\partial}{\partial z}\right)_p + d\bar{z}(v)\left(\frac{\partial}{\partial \bar{z}}\right)_p,$$

$$\begin{aligned}\alpha &= \alpha((\partial/\partial x)_p)(dx)_p + \alpha((\partial/\partial y)_p)(dy)_p \\ &= \alpha((\partial/\partial z)_p)(dz)_p + \alpha((\partial/\partial \bar{z})_p)(d\bar{z})_p,\end{aligned}$$

$$(df)_p = \frac{\partial f}{\partial z}(p)(dz)_p + \frac{\partial f}{\partial \bar{z}}(p)(d\bar{z})_p.$$

In particular, f is holomorphic if and only if $df = (\partial f/\partial z) \cdot dz$. If $\Psi \colon X \to Y$ is a holomorphic mapping into a Riemann surface Y, (V, w) is a local holomorphic coordinate neighborhood of $\Psi(p)$ in Y, and $g = w \circ \Psi$, then for all $a, b \in \mathbb{C}$,

$$(\Psi_*)_p\left[a\left(\frac{\partial}{\partial z}\right)_p + b\left(\frac{\partial}{\partial \bar{z}}\right)_p\right] = a\frac{\partial g}{\partial z}(p)\left(\frac{\partial}{\partial w}\right)_{\Psi(p)} + b\,\overline{\frac{\partial g}{\partial z}(p)}\left(\frac{\partial}{\partial \bar{w}}\right)_{\Psi(p)}.$$

Consequently, if any one of the following linear mappings is nontrivial (i.e., not the zero mapping), then all of the mappings are isomorphisms:

$$\begin{array}{ll} (\Psi_*)_p \colon (T_pX)_{\mathbb{C}} \to (T_{\Phi(p)}Y)_{\mathbb{C}}, & (\Psi_*)_p \colon T_pX \to T_{\Phi(p)}Y, \\ (\Psi_*)_p \colon (T_pX)^{1,0} \to (T_{\Phi(p)}Y)^{1,0}, & (\Psi_*)_p \colon (T_pX)^{0,1} \to (T_{\Phi(p)}Y)^{0,1}, \\ (dg)_p \colon (T_pX)^{1,0} \to \mathbb{C}. & \end{array}$$

2. We express the decomposition of the complexified tangent and cotangent bundles by writing $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ and $(T^*X)_{\mathbb{C}} = (T^*X)^{1,0} \oplus (T^*X)^{0,1}$, and for each pair $(r, s) \in \{(1, 0), (0, 1)\}$, we let

$$\mathcal{P}^{r,s} \colon (TX)_{\mathbb{C}} \to (TX)^{r,s} \quad \text{and} \quad \mathcal{P}^{r,s} \colon (T^*X)_{\mathbb{C}} \to (T^*X)^{r,s}$$

denote the mappings for which the restrictions

$$\mathcal{P}^{r,s}\restriction_{(T_pX)_{\mathbb{C}}} \colon (T_pX)_{\mathbb{C}} = (T_pX)^{1,0} \oplus (T_pX)^{0,1} \to (T_pX)^{r,s}$$

and

$$\mathcal{P}^{r,s}\restriction_{(T_p^*X)_{\mathbb{C}}} \colon (T_p^*X)_{\mathbb{C}} = (T_p^*X)^{1,0} \oplus (T_p^*X)^{0,1} \to (T_p^*X)^{r,s}$$

are the corresponding vector space projections for each point $p \in X$. For any element ξ of $(T_pX)_{\mathbb{C}}$ or $(T_p^*X)_{\mathbb{C}}$, we call $\mathcal{P}^{r,s}\xi$ the (r, s) *part* of ξ.

3. The holomorphic tangent bundle $(TX)^{1,0}$ and cotangent bundle $(T^*X)^{1,0}$ are examples of holomorphic line bundles (see Example 3.1.4). The (0, 1) tangent bundle $(TX)^{0,1}$ and cotangent bundle $(T^*X)^{0,1}$ are examples of $\mathcal{C}^\infty$ line bundles.

4. A reader familiar with vector bundles will recognize the decomposition $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ as a decomposition of the $\mathcal{C}^\infty$ vector bundle $(TX)_{\mathbb{C}}$ into a sum of $\mathcal{C}^\infty$ subbundles (as is the decomposition of $(T^*X)_{\mathbb{C}}$).

5. For $(r,s) \in \{(1,0),(0,1)\}$ and for any open subset Ω of a complex 1-manifold X with inclusion mapping $\iota\colon \Omega \hookrightarrow X$, we identify $(T\Omega)^{r,s}$ and $(T^*\Omega)^{r,s}$ with the sets $\Pi^{-1}_{(TX)^{r,s}}(\Omega) \subset (TX)^{r,s}$ and $\Pi^{-1}_{(T^*X)^{r,s}}(\Omega) \subset (T^*X)^{r,s}$, respectively, under the bijections $\iota_*\colon (T\Omega)^{r,s} \to \Pi^{-1}_{(TX)^{r,s}}(\Omega)$ and $\iota^*\colon \Pi^{-1}_{(T^*X)^{r,s}}(\Omega) \to (T^*\Omega)^{r,s}$, respectively.

Guided by Definition 9.4.5, we make the following definition:

Definition 2.4.3 Let X be a complex 1-manifold.

(a) The *coefficient functions* (or simply the *coefficients*) of a vector field v on a set $S \subset X$ with respect to (or in) a local holomorphic coordinate neighborhood (U,z) are the functions $dz(v) = v(z)$ and $d\bar{z}(v) = v(\bar{z})$ on $S \cap U$.
(b) We call a vector field v of type $(1,0)$ on an open set $\Omega \subset X$ (that is, v_p is of type $(1,0)$ for each point p) a *holomorphic vector field* if the coefficient function $f = dz(v) = v(z)\colon \Omega \cap U \to \mathbb{C}$ in every local holomorphic coordinate neighborhood (U,z) is holomorphic (we have $d\bar{z}(v) = v(\bar{z}) = 0$, since v is of type $(1,0)$); that is, $v = f \cdot (\partial/\partial z)$ on $\Omega \cap U$ for some function $f \in \mathcal{O}(\Omega \cap U)$.

Remark A vector field on a set $S \subset X$ is of class $\mathcal{C}^k$ if and only if its coefficient functions with respect to every local holomorphic coordinate neighborhood (or equivalently, for each point $p \in S$, with respect to *some* local holomorphic coordinate neighborhood of p) are of class $\mathcal{C}^k$ (see Exercise 2.4.1).

We close this section with the observation that the holomorphic inverse function theorem for domains in the plane (Theorem 1.5.1) gives the following analogue for Riemann surfaces:

Theorem 2.4.4 (Holomorphic inverse function theorem for Riemann surfaces) *Let $\Phi\colon X \to Y$ be a holomorphic mapping of Riemann surfaces.*

(a) *If $p \in X$ and $(\Phi_*)_p \neq 0$, then Φ maps some neighborhood of p biholomorphically onto a neighborhood of $q \equiv \Phi(p)$. In particular, if f is a holomorphic function on a neighborhood of a point $p \in X$ and $(df)_p \neq 0$, then $(U, f\restriction_U)$ is a local holomorphic coordinate neighborhood for some neighborhood U of p.*
(b) *If Φ is one-to-one, then Φ maps X biholomorphically onto an open subset $\Phi(X)$ of Y.*

The proof is left to the reader (see Exercise 2.4.2).

Corollary 2.4.5 *Let f be a holomorphic function on a Riemann surface X, let r be a positive regular value* (*see Definition* 9.4.6) *of the function $\rho \equiv |f|$ ($|f|$ is of class $\mathcal{C}^\infty$ on the complement of its zero set*), *and let $\Omega \equiv \{p \in X \mid |f(p)| < r\}$. Then every point $p \in \partial\Omega$ admits a local holomorphic coordinate neighborhood* $(U, z = x + iy)$ *in which*

$$\Omega \cap U = \{q \in U \mid x(q) < 0\}.$$

Remark In particular, Ω is a $\mathcal{C}^\infty$ open set (see Definition 9.7.14).

Proof We have $2\rho\, d\rho = d\rho^2 = \bar{f}\, df + f\, d\bar{f}$ on $X \setminus f^{-1}(0)$, and hence, given a point $p \in \partial\Omega$, we have $(df)_p \neq 0$. Therefore, by Theorem 2.4.4, there exists a local holomorphic chart of the form $(U, f\restriction_U, U')$ with $p \in U$. Moreover, we may choose U and U' so small that there exists a holomorphic branch g of the logarithmic function on U' (see Example 1.6.2). Since g is a biholomorphism (with holomorphic inverse function $\zeta \mapsto e^\zeta$), we get a local holomorphic coordinate neighborhood

$$(U, z = x + iy \equiv -\log r + g \circ f)$$

in which $x = \operatorname{Re} z = \log |f/r|\restriction_U$. In particular, $\Omega \cap U = \{q \in U \mid x(q) < 0\}$. $\square$

Exercises for Sect. 2.4

2.4.1 Let X be a Riemann surface.
  (a) For a vector field v on X and for $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, prove that the following are equivalent (cf. Exercise 9.4.2):
    (i) The vector field v is of class $\mathcal{C}^k$.
    (ii) The coefficients of v in every local holomorphic coordinate neighborhood are of class $\mathcal{C}^k$.
    (iii) For every point in X, there exists a local holomorphic coordinate neighborhood with respect to which the coefficients of v are of class $\mathcal{C}^k$.
    (iv) The $(1, 0)$ and $(0, 1)$ parts of v are of class $\mathcal{C}^k$.
  (b) For a vector field v of type $(1, 0)$ on X, prove that the following are equivalent:
    (i) The vector field v is holomorphic.
    (ii) The coefficient $dz(v)$ of v in some local holomorphic coordinate neighborhood (U, z) of each point in X is holomorphic.
    (iii) For every holomorphic function f on an open set $U \subset X$, the function $df(v)\colon\ p \mapsto df(v_p) = v_p(f)$ is holomorphic.
  (c) Prove that any sum of holomorphic vector fields, and any product of a holomorphic function and a holomorphic vector field, are holomorphic (in particular, the set of holomorphic vector fields on X is a complex vector space).

2.4.2 Prove Theorem 2.4.4.

2.4.3 Let X be a Riemann surface.
  (a) Prove that there is a unique 2-dimensional holomorphic structure on $(TX)^{1,0}$ such that for each local holomorphic chart $(U, \Phi = z, U')$ in X, the triple

$$\left(\Pi^{-1}_{(TX)^{1,0}}(U), (\Phi \circ \Pi_{(TX)^{1,0}}, dz), U' \times \mathbb{C}\right)$$

  is a local holomorphic chart in $(TX)^{1,0}$ (see Exercise 2.2.6 for the definition of a complex manifold, and cf. Exercise 9.4.3).

(b) Prove that a vector field v of type $(1,0)$ on X is holomorphic if and only if the mapping $v\colon X \to (TX)^{1,0}$ is holomorphic as a mapping of complex manifolds (cf. Exercise 9.4.5).

(c) Prove that there is a unique 2-dimensional holomorphic structure on $(T^*X)^{1,0}$ such that for each local holomorphic chart $(U, \Phi = z, U')$ in X, the triple

$$\left(\Pi^{-1}_{(T^*X)^{1,0}}(U), \Psi, U' \times \mathbb{C}\right),$$

where the map $\Psi\colon \Pi^{-1}_{(T^*X)^{1,0}}(U) \to U' \times \mathbb{C}$ is given by $\alpha \mapsto (z(p), \alpha(\frac{\partial}{\partial z}))$ for each point $p \in U$ and each element $\alpha \in (T_p^*X)^{1,0}$, is a local holomorphic chart in $(T^*X)^{1,0}$.

(d) Find (natural) $\mathcal{C}^\infty$ structures in $(TX)^{0,1}$ and $(T^*X)^{0,1}$.

2.5 Differential Forms on a Riemann Surface

We recall that a differential form α of degree r on a subset S of a $\mathcal{C}^\infty$ manifold M is a mapping of S into $\Lambda^r(T^*M)_{\mathbb{C}}$ with $\alpha_p \in \Lambda^r(T_p^*M)_{\mathbb{C}}$ for each point $p \in S$ (see Sect. 9.5). On a Riemann surface, the decomposition of the complexified tangent space into $(1,0)$ and $(0,1)$ parts induces a decomposition of differential forms.

Definition 2.5.1 Let X be a complex 1-manifold, and let $(r,s) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

(a) We set

$$\Lambda^{r,s}T^*X \equiv \begin{cases} \Lambda^0(T^*X)_{\mathbb{C}} = X \times \mathbb{C} & \text{if } (r,s) = (0,0), \\ (T^*X)^{r,s} & \text{if } (r,s) = (1,0) \text{ or } (0,1), \\ \Lambda^2(T^*X)_{\mathbb{C}} & \text{if } (r,s) = (1,1), \\ X \times \{0\} & \text{if } r \geq 2 \text{ or } s \geq 2, \end{cases}$$

we let $\Pi_{\Lambda^{r,s}T^*X}\colon \Lambda^{r,s}T^*X \to X$ be the corresponding projection, and we let $\Lambda^{r,s}T_p^*X = \Pi^{-1}_{\Lambda^{r,s}T^*X}(p)$ for each point $p \in X$.

(b) Elements of $\Lambda^{r,s}T^*X$ are said to be of *type* (r,s). A differential form α on a set $S \subset X$ is of *type* (r,s) if $\alpha_p \in \Lambda^{r,s}T_p^*X$ for each point $p \in S$. We also call α a *(differential)* (r,s)*-form*.

(c) For each open set $\Omega \subset X$, $\mathcal{E}^{r,s}(\Omega)$ denotes the vector space of $\mathcal{C}^\infty$ differential forms of type (r,s). The vector space of $\mathcal{C}^\infty$ (r,s)-forms with compact support in Ω is denoted by $\mathcal{D}^{r,s}(\Omega)$.

(d) Let α be a differential form of degree q on a set $S \subset X$. The *coefficient function(s)* (or simply the *coefficient(s)*) of α with respect to (or in) a local holomorphic coordinate neighborhood (U, z) is (are) the function(s) on $S \cap U$ given by

$$\begin{cases} \alpha & \text{if } q = 0, \\ a \equiv \alpha(\partial/\partial z), \qquad b \equiv \alpha(\partial/\partial \bar{z}) \quad (\text{i.e., } \alpha = a\,dz + b\,d\bar{z}) & \text{if } q = 1, \\ a \equiv \alpha(\partial/\partial z, \partial/\partial \bar{z}) = \dfrac{\alpha}{dz \wedge d\bar{z}} \quad (\text{i.e., } \alpha = a\,dz \wedge d\bar{z}) & \text{if } q = 2, \\ 0 & \text{if } q > 2. \end{cases}$$

Remarks 1. For each point $p \in X$, we have $\alpha \wedge \beta = \bar{\alpha} \wedge \bar{\beta} = 0$ for all $\alpha, \beta \in \Lambda^{1,0} T_p^* X$. Observe also that $\xi \wedge \zeta$ is of type $(r+t, s+u)$ if $\xi \in \Lambda^{r,s} T_p^* X$ and $\zeta \in \Lambda^{t,u} T_p^* X$.

2. On any local holomorphic coordinate neighborhood $(U, z = x + iy)$, we have $(i/2)\,dz \wedge d\bar{z} = dx \wedge dy$.

3. For all $r, s \in \mathbb{Z}_{\geq 0}$, conjugation gives $\overline{\Lambda^{r,s} T^* X} = \Lambda^{s,r} T^* X$; that is, $\bar{\alpha} \in \Lambda^{s,r} T^* X$ for each element $\alpha \in \Lambda^{r,s} T^* X$ (see Exercise 2.5.1).

4. Clearly, a differential form of type (r, s) is of degree $r + s$.

5. Exercises 9.5.4 and 2.4.3 provide a natural $\mathcal{C}^\infty$ structure on $\Lambda^{r,s} T^* X$.

Guided by Definition 9.5.2 and Definition 2.4.3, we make the following definition:

Definition 2.5.2 Let Ξ be an open subset of a complex 1-manifold X.

(a) A *holomorphic* 1*-form* on Ξ is a differential form θ of type $(1, 0)$ on Ξ such that for every local holomorphic coordinate neighborhood (U, z) in X, the coefficient function $\theta(\partial/\partial z) = \theta/dz$ is holomorphic on $\Xi \cap U$; that is, for some function $f \in \mathcal{O}(\Xi \cap U)$, we have $\theta = f\,dz$ on $\Xi \cap U$. The vector space of holomorphic 1-forms on Ξ is denoted by $\Omega_X(\Xi)$ or $\Omega(\Xi)$.

(b) A *meromorphic* 1*-form* on Ξ is a differential form θ of type $(1, 0)$ defined on the complement in Ξ of a discrete subset P such that for every local holomorphic coordinate neighborhood (U, z) in X, the coefficient function θ/dz is meromorphic on $\Xi \cap U$ with pole set $P \cap U$; that is, for some function $f \in \mathcal{M}(\Xi \cap U)$, we have $\theta = f\,dz$ on $\Xi \cap U \setminus f^{-1}(\infty) = \Xi \cap U \setminus P$ (we normally say simply $\theta = f\,dz$ on $\Xi \cap U$).

(c) A holomorphic (meromorphic) function on Ξ is also called a *holomorphic* (respectively, *meromorphic*) 0-form.

(d) A meromorphic 1-form θ on Ξ has a *zero* (a *pole*) of order ν at a point $p \in \Xi$ if for every local holomorphic coordinate neighborhood (U, z) of p, the meromorphic function θ/dz has a zero (respectively, a pole) of order ν at p. If $\nu = 1$, then we also say that θ has a *simple zero* (respectively, a *simple pole*) at p.

Remarks 1. A holomorphic 1-form is of class $\mathcal{C}^\infty$ (see Proposition 2.5.3 below).

2. For a nontrivial (i.e., not everywhere zero) meromorphic 1-form on a Riemann surface, the set of zeros (i.e., the set of points at which the 1-form is equal to 0) is discrete (see Exercise 2.5.2).

The proof of the following characterization of continuous, $\mathcal{C}^k$, holomorphic, and meromorphic differential forms is left to the reader (see Exercise 2.5.3):

Proposition 2.5.3 (Cf. Proposition 9.5.3 and Definition 9.7.12) *Let α be a differential form of degree r defined at points of a subset S of a Riemann surface X, let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, and let $d \in [1, \infty]$. Then*:

(a) *The following are equivalent*:
  (i) *The differential form α is continuous* (*of class $\mathcal{C}^k$, measurable, in L^d_{loc}*).
  (ii) *The coefficients of α in every local holomorphic coordinate neighborhood are continuous* (*respectively, of class $\mathcal{C}^k$, measurable, in L^d_{loc}*).
  (iii) *For every point in S, there exists a local holomorphic coordinate neighborhood with respect to which the coefficients of α are continuous* (*respectively, of class $\mathcal{C}^k$, measurable, in L^d_{loc}*).

  Moreover, for $r = 1$, α is continuous (*of class $\mathcal{C}^k$, measurable, in L^d_{loc}*) *if and only if the* $(1,0)$ *and* $(0,1)$ *parts of α are continuous* (*respectively, of class $\mathcal{C}^k$, measurable, in L^d_{loc}*).

(b) *For S open and α of type* $(1,0)$, *α is holomorphic if and only if for every point in S, there exists a local holomorphic coordinate neighborhood* (U, z) *with respect to which the coefficient $\alpha(\partial/\partial z) = \alpha/dz$ is holomorphic.*

(c) *Suppose α is of type* $(1,0)$ *and $S = \Xi \setminus P$ for some discrete set P in an open set $\Xi \subset X$. Then α is a meromorphic* 1*-form on Ξ with pole set P if and only if for each point $p \in \Xi$, there exists a local holomorphic coordinate neighborhood* (U, z) *of p such that the coefficient $\alpha(\partial/\partial z) = \alpha/dz$ is a meromorphic function on $\Xi \cap U$ with pole set $\Xi \cap U \cap P$. Moreover, if α is a meromorphic* 1*-form on Ξ, then α has a zero* (*a pole*) *of order ν at a point $p \in \Xi$ if and only if there exists a local holomorphic coordinate neighborhood* (U, z) *of p such that the coefficient α/dz has a zero* (*respectively, a pole*) *of order ν at p.*

The decomposition of $\Lambda^1(T^*X)_{\mathbb{C}}$ yields a decomposition of the exterior derivative operator d (see Definition 9.5.5):

Definition 2.5.4 Let X be a complex 1-manifold, let α be a differential form of degree r on an open subset Ω of X, and let $\mathcal{P}^{p,q} : \Lambda^1(T^*X)_{\mathbb{C}} \to \Lambda^{p,q}T^*X$ be the associated projection for each pair $(p,q) \in \{(1,0), (0,1)\}$.

(a) If α is of class $\mathcal{C}^1$, then we define

$$\partial\alpha = \begin{cases} \mathcal{P}^{1,0}(d\alpha) & \text{if } r = 0, \\ d(\mathcal{P}^{0,1}\alpha) & \text{if } r = 1, \\ 0 & \text{if } r \geq 2, \end{cases} \quad \text{and} \quad \bar\partial\alpha = \begin{cases} \mathcal{P}^{0,1}(d\alpha) & \text{if } r = 0, \\ d(\mathcal{P}^{1,0}\alpha) & \text{if } r = 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

(b) If α is of class $\mathcal{C}^1$ and we have $\bar\partial\alpha = 0$ ($\partial\alpha = 0$), then we say that α is $\bar\partial$-*closed* (respectively, ∂-*closed*).

(c) If $\alpha = \bar{\partial}\beta$ ($\alpha = \partial\beta$) for some $\mathcal{C}^1$ differential form β of degree $r-1$ on Ω (in particular, $r > 0$), then we say that α is *$\bar{\partial}$-exact* (respectively, *∂-exact*). If β may be chosen to be of class $\mathcal{C}^k$ with $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then we also say that α is *$\mathcal{C}^k$ $\bar{\partial}$-exact* (respectively, *$\mathcal{C}^k$ ∂-exact*). If for each point in Ω, there exists a $\mathcal{C}^1$ differential form β_0 of degree $r-1$ on a neighborhood U_0 with $\bar{\partial}\beta_0 = \alpha\restriction_{U_0}$ ($\partial\beta_0 = \alpha\restriction_{U_0}$), then we say that α is *locally $\bar{\partial}$-exact* (respectively, *locally ∂-exact*). If for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, each of the local forms β_0 may be chosen to be of class $\mathcal{C}^k$, then we also say that α is *locally $\mathcal{C}^k$ $\bar{\partial}$-exact* (respectively, *locally $\mathcal{C}^k$ ∂-exact*). It is also convenient to consider the trivial 0-form $\alpha \equiv 0$ to be $\mathcal{C}^\infty$ ∂-exact and $\mathcal{C}^\infty$ $\bar{\partial}$-exact, and to write $0 = \partial 0 = \bar{\partial} 0$.

Remark If α is a $\bar{\partial}$-exact (∂-exact) r-form on a Riemann surface X, then we may choose a $\mathcal{C}^1$ differential form β with $\bar{\partial}\beta = \alpha$ (respectively, $\partial\beta = \alpha$). For $r = 1$, β is of type $(0,0)$ and α is of type $(0,1)$ (respectively, α is of type $(1,0)$). For $r = 2$, α is of type $(1,1)$ and we have $\bar{\partial}\mathcal{P}^{1,0}\beta = \bar{\partial}\beta = \alpha$ (respectively, $\partial\mathcal{P}^{0,1}\beta = \partial\beta = \alpha$). Thus, in either case ($r = 1$ or 2), α is of type $(p, q+1)$ (respectively, of type $(p+1, q)$) with $p + q + 1 = r$, and we may choose β to be of type (p, q). Moreover, according to part (c) of Proposition 2.5.5 below, if α is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then *any* such β of type (p, q) is also of class $\mathcal{C}^k$.

Proposition 2.5.5 *For any $\mathcal{C}^1$ differential form α of degree r on an open subset Ω of a Riemann surface X, we have the following*:

(a) *The exterior derivative satisfies $d\alpha = \partial\alpha + \bar{\partial}\alpha$, and if α is of class $\mathcal{C}^2$, then $d^2\alpha = \bar{\partial}^2\alpha = \partial^2\alpha = \partial\bar{\partial}\alpha + \bar{\partial}\partial\alpha = 0$; in other words,*

$$d = \partial + \bar{\partial} \text{ on } \mathcal{C}^1 \text{ forms}$$

and

$$d^2 = \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0 \text{ on } \mathcal{C}^2 \text{ forms.}$$

We also have $\overline{\partial\alpha} = \bar{\partial}\bar{\alpha}$ and $\overline{\bar{\partial}\alpha} = \partial\bar{\alpha}$. Consequently, α is $\bar{\partial}$-closed (∂-closed) if and only if $\bar{\alpha}$ is ∂-closed (respectively, $\bar{\partial}$-closed); and if $\alpha = \bar{\partial}\beta$ ($\alpha = \partial\beta$) for some differential form β of class $\mathcal{C}^2$, then α is $\bar{\partial}$-closed (respectively, ∂-closed).

(b) *If α is of type (p, q), then $\partial\alpha$ is of type $(p+1, q)$ and $\bar{\partial}\alpha$ is of type $(p, q+1)$. In particular, $\partial\alpha = 0$ if $p \geq 1$ and $\bar{\partial}\alpha = 0$ if $q \geq 1$.*

(c) *If α is $\bar{\partial}$-exact (∂-exact) and of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is $\bar{\partial}$-closed (respectively, ∂-closed) and of type $(p, q+1)$ (respectively, of type $(p+1, q)$) for some pair of integers (p, q). Moreover, there exists a $\mathcal{C}^1$ differential form β of type (p, q) such that $\alpha = \bar{\partial}\beta$ (respectively, $\alpha = \partial\beta$), and any such form β is actually of class $\mathcal{C}^k$.*

(d) *Let (U, z) be a local holomorphic coordinate neighborhood in Ω. If $r = 0$, then on U,*

$$d\alpha = \frac{\partial\alpha}{\partial z}\,dz + \frac{\partial\alpha}{\partial\bar{z}}\,d\bar{z}, \qquad \partial\alpha = \frac{\partial\alpha}{\partial z}\,dz, \quad \text{and} \quad \bar{\partial}\alpha = \frac{\partial\alpha}{\partial\bar{z}}\,d\bar{z}.$$

If $r = 1$ and $\alpha = a\,dz + b\,d\bar{z}$ on U, then on U, we have

$$d\alpha = \left(\frac{\partial b}{\partial z} - \frac{\partial a}{\partial \bar{z}}\right) dz \wedge d\bar{z},$$

and

$$\partial\alpha = \frac{\partial b}{\partial z}\,dz \wedge d\bar{z} \quad \textit{and} \quad \bar{\partial}\alpha = \frac{\partial a}{\partial \bar{z}}\,d\bar{z} \wedge dz = -\frac{\partial a}{\partial \bar{z}}\,dz \wedge d\bar{z}.$$

(e) *For any $\mathcal{C}^1$ differential form β on Ω, we have*

$$\partial(\alpha \wedge \beta) = (\partial\alpha) \wedge \beta + (-1)^r \alpha \wedge \partial\beta$$

and

$$\bar{\partial}(\alpha \wedge \beta) = (\bar{\partial}\alpha) \wedge \beta + (-1)^r \alpha \wedge \bar{\partial}\beta.$$

(f) *If α is of type $(0,0)$ or $(1,0)$, then α is a holomorphic r-form if and only if $\bar{\partial}\alpha = 0$. If α is a holomorphic 0-form (i.e., a holomorphic function), then $\partial\alpha$ is a holomorphic 1-form.*

(g) *If $\Phi\colon Y \to X$ is a holomorphic mapping of a Riemann surface Y into X, then $\partial\Phi^*\alpha = \Phi^*\partial\alpha$ and $\bar{\partial}\Phi^*\alpha = \Phi^*\bar{\partial}\alpha$ on $\Phi^{-1}(\Omega)$. In particular, if α is a holomorphic r-form, then $\Phi^*\alpha$ is also a holomorphic r-form. If α is a meromorphic r-form and $\Phi(Y)$ is not contained in the pole set of α, then $\Phi^*\alpha$ is a meromorphic r-form.*

Proof Let (U, z) be a local holomorphic coordinate neighborhood in Ω. If $r = 0$, then, by (the remarks following) Proposition 2.4.2,

$$d\alpha = \frac{\partial\alpha}{\partial z}\,dz + \frac{\partial\alpha}{\partial \bar{z}}\,d\bar{z};$$

and since the first summand on the right-hand side is of type $(1,0)$ and the second is of type $(0,1)$, we get the expressions in (d) for $\partial\alpha$ and $\bar{\partial}\alpha$. If $r = 1$ and $\alpha = a\,dz + b\,d\bar{z}$ on U, then since $d^2 = 0$ (on class $\mathcal{C}^2$ forms), $dz \wedge dz = d\bar{z} \wedge d\bar{z} = 0$, and $dz \wedge d\bar{z} = -d\bar{z} \wedge dz$, we have

$$d\alpha = d(\mathcal{P}^{0,1}\alpha + \mathcal{P}^{1,0}\alpha) = d(\mathcal{P}^{0,1}\alpha) + d(\mathcal{P}^{1,0}\alpha) = \partial\alpha + \bar{\partial}\alpha,$$

$$\begin{aligned}\partial\alpha &= d(\mathcal{P}^{0,1}\alpha) = d(b\,d\bar{z}) = db \wedge d\bar{z} \\ &= \left(\frac{\partial b}{\partial z}\,dz + \frac{\partial b}{\partial \bar{z}}\,d\bar{z}\right) \wedge d\bar{z} = \frac{\partial b}{\partial z}\,dz \wedge d\bar{z},\end{aligned}$$

and

$$\begin{aligned}\bar{\partial}\alpha &= d(\mathcal{P}^{1,0}\alpha) = d(a\,dz) = da \wedge dz \\ &= \left(\frac{\partial a}{\partial z}\,dz + \frac{\partial a}{\partial \bar{z}}\,d\bar{z}\right) \wedge dz = -\frac{\partial a}{\partial \bar{z}}\,dz \wedge d\bar{z}.\end{aligned}$$

In particular, we get (d). The verifications of the remaining claims are left to the reader (see Exercise 2.5.5). $\square$

We have the following analogue of the Poincaré lemma (Lemma 9.5.7) for the operators ∂ and $\bar\partial$:

Lemma 2.5.6 (Dolbeault lemma) *Let p be a point in a Riemann surface X. Then there exists a neighborhood D of p in X such that for every $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, all $r, s \in \mathbb{Z}_{\geq 0}$, and every $\mathcal{C}^k$ differential form β of type $(r, s+1)$ (respectively, $(r+1, s)$) on a neighborhood of $\overline{D}$, there is a $\mathcal{C}^k$ differential form α of type (r, s) on D with $\bar\partial\alpha = \beta$ (respectively, $\partial\alpha = \beta$) on D. Consequently, every $\mathcal{C}^k$ differential form of type $(r, s+1)$ (of type $(r+1, s)$) is locally $\mathcal{C}^k$ $\bar\partial$-exact (respectively, locally $\mathcal{C}^k$ ∂-exact).*

Proof We may fix a local holomorphic coordinate neighborhood (U, z) of p in X and a neighborhood $D \Subset U$ such that $z(D)$ is a disk in $\mathbb{C}$. Suppose $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, $r, s \in \mathbb{Z}_{\geq 0}$, and β is a $\mathcal{C}^k$ differential form of type $(r, s+1)$ on a neighborhood of $\overline{D}$, which we may assume to be U. Clearly, $\beta \equiv 0 = \bar\partial 0$ if $r > 1$ or $s > 0$, so we may also assume that $r = 0$ or 1 and that $s = 0$. Thus we have either $\beta = b\,d\bar z$ or $\beta = b\,dz \wedge d\bar z$ on U for some function $b \in \mathcal{C}^k(U)$. By Lemma 1.2.2, there exists a function $a \in \mathcal{C}^k(D)$ with $\partial a/\partial\bar z = b$ on D. Thus

$$\bar\partial a = b\,d\bar z \quad \text{and} \quad \bar\partial(-a\,dz) = b\,dz \wedge d\bar z,$$

and it follows that $\beta = \bar\partial\alpha$ on D, where $\alpha = a$ if $r = 0$ and $\alpha = -a\,dz$ if $r = 1$.

If instead, we take β to be of type $(r+1, s)$, then $\bar\beta$ is of type $(s, r+1)$. Hence, by the above, there is a $\mathcal{C}^k$ differential form α of type (s, r) such that $\bar\partial\alpha = \bar\beta$ on D. Thus the $\mathcal{C}^k$ form $\bar\alpha$ is of type (r, s) and $\partial\bar\alpha = \overline{\bar\partial\alpha} = \bar{\bar\beta} = \beta$ on D. $\square$

We close this section with some remarks concerning integration on a Riemann surface. We first observe that line integrals (Definition 9.7.18) may be expressed in terms of local holomorphic coordinates. For example, if (U, z) is a local holomorphic coordinate neighborhood in a complex 1-manifold X, $\alpha = f\,dz + g\,d\bar z$ is a continuous differential form of degree 1 on U, and $\gamma\colon [a, b] \to U$ is a piecewise $\mathcal{C}^1$ path, then

$$\int_\gamma \alpha = \int_a^b \left(f(\gamma(t))\frac{d}{dt}\big[z(\gamma(t))\big] + g(\gamma(t))\frac{d}{dt}\big[\overline{z(\gamma(t))}\big] \right) dt.$$

For integration of 2-forms, observe that there is a natural orientation (see Sect. 9.7) in a complex 1-manifold X determined by the local holomorphic charts. For if $(U, \Phi = z = x + iy \leftrightarrow (x, y), U')$ and $(V, \Psi = w = u + iv \leftrightarrow (u, v), V')$ are two local holomorphic charts, then on $U \cap V$,

$$\mathcal{J}_{\Psi\circ\Phi^{-1}} \circ \Phi = \frac{du \wedge dv}{dx \wedge dy} = \frac{dw \wedge d\bar w}{dz \wedge d\bar z} = \left|\frac{\partial w}{\partial z}\right|^2 > 0.$$

In particular, $(i/2)\,dz \wedge d\bar{z} = dx \wedge dy$ is a positive $\mathcal{C}^\infty$ $(1,1)$-form (i.e., a positive $\mathcal{C}^\infty$ 2-form) on U. A positive $\mathcal{C}^\infty$ $(1,1)$-form ω on X is also called a *Kähler form*. By part (g) of Proposition 9.7.9, for any nonnegative measurable $(1,1)$-form α defined on a measurable set $S \subset X$, we have

$$\int_S \alpha = \sup \sum_{j=1}^m \int_{S_j} \alpha,$$

where the supremum is taken over all choices of disjoint measurable subsets $S_1, \ldots, S_m$ of S each of which is contained in a local holomorphic coordinate neighborhood.

Remark By definition, a positive $\mathcal{C}^\infty$ $(1,1)$-form on a complex manifold of arbitrary dimension is called a *Kähler form* if it is d-closed. This condition holds automatically in complex dimension 1, since every 2-form on a $\mathcal{C}^\infty$ manifold of real dimension 2 is closed.

We make the following observation:

Proposition 2.5.7 (Cf. Exercise 2.5.8 and Exercise 6.7.1) *For a nontrivial meromorphic function f (i.e., f is not everywhere zero) on a compact Riemann surface X, counting multiplicities, the number of zeros is equal to the number of poles. More precisely, if Z is the set of zeros, P is the set of poles, μ_p is the order of the zero at each point $p \in Z$, and ν_p is the order of the pole at each point $p \in P$, then*

$$\sum_{p \in Z \cup P} \operatorname{ord}_p f = \sum_{p \in Z} \mu_p - \sum_{p \in P} \nu_p = 0.$$

In particular, if f has exactly one (simple) pole, then f maps X biholomorphically onto $\mathbb{P}^1$.

Proof For $r > 0$ sufficiently small, the local representation of meromorphic functions provided by Proposition 2.2.6 allows us to fix disjoint local holomorphic charts $\{(U_p, \Phi_p = z_p, \Delta(0; 2r))\}_{p \in Z \cup P}$ in X such that for each point $p \in Z \cup P$, we have $z_p(p) = 0$ and on U_p,

$$f = \begin{cases} z_p^{\mu_p} & \text{if } p \in Z, \\ z_p^{-\nu_p} & \text{if } p \in P. \end{cases}$$

Thus the meromorphic 1-form $\theta \equiv df/f$ satisfies, on U_p,

$$\theta = \begin{cases} \mu_p \cdot z_p^{-1} \cdot dz_p & \text{if } p \in Z, \\ -\nu_p \cdot z_p^{-1} \cdot dz_p & \text{if } p \in P. \end{cases}$$

Stokes' theorem now gives

$$\sum_{p\in Z} 2\pi i\mu_p - \sum_{p\in P} 2\pi i\nu_p = \sum_{p\in Z\cup P} \int_{\partial\Phi_p^{-1}(\Delta(0;r))} \theta$$
$$= -\int_{X\setminus\bigcup_{p\in Z\cup P}\Phi_p^{-1}(\overline{\Delta(0;r)})} d\theta = 0.$$

In particular, if f is holomorphic except for a simple pole at $p\in X$ (i.e., $P=\{p\}$ and $\nu_p=1$), then $f^{-1}(\infty)=\{p\}$, and for each point $q\in X\setminus\{p\}$, the meromorphic function $f-f(q)$ is nonvanishing except for a simple zero at q. Thus $f\colon X\to\mathbb{P}^1$ is injective, and by the holomorphic inverse function theorem (Theorem 2.4.4), f maps X biholomorphically onto an open subset of $\mathbb{P}^1$. On the other hand, $f(X)$ is compact, hence closed, so we must have $f(X)=\mathbb{P}^1$. □

Exercises for Sect. 2.5

2.5.1 Verify that for all $r,s\in\mathbb{Z}_{\geq 0}$, conjugation gives $\overline{\Lambda^{r,s}T^*X}=\Lambda^{s,r}T^*X$ on a Riemann surface X.

2.5.2 Let θ be a nontrivial meromorphic 1-form on a Riemann surface X. Verify that $\{x\in X \mid \theta_x=0\}$ is discrete.

2.5.3 Prove Proposition 2.5.3.

2.5.4 Prove that $\mathbb{P}^1$ does not admit any nontrivial holomorphic 1-forms.

2.5.5 Prove parts (a)–(c) and (e)–(g) of Proposition 2.5.5.

2.5.6 Let α be a differential form of type $(1,0)$ on a Riemann surface X. Prove that α is a holomorphic 1-form if and only if α is holomorphic as a mapping of X into the 2-dimensional complex manifold $(T^*X)^{1,0}$ (see Exercise 2.4.3 for a description of the natural holomorphic structure on $(T^*X)^{1,0}$).

2.5.7 Let X be a complex torus (Example 2.1.6). Prove that the vector space of holomorphic 1-forms on X is 1-dimensional.

2.5.8 Let Ω be a (nonempty) smooth relatively compact domain in a Riemann surface X, and for any given nontrivial meromorphic function h on X that does not have any zeros or poles in $\partial\Omega$, let μ_h denote the number of zeros of h in Ω, and ν_h the number of poles of h in Ω, counting multiplicities.

(a) Prove the following version of the *argument principle* (cf. Proposition 2.5.7 and Exercises 5.1.6, 5.1.7, 5.9.5, 6.7.1, and 6.7.6): If f is a nontrivial meromorphic function on X that does not have any zeros or poles in $\partial\Omega$, then

$$\mu_f-\nu_f=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{df}{f}.$$

(b) Prove the following version of *Rouché's theorem*: If f and g are nontrivial meromorphic functions on X that do not have any zeros or poles in $\partial\Omega$, and $|g|<|f|$ on $\partial\Omega$, then $\mu_f-\nu_f=\mu_{f+g}-\nu_{f+g}$.

Hint. Using part (a), show that the integer-valued function $t\mapsto\mu_{f+tg}-\nu_{f+tg}$ is continuous on the interval [0, 1].

2.5.9 If X is a complex 1-manifold, $p \in X$, θ is a holomorphic 1-form on $V \setminus \{p\}$ for some neighborhood V of p in X, $(U, \Phi = z, \Delta(z_0; R))$ is a local holomorphic coordinate neighborhood of p with $\Phi(p) = z_0$ and $U \subset V$, $f = \theta/dz$ (i.e., $\theta = f\,dz$ on $U \setminus \{p\}$), and $f = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$ is the corresponding Laurent series representation of f, then the coefficient c_{-1} is called the *residue* of θ at p and is denoted by $\operatorname{res}_p\theta$. Equivalently, if $r \in (0, R)$ and $D \equiv \Phi^{-1}(\Delta(0; r))$, then

$$\operatorname{res}_p \theta = \frac{1}{2\pi i} \int_{\partial D} \theta.$$

(a) Prove that the residue of a holomorphic 1-form at an isolated singularity is well defined (that is, prove that the above definition is independent of the choice of the local holomorphic coordinate).

(b) Prove the following version of the *residue theorem* (cf. Exercises 5.1.6, 5.1.7, 5.9.5, 6.7.1, and 6.7.6): If Ω is a (nonempty) smooth relatively compact domain in a Riemann surface X, S is a finite subset of Ω, and θ is a holomorphic 1-form on $X \setminus S$, then

$$\frac{1}{2\pi i} \int_{\partial\Omega} \theta = \sum_{p \in S} \operatorname{res}_p \theta.$$

In particular, if X is a compact Riemann surface, S is a finite subset of X, and θ is a holomorphic 1-form on $X \setminus S$, then $\sum_{p\in S} \operatorname{res}_p \theta = 0$.

2.6 L^2 Scalar-Valued Differential Forms on a Riemann Surface

Throughout this section, X denotes a complex 1-manifold. The goal of this section is the development of a suitable L^2 space of differential forms. Since the objects that we integrate on oriented surfaces are the 2-forms (see Sect. 9.7) and we wish to pair forms of the same degree and integrate in order to get an inner product, it is natural to consider a pointwise pairing that gives a 2-form. Let us first consider integration on $\mathbb{C}$. We have the natural unit-length $(1,0)$-form $\theta = (1/\sqrt{2})\,dz$ and the natural positive $(1,1)$-form $\omega = \sqrt{-1}\theta \wedge \bar{\theta} = (\sqrt{-1}/2)\,dz \wedge d\bar{z}$. In the notation of Sect. 9.7, the Lebesgue measure λ in $\mathbb{C}$ is equal to the measure λ_ω associated to ω (see Definition 9.7.10). If $\alpha = a\theta$ and $\beta = b\theta$ are differential forms of type $(1,0)$ with L^2 coefficients, then

$$\langle a, b\rangle_{L^2(\mathbb{C})} = \int_{\mathbb{C}} a\bar{b}\,d\lambda = \int_{\mathbb{C}} a\bar{b}\,\omega = \int_{\mathbb{C}} a\bar{b}\,i\theta \wedge \bar{\theta} = \int_{\mathbb{C}} i\alpha \wedge \bar{\beta}.$$

If α and β are L^2 differential forms of type $(0,0)$ (i.e., L^2 functions), then

$$\langle \alpha, \beta\rangle_{L^2(\mathbb{C})} = \int_{\mathbb{C}} \alpha\bar{\beta}\,d\lambda = \int_{\mathbb{C}} \alpha\bar{\beta}\,\omega.$$

Finally, if $\alpha = a\omega$ and $\beta = b\omega$ are differential forms of type $(1,1)$ with L^2 coefficients, then

$$\langle a, b\rangle_{L^2(\mathbb{C})} = \int_{\mathbb{C}} a\bar{b}\, d\lambda = \int_{\mathbb{C}} a\bar{b}\, \omega = \int_{\mathbb{C}} \frac{\alpha}{\omega} \cdot \overline{\frac{\beta}{\omega}} \cdot \omega.$$

The right-hand sides in the above point to a reasonable definition for the L^2 inner product of differential forms on the complex 1-manifold X. Observe that the expression for the inner product of a pair of forms of type $(0,0)$ and $(1,1)$ involves the Kähler form ω, although that for the inner product of a pair of $(1,0)$-forms does *not*. Thus, on X, for forms of type $(0,0)$ and $(1,1)$, we must define the inner product with respect to some choice of a positive $(1,1)$-form. For functions, it turns out to be useful to weaken the requirement on the $(1,1)$-form by allowing it to be only nonnegative. After all, it is not even a priori clear that X admits a global Kähler form; although, as will be shown in Sect. 2.11, it turns out that every Riemann surface is second countable and therefore that X *does* admit a Kähler form (see Corollary 2.11.3). As will be seen in Sect. 2.9, it is also useful to include a *weight function* $e^{-\varphi}$ (in an abuse of language, we also call φ a *weight function*). For example, a Riemann surface need not admit any L^2 holomorphic 1-forms, but given a holomorphic 1-form, one may construct a $\mathcal{C}^\infty$ function φ that grows so large at infinity that the holomorphic 1-form becomes L^2 with respect to the weight function $e^{-\varphi}$. Finally, observe that $i\alpha \wedge \bar{\alpha} > 0$ for every nonzero $\alpha \in \Lambda^{1,0}T^*X$. Based on these considerations, we make the following definition:

Definition 2.6.1 Let S be a measurable subset of X, let φ be a measurable real-valued function that is defined on S, and let α and β be measurable differential forms of type (p,q) that are defined on S.

(a) For $(p,q) = (1,0)$, we define

$$\|\alpha\|_{L^2_{1,0}(S,\varphi)} \equiv \left[\int_S \sqrt{-1}\alpha \wedge \bar{\alpha} \cdot e^{-\varphi}\right]^{1/2} \in [0,\infty].$$

If $\sqrt{-1}\alpha \wedge \bar{\beta} \cdot e^{-\varphi}$ is integrable on S, then we define

$$\langle \alpha, \beta\rangle_{L^2_{1,0}(S,\varphi)} \equiv \int_S \sqrt{-1}\alpha \wedge \bar{\beta} \cdot e^{-\varphi} \in \mathbb{C}.$$

(b) For $(p,q) = (0,1)$, we define

$$\|\alpha\|_{L^2_{0,1}(S,\varphi)} \equiv \left[-\int_S \sqrt{-1}\alpha \wedge \bar{\alpha} \cdot e^{-\varphi}\right]^{1/2} \in [0,\infty].$$

If $-\sqrt{-1}\alpha \wedge \bar{\beta} \cdot e^{-\varphi}$ is integrable on S, then we define

$$\langle \alpha, \beta\rangle_{L^2_{0,1}(S,\varphi)} \equiv -\int_S \sqrt{-1}\alpha \wedge \bar{\beta} \cdot e^{-\varphi} \in \mathbb{C}.$$

(c) If $(p, q) = (0, 0)$ and ω is a *nonnegative* measurable form of type $(1, 1)$ defined on S, then we define

$$\|\alpha\|_{L^2_{0,0}(S,\omega,\varphi)} \equiv \left[\int_S |\alpha|^2 e^{-\varphi}\omega\right]^{1/2} \in [0, \infty].$$

If $\alpha\bar{\beta} \cdot e^{-\varphi}\omega$ is integrable on S, then we define

$$\langle \alpha, \beta\rangle_{L^2_{0,0}(S,\omega,\varphi)} \equiv \int_S \alpha\bar{\beta} \cdot e^{-\varphi}\omega \in \mathbb{C}.$$

(d) If $(p, q) = (1, 1)$ and ω is a *positive* measurable form of type $(1, 1)$ defined on S, then we define

$$\|\alpha\|_{L^2_{1,1}(S,\omega,\varphi)} \equiv \left[\int_S \left|\frac{\alpha}{\omega}\right|^2 e^{-\varphi}\omega\right]^{1/2} = \|\alpha/\omega\|_{L^2_{0,0}(S,\omega,\varphi)} \in [0, \infty].$$

If the form

$$\frac{\alpha}{\omega} \cdot \overline{\frac{\beta}{\omega}} \cdot e^{-\varphi}\omega$$

is integrable on S, then we define

$$\langle \alpha, \beta\rangle_{L^2_{1,1}(S,\omega,\varphi)} \equiv \int_S \frac{\alpha}{\omega} \cdot \overline{\frac{\beta}{\omega}} \cdot e^{-\varphi}\omega = \left\langle \frac{\alpha}{\omega}, \frac{\beta}{\omega}\right\rangle_{L^2_{0,0}(S,\omega,\varphi)} \in \mathbb{C}.$$

(e) When there is no danger of confusion (for example, when the choice of (p, q), S, ω, or φ is understood from the context), we will suppress parts of the notation. For example, we will often write $\|\alpha\|_{L^2_{p,q}(S,\omega,\varphi)}$ simply as $\|\alpha\|_{S,\omega,\varphi}$, $\|\alpha\|_{S,\varphi}$, $\|\alpha\|_{S,\omega}$, $\|\alpha\|_{\omega,\varphi}$, $\|\alpha\|_{\varphi}$, $\|\alpha\|_{\omega}$, $\|\alpha\|_{L^2(S,\omega,\varphi)}$, $\|\alpha\|_{L^2(S,\varphi)}$, $\|\alpha\|_{L^2(\omega,\varphi)}$, $\|\alpha\|_{L^2(\varphi)}$, $\|\alpha\|_{L^2(\omega)}$, or $\|\alpha\|$. We will also use the analogous simplified notation for $\langle \alpha, \beta\rangle_{L^2_{p,q}(S,\omega,\varphi)}$, for $\|\alpha\|_{L^2_{p,q}(S,\varphi)}$, and for $\langle \alpha, \beta\rangle_{L^2_{p,q}(S,\varphi)}$. Moreover, when no mention of a weight function appears in the context, then the weight function will be assumed to be $\varphi \equiv 0$ and we will write $\|\alpha\|_{L^2_{p,q}(S,\omega,0)}$ simply as $\|\alpha\|_{L^2_{p,q}(S,\omega)}$ and so on.

(f) Let γ and η be measurable differential 1-forms defined on S with (r, s) parts $\gamma^{r,s}$ and $\eta^{r,s}$, respectively, for each $(r, s) \in \{(1, 0), (0, 1)\}$. Then we set

$$\|\gamma\|^2_{L^2_1(S,\varphi)} \equiv \|\gamma^{1,0}\|^2_{\varphi} + \|\gamma^{0,1}\|^2_{\varphi}.$$

We also set $\langle \gamma, \eta\rangle_{L^2_1(S,\varphi)} \equiv \langle \gamma^{1,0}, \eta^{1,0}\rangle_{\varphi} + \langle \gamma^{0,1}, \eta^{0,1}\rangle_{\varphi}$, provided each of the summands on the right-hand side is defined. We also use the simplified notation analogous to that appearing in (e).

Definition 2.6.2 Let S be a measurable subset of X, and let φ be a measurable real-valued function that is defined on S.

(a) For $(p, q) = (1, 0)$ or $(0, 1)$, $L^2_{p,q}(S, \varphi)$ consists of all equivalence classes of measurable differential forms α of type (p, q) on S with $\|\alpha\|_{L^2(S,\varphi)} < \infty$, where we identify any two elements that are equal almost everywhere. The set $L^2_1(S, \varphi)$ consists of all equivalence classes of measurable 1-forms α with $\|\alpha\|_{L^2(S,\varphi)} < \infty$, where again, we identify any two elements that are equal almost everywhere.

(b) For ω a *nonnegative* measurable differential form of type $(1, 1)$ that is defined on S, $L^2_{0,0}(S, \omega, \varphi)$ consists of all equivalence classes of measurable functions α on S with $\|\alpha\|_{L^2(S,\omega,\varphi)} < \infty$, where we identify any two elements that are equal almost everywhere.

(c) For ω a *positive* measurable differential form of type $(1, 1)$ that is defined on S, $L^2_{1,1}(S, \omega, \varphi)$ consists of all equivalence classes of measurable differential forms α of type $(1, 1)$ on S with $\|\alpha\|_{L^2(S,\omega,\varphi)} < \infty$, where we identify any two elements that are equal almost everywhere.

(d) When no mention of a weight function appears in the context and there is no danger of confusion, then the weight function will be assumed to be $\varphi \equiv 0$ and we will write $L^2_{p,q}(S, 0)$ simply as $L^2_{p,q}(S)$, and $L^2_{p,q}(S, \omega, 0)$ simply as $L^2_{p,q}(S, \omega)$.

Remark For our purposes, we will need only weight functions φ and nonnegative (or positive) $(1, 1)$-forms ω that are defined and of class $\mathcal{C}^\infty$ on an open subset of X.

Proposition 2.6.3 *Let S be a measurable subset of X, and let φ be a continuous real-valued function that is defined on S.*

(a) *The pair $(L^2_1(S, \varphi), \langle\cdot, \cdot\rangle_{L^2(S,\varphi)})$, and the pair $(L^2_{p,q}(S, \varphi), \langle\cdot, \cdot\rangle_{L^2(S,\varphi)})$ for $(p, q) = (1, 0)$ or $(0, 1)$, are Hilbert spaces (where the inner product of any two equivalence classes is given by the pairing of any representatives). Moreover, we have the Hilbert space orthogonal decomposition*

$$L^2_1(S, \varphi) = L^2_{1,0}(S, \varphi) \oplus L^2_{0,1}(S, \varphi).$$

(b) *For ω a continuous positive differential form of type $(1, 1)$ that is defined on S and for $(p, q) = (0, 0)$ or $(1, 1)$, $(L^2_{p,q}(S, \omega, \varphi), \langle\cdot, \cdot\rangle_{L^2(S,\omega,\varphi)})$ is a Hilbert space (where the inner product of any two equivalence classes is given by the pairing of any representatives).*

Moreover, in each of the above spaces, any sequence converging to an element α admits a subsequence that converges to α pointwise almost everywhere in S.

Proof Let $p, q \in \{0, 1\}$. We set

$$L^2_{p,q} \equiv \begin{cases} L^2_{p,q}(S, \varphi) \text{ as in (a)} & \text{if } (p, q) = (1, 0) \text{ or } (0, 1), \\ L^2_{p,q}(S, \omega, \varphi) \text{ as in (b)} & \text{if } (p, q) = (0, 0) \text{ or } (1, 1); \end{cases}$$

and for each point $r \in S$ and each pair of elements $\eta, \theta \in \Lambda^{p,q} T_r^* X$, we set

$$H(\eta, \theta) \equiv \begin{cases} \eta\bar{\theta} \cdot e^{-\varphi(r)} \cdot \omega_r & \text{if } (p,q) = (0,0), \\ i\eta \wedge \bar{\theta} \cdot e^{-\varphi(r)} & \text{if } (p,q) = (1,0), \\ -i\eta \wedge \bar{\theta} \cdot e^{-\varphi(r)} & \text{if } (p,q) = (0,1), \\ \dfrac{\eta}{\omega_r} \cdot \overline{\dfrac{\theta}{\omega_r}} \cdot e^{-\varphi(r)} \omega_r & \text{if } (p,q) = (1,1). \end{cases}$$

We first show that $(L^2_{p,q}, \langle \cdot, \cdot \rangle)$ is an inner product space. For this, observe that if $\alpha, \beta \in L^2_{p,q}$, then the $(1,1)$-form $H(\alpha, \beta)\colon r \mapsto H(\alpha_r, \beta_r)$ is measurable. Suppose $(U, \Phi = z = x + iy, U')$ is a local holomorphic chart and $\omega_\Phi \equiv (i/2)\, dz \wedge d\bar{z} = dx \wedge dy$. Then the map

$$h\colon (\eta, \theta) \mapsto h(\eta, \theta) \equiv \frac{H(\eta, \theta)}{(\omega_\Phi)_r} \in \mathbb{C} \quad \forall \eta, \theta \in \Lambda^{p,q} T_r^* X$$

is a Hermitian inner product in $\Lambda^{p,q} T_r^* X$ for each point $r \in S \cap U$. We also have the corresponding pointwise norm $\eta \mapsto |\eta|_h = [h(\eta, \eta)]^{1/2}$. Hence, for each measurable set $R \subset S \cap U$ and each complex number ζ with $|\zeta| = 1$, we have (by the Schwarz inequality)

$$\begin{aligned} \int_R [\mathrm{Re}(\zeta H(\alpha, \beta))]^+ &= \int_R [\mathrm{Re}(\zeta h(\alpha, \beta))]^+ \, d\lambda_{\omega_\Phi} \leq \int_R |\alpha|_h |\beta|_h \, d\lambda_{\omega_\Phi} \\ &\leq \left[\int_R |\alpha|_h^2 \, d\lambda_{\omega_\Phi} \right]^{1/2} \left[\int_R |\beta|_h^2 \, d\lambda_{\omega_\Phi} \right]^{1/2} \\ &= \left[\int_R H(\alpha, \alpha) \right]^{1/2} \cdot \left[\int_R H(\beta, \beta) \right]^{1/2} \end{aligned}$$

(where λ_{ω_Φ} is the positive measure associated to ω_Φ as in Definition 9.7.10). Thus, if $S_1, \dots, S_m$ are disjoint measurable subsets of S each of which lies in some local holomorphic coordinate neighborhood, then, by the Schwarz inequality for sums,

$$\begin{aligned} \sum_{j=1}^m \int_{S_j} [\mathrm{Re}(\zeta H(\alpha, \beta))]^+ &\leq \sum_{j=1}^m \left[\int_{S_j} H(\alpha, \alpha) \right]^{1/2} \cdot \left[\int_{S_j} H(\beta, \beta) \right]^{1/2} \\ &\leq \left[\sum_{j=1}^m \int_{S_j} H(\alpha, \alpha) \right]^{1/2} \cdot \left[\sum_{j=1}^m \int_{S_j} H(\beta, \beta) \right]^{1/2} \\ &\leq \left[\int_S H(\alpha, \alpha) \right]^{1/2} \cdot \left[\int_S H(\beta, \beta) \right]^{1/2} = \|\alpha\| \cdot \|\beta\|. \end{aligned}$$

Passing to the supremum, we get

$$\int_S [\mathrm{Re}(\zeta H(\alpha, \beta))]^+ \leq \|\alpha\| \cdot \|\beta\| < \infty.$$

Taking $\zeta = \pm 1, \pm i$, we see that $H(\alpha, \beta)$ is integrable and therefore that $\langle \alpha, \beta\rangle = \int_S H(\alpha, \beta)$ is defined. Furthermore, choosing $\zeta \in \mathbb{C}$ so that $|\zeta| = 1$ and

$$\zeta \cdot \langle \alpha, \beta\rangle = |\langle \alpha, \beta\rangle|,$$

we get

$$\begin{aligned}|\langle \alpha, \beta\rangle| = \int_S \zeta H(\alpha, \beta) &= \int_S [\mathrm{Re}(\zeta H(\alpha, \beta))]^+ - \int_S [\mathrm{Re}(\zeta H(\alpha, \beta))]^- \\ &\leq \int_S [\mathrm{Re}(\zeta H(\alpha, \beta))]^+ \leq \|\alpha\| \cdot \|\beta\|.\end{aligned}$$

For each constant $s \in \mathbb{C}$, we have

$$H(s\alpha + \beta, s\alpha + \beta) = |s|^2 H(\alpha, \alpha) + 2\,\mathrm{Re}(sH(\alpha, \beta)) + H(\beta, \beta),$$

so $H(s\alpha + \beta, s\alpha + \beta)$ is integrable. If $\alpha = \alpha'$ a.e. (almost everywhere) in S, then clearly, $H(\alpha', \alpha') = H(\alpha, \alpha)$ and $H(\alpha', \beta) = H(\alpha, \beta)$ a.e. Thus $L^2_{p,q}$ is a well-defined vector space, and as is now easy to verify, $\langle \cdot, \cdot\rangle$ is a well-defined Hermitian inner product.

It remains to show that $(L^2_{p,q}, \langle \cdot, \cdot\rangle)$ is complete with respect to the norm $\|\alpha\| = \langle \alpha, \alpha\rangle^{1/2}$. We again apply (appropriately modified) standard arguments (cf., for example, [Rud1]). We must show that a given Cauchy sequence $\{\alpha_\nu\}$ in $L^2_{p,q}$ converges. For this, it suffices to show that some subsequence converges. Hence, after replacing the sequence with a suitable subsequence, we may assume without loss of generality that $\sum \|\alpha_{\nu+1} - \alpha_\nu\| < 1$. For a local holomorphic chart $(U, \Phi = z, U')$, for $\omega_\Phi = dx \wedge dy = (i/2)\, dz \wedge d\bar z$ and for $h(\cdot, \cdot) = H(\cdot, \cdot)/\omega_\Phi$, as before, let us set

$$\varphi_N \equiv \sum_{\nu=1}^{N} |\alpha_{\nu+1} - \alpha_\nu|_h \quad \forall N \in \mathbb{Z}_{>0} \quad \text{and} \quad \varphi \equiv \sum_{\nu=1}^{\infty} |\alpha_{\nu+1} - \alpha_\nu|_h.$$

For each $N \in \mathbb{Z}_{>0}$, we have

$$\|\varphi_N\|_{L^2(S\cap U, \lambda_{\omega_\Phi})} \leq \sum_{\nu=1}^{N} \||\alpha_{\nu+1} - \alpha_\nu|_h\|_{L^2(S\cap U, \lambda_{\omega_\Phi})} \leq \sum_{\nu=1}^{N} \|\alpha_{\nu+1} - \alpha_\nu\| < 1.$$

Fatou's lemma now implies that $\|\varphi\|_{L^2(S\cap U, \lambda_{\omega_\Phi})} \leq 1$. In particular, $\varphi < \infty$ almost everywhere in $S \cap U$. It follows that the series

$$\alpha_1 + \sum_{\nu=1}^{\infty} (\alpha_{\nu+1} - \alpha_\nu)$$

converges pointwise almost everywhere in S to a measurable differential form α of type (p, q), and hence $\alpha_\nu \to \alpha$ pointwise a.e.

It remains to show that $\alpha \in L^2_{p,q}$ and that $\|\alpha - \alpha_\nu\| \to 0$. Given $\epsilon > 0$, we may choose $N \in \mathbb{Z}_{>0}$ so that $\|\alpha_\mu - \alpha_\nu\| < \epsilon$ for all $\mu, \nu > N$. For any $\nu > N$, Fatou's lemma (Theorem 9.7.11) gives

$$\|\alpha - \alpha_\nu\|^2 = \int_S H(\alpha - \alpha_\nu, \alpha - \alpha_\nu) \leq \liminf_{\mu \to \infty} \int_S H(\alpha_\mu - \alpha_\nu, \alpha_\mu - \alpha_\nu) \leq \epsilon^2.$$

Thus $\alpha = (\alpha - \alpha_\nu) + \alpha_\nu \in L^2_{p,q}$ and $\|\alpha - \alpha_\nu\| \leq \epsilon$. Therefore $\alpha_\nu \to \alpha$ in $L^2_{p,q}$ (and pointwise almost everywhere in S) and $L^2_{p,q}$ is a Hilbert space. The proposition (including the orthogonal decomposition in (a)) now follows. □

We have the following useful version of Theorem 1.2.4:

Theorem 2.6.4 *Let φ be a continuous real-valued function and let ω be a continuous positive differential form of type $(1, 1)$ on X. Then, for every compact set $K \subset X$, there is a constant $C = C(X, K, \omega, \varphi) > 0$ such that*

$$\max_K |f| \leq C \|f\|_{L^2(X,\omega,\varphi)} \quad \forall f \in \mathcal{O}(X).$$

Consequently, $\mathcal{O}(X) \cap L^2_{0,0}(X, \omega, \varphi)$ is a closed subspace of $L^2_{0,0}(X, \omega, \varphi)$.

Proof Given a point $p \in K$, we may fix a relatively compact local holomorphic coordinate neighborhood (U, z) on which the function $(i/2)(dz \wedge d\bar{z})/\omega$ is bounded. Therefore, for some constant $R > 0$, we have, for every holomorphic function f on U,

$$\int_U |f|^2 \frac{i}{2} dz \wedge d\bar{z} = \int_U |f|^2 e^\varphi \frac{(i/2)\, dz \wedge d\bar{z}}{\omega} \cdot e^{-\varphi} \cdot \omega \leq R \|f\|^2_{L^2(U,\omega,\varphi)}.$$

Fixing a relatively compact neighborhood V_p of p in U and applying Theorem 1.2.4, we get a constant $C_p > 0$ such that

$$\sup_{V_p} |f| \leq C_p \|f\|_{L^2(U,\omega,\varphi)} \leq C_p \|f\|_{L^2(X,\omega,\varphi)} \quad \forall f \in \mathcal{O}(X).$$

Covering K by finitely many such neighborhoods V_p and taking C to be the maximum of the associated constants C_p, we get the claim. The proof that $\mathcal{O}(X) \cap L^2_{0,0}(X, \omega, \varphi)$ is a closed subspace is left to the reader (see Exercise 2.6.3). □

Exercises for Sect. 2.6

2.6.1 Show that there are no nontrivial L^2 holomorphic 1-forms on $\mathbb{C}$ (cf. Exercise 2.5.4).

2.6.2 For the function $\varphi \colon z \mapsto |z|^2$ on $\mathbb{C}$, show that there exists a nontrivial holomorphic 1-form in $L^2_{1,0}(\mathbb{C}, \varphi)$.

2.6.3 Let φ be a continuous real-valued function on a Riemann surface X. Prove that the vector space of holomorphic 1-forms in $L^2_{1,0}(X,\varphi)$ is a closed subspace of $L^2_{1,0}(X,\varphi)$. Also prove that if ω is a continuous positive $(1,1)$-form on X, then the vector space of holomorphic functions in $L^2_{0,0}(X,\omega,\varphi)$ is a closed subspace (i.e., prove the second part of Theorem 2.6.4).

2.7 The Distributional $\bar{\partial}$ Operator on Scalar-Valued Forms

Throughout this section, X again denotes a complex 1-manifold. We now develop a suitable distributional version of the $\bar{\partial}$ operator. In particular, we are led to consider a modified version of the exterior derivative d called the *canonical* (or *Chern*) *connection*.

For locally integrable functions f and g on a local holomorphic coordinate neighborhood (U,z), we have $(\partial f/\partial\bar{z})_{\text{distr}} = g$ (see Definition 9.8.2 and Proposition 9.8.3) if and only if for each function $u \in \mathcal{D}(U)$, we have

$$\int_U f\overline{\left(-\frac{\partial u}{\partial z}\right)}\frac{i}{2}\,dz\wedge d\bar{z} = \int_U g\bar{u}\frac{i}{2}\,dz\wedge d\bar{z}$$

(equivalently, $\int_U f\overline{(-\partial u/\partial z)}\,dz\wedge d\bar{z} = \int_U g\bar{u}\,dz\wedge d\bar{z}$). One natural definition for the distributional $\bar{\partial}$ operator is the following:

Definition 2.7.1 Let α and β be locally integrable differential forms on an open set $\Omega\subset X$. We write $\bar{\partial}_{\text{distr}}\alpha = \beta$ if for every local holomorphic coordinate neighborhood (U,z), one of the following holds:

(i) On $U\cap\Omega$, α is a 0-form, $\beta = b\,d\bar{z}$, and $(\partial\alpha/\partial\bar{z})_{\text{distr}} = b$;
(ii) On $U\cap\Omega$, $\alpha = a_1\,dz + a_2\,d\bar{z}$, $\beta = b\,dz\wedge d\bar{z}$, and $(\partial a_1/\partial\bar{z})_{\text{distr}} = -b$;
(iii) The form α is of degree >1 and $\beta\equiv 0$.

Remark Given a locally integrable differential form α, $\bar{\partial}_{\text{distr}}\alpha$ need not exist as a form (see the remarks following Definition 7.4.2). For example, let u be the characteristic function of the unit disk $\Delta\equiv\Delta(0;1)$. Then $\bar{\partial}u\equiv 0$ on the complement $\mathbb{C}\setminus\partial\Delta$ of the measure-zero set $\partial\Delta$. Thus, if $\bar{\partial}_{\text{distr}}u$ were to exist on $\mathbb{C}$, then it would be the zero form, and hence by the regularity theorem (Theorem 1.2.8), u would be holomorphic, which it is not. It also follows that for the measurable differential form $\alpha = u\,dz$, $\bar{\partial}_{\text{distr}}\alpha$ does not exist as a form.

It is often more convenient to work with an intrinsic form of an operator, so we look for an intrinsic description of $\bar{\partial}_{\text{distr}}$. Distributional differential operators in $\mathbb{R}^n$ are defined via integration against test functions (as in Sect. 7.4.2). On a surface, the objects that we integrate on open sets (or, more generally, measurable sets) are 2-forms, and for a differential form, its wedge product with a form of complementary degree is a 2-form. Thus, in the present context, it is natural to integrate against *test forms* of complementary degree. For example, suppose α and β are $\mathcal{C}^\infty$ differential forms of type $(1,0)$ and $(1,1)$, respectively, and $\bar{\partial}\alpha = \beta$. Since we integrate $\mathcal{C}^\infty$ forms of type $(1,1)$ (i.e., 2-forms) on open sets in X, it is natural to integrate the wedge product of α and the conjugate of a $(1,0)$-form, and the scalar product (i.e., the wedge product) of β and a function (i.e., a $(0,0)$-form). Given a function $f \in \mathcal{D}(X)$, we have

$$\begin{aligned}\beta \cdot \bar{f} &= (\bar{\partial}\alpha) \cdot \bar{f} = (d\alpha) \cdot \bar{f} = d(\alpha \cdot \bar{f}) + \alpha \wedge d\bar{f}\\ &= d(\alpha \cdot \bar{f}) + \alpha \wedge \partial \bar{f} + \alpha \wedge \bar{\partial} \bar{f}\\ &= d(\alpha \cdot \bar{f}) + \alpha \wedge \overline{\partial f}.\end{aligned}$$

Integrating and applying Stokes' theorem, we get

$$\int_X \beta \cdot \bar{f} = \int_X \alpha \wedge \overline{\partial f}.$$

It turns out to be useful to insert a weight function φ (more precisely, $e^{-\varphi}$), where φ is a real-valued $\mathcal{C}^\infty$ (usually) function on X. Doing so allows one to obtain estimates on L^2 norms that lead to solutions of the Cauchy–Riemann equation as well as certain bounds on the L^2 norms of the solutions (see Sect. 2.9). Moreover, a Hermitian metric in a holomorphic line bundle (see Chap. 3) is locally represented by such weight functions, and this point of view yields results that generalize readily to that context. For α, β, and f as above, we have

$$\begin{aligned}\beta \cdot \bar{f} \cdot e^{-\varphi} &= (\bar{\partial}\alpha) \cdot \bar{f} \cdot e^{-\varphi} = (d\alpha) \cdot (\bar{f} e^{-\varphi}) = d(\alpha \cdot e^{-\varphi} \bar{f}) + \alpha \wedge d(e^{-\varphi} \bar{f})\\ &= d(\alpha \cdot e^{-\varphi} \bar{f}) + \alpha \wedge \bar{\partial}(e^{-\varphi} \bar{f})\\ &= d(\alpha \cdot e^{-\varphi} \bar{f}) + \alpha \wedge \overline{e^{\varphi} \partial (e^{-\varphi} f)} \cdot e^{-\varphi}.\end{aligned}$$

Thus

$$\int_X \beta \cdot \bar{f} \cdot e^{-\varphi} = \int_X \alpha \wedge \overline{e^{\varphi} \partial (e^{-\varphi} f)} \cdot e^{-\varphi}.$$

The above suggests a natural intrinsic form of the definition of $\bar{\partial}_{\mathrm{distr}}$ on forms of type $(1,0)$. A similar computation applies to forms of type $(0,0)$. Based on the above, we make the following definition:

Definition 2.7.2 Given a real-valued function $\varphi \in \mathcal{C}^\infty(X)$, the associated *canonical connection* (or the *Chern connection*) is the operator $D = D_\varphi = D' + D''$, where for each $\mathcal{C}^1$ differential form α on an open subset of X, we define

$$D'\alpha = D'_\varphi \alpha \equiv e^{\varphi} \partial (e^{-\varphi} \alpha) = \partial \alpha + e^{\varphi} (\partial e^{-\varphi}) \wedge \alpha = \partial \alpha - \partial \varphi \wedge \alpha$$

and

$$D''\alpha = D''_\varphi \alpha \equiv \bar{\partial} \alpha.$$

We call D' and D'', respectively, the $(1,0)$ *part* and $(0,1)$ *part* of the connection.

Remarks 1. Observe that $D'' = \bar{\partial}$ does not depend on the choice of φ.

2. We have $D = d + e^{\varphi}(\partial e^{-\varphi}) \wedge (\cdot) = d - \partial\varphi \wedge (\cdot)$.

3. If α is of type (p, q), then $D'\alpha$ is of type $(p+1, q)$ (in particular, $D'\alpha = 0$ if $p \geq 1$) and $D''\alpha$ is of type $(p, q+1)$ ($D''\alpha = 0$ if $q \geq 1$).

4. If α and β are $\mathcal{C}^1$ differential forms and α is of degree p, then (see Exercise 2.7.1)

$$D(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge D\beta = (D\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

5. D_φ, D'_φ, and D''_φ are defined in the same way as above for $\varphi \in \mathcal{C}^1(X)$.

We now get the following equivalent form for the definition of $\bar{\partial}_{\mathrm{distr}}$ (which we also denote by D''_{distr}):

Proposition 2.7.3 *Let φ be a real-valued $\mathcal{C}^\infty$ function on X, and let α and β be locally integrable differential forms on an open set $\Omega \subset X$.*

(a) *If α is of type $(1, 0)$ and β is of type $(1, 1)$, then $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$ if and only if*

$$\int_\Omega \alpha \wedge \overline{D' f} e^{-\varphi} = \int_\Omega \beta \cdot \bar{f} \cdot e^{-\varphi} \quad \forall f \in \mathcal{D}(\Omega).$$

(b) *If α is of type $(0, 0)$ and β is of type $(0, 1)$, then $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$ if and only if*

$$\int_\Omega \alpha \cdot \overline{(-D'\gamma)} \cdot e^{-\varphi} = \int_\Omega \beta \wedge \overline{\gamma} \cdot e^{-\varphi} \quad \forall \gamma \in \mathcal{D}^{0,1}(\Omega).$$

(c) *If α is of type (p, q) with $p \geq 2$ or $q \geq 1$, then $\bar{\partial}_{\mathrm{distr}}\alpha = 0$.*

Remark It will follow from the proof that if the conditions in Definition 2.7.1 hold for the forms α and β in *some* local holomorphic coordinate neighborhood of every point, then they hold in *every* local holomorphic coordinate neighborhood.

Proof of Proposition 2.7.3 Suppose $\alpha = a\,dz$ and $\beta = b\,dz \wedge d\bar{z}$ in some local holomorphic coordinate neighborhood (U, z) with $U \subset \Omega$. Then, for every function $f \in \mathcal{D}(U)$ (which we may view as a $\mathcal{C}^\infty$ function with compact support in Ω), we have

$$\int_\Omega \beta \bar{f} e^{-\varphi} = \int_U b\overline{(e^{-\varphi} f)}\, dz \wedge d\bar{z}$$

and

$$\int_\Omega \alpha \wedge \overline{D' f} \cdot e^{-\varphi} = \int_U \alpha \wedge \overline{\partial(e^{-\varphi} f)} = \int_U a \cdot \overline{\frac{\partial}{\partial z}(e^{-\varphi} f)}\, dz \wedge d\bar{z}.$$

Given $u \in \mathcal{D}(U)$, setting $f = e^{\varphi} u$, we see that if the above left-hand sides are always equal, then we have $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$. Conversely, if $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$ and $f \in \mathcal{D}(\Omega)$, then, choosing finitely many $\mathcal{C}^\infty$ functions $\{\eta_\nu\}_{\nu=1}^n$ on Ω such that $\sum \eta_\nu \equiv 1$ on supp f

and such that for each ν, the support of η_ν is contained in some local holomorphic coordinate neighborhood U_ν, the above gives

$$\int_\Omega \alpha \wedge \overline{D'f} \cdot e^{-\varphi} = \sum_\nu \int_\Omega \alpha \wedge \overline{D'(\eta_\nu \cdot f)} \cdot e^{-\varphi} = \sum_\nu \int_{U_\nu} \alpha \wedge \overline{D'(\eta_\nu \cdot f)} \cdot e^{-\varphi}$$
$$= \sum_\nu \int_{U_\nu} \beta \overline{\eta_\nu f} e^{-\varphi} = \sum_\nu \int_\Omega \beta \overline{\eta_\nu f} e^{-\varphi} = \int_\Omega \beta \bar{f} \cdot e^{-\varphi}.$$

Thus part (a) is proved.

Part (c) is obvious and the proof of part (b) is left to the reader (see Exercise 2.7.2). □

The regularity theorem (Theorem 1.2.8) gives the following in this context:

Theorem 2.7.4 *If $p \in \{0, 1\}$, α is a locally integrable form of type $(p, 0)$ on X, and $\bar\partial_{\mathrm{distr}}\alpha = \beta$ for a form β of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class $\mathcal{C}^k$. In particular, if $\bar\partial_{\mathrm{distr}}\alpha = 0$, then α is a holomorphic p-form.*

Exercises for Sect. 2.7

2.7.1 Let φ be a $\mathcal{C}^\infty$ real-valued function on a Riemann surface X. Show that if α and β are $\mathcal{C}^1$ differential forms on X and α is of degree p, then

$$D_\varphi(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge D_\varphi \beta = (D_\varphi \alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

2.7.2 Prove part (b) of Proposition 2.7.3.

2.7.3 Let X be a Riemann surface, let φ be a real-valued $\mathcal{C}^\infty$ function on X, let $\{\theta_\nu\}$ be a sequence of holomorphic 1-forms in $L^2_{1,0}(X, \varphi)$, and let $\theta \in L^2_{1,0}(X, \varphi)$. Assume that for each element $\alpha \in L^2_{1,0}(X, \varphi)$, we have

$$\lim_{\nu \to \infty} \langle \theta_\nu, \alpha \rangle_{L^2_{1,0}(X,\varphi)} = \langle \theta, \alpha \rangle_{L^2_{1,0}(X,\varphi)}$$

(that is, $\{\theta_\nu\}$ converges *weakly* to θ in $L^2_{1,0}(X, \varphi)$). Prove that θ is a holomorphic 1-form.

2.8 Curvature and the Fundamental Estimate for Scalar-Valued Forms

Throughout this section, X again denotes a complex 1-manifold. Suppose φ is a real-valued $\mathcal{C}^\infty$ function on X. We recall that for any $\mathcal{C}^\infty$ differential form α on an open subset of X, we have $D_\varphi \alpha = D\alpha = D'\alpha + D''\alpha$, where

$$D'\alpha = e^\varphi \partial[e^{-\varphi}\alpha] = \partial\alpha - (\partial\varphi) \wedge \alpha \quad \text{and} \quad D''\alpha = \bar\partial\alpha.$$

Consequently, $(D')^2 = (D'')^2 = 0$ and $D^2 = D'D'' + D''D'$. Since $\partial\bar\partial + \bar\partial\partial = 0$, we have

$$\begin{aligned} D^2\alpha &= \partial\bar\partial\alpha - (\partial\varphi)\wedge\bar\partial\alpha + \bar\partial\partial\alpha - \bar\partial[(\partial\varphi)\wedge\alpha] \\ &= -(\partial\varphi)\wedge\bar\partial\alpha - (\bar\partial\partial\varphi)\wedge\alpha + (\partial\varphi)\wedge\bar\partial\alpha = \Theta_\varphi\wedge\alpha, \end{aligned}$$

where $\Theta_\varphi \equiv \partial\bar\partial\varphi$ is a differential form of type $(1, 1)$ (of course, $\theta_\varphi \wedge \alpha = 0$ if $\deg\alpha > 0$).

Definition 2.8.1 The *curvature* (or *curvature form* or *Levi form*) associated to a real-valued $\mathcal{C}^\infty$ function φ on X is the differential form $\Theta = \Theta_\varphi \equiv \partial\bar\partial\varphi$. In other words, Θ is defined by

$$D^2 = D'D'' + D''D' = \Theta \wedge (\cdot).$$

The function φ is called *subharmonic* (*strictly subharmonic*, *harmonic*, *superharmonic*, *strictly superharmonic*) if the real $(1, 1)$-form $i\Theta_\varphi$ satisfies $i\Theta_\varphi \geq 0$ (respectively, $i\Theta_\varphi > 0$, $i\Theta_\varphi = 0$, $i\Theta_\varphi \leq 0$, $i\Theta_\varphi < 0$). In a slight abuse of language, we also say that φ has *nonnegative* (respectively, *positive*, *zero*, *nonpositive*, *negative*) *curvature*.

Remarks 1. The above terminology, with the same definitions, is also applied to $\mathcal{C}^2$ functions. There is also a natural and useful notion of a *continuous* subharmonic function (see, for example, [Ns5]), but continuous subharmonic functions are not considered in this book.

2. For any local holomorphic coordinate neighborhood $(U, z = x + iy)$ in X and for any real-valued $\mathcal{C}^\infty$ function φ on U, we have

$$i\Theta_\varphi = \frac{\partial^2\varphi}{\partial z\partial\bar z}\, i\, dz \wedge d\bar z = 2\frac{\partial^2\varphi}{\partial z\partial\bar z}\, dx \wedge dy = \frac{1}{2}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) dx \wedge dy.$$

Thus φ is subharmonic (strictly subharmonic, harmonic) on U if and only if

$$\frac{\partial^2\varphi}{\partial z\partial\bar z} \geq 0 \quad (\text{respectively, } > 0, \quad = 0).$$

3. It is easy to see that a strictly subharmonic function cannot attain a local maximum (see Exercise 2.8.2). In fact, one can show that subharmonic functions satisfy a strong maximum principle (see, for example, [Ns5]). In particular, every subharmonic function on a compact Riemann surface is constant.

Proposition 2.8.2 (Fundamental estimate for scalar-valued forms) *Let φ be a real-valued $\mathcal{C}^\infty$ function on X, and let $D' = D'_\varphi$, $\Theta = \Theta_\varphi$, and $D'' = \bar\partial$. Then, for all functions $u, v \in \mathcal{D}(X)$, we have*

$$\langle D'u, D'v\rangle_{L^2(X,\varphi)} = \langle D''u, D''v\rangle_{L^2(X,\varphi)} + \int_X i\Theta u\bar v e^{-\varphi}.$$

In particular, $\|D'u\|^2_{L^2(X,\varphi)} = \|D''u\|^2_{L^2(X,\varphi)} + \int_X i\Theta|u|^2 e^{-\varphi} \geq \int_X i\Theta|u|^2 e^{-\varphi}$.

Remark If $i\Theta \geq 0$, then we get

$$\langle D'u, D'v\rangle_{L^2(X,\varphi)} = \langle D''u, D''v\rangle_{L^2(X,\varphi)} + \langle u, v\rangle_{L^2(X,i\Theta,\varphi)}$$

and $\|D'u\|^2_{L^2(X,\varphi)} = \|D''u\|^2_{L^2(X,\varphi)} + \|u\|^2_{L^2(X,i\Theta,\varphi)} \geq \|u\|^2_{L^2(X,i\Theta,\varphi)}$.

Proof of Proposition 2.8.2 For each pair of functions $u, v \in \mathcal{D}(X)$, Proposition 2.7.3 gives

$$\begin{aligned}
\langle D'u, D'v\rangle_{L^2(X,\varphi)} &= \int_X iD'u \wedge \overline{D'v}e^{-\varphi} = \int_X i(D''D'u)\cdot \bar v e^{-\varphi} \\
&= -\int_X i(D'D''u)\cdot \bar v e^{-\varphi} + \int_X i\Theta u\bar v e^{-\varphi} \\
&= -i\overline{\left[\int_X v\overline{D'D''u}\cdot e^{-\varphi}\right]} + \int_X i\Theta u\bar v e^{-\varphi} \\
&= i\overline{\left[\int_X (D''v)\wedge \overline{D''u}\cdot e^{-\varphi}\right]} + \int_X i\Theta u\bar v e^{-\varphi} \\
&= -i\int_X (D''u)\wedge \overline{D''v}\cdot e^{-\varphi} + \int_X i\Theta u\bar v e^{-\varphi} \\
&= \langle D''u, D''v\rangle_{L^2(X,\varphi)} + \int_X i\Theta u\bar v e^{-\varphi}.
\end{aligned}$$
□

Exercises for Sect. 2.8

2.8.1 Show that the function $z \mapsto |z|^2$ on $\mathbb{C}$ is strictly subharmonic.
2.8.2 Show that if φ is a $\mathcal{C}^2$ strictly subharmonic function (i.e., $i\partial\bar\partial\varphi > 0$) on a Riemann surface X, then φ cannot attain a local maximum.

2.9 The L^2 $\bar\partial$-Method for Scalar-Valued Forms of Type $(1,0)$

The following theorem (in various guises) is the main tool in this book:

Theorem 2.9.1 *Let X be a Riemann surface, let φ be a real-valued $\mathcal{C}^\infty$ function on X with $i\Theta = i\Theta_\varphi = i\partial\bar\partial\varphi \geq 0$, and let $Z = \{x \in X \mid \Theta_x = 0\}$. Then, for every measurable $(1,1)$-form β on X with $\beta\restriction_{X\setminus Z} \in L^2_{1,1}(X\setminus Z, i\Theta, \varphi)$ and $\beta = 0$ a.e. in Z (in particular, β is in L^2_{loc} on X), there exists a form $\alpha \in L^2_{1,0}(X,\varphi)$ such that*

$$D''_{\mathrm{distr}}\alpha = \bar\partial_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,\varphi)} \leq \|\beta\|_{L^2(X\setminus Z,i\Theta,\varphi)}.$$

In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0}\cup\{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar\partial\alpha = \beta$.

Remark The form β in the above theorem is in L^2_{loc} because β vanishes on Z, while for any compact set K in any local holomorphic coordinate neighborhood (U, z), we have

$$\|\cdot\|_{L^2_{1,1}(K\setminus Z,(i/2)\,dz\wedge d\bar{z})} \leq A\|\cdot\|_{L^2_{1,1}(K\setminus Z,i\Theta,\varphi)},$$

where

$$A \equiv \left(\max_K \frac{i\Theta}{(i/2)\,dz\wedge d\bar{z}}e^{\varphi}\right)^{1/2} < \infty.$$

Proof of Theorem 2.9.1 Let $N \equiv \|\beta\|_{L^2(X\setminus Z,i\Theta,\varphi)} = \|\beta/(i\Theta)\|_{L^2(X\setminus Z,i\Theta,\varphi)}$. For each function $f \in \mathcal{D}(X)$, the Schwarz inequality and the fundamental estimate (Proposition 2.8.2) give

$$\begin{aligned}
\left|\int_X f\bar{\beta}e^{-\varphi}\right| &= \left|\int_{X\setminus Z} f\cdot\frac{\bar{\beta}}{i\Theta}\cdot e^{-\varphi}\cdot i\Theta\right| = \left|\langle f, \beta/(i\Theta)\rangle_{L^2(X\setminus Z,i\Theta,\varphi)}\right| \\
&\leq \|f\|_{L^2(X\setminus Z,i\Theta,\varphi)}\cdot\|\beta/(i\Theta)\|_{L^2(X\setminus Z,i\Theta,\varphi)} \\
&\leq N\cdot\|f\|_{L^2(X,i\Theta,\varphi)} \leq N\cdot\|D'f\|_{L^2(X,\varphi)}.
\end{aligned}$$

It follows that the mapping $\Upsilon\colon [D'f] \mapsto -i\int_X f\bar{\beta}e^{-\varphi}$ is a bounded complex linear functional on the subspace $D'[\mathcal{D}(X)]$ of $L^2_{1,0}(X,\varphi)$. For by the above inequality, Υ is well defined, and for each $f \in \mathcal{D}(X)$, we have $|\Upsilon[D'f]| \leq N\cdot\|D'f\|_{L^2(X,\varphi)}$. In particular, $\|\Upsilon\| \leq N$. By the Hahn–Banach theorem (Theorem 7.5.11), there exists a bounded linear functional $\widehat{\Upsilon}$ on $L^2_{1,0}(X,\varphi)$ such that $\widehat{\Upsilon}\restriction_{D'[\mathcal{D}(X)]} = \Upsilon$ and $\|\widehat{\Upsilon}\| = \|\Upsilon\|$. Therefore, by Theorem 7.5.10, there exists a (unique) element $\alpha \in L^2_{1,0}(X,\varphi)$ such that $\|\alpha\|_{L^2(X,\varphi)} = \|\widehat{\Upsilon}\| \leq N$ and $\widehat{\Upsilon}(\cdot) = \langle\cdot,\alpha\rangle_{L^2(X,\varphi)}$. Moreover, for each $f \in \mathcal{D}(X)$, we have

$$\int_X i\alpha\wedge\overline{D'f}e^{-\varphi} = \overline{\Upsilon(D'f)} = \overline{\int_X(-i)f\bar{\beta}e^{-\varphi}} = \int_X i\beta\bar{f}e^{-\varphi}.$$

Therefore, by Proposition 2.7.3, $D''_{\mathrm{distr}}\alpha = \beta$, as required. Finally, the regularity statement at the end follows from Theorem 2.7.4. □

It is often more convenient to apply Theorem 2.9.1 in one of the following forms, the proofs of which are left to the reader (see Exercises 2.9.1 and 2.9.2):

Corollary 2.9.2 *Suppose that X is a Riemann surface, ω is a Kähler form on X, φ is a real-valued $\mathcal{C}^\infty$ function on X, ρ is a nonnegative measurable function on X with $i\Theta_\varphi \geq \rho\omega$, and $Z = \{x \in X \mid \rho(x) = 0\}$. Then, for every measurable* $(1,1)$*-form β on X with $\beta = 0$ a.e. in Z and*

$$\beta\restriction_{X\setminus Z} \in L^2_{1,1}(X\setminus Z,\rho\omega,\varphi) = L^2_{1,1}(X\setminus Z,\omega,\varphi+\log\rho)$$

(*in particular, β is in L^2_{loc} on X*), *there exists a form $\alpha \in L^2_{1,0}(X, \varphi)$ such that*

$$D''_{\mathrm{distr}}\alpha = \bar{\partial}_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,\varphi)} \leq \|\beta\|_{L^2(X\setminus Z,\rho\omega,\varphi)}.$$

In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.

Remark The main point is that $\|\beta\|_{L^2(X\setminus Z,i\Theta_\varphi,\varphi)} \leq \|\beta\|_{L^2(X\setminus Z,\rho\omega,\varphi)}$ if $i\Theta_\varphi \geq \rho\omega$.

Corollary 2.9.3 *Suppose that X is a Riemann surface, ω is Kähler form on X, φ is a real-valued $\mathcal{C}^\infty$ function on X, and C is a positive constant with $i\Theta_\varphi \geq C^2\omega$. Then, for every form $\beta \in L^2_{1,1}(X, \omega, \varphi)$, there exists a form $\alpha \in L^2_{1,0}(X, \varphi)$ such that*

$$D''_{\mathrm{distr}}\alpha = \bar{\partial}_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,\varphi)} \leq C^{-1}\|\beta\|_{L^2(X,\omega,\varphi)}.$$

In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.

Remark In Sect. 2.11, we will show that every Riemann surface admits a Kähler form.

Exercises for Sect. 2.9

2.9.1 Prove Corollary 2.9.2.
2.9.2 Prove Corollary 2.9.3.
2.9.3 Let X be a Riemann surface, let φ be a real-valued $\mathcal{C}^\infty$ function on X with $i\Theta = i\Theta_\varphi \geq 0$, and let $Z = \{x \in X \mid \Theta_x = 0\}$. Prove that for every $\mathcal{C}^\infty$ (1, 1)-form β on X with $\beta = 0$ a.e. in Z and $\beta\restriction_{X\setminus Z} \in L^2_{1,1}(X \setminus Z, i\Theta, \varphi)$, there exists a $\mathcal{C}^\infty$ form $\alpha \in L^2_{0,1}(X, \varphi)$ such that

$$\partial\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2_{0,1}(X,\varphi)} \leq \|\beta\|_{L^2_{1,1}(X\setminus Z,i\Theta,\varphi)}.$$

2.9.4 Let X be a Riemann surface, let φ be a real-valued $\mathcal{C}^\infty$ function on X, let ω be a nonnegative $\mathcal{C}^\infty$ (1, 1)-form on X, let $Z = \{x \in X \mid \omega_x = 0\}$, and let β be a $\mathcal{C}^\infty$ (1, 1)-form on X such that $\beta = 0$ a.e. in Z and $\beta\restriction_{X\setminus Z} \in L^2_{1,1}(X \setminus Z, \omega, \varphi)$. Prove that $\beta \equiv 0$ on Z (hence in the $\mathcal{C}^\infty$ case of Theorem 2.9.1 and in Exercise 2.9.3, there is no loss of generality if we assume that $\beta \equiv 0$ on Z).

2.10 Existence of Meromorphic 1-Forms and Meromorphic Functions

The power of Theorem 2.9.1 is demonstrated by the following important application, which will play a crucial role in the proof of such central facts as Radó's theorem on second countability (see Sect. 2.11) and the Riemann mapping theorem (see Chap. 5):

Theorem 2.10.1 *For every point in a Riemann surface, there exists a meromorphic 1-form that is holomorphic except for a pole of arbitrary prescribed order ≥ 2 at the point. In fact, for each integer $m \geq 2$, there exists a universal constant $C_m > 0$ such that if X is any Riemann surface, p is any point in X, and $(D, \Phi = z, \Delta(0; 1))$ is any local holomorphic chart with $p \in D$ and $\Phi(p) = z(p) = 0$, then there exists a meromorphic 1-form θ on X with the following properties:*

(i) *The meromorphic 1-form θ is holomorphic on $X \setminus \{p\}$ and has a pole of order m at p;*
(ii) *We have $\|\theta\|_{L^2(X\setminus D)} \leq C_m$;*
(iii) *The meromorphic 1-form $\theta - z^{-m}\, dz$ on D has at worst a simple pole at p; and*
(iv) *We have $\|z\theta - z^{-m+1}\, dz\|_{L^2(D)} \leq C_m$.*

Remark The value for the constant C_m that we will obtain is far from optimal.

Lemma 2.10.2 *For any choice of constants $a, b, c \in \mathbb{R}$ with $a < b < c$, there exists a $\mathcal{C}^\infty$ function $\chi : \mathbb{R} \to \mathbb{R}$ such that*

(i) *For each $t \in \mathbb{R}$, $\chi'(t) \geq 0$ and $\chi''(t) \geq 0$;*
(ii) *For each $t \leq a$, $\chi(t) = 0$; and*
(iii) *For each $t \geq c$, $\chi(t) = t - b$.*

Proof For $s > 0$, the function

$$\rho_s(t) \equiv \begin{cases} \frac{s}{s+\exp(\frac{1}{t-a}+\frac{1}{t-c})} = \frac{s\exp(\frac{1}{a-t})}{s\exp(\frac{1}{a-t})+\exp(\frac{1}{t-c})} & \text{if } a < t < c, \\ 0 & \text{if } t \leq a, \\ 1 & \text{if } c \leq t, \end{cases}$$

is of class $\mathcal{C}^\infty$, $\rho_s \geq 0$, and $\rho_s' \geq 0$. Moreover, the function $\mu \colon s \mapsto \int_a^c \rho_s(t)\, dt$ is continuous on $(0, \infty)$, $\mu(s) \to 0$ as $s \to 0^+$, and $\mu(s) \to c - a > c - b$ as $s \to \infty$ (for example, by the dominated convergence theorem). Therefore, by the intermediate value theorem, there is a number $s_0 > 0$ such that $\mu(s_0) = c - b$. The function $t \mapsto \chi(t) \equiv \int_a^t \rho_{s_0}(u)\, du$ then has the required properties. □

Lemma 2.10.3 *Let $r > 0$ be a constant and let ψ be a $\mathcal{C}^\infty$ strictly subharmonic function on $\Delta(0; r)$. Then there is a constant $b_0 = b_0(r, \psi) > 0$ such that for every constant $b > b_0$, there exist a constant $R = R(b, r, \psi) \in (0, r)$ and a nonnegative $\mathcal{C}^\infty$ subharmonic function φ on $\mathbb{C}^*$ with $\varphi \equiv 0$ on a neighborhood of $\mathbb{C} \setminus \Delta(0; r)$ and*

$$\varphi(z) = \psi(z) - \log|z|^2 - b \quad \forall z \in \Delta^*(0; R).$$

Proof Setting $\rho(z) \equiv \psi(z) - \log|z|^2$ for all $z \in \Delta^*(0; r)$, we may fix positive constants R_0 and R_1 with $0 < R_0 < R_1 < r$ and a positive constant $b_0 > \sup_{\Delta(0; R_0, R_1)} \rho$. Given a constant $b > b_0$, we may fix constants a and c with

$c > b > a > b_0 > 0$, and applying Lemma 2.10.2, we get a $\mathcal{C}^\infty$ function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi' \geq 0$ and $\chi'' \geq 0$ on $\mathbb{R}$, $\chi(t) = 0$ for $t \leq a$, and $\chi(t) = t - b$ for $t \geq c$. For $R \in (0, r)$ sufficiently small, we have $\rho > c$ on $\Delta^*(0; R)$. This constant R and the function φ on $\mathbb{C}^*$ given by

$$\varphi \equiv \begin{cases} \chi(\rho) & \text{on } \Delta^*(0; R_1), \\ 0 & \text{on } \mathbb{C} \setminus \Delta(0; R_1), \end{cases}$$

then have the required properties. For the choice of χ guarantees that φ is nonnegative and of class $\mathcal{C}^\infty$, φ vanishes on $\mathbb{C} \setminus \Delta(0; R_0)$, and $\varphi = \rho - b$ on $\Delta^*(0; R)$. Furthermore, on $\Delta^*(0; R_1)$, we have $i\Theta_\rho = i\Theta_\psi > 0$ and hence

$$i\Theta_\varphi = \chi'(\rho) \cdot i\Theta_\psi + \chi''(\rho) \cdot i\partial\rho \wedge \overline{\partial\rho} \geq 0. \qquad \square$$

Proof of Theorem 2.10.1 The idea of the proof (an idea that is applied in various guises throughout this book) is to first produce a $\mathcal{C}^\infty$ solution, that is, a $\mathcal{C}^\infty$ 1-form τ that has all of the required properties except that it is not meromorphic (away from p). Forming a suitable L^2 solution α of the equation $\bar\partial\alpha = \bar\partial\tau$, one gets a meromorphic 1-form $\theta = \tau - \alpha$ with the required properties.

Letting ζ denote the (standard) coordinate function on $\mathbb{C}$, setting $\psi_0 \equiv |\zeta|^2$, and applying Lemma 2.10.3, we get positive constants b, R_0, and R_1 and a nonnegative $\mathcal{C}^\infty$ subharmonic function φ_0 on $\mathbb{C}^*$ such that $R_0 < R_1 < 1$, $\varphi_0 \equiv 0$ on $\mathbb{C} \setminus \Delta(0; R_1)$, and

$$\varphi_0 = |\zeta|^2 - \log|\zeta|^2 - b \quad \text{on } \Delta^*(0; R_0).$$

In particular, $i\Theta_{\varphi_0} = i\Theta_{\psi_0} = i\,d\zeta \wedge d\bar\zeta > 0$ on $\Delta^*(0; R_0)$. We may also fix a constant $R_2 \in (0, R_0)$ and a function $\eta_0 \in \mathcal{D}(\Delta(0; R_0))$ such that $\eta_0 \equiv 1$ on $\Delta(0; R_2)$.

Given an integer $m \geq 2$, we have the meromorphic 1-form $\gamma_m \equiv \zeta^{-m} d\zeta$ on $\mathbb{C}$ and the $\mathcal{C}^\infty$ form $\tau_m \equiv \eta_0\gamma_m$ of type $(1, 0)$ on $\mathbb{C}^*$. In particular, $\tau_m \equiv 0$ on a neighborhood of $\mathbb{C} \setminus \Delta(0; R_0)$ and τ_m is holomorphic on $\Delta^*(0; R_2)$. Hence $\beta_m \equiv \bar\partial\tau_m = \bar\partial\eta_0 \wedge \gamma_m$ is a $\mathcal{C}^\infty$ 2-form that vanishes on $\mathbb{C}^* \setminus \Delta(0; R_2, R_0)$. Observe that

$$A_m \equiv \|\beta_m\|_{L^2(\Delta^*(0;R_0), i\Theta_{\varphi_0}, \varphi_0)} = \|\beta_m\|_{L^2(\Delta(0;R_2,R_0), i\,d\zeta\wedge d\bar\zeta, \varphi_0)} < \infty$$

and

$$B_m \equiv \|\zeta\tau_m - \zeta\gamma_m\|_{L^2(\Delta^*(0;1))} = \|(\eta_0 - 1)d\zeta/\zeta^{m-1}\|_{L^2(\Delta(0;R_2,1))} < \infty.$$

By construction, the function $\varphi_0 + \log|\zeta|^2$ is bounded on $\Delta^*(0; 1)$, so we may define a positive constant C_m by

$$C_m \equiv A_m + B_m + \sup_{\Delta^*(0;1)} e^{(\varphi_0 + \log|\zeta|^2)/2} \cdot A_m < \infty.$$

Suppose now that p is a point in a Riemann surface X $(D, \Phi = z, \Delta(0; 1))$ is a local holomorphic chart with $p \in D$ and $\Phi(p) = z(p) = 0$, and $Y \equiv X \setminus \{p\}$. The form τ on Y that is equal to $\Phi^*\tau_m$ on $D \setminus \{p\}$ and 0 elsewhere is a $\mathcal{C}^\infty$ differential form of type $(1,0)$ that vanishes on $X \setminus \Phi^{-1}(\Delta(0; R_0))$ and that is holomorphic on $\Phi^{-1}(\Delta^*(0; R_2))$. The function φ on Y given by $\varphi = \varphi_0(z)$ on $D \setminus \{p\}$ and 0 elsewhere is nonnegative, subharmonic, and of class $\mathcal{C}^\infty$. Moreover, φ vanishes on $X \setminus \Phi^{-1}(\Delta(0; R_1))$ and $i\Theta_\varphi = i\,dz \wedge d\bar{z} > 0$ on $\Phi^{-1}(\Delta^*(0; R_0))$. The $\mathcal{C}^\infty$ $(1,1)$-form $\beta \equiv \bar{\partial}\tau$ on Y is equal to $\Phi^*\beta_m$ on $D \setminus \{p\}$, vanishes on

$$Y \setminus \Phi^{-1}(\Delta(0; R_2, R_0)) \supset Z \equiv \left\{q \in Y \mid i(\Theta_\varphi)_q = 0\right\},$$

and satisfies

$$\|\beta\|_{L^2(Y \setminus Z, i\Theta_\varphi, \varphi)} = \|\beta_m\|_{L^2(\Delta^*(0; R_0), i\Theta_{\varphi_0}, \varphi_0)} = A_m < \infty.$$

Therefore, by Theorem 2.9.1, there exists a $\mathcal{C}^\infty$ $(1,0)$-form α on Y such that

$$\bar{\partial}\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(Y,\varphi)} \leq A_m.$$

Thus $\mathcal{C}^\infty$ $(1,0)$-form $\theta \equiv \tau - \alpha$ on Y satisfies $\bar{\partial}\theta = 0$ and is therefore holomorphic on Y. Since τ and therefore α are holomorphic on $\Phi^{-1}(\Delta^*(0; R_2))$, we have $\alpha \restriction_{\Phi^{-1}(\Delta^*(0; R_2))} = f(z)dz$ for some function $f \in \mathcal{O}(\Delta^*(0; R_2))$. In particular,

$$\begin{aligned}\int_{\Delta^*(0; R_2)} |\zeta f(\zeta)|^2 \frac{i}{2} d\zeta \wedge d\bar{\zeta} &= \int_{\Phi^{-1}(\Delta^*(0; R_2))} \frac{i}{2}\alpha \wedge \bar{\alpha} \cdot e^{-\varphi} \cdot e^{|z|^2 - b} \\ &\leq \frac{1}{2} e^{R_2^2 - b} \cdot \|\alpha\|^2_{L^2(Y,\varphi)} < \infty.\end{aligned}$$

Hence, by Riemann's extension theorem (Theorem 1.2.10), the function $\zeta \mapsto \zeta f(\zeta)$ extends to a (unique) holomorphic function on $\Delta(0; R_2)$; that is, f extends to a meromorphic function with at worst a simple pole at 0. Therefore, since $\eta_0 \equiv 1$ on $\Delta(0; R_2)$ and γ_m is a meromorphic 1-form with a pole of order $m > 1$ at 0, we see that $\theta = \tau - \alpha$ determines a meromorphic 1-form on X that is holomorphic on Y and that has a pole of order m at p.

For the bounds, we observe that

$$\|\theta\|_{L^2(X \setminus D)} = \|-\alpha\|_{L^2(X \setminus D)} = \|\alpha\|_{L^2(X \setminus D, \varphi)} \leq A_m \leq C_m$$

and that

$$\begin{aligned}\|z\theta - z^{-m+1}dz\|_{L^2(D)} &\leq \|z\tau - z^{-m+1}dz\|_{L^2(D)} + \|z\alpha\|_{L^2(D)} \\ &\leq B_m + \sup_{\Delta^*(0;1)} e^{(\varphi_0 + \log|\zeta|^2)/2} \cdot \|\alpha\|_{L^2(D,\varphi)} \\ &\leq B_m + \sup_{\Delta^*(0;1)} e^{(\varphi_0 + \log|\zeta|^2)/2} \cdot \|\alpha\|_{L^2(Y,\varphi)} \\ &\leq B_m + \sup_{\Delta^*(0;1)} e^{(\varphi_0 + \log|\zeta|^2)/2} \cdot A_m \leq C_m.\end{aligned}$$

□

Corollary 2.10.4 *Every Riemann surface X admits a nonconstant meromorphic function.*

Proof We may choose two distinct points p and q in X. Applying Theorem 2.10.1, we get two meromorphic 1-forms η and θ that are holomorphic except for poles of order 2 at p and q, respectively. The quotient $f = \eta/\theta$ then determines a nonconstant meromorphic function on X. □

Exercises for Sect. 2.10

2.10.1 Show that there exists a constant $A > 0$ such that for each integer $m \geq 2$, the constant $C_m \equiv A^m$ has the properties described in Theorem 2.10.1.

2.10.2 Prove the following generalization of Theorem 2.10.1. Let X_0 be a Riemann surface, let θ_0 be a meromorphic 1-form on X_0 that is holomorphic except for a pole of order $m \geq 2$ at some point $p_0 \in X_0$, let Ω_0 be a relatively compact neighborhood of p_0 in X_0, and let f_0 be a holomorphic function on X_0 that vanishes at p_0. Then there exists a constant $C = C(X_0, \theta_0, \Omega_0, f_0) > 0$ such that if X is any Riemann surface, p is a point in X, and $\Phi\colon \Omega \to \Omega_0$ is a biholomorphic mapping of a neighborhood Ω of p onto Ω_0 with $\Phi(p) = p_0$, then there exists a meromorphic 1-form θ on X with the following properties:

   (i) The meromorphic 1-form θ is holomorphic $X \setminus \{p\}$ and has a pole of order m at p;

   (ii) We have $\|\theta\|_{L^2(X\setminus\Omega)} \leq C$;

   (iii) The meromorphic 1-form $\theta - \Phi^*\theta_0$ on Ω has at worst a simple pole at p; and

   (iv) We have $\|f_0(\Phi)\cdot\theta - f_0(\Phi)\cdot\Phi^*\theta_0\|_{L^2(\Omega)} \leq C$.

2.11 Radó's Theorem on Second Countability

The goal of this section is the following important theorem:

Theorem 2.11.1 (Radó) *Every Riemann surface X is second countable*; *that is*, *the topology in X admits a countable basis.*

The proof considered here is similar to that in [Sp].

Lemma 2.11.2 *Every nonempty open subset Ω of a Riemann surface X has only countably many connected components.*

Proof Fixing a point $p \in \Omega$ and a connected neighborhood U of p in Ω, and applying Theorem 2.10.1, we get a nontrivial (i.e., not everywhere zero) holomorphic 1-form θ on $X \setminus \{p\}$ with $\|\theta\|_{L^2(X\setminus U)} < \infty$. If $\{\Omega_i\}_{i\in I}$ is the family of (distinct) connected components of Ω that do *not* meet (i.e., which do not contain) U and $I \neq \emptyset$, then for every finite set $J \subset I$, we have

$$\|\theta\|^2_{L^2(X\setminus U)} \geq \|\theta\|^2_{L^2(\bigcup_{i\in J}\Omega_i)} = \sum_{i\in J} \|\theta\|^2_{L^2(\Omega_i)}.$$

Therefore $\infty > \sum_{i\in I} \|\theta\|^2_{L^2(\Omega_i)}$, and hence, since the right-hand side is an unordered sum of (strictly) positive terms (see Example 7.1.3), the index set I must be countable. □

Proof of Radó's theorem Corollary 2.10.4 provides a nonconstant meromorphic function, that is, a nonconstant holomorphic mapping $f\colon X \to \mathbb{P}^1$. Let $\mathcal{P}$ be the countable collection of all sets $D \subset \mathbb{P}^1$, where D is either a disk in $\mathbb{C}$ with rational radius and center in $\mathbb{Q} + i\mathbb{Q}$, or $D = \mathbb{P}^1 \setminus \overline{\Delta(0;r)}$ for some $r \in \mathbb{Q}_{>0}$. Let $\mathcal{B}$ be the collection of all relatively compact open subsets B of X such that B is a connected component of $f^{-1}(D)$ for some set $D \in \mathcal{P}$. It follows that $\mathcal{B}$ is a basis for the topology in X (see Exercise 2.11.1). Moreover, by Lemma 2.11.2, the set $f^{-1}(D)$ has only countably many connected components for each $D \in \mathcal{P}$, and therefore the basis $\mathcal{B}$ must be countable. □

Radó's theorem and Proposition 9.7.6 together give the following:

Corollary 2.11.3 *Every Riemann surface X admits a Kähler form* (*i.e., a positive $\mathcal{C}^\infty$ differential form of type* $(1,1)$). *In fact, given a continuous real* $(1,1)$*-form τ on X, there exists a Kähler form ω with $\omega \geq \tau$ on X.*

Corollary 2.11.4 (Montel's theorem for a Riemann surface) *If $\{f_n\}$ is a sequence of holomorphic functions on a Riemann surface X and $\{f_n\}$ is uniformly bounded on each compact subset of X, then some subsequence of $\{f_n\}$ converges uniformly on compact subsets of X to a holomorphic function on X.*

Proof We may choose a countable open covering $\{U_\nu\}$ of X such that for each ν, U_ν is relatively compact in some local holomorphic coordinate neighborhood. Montel's theorem in the plane (Corollary 1.2.7) and Cantor's diagonal process together yield a subsequence $\{f_{n_k}\}$ such that the sequence $\{f_{n_k}\restriction_{U_\nu}\}$ converges uniformly to a holomorphic function on U_ν for each ν. It follows that $\{f_{n_k}\}$ converges uniformly on compact subsets of X to some holomorphic function on X. □

Remark A $\mathcal{C}^\infty$ surface need not be second countable. Moreover, Radó's theorem is false in higher dimensions; that is, there exist connected 2-dimensional complex manifolds that are *not* second countable, provided, of course, one does not include second countability as part of the definition (see, for example, [Hu]).

Exercises for Sect. 2.11 In Exercises 2.11.1 and 2.11.2 below, as in the proof of Radó's theorem (Theorem 2.11.1), X denotes a Riemann surface; $f\colon X \to \mathbb{P}^1$ denotes a nonconstant holomorphic mapping; $\mathcal{P}$ denotes the collection of all sets

$D \subset \mathbb{P}^1$, where D is either a disk in $\mathbb{C}$ with rational radius and center in $\mathbb{Q} + i\mathbb{Q}$, or $D = \mathbb{P}^1 \setminus \overline{\Delta(0;r)}$ for some $r \in \mathbb{Q}_{>0}$; and $\mathcal{B}$ denotes the collection of all relatively compact open subsets B of X such that B is a connected component of $\Phi^{-1}(D)$ for some set $D \in \mathcal{P}$.

2.11.1 Prove that $\mathcal{B}$ is a basis for the topology in X (this fact was used in the proof of Radó's theorem).

2.11.2 Another way to see that $\mathcal{B}$ is countable is to observe that each element $B \in \mathcal{B}$ meets at most countably many connected components of $f^{-1}(D)$ for each set $D \in \mathcal{P}$. Prove this observation, and using it in place of Lemma 2.11.2, prove Radó's theorem.

Remark This argument is essentially the proof of a special case of the Poincaré–Volterra theorem (see, for example, [Ns5]).

2.12 The L^2 $\bar\partial$-Method for Scalar-Valued Forms of Type (0, 0)

It is also useful to consider solutions of the inhomogeneous Cauchy–Riemann equation $\bar\partial\alpha = \beta$ in which α is a function and β is a differential form of type $(0,1)$. In fact, this problem is essentially equivalent to the problem considered in Sect. 2.9. In order to obtain the L^2 solution, we will have to consider the curvature form associated to a Kähler form ω (i.e., a $\mathcal{C}^\infty$ positive differential form of type $(1,1)$) on a complex 1-manifold X. Suppose θ_1 and θ_2 are two nonvanishing holomorphic 1-forms on an open set $U \subset X$ and $G_j \equiv \omega/(i\theta_j \wedge \bar\theta_j)$ for $j = 1,2$. Then the difference

$$(-\log G_1) - (-\log G_2) = \log|\theta_1/\theta_2|^2$$

is a harmonic function; that is, $\partial\bar\partial[-\log G_1] = \partial\bar\partial[-\log G_2]$. Thus we may make the following definition:

Definition 2.12.1 For any Kähler form ω on a complex 1-manifold X, the associated *curvature* is the unique differential form $\Theta_{(X,\omega)} = \Theta_\omega$ that satisfies

$$\Theta_\omega\restriction_U = \Theta_{-\log(\omega/(i\theta\wedge\bar\theta))} = \partial\bar\partial[-\log(\omega/(i\theta \wedge \bar\theta))]$$

for every nonvanishing local holomorphic 1-form θ on an open set $U \subset X$.

Remarks 1. Equivalently, if $\omega = G \cdot (i/2)\,dz \wedge d\bar z$ (i.e., $G = -2i\omega(\partial/\partial z, \partial/\partial\bar z) = \omega(\partial/\partial x, \partial/\partial y)$) in a local holomorphic coordinate neighborhood $(U, z = x + iy)$, then $\Theta_\omega = \Theta_{-\log G} = \partial\bar\partial[-\log G]$ on U.

2. If φ is a real-valued $\mathcal{C}^\infty$ function on X, then $\Theta_{e^{-\varphi}\omega} = \Theta_\omega + \Theta_\varphi$.

Example 2.12.2 For the Euclidean Kähler form $\omega \equiv (i/2)\,dz \wedge d\bar z$ on $\mathbb{C}$, $\Theta_\omega \equiv 0$.

Example 2.12.3 The *chordal Kähler form* on the Riemann sphere $\mathbb{P}^1$ is the unique Kähler form ω for which $\omega\restriction_{\mathbb{C}} = 2i(1+|z|^2)^{-2}dz \wedge d\bar z$. The corresponding curvature form satisfies $i\Theta_\omega = \omega$ (see Exercise 2.12.1).

The main goal of this section is the following:

Theorem 2.12.4 *Let X be a Riemann surface, let ω be a Kähler form on X, let φ be a real-valued $\mathcal{C}^\infty$ function on X with*

$$i\Theta \equiv i\Theta_{e^{-\varphi}\omega} = i\Theta_\omega + i\Theta_\varphi \geq 0,$$

let $\rho \equiv i\Theta/\omega$, and let $Z = \{x \in X \mid \Theta_x = 0\} = \{x \in X \mid \rho(x) = 0\}$. Then, for every measurable $(0,1)$-form β on X with $\beta = 0$ a.e. in Z and $\beta\restriction_{X\setminus Z} \in L^2_{0,1}(X \setminus Z, \varphi + \log\rho)$ (in particular, β is in L^2_{loc} on X), there exists a function $\alpha \in L^2_{0,0}(X, \omega, \varphi)$ such that

$$D''_{\mathrm{distr}}\alpha = \bar{\partial}_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,\omega,\varphi)} \leq \|\beta\|_{L^2(X\setminus Z,\varphi+\log\rho)}.$$

In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.

Remarks 1. In order to apply Theorem 2.12.4, one needs a Kähler form ω. Fortunately, by Corollary 2.11.3, a Kähler form always exists. In fact, this version is actually equivalent to Theorem 2.9.1.

2. Although it is possible to obtain the theorem directly, we will instead prove the theorem by first forming the exterior product of the given form and a meromorphic 1-form (provided by Theorem 2.10.1), and then applying the solution for forms of type $(1,0)$ (Theorem 2.9.1).

3. The proof that β is in L^2_{loc} in the above (as well as in Corollary 2.12.5 below) is similar to that in the remark following the statement of Theorem 2.9.1.

It is often more convenient to apply Theorem 2.12.4 in one of the following forms, the proofs of which are left to the reader (see Exercises 2.12.2 and 2.12.3):

Corollary 2.12.5 *Suppose that X is a Riemann surface, ω is a Kähler form on X, φ is a real-valued $\mathcal{C}^\infty$ function on X, ρ is a nonnegative measurable function on X with $i\Theta_\omega + i\Theta_\varphi \geq \rho\omega$, and $Z = \{x \in X \mid \rho(x) = 0\}$. Then, for every measurable $(0,1)$-form β on X with $\beta = 0$ a.e. in Z and $\beta\restriction_{X\setminus Z} \in L^2_{0,1}(X \setminus Z, \varphi + \log\rho)$ (in particular, β is in L^2_{loc} on X), there exists a function $\alpha \in L^2_{0,0}(X, \omega, \varphi)$ such that*

$$D''_{\mathrm{distr}}\alpha = \bar{\partial}_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,\omega,\varphi)} \leq \|\beta\|_{L^2(X\setminus Z,\varphi+\log\rho)}.$$

In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.

Corollary 2.12.6 *Suppose that X is a Riemann surface, ω is a Kähler form on X, φ is a real-valued $\mathcal{C}^\infty$ function on X, and C is a positive constant for which $i\Theta_\omega + i\Theta_\varphi \geq C^2\omega$ on X. Then, for every form $\beta \in L^2_{0,1}(X, \varphi)$, there exists a func-*

tion $\alpha \in L^2_{0,0}(X, \omega, \varphi)$ *such that*

$$D''_{\mathrm{distr}}\alpha = \bar{\partial}_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,\omega,\varphi)} \leq C^{-1}\|\beta\|_{L^2(X,\varphi)}.$$

In particular, if β *is of class* $\mathcal{C}^k$ *for some* $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, *then* α *is also of class* $\mathcal{C}^k$ *and* $\bar{\partial}\alpha = \beta$.

Proof of Theorem 2.12.4 According to Theorem 2.10.1, we may fix a nontrivial meromorphic 1-form θ on X, and we may let Q be the (discrete) set of zeros and poles of θ. On the Riemann surface $Y \equiv X \setminus Q$, the function $\tau \equiv \omega/(i\theta \wedge \bar{\theta})$ is then positive and of class $\mathcal{C}^\infty$, and we have $\Theta_{-\log\tau} = \Theta_\omega$; that is, $i\Theta_{\varphi-\log\tau} = i\Theta = \rho\omega$. The form $\beta_0 \equiv \theta \wedge \beta$ is a measurable differential form of type (1, 1) that vanishes on $Z \cap Y$ and that satisfies (since the set Q is discrete, and therefore of measure 0)

$$\begin{aligned}
&\|\beta_0\|^2_{L^2(Y\setminus Z, i\Theta_{\varphi-\log\tau}, \varphi-\log\tau)} \\
&\quad = \|\beta_0\|^2_{L^2(Y\setminus Z,\rho\omega,\varphi-\log\tau)} = \int_{Y\setminus Z}\left|\frac{\theta\wedge\beta}{\rho\omega}\right|^2 e^{-(\varphi-\log\tau)}\rho\omega \\
&\quad = \int_{Y\setminus Z}\frac{\theta\wedge\beta}{\omega}\cdot\frac{\bar{\theta}\wedge\bar{\beta}}{\omega}\cdot\frac{\omega}{i\theta\wedge\bar{\theta}}\cdot e^{-(\varphi+\log\rho)}\omega \\
&\quad = \int_{Y\setminus Z} i\theta\wedge\beta\cdot\frac{i\theta\wedge\bar{\theta}}{i\theta\wedge\bar{\theta}}\cdot\frac{\bar{\beta}}{\theta}\cdot e^{-(\varphi+\log\rho)} \\
&\quad = \int_{X\setminus Z}(-i)\beta\wedge\bar{\beta}\cdot e^{-(\varphi+\log\rho)} = \|\beta\|^2_{L^2(X\setminus Z,\varphi+\log\rho)} < \infty.
\end{aligned}$$

Therefore, β_0 is in L^2_{loc} on Y, and by Theorem 2.9.1, there exists a form $\alpha_0 \in L^2_{1,0}(Y, \varphi - \log\tau)$ such that $D''_{\mathrm{distr}}\alpha_0 = \bar{\partial}_{\mathrm{distr}}\alpha_0 = \beta_0$ in Y and

$$\|\alpha_0\|_{L^2(Y,\varphi-\log\tau)} \leq \|\beta_0\|_{L^2(Y\setminus Z, i\Theta_{\varphi-\log\tau},\varphi-\log\tau)} = \|\beta\|_{L^2(X\setminus Z,\varphi+\log\rho)}.$$

The measurable function $\alpha \equiv -\alpha_0/\theta$ on X (again, Q is a set of measure 0) then satisfies

$$\begin{aligned}
\|\alpha\|^2_{L^2(X,\omega,\varphi)} &= \int_Y |\alpha|^2 e^{-\varphi}\omega = \int_Y |\alpha_0/\theta|^2 e^{-\varphi}\omega = \int_Y \frac{i\alpha_0\wedge\bar{\alpha}_0}{i\theta\wedge\bar{\theta}}e^{-\varphi}\omega \\
&= \int_Y i\alpha_0\wedge\bar{\alpha}_0\cdot e^{-(\varphi-\log\tau)} \\
&= \|\alpha_0\|^2_{L^2(Y,\varphi-\log\tau)} \leq \|\beta\|^2_{L^2(X\setminus Z,\varphi+\log\rho)}.
\end{aligned}$$

In particular, α is in L^2_{loc} on X, and it remains to show that $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$ on X. For each point $p \in X$, we may choose a local holomorphic coordinate neighborhood $(U, \Phi = z, \Delta(0; 2))$ such that $U \cap Q \subset \{p\}$ and $z(p) = 0$. The meromorphic function $f \equiv \theta/dz$ on U is then nonvanishing and holomorphic on $U \setminus \{p\} \subset Y$, and the measurable functions $a \equiv \alpha_0/dz$ and $b \equiv \beta/d\bar{z}$ on U are in $L^2_{\mathrm{loc}} \subset L^1_{\mathrm{loc}}$. In particular, $\alpha = -a/f$ and $\beta_0 = fb\,dz \wedge d\bar{z}$ on U. We may also choose a nonnegative $\mathcal{C}^\infty$ function χ such that $\chi \equiv 1$ on $\mathbb{C} \setminus \Delta(0;2)$ and $\chi \equiv 0$ on $\Delta(0;1)$. For every $\mathcal{C}^\infty$ function η with compact support in $D \equiv \Phi^{-1}(\Delta(0;1))$, the dominated convergence theorem gives (here, we denote the coordinate on $\mathbb{C}$ by w)

$$\begin{aligned}
&\int_D \alpha \cdot \frac{\partial \eta}{\partial \bar{z}} \cdot \frac{i}{2}\, dz \wedge d\bar{z} \\
&= \lim_{\epsilon \to 0^+} \int_D \chi(z/\epsilon)\alpha \cdot \frac{\partial \eta}{\partial \bar{z}} \cdot \frac{i}{2}\, dz \wedge d\bar{z} \\
&= \lim_{\epsilon \to 0^+} \int_D \chi(z/\epsilon)\frac{-a}{f} \cdot \frac{\partial \eta}{\partial \bar{z}} \cdot \frac{i}{2}\, dz \wedge d\bar{z} \\
&= \lim_{\epsilon \to 0^+} \left[-\int_D a \cdot \frac{\partial}{\partial \bar{z}}\left(\chi(z/\epsilon)\frac{\eta}{f} \right) \cdot \frac{i}{2}\, dz \wedge d\bar{z} \right. \\
&\qquad \left. + \int_D a \frac{1}{\epsilon}\frac{\partial \chi}{\partial \bar{w}}(z/\epsilon)\frac{\eta}{f} \cdot \frac{i}{2}\, dz \wedge d\bar{z} \right] \\
&= \lim_{\epsilon \to 0^+} \left[-\int_D b \cdot \chi(z/\epsilon)\eta \cdot \frac{i}{2}\, dz \wedge d\bar{z} \right. \\
&\qquad \left. - \int_D \alpha \frac{1}{\epsilon}\frac{\partial \chi}{\partial \bar{w}}(z/\epsilon)\eta \cdot \frac{i}{2}\, dz \wedge d\bar{z} \right] \\
&= -\int_D b\eta \cdot \frac{i}{2}\, dz \wedge d\bar{z} - \lim_{\epsilon \to 0^+} \left[\int_D \alpha \frac{1}{\epsilon}\frac{\partial \chi}{\partial \bar{w}}(z/\epsilon)\eta \cdot \frac{i}{2}\, dz \wedge d\bar{z} \right].
\end{aligned}$$

On the other hand, since $\alpha \in L^2_{\mathrm{loc}}(X)$ and $\partial\chi/\partial\bar{w} \equiv 0$ on $\mathbb{C} \setminus \Delta(0;1,2)$, we have, for some constant $C > 0$,

$$\left| \int_D \alpha \frac{1}{\epsilon}\frac{\partial \chi}{\partial \bar{w}}(z/\epsilon)\eta \cdot \frac{i}{2}\, dz \wedge d\bar{z} \right| \leq C\|\alpha \circ \Phi^{-1}\|_{L^2(\Delta(0;\epsilon,2\epsilon))} \to 0 \quad \text{as } \epsilon \to 0^+.$$

It follows that $\bar\partial_{\mathrm{distr}}\alpha = \beta$. □

Exercises for Sect. 2.12

2.12.1 Verify that (see Example 2.12.3) there is a unique Kähler form ω on $\mathbb{P}^1$ such that

$$\omega\restriction_{\mathbb{C}} = 2i(1 + |z|^2)^{-2}\, dz \wedge d\bar{z}.$$

Verify also that $i\Theta_\omega = \omega$.

2.12.2 Prove Corollary 2.12.5.

2.12.3 Prove Corollary 2.12.6.

2.12.4 Let X be a Riemann surface, let ω be a Kähler form on X, let φ be a real-valued $\mathcal{C}^\infty$ function on X with $i\Theta \equiv i\Theta_{e^{-\varphi}\omega} = i\Theta_\omega + i\Theta_\varphi \geq 0$, let

$\rho \equiv i\Theta/\omega$, and let $Z = \{x \in X \mid \Theta_x = 0\} = \{x \in X \mid \rho(x) = 0\}$. Prove that for every $\mathcal{C}^\infty$ $(1,0)$-form β on X with $\beta = 0$ a.e. in Z and $\beta\restriction_{X\setminus Z} \in L^2_{1,0}(X \setminus Z, \varphi + \log\rho)$, there exists a $\mathcal{C}^\infty$ function $\alpha \in L^2_{0,0}(X, \omega, \varphi)$ such that

$$\partial\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(X,\omega,\varphi)} \leq \|\beta\|_{L^2(X\setminus Z,\varphi+\log\rho)}$$

(cf. Exercises 2.9.3 and 2.9.4).

2.13 Topological Hulls and Chains to Infinity

As suggested by Theorem 2.9.1 and its applications (see also Theorem 3.9.1 and its applications), it is useful to have available a large supply of $\mathcal{C}^\infty$ strictly subharmonic functions on an (open) Riemann surface, i.e., $\mathcal{C}^\infty$ functions φ for which $i\Theta_\varphi > 0$ (see Definition 2.8.1). In Sect. 2.14, it will be shown that every open Riemann surface admits a $\mathcal{C}^\infty$ strictly subharmonic exhaustion function. Recall that a function ρ on a Hausdorff space X is an *exhaustion function* if $\{x \in X \mid \rho(x) < a\} \Subset X$ for each $a \in \mathbb{R}$ (see Definition 9.3.10). The first proof of the existence of a $\mathcal{C}^\infty$ strictly subharmonic exhaustion function on a connected noncompact Riemannian manifold was obtained by Greene and Wu [GreW]. Demailly [De2] provided an elementary proof using a local construction. Demailly's proof may be modified (as well as simplified) to give an exhausting $\mathcal{C}^\infty$ strict subsolution for an arbitrary second-order linear elliptic differential operator with continuous coefficients on a second countable noncompact $\mathcal{C}^\infty$ manifold (for the details see [NR]). The proof of the existence of a $\mathcal{C}^\infty$ strictly subharmonic exhaustion function on an open Riemann surface appearing in this book is adapted from [NR]. In this section, we first consider some of the topological facts required for the proof.

Definition 2.13.1 For a subset A of a Hausdorff space X, the *topological hull* $\mathfrak{h}_X(A)$ *of A in X* is the union of A with all of the connected components of $X \setminus A$ that are relatively compact in X. An open subset Ω of X is called *topologically Runge in X* if $\mathfrak{h}_X(\Omega) = \Omega$.

Remark Intuitively, one obtains $\mathfrak{h}_X(A)$ by filling in the holes of A (see Fig. 2.5).

The proofs of the following properties are left to the reader (see Exercise 2.13.1):

Lemma 2.13.2 *Let X be a Hausdorff space.*

(a) *For every set $A \subset X$, we have $\mathfrak{h}_X(\mathfrak{h}_X(A)) = \mathfrak{h}_X(A) \supset A$.*
(b) *If $A \subset B \subset X$, then $\mathfrak{h}_X(A) \subset \mathfrak{h}_X(B)$.*
(c) *If X is locally connected and A is a closed subset of X, then $\mathfrak{h}_X(A)$ is closed.*

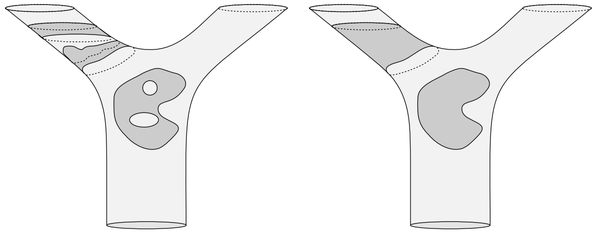

Fig. 2.5 A subset and its topological hull

For a compact subset of a suitable topological space (such as a manifold), the topological hull has a nice form (cf. [Mal] and [Ns5]):

Lemma 2.13.3 *If K is a compact subset of a connected, noncompact, locally connected, locally compact Hausdorff space X, then $\mathfrak{h}_X(K)$ is compact; $X \setminus \mathfrak{h}_X(K)$ has only finitely many components; and for each point $p \in X \setminus \mathfrak{h}_X(K)$, there is a connected noncompact closed subset C of X with $p \in C \subset X \setminus \mathfrak{h}_X(K)$.*

Remark One gets a version in which X has more than one, but only finitely many, connected components by applying the lemma to each connected component (see Exercise 2.13.2).

Proof of Lemma 2.13.3 The lemma is trivial for K empty, so we may assume that $K \neq \emptyset$. Since X is locally connected, $\mathfrak{h}_X(K)$ is a closed set whose complement has no relatively compact components (by Lemma 2.13.2). Since X is locally compact Hausdorff, we may choose a relatively compact neighborhood Ω of K in X. The components of $X \setminus K$ are open and disjoint, so only finitely many meet the compact set $\partial\Omega \subset X \setminus K$. By replacing Ω with the union of Ω and all of the relatively compact components of $X \setminus K$ meeting $\partial\Omega$, we may assume that no relatively compact component of $X \setminus K$ meets $\partial\Omega$. On the other hand, every component E of $X \setminus K$ must satisfy

$$\overline{E} \cap K = \partial E \neq \emptyset.$$

For E is open and closed in $X \setminus K$, so $\partial E \subset K$, while $E \neq X$, so $\partial E = \overline{E} \setminus E \neq \emptyset$ (E cannot be both open and closed in the connected space X). It follows that if E meets $X \setminus \Omega$, then E meets $\partial\Omega$, and hence E is *not* relatively compact in X. Thus

$$X \setminus \Omega \subset E_1 \cup \cdots \cup E_m$$

for finitely many components $E_1, \dots, E_m$ of $X \setminus K$, none of which are relatively compact in X. It follows that $\mathfrak{h}_X(K) \subset \Omega$ and $X \setminus \mathfrak{h}_X(K) = E_1 \cup \cdots \cup E_m$.

Applying the above argument with $K' = \overline{\Omega}$ in place of K, we get a relatively compact neighborhood Ω' of K' in X such that $X \setminus \Omega' \subset X \setminus \mathfrak{h}_X(K') = E'_1 \cup \cdots \cup E'_k$, where $E'_1, \dots, E'_k$ are distinct components of $X \setminus K'$, none of which are relatively compact in X. For each $i = 1, \dots, m$, we get $\overline{E'_j} \subset E_i$ for some

$j \in \{1, \dots, k\}$. Thus, if A_i is the set of points in E_i that lie some connected noncompact closed subset of X that is contained in E_i, then $A_i \neq \emptyset$. Given a point $p \in E_i \cap \overline{A}_i$, we may choose a relatively compact connected neighborhood V of p in E_i. We may then choose a point $q \in A_i \cap V$ and a connected noncompact closed subset B of X with $q \in B \subset E_i$. The set $C \equiv \overline{V} \cup B$ is then a connected noncompact closed subset of X that lies in E_i and that contains V. Thus $V \subset A_i$, and hence the nonempty set A_i is both open and closed in the connected set E_i. Therefore, $A_i = E_i$, and the lemma follows. □

Lemma 2.13.4 *Let X be a second countable, noncompact, connected, locally connected, locally compact Hausdorff space. Then there is a sequence of compact sets $\{K_\nu\}_{\nu=1}^{\infty}$ such that $X = \bigcup_{\nu=1}^{\infty} K_\nu$, and for each ν, $K_\nu \subset \overset{\circ}{K}_{\nu+1}$ and $\mathfrak{h}_X(K_\nu) = K_\nu$.*

Proof By Lemma 9.3.6, we may fix a sequence of compact sets $\{H_\nu\}$ with $X = \bigcup_{\nu=1}^{\infty} H_\nu$. Set $K_1 \equiv \mathfrak{h}_X(H_1)$. Given K_ν, we may choose a compact set $K'_{\nu+1}$ with $H_\nu \cup K_\nu \subset \overset{\circ}{K'}_{\nu+1}$, and we may set $K_{\nu+1} \equiv \mathfrak{h}_X(K'_{\nu+1})$. This yields the desired sequence. □

Lemma 2.13.5 *Let X be a connected, locally connected, locally compact Hausdorff space; let $\mathcal{B}$ be a countable collection of connected open subsets that is a basis for the topology in X; let K be a compact subset of X; and let U be a connected component of $X \setminus \mathfrak{h}_X(K)$. Then, for each point $p \in U$, there exists a sequence of basis elements $\{B_j\}$ that tends to infinity (i.e., $\{B_j\}$ is a locally finite family in X) such that $p \in B_1$, and for each j, $B_j \Subset U$ and $B_j \cap B_{j+1} \neq \emptyset$.*

Proof Lemma 2.13.3 provides a connected noncompact closed subset C of X with $p \in C \subset U$, and Lemma 9.3.6 provides a countable locally finite (in X) covering $\mathcal{A}$ of C by basis elements that are relatively compact in U. For each point $q \in C$, there is a finite sequence of elements $B_1, \dots, B_k$ of $\mathcal{A}$ that forms a *chain from p to q*; that is, $p \in B_1$, $q \in B_k$, and $B_j \cap B_{j+1} \neq \emptyset$ for $j = 1, \dots, k-1$ (we will call k the *length* of the chain). For the set E of points q in C for which there is a chain from p to q is clearly nonempty and open in C. On the other hand, E is also closed, because if $q \in \overline{E}$, then $q \in B$ for some set $B \in \mathcal{A}$ and there must be some point $r \in B \cap E$. A chain $B_1, \dots, B_k$ from p to r yields the chain $B_1, \dots, B_k, B$ from p to q. Thus $E = C$. Observe that if $q \in C$ and $B_1, \dots, B_k$ is a chain of minimal length from p to q, then the sets $B_1, \dots, B_k$ are distinct.

Now since C is noncompact and closed, we may choose a locally finite sequence of points $\{q_\nu\}$ in C; i.e., $q_\nu \to \infty$ in X (for example, we may fix an increasing sequence $\{\Omega_\nu\}$ of relatively compact open subsets of X with union X and choose $q_\nu \in C \setminus \Omega_\nu$ for each ν). For each ν, we may choose a chain $B_1^{(\nu)}, \dots, B_{k_\nu}^{(\nu)}$ of minimal length from p to q_ν. Since the elements of $\mathcal{A}$ are relatively compact in U and $\mathcal{A}$ is locally finite in X, there are only finitely many possible choices for $B_j^{(\nu)}$ for each j (only finitely many elements of $\mathcal{A}$ are in some chain of length j from p). Moreover, for each fixed $j \in \mathbb{Z}_{>0}$, we have $k_\nu > j$ for $\nu \gg 0$, because the set of points in C joined to p by a chain of length $\leq j$ is relatively compact in C while $q_\nu \to \infty$. Therefore, after applying a diagonal process and passing to the associated subsequence of $\{q_\nu\}$, we may assume that for each j, there is an element $B_j \in \mathcal{A}$ with $B_j^{(\nu)} = B_j$ for all $\nu \gg 0$. Thus we get an infinite chain of distinct elements $\{B_j\}$ from p to infinity as required in (ii) (local finiteness in X is guaranteed since $\mathcal{A}$ is locally finite and the elements $\{B_j\}$ are distinct). □

Remark The above lemma actually holds even if the basis elements are not necessarily connected, but it is easier to picture chains of connected sets.

Exercises for Sect. 2.13

2.13.1 Prove Lemma 2.13.2.

2.13.2 Let K be a compact subset of a noncompact, locally connected, locally compact Hausdorff space X. Prove that if X has only finitely many connected components, then $\mathfrak{h}_X(K)$ is compact, $X \setminus \mathfrak{h}_X(K)$ has only finitely many components, and for each point $p \in X \setminus \mathfrak{h}_X(K)$, there is a connected noncompact closed subset C of X with $p \in C \subset X \setminus \mathfrak{h}_X(K)$.

2.13.3 Let K be a compact subset of a connected, noncompact, locally connected, locally compact Hausdorff space X.

(a) Prove that $\mathfrak{h}_X(K)$ is equal to the intersection of all topologically Runge neighborhoods of K.

(b) Prove that for every neighborhood U of $\mathfrak{h}_X(K)$ in X, there exists a topologically Runge open set Ω with $\mathfrak{h}_X(K) \subset \Omega \subset U$.

Hint. Show that one may assume without loss of generality that $U \Subset X$ and that there exists a finite covering of ∂U by connected noncompact closed subsets of X that lie in $X \setminus \mathfrak{h}_X(K)$. Then set Ω equal to the complement of these sets in U.

2.13.4 Prove that if X is a locally connected Hausdorff space and A is a connected subset of X, then $\mathfrak{h}_X(A)$ is connected.

2.13.5 Let Ω be a relatively compact open subset of a connected, noncompact, locally connected, locally compact Hausdorff space X. Prove that $\mathfrak{h}_X(\Omega)$ is a relatively compact open subset of X and that $X \setminus \mathfrak{h}_X(\Omega)$ has only finitely many connected components (cf. Exercise 2.13.6 below).

2.13.6 This exercise will be applied in Exercise 2.16.5 (cf. [Ns5]).

(a) Prove that if Y is a locally compact Hausdorff space, C is a compact connected component of Y, and U is a neighborhood of C in Y, then there exists a set A such that $C \subset A \subset U$ and A is open and closed in Y.

Hint. First work in a relatively compact neighborhood V of C in Y. Let D be the intersection of all subsets of $\overline{V}$ that are open and closed in $\overline{V}$ and that contain C. Show that each neighborhood of D in $\overline{V}$ contains a set $A \supset D$ that is open and closed in $\overline{V}$. Using this observation, prove that D is connected and conclude that $D = C$. Finally, construct the desired open and closed subset of Y for a given neighborhood U.

(b) Let Ω be an open subset of a locally compact Hausdorff space X, and let C be a compact connected component of $X \setminus \Omega$. Prove that for every neighborhood U of C in X, there is a neighborhood V of C in U with $\partial V \subset \Omega$.

(c) Let Ω be an open subset of a locally compact Hausdorff space X. Prove that $\mathfrak{h}_X(\Omega)$ is open.

2.13.7 Let X be a connected, locally *path* connected, locally compact Hausdorff space; let $\mathcal{B}$ be a countable collection of connected open subsets that is a basis for the topology in X; let K be a compact subset of X; and let U be a connected component of $X \setminus \mathfrak{h}_X(K)$. Prove that for each point $p \in U$, there is a proper continuous map $\gamma : [0, \infty) \to X$ with $\gamma(0) = p$ and $\gamma([0, \infty)) \subset U$ (i.e., a path in U from p to ∞).

In the remaining exercises in this section, and in some exercises in later sections, generalizations in various contexts are obtained by considering a different hull as follows. For a subset A of a Hausdorff space X, the *extended topological hull* $\mathfrak{h}^*_X(A)$ *of* A *in* X is the union of A with all of the connected components of $X \setminus A$ that do not contain any connected noncompact closed subsets of X.

2.13.8 Prove that Lemma 2.13.2 holds with the extended topological hull in place of the topological hull. Prove also that $\mathfrak{h}_X(A) \subset \mathfrak{h}^*_X(A)$.

2.13.9 Give an example of a closed set K in a Hausdorff space X with $\mathfrak{h}^*_X(K) \neq \mathfrak{h}_X(K)$.

2.13.10 Prove that if Ω is an open subset of a Hausdorff space X, then $\mathfrak{h}^*_X(\Omega) = \mathfrak{h}_X(\Omega)$.

2.13.11 Prove that if K is a compact subset of a connected, noncompact, locally connected, locally compact Hausdorff space X, then $\mathfrak{h}^*_X(K) = \mathfrak{h}_X(K)$.

2.13.12 Let X be a second countable connected, locally connected, locally compact Hausdorff space, and let K be a closed subset of X.

(a) Prove that if $\mathcal{B}$ is a countable collection of connected open subsets that is a basis for the topology in X and U is a connected component of $X \setminus \mathfrak{h}^*_X(K)$, then for each point $p \in U$, there exists a sequence of basis elements $\{B_j\}$ that tends to infinity (i.e., $\{B_j\}$ is a locally finite family in X) such that $p \in B_1$ and for each j, $B_j \Subset U$ and $B_j \cap B_{j+1} \neq \emptyset$ (cf. Lemma 2.13.5).

(b) Prove that if $D \subset X \setminus K$ is a closed subset of X with no compact connected components, then there exist a countable locally finite (in X) family of disjoint connected noncompact closed sets $\{C_\lambda\}_{\lambda\in\Lambda}$ and a locally finite (in X) family of disjoint connected open sets $\{U_\lambda\}_{\lambda\in\Lambda}$ such that

$$D \subset C \equiv \bigcup_{\lambda\in\Lambda} C_\lambda \quad \text{and} \quad C_\lambda \subset U_\lambda \subset \overline{U}_\lambda \subset X \setminus K \quad \forall \lambda \in \Lambda.$$

Hint. First show that there is a countable locally finite covering $\mathcal{A}$ of D by connected open relatively compact subsets of $X \setminus K$ each of which meets D, that $X \setminus K$ contains the closure $C \equiv \overline{V}$ of the union V

of the elements of $\mathcal{A}$, that the family of components $\{V_\gamma\}_{\gamma\in\Gamma}$ of V is countable and locally finite, and that $\overline{V_\gamma}$ is noncompact for each $\gamma \in \Gamma$. Choosing suitable neighborhoods $\{U_\lambda\}_{\lambda\in\Lambda}$ of the connected components $\{C_\lambda\}_{\lambda\in\Lambda}$ of C, one gets the claim.

2.13.13 Let K be a subset of a second countable, connected, noncompact, locally connected, locally compact Hausdorff space X.

(a) Prove that, if K is closed, then $\mathfrak{h}^*_X(K)$ is equal to the intersection of all topologically Runge neighborhoods of K (cf. Exercise 2.13.3). Give an example that shows that this need not be the case if K is not closed.

(b) Give an example of such a space X and a closed subset K such that $K = \mathfrak{h}_X(K) = \mathfrak{h}^*_X(K)$, but not every neighborhood of K contains a neighborhood of K that is topologically Runge in X.

2.14 Construction of a Subharmonic Exhaustion Function

This section contains the construction, alluded to in the beginning of Sect. 2.13, of a $\mathcal{C}^\infty$ strictly subharmonic exhaustion function on an open Riemann surface. The main goal is the following:

Theorem 2.14.1 *Suppose X is an open Riemann surface, K is a compact subset of X satisfying $\mathfrak{h}_X(K) = K$, τ is a continuous real-valued function on X, θ is a continuous real $(1,1)$-form on X, and Ω is a neighborhood of K in X. Then there exists a $\mathcal{C}^\infty$ exhaustion function φ on X such that*

(i) *On X, $\varphi \geq 0$ and $i\Theta_\varphi \geq 0$;*
(ii) *On $X \setminus \Omega$, $\varphi > \tau$ and $i\Theta_\varphi \geq \theta$; and*
(iii) *We have* $\operatorname{supp}\varphi \subset X \setminus K$.

Remark It suffices to consider the case $\tau \geq 0$ and $\theta \geq 0$, since in general, one may construct φ with the nonnegative function $|\tau| \geq \tau$ and the nonnegative form $\theta^+ + \theta^- \geq \theta$ in place of τ and θ, respectively.

Setting $K = \Omega = \emptyset$ and choosing $\theta > 0$ (as we may by Corollary 2.11.3), we get the following:

Corollary 2.14.2 *Every open Riemann surface admits a $\mathcal{C}^\infty$ strictly subharmonic exhaustion function.*

After fixing a Kähler form, one may work with a convenient 0-form in place of the curvature form:

Definition 2.14.3 The *Laplace operator* associated to a Kähler form ω on a complex 1-manifold X is the second-order linear differential operator Δ_ω with $\mathcal{C}^\infty$ coefficients given by

$$\Delta_\omega \varphi \equiv \frac{i\partial\bar{\partial}\varphi}{\omega}$$

for every $\mathcal{C}^2$ function φ. In particular, for φ real-valued, we have $\Delta_\omega \varphi = i\Theta_\varphi/\omega$.

Remarks 1. Writing $\omega = G \cdot (i/2)\, dz \wedge d\bar{z} = G\, dx \wedge dy$ with respect to a local holomorphic coordinate $z = x + iy$, we get

$$\Delta_\omega = \frac{2}{G}\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2G}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

For example, if $\omega = (i/2)\, dz \wedge d\bar{z} = dx \wedge dy$ is the standard Euclidean volume form on $\mathbb{C}$, then

$$\Delta_\omega = 2\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

which is $1/2$ the standard Laplace operator.

2. Clearly, a real-valued $\mathcal{C}^\infty$ (or $\mathcal{C}^2$) function φ on an open set $\Omega \subset X$ is subharmonic (strictly subharmonic) if and only if $\Delta_\omega \varphi \geq 0$ (respectively, $\Delta_\omega \varphi > 0$) on Ω.

For the rest of this section, X will denote an *open* Riemann surface. According to Radó's theorem, X is second countable, and applying Corollary 2.11.3, we get a Kähler form ω on X. Instead of proving Theorem 2.14.1 directly, we will prove the following equivalent version (in Exercise 2.14.1, the reader is asked to verify that the two versions are equivalent):

Theorem 2.14.4 (Cf. [GreW], [De2], and [NR]) *Suppose K is a compact subset of X satisfying $\mathfrak{h}_X(K) = K$, ρ is a continuous real-valued function on X, and Ω is a neighborhood of K in X. Then there exists a $\mathcal{C}^\infty$ exhaustion function φ on X such that*

(i) *On X, $\varphi \geq 0$ and $\Delta_\omega \varphi \geq 0$;*
(ii) *On $X \setminus \Omega$, $\varphi > \rho$ and $\Delta_\omega \varphi > \rho$; and*
(iii) *We have* $\operatorname{supp} \varphi \subset X \setminus K$.

The main step in the proof of Theorem 2.14.4 is the following:

Proposition 2.14.5 *Let K be a compact subset of X satisfying $\mathfrak{h}_X(K) = K$. Then, for each point $p \in X \setminus K$, there is a nonnegative $\mathcal{C}^\infty$ function α on X such that $\Delta_\omega \alpha \geq 0$ on X,* $\operatorname{supp} \alpha \subset X \setminus K$, *$\alpha(p) > 0$, and $\Delta_\omega \alpha(p) > 0$.*

Remark According to the maximum principle for subharmonic functions (see, for example, [Ns5]), a nonconstant subharmonic function on a domain cannot attain a maximum value (in particular, any subharmonic function on a compact Riemann surface is constant). Therefore, the converse of the proposition also holds; that is, if such a function α exists for some point $p \in X \setminus K$, then the component of $X \setminus K$ containing p is not relatively compact in X.

Assuming Proposition 2.14.5 for now, we may prove Theorem 2.14.4.

Proof of Theorem 2.14.4 By Proposition 9.3.11, we may assume that ρ is a positive exhaustion function. Let $K_0 = K$. By Lemma 2.13.4, we may choose a sequence of nonempty compact sets $\{K_\nu\}$ such that $X = \bigcup_{\nu=1}^\infty K_\nu$ and such that for each $\nu = 1, 2, 3, \dots, K_{\nu-1} \subset \overset{\circ}{K}_\nu$ and $\mathfrak{h}_X(K_\nu) = K_\nu$.

Given a point $p \in X \setminus \Omega$, there is a unique $\nu = \nu(p) > 0$ with $p \in K_\nu \setminus K_{\nu-1}$, and Proposition 2.14.5 provides a nonnegative $\mathcal{C}^\infty$ function α_p and a relatively compact neighborhood V_p of p in $X \setminus K_{\nu-1}$ such that $\Delta_\omega \alpha_p \geq 0$ on X, $\operatorname{supp} \alpha_p \subset X \setminus K_{\nu-1}$, and $\alpha_p > \rho$ and $\Delta_\omega \alpha_p > \rho$ on V_p (one obtains the last two conditions by multiplying by a sufficiently large positive constant). We may choose a sequence of points $\{p_k\}$ in X, and corresponding functions $\{\alpha_{p_k}\}$ and neighborhoods $\{V_{p_k}\}$, so that $\{V_{p_k}\}$ forms a locally finite covering of $X \setminus \Omega$ (for example, we may take $\{p_k\}$ to be an enumeration of the countable set $\bigcup_{\nu=0}^\infty Z_\nu$, where for each ν, Z_ν is a finite set of points in $X \setminus [\overset{\circ}{K}_{\nu-1} \cup \Omega]$ such that $\{V_p\}_{p \in Z_\nu}$ covers $K_\nu \setminus [\overset{\circ}{K}_{\nu-1} \cup \Omega]$). The collection $\{\operatorname{supp} \alpha_{p_k}\}$ is then locally finite in X, since $\operatorname{supp} \alpha_{p_k} \subset X \setminus K_{\nu-1}$ whenever $p_k \notin K_{\nu-1}$. Hence the sum $\sum_{k=1}^\infty \alpha_{p_k}$ is locally finite and therefore convergent to a $\mathcal{C}^\infty$ function φ on X satisfying $\varphi \geq \alpha_{p_k} > \rho$ and $\Delta_\omega \varphi \geq \Delta_\omega \alpha_{p_k} > \rho$ on V_{p_k} for each k. Therefore, since $\{V_{p_k}\}$ covers $X \setminus \Omega$, we get $\varphi > \rho$ and $\Delta_\omega \varphi > \rho$ on $X \setminus \Omega$. In particular, φ is an exhaustion function. Similarly, we also have $\operatorname{supp} \varphi \subset X \setminus K$, and $\varphi \geq 0$ and $\Delta_\omega \varphi \geq 0$ on X. □

It remains to prove Proposition 2.14.5. The idea is to form a sequence of functions, each of which is subharmonic outside a small set and strictly subharmonic on the bad set of the previous function. Multiplying each function by a sufficiently large positive constant (obtained inductively), one pushes these small bad sets off to infinity.

Lemma 2.14.6 *There exists a nonnegative $\mathcal{C}^\infty$ function χ on $\mathbb{R}$ such that $\chi \equiv 0$ on $(-\infty, 0]$ and $\chi, \chi', \chi'' > 0$ on $(0, \infty)$.*

Proof For example, the $\mathcal{C}^\infty$ function

$$\chi(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \exp(t - (1/t)) & \text{if } t > 0, \end{cases}$$

has the required properties. The verification is left to the reader (see Exercise 2.14.3). □

Lemma 2.14.7 *For every $r \in (0, 1)$, there exists a nonnegative $\mathcal{C}^\infty$ function α on $\mathbb{P}^1$ such that*

(i) *We have* $\alpha > 0$ *on* $\Delta(0; 1)$ *and* $\alpha \equiv 0$ *on* $\mathbb{P}^1 \setminus \Delta(0; 1)$; *and*
(ii) *The function* α *is strictly subharmonic on the annulus* $\Delta(0; r, 1)$ (*and therefore subharmonic on* $\mathbb{P}^1 \setminus \overline{\Delta(0; r)}$).

Proof Letting $\chi : \mathbb{R} \to [0, \infty)$ be the function provided by Lemma 2.14.6, it is easy to see that the function given by $\alpha(\infty) = 0$ and

$$\alpha(z) \equiv \chi\left(e^{-|z|^2/r^2} - e^{-1/r^2}\right) \quad \forall z \in \mathbb{C}$$

has the required properties. Again, the verification is left to the reader (see Exercise 2.14.4). □

Proposition 2.14.8 *Let* $c \in \Delta \equiv \Delta(0; 1)$, *and let* $\Phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ *be the map given by* $z \mapsto (z - c)/(1 - \bar{c}z)$ (*with* $\Phi(1/\bar{c}) = \infty$ *and* $\Phi(\infty) = -1/\bar{c}$). *Then we have the following*:

(i) Φ *is an automorphism of* $\mathbb{P}^1$ (*i.e., a biholomorphic mapping of* $\mathbb{P}^1$ *onto itself*);
(ii) $\Phi(c) = 0$, $\Phi'(0) = 1 - |c|^2$, *and* $\Phi'(c) = 1/(1 - |c|^2)$; *and*
(iii) $\Phi(\Delta) = \Delta$ *and* $\Phi(\partial\Delta) = \partial\Delta$.

The proof is left to the reader (see Exercise 2.14.5). It follows from the above that $\Phi\restriction_\Delta$ is an automorphism of Δ. In Chapter 5, we will see that in fact, every automorphism of Δ is of the form $b\Phi\restriction_\Delta$ for some $b \in \partial\Delta$ and some Φ as above (see Theorem 5.8.2).

Lemma 2.14.9 *For every nonempty relatively compact open subset* U *of the unit disk* $\Delta \equiv \Delta(0; 1)$, *there exists a nonnegative* $\mathcal{C}^\infty$ *function* β *on* $\mathbb{P}^1$ *such that*

(i) *We have* $\beta > 0$ *on* Δ *and* $\beta \equiv 0$ *on* $\mathbb{P}^1 \setminus \Delta$; *and*
(ii) *The function* β *is strictly subharmonic on the set* $\Delta \setminus \overline{U}$ (*and therefore subharmonic on* $\mathbb{P}^1 \setminus \overline{U}$).

Proof Fixing a point $c \in U$, we get the automorphism $\Phi\colon z \mapsto (z - c)/(1 - \bar{c}z)$ of $\mathbb{P}^1$ mapping c to 0 as in Proposition 2.14.8, and for $r \in (0, 1)$ sufficiently small, we have $\Phi^{-1}(\Delta(0; r)) \subset U$. By Lemma 2.14.7, there exists a nonnegative $\mathcal{C}^\infty$ function α on $\mathbb{P}^1$ such that $\alpha > 0$ on Δ, $\alpha \equiv 0$ on $\mathbb{P}^1 \setminus \Delta$, and α is strictly subharmonic on $\Delta(0; r, 1)$. The function $\beta \equiv \alpha(\Phi)$ then has the required properties. □

Proof of Proposition 2.14.5 By Lemma 2.13.5, given a point $p \in X \setminus K$, there is a locally finite (in X) sequence of relatively compact open subsets $\{V_m\}$ of $X \setminus K$ such that $p \in V_1$ and such that for each m, we have $V_m \cap V_{m+1} \neq \emptyset$ and there is a biholomorphism of a neighborhood of the closure $\overline{V}_m$ of V_m onto a neighborhood of the closure $\overline{\Delta}$ of the unit disk $\Delta \equiv \Delta(0; 1)$ that maps V_m onto Δ. Hence we may choose a sequence of nonempty open sets $\{W_m\}_{m=0}^\infty$ with disjoint closures such that $p \in W_0 \Subset V_1$ and such that for each $m \geq 1$, $W_m \Subset V_m \cap V_{m+1}$ (see Fig. 2.6).

By Lemma 2.14.9, there is a sequence of nonnegative $\mathcal{C}^\infty$ functions $\{\beta_m\}_{m=1}^\infty$ such that for each m, $\beta_m \equiv 0$ on $X \setminus V_m$, $\Delta_\omega\beta_m \geq 0$ on $X \setminus W_m$, and

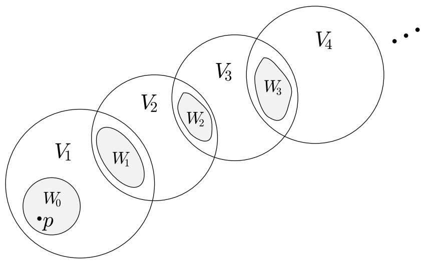

Fig. 2.6 Construction of a subharmonic function that is strictly subharmonic near the point p

$\beta_m > 0$ and $\Delta_\omega \beta_m > 0$ on $\overline{W}_{m-1}$. We will choose positive constants $\{R_m\}_{m=1}^\infty$ inductively so that for each $m = 1, 2, 3, \ldots,$

$$\Delta_\omega\left(\sum_{j=1}^{m} R_j\beta_j\right)\begin{cases} \geq 0 & \text{on } X \setminus W_m, \\ > 0 & \text{on } \overline{W}_0. \end{cases}$$

Fix $R_1 > 0$. Given $R_1, \ldots, R_{m-1} > 0$ with the above property, using the fact that $\Delta_\omega \beta_m > 0$ on $\overline{W}_{m-1}$, we get, for $R_m \gg 0$,

$$\Delta_\omega\left(\sum_{j=1}^{m} R_j\beta_j\right) = R_m\Delta_\omega\beta_m + \Delta_\omega\left(\sum_{j=1}^{m-1} R_j\beta_j\right) > 0 \quad \text{on } \overline{W}_{m-1}.$$

On $X \setminus (W_{m-1} \cup W_m)$, we have $\Delta_\omega \beta_m \geq 0$ and hence

$$\Delta_\omega\left(\sum_{j=1}^{m} R_j\beta_j\right) \geq \Delta_\omega\left(\sum_{j=1}^{m-1} R_j\beta_j\right) \geq 0.$$

On $\overline{W}_0 \subset X \setminus (W_{m-1} \cup W_m)$, the above middle expression, and hence the expression on the left, is positive. Moreover, $\sum_{j=1}^m R_j\beta_j \geq R_1\beta_1 > 0$ on $\overline{W}_0$. Proceeding, we get the sequence $\{R_m\}$. The sum $\sum R_m\beta_m$ is locally finite in X and the sequence of sets $\{W_m\}$ is locally finite in X, so the sum converges to a function α with the required properties. □

Remark When second countability of a particular open Riemann surface X is evident (for example, when X is a proper nonempty open subset of a compact Riemann surface), one gets Theorem 2.14.4 without any reliance on Radó's theorem, and hence with almost no reliance on Sects. 2.6–2.12.

The existence of strictly subharmonic exhaustion functions allows one to use the full power of the L^2 $\bar{\partial}$-method. For example, we have the following:

Theorem 2.14.10 *Let $p \in \{0, 1\}$ and let β be a locally square-integrable differential form of type $(p, 1)$ on the open Riemann surface X. Then there exists a locally* *square-integrable differential form α of type $(p, 0)$ with $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$. In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.*

Proof We may fix a Kähler form ω on X. Applying Theorem 2.14.4, with the compact set given by $K = \emptyset$ and the function ρ chosen (with the help of Lemma 9.7.13) so large that $\beta \in L^2_{0,1}(X, \rho)$ or $L^2_{1,1}(X, \omega, \rho)$ and $\rho > 1 + |i\Theta_\omega/\omega|$, we get a positive $\mathcal{C}^\infty$ strictly subharmonic (exhaustion) function φ on X such that

$$\begin{aligned} i\Theta_\omega + i\Theta_\varphi \geq \omega \quad &\text{and} \quad \|\beta\|_{L^2(X,\varphi)} < \infty \quad \text{if } p = 0, \\ i\Theta_\varphi \geq \omega \quad &\text{and} \quad \|\beta\|_{L^2(X,\omega,\varphi)} < \infty \quad \text{if } p = 1. \end{aligned}$$

Corollary 2.12.6 and Corollary 2.9.3 now give the desired differential form α. □

In Sects. 2.15 and 2.16, we will consider other applications to open Riemann surfaces.

Exercises for Sect. 2.14

2.14.1 Prove that Theorem 2.14.1 and Theorem 2.14.4 are equivalent.

2.14.2 Let φ be a real-valued $\mathcal{C}^\infty$ function on a Riemann surface X.

(a) Prove that if χ is a real-valued $\mathcal{C}^\infty$ function on $\mathbb{R}$, then

$$\Theta_{\chi(\varphi)} = \chi'(\varphi)\Theta_{\chi(\varphi)} + \chi''(\varphi)\partial\varphi \wedge \bar{\partial}\varphi.$$

From this conclude that if φ is subharmonic and χ' and χ'' are nonnegative, then $\chi(\varphi)$ is subharmonic. Show also that if φ is strictly subharmonic and $\chi' > 0$ and $\chi'' \geq 0$, then $\chi(\varphi)$ is strictly subharmonic.

(b) Prove that if φ is a strictly subharmonic exhaustion function on X, τ is a continuous real-valued function on X, and θ is a continuous real $(1, 1)$-form on X, then there exists a $\mathcal{C}^\infty$ function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(\varphi) > \tau$ and $i\Theta_{\chi(\varphi)} \geq \theta$.

2.14.3 Verify that the function constructed in the proof of Lemma 2.14.6 has the required properties.

2.14.4 Verify that the function constructed in the proof of Lemma 2.14.7 has the required properties.

2.14.5 Prove Proposition 2.14.8.

2.14.6 Let X be an open Riemann surface.

(a) Prove that if $q \in \{0, 1\}$ and β is a $\mathcal{C}^\infty$ differential form of type $(1, q)$ on X, then there exists a $\mathcal{C}^\infty$ differential form α of type $(0, q)$ with $\partial\alpha = \beta$.

(b) Prove that if ω is a Kähler form on X, then there exists a $\mathcal{C}^\infty$ strictly subharmonic function φ on X such that $i\Theta_\varphi = \omega$.

2.14.7 This exercise requires Exercises 2.13.8 and 2.13.12. Let X be an open Riemann surface, let ω be a Kähler form on X, and let K be a closed subset of X satisfying $\mathfrak{h}^*_X(K) = K$.

(a) Prove that for each connected component U of $X \setminus K$ and each point $p \in X \setminus K$, there is a nonnegative $\mathcal{C}^\infty$ function α on X such that $\Delta_\omega \alpha \geq 0$ on X, $\operatorname{supp} \alpha \subset U$, $\alpha(p) > 0$, and $\Delta_\omega \alpha(p) > 0$ (cf. Proposition 2.14.5).

(b) Suppose that U is a connected component of $X \setminus K$, C is a connected noncompact closed subset of X contained in U, and ρ is a continuous real-valued function on X. Prove that there exists a $\mathcal{C}^\infty$ function φ on X such that
   (i) We have $\varphi \geq 0$ and $\Delta_\omega \varphi \geq 0$ on X;
   (ii) We have $\varphi > \rho$ and $\Delta_\omega \varphi > \rho$ on C; and
   (iii) We have $\operatorname{supp} \varphi \subset U$.

   Hint. First show that there exists a connected *locally connected* closed set D in X with $C \subset D \subset U$ (for example, by forming the union of the closures of elements of a suitable locally finite covering of C by coordinate disks each of which meets C). Then show that there is a sequence of compact sets $\{K_\nu\}$ such that $X = \bigcup_\nu K_\nu$ and such that for each ν, $K_\nu \subset \mathring{K}_{\nu+1}$ and $\mathfrak{h}_D(K_\nu \cap D) = K_\nu \cap D$. Proceed now as in the proof of Theorem 2.14.4.

(c) Suppose that $D \subset X \setminus K$ is a closed subset of X with no compact connected components and ρ is a real-valued continuous function on X. Prove that there exists a $\mathcal{C}^\infty$ function φ on X such that
   (i) We have $\varphi \geq 0$ and $\Delta_\omega \varphi \geq 0$ on X;
   (ii) We have $\varphi > \rho$ and $\Delta_\omega \varphi > \rho$ on D; and
   (iii) We have $\operatorname{supp} \varphi \subset X \setminus K$.

2.15 The Mittag-Leffler Theorem

One of the main applications of the L^2 $\bar{\partial}$-method is the construction of a holomorphic or meromorphic function (or section of a holomorphic line bundle) with prescribed values on a given discrete set, or as in the generalization of the classical Mittag-Leffler theorem below, with some prescribed Laurent series terms (see Theorem 1.3.6). This generalization is due to Behnke–Stein [BehS] (see also Florack [Fl]):

Theorem 2.15.1 (Mittag-Leffler theorem) *Let X be an open Riemann surface, let P be a discrete subset of X (i.e., a closed set with no limit points in X), and for each point $p \in P$, let U_p be a neighborhood of p with $U_p \cap P = \{p\}$, let f_p be a holomorphic function on $U_p \setminus \{p\}$, and let m_p be a positive integer. Then there exists a holomorphic function f on $X \setminus P$ such that for each point $p \in P$, $f - f_p$ extends to a holomorphic function on U_p that either vanishes on a neighborhood of p or has a zero of order at least m_p at p (in other words, if z is a local holomorphic coordinate on a neighborhood of p, and $f = \sum_{n \in \mathbb{Z}} a_n (z - z(p))^n$ and $f_p = \sum_{n \in \mathbb{Z}} b_n (z - z(p))^n$ are the corresponding Laurent series expansions centered at p, then $a_{m_p - n} = b_{m_p - n}$ for $n = 1, 2, 3, \dots$).*

Remark It follows that one may actually choose the above function $f \in \mathcal{O}(X \setminus P)$ so that for each point $p \in P$, $f - f_p$ extends to a holomorphic function on U_p with a zero of order *equal to* m_p at p (see Exercise 2.15.1).

The proofs of Corollaries 2.15.2, 2.15.3, and 2.15.4 below are left to the reader (see Exercises 2.15.2, 2.15.3, and 2.15.4).

Corollary 2.15.2 *Let P be a discrete subset of an open Riemann surface X.*

(a) *If f_p is a meromorphic function on a neighborhood of p and $m_p \in \mathbb{Z}_{>0}$ for each point $p \in P$, then there exists a function $f \in \mathcal{M}(X)$ such that f is holomorphic on $X \setminus P$ and $\operatorname{ord}_p(f - f_p) \geq m_p$ for every $p \in P$.*
(b) *If f_p is a holomorphic function on a neighborhood of p and $m_p \in \mathbb{Z}_{>0}$ for each point $p \in P$, then there exists a function $f \in \mathcal{O}(X)$ with $\operatorname{ord}_p(f - f_p) \geq m_p$ for every $p \in P$.*
(c) *If $\zeta_p \in \mathbb{C}$ for each point $p \in P$, then there exists a function $f \in \mathcal{O}(X)$ with $f(p) = \zeta_p$ for every $p \in P$.*

Corollary 2.15.3 (Behnke–Stein theorem [BehS]) *Every open Riemann surface X is Stein*; *that is, X has the following properties*:

(i) (*Holomorphic convexity*) *If P is any infinite discrete subset of X, then there exists a holomorphic function on X that is unbounded on P*;
(ii) (*Separation of points*) *If $p, q \in X$ and $p \neq q$, then there exists a holomorphic function f on X such that $f(p) \neq f(q)$*; *and*
(iii) (*Global functions give local coordinates*) *For each point $p \in X$, there exists a holomorphic function f on X such that $(df)_p \neq 0$.*

Corollary 2.15.4 *Let X be an open Riemann surface, let P be a discrete subset of X, and for each point $p \in P$, let U_p be a neighborhood of p with $U_p \cap P = \{p\}$, let θ_p be a holomorphic 1-form on $U_p \setminus \{p\}$, and let m_p be a positive integer. Then there exists a holomorphic 1-form θ on $X \setminus P$ such that for each point $p \in P$, $\theta - \theta_p$ extends to a holomorphic 1-form on U_p that either vanishes on a neighborhood of p or has a zero of order at least m_p at p.*

Proof of Theorem 2.15.1 Let $Y = X \setminus P$. We may choose a Kähler form ω on X, and we may choose a locally finite family of disjoint local holomorphic coordinate neighborhoods $\{(V_p, z_p)\}_{p \in P}$ and a family of open sets $\{W_p\}_{p \in P}$ such that for each point $p \in P$, we have $p \in W_p \Subset V_p \Subset U_p$ and $z_p(p) = 0$. Let $V \equiv \bigcup_{p \in P} V_p$ and $W \equiv \bigcup_{p \in P} W_p$.

By cutting off, we may construct a real-valued $\mathcal{C}^\infty$ function ρ_0 on Y such that $\operatorname{supp} \rho_0 \subset V$ and $\rho_0 = m_p \log |z_p|^2$ on $W_p \setminus \{p\}$ for each $p \in P$. We may also fix a $\mathcal{C}^\infty$ function γ on Y such that $\operatorname{supp} \gamma \subset V$ and $\gamma = f_p$ on $W_p \setminus \{p\}$ for each $p \in P$. The form $\beta = \bar{\partial}\gamma$ is then a $\mathcal{C}^\infty$ differential form of type $(0, 1)$ on Y with $\operatorname{supp} \beta \subset V \setminus W$. Since $\log |z_p|^2$ is a harmonic function on $V_p \setminus \{p\}$ for each $p \in P$, by applying Theorem 2.14.4 (or Theorem 2.14.1), with the compact set equal to the empty set and the function ρ chosen (with the help of Lemma 9.7.13) so large that $\beta \in L^2_{0,1}(Y, \rho - |\rho_0|)$ and $\rho > 1 + |i\Theta_\omega/\omega| + |\Delta_\omega \rho_0|$ on Y, we get a positive $\mathcal{C}^\infty$ strictly subharmonic (exhaustion) function ρ_1 on X such that $i\Theta_\omega + i\Theta_{\rho_0} + i\Theta_{\rho_1} \geq \omega$ on Y and such that $\|\beta\|_{L^2(Y,\rho_0+\rho_1)} < \infty$.

Corollary 2.12.6 now provides a $\mathcal{C}^\infty$ function α on Y such that

$$\bar{\partial}\alpha = \beta \quad \text{and} \quad \|\alpha\|_{L^2(Y,\omega,\rho_0+\rho_1)} < \infty.$$

In particular, the $\mathcal{C}^\infty$ function $f \equiv \gamma - \alpha$ is holomorphic on Y. Moreover, for each $p \in P$, we have $f - f_p = -\alpha$ on $W_p \setminus \{p\}$. On the other hand, for some positive constant C (depending on p), we have

$$\begin{aligned}
&\|z_p^{-m_p}\alpha\|_{L^2(W_p\setminus\{p\},(i/2)dz_p\wedge d\bar{z}_p)} \\
&\quad = \|\alpha\|_{L^2(W_p\setminus\{p\},(i/2)\,dz_p\wedge d\bar{z}_p,\rho_0)} \leq C\|\alpha\|_{L^2(W_p\setminus\{p\},\omega,\rho_0+\rho_1)} \\
&\quad \leq C\|\alpha\|_{L^2(Y,\omega,\rho_0+\rho_1)} < \infty.
\end{aligned}$$

Hence the holomorphic function $z_p^{-m_p}\alpha$ on $W_p \setminus \{p\}$ is square-integrable, and therefore, by Riemann's extension theorem (Theorem 1.2.10), this function extends to a holomorphic function on W_p. Thus α extends holomorphically past p with order at least m_p at p, and therefore $f - f_p$ extends to a holomorphic function on U_p with order at least m_p at p. □

Exercises for Sect. 2.15

2.15.1 Prove that in the Mittag-Leffler theorem (Theorem 2.15.1), one may actually choose the function $f \in \mathcal{O}(X \setminus P)$ so that for each point $p \in P$, $f - f_p$ extends to a holomorphic function on U_p with a zero of order *equal to* m_p at p.

2.15.2 Prove Corollary 2.15.2.

2.15.3 Prove the Behnke–Stein theorem (Corollary 2.15.3).

2.15.4 Prove Corollary 2.15.4.

2.15.5 Let f be a meromorphic function on an open Riemann surface X. Prove that there exist holomorphic functions g and h on X such that h is not the zero function and $f = g/h$.

2.15.6 The goal of this exercise is a generalization of the Mittag-Leffler theorem that, in higher dimensions, is known as the solution of the *additive Cousin problem* (or the *Cousin problem* I). Let X be an open Riemann surface, let P be a discrete subset of X, let $\{m_p\}_{p\in P}$ be a collection of positive integers, let $\{U_i\}_{i\in I}$ be an open covering of X, and for each pair of indices $i, j \in I$, let f_{ij} be a holomorphic function on $U_i \cap U_j$ with $\mathrm{ord}_p f_{ij} \geq m_p$ for each point $p \in P \cap U_i \cap U_j$. Assume that the family $\{f_{ij}\}$ satisfies the (additive) *cocycle relation*

$$f_{ik} = f_{ij} + f_{jk} \text{ on } U_i \cap U_j \cap U_k \quad \forall i, j, k \in I.$$

Prove that there exist functions $\{g_i\}_{i\in I}$ such that

(i) For each index $i \in I$, $g_i \in \mathcal{O}(U_i)$ and $\mathrm{ord}_p\, g_i \geq m_p$ for each point $p \in P \cap U_i$; and

(ii) For each pair of indices $i, j \in I$, we have $f_{ij} = g_j - g_i$ on $U_i \cap U_j$.

Prove also that the above implies the standard Mittag-Leffler theorem (Theorem 2.15.1).

Hint. Using a $\mathcal{C}^\infty$ partition of unity $\{\lambda_\nu\}$ such that each point in P lies in $\mathrm{supp}\, \lambda_\nu$ for exactly one index ν and such that for each ν, $\mathrm{supp}\, \lambda_\nu \subset U_{k_\nu}$ for some index $k_\nu \in I$, one may form a $\mathcal{C}^\infty$ solution of the problem of the form $v_i = \sum_\nu \lambda_\nu \cdot f_{k_\nu i}$. In particular, the forms $\{\bar\partial v_i\}$ agree on the overlaps and therefore determine a well-defined $(0,1)$-form β on X. Suitable weight functions (as in the proof of Theorem 2.15.1) now give a suitable solution of $\bar\partial \alpha = \beta$.

2.16 The Runge Approximation Theorem

According to the Mittag-Leffler theorem (Theorem 2.15.1), on an open Riemann surface X, one may prescribe values for a holomorphic function (to arbitrary order) at the points in a given discrete set. The identity theorem implies that it is not possible to prescribe values on a set that is not discrete. However, for a compact set K with $\mathfrak{h}_X(K) = K$, one can *uniformly approximate* a holomorphic function on a neighborhood of K by a global holomorphic function. In other words, we have the following Riemann surface analogue, due to Behnke and Stein [BehS], of the classical Runge approximation theorem [Run] for domains in the plane:

Theorem 2.16.1 (Runge approximation theorem) *Suppose K is a compact subset of an open Riemann surface X with $\mathfrak{h}_X(K) = K$, f_0 is a holomorphic function on a neighborhood of K in X, and $\epsilon > 0$. Then there exists a holomorphic function f on X such that $|f - f_0| < \epsilon$ on K.*

The converse is also true (see Exercise 2.16.4). Until now, we have not applied, in an essential way, the L^2 estimate in Theorem 2.9.1; but this estimate will play an important role in the proof of this approximation theorem. We will also consider the more general version Theorem 2.16.3, so the reader may wish to skip the proof of Theorem 2.16.1 below and instead, consider only the proof of Theorem 2.16.3.

Proof of Theorem 2.16.1 Multiplying f_0 by a $\mathcal{C}^\infty$ function that has support contained in a small neighborhood of K but that is equal to 1 on some smaller neighborhood, and then extending by 0 to all of X and choosing suitable neighborhoods, we get a $\mathcal{C}^\infty$ function τ on X and open sets Ω_0 and Ω_1 such that $K \subset \Omega_0 \Subset \Omega_1 \Subset X$, $\tau = f_0$ on Ω_0, and $\mathrm{supp}\, \tau \subset \Omega_1$.

We may also fix a Kähler form ω on X (by Corollary 2.11.3). Applying Theorem 2.14.1 (with the compact set and its neighborhood given by the empty set, and the nonnegative $(1,1)$-form chosen so that its sum with $i\Theta_\omega$ is greater than or equal to ω), we get a positive $\mathcal{C}^\infty$ strictly subharmonic (exhaustion) function ρ_0 on

X such that $i\Theta_\omega + i\Theta_{\rho_0} \geq \omega$ on X. Applying the theorem again (this time with the compact set given by K), we get a nonnegative $\mathcal{C}^\infty$ subharmonic (exhaustion) function ρ_1 on X such that $\operatorname{supp}\rho_1 \subset X \setminus K$ and $\rho_1 > 0$ on $X \setminus \Omega_0$. In particular, $r \equiv \inf_{\Omega_1\setminus\Omega_0} \rho_1 > 0$.

For each positive integer ν, let $\varphi_\nu \equiv \rho_0 + \nu\rho_1$. The form $\beta \equiv \bar{\partial}\tau$ is a compactly supported $\mathcal{C}^\infty$ differential form of type (0, 1) on X. Since we have

$$i\Theta_\omega + i\Theta_{\varphi_\nu} = i\Theta_\omega + i\Theta_{\rho_0} + \nu i\Theta_{\rho_1} \geq \omega,$$

Corollary 2.12.6 provides a $\mathcal{C}^\infty$ function α on X such that $\bar{\partial}\alpha = \beta$ and

$$\|\alpha\|^2_{L^2(X,\omega,\varphi_\nu)} \leq \|\beta\|^2_{L^2(X,\varphi_\nu)} = \|\bar{\partial}\tau\|^2_{L^2(\Omega_1\setminus\Omega_0,\varphi_\nu)} \leq e^{-\nu r}\|\bar{\partial}\tau\|^2_{L^2(\Omega_1\setminus\Omega_0,\rho_0)}.$$

By construction, we have $\rho_1 \equiv 0$ on some relatively compact neighborhood Ω_2 of K in Ω_0, and hence

$$\|\alpha\|^2_{L^2(\Omega_2,\omega,\rho_0)} = \|\alpha\|^2_{L^2(\Omega_2,\omega,\varphi_\nu)} \leq \|\alpha\|^2_{L^2(X,\omega,\varphi_\nu)}.$$

Therefore, by choosing ν sufficiently large, we can make $\|\alpha\|_{L^2(\Omega_2,\omega,\rho_0)}$ arbitrarily small. Furthermore, the $\mathcal{C}^\infty$ function $f \equiv \tau - \alpha$ on X is actually holomorphic (since $\bar{\partial} f = 0$), and we have $\|f - f_0\|_{L^2(\Omega_2,\omega,\rho_0)} = \|\alpha\|_{L^2(\Omega_2,\omega,\rho_0)}$. Applying Theorem 2.6.4 and choosing $\nu \gg 0$, we get $|f - f_0| < \epsilon$ on K. □

We have the following consequence (the converse, which also holds, is considered in Exercise 2.16.5):

Corollary 2.16.2 *Let Ω be a topologically Runge open subset of an open Riemann surface X. Then, for every holomorphic function f_0 on Ω, for every compact set $K \subset \Omega$, and for every $\epsilon > 0$, there exists a holomorphic function f on X such that $|f - f_0| < \epsilon$ on K.*

Proof For every compact set $K \subset \Omega$, we have $\mathfrak{h}_X(K) \subset \mathfrak{h}_X(\Omega) = \Omega$. Theorem 2.16.1 now gives the claim. □

We also have the following more general version of Theorem 2.16.1:

Theorem 2.16.3 (Runge approximation with poles at prescribed points) *Suppose K is a compact subset of a Riemann surface X, P is a finite subset of $X \setminus K$, $Y = X \setminus P$, $\mathfrak{h}_Y(K) = K$, f_0 is a holomorphic function on a neighborhood of K in X, and $\epsilon > 0$. Then there exists a meromorphic function f on X such that f is holomorphic on $X \setminus P = Y$, f has a pole at each point in P, and $|f - f_0| < \epsilon$ on K.*

For the proof, we will need the following combined version of Lemma 2.10.3 and Theorem 2.14.1:

Lemma 2.16.4 *Suppose X is a Riemann surface, K is a compact subset of X, P is a finite subset of $X \setminus K$, $Y = X \setminus P$, $\mathfrak{h}_Y(K) = K$, $\{(U_p, z_p)\}_{p\in P}$ is a collection of disjoint local holomorphic coordinate neighborhoods in X with $p \in U_p$ and $z_p(p) = 0$ for each $p \in P$, ω is a Kähler form on X, and Ω is a neighborhood of K in X. Then, for every sufficiently large positive constant b, there exist open sets $\{V_p\}_{p\in P}$ with $p \in V_p \Subset U_p$ for each point $p \in P$ such that for every sufficiently large positive constant R (depending on the above choices), there exists a nonnegative $\mathcal{C}^\infty$ subharmonic exhaustion function φ on Y with the following properties*:

(i) *On $Y \setminus \Omega$, $\varphi > 0$ and $i\Theta_\varphi \geq \omega$;*
(ii) *We have* $\operatorname{supp}\varphi \subset Y \setminus K$; *and*
(iii) *For each $p \in P$, we have $\varphi = R \cdot (|z_p|^2 - \log|z_p|^2 - b)$ on $V_p \setminus \{p\}$.*

Proof If $P = \emptyset$ and X is compact, then we have $K = X$ and the claim is trivial. Thus we may assume without loss of generality that Y is noncompact. By shrinking Ω and the sets $\{(U_p, z_p)\}_{p\in P}$ if necessary, we may also assume without loss of generality that $U_p \Subset X \setminus \overline{\Omega}$ for each point $p \in P$. We have $\mathfrak{h}_Y(K) = K$, so Theorem 2.14.1 provides a nonnegative $\mathcal{C}^\infty$ subharmonic exhaustion function α on Y such that $\operatorname{supp}\alpha \subset Y \setminus K$ and such that $\alpha > 0$ and $i\Theta_\alpha \geq \omega$ on $Y \setminus \Omega$. For $b \gg 0$, Lemma 2.10.3 provides, for each point $p \in P$, a nonnegative $\mathcal{C}^\infty$ subharmonic function β_p on $X \setminus \{p\}$ such that $\beta_p \equiv 0$ on $X \setminus U_p$ and $\beta_p = |z_p|^2 - \log|z_p|^2 - b > 0$ on $W_p \setminus \{p\}$ for some relatively compact neighborhood W_p of p in U_p (the finiteness of P allows us to choose a single sufficiently large constant b that works for each of the points $p \in P$). Choosing a relatively compact neighborhood V_p of p in W_p for each $p \in P$, setting $V \equiv \bigcup_{p\in P} V_p \Subset W \equiv \bigcup_{p\in P} W_p$, and choosing a nonnegative $\mathcal{C}^\infty$ function η on X such that $\eta \equiv 1$ on a neighborhood of $X \setminus W$ and $\operatorname{supp}\eta \subset X \setminus V$, we see that if $R \gg 0$, then the function $\varphi = \eta \cdot \alpha + \sum_{p\in P} R \cdot \beta_p$ has the required properties. $\square$

Proof of Theorem 2.16.3 We have $Y = X \setminus P$ and $\mathfrak{h}_Y(K) = K$, and as in the proof of Lemma 2.16.4, we may assume without loss of generality that Y is noncompact. Multiplying f_0 by a $\mathcal{C}^\infty$ function that has support contained in a small neighborhood of K but that is equal to 1 on some smaller neighborhood, and then extending by 0 to all of X and choosing suitable neighborhoods, we get a $\mathcal{C}^\infty$ function τ on X and open sets Ω_0 and Ω_1 such that $K \subset \Omega_0 \Subset \Omega_1 \Subset X \setminus P$, $\tau = f_0$ on Ω_0, and $\operatorname{supp}\tau \subset \Omega_1$. We may also choose disjoint local holomorphic coordinate neighborhoods $\{(U_p, z_p)\}_{p\in P}$ in X such that for each $p \in P$, we have $p \in U_p \Subset X \setminus \overline{\Omega}_1$ and $z_p(p) = 0$. We may also fix a Kähler form ω on X (by Corollary 2.11.3).

By applying Lemma 2.16.4 (with the compact set and its neighborhood given by the empty set, and the Kähler form chosen so that its sum with $i\Theta_\omega$ is greater than or equal to ω), we get a positive constant b_0, open sets $\{V_p\}_{p\in P}$ with $p \in V_p \Subset U_p$ for each point $p \in P$, a positive integer k, and a positive $\mathcal{C}^\infty$ strictly subharmonic (exhaustion) function ρ_0 on Y such that $i\Theta_\omega + i\Theta_{\rho_0} \geq \omega$ on Y and such that for each point $p \in P$, $\rho_0 = k(|z_p|^2 - \log|z_p|^2 - b_0)$ on $V_p \setminus \{p\}$. Applying Lemma 2.16.4 again (this time with the compact set given by K) and shrinking each of the neighborhoods V_p for $p \in P$, we get a positive constant b_1 and a nonnegative $\mathcal{C}^\infty$ subharmonic (exhaustion) function ρ_1 on Y such that $\operatorname{supp} \rho_1 \subset Y \setminus K$, $\rho_1 > 0$ on $X \setminus \Omega_0$, and $\rho_1 = |z_p|^2 - \log |z_p|^2 - b_1$ on $V_p \setminus \{p\}$ for each $p \in P$ (here, we apply the lemma to get constants b and R, and we then divide the resulting function by R and set $b_1 \equiv b/R$). In particular, $r \equiv \inf_{\Omega_1 \setminus \Omega_0} \rho_1 > 0$. We may also fix a $\mathcal{C}^\infty$ function η with compact support in the neighborhood $\bigcup_{p \in P} V_p$ of P such that $\eta \equiv 1$ on a neighborhood of P.

Now for each positive integer ν, let $\varphi_\nu \equiv \rho_0 + \nu\rho_1$. Given a positive integer μ and a constant $\delta > 0$, we get a $\mathcal{C}^\infty$ function γ on Y by setting $\gamma = \delta\eta/z_p^{(\mu+\nu+k)}$ on $U_p \setminus \{p\}$ for each $p \in P$ and $\gamma = \tau$ elsewhere. The form $\beta \equiv \bar{\partial}\gamma$ is then a compactly supported $\mathcal{C}^\infty$ differential form of type (0, 1) on Y. Since we have

$$i\Theta_\omega + i\Theta_{\varphi_\nu} = i\Theta_\omega + i\Theta_{\rho_0} + \nu i\Theta_{\rho_1} \geq \omega,$$

Corollary 2.12.6 provides a $\mathcal{C}^\infty$ function α on Y such that $\bar{\partial}\alpha = \beta$ and

$$\begin{aligned} \|\alpha\|^2_{L^2(Y,\omega,\varphi_\nu)} \leq \|\beta\|^2_{L^2(Y,\varphi_\nu)} &= \|\bar{\partial}\tau\|^2_{L^2(\Omega_1\setminus\Omega_0,\varphi_\nu)} + \sum_{p\in P} \delta^2 \|z_p^{-(\mu+\nu+k)}\bar{\partial}\eta\|^2_{L^2(V_p,\varphi_\nu)} \\ &\leq e^{-\nu r}\|\bar{\partial}\tau\|^2_{L^2(\Omega_1\setminus\Omega_0,\rho_0)} + \sum_{p\in P} \delta^2 \|z_p^{-(\mu+\nu+k)}\bar{\partial}\eta\|^2_{L^2(V_p,\varphi_\nu)}. \end{aligned}$$

By construction, we have $\rho_1 \equiv 0$ on some relatively compact neighborhood Ω_2 of K in Ω_0, and hence

$$\|\alpha\|^2_{L^2(\Omega_2,\omega,\rho_0)} = \|\alpha\|^2_{L^2(\Omega_2,\omega,\varphi_\nu)} \leq \|\alpha\|^2_{L^2(Y,\omega,\varphi_\nu)}.$$

Therefore, by choosing ν sufficiently large and then choosing $\delta > 0$ sufficiently small (depending on ν), we can make $\|\alpha\|_{L^2(\Omega_2,\omega,\rho_0)}$ arbitrarily small.

The $\mathcal{C}^\infty$ function $f \equiv \gamma - \alpha$ on Y is actually holomorphic, since $\bar{\partial} f = 0$. Near each point $p \in P$, the function $z_p^{\nu+k}\alpha = \delta z_p^{-\mu} - z_p^{\nu+k} f$ is holomorphic except for an isolated singularity at p, and $|z_p^{\nu+k}\alpha|^2 = |\alpha|^2 e^{-(-(\nu+k)\log|z_p|^2)}$ is locally integrable near p by the choice of φ_ν. Therefore, by Riemann's extension theorem (Theorem 1.2.10), $z_p^{\nu+k}\alpha$ extends to a holomorphic function in a neighborhood of p, and hence the function $-\alpha = f - \gamma$, which is equal to $f - \delta z_p^{-(\mu+\nu+k)}$ near p, is meromorphic on a neighborhood of p with at worst a pole of order $\nu + k$ at p. It follows that f extends to a meromorphic function on X that is holomorphic on $X \setminus P$ and that has a pole of order $\mu + \nu + k$ at each point $p \in P$. Finally, since $\|f - f_0\|_{L^2(\Omega_2,\omega,\rho_0)} = \|\alpha\|_{L^2(\Omega_2,\omega,\rho_0)}$, Theorem 2.6.4 implies that for $\nu \gg 0$ and δ sufficiently small (depending on ν), we have $|f - f_0| < \epsilon$ on K. □

Remarks 1. Note that we may choose the function f in the above proof to have a pole of any order $> \nu + k$ at each point in P (see Exercise 2.16.3).

2. A version in which the discrete set P may be infinite is considered in Exercise 2.16.6.

We close this section with a consequence of Theorem 2.16.3 that will play an important role in the proof of the Riemann mapping theorem in Chap. 5 (see Sect. 5.4).

Lemma 2.16.5 *Given a compact subset K of an open Riemann surface X, there exist a holomorphic function f on a neighborhood Y of K in X, a positive regular value r of the function $|f|$, and a $\mathcal{C}^\infty$ domain Ω such that Ω is a connected component of $\{x \in Y \mid |f(x)| < r\}$ and $K \subset \Omega \Subset Y$.*

Proof Clearly, we may assume that K is nonempty and connected. Thus we may choose a compact set $K_1 \subsetneqq X$ with $K \subset K_1$ and $\mathfrak{h}_X(K_1) = K_1$ (in fact, we could replace K with $\mathfrak{h}_X(K)$ and set $K = K_1$, but this would require the fact, considered in Exercise 2.13.4, that the topological hull of a connected set in a manifold is connected, and is not really necessary here). Similarly, we may choose a relatively compact neighborhood Ω_0 of K_1 in X and a compact set K_2 such that $\partial\Omega_0 \subset K_2 \subset X \setminus K_1$ and $\mathfrak{h}_{X\setminus K_1}(K_2) = K_2$ (see Exercise 2.13.2). In particular, the set $X \setminus (K_1 \cup K_2) = (X \setminus K_1) \setminus K_2$ has only finitely many components, and hence we may choose a finite set $P \subset X \setminus (K_1 \cup K_2)$ that meets each of these components. The domain $Y \equiv X \setminus P$ then contains $K_1 \cup K_2$, and furthermore, $\mathfrak{h}_Y(K_1 \cup K_2) = K_1 \cup K_2$. For each component of $Y \setminus (K_1 \cup K_2)$ is of the form $U \setminus P$, where U is a component of $X \setminus (K_1 \cup K_2)$, and by construction, P must meet U. Thus $U \setminus P$ must contain points arbitrarily close to $P = X \setminus Y$, and hence $U \setminus P$ cannot be relatively compact in Y.

Now, by applying Theorem 2.16.3 to a locally constant function on a neighborhood of $K_1 \cup K_2$ that is equal to 0 on K_1 and 3 on K_2, we get a meromorphic function g on X such that g is holomorphic on $X \setminus P = Y$, g has a pole at each point in P, $|g| < 1$ on $K_1 \subset \Omega_0$, and $|g| > 2$ on $K_2 \supset \partial\Omega_0$. Since g is nonconstant, the set of positive critical values of the function $\rho \equiv |g|\restriction_Y$ is countable ($dg = 0$ at any critical point of ρ in $Y \setminus g^{-1}(0) = X \setminus g^{-1}(\{0, \infty\})$ since $2\rho\, d\rho = d\rho^2 = \bar{g}\, dg + g\, d\bar{g}$ on this set). Thus we may fix a regular value $r \in (1, 2)$. The connected component Ω of $\{x \in Y \mid \rho(x) < r\}$ containing the connected compact set K must then be a nonempty $\mathcal{C}^\infty$ (by Corollary 2.4.5) domain that is relatively compact in $\Omega_0 \setminus P = \Omega_0 \cap Y$, since $\rho \to \infty$ at P and $\rho > 2$ on $\partial\Omega_0$. Setting $f \equiv g\restriction_Y$, we get the desired objects. □

Exercises for Sect. 2.16

2.16.1 Let X be an open Riemann surface. Using the Runge approximation theorem (not the results of Sect. 2.15), prove the following (cf. Corollary 2.15.3):
  (i) *Separation of points*. If $p, q \in X$ and $p \neq q$, then there exists a holomorphic function f on X such that $f(p) \neq f(q)$; and
  (ii) *Global holomorphic functions give local coordinates*. For each point $p \in X$, there exists a holomorphic function f on X such that $(df)_p \neq 0$.

2.16.2 Let P be a finite subset of a compact Riemann surface X, and let U be a neighborhood of P. Prove that there is a meromorphic function f on X

such that f is holomorphic on $X \setminus P$, f has a pole at each point in P, and the set of zeros of f is contained in U.

2.16.3 Verify that in the proof of the Theorem 2.16.3, the constructed function f may be chosen to have a pole of any order $> \nu + k$ at each point in P.

2.16.4 Let X be an open Riemann surface and let K be a nonempty compact subset of X.

(a) Prove that $\mathfrak{h}_X(K) = \{p \in X \mid |f(p)| \leq \max_K |f| \; \forall f \in \mathcal{O}(X)\}$.

Hint. Show that for each point $p \in X \setminus \mathfrak{h}_X(K)$, we have $\mathfrak{h}_X(\mathfrak{h}_X(K) \cup \{p\}) = \mathfrak{h}_X(K) \cup \{p\}$. Then form a holomorphic approximation to the function that is equal to 0 near $\mathfrak{h}_X(K)$ and 1 near p. Given a point $p \in \mathfrak{h}_X(K) \setminus K$, apply the maximum principle.

(b) Prove that $\mathfrak{h}_X(K) = K$ if and only if for every holomorphic function f_0 on a neighborhood of K in X and for every $\epsilon > 0$, there exists a holomorphic function f on X such that $|f - f_0| < \epsilon$ on K.

Hint. Assuming the approximation condition, suppose U is a connected component of $X \setminus K$ with $U \Subset X$. Fixing a point $p \in U$, the results of Sect. 2.16 (or Sect. 2.15) provide a function $f \in \mathcal{M}(X)$ that is holomorphic except for a pole at p. Show that there is a sequence $\{g_n\}$ in $\mathcal{O}(X)$ that converges uniformly to f on K. Applying the maximum principle to the sequence $\{g_n \restriction_{\overline{U}}\}$ (along with the Cauchy criterion), one gets a continuous function on $\overline{U}$ that is holomorphic on U and that is equal to f on ∂U. This leads to a contradiction.

2.16.5 This exercise requires facts proved in Exercise 2.13.6. Let Ω be a nonempty open subset of an open Riemann surface X. Prove that the following are equivalent:

(i) Ω is topologically Runge.
(ii) For every compact set $K \subset \Omega$, we have $\mathfrak{h}_X(K) \subset \Omega$.
(iii) For every holomorphic function f_0 on Ω, every compact set $K \subset \Omega$, and every $\epsilon > 0$, there exists a holomorphic function f on X such that $|f - f_0| < \epsilon$ on K (that is, Ω is *holomorphically Runge*).

Hint. The proof of (iii)$\Rightarrow$(i) is similar to that of part (b) of Exercise 2.16.4.

2.16.6 Let X be a Riemann surface, let K be a compact subset of X, let P be a discrete subset of X with $P \subset X \setminus K$, and let $Y = X \setminus P$. Assume that $\mathfrak{h}_Y(K) = K$.

(a) Suppose that $\{(U_p, z_p)\}_{p \in P}$ is a locally finite collection of disjoint local holomorphic coordinate neighborhoods in X with $p \in U_p$ and $z_p(p) = 0$ for each $p \in P$, ω is a Kähler form on X, and Ω is a neighborhood of K in X. Prove that for every collection of sufficiently large positive constants $\{b_p\}_{p \in P}$, there exist open sets $\{V_p\}_{p \in P}$ with $p \in V_p \Subset U_p$ for each point $p \in P$ such that for every collection of sufficiently large positive constants $\{R_p\}_{p \in P}$ (depending on the above choices), there exists a nonnegative $\mathcal{C}^\infty$ subharmonic exhaustion function φ on Y with the following properties (cf. Lemma 2.16.4):

(i) On $Y \setminus \Omega$, $\varphi > 0$ and $i\Theta_\varphi \geq \omega$;
(ii) We have $\operatorname{supp}\varphi \subset Y \setminus K$; and
(iii) For each $p \in P$, we have $\varphi = R_p \cdot (|z_p|^2 - \log|z_p|^2 - b_p)$ on $V_p \setminus \{p\}$.

(b) Suppose that f_0 is a holomorphic function on a neighborhood of K in X and $\epsilon > 0$. Prove that there exists a meromorphic function f on X such that f is holomorphic on $X \setminus P = Y$, f has a pole at each point in P, and $|f - f_0| < \epsilon$ on K (cf. Theorem 2.16.3).

2.16.7 Let X be an open Riemann surface, let $K \subset X$ be a compact subset with $\mathfrak{h}_X(K) = K$, let $P \subset X$ be a discrete subset with $P \subset X \setminus K$, let f_0 be a holomorphic function on a neighborhood of K in X, and for each point $p \in P$, let f_p be a holomorphic function on $U_p \setminus \{p\}$ for some neighborhood U_p of p in X with $U_p \cap P = \{p\}$, and let m_p be a positive integer. Prove that for every $\epsilon > 0$, there exists a holomorphic function f on $X \setminus P$ such that $|f - f_0| < \epsilon$ on K and such that for each point $p \in P$, $f - f_p$ extends to a holomorphic function on U_p that either vanishes on a neighborhood of p or has a zero of order at least m_p at p (note that this is a combined version of the Mittag-Leffler theorem and the Runge approximation theorem).

2.16.8 Suppose K is compact subset of an open Riemann surface X with $\mathfrak{h}_X(K) = K$ and θ_0 is a holomorphic 1-form on a neighborhood of K in X. Prove that there exists a sequence of holomorphic 1-forms $\{\theta_\nu\}$ on X such that for every local holomorphic coordinate neighborhood (U, z) in X, $\theta_\nu/dz \to \theta_0/dz$ uniformly on compact subsets of $K \cap U$.

2.16.9 This exercise requires Exercises 2.13.8, 2.13.10, 2.13.12, and 2.14.7. Let Ω be a topologically Runge open subset of an open Riemann surface X.

(a) Suppose that ω is a Kähler form on X and φ is a real-valued $\mathcal{C}^\infty$ function on X with $i\Theta_\omega + i\Theta_\varphi \geq 0$ on X. Prove that for every holomorphic function f_0 on Ω, every closed set $K \subset \Omega$, and every $\epsilon > 0$, there exists a holomorphic function f on X such that $\|f - f_0\|_{L^2_{0,0}(K,\omega,\varphi)} < \epsilon$.

(b) Suppose that φ is a $\mathcal{C}^\infty$ subharmonic function on X. Prove that for every holomorphic 1-form θ_0 on Ω, every closed set $K \subset \Omega$, and every $\epsilon > 0$, there exists a holomorphic 1-form θ on X such that $\|\theta - \theta_0\|_{L^2_{1,0}(K,\varphi)} < \epsilon$.

2.16.10 The goal of this exercise is a special case of the Mergelyan–Bishop theorem. Let λ denote Lebesgue measure on $\mathbb{C}$.

(a) Prove that if ρ_0 is a continuous complex-valued function on a compact set $K \subset \mathbb{C}$ and $\epsilon > 0$, then there exists a $\mathcal{C}^\infty$ function ρ on $\mathbb{C}$ such that $|\rho - \rho_0| < \epsilon$ on K.

Hint. Patch together local constant approximations using a $\mathcal{C}^\infty$ partition of unity.

(b) Prove that if S is a measurable subset of $\mathbb{C}$ and $z \in \mathbb{C}$, then

$$\int_S \frac{1}{|\zeta - z|}\, d\lambda(\zeta) \leq 2\pi R + \frac{\lambda(S)}{R} \quad \forall R > 0.$$

Conclude from this that in particular, $\int_S (1/|\zeta - z|)\, d\lambda(\zeta) \leq \sqrt{8\pi\lambda(S)}$.

(c) Prove that if $f \in \mathcal{C}^\infty(\mathbb{C})$ and Ω is a smooth relatively compact domain in $\mathbb{C}$, then

$$\left| f(z) - \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| \leq \sup_\Omega \left|\frac{\partial f}{\partial\bar{\zeta}}\right| \cdot \sqrt{8\lambda(\Omega)/\pi}.$$

(d) *Hartogs–Rosenthal theorem* (see [HarR]). Suppose K is a compact set of measure 0 in $\mathbb{C}$. Prove that if f_0 is a continuous function on K and $\epsilon > 0$, then there exists a holomorphic function f on a neighborhood of K in $\mathbb{C}$ such that $|f - f_0| < \epsilon$ on K. Prove also that if in addition, $\mathbb{C} \setminus K$ is connected, then there actually exists an entire function f such that $|f - f_0| < \epsilon$ on K.

(e) Prove that given an open set $\Omega \subset \mathbb{C}$ and a compact set $K \subset \Omega$, there exists a constant $C = C(K, \Omega) > 0$ such that

$$\max_K |f| \leq C\left[\|f\|_{L^2(\Omega)} + \sup_\Omega \left|\frac{\partial f}{\partial \bar{\zeta}}\right|\right] \quad \forall f \in \mathcal{C}^\infty(\Omega).$$

Hint. First consider the case in which $K = \overline{\Delta(z_0; R)}$ and $\Omega = \Delta(z_0; 4R)$ for some point $z_0 \in \mathbb{C}$ and some $R > 0$. For this, apply the Cauchy integral formula (Lemma 1.2.1) to the disk $\Delta(z_0; 3R)$ and to the function ηf, where $\eta \in \mathcal{D}(\Delta(z_0; 3R))$ and $\eta \equiv 1$ on $\Delta(z_0; 2R)$. For the general case, cover K by finitely many disks of the form $\Delta(z_0; R)$ with $\Delta(z_0; 4R) \subset \Omega$.

(f) Let X be a Riemann surface. Prove that given an open set $\Omega \subset X$, a compact set $K \subset \Omega$, a Kähler form ω on X, and a real-valued $\mathcal{C}^\infty$ function φ on X, there exists a constant $C = C(K, \Omega, \omega, \varphi) > 0$ such that

$$\max_K |f| \leq C\left[\|f\|_{L^2(\Omega,\omega,\varphi)} + \sup_\Omega \left|\frac{\bar{\partial} f \wedge \overline{\bar{\partial} f}}{\omega}\right|^{1/2}\right] \quad \forall f \in \mathcal{C}^\infty(\Omega).$$

(g) *Bishop–Kodama localization theorem* (see [Bis] and [Kod]). Let X be an open Riemann surface, let K be a compact subset of X, and let f_0 be a continuous function on K. Assume that each point $p \in K$ admits a neighborhood U in X such that for every $\epsilon > 0$, there exists a holomorphic function f on a neighborhood of the compact set $K' \equiv K \cap \overline{U}$ in X with $|f - f_0| < \epsilon$ on K'. Prove that for every $\epsilon > 0$, there exists a holomorphic function f on a neighborhood of K in X such that $|f - f_0| < \epsilon$ on K.

Hint. By replacing X with a large domain, one may assume that there are a Kähler metric ω and a $\mathcal{C}^\infty$ strictly subharmonic function φ on X as in Corollary 2.12.6. Fix a finite covering of K by relatively compact open subsets of X with the above approximation property, and fix a suitable partition of unity. Given $\delta > 0$, one may form local approximations to within δ on each of these sets. Patching these local approximations using the partition of unity, one gets a $\mathcal{C}^\infty$ function τ; and the $(0,1)$-form $\bar{\partial}\tau$ is controlled by δ at points in K. Multiplying $\bar{\partial}\tau$ by a $\mathcal{C}^\infty$ (cutoff) function that is equal to 1 on K and that vanishes outside a small neighborhood of K (this function depends on the choice of δ), one gets a $(1,0)$-form β that is controlled by δ *everywhere* in X. Corollary 2.12.6 then provides a solution of the equation $\bar{\partial}\alpha = \beta$ along with an L^2 estimate. Guided by part (f), one sees that if $\delta > 0$ is sufficiently small, then the restriction of $\tau - \alpha$ to a small neighborhood of K (which depends on the choice of δ) has the required properties.

(h) Let X be an open Riemann surface, and let $K \subset X$ be a compact set of measure 0. Prove that for every continuous function f_0 on K and for every $\epsilon > 0$, there exists a holomorphic function f on a neighborhood of K in X such that $|f - f_0| < \epsilon$ on K. Prove also that if in addition, $\mathfrak{h}_X(K) = K$, then there actually exists a function $f \in \mathcal{O}(X)$ such that $|f - f_0| < \epsilon$ on K.

Remarks According to Mergelyan's theorem (see [Me] and [Rud1]), given a compact set $K \subset \mathbb{C}$ with connected complement, a continuous function f_0 on K that is holomorphic on the interior of K, and a constant $\epsilon > 0$, there exists an entire function f with $|f - f_0| < \epsilon$ on K. The Mergelyan–Bishop theorem includes the natural analogue for an open Riemann surface X: Given a compact set $K \subset X$ with $\mathfrak{h}_X(K) = K$, a continuous function f_0 on K that is holomorphic on the interior of K, and a constant $\epsilon > 0$, there exists a holomorphic function f on X with $|f - f_0| < \epsilon$ on K. The proof of the Hartogs–Rosenthal theorem outlined in parts (b)–(d) is due to Hartogs and Rosenthal. The proof of the Bishop–Kodama localization theorem outlined in parts (e)–(g) is due to Jarnicki and Pflug (see [JP] and [Ga]).

Chapter 3
The L^2 $\bar{\partial}$-Method in a Holomorphic Line Bundle

In this chapter, we consider a useful generalization of the notion of a holomorphic function, namely, that of a holomorphic section of a holomorphic line bundle. We first consider the basic properties of holomorphic line bundles as well as those of sheaves and divisors. We then proceed with a discussion of the solution of the inhomogeneous Cauchy–Riemann equation with L^2 estimates in this more general setting. In this setting, there is a natural generalization of Theorem 2.9.1 for Hermitian holomorphic line bundles (E, h) with positive curvature; that is, $i\Theta_h > 0$, where Θ_h is a natural generalization of the curvature form $\Theta_\varphi = \partial\bar{\partial}\varphi$ considered in Sect. 2.8. In fact, Sects. 3.6–3.9 may be read in place of most of the material in Sects. 2.6–2.9. We then consider applications, mostly to the study of holomorphic line bundles on *open* Riemann surfaces (holomorphic line bundles on compact Riemann surfaces are considered in greater depth in Chap. 4). For example, in Sect. 3.11, we prove that every holomorphic line bundle on an open Riemann surface admits a positive-curvature Hermitian metric (this follows easily from the results of Sect. 2.14); and we then obtain a slightly more streamlined proof of (a generalization of) the Mittag-Leffler theorem (Theorem 2.15.1). In Sect. 3.12, we prove the Weierstrass theorem (Theorem 3.12.1), according to which every holomorphic line bundle on an open Riemann surface is actually holomorphically trivial.

3.1 Holomorphic Line Bundles

Throughout this section, X denotes a complex 1-manifold. In this section, we consider the basic properties of holomorphic line bundles and differential forms with values in a holomorphic line bundle. In order to introduce some of the terminology, we first consider a set-theoretic version.

Definition 3.1.1 Let Y be a topological space.

(a) A (*set-theoretic*) *complex line bundle* over (or on) Y consists of a set E, a surjective mapping $\Pi\colon E \to Y$, and a choice of a 1-dimensional complex vector

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space structure in the fiber $E_p \equiv \Pi^{-1}(p)$ for each point $p \in Y$. We usually let the *total space* E or the *projection map* $\Pi\colon E \to Y$ represent the line bundle, rather than referring to the triple consisting of the set E, the map Π, and the choice of vector space structures in the fibers. A *local trivialization* of E is a bijection of the form $\Psi = (\Pi, \Phi)\colon \Pi^{-1}(U) \to U \times \mathbb{C}$, where U is an open subset of Y and $\Phi\colon \Pi^{-1}(U) \to \mathbb{C}$ is a surjective mapping for which the restriction $\Phi_p \equiv \Phi\restriction_{E_p}\colon E_p \to \mathbb{C}$ is a complex linear isomorphism for each point $p \in U$. In other words, we have a commutative diagram

$$\begin{tikzcd} \Pi^{-1}(U) \arrow[r, "\Psi"] \arrow[d, "\Pi"'] & U \times \mathbb{C} \arrow[dl, "\mathrm{pr}_U"] \\ U & \end{tikzcd}$$

in which Ψ is a bijection that is linear on each fiber. A local trivialization of the form (X, Ψ) is called a *global trivialization*.

(b) Given two local trivializations $(U_1, \Psi_1 = (\Pi, \Phi_1))$ and $(U_2, \Psi_2 = (\Pi, \Phi_2))$ of a complex line bundle E over Y, the map $g\colon U_1 \cap U_2 \to \mathbb{C}^*$ determined by $\Psi_1 \circ \Psi_2^{-1}(x, \zeta) = (x, g(x) \cdot \zeta)$ for all $(x, \zeta) \in (U_1 \cap U_2) \times \mathbb{C}$ (that is, for each point $x \in U_1 \cap U_2$, $g(x) = (\Phi_1)_x/(\Phi_2)_x$ is the nonzero scalar representing the linear isomorphism $\zeta \mapsto \Phi_1(\Psi_2^{-1}(x, \zeta))$) is called a *transition function* (from Φ_2 to Φ_1 or from Ψ_2 to Ψ_1).

(c) A collection of local trivializations $\mathcal{A} = \{(U_i, \Psi_i)\}_{i \in I}$ of a complex line bundle E on Y with $Y = \bigcup_i U_i$ is called a *line bundle atlas* for E.

Remark A more standard approach is to consider only line bundles for which the total space E is a topological space, the projection Π is continuous, and the local trivializations are homeomorphisms. The above weaker definition is more convenient for our purposes.

A holomorphic structure allows one to consider a holomorphic version:

Definition 3.1.2 Let $\Pi\colon E \to X$ be a complex line bundle over X.

(a) Two local trivializations $(U_1, \Psi_1 = (\Pi, \Phi_1))$ and $(U_2, \Psi_2 = (\Pi, \Phi_2))$ of E are *holomorphically compatible* if the transition functions $g_{12}, g_{21}\colon U_1 \cap U_2 \to \mathbb{C}^*$, given by

$$g_{12}(p) = \frac{1}{g_{21}(p)} = \frac{(\Phi_1)_p}{(\Phi_2)_p} \quad \forall p \in U_1 \cap U_2,$$

are holomorphic.

(b) A line bundle atlas for E for which the transition functions are holomorphically compatible is called a *holomorphic line bundle atlas*.

(c) Two holomorphic line bundle atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ for E are *holomorphically equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a holomorphic line bundle atlas (this is an equivalence relation).

(d) An equivalence class $\mathcal{S}$ of holomorphic line bundle atlases for E is called a *holomorphic line bundle structure* in E over X. The pair $(E, \mathcal{S})$ (usually denoted simply by E) is called a *holomorphic line bundle* over (or on) X.

(e) If E is equipped with a holomorphic line bundle structure, then any local trivialization (U, Ψ) in any holomorphic line bundle atlas in the holomorphic line bundle structure is called a *local holomorphic trivialization* for E. If we have $U = X$, then (X, Ψ) is a (*global*) *holomorphic trivialization* for E and E is *holomorphically trivial*. The *trivial line bundle* over X is the line bundle $1_X \equiv X \times \mathbb{C} \to X$.

Remarks 1. A *topological line bundle* over a topological space and a $\mathcal{C}^\infty$ *line bundle* over a $\mathcal{C}^\infty$ manifold are defined analogously. The natural higher-rank analogues, in which the local trivializations are fiberwise linear bijections of the form $\Pi^{-1}(U) \to U \times \mathbb{C}^r$, are called *vector bundles*.

2. A holomorphic line bundle on X has a natural underlying $\mathcal{C}^\infty$ structure, an underlying $\mathcal{C}^\infty$ line bundle structure, and a 2-dimensional holomorphic structure (see Exercise 2.2.6 for the definition of a complex manifold and Exercise 3.1.9 for the verification).

Definition 3.1.3 Let $\Pi\colon E \to X$ be a holomorphic line bundle over X.

(a) A *section* of E on a set $B \subset X$ is a mapping $s\colon B \to E$ such that $\Pi \circ s = \mathrm{Id}_B$. For each point $p \in B$, we usually denote the value $s(p) \in E_p$ by s_p. If B is an open set, then we also call s a *local section* of E. If $B = X$, then we also call s a *global section*.

(b) Let s be a section of E on a set B. Given a local trivialization $(U, (\Pi, \Phi))$, the function $\Phi(s)\colon B \cap U \to \mathbb{C}$ (i.e., the function $p \mapsto \Phi(s_p)$) is called the *representation* of s in the local trivialization.

(c) We say that a section s of E on a set B is *continuous* (*of class* $\mathcal{C}^k$ with $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, *holomorphic*, *measurable*, L^d_{loc} with $d \in [1, \infty]$) if the representation $\Phi(s)$ of s in every local holomorphic trivialization $(U, (\Pi, \Phi))$ (equivalently, in *some* local holomorphic trivialization in a neighborhood of each point in B) is a continuous (respectively, $\mathcal{C}^k$, holomorphic, measurable, L^d_{loc}) function.

(d) For $B \subset X$ an open set, we call a section s of E on the complement in B of a discrete subset P of B a *meromorphic* section of E on B if the representation $\Phi(s)$ of s in every local holomorphic trivialization $(U, (\Pi, \Phi))$ (equivalently, in *some* local holomorphic trivialization in a neighborhood of each point in B) is a meromorphic function on $B \cap U$ with set of poles $P \cap U$. We say that s has a *zero* (a *pole*) *of order* m *at a point* $p \in B$ if the representation $\Phi(s)$ of s in every (equivalently, in some) local holomorphic trivialization $(U, (\Pi, \Phi))$ over a neighborhood U of p has a zero (respectively, a pole) of order m at p.

For any point $p \in B$, the *order of s at p* is given by

$$\operatorname{ord}_p s = \begin{cases} 0 & \text{if } p \text{ is neither a zero nor a pole of } s, \\ m & \text{if } s \text{ has a zero of order } m \text{ at } p, \\ -m & \text{if } s \text{ has a pole of order } m \text{ at } p, \\ \infty & \text{if } s \equiv 0 \text{ on a neighborhood of } p. \end{cases}$$

We say that s has a *simple zero* (*simple pole*) *at p* if $\operatorname{ord}_p s = 1$ (respectively, $\operatorname{ord}_p s = -1$).

(e) For any open set $U \subset X$, the set of holomorphic sections of E on U is denoted by $\mathcal{O}(E)(U)$ or $\Gamma(U, \mathcal{O}(E))$ (or $H^0_{\mathrm{Dol}}(U, E)$); the set of meromorphic sections of E on U is denoted by $\mathcal{M}(E)(U)$ or $\Gamma(U, \mathcal{M}(E))$; and the set of $\mathcal{C}^\infty$ sections of E on U is denoted by $\mathcal{E}(E)(U)$ or $\Gamma(U, \mathcal{E}(E))$. The set of $\mathcal{C}^\infty$ sections with compact support in U is denoted by $\mathcal{D}(E)(U)$ or $\mathcal{D}(U, E)$.

Remarks 1. It is not hard to verify that, as noted in the above definition, if a section of a holomorphic line bundle on X has a continuous ($\mathcal{C}^k$, holomorphic, meromorphic, measurable, L^d_{loc}) representation in *some* local holomorphic trivialization in a neighborhood of each point, then the representation in *every* local holomorphic trivialization is continuous (respectively, $\mathcal{C}^k$, holomorphic, meromorphic, measurable, L^d_{loc}). The analogous statement holds for zeros and poles of a meromorphic section. The verifications are left to the reader (see Exercise 3.1.1).

2. For a nontrivial (i.e., not everywhere zero) meromorphic section of a holomorphic line bundle over a Riemann surface, the set of zeros (i.e., the set of points at which the section is equal to the zero element of the fiber) is discrete (see Exercise 3.1.2).

3. One goal of this chapter is a proof that every holomorphic line bundle on an *open* Riemann surface admits a great many (nontrivial) holomorphic sections. On the other hand, a holomorphic line bundle on a compact Riemann surface need not admit a nontrivial holomorphic section (see Exercise 3.1.6). Much of Chap. 4, and part of this chapter, are concerned with determining when (and how many) nontrivial holomorphic sections exist.

4. We may identify a section $x \mapsto (x, f(x))$ of the trivial line bundle $1_X = X \times \mathbb{C} \to X$ with the complex-valued function f.

Example 3.1.4 The holomorphic tangent bundle $\Pi_{(TX)^{1,0}}\colon (TX)^{1,0} \to X$ and holomorphic cotangent bundle $\Pi_{(T^*X)^{1,0}}\colon (T^*X)^{1,0} \to X$ have natural holomorphic line bundle structures. For given two local holomorphic coordinate neighborhoods (U_1, z_1) and (U_2, z_2) in X, we have the local trivialization $(U_j, (\Pi_{(TX)^{1,0}}, dz_j))$ of $(TX)^{1,0}$ over U_j for $j = 1, 2$, and we have holomorphic transition functions

$$g_{12} = \frac{dz_1}{dz_2} = \frac{\partial z_1}{\partial z_2} \quad \text{and} \quad g_{21} = \frac{dz_2}{dz_1} = \frac{\partial z_2}{\partial z_1} = 1/g_{12}.$$

For each $j = 1, 2$, we also have the local trivialization $(U_j, (\Pi_{(T^*X)^{1,0}}, \rho_j))$ of $(T^*X)^{1,0}$ over U_j, where $\rho_j(\alpha) = \alpha((\partial/\partial z_j)_p)$ for each point $p \in U_j$ and each element $\alpha \in (T_p^* X)^{1,0}$. The transition functions are the holomorphic functions

$$g_{12} = \frac{\rho_1}{\rho_2} = \frac{\partial/\partial z_1}{\partial/\partial z_2} = \frac{\partial z_2}{\partial z_1} \quad \text{and} \quad g_{21} = \frac{\partial/\partial z_2}{\partial/\partial z_1} = \frac{\partial z_1}{\partial z_2} = 1/g_{12}.$$

The local holomorphic sections of $(TX)^{1,0}$ are precisely the local holomorphic vector fields, and the local holomorphic sections of $(T^*X)^{1,0}$ are precisely the local holomorphic 1-forms (and the analogous statements hold for continuous, $\mathcal{C}^k$, measurable, meromorphic, and L^d_{loc} vector fields and forms). The holomorphic cotangent bundle $(T^*X)^{1,0}$ is also called the *canonical line bundle* (or *canonical bundle*) and is denoted by $\Pi_{K_X} : K_X \to X$.

Example 3.1.5 For the open subsets $U_0 \equiv \mathbb{C}$ and $U_\infty \equiv \mathbb{P}^1 \setminus \{0\}$ of $\mathbb{P}^1$, let

$$E \equiv [(U_0 \times \mathbb{C}) \sqcup (U_\infty \times \mathbb{C})]/\sim,$$

where for $(z, \zeta) \in (U_0 \setminus \{0\}) \times \mathbb{C}$ and $(w, \xi) \in (U_\infty \setminus \{\infty\}) \times \mathbb{C}$,

$$(z, \zeta) \sim (w, \xi) \quad \Longleftrightarrow \quad z = w \quad \text{and} \quad \zeta = z \cdot \xi;$$

let $\rho\colon (U_0 \times \mathbb{C}) \sqcup (U_\infty \times \mathbb{C}) \to E$ be the quotient map; and let $\Pi_E : E \to \mathbb{P}^1$ be the (well-defined) surjective mapping given by $\Pi_E : \rho(z, \zeta) \mapsto z$. Then $\Pi_E : E \to \mathbb{P}^1$ is a line bundle, which is called the *hyperplane bundle*. Moreover, for the local trivializations

$$(\Pi_E, \Phi_0) \equiv [\rho\restriction_{U_0 \times \mathbb{C}}]^{-1} : \Pi_E^{-1}(U_0) = \rho(U_0 \times \mathbb{C}) \to U_0 \times \mathbb{C}$$

and

$$(\Pi_E, \Phi_\infty) \equiv [\rho\restriction_{U_\infty \times \mathbb{C}}]^{-1} : \Pi_E^{-1}(U_\infty) = \rho(U_\infty \times \mathbb{C}) \to U_\infty \times \mathbb{C},$$

we have the holomorphic transition functions $g_{0\infty}$ from Φ_∞ to Φ_0 and $g_{\infty 0}$ from Φ_0 to Φ_∞ determined by

$$g_{0\infty} = \frac{\Phi_0}{\Phi_\infty} = \frac{1}{g_{\infty 0}} : \quad z \mapsto z.$$

Thus E, together with these local trivializations, is a holomorphic line bundle. The holomorphic section s of E on U_∞ determined by $z \mapsto \rho(z, 1) = (\Pi_E, \Phi_\infty)^{-1}(z, 1)$ for $(z, 1) \in U_\infty \times \mathbb{C}$ (i.e., the section that is represented by the constant function $\Phi_\infty(s) : z \mapsto 1$ in the local holomorphic trivialization $(U_\infty, (\Pi_E, \Phi_\infty))$) is represented in the local holomorphic trivialization $(U_0, (\Pi_E, \Phi_0))$ by the holomorphic function $\Phi_0(s) : z \mapsto z$. It follows that s extends to a unique holomorphic section of E on $\mathbb{P}^1$ that is nonvanishing except for a simple zero at the point 0.

Section 3.3 contains a simple method for producing many (in fact, it turns out, *all*) examples of holomorphic line bundles on a Riemann surface.

Given a local holomorphic trivialization $(U, \Psi = (\Pi, \Phi))$ of a holomorphic line bundle $\Pi \colon E \to X$, we get a nonvanishing holomorphic section t on U given by $t_p = \Phi_p^{-1}(1) = \Psi^{-1}(p, 1)$ for all $p \in U$; that is, t is the section with local representation $p \mapsto 1$. Conversely, according to Proposition 3.1.6 below, a nonvanishing local holomorphic section provides a local holomorphic trivialization. In other words, local holomorphic trivializations are equivalent to nonvanishing local holomorphic sections; and the latter point of view is often more convenient than the former. Further characterizations of continuity, continuous differentiability, etc., of sections in terms of local holomorphic sections are also contained in the proposition.

Proposition 3.1.6 *Suppose that* $\Pi \colon E \to X$ *is a holomorphic line bundle,* $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, *and* $d \in [1, \infty]$. *Then we have the following*:

(a) *If* t *is a nonvanishing holomorphic section of* E *on an open set* $U \subset X$, *then the mapping* $\Phi \colon \Pi^{-1}(U) \to \mathbb{C}$ *given by* $\xi \mapsto \xi/t(p)$ *for all* $p \in U$ *and* $\xi \in E_p$ *determines a local holomorphic trivialization* $(U, \Psi = (\Pi, \Phi))$ *of* E (*with* $\Phi(t) \equiv 1$).

(b) *The product* fs *of a continuous* ($\mathcal{C}^k$, *holomorphic, meromorphic, measurable*) *function* f *and a continuous* (*respectively,* $\mathcal{C}^k$, *holomorphic, meromorphic, measurable*) *section* s *of* E *and the sum* $s_1 + s_2$ *of two continuous* (*respectively,* $\mathcal{C}^k$, *holomorphic, meromorphic, measurable*) *sections* s_1 *and* s_2 *of* E *are continuous* (*respectively,* $\mathcal{C}^k$, *holomorphic, meromorphic, measurable*) *sections. The sum of two* L^d_{loc} *sections is in* L^d_{loc}, *and for* $d' \in [1, \infty]$ *with* $(1/d) + (1/d') = 1$, *the product of an* L^d_{loc} *function and an* $L^{d'}_{\mathrm{loc}}$ *section is in* L^1_{loc}.

(c) *A section* $s \colon S \to E$ *on a set* $S \subset X$ *is continuous* ($\mathcal{C}^k$, *holomorphic, meromorphic, measurable,* L^d_{loc}) *if and only if the quotient* $s/t \colon p \mapsto s(p)/t(p)$ *is continuous* (*respectively,* $\mathcal{C}^k$, *holomorphic, meromorphic, measurable,* L^d_{loc}) *for every nonvanishing local holomorphic section* t *of* E (*equivalently, for some nonvanishing local holomorphic section* t *on a neighborhood of each point in* S). *Furthermore, if* s *is meromorphic, then* s *has a zero* (*a pole*) *of order* m *at a point* $p \in S$ *if and only if the meromorphic function* s/t *has a zero* (*respectively, a pole*) *of order* m *at* p *for every* (*equivalently, for some*) *nonvanishing local holomorphic section* t *of* E *on a neighborhood of* p.

Proof Given a nonvanishing holomorphic section t of E on an open set $U \subset X$, the mapping $\Phi \colon \Pi^{-1}(U) \to \mathbb{C}$ given by $\xi \mapsto \xi/t(p)$ for all $p \in U$ and $\xi \in E_p$ determines a bijection $\Psi = (\Pi, \Phi) \colon \Pi^{-1}(U) \to U \times \mathbb{C}$. Moreover, if $(V, \Lambda = (\Pi, \Upsilon))$ is a local holomorphic trivialization of E, then (by Definition 3.1.3) the representing function $g = \Upsilon(t) \colon U \cap V \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is holomorphic and we have

$$\Upsilon(\xi) = g(p) \cdot \Phi(\xi) \quad \forall p \in U \cap V,\ \xi \in E_p.$$

Thus g is a holomorphic transition function from the local trivialization (Π, Φ) to (Π, Υ), and hence (Π, Φ) is a local holomorphic trivialization. One gets part (b) by considering local representations of the sections. Finally, (c) follows from parts (a) and (b). $\square$

Definition 3.1.7 Let $\Pi_E : E \to X$ and $\Pi_{E'} : E' \to X'$ be holomorphic line bundles over complex 1-manifolds X and X', let $\Upsilon : X \to X'$ be a holomorphic map, and let $\Lambda : E \to E'$ be a mapping such that $\Pi_{E'} \circ \Lambda = \Upsilon \circ \Pi_E$ and such that for each point $p \in X$, the mapping $\Lambda\restriction_{E_p} : E_p \to E'_{\Upsilon(p)}$ is linear. Then:

(a) The map Λ is called a *line bundle homomorphism* (or *line bundle map*) *along* Υ. Unless otherwise indicated, we will assume that a given line bundle homomorphism is taken along the identity map.
(b) We call Λ a *continuous* ($\mathcal{C}^k$, *holomorphic*) line bundle homomorphism if the function $\Phi \circ \Lambda(s)$ is a continuous (respectively, $\mathcal{C}^k$, holomorphic) function for every local holomorphic section s of E and every local holomorphic trivialization $(U, (\Pi_{E'}, \Phi))$ of E'.
(c) A bijective holomorphic line bundle homomorphism along the identity with holomorphic inverse line bundle homomorphism is called a *holomorphic line bundle isomorphism* (and the two bundles are said to be *isomorphic*).

Example 3.1.8 If $\Phi : X \to Y$ is a holomorphic mapping (a $\mathcal{C}^k$ mapping with $k \in \mathbb{Z}_{>0} \cup \{\infty\}$) of X into a complex 1-manifold Y, then the associated tangent mapping $\Phi_* : (TX)^{1,0} \to (TY)^{1,0}$ and pullback mapping $\Phi^* : (T^*Y)^{1,0} \to (T^*X)^{1,0}$ are holomorphic (respectively, $\mathcal{C}^{k-1}$) line bundle homomorphisms (see Exercise 3.1.3).

Remarks 1. A holomorphic line bundle $E \to X$ is holomorphically trivial if and only if $E \cong 1_X$.

2. As we will see, every holomorphic line bundle on an *open* Riemann surface is holomorphically trivial (Theorem 3.12.1). However, even in that context, the abstract point of view is still useful, since, for example, there is usually no natural choice of a global holomorphic trivialization.

3. A *real linear* (or *conjugate linear*) homomorphism of holomorphic line bundles is defined analogously, with the mapping real linear (respectively, conjugate linear) on each fiber. Continuous and $\mathcal{C}^k$ real linear and conjugate linear line bundle homomorphisms and isomorphisms are defined analogously.

4. We often identify two isomorphic line bundles without comment, although, occasionally one must proceed with some caution in doing so.

Definition 3.1.9 For holomorphic line bundles $\Pi_E : E \to X$ and $\Pi_{E'} : E' \to X$:

(a) The *dual bundle* of E is given by $E^* \equiv \bigcup_{p\in X} E^*_p$, and the corresponding *projection* $\Pi_{E^*} : E^* \to X$ is determined by $\alpha \mapsto p$ for each point $p \in X$ and each linear functional $\alpha \in E^*_p$.
(b) The *tensor product bundle* of E and E' is given by $E \otimes E' \equiv \bigcup_{p\in X} E_p \otimes E'_p$ (see Sect. 8.3), and the corresponding *projection* $\Pi_{E\otimes E'} : E \otimes E' \to X$ is determined by $\xi \mapsto p$ for each point $p \in X$ and each element $\xi \in E_p \otimes E'_p$.

We will assume that any given dual or tensor product bundle as above has the holomorphic line bundle structure provided by the following:

Proposition 3.1.10 *Let $\Pi_E\colon E \to X$, $\Pi_{E'}\colon E' \to X$, and $\Pi_{E''}\colon E'' \to X$ be holomorphic line bundles. Then we have the following*:

(a) *There is a* (*unique*) *holomorphic line bundle structure in $\Pi_{E^*}\colon E^* \to X$ such that for each nonvanishing holomorphic section t of E on an open set $U \subset X$, the map $\Psi\colon \Pi_{E^*}^{-1}(U) \to U \times \mathbb{C}$ given by $\alpha \mapsto (p, \alpha(t(p)))$ for all $p \in U$ and $\alpha \in E_p^*$ is a local holomorphic trivialization.*

(b) *There is a unique holomorphic line bundle structure in $E \otimes E'$ such that for each pair of holomorphic sections t of E and t' of E' on an open set $U \subset X$, the tensor product section $t \otimes t'$ (where $(s \otimes s')_p = s_p \otimes s'_p$ for each point p and each pair of sections s, s') is a local holomorphic section of $E \otimes E'$.*

(c) *For the above holomorphic line bundle structures*, *we have the* (*natural*) *holomorphic line bundle isomorphisms*
   (i) *$E \to (E^*)^*$ given by $\xi \mapsto \lambda_\xi$, where $\lambda_\xi(\alpha) = \alpha(\xi)$ for all $\xi \in E_p$ and $\alpha \in E_p^*$ with $p \in X$;*
   (ii) *$E^* \otimes E \to 1_X = X \times \mathbb{C}$ given by $\alpha \otimes \xi \mapsto \alpha(\xi)$ for all $\alpha \otimes \xi \in E_p^* \otimes E_p$ with $p \in X$;*
   (iii) *$1_X \otimes E \to E$ given by $(p, \zeta) \otimes \xi \mapsto \zeta\xi$ for all $p \in X$, $\zeta \in \mathbb{C}$, and $\xi \in E_p$;*
   (iv) *$E \otimes E' \to E' \otimes E$ given by $\eta \otimes \xi \mapsto \xi \otimes \eta$ for all $\eta \otimes \xi \in E_p \otimes E'_p$ with $p \in X$; and*
   (v) *$(E \otimes E') \otimes E'' \to E \otimes (E' \otimes E'')$ given by $(\eta \otimes \xi) \otimes \tau \mapsto \eta \otimes (\xi \otimes \tau)$ for all $\eta \otimes \xi \in E_p$, $\xi \in E'_p$, and $\tau \in E''_p$ with $p \in X$.*

Proof For the proof of (a), we consider two nonvanishing holomorphic sections t and u of E on open sets U and V, respectively. Proposition 3.1.6 then implies that the function $g \equiv t/u\colon U \cap V \to \mathbb{C}^*$ is holomorphic. Thus, for each element $\alpha \in E_p^*$ with $p \in U \cap V$, we have $\alpha(t(p)) = g(p)\alpha(u(p))$. It follows that the expression in (a) yields a bijection and that the transition functions for any two such bijections are holomorphic. Thus these mappings form a holomorphic line bundle atlas that determines a holomorphic line bundle structure on E^*, and (a) follows.

The proofs of (b) and (c) are left to the reader (see Exercise 3.1.4). □

Example 3.1.11 The canonical line bundle $K_X = (T^*X)^{1,0}$ is the dual of the holomorphic tangent bundle $(TX)^{1,0}$.

Example 3.1.12 The dual bundle $E^* \to \mathbb{P}^1$ of the hyperplane bundle $E \to \mathbb{P}^1$ (Example 3.1.5) is called the *tautological bundle*.

Remarks Let E, E', E'' be holomorphic line bundles on X.

1. Guided by part (c) of Proposition 3.1.10, we identify $(E^*)^*$ with E, $E^* \otimes E$ with 1_X, $1_X \otimes E$ with E, $E \otimes E'$ with $E' \otimes E$, and $(E \otimes E') \otimes E''$ with $E \otimes (E' \otimes E'')$.

2. We denote $(E \otimes E') \otimes E'' = E \otimes (E' \otimes E'')$ simply by $E \otimes E' \otimes E''$; we set $E^0 = 1_X$; and for any positive integer r, we set

$$E^r \equiv \overbrace{E \otimes \cdots \otimes E}^{r \text{ factors}}.$$

3. One may treat E^* as a multiplicative inverse of E, and the trivial line bundle 1_X as a multiplicative identity.

The proof of the following is left to the reader (see Exercise 3.1.5):

Proposition 3.1.13 *Let $\Pi_E : E \to X$ and $\Pi_{E'} : E' \to X$ be holomorphic line bundles, let $S \subset X$, let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, and let $d, d' \in [1, \infty]$ with $(1/d) + (1/d') = 1$. Then we have the following:*

(a) *For any section α of E^* on S, the following are equivalent:*
  (i) *The section α is continuous (of class $\mathcal{C}^k$, holomorphic, measurable).*
  (ii) *The function $\alpha(s)$ is continuous (respectively, of class $\mathcal{C}^k$, holomorphic, measurable) for every local continuous (respectively, class $\mathcal{C}^k$, holomorphic, measurable) section s of E.*
  (iii) *For each point $p \in S$, the function $\alpha(s)$ is continuous (respectively, of class $\mathcal{C}^k$, holomorphic, measurable) for some nonvanishing continuous (respectively, $\mathcal{C}^k$, holomorphic, measurable) section s of E on a neighborhood of p.*

(b) *For any section α of E^* on S, the following are equivalent:*
  (i) *The section α is in L^d_{loc}.*
  (ii) *The function $\alpha(s)$ is in L^d_{loc} for every local continuous section s of E.*
  (iii) *For each point $p \in S$, the function $\alpha(s)$ is in L^d_{loc} for some nonvanishing continuous section s of E on a neighborhood of p.*

  Moreover, if α is in L^d_{loc} and s is a section of E in $L^{d'}_{\mathrm{loc}}$ on S, then the function $\alpha(s)$ is in L^1_{loc}.

(c) *The tensor product $s \otimes s'$ of continuous ($\mathcal{C}^k$, holomorphic, meromorphic, measurable) sections s of E and s' of E' on S is continuous (respectively, $\mathcal{C}^k$, holomorphic, meromorphic, measurable). Furthermore, if s is nonvanishing, then the section s^{-1} of E^* is continuous (respectively, $\mathcal{C}^k$, holomorphic, meromorphic, measurable). If t is a meromorphic section of E on X that is not identically zero on any connected component of X, then t^{-1} is a meromorphic section of E^*. Finally, if u is a section of E in L^d_{loc} on S and u' is a section of E' in $L^{d'}_{\mathrm{loc}}$ on S, then $u \otimes u'$ is in L^1_{loc}.*

(d) *Any section s of $E^* \otimes E'$ on X determines a line bundle homomorphism $\Psi : E \to E'$ given by $\xi \mapsto (s_p/t)(\xi) \cdot t$ for each point $p \in X$, each element $t \in E'_p \setminus \{0\}$, and each element $\xi \in E_p$; and Ψ is continuous (of class $\mathcal{C}^k$, holomorphic) if and only if the section s is continuous (respectively, of class $\mathcal{C}^k$, holomorphic). Conversely, any line bundle homomorphism $\Psi : E \to E'$ determines a section s of $E^* \otimes E'$ on X given by $s_p \equiv t^{-1} \otimes \Psi(t)$ for each point $p \in X$ and each element $t \in E_p \setminus \{0\}$; and s is continuous (of class $\mathcal{C}^k$, holomorphic) if and only if Ψ is continuous (respectively, of class $\mathcal{C}^k$, holomorphic). Moreover, the above associations are inverse mappings.*

We now consider differential forms with values in a holomorphic line bundle.

Definition 3.1.14 Let $\Pi_E\colon E \to X$ be a holomorphic line bundle. For each pair $p, q \in \mathbb{Z}_{\geq 0}$ and for $r = p + q$, we define

$$\Lambda^{p,q}T^*X \otimes E \equiv \bigcup_{x\in X} \Lambda^{p,q}T_x^*X \otimes E_x \subset \Lambda^r(T^*X)_{\mathbb{C}} \otimes E \equiv \bigcup_{x\in X} \Lambda^r(T_x^*X)_{\mathbb{C}} \otimes E_x$$

($\Lambda^{p,q}T^*X \otimes E = X \times \{0\}$ for $p > 1$ or $q > 1$, $\Lambda^r(T^*X)_{\mathbb{C}} \otimes E = X \times \{0\}$ for $r > 2$). The corresponding *projections* are the (surjective) mappings

$$\Pi_{\Lambda^{p,q}T^*X\otimes E}\colon \Lambda^{p,q}T^*X \otimes E \to X \quad \text{and} \quad \Pi_{\Lambda^r(T^*X)_{\mathbb{C}}\otimes E}\colon \Lambda^r(T^*X)_{\mathbb{C}} \otimes E \to X$$

given by $\alpha \mapsto x$ for each $x \in X$ and each $\alpha \in \Lambda^{p,q}T_x^*X \otimes E$ or $\Lambda^r(T_x^*X)_{\mathbb{C}} \otimes E$. The elements of $\Lambda^{p,q}T^*X \otimes E$ are said to be of *type* (p,q). For each point $x \in X$, we also set

$$[\Lambda^{p,q}T^*X \otimes E]_x \equiv \Lambda^{p,q}T_x^*X \otimes E_x \subset [\Lambda^r(T^*X)_{\mathbb{C}} \otimes E]_x \equiv \Lambda^r(T_x^*X)_{\mathbb{C}} \otimes E_x.$$

Given a nonnegative integer s, another holomorphic line bundle $\Pi_F\colon F \to X$, and a point $x \in X$, the mapping

$$[\Lambda^r(T^*X)_{\mathbb{C}} \otimes E]_x \times [\Lambda^s(T^*X)_{\mathbb{C}} \otimes F]_x \to [\Lambda^{r+s}(T^*X)_{\mathbb{C}} \otimes E \otimes F]_x$$

given by

$$(\alpha, \beta) \mapsto \alpha \wedge \beta \equiv \frac{\alpha}{t} \wedge \frac{\beta}{u} \otimes t \otimes u,$$

for any choice of $t \in E_x \setminus \{0\}$ and $u \in F_x \setminus \{0\}$, is called the *exterior product* (or *wedge product*).

Remarks 1. For any holomorphic line bundle E on X, the spaces

$$\Lambda^{0,0}T^*X \otimes E = 1_X \otimes E \cong E \quad \text{and} \quad K_X \otimes E = \Lambda^{1,0}T^*X \otimes E$$

are actually tensor products of holomorphic line bundles, so they have natural holomorphic line bundle structures (provided by Proposition 3.1.10). For $p > 1$ or $q > 1$,

$$\Lambda^{p,q}T^*X \otimes E = X \times \{0\} \cong X$$

has the natural holomorphic structure inherited from X.

2. $\Lambda^{p,q}T^*X \otimes E$ and $\Lambda^r T^*X \otimes E$ have natural $\mathcal{C}^\infty$ structures (see Exercise 3.1.10). Moreover, $\Lambda^{p,q}T^*X \otimes E$ and $\Lambda^r T^*X \otimes E$ are $\mathcal{C}^\infty$ vector bundles (see, for example, [Wel]).

3. The above definition of the exterior product is independent of the choice of the nonzero elements of the fibers of the line bundles (see Exercise 3.1.7).

Guided by Definition 9.5.2, Definition 9.7.12, Definition 2.5.1, Definition 2.5.2, and Proposition 3.1.6, we make the following definition:

Definition 3.1.15 Let $\Pi_E \colon E \to X$ be a holomorphic line bundle, let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, and let $d \in [1, \infty]$.

(a) For each $r \in \mathbb{Z}_{\geq 0}$, an *$E$-valued differential form of degree r* (or an *r-form with values in E* or an *E-valued r-form*) in X on a set $S \subset X$ is a mapping α of S into $\Lambda^r(T^*X)_{\mathbb{C}} \otimes E$ such that $\Pi_{\Lambda^r(T^*X)_{\mathbb{C}} \otimes E} \circ \alpha = \mathrm{Id}_S$ (in particular, α/s is a scalar-valued differential form of degree r for every nonvanishing local holomorphic section s of E). We usually denote the value of α at x by α_x for each point $x \in S$. If $\alpha_x \in \Lambda^{p,q} T_x^* X \otimes E_x$ for each point $x \in S$, then we say that α is of *type* (p, q) or that α is an E-valued (p, q)*-form*.

(b) An E-valued differential form α is *continuous* (of *class $\mathcal{C}^k$*, *holomorphic*, *meromorphic*, *measurable*, *in L^d_{loc}*) if the scalar-valued differential form α/s is continuous (respectively, of class $\mathcal{C}^k$, holomorphic, meromorphic, measurable, in L^d_{loc}) for every nonvanishing local holomorphic section s of E. A sequence of L^d_{loc} differential forms $\{\alpha_\nu\}$ with values in E *converges in L^d_{loc}* to an E-valued differential form α if $\{\alpha_\nu/s\}$ converges to α/s in L^d_{loc} for every nonvanishing local holomorphic section s of E.

(c) For each open set $U \subset X$ and each nonnegative integer r, the set of $\mathcal{C}^\infty$ E-valued r-forms on U is denoted by $\mathcal{E}^r(U, E)$ or $\mathcal{E}^r(E)(U)$. The set of $\mathcal{C}^\infty$ E-valued r-forms with compact support in Ω is denoted by $\mathcal{D}^r(U, E)$ or $\mathcal{D}^r(E)(U)$. Similarly, for $p, q \in \mathbb{Z}_{\geq 0}$, the set of $\mathcal{C}^\infty$ E-valued (p, q)-forms on U is denoted by $\mathcal{E}^{p,q}(U, E)$ or $\mathcal{E}^{p,q}(E)(U)$, and the set of $\mathcal{C}^\infty$ E-valued (p, q)-forms with compact support in U is denoted by $\mathcal{D}^{p,q}(U, E)$ or $\mathcal{D}^{p,q}(E)(U)$. The set of E-valued holomorphic 1-forms on U is denoted by $\Omega(U, E)$ or $\Omega(E)(U)$.

Remarks 1. Let α be an r-form with values in a holomorphic line bundle E on a subset S of X. Given a nonvanishing holomorphic section s of E on an open set U, we get the scalar-valued form $\beta \equiv \alpha/s$ with $\alpha\restriction_{S\cap U} = \beta \otimes s$. If α is of type $(1, 0)$, then α is actually a section of the holomorphic line bundle $\Lambda^{1,0}T^*X \otimes E$. If $r = 1$, then for each point $x \in S$, we may also identify α_x with the linear mapping of the tangent space at x into E_x given by

$$v \mapsto \frac{\alpha_x}{t}(v) \cdot t,$$

for any choice of $t \in E_x \setminus \{0\}$ (the above is independent of the choice of t). For $r = 2$ and $x \in S$, we may identify α_x with the *skew-symmetric bilinear* pairing of the tangent space at x into E_x given by

$$(u, v) \mapsto \frac{\alpha_x}{t}(u, v) \cdot t$$

for any choice of $t \in E_x \setminus \{0\}$. The existence of the above identifications is the reason one calls α a differential form with *values in E*. For s a nonvanishing holomorphic section on an open set U and $\beta = \alpha/s$ on $S \cap U$, the above also leads us to denote $\alpha\restriction_{S\cap U} = \beta \otimes s$ simply by $\beta \cdot s$ or by $s \otimes \beta$ or $s \cdot \beta$ (see Sect. 8.3).

2. For α and β differential forms with values in holomorphic line bundles E and F, respectively, we may form the exterior product $\alpha \wedge \beta$ with values in $E \otimes F$. Identifying $E \otimes F$ with $F \otimes E$, we see that if α and β are of degree r and m, respectively, then $\alpha \wedge \beta = (-1)^{rm} \beta \wedge \alpha$.

3. Let α be an E-valued r-form on a set S, let (U, z) be a local holomorphic coordinate neighborhood, and let s be a nonvanishing holomorphic section of E on an open set V. Then, on $S \cap U \cap V$, we have $\alpha = as$, where $a = \alpha/s$, if $r = 0$; $\alpha = a\,dz \otimes s + b\,d\bar{z} \otimes s$, where a and b are the coefficients of α/s, if $r = 1$; and $\alpha = a\,dz \wedge d\bar{z} \otimes s$, where a is the coefficient of α/s, if $r = 2$.

4. A holomorphic 1-form θ with values in a holomorphic line bundle E over X may be viewed as a holomorphic section of the holomorphic line bundle $K_X \otimes E$. We also have an identification of $\Lambda^{1,1} T^* X \otimes E$ with $\Lambda^{0,1} T^* X \otimes K_X \otimes E$ under the mapping

$$\alpha = \theta \wedge \beta \cdot s = -\beta \wedge \theta \cdot s \mapsto -\beta \cdot (\theta \otimes s);$$

so a $(1, 1)$-form with values in E may be identified with a $(0, 1)$-form with values in $K_X \otimes E$ (see Sect. 3.10). On the other hand, certain operators and pairings to be considered in this chapter and Chap. 4 are not completely preserved under this mapping, so one must proceed with some caution when making this identification.

The proof of the following is left to the reader (see Exercise 3.1.8):

Proposition 3.1.16 *Let E and F be holomorphic line bundles on X, let $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, and let $d, d' \in [1, \infty]$. Then we have the following*:

(a) *A differential form α on a set $S \subset X$ with values in E is continuous* (*of class $\mathcal{C}^k$, holomorphic, meromorphic, measurable, in L^d_{loc}*) *if and only if for each point in S, the scalar-valued differential form α/s is continuous* (*respectively, of class $\mathcal{C}^k$, holomorphic, meromorphic, measurable, in L^d_{loc}*) *for some nonvanishing local holomorphic section s of E on a neighborhood of the point. A sequence of $L^d_{\mathrm{loc}}(S)$ differential forms $\{\alpha_\nu\}$ with values in E converges in $L^d_{\mathrm{loc}}(S)$ to an E-valued differential form α if and only if for each point in S, $\{\alpha_\nu/s\}$ converges to α/s in L^d_{loc} for some nonvanishing local holomorphic section s of E on a neighborhood of the point.*

(b) *Let α and β be differential forms with values in E and F, respectively. If α and β are continuous* (*of class $\mathcal{C}^k$, holomorphic, meromorphic, measurable*), *then $\alpha + \beta$ and $\alpha \wedge \beta$ are continuous* (*respectively, of class $\mathcal{C}^k$, holomorphic, meromorphic, measurable*). *If α and β are in L^d_{loc}, then $\alpha + \beta$ is in L^d_{loc}. If α and β are in L^d_{loc} and $L^{d'}_{\mathrm{loc}}$, respectively, and $(1/d) + (1/d') = 1$, then $\alpha \wedge \beta$ is locally integrable* (*i.e., $\alpha \wedge \beta$ is in L^1_{loc}*).

Remark It follows from the above proposition that the set of continuous ($\mathcal{C}^k$, holomorphic, meromorphic, measurable, L^d_{loc}) differential forms with values in a given holomorphic line bundle over a subset of a complex 1-manifold is a vector space.

Exercises for Sect. 3.1

3.1.1 Let $\Pi\colon E \to X$ be a holomorphic line bundle over a Riemann surface X.
(a) Verify that, as noted in Definition 3.1.3, if a section of E has a continuous ($\mathcal{C}^k$, holomorphic, meromorphic, measurable, L^d_{loc}) representation in *some* local holomorphic trivialization in a neighborhood of each point in X, then the representation in *every* local holomorphic trivialization is continuous (respectively, $\mathcal{C}^k$, holomorphic, meromorphic, measurable, L^d_{loc}).
(b) Verify that if $p \in X$ and s is a holomorphic section of E on X such that the representation of s in *some* local holomorphic trivialization in a neighborhood of p has a zero of order m at p, then s has a zero of order m at p (i.e., the representation in *every* local holomorphic trivialization in a neighborhood of p has a zero of order m at p).
(c) Verify that if $p \in X$ and s is a holomorphic section of E on $X \setminus \{p\}$ such that the representation of s in *some* local holomorphic trivialization in a neighborhood of p has a pole of order m at p, then s is meromorphic with a pole of order m at p (i.e., the representation in *every* local holomorphic trivialization in a neighborhood of p is meromorphic with a pole of order m at p).

3.1.2 Let s be a nontrivial meromorphic section of a holomorphic line bundle E on a Riemann surface X. Verify that $\{x \in X \mid s_x = 0\}$ is discrete.

3.1.3 Prove that if $\Phi\colon X \to Y$ is a holomorphic mapping (a $\mathcal{C}^k$ mapping with $k \in \mathbb{Z}_{>0} \cup \{\infty\}$) of Riemann surfaces X and Y, then the associated tangent mapping $\Phi_*\colon (TX)^{1,0} \to (TY)^{1,0}$ and pullback mapping $\Phi^*\colon (T^*Y)^{1,0} \to (T^*X)^{1,0}$ are holomorphic (respectively, $\mathcal{C}^{k-1}$) line bundle homomorphisms.

3.1.4 Prove parts (b) and (c) of Proposition 3.1.10.

3.1.5 Prove Proposition 3.1.13.

3.1.6 Let E be a nontrivial holomorphic line bundle on a compact Riemann surface X. Prove that $\Gamma(X, \mathcal{O}(E)) = 0$ or $\Gamma(X, \mathcal{O}(E^*)) = 0$.

3.1.7 Verify that the exterior product as given in Definition 3.1.14 is well defined (i.e., independent of the choice of the nonzero elements of the fibers of the line bundles).

3.1.8 Prove Proposition 3.1.16.

3.1.9 Let $\Pi\colon E \to X$ be a holomorphic line bundle over a complex 1-manifold X. Prove that there is a unique structure of a 2-dimensional complex manifold on E for which Π is a holomorphic mapping and $\Phi\colon \Pi^{-1}(U) \to \mathbb{C}$ is a holomorphic function for every local holomorphic trivialization $(U, (\Pi, \Phi))$ (see Exercise 2.2.6 for the definition of a complex manifold and a holomorphic mapping). Prove also that a section of E is holomorphic if and only if it is holomorphic as a mapping of complex manifolds.

3.1.10 Let $\Pi_E\colon E \to X$ be a holomorphic line bundle over a complex 1-manifold X.
(a) Prove that there is a (unique) structure of a $\mathcal{C}^\infty$ manifold on $\Lambda^1(T^*X)_{\mathbb{C}} \otimes E$ such that for each local holomorphic chart $(U, \Phi = z, U')$ in X and

each local holomorphic trivialization $(U, (\Pi_E, \Upsilon))$, we get a local $\mathcal{C}^\infty$ chart

$$\left(\Pi^{-1}_{\Lambda^1(T^*X)_{\mathbb{C}}\otimes E}(U), \Psi, U' \times \mathbb{C}^2 = U' \times \mathbb{R}^4\right)$$

in $\Lambda^1(T^*X)_{\mathbb{C}} \otimes E$, where

$$\Psi\colon \alpha \otimes \xi \mapsto \left(z(c), \Upsilon(\xi)\cdot\alpha\left(\left(\frac{\partial}{\partial z}\right)_c\right), \Upsilon(\xi)\cdot\alpha\left(\left(\frac{\partial}{\partial \bar{z}}\right)_c\right)\right)$$

for each point $c \in U$, each element $\xi \in E_c$, and each element $\alpha \in \Lambda^1(T^*_c X)_{\mathbb{C}}$. In other words, setting $\xi = (\Pi_E, \Upsilon)^{-1}(c, 1)$, we have

$$\Psi((a(dz)_c + b(d\bar{z})_c) \otimes \xi) = (z(c), a, b) \quad \forall (a, b) \in \mathbb{C}^2.$$

(b) Prove that there is a (unique) structure of a $\mathcal{C}^\infty$ manifold on $\Lambda^{0,1}T^*X \otimes E$ such that for each local holomorphic chart $(U, \Phi = z, U')$ in X and each local holomorphic trivialization $(U, (\Pi_E, \Upsilon))$, we get a local $\mathcal{C}^\infty$ chart

$$\left(\Pi^{-1}_{\Lambda^{0,1}T^*X\otimes E}(U), \Psi, U' \times \mathbb{C} = U' \times \mathbb{R}^2\right)$$

in $\Lambda^{0,1}(T^*X)_{\mathbb{C}} \otimes E$, where

$$\Psi\colon \alpha \otimes \xi \mapsto \left(z(c), \Upsilon(\xi)\cdot\alpha\left(\left(\frac{\partial}{\partial \bar{z}}\right)_c\right)\right)$$

for each point $c \in U$, each element $\xi \in E_c$, and each element $\alpha \in \Lambda^{0,1}T^*_c X$. Note that the holomorphic line bundle $K_X \otimes E = (T^*X)^{1,0} \otimes E$ has a $\mathcal{C}^\infty$ structure provided by Exercise 3.1.9.

(c) Prove that there is a (unique) structure of a $\mathcal{C}^\infty$ manifold on $\Lambda^{1,1}T^*X \otimes E = \Lambda^2(T^*X)_{\mathbb{C}} \otimes E$ such that for each local holomorphic chart $(U, \Phi = z, U')$ in X and each local holomorphic trivialization $(U, (\Pi_E, \Upsilon))$, we get a local $\mathcal{C}^\infty$ chart

$$\left(\Pi^{-1}_{\Lambda^{1,1}T^*X\otimes E}(U), \Psi, U' \times \mathbb{C} = U' \times \mathbb{R}^2\right)$$

in $\Lambda^{1,1}T^*X \otimes E$, where

$$\Psi\colon \alpha \otimes \xi \mapsto \left(z(c), \Upsilon(\xi)\cdot\alpha\left(\left(\frac{\partial}{\partial z}\right)_c, \left(\frac{\partial}{\partial \bar{z}}\right)_c\right)\right)$$

for each point $c \in U$, each element $\xi \in E_c$, and each element $\alpha \in \Lambda^{1,1}T^*_c X$.

(d) Prove that the projection mappings to X in the above are $\mathcal{C}^\infty$ mappings.

(e) Prove that an E-valued differential form is continuous (of class $\mathcal{C}^k$) if and only if it is continuous (respectively, of class $\mathcal{C}^k$) as a mapping.

3.2 Sheaves Associated to a Holomorphic Line Bundle

Sheaf theory is a convenient and powerful tool for describing local objects and for passing to global objects. In this book, rather than consider the general theory of sheaves, we instead mostly consider and apply a few specific types of sheaves associated to a holomorphic line bundle. We first consider some examples before considering a formal definition. Throughout this section, $\Pi\colon E \to X$ denotes a holomorphic line bundle over a complex 1-manifold X.

Example 3.2.1 The *sheaf* $\mathcal{O}(E)$ *of holomorphic sections of* E consists of the assignment

$$U \mapsto \mathcal{O}(E)(U) = \Gamma(U, \mathcal{O}(E))$$

of the collection of holomorphic sections $\mathcal{O}(E)(U)$ to each open set $U \subset X$ together with the associated restriction maps $\rho^U_V\colon \Gamma(U, \mathcal{O}(E)) \to \Gamma(V, \mathcal{O}(E))$ given by $s \mapsto s\restriction_V$ for open sets $U \supset V$ ($\mathcal{O}(E)$ is also called a *locally free analytic sheaf of rank* 1). The *sheaf of holomorphic functions* $\mathcal{O}$ is given by $U \mapsto \mathcal{O}(U)$, and we identify $\mathcal{O}$ with $\mathcal{O}(1_X)$.

Let $p \in X$ and let $\sim_p$ be the equivalence relation on the set of local holomorphic sections of E that are defined on a neighborhood of p determined by

$$s \sim_p t \quad \Longleftrightarrow \quad s = t \text{ on some neighborhood of } p.$$

Each of the associated equivalence classes is called a *germ of a holomorphic section of* E *at* p. The set $\mathcal{O}(E)_p$ of germs at p is called the *stalk of* $\mathcal{O}(E)$ *at* p. We denote by $\mathrm{germ}_p s$ the germ represented by a local holomorphic section s of E on a neighborhood of p.

For each open set $U \subset X$, $\Gamma(U, \mathcal{O}(E))$ is a module over $\Gamma(U, \mathcal{O})$ (we view $\Gamma(\emptyset, \mathcal{O}(E))$ as the trivial module $\{0\}$). The operations descend to the level of germs, making $\mathcal{O}(E)_p$ a module over the ring $\mathcal{O}_p$ for each point $p \in X$. More precisely, for neighborhoods U, V, and W of p, sections $s \in \Gamma(U, \mathcal{O}(E))$ and $t \in \Gamma(V, \mathcal{O}(E))$, and a function $f \in \mathcal{O}(W)$, we define

$$\mathrm{germ}_p f \cdot \mathrm{germ}_p s \equiv \mathrm{germ}_p (f \cdot s),$$

$$\mathrm{germ}_p s + \mathrm{germ}_p t \equiv \mathrm{germ}_p (s + t),$$

$$\mathrm{germ}_p s - \mathrm{germ}_p t \equiv \mathrm{germ}_p (s - t).$$

Example 3.2.2 The definitions of the *sheaf* $\mathcal{M} = \mathcal{M}(1_X)$ *of meromorphic functions*, the *sheaf* $\mathcal{M}(E)$ *of meromorphic sections of* E (a sheaf of modules over $\mathcal{M}$), the *germ of a meromorphic section of* E, and the *stalk* $\mathcal{M}(E)_p$ *of* $\mathcal{M}(E)$ at $p \in X$ are analogous to the definitions in Example 3.2.1.

Example 3.2.3 The definitions of the *sheaf* $\Omega = \Omega(1_X) \cong \mathcal{O}(K_X)$ *of holomorphic* 1*-forms* (a sheaf of $\mathcal{O}$-modules), the *sheaf* $\Omega(E) \cong \mathcal{O}(K_X \otimes E)$ *of* E*-valued holomorphic* 1*-forms* (a sheaf of $\mathcal{O}$-modules), the *sheaf of meromorphic* 1*-forms* (which is isomorphic to $\mathcal{M}(K_X)$ as a sheaf of $\mathcal{M}$-modules), the *sheaf of* E*-valued meromorphic* 1*-forms* (which is isomorphic to $\mathcal{M}(K_X \otimes E)$ as a sheaf of $\mathcal{M}$-modules), and the associated germs and stalks, are analogous to the definitions in Example 3.2.1.

Example 3.2.4 The definitions of the sheaves $\mathcal{E} = \mathcal{E}^0$, $\mathcal{E}^r$, $\mathcal{E}^{p,q}$, $\mathcal{E}(E)$, $\mathcal{E}^r(E)$, and $\mathcal{E}^{p,q}(E)$ (which are sheaves of modules over $\mathcal{E}$), and the associated germs and stalks, are analogous to the definitions in Example 3.2.1.

Example 3.2.5 Given a holomorphic line bundle homomorphism $\Psi\colon E \to F$ (along the identity), we get corresponding *sheaf morphisms* $\Psi_*\colon \mathcal{O}(E) \to \mathcal{O}(F)$ and $\Psi_*\colon \mathcal{M}(E) \to \mathcal{M}(F)$ given by $s \mapsto \Psi(s)$. Similarly, a $\mathcal{C}^\infty$ line bundle homomorphism $\Psi\colon E \to F$ induces sheaf morphisms $\mathcal{E}^r(E) \to \mathcal{E}^r(F)$ and $\mathcal{E}^{p,q}(E) \to \mathcal{E}^{p,q}(F)$ (locally, the mappings are given by $\alpha \mapsto (\alpha/t) \otimes \Psi(t)$ for any nonvanishing local holomorphic section t of E).

Example 3.2.6 Let $p \in X$. The set $\mathfrak{m}_p$ of germs at p of holomorphic functions that vanish at p is the maximal ideal in the ring $\mathcal{O}_p$, and for each positive integer r, $\mathfrak{m}_p^r$ is precisely the set of germs at p of holomorphic functions that vanish at p to order at least r. For each open set $U \subset X$, let $\mathcal{F}(U)$ be the $\mathcal{O}(U)$-submodule of $\mathcal{O}(E)(U)$ consisting of the holomorphic sections that vanish at p to order at least r (in particular, $\mathcal{F}(U) = \mathcal{O}(E)(U)$ if $p \notin U$). Together with the given restriction mappings, these submodules form a *subsheaf* $\mathcal{F}$ of $\mathcal{O}(E)$. For each point $x \in X$, the *stalk* $\mathcal{F}_x$ *at* x is the collection of germs that are represented by functions in $\mathcal{F}(U)$ for some neighborhood U of x. Thus

$$\mathcal{F}_x = \begin{cases} \mathcal{O}(E)_x & \text{if } x \neq p, \\ \mathfrak{m}_p^r \cdot \mathcal{O}(E)_p & \text{if } x = p. \end{cases}$$

The $\mathcal{O}_p$-submodule $\mathfrak{m}_p^r \cdot \mathcal{O}(E)_p$ of $\mathcal{O}(E)_p$ consists of the germs at p of holomorphic sections s of E with a zero of order at least r at p. The quotient

$$\mathcal{O}(E)_p/\mathfrak{m}_p^r \cdot \mathcal{O}(E)_p = \{\xi + \mathfrak{m}_p^r \cdot \mathcal{O}(E)_p \mid \xi \in \mathcal{O}(E)_p\}$$

is a complex vector space of dimension r. For in terms of a local holomorphic coordinate z vanishing at r and a nonvanishing holomorphic section t of E in a neighborhood of p, the elements $\xi_0, \dots, \xi_{r-1}$ of $\mathcal{V}$ represented by $t, zt, z^2t, \dots, z^{r-1}t$, respectively, form a basis.

For each open set $U \subset X$, let

$$\mathcal{S}(U) \equiv \begin{cases} \mathcal{O}(E)_p/\mathfrak{m}_p^r \cdot \mathcal{O}(E)_p & \text{if } p \in U, \\ 0 & \text{if } p \notin U, \end{cases}$$

and for any open set $V \subset U$, let $\rho_V^U\colon \mathcal{S}(U) \to \mathcal{S}(V)$ be the obvious restriction map; i.e., ρ_V^U is the identity if $p \in V$ (so that $\mathcal{S}(U) = \mathcal{S}(V)$) and ρ_V^U is identically 0 if

$p \notin V$. Then $\mathcal{S}$ is a sheaf of modules over $\mathcal{O}$. We may form germs and stalks at a point x by identifying elements that have the same image under some restriction map in some neighborhood of x. Thus $\mathcal{S}_x = \mathcal{O}(E)_x \big/ \mathcal{F}_x$ at each point $x \in X$. Because there is this single nonzero r-dimensional vector space stalk at p, $\mathcal{S}$ is called a *skyscraper sheaf.*

The induced maps $\mathcal{O}(E)(U) \to \mathcal{S}(U)$ given by

$$t \mapsto \begin{cases} \operatorname{germ}_p t + \mathfrak{m}_p^r \cdot \mathcal{O}(E)_p \in \mathcal{O}(E)_p/\mathfrak{m}_p^r \cdot \mathcal{O}(E)_p & \text{if } p \in U, \\ 0 & \text{if } p \notin U, \end{cases}$$

determine a sheaf morphism, which we denote by $\mathcal{O}(E) \to \mathcal{S}$. This morphism may also be identified with the (quotient) module maps $\mathcal{O}(E)_x \to \mathcal{S}_x = \mathcal{O}(E)_x \big/ \mathcal{F}_x$ at the level of stalks.

Example 3.2.7 Let Y be a topological space, and let R be a ring. Then the associated *constant sheaf* $\mathcal{R}$ is given by $\mathcal{R}(U) \equiv R$ for each nonempty open set $U \subset Y$, and $\mathcal{R}(\emptyset) \equiv 0$. We also denote $\mathcal{R}$ by R. In particular, we have the *zero sheaf* $U \mapsto \{0\}$, which is denoted by 0. Constant sheaves of modules, groups, etc., are defined analogously.

We now consider the formal definitions.

Definition 3.2.8 Let Y be a topological space.

(a) A *sheaf* $\mathcal{F}$ on Y is an assignment of a set $\mathcal{F}(U)$ to each open set $U \subset Y$ and a *restriction mapping* $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ to each pair of open sets $U \supset V$ such that
  (i) For each open set $U \subset Y$, we have $\rho_U^U = \mathrm{Id}_U$ on U;
  (ii) For open sets $U \supset V \supset W$, we have $\rho_W^U = \rho_W^V \circ \rho_V^U$;
  (iii) If $\{U_i\}_{i \in I}$ is a family of open subsets of Y, $U = \bigcup_{i \in I} U_i$, and $s, t \in \mathcal{F}(U)$ with $\rho_{U_i}^U(s) = \rho_{U_i}^U(t)$ for each $i \in I$, then $s = t$; and
  (iv) If $\{U_i\}_{i \in I}$ is a family of open subsets of Y, $U = \bigcup_{i \in I} U_i$, $s_i \in \mathcal{F}(U_i)$ for each $i \in I$, and $\rho_{U_i \cap U_j}^{U_i}(s_i) = \rho_{U_i \cap U_j}^{U_j}(s_j)$ for every pair of indices $i, j \in I$, then there exists an $s \in \mathcal{F}(U)$ such that $\rho_{U_i}^U(s) = s_i$ for each $i \in I$.

  The elements of $\mathcal{F}(U)$ are called *sections of* $\mathcal{F}$ *over* U. The set $\mathcal{F}(U)$ is also denoted by $\Gamma(U, \mathcal{F})$ or $H^0(U, \mathcal{F})$.

(b) A sheaf $\mathcal{R}$ on Y together with the assignment of a ring structure to $\mathcal{R}(U)$ for each open set U such that the restriction mappings are homomorphisms with respect to these ring structures is called a *sheaf of rings*. Given a sheaf of rings $\mathcal{R}$ on Y, a *sheaf of* $\mathcal{R}$*-modules* on Y is a sheaf $\mathcal{F}$ on Y together with the assignment of a module structure to $\mathcal{F}(U)$ over the ring $\mathcal{R}(U)$ for each open set $U \subset Y$ such that the restriction mappings commute with the associated sum and product operations. Sheaves of groups, ideals, vector spaces, and other algebraic structures are defined analogously.

(c) A *morphism* (or *sheaf mapping*) $\Psi\colon \mathcal{F} \to \mathcal{G}$ of sheaves $\mathcal{F}$ and $\mathcal{G}$ is a choice of a morphism $\Psi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ (i.e., a map preserving the algebraic structures) for each open set U such that for open sets $U \supset V$, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\Psi_U} & \mathcal{G}(U) \\ {\scriptstyle \rho^U_V}\downarrow & & \downarrow{\scriptstyle \rho^U_V} \\ \mathcal{F}(V) & \xrightarrow{\Psi_V} & \mathcal{G}(V) \end{array}$$

If each of the above maps Ψ_U is an inclusion, then we call $\mathcal{F}$ a *subsheaf* of $\mathcal{G}$ and we call Ψ an *inclusion*. If each of the above maps is an isomorphism, then we call Ψ a *sheaf isomorphism*.

(d) Let $\mathcal{F}$ be a sheaf on Y, and for each point $p \in Y$, let $\sim_p$ be the equivalence relation on the set of sections of $\mathcal{F}$ over neighborhoods of p defined as follows: Given $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ with $p \in U \cap V$, we have $s \sim_p t$ if and only if there is an open set $W \subset U \cap V$ with $\rho^U_W(s) = \rho^V_W(t)$. For each section s of $\mathcal{F}$ over a neighborhood U of p, we call the equivalence class represented by s the *germ of* s *at* p and we denote this germ by $\mathrm{germ}_p s$. The collection of germs at p is called the *stalk* at p and is denoted by $\mathcal{F}_p$. If $\mathcal{F}$ is a sheaf of $\mathcal{R}$-modules on Y for some sheaf of rings $\mathcal{R}$, then for each point $p \in Y$, the stalk $\mathcal{R}_p$ is a ring and the stalk $\mathcal{F}_p$ is a module over $\mathcal{R}_p$ with the natural induced operations. A sheaf morphism $\Psi\colon \mathcal{F} \to \mathcal{G}$ induces a well-defined morphism $\Psi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ given by $\Psi_p(\mathrm{germ}_p t) \equiv \mathrm{germ}_p \Psi_U(t)$ for each point $p \in Y$, each neighborhood U, and each section $t \in \mathcal{F}(U)$. These induced morphisms are inclusions (isomorphisms) if and only if Ψ_U is injective (respectively, bijective) for each open set U (see Exercise 3.2.2). If Ψ_U is surjective for each open set U, then $\Psi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is surjective for each point $p \in M$. However, the converse is false (see Exercise 3.2.3).

Remarks 1. Given a sheaf morphism $\Psi\colon \mathcal{F} \to \mathcal{G}$, the *kernel* $\mathcal{K}$ is the subsheaf with sections over a nonempty open set U given by $\ker \Psi_U$ and the restriction mappings inherited from $\mathcal{F}$. In general, the assignment $U \to \operatorname{im} \Psi_U$ need not be a sheaf. It is an example of what is called a *presheaf*, and it therefore determines a sheaf (called the *image*), but we will not consider presheaves and their associated sheaves in this book.

2. The skyscraper sheaf is an example of a quotient sheaf. Again, to define quotient sheaves in general, one must consider presheaves and their associated sheaves.

3. For a sheaf $\mathcal{F}$, since we may write $\emptyset$ as a union of the form $\bigcup_{i\in\emptyset} U_i$, and since for $s, t \in \mathcal{F}(\emptyset)$, the equality $\rho^{\emptyset}_{U_i}(s) = \rho^{\emptyset}_{U_i}(t)$ then holds vacuously for $i \in \emptyset$, the axiom (iii) implies that $\mathcal{F}(\emptyset)$ contains at most one element. Similarly, the condition (iv) implies that $\mathcal{F}(\emptyset)$ is nonempty and therefore that $\mathcal{F}(\emptyset)$ is a singleton. In particular, for $\mathcal{F}$ a sheaf of modules, we have $\mathcal{F}(\emptyset) = 0$.

In many contexts, it is often convenient to consider exact sequences of sheaves. However, this is done at the level of stalks, not sections:

Definition 3.2.9 A sequence of sheaf morphisms

$$\mathcal{F}_0 \xrightarrow{\Psi_1} \mathcal{F}_1 \xrightarrow{\Psi_2} \mathcal{F}_2 \xrightarrow{\Psi_3} \cdots \xrightarrow{\Psi_m} \mathcal{F}_m$$

on a topological space Y is called *exact* if for each point $p \in Y$ and each index $j = 2, \dots, m$, we have

$$\ker(\Psi_j)_p = \operatorname{im}(\Psi_{j-1})_p.$$

An exact sequence of sheaves does not always yield an exact sequence at the level of sections on nonempty open sets (see Exercise 3.2.3).

Exercises for Sect. 3.2

3.2.1 Verify that the examples given in this section are indeed sheaves.

3.2.2 Verify that the morphisms on stalks induced by a sheaf mapping Ψ are well defined and that they preserve the given algebraic structures (group, ring, or module). Also verify that these morphisms are inclusions (isomorphisms) if and only if Ψ_U is injective (respectively, bijective) for each open set U.

3.2.3 Prove that if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves on a topological space Y, then for every nonempty open set $U \subset Y$, the sequence

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. Give an example to show that the analogous statement for an exact sequence $\mathcal{F} \to \mathcal{G} \to 0$ need not hold.

3.3 Divisors

Divisors are convenient objects for encoding zeros and poles of meromorphic functions and sections. In particular, they allow one to produce many (in fact, it turns out, *all*) examples of holomorphic line bundles on a Riemann surface. Throughout this section, X denotes a complex 1-manifold.

Definition 3.3.1 For a given complex 1-manifold X:

(a) A *divisor* on X is a mapping $D\colon X \to \mathbb{Z}$ with discrete support; that is, the set $D^{-1}(\mathbb{Z} \setminus \{0\}) \cap K$ is finite for every compact set $K \subset X$. The Abelian group consisting of the set of divisors on X (together with the natural addition) is denoted by $\operatorname{Div}(X)$.

(b) Given a meromorphic section s of a holomorphic line bundle on X, the function $\operatorname{div}(s)\colon X \to \mathbb{Z} \cup \{+\infty\}$ is given by $p \mapsto \operatorname{ord}_p s$. In particular, if s does not vanish identically on any nonempty open subset of X, then $D = \operatorname{div}(s)$ is a divisor that is called the *divisor of* s and s is called a *defining section* for D. The divisor $D = \operatorname{div}(f)$ of a meromorphic function f that does not vanish identically on any nonempty open subset of X is called a *principal divisor* and f is called a *solution* of D (or a *defining function* for D).

(c) If the difference $D - D'$ of two divisors D and D' on X is a principal divisor, then we write $D \sim D'$ and we say that D and D' are *linearly equivalent*.
(d) A divisor D is called *effective* if $D \geq 0$.

Remarks 1. We may identify $\mathrm{Div}(X)$ with the group of formal sums $\sum_{p \in S} \nu_p \cdot p$, where S is a discrete subset of X and $\{\nu_p\}_{p \in S}$ is a collection of integers (we identify two such sums $\sum_{p \in S} \nu_p \cdot p$ and $\sum_{p \in T} \mu_p \cdot p$ if and only if $\nu_p = \mu_p$ whenever $p \in S \cap T$, $\nu_p = 0$ if $p \in S \setminus T$, and $\mu_p = 0$ if $p \in T \setminus S$). More precisely, to each such sum $\sum_{p \in S} \nu_p \cdot p$ we associate the divisor D given by $D(p) = \nu_p$ for each point $p \in S$ and $D(q) = 0$ for each point $q \in X \setminus S$. Conversely, to each divisor D we associate the sum $\sum_{p \in \mathrm{supp}\, D} D(p) \cdot p$ (to the zero divisor we associate the zero sum).

2. Given meromorphic sections s and t of holomorphic line bundles E and F, respectively, on X that are not identically zero on any nonempty open subset of X, we have

$$\mathrm{div}(s \otimes t) = \mathrm{div}(s) + \mathrm{div}(t) \quad \text{and} \quad \mathrm{div}(s/t) = \mathrm{div}(s) - \mathrm{div}(t)$$

(where $s/t = s \otimes t^{-1}$ is the associated meromorphic section of $E \otimes F^*$). Consequently, linear equivalence of divisors is an equivalence relation (see Exercise 3.3.1).

3. If $\Psi \colon E \to F$ is a holomorphic isomorphism of holomorphic line bundles and $s \in \Gamma(X, \mathcal{O}(E))$, then $\mathrm{div}(\Psi(s)) = \mathrm{div}(s)$.

Proposition 3.3.2 *Let D be a divisor on X; let $\mathcal{F}$ be the set of pairs $\lambda = (U, f)$ consisting of an open set $U \subset X$ and a meromorphic function f on U with* $\mathrm{div}(f) = D{\restriction_U}$ (*i.e., f is a* local defining function *for D*); *let $\Omega_\lambda \equiv U \times \mathbb{C}$ for each element $\lambda = (U, f) \in \mathcal{F}$; let $[D] \equiv \bigsqcup_{\lambda \in \mathcal{F}} \Omega_\lambda/\sim$, where $\sim$ is the equivalence relation determined by*

$$(p_0, \zeta_0) \sim (p_1, \zeta_1) \quad \Longleftrightarrow \quad p_0 = p_1 \quad \text{and} \quad \zeta_0 = \frac{f_0}{f_1}(p_0) \cdot \zeta_1$$

for elements $\lambda_j = (U_j, f_j) \in \mathcal{F}$ and $(p_j, \zeta_j) \in \Omega_{\lambda_j}$ for $j = 0, 1$ (here, f_0/f_1 is the unique extension of this quotient to a nonvanishing holomorphic function on $U_0 \cap U_1$); and let $\rho \colon \bigsqcup_{\lambda \in \mathcal{F}} \Omega_\lambda \to [D]$ be the corresponding quotient map. Then we have the following:

(a) *The mapping $\Pi \colon [D] \to X$ given by $\rho(p, \zeta) \mapsto p$, and for each $\lambda = (U, f) \in \mathcal{F}$, the mapping $\Phi_\lambda \colon \Pi^{-1}(U) = \rho(\Omega_\lambda) \to \mathbb{C}$ given by $\rho(p, \zeta) \mapsto \zeta$ for $(p, \zeta) \in \Omega_\lambda$, are well defined. Moreover, $\Pi \colon [D] \to X$ is a holomorphic line bundle with holomorphic line bundle atlas $\{(U, (\Pi, \Phi_\lambda))\}_{\lambda = (U, f) \in \mathcal{F}}$, and the section s of $[D]$ on $X \setminus D^{-1}((-\infty, 0))$ determined by $\Phi_\lambda(s{\restriction_U}) = f$ for each element $\lambda = (U, f) \in \mathcal{F}$ is a well-defined meromorphic section with* $\mathrm{div}(s) = D$. *In particular, $[D]$ is (holomorphically) trivial if and only if D is a principal divisor (i.e., $D \sim 0$), and s is holomorphic if and only if D is effective.*

(b) *For any holomorphic line bundle E on X with a defining section t for D (i.e., t is a meromorphic section of E with* $\operatorname{div}(t) = D$*), there is a (natural) isomorphism $E \to [D]$ given by multiplication by the nonvanishing holomorphic section s/t of $E^* \otimes [D]$; that is, the isomorphism is given by*

$$\xi \mapsto \xi \cdot \frac{s}{t}(p) \in (E \otimes E^* \otimes [D])_p = [D]_p \quad \forall p \in X, \xi \in E_p.$$

Moreover, this is the unique isomorphism mapping t to s.

(c) *For any holomorphic line bundle E on X and any meromorphic section t of E with divisor $D' =$* $\operatorname{div}(t)$*, we have $E \cong [D]$ if and only if $D' \sim D$.*

Proof It is easy to see that Π is a well-defined map, and for each $\lambda = (U, f) \in \mathcal{F}$, $(\Pi, \Phi_\lambda)\colon \Pi^{-1}(U) = \rho(\Omega_\lambda) \to U \times \mathbb{C}$ is a well-defined bijection. Since $\operatorname{supp} D$ is discrete, a local defining function for D exists in a neighborhood of each point, and hence Π is surjective. Moreover, for each point $p \in X$, $[D]_p = \Pi^{-1}(p)$ has a natural well-defined 1-dimensional complex vector space structure determined by

$$\rho(p, \zeta) + \eta \cdot \rho(p, \xi) = \rho(p, \zeta + \eta \cdot \xi)$$

for each $\lambda = (U, f) \in \mathcal{F}$ with $p \in U$, each pair of elements $(p, \zeta), (p, \xi) \in \Omega_\lambda$, and each scalar $\eta \in \mathbb{C}$; and for each $\lambda = (U, f) \in \mathcal{F}$ with $p \in U$, $\Phi_\lambda \restriction_{[D]_p} \colon [D]_p \to \mathbb{C}$ is a complex linear isomorphism. Thus $[D]$ is a complex line bundle with local trivializations $\{(U, (\Pi, \Phi_\lambda))\}_{\lambda=(U,f)\in\mathcal{F}}$. If $\lambda_j = (U_j, f_j) \in \mathcal{F}$ for $j = 0, 1$, then the corresponding transition functions are the nonvanishing holomorphic functions

$$\frac{\Phi_{\lambda_0}}{\Phi_{\lambda_1}} = \frac{f_0}{f_1}\colon U_0 \cap U_1 \to \mathbb{C} \quad \text{and} \quad \frac{\Phi_{\lambda_1}}{\Phi_{\lambda_0}} = \frac{f_1}{f_0}\colon U_0 \cap U_1 \to \mathbb{C},$$

so these local trivializations determine a holomorphic line bundle structure in $[D]$. Moreover, for each point $p \in U_0 \cap U_1$ with $D(p) \geq 0$, we have

$$\Phi_{\lambda_0}((\Pi, \Phi_{\lambda_1})^{-1}(p, f_1(p))) = \frac{f_0}{f_1}(p) \cdot f_1(p) = f_0(p).$$

Thus the section s is well defined, and since $\Phi_\lambda(s) = f \in \mathcal{M}(U)$ for each $\lambda = (U, f) \in \mathcal{F}$, s is a meromorphic section with $\operatorname{div}(s) = D$ ($\operatorname{div}(s)\restriction_U = \operatorname{div}(f) = D\restriction_U$ for each such λ). Thus (a) is proved, and part (b) follows.

For the proof of part (c), suppose E is a holomorphic line bundle on X, t is a meromorphic section of E on X that does not vanish identically on any open subset of X, and $D' = \operatorname{div}(t)$. If $E \cong [D]$, then the image of t is a meromorphic section t' of $[D]$ with divisor D', and t'/s is a meromorphic function with divisor $\operatorname{div}(t'/s) = D' - D$, and hence $D' \sim D$. Conversely, if $D' \sim D$ and f is a defining function for $D' - D$, then t/f is a meromorphic section of E with divisor $\operatorname{div}(t/f) = D' - (D' - D) = D$, and part (b) implies that $E \cong [D]$. □

Definition 3.3.3 Given a divisor D on X, the holomorphic line bundle $[D]$ and the defining section provided by Proposition 3.3.2 are called the *holomorphic line bun-*

dle associated to D and the *associated defining section*, respectively. Given a holomorphic line bundle E over X, $\mathcal{O}_D(E)$ is the subsheaf of $\mathcal{M}(E)$, considered as a sheaf of $\mathcal{O}$-modules, for which for every open set $U \subset X$, $\mathcal{O}_D(E)(U) \subset \mathcal{M}(E)(U)$ is the $\mathcal{O}(U)$-submodule of sections $s \in \mathcal{M}(E)(U)$ with $\operatorname{div}(s) + D\restriction_U \geq 0$ (in particular, for $D \geq 0$, $\mathcal{O}_{-D}(E)$ is a subsheaf of $\mathcal{O}(E)$). For E the trivial line bundle, we let $\mathcal{O}_D \equiv \mathcal{O}_D(E)$.

Remarks 1. Given two divisors D and D' on X with associated holomorphic line bundles $[D]$ and $[D']$, respectively, and associated defining sections s and s', respectively, $s \otimes s'$ and s^{-1} are meromorphic sections of $[D] \otimes [D']$ and $[D]^*$, respectively, with divisors $\operatorname{div}(s \otimes s') = D + D'$ and $\operatorname{div}(s^{-1}) = -D$. Thus Proposition 3.3.2 gives natural isomorphisms $[D] \otimes [D'] \cong [D + D']$ and $[D]^* \cong [-D]$.

2. According to Proposition 3.3.2, if E is a holomorphic line bundle on X and t is a meromorphic section with divisor $D = \operatorname{div}(t)$ (note that the product of t and any nonvanishing holomorphic function on X also has divisor D), then we may identify the pair (E, t) with $([D], s)$, where s is the associated defining section for D. Thus a divisor on X may be viewed as a holomorphic line bundle together with a choice of a meromorphic section. In particular, a holomorphic line bundle on a Riemann surface is isomorphic to the line bundle associated to some divisor if and only if E admits a nontrivial meromorphic section. It turns out that *every* holomorphic line bundle on a Riemann surface has this property (see Corollary 3.11.7 and Theorem 4.2.3).

3. Let D be a divisor on a Riemann surface X, and let s be the associated defining section of $[D]$. Then $\mathcal{O}_D$ and $\mathcal{O}([D])$ are isomorphic as sheaves of $\mathcal{O}$-modules under the sheaf morphism $f \mapsto f \cdot s$. If E is a holomorphic line bundle on X, then $\mathcal{O}_D(E)$ and $\mathcal{O}(E \otimes [D])$ are isomorphic as sheaves of $\mathcal{O}$-modules under the sheaf morphism $t \mapsto t \otimes s$ (see Exercise 3.3.4).

Example 3.3.4 The hyperplane bundle $E \to \mathbb{P}^1$ (see Example 3.1.5) is equal to the line bundle associated to the divisor D with $D(0) = 1$ and $D(q) = 0$ for $q \in \mathbb{P}^1 \setminus \{0\}$. In fact, for any point $p \in \mathbb{P}^1 \setminus \{0\}$, the meromorphic function f given by $z \mapsto (z - p)/z$ if $p \in \mathbb{C}^*$ and $z \mapsto 1/z$ if $p = \infty$ has a simple pole at 0 and a simple zero at p. Thus, for $D_p = 1 \cdot p$, we have $D_p = D + \operatorname{div}(f)$, and hence $[D_p] \cong E$. The tautological bundle is $E^* = [-D_p]$.

Definition 3.3.5 Let D be an effective divisor and let E be a holomorphic line bundle on X. The *skyline sheaf of E along D* is the sheaf $\mathcal{Q}_D(E)$ of $\mathcal{O}$-modules with sections over any nonempty open set U given by $\mathcal{Q}_D(E)(U) = 0$ if $U \cap \operatorname{supp} D = \emptyset$ and otherwise by

$$\begin{aligned}
\mathcal{Q}_D(E)(U) &= \bigoplus_{p \in U \cap \operatorname{supp} D} \mathcal{O}(E)_p / \mathcal{O}_{-D}(E)_p \\
&= \bigoplus_{p \in U \cap \operatorname{supp} D} \mathcal{O}(E)_p \big/ \mathfrak{m}_p^{D(p)} \cdot \mathcal{O}(E)_p \\
&\cong \bigoplus_{p \in U \cap \operatorname{supp} D} \mathbb{C}^{D(p)} \cong \mathbb{C}^{\sum_{p \in U \cap \operatorname{supp} D} (D(p))},
\end{aligned}$$

and with each restriction mapping $\rho_V^U \colon \mathcal{Q}_D(E)(U) \to \mathcal{Q}_D(E)(V)$ for $U \supset V$ given by the module projection

$$s = \{s_p\}_{p \in U \cap \operatorname{supp} D} \mapsto \{s_p\}_{p \in V \cap \operatorname{supp} D}$$

for $\operatorname{supp} V \cap D \neq \emptyset$, and $s \mapsto 0$ otherwise. We also denote $\mathcal{Q}_D(E)$ by $\mathcal{O}(E)/\mathcal{O}_{-D}(E)$.

Remarks 1. It is easy to verify that $\mathcal{Q}_D(E)$ is a sheaf of $\mathcal{O}$-modules (see Exercise 3.3.3). At the level of stalks, for each point $p \in X$, we have the natural isomorphism $\mathcal{Q}_D(E)_p \cong \mathcal{O}(E)_p/\mathcal{O}_{-D}(E)_p$.

2. A *skyscraper sheaf* is a skyline sheaf for a divisor $D = p$ for some point p.

3. The sheaf $\mathcal{Q}_D(E)$ is an example of a quotient sheaf.

Observation 3.3.6 Let E be a holomorphic line bundle on X. Given an effective divisor D on X with associated holomorphic defining section s, we get the holomorphic line bundle map $\Psi \colon E \to E \otimes [D]$ given by $\xi \mapsto \Psi(\xi) = \xi \otimes s_p$ for $p \in X$ and $\xi \in E_p$. We also get the corresponding sheaf morphism $\Psi_* \colon \mathcal{O}(E) \to \mathcal{O}(E \otimes [D])$ given by $\xi \mapsto \xi \otimes s$ for any local holomorphic section ξ. We also denote $D = \operatorname{div}(s)$ by $\operatorname{div}(\Psi)$. We also get the natural induced sheaf morphism $\mathcal{O}(E \otimes [D]) \to \mathcal{Q}_D(E \otimes [D])$. This yields the corresponding exact sequence of sheaves (see Exercise 3.3.5)

$$0 \to \mathcal{O}(E) \to \mathcal{O}(E \otimes [D]) \to \mathcal{Q}_D(E \otimes [D]) \to 0.$$

If G is any holomorphic line bundle on X and t is any holomorphic section of G with $\operatorname{div}(t) = D$, then we may identify (G, t) with $([D], s)$, and hence we may identify the above exact sequence with the exact sequence

$$0 \to \mathcal{O}(E) \to \mathcal{O}(E \otimes G) \to \mathcal{Q}_D(E \otimes G) \to 0$$

associated to the holomorphic line bundle map given by $\xi \mapsto \Psi(\xi) = \xi \otimes t_p$ for $p \in X$ and $\xi \in E_p$. In fact, any holomorphic homomorphism $\Psi \colon E \to F \cong E \otimes E^* \otimes F$ of holomorphic line bundles that is not indentically zero over any nonempty open subset of X may be viewed as above. For Proposition 3.1.13 provides a corresponding holomorphic section t of $E^* \otimes F$, and Ψ may be identified with the holomorphic line bundle mapping $\Psi \colon \xi \mapsto \xi \otimes t(p)$ for $p \in X$ and $\xi \in E_p$. Setting $D = \operatorname{div}(\Psi) = \operatorname{div}(t)$, we get the exact sequence

$$0 \to \mathcal{O}(E) \to \mathcal{O}(F) \to \mathcal{Q}_D(F) \to 0,$$

which we may identify with the exact sequence

$$0 \to \mathcal{O}(E) \to \mathcal{O}(E \otimes E^* \otimes F) \to \mathcal{Q}_D(E \otimes E^* \otimes F) \to 0,$$

given by multiplication by t, as well as with the exact sequence

$$0 \to \mathcal{O}(E) \to \mathcal{O}(E \otimes [D]) \to \mathcal{Q}_D(E \otimes [D]) \to 0,$$

given by multiplication by the associated defining section for D.

Definition 3.3.7 For X a compact Riemann surface, the *degree* of a divisor D on X is the integer

$$\deg D \equiv \sum_{p \in \operatorname{supp} D} D(p) \quad (\deg D = 0 \text{ if } D = 0).$$

Remarks 1. The mapping $D \mapsto \deg D$ gives a homomorphism $\deg\colon \operatorname{Div}(X) \to \mathbb{Z}$ for X a compact Riemann surface.

2. By Proposition 2.5.7, for X a compact Riemann surface, any principal divisor has degree 0. Consequently, if D and D' are two divisors with isomorphic associated line bundles, then $\deg D = \deg D'$. Thus, for any holomorphic line bundle E on X that admits a nontrivial meromorphic section s, we may define the *degree of* E to be the integer $\deg E \equiv \deg(\operatorname{div}(s))$, which is independent of the choice of s (and depends only on the isomorphism equivalence class of E).

3. If X is a compact Riemann surface and D is a divisor of *negative* degree on X, then $[D]$ has no nontrivial global holomorphic sections, since the divisor of such a section must have *nonnegative* degree. In particular, if D is a nontrivial effective divisor, then D has a nontrivial holomorphic section, while $[-D] = [D]^*$ has no nontrivial holomorphic sections (of course, for the trivial case $D \equiv 0$, $\Gamma(X, \mathcal{O}([D])) = \Gamma(X, \mathcal{O}([-D])) = \Gamma(X, \mathcal{O}) \cong \mathbb{C}$).

Exercises for Sect. 3.3

3.3.1 Verify that linear equivalence of divisors is an equivalence relation.

3.3.2 Let G be the set of isomorphism equivalence classes of holomorphic line bundles on a Riemann surface X.
  (a) Show that G together with the product $\otimes$ and the identity element 1_X is an Abelian group.
  (b) Show that the mapping in $D \mapsto [D]$ determines an injective homomorphism into G of the quotient group of $\operatorname{Div}(X)$ by the subgroup of principal divisors (it will later follow that this mapping is actually an isomorphism).

3.3.3 Verify that for any divisor D and any holomorphic line bundle E, $\mathcal{O}_D(E)$ (see Definition 3.3.3) is a subsheaf of $\mathcal{O}$-modules of $\mathcal{M}(E)$. Verify also that for $D \geq 0$, $\mathcal{Q}_D(E)$ (see Definition 3.3.5) is a sheaf of $\mathcal{O}$-modules.

3.3.4 Prove that if D is a divisor on a Riemann surface X, s is a meromorphic section of $[D]$ with $\operatorname{div}(s) = D$, and E is a holomorphic line bundle on X, then $\mathcal{O}_D(E)$ and $\mathcal{O}(E \otimes [D])$ are isomorphic as sheaves of $\mathcal{O}$-modules under the sheaf morphism $t \mapsto t \otimes s$ (in particular, $\mathcal{O}_D \cong \mathcal{O}([D])$ under $f \mapsto fs$).

3.3.5 Verify that the (first) associated sequence of sheaf mappings in Observation 3.3.6 is exact.

3.4 The $\bar{\partial}$ Operator and Dolbeault Cohomology

Throughout this section, E and F denote holomorphic line bundles over a complex 1-manifold X. There is no canonical way to generalize the exterior derivative operator d and the operator ∂ on scalar-valued forms to an operator on E-valued forms, although there is a very useful generalization of the canonical connection that depends on the choice of a Hermitian metric in E (see Sect. 3.7). On the other hand, the holomorphic structure in E provides a natural generalization of the operator $\bar{\partial}$.

Definition 3.4.1 Let α be an E-valued r-form on an open set $\Omega \subset X$.

(a) If α is of class $\mathcal{C}^1$, then $\bar{\partial}\alpha$ is the unique E-valued $(r+1)$-form that satisfies

$$\bar{\partial}\alpha = \bar{\partial}(\alpha/s) \cdot s \quad \text{on } U$$

for every nonvanishing holomorphic section s of E on an open subset U of Ω.

(b) If α is of class $\mathcal{C}^1$ and $\bar{\partial}\alpha = 0$, then we say that α is $\bar{\partial}$-*closed*.

(c) If $\alpha = \bar{\partial}\beta$ for some $\mathcal{C}^1$ E-valued $(r-1)$-form β on Ω, then we say that α is $\bar{\partial}$-*exact*. If β may be chosen to be of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then we also say that α is $\mathcal{C}^k$ $\bar{\partial}$-*exact*. If for each point in Ω, there exists a $\mathcal{C}^1$ E-valued $(r-1)$-form β_0 on a neighborhood U_0 such that $\bar{\partial}\beta_0 = \alpha\restriction_{U_0}$, then we say that α is *locally* $\bar{\partial}$-*exact*. If for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, each of the local forms β_0 may be chosen to be of class $\mathcal{C}^k$, then we also say that α is *locally* $\mathcal{C}^k$ $\bar{\partial}$-*exact*. It is also convenient to consider the trivial 0-form $\alpha \equiv 0$ to be $\mathcal{C}^\infty$ $\bar{\partial}$-exact and to write $0 = \bar{\partial}0$.

The operator $\bar{\partial}$ is well defined because if α is a $\mathcal{C}^1$ differential form and s and t are nonvanishing holomorphic sections of E on an open set U, then s/t is a holomorphic function and hence

$$\bar{\partial}(\alpha/t) \cdot t = \bar{\partial}\left(\frac{s}{t} \cdot \frac{\alpha}{s}\right) \cdot t = \frac{s}{t} \cdot \bar{\partial}\left(\frac{\alpha}{s}\right) \cdot t = \bar{\partial}(\alpha/s) \cdot s.$$

The Dolbeault lemma (applied to a local representation of the form) implies that for $k \in \mathbb{Z}_{>0} \cup \{\infty\}$ and $p, q \in \mathbb{Z}_{\geq 0}$, every E-valued $\bar{\partial}$-closed $(p, q+1)$-form of class $\mathcal{C}^k$ is locally $\mathcal{C}^k$ $\bar{\partial}$-exact. Proposition 2.5.5 has the following analogue, the proof of which is left to the reader (see Exercise 3.4.1):

Proposition 3.4.2 *For any $\mathcal{C}^1$ E-valued r-form α on an open set $\Omega \subset X$, we have the following*:

(a) *If α is of class $\mathcal{C}^2$, then $\bar{\partial}^2\alpha = 0$.*

(b) *If α is of type (p, q), then $\bar{\partial}\alpha$ is of type $(p, q+1)$. In particular, $\bar{\partial}\alpha = 0$ if $q \geq 1$.*

(c) *If α is $\bar{\partial}$-exact and of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is $\bar{\partial}$-closed and of type $(p, q+1)$ for some pair of integers (p, q). Moreover, there exists a $\mathcal{C}^1$ E-valued differential form β of type (p, q) such that $\alpha = \bar{\partial}\beta$, and any such form β is actually of class $\mathcal{C}^k$.*

(d) *If (U, z) is a local holomorphic coordinate neighborhood, s is a nonvanishing holomorphic section of E on U, and $a, b \in \mathcal{C}^1(U)$, then*

$$\bar{\partial}(as) = \frac{\partial a}{\partial \bar{z}} d\bar{z} \cdot s \quad \textit{and} \quad \bar{\partial}(a\,dz + b\,d\bar{z}) \cdot s = -\frac{\partial a}{\partial \bar{z}} dz \wedge d\bar{z} \cdot s.$$

(e) *If* α *is of type* $(0,0)$ *or* $(1,0)$, *then* α *is an* E*-valued holomorphic* r*-form if and only if* $\bar{\partial}\alpha = 0$.

Definition 3.4.3 For any $q \in \mathbb{Z}_{\geq 0}$, the *qth Dolbeault cohomology of E over X* is the quotient vector space

$$H^q_{\mathrm{Dol}}(X,E) \equiv \ker\bigl(\mathcal{E}^{0,q}(X,E) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,q+1}(X,E)\bigr)\big/\mathrm{im}\bigl(\mathcal{E}^{0,q-1}(X,E) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,q}(X,E)\bigr)$$

(here, we set $\bar{\partial} = 0$ on $\mathcal{E}^{0,-1}(X,E) \equiv 0$). For the trivial line bundle 1_X (i.e., for scalar-valued $(0,q)$-forms), we set $H^q_{\mathrm{Dol}}(X) \equiv H^q_{\mathrm{Dol}}(X,1_X)$. For each $\bar{\partial}$-closed form $\alpha \in \mathcal{E}^{0,q}(X,E)$, we call the corresponding equivalence class in $H^q_{\mathrm{Dol}}(X,E)$ the *Dolbeault cohomology class* of α, and we denote this class by $[\alpha]_{H^q_{\mathrm{Dol}}(X,E)}$, by $[\alpha]_{\mathrm{Dol}}$, or simply by $[\alpha]$. We say that two $\bar{\partial}$-closed $\mathcal{C}^\infty$ $(0,q)$-forms are *Dolbeault cohomologous* if they represent the same Dolbeault cohomology class (it follows from part (c) of Proposition 3.4.2 that two $\mathcal{C}^\infty$ $\bar{\partial}$-closed E-valued $(0,q)$-forms are cohomologous if and only if their difference is $\bar{\partial}$-exact).

In other words, $H^q_{\mathrm{Dol}}(X,E)$ is given by the $\bar{\partial}$-closed $\mathcal{C}^\infty$ E-valued $(0,q)$-forms, where we identify two such forms α and β if and only if $\alpha - \beta$ is $\bar{\partial}$-exact. Clearly, $H^0_{\mathrm{Dol}}(X,E) = \Gamma(X,\mathcal{O}(E))$ and $H^q_{\mathrm{Dol}}(X,E) = 0$ for $q > 1$.

The Dolbeault Exact Sequence Let $\Psi\colon E \to F$ be a holomorphic line bundle mapping that is not identically zero over any nonempty open subset of X. Equivalently, for the corresponding holomorphic section s of $E^* \otimes F$ and the effective divisor $D = \mathrm{div}(\Psi) = \mathrm{div}(s)$ (s is the associated defining section for D under the identification of $E^* \otimes F$ with $[D]$), we have $F = E \otimes [D]$ and $\Psi(\xi) = \xi \otimes s(p)$ for each point $p \in X$ and each vector $\xi \in E_p$ (see Observation 3.3.6). We get the associated exact sequence of sheaves

$$0 \to \mathcal{O}(E) \xrightarrow{\Psi_*} \mathcal{O}(F) \to \mathcal{Q}_D(F) \to 0.$$

We will now see that Ψ also induces an exact sequence of linear maps of vector spaces

$$\begin{aligned} 0 &\to \Gamma(X,\mathcal{O}(E)) \to \Gamma(X,\mathcal{O}(F)) \to \Gamma(X,\mathcal{Q}_D(F)) \\ &\to H^1_{\mathrm{Dol}}(X,E) \to H^1_{\mathrm{Dol}}(X,F) \to 0. \end{aligned}$$

The second map $\Gamma(X,\mathcal{O}(E)) \to \Gamma(X,\mathcal{O}(F))$ is given by $t \mapsto \Psi(t) = t \otimes s$, which is clearly injective. The third map

$$\Gamma(X,\mathcal{O}(F)) \to \Gamma(X,\mathcal{Q}_D(F)) = \bigoplus_{p \in \mathrm{supp}\, D} \mathcal{O}(F)_p/\mathfrak{m}_p^{D(p)}\mathcal{O}(F)_p$$

is given by

$$t \mapsto \{\mathrm{germ}_p t + \mathfrak{m}_p^{D(p)}\mathcal{O}(F)_p\}_{p \in \mathrm{supp}\, D}.$$

The section t maps to 0 if and only if t vanishes to order at least $D(p)$ at each point $p \in \operatorname{supp} D$, that is, if and only if t/s is a holomorphic section of E. Thus the kernel of this third map is the image of the second map.

For the fourth map $\Gamma(X, \mathcal{Q}_D(F)) \to H^1_{\mathrm{Dol}}(X, E)$, which is also called the *connecting homomorphism*, suppose ξ is a nonzero element of $\Gamma(X, \mathcal{Q}_D(F))$. We may choose a $\mathcal{C}^\infty$ section t of $F = E \otimes [D]$ such that t is holomorphic on a neighborhood U of $\operatorname{supp} D$ and $\xi = \{\mathrm{germ}_p t + \mathfrak{m}_p^{D(p)} \mathcal{O}(F)_p\}_{p \in \operatorname{supp} D}$ (for we may form a representing holomorphic section in a neighborhood of $\operatorname{supp} D$ and then cut off this section). Thus $\bar{\partial} t$ is a $\mathcal{C}^\infty$ form of type $(0, 1)$ with values in $E \otimes [D]$ that vanishes on a neighborhood of $\operatorname{supp} D = \{s = 0\}$. Thus $\bar{\partial} t/s$ extends to a unique $\mathcal{C}^\infty$ form θ of type $(0, 1)$ with values in E on X, and we get the associated Dolbeault cohomology class $\eta \equiv [\theta]_{\mathrm{Dol}}$. The map $\xi \mapsto \eta$ is well defined. For if $t' \in \mathcal{E}(F)(X)$ with $t' \restriction_{U'} \in \mathcal{O}(F)(U')$ is another such representing section, θ' is the extension of $\bar{\partial} t'/s$, and $\eta' \equiv [\theta']_{\mathrm{Dol}}$, then $(t - t')/s$ extends to a unique $\mathcal{C}^\infty$ section u of E on X. Thus $\theta - \theta' = \bar{\partial} u$, and hence $\eta = \eta'$. Furthermore, if we may choose the representing section t for ξ to be holomorphic on X, then we get $\theta = \bar{\partial} t/s \equiv 0$, and hence $\eta = [\theta]_{\mathrm{Dol}} = 0$. Conversely, if $\xi \mapsto \eta = 0$, then for t and θ as above, $\theta = \bar{\partial} v$ for some $\mathcal{C}^\infty$ section v of E on X, and hence $(t/s) - v$ is a holomorphic section of E on $X \setminus \operatorname{supp} D$. Since t is holomorphic near $\operatorname{supp} D$, Riemann's extension theorem (Theorem 1.2.10) implies that v is holomorphic near $\operatorname{supp} D$, and hence the section $t' \equiv t - v \otimes s$ of F must be holomorphic on X. Moreover, we have $t' \mapsto \xi$, so we get exactness at this step.

The fifth map $H^1_{\mathrm{Dol}}(X, E) \to H^1_{\mathrm{Dol}}(X, F)$ is given by $\eta = [\theta]_{\mathrm{Dol}} \mapsto [\theta \otimes s]_{\mathrm{Dol}}$. This is a well-defined linear map because for any $\mathcal{C}^\infty$ section u of E on X, we have $(\bar{\partial} u) \otimes s = \bar{\partial}(u \otimes s)$. Given a class $\eta \in H^1_{\mathrm{Dol}}(X, E)$ with a representing $\mathcal{C}^\infty$ form θ, we may apply the Dolbeault lemma (Lemma 2.5.6) and cut off to get a $\mathcal{C}^\infty$ section v of E on X such that $\bar{\partial} v = \theta$ on some neighborhood of $\operatorname{supp} D$. Thus, by replacing θ with $\theta - \bar{\partial} v$, we see that we may choose the representing form θ to vanish near $\operatorname{supp} D$. If $[\theta \otimes s]_{\mathrm{Dol}} = 0$, then we have $\theta \otimes s = \bar{\partial} t$ for some $\mathcal{C}^\infty$ section t of F on X that is holomorphic near $\operatorname{supp} D$. Setting

$$\xi \equiv \{\mathrm{germ}_p t + \mathfrak{m}_p^{D(p)} \mathcal{O}(F)_p\}_{p \in \operatorname{supp} D} \in \Gamma(X, \mathcal{Q}_D(F)),$$

we get $\xi \mapsto \eta$. Conversely, if t is any $\mathcal{C}^\infty$ section of F that is holomorphic near $\operatorname{supp} D$ and that represents an element ξ of $\Gamma(X, \mathcal{Q}_D(F))$, θ is the unique extension of $\bar{\partial} t/s$ to a $\mathcal{C}^\infty$ form on X, and $\eta = [\theta]_{\mathrm{Dol}}$, then we have $\theta \otimes s = \bar{\partial} t$ on X, and hence $\eta \mapsto [\theta \otimes s]_{\mathrm{Dol}} = 0$.

It remains to show that the map $H^1_{\mathrm{Dol}}(X, E) \to H^1_{\mathrm{Dol}}(X, F)$ is surjective. Suppose τ is a $\mathcal{C}^\infty$ form of type $(0, 1)$ with values in F on X and $\lambda \equiv [\tau]_{\mathrm{Dol}}$. By applying the Dolbeault lemma and cutting off as above, we may assume that τ vanishes on some neighborhood of $\operatorname{supp} D$ in X. Hence τ/s extends to a unique $\mathcal{C}^\infty$ form θ of type $(0, 1)$ with values in E on X, and we have $\eta \equiv [\theta]_{\mathrm{Dol}} \mapsto \lambda$.

We may summarize the above remarks as follows:

Theorem 3.4.4 *Any holomorphic line bundle mapping $\Psi \colon E \to F$ over X with divisor $D = \operatorname{div}(\Psi)$ (in particular, Ψ is not identically zero over any nonempty open* *subset of* X) *induces an exact sequence of linear maps*

$$0 \to \Gamma(X, \mathcal{O}(E)) \to \Gamma(X, \mathcal{O}(F)) \to \Gamma(X, \mathcal{Q}_D(F))$$
$$\to H^1_{\mathrm{Dol}}(X, E) \to H^1_{\mathrm{Dol}}(X, F) \to 0.$$

Definition 3.4.5 We call the sequence provided by Theorem 3.4.4 the *Dolbeault exact sequence* corresponding to the line bundle map.

Exercises for Sect. 3.4

3.4.1 Prove Proposition 3.4.2 (cf. Proposition 2.5.5 and Exercise 2.5.5).

3.5 Hermitian Holomorphic Line Bundles

Throughout this section, X denotes a complex 1-manifold. In order to consider L^2 forms with values in a holomorphic line bundle, we need some notion of a pointwise norm (or length) of the values. This is provided by a Hermitian metric.

Definition 3.5.1 A *Hermitian metric* h in a holomorphic line bundle $\Pi: E \to X$ on X is a choice of a Hermitian inner product $h_p(\cdot, \cdot)$ in the fiber E_p of E over each point $p \in X$ such that for each pair of local $\mathcal{C}^\infty$ sections s and t of E, the function $h(s,t): p \mapsto h_p(s_p, t_p)$ is of class $\mathcal{C}^\infty$. For each point $p \in X$ and each pair of elements $u, v \in E_p$, we also write $h(u,v) \equiv h_p(u,v)$ and $|u|^2_h \equiv h(u,u)$. If E is equipped with a Hermitian metric h, then we call (E, h) or E a *Hermitian holomorphic line bundle*.

Remarks 1. According to Proposition 3.11.1 in Sect. 3.11 below, every holomorphic line bundle on a Riemann surface admits a Hermitian metric. The main point is that according to Radó's theorem (Theorem 2.11.1), every Riemann surface is second countable, after which the construction of a Hermitian metric (using a partition of unity) is standard. For now, we will avoid using Proposition 3.11.1 (until Sect. 3.11), and thereby avoid any reliance on Radó's theorem. This will also allow us to avoid using (for now) most of the material from Sects. 2.6–2.9 (in fact, Sects. 3.6–3.9 below may be read in place of most of the material in Sects. 2.6–2.9).

2. If (E, h) is a Hermitian holomorphic line bundle on X and s is a nonvanishing local holomorphic section of E, then, setting $\varphi \equiv -\log|s|^2_h$, we get

$$h(u,v) = \left(\frac{u}{s}\right) \cdot \overline{\left(\frac{v}{s}\right)} \cdot e^{-\varphi}.$$

Thus such a nonvanishing local holomorphic section s allows one to locally represent the Hermitian metric h by the weight function φ (i.e., by $e^{-\varphi}$). In particular, it follows that if u and v are sections of E that are continuous ($\mathcal{C}^k$ with $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, measurable), then the function $h(u,v)$ is continuous (respectively, $\mathcal{C}^k$, measurable).

For the trivial line bundle $1_X = X \times \mathbb{C}$, the Hermitian metrics are precisely the Hermitian metrics of the form $e^{-\varphi}h_0$, where φ is a real-valued $\mathcal{C}^\infty$ function on X and h_0 is the standard Hermitian metric given by $h_0((x,\zeta),(x,\xi)) = \zeta\cdot\bar{\xi}$ for all $x\in X$ and $\zeta,\xi\in\mathbb{C}$; in other words, a Hermitian metric in the trivial line bundle is equivalent to a $\mathcal{C}^\infty$ weight function φ.

Example 3.5.2 Let $\Pi\colon E\to\mathbb{P}^1$ be the hyperplane bundle, and let s be the holomorphic section of E considered in Example 3.1.5, which is nonvanishing except for a simple zero at 0. Then there is a unique Hermitian metric h in E with $|s_z|_h^2 = |z|^2/(1+|z|^2)$ for $z\in\mathbb{P}^1$ ($|s_\infty|_h^2 = 1$). For the section s/z extends to a holomorphic section t on $\mathbb{P}^1$ that is nonvanishing except for a simple zero at ∞ (in the notation of Example 3.1.5, $t_z = (\Pi_E,\Phi_0)^{-1}(z,1)$ for $z\in\mathbb{C}$). For $z\in\mathbb{P}^1$ and $u,v\in E_z$, we may then set

$$h(u,v) = h_z(u,v) \equiv \begin{cases} \frac{(u/t(z))\cdot\overline{(v/t(z))}}{1+|z|^2} & \text{if } z\in\mathbb{C}, \\ \frac{(u/s(z))\cdot\overline{(v/s(z))}}{1+|1/z|^2} & \text{if } z\in\mathbb{P}^1\setminus\{0\} \end{cases}$$

(with $1/z = 0$ for $z=\infty$). One may now verify that h is well defined on the overlap $\mathbb{C}^*$ and that h is a Hermitian metric with the required properties (see Exercise 3.5.1).

Lemma 3.5.3 *A Kähler form on X is equivalent to a Hermitian metric in the holomorphic tangent bundle. More precisely*:

(a) *If ω is a Kähler form on X, then there is a (unique) Hermitian metric g in $(TX)^{1,0}$ such that $g(u,v) = -i\omega(u,\bar{v})$ for each point $p\in X$ and each pair of tangent vectors $u,v\in(T_pX)^{1,0}$.*

(b) *If g is a Hermitian metric in $(TX)^{1,0}$, then there is a (unique) Kähler form ω on X such that $\omega(u,\bar{v}) = ig(u,v)$ for each point $p\in X$ and each pair of tangent vectors $u,v\in(T_pX)^{1,0}$.*

Proof Given a Kähler form ω, we may define g (as in (a)) by $g(u,v) = g_p(u,v) \equiv -i\omega(u,\bar{v})$ for each $p\in X$ and all $u,v\in(T_pX)^{1,0}$. Given a point $p\in X$ and a local holomorphic coordinate neighborhood (U,z) of p, we have $\omega = G(i/2)\,dz\wedge d\bar{z}$ on U for some positive $\mathcal{C}^\infty$ function G, and hence for each pair of tangent vectors $u_0,v_0\in(T_pX)^{1,0}$, we have $g_p(u_0,v_0) = (G(p)/2)\cdot dz(u_0)\overline{dz(v_0)}$. It follows easily that g_p is a Hermitian inner product and that the function $g(u,v)$ is of class $\mathcal{C}^\infty$ for each choice of local $\mathcal{C}^\infty$ vector fields u and v of type $(1,0)$. Thus g is a Hermitian metric, and it is clear that g is uniquely determined.

Conversely, if g is a Hermitian metric in $(TX)^{1,0}$, then we may define a $\mathcal{C}^\infty$ differential 2-form ω by

$$\omega(u,v) = \omega_p(u,v) \equiv ig(u^{1,0},\overline{v^{0,1}}) - ig(v^{1,0},\overline{u^{0,1}})$$

for each point $p\in X$ and each pair of complex tangent vectors u and v at p with $(1,0)$ parts $u^{1,0}$ and $v^{1,0}$, respectively, and $(0,1)$ parts $u^{0,1}$ and $v^{0,1}$, respectively.

Clearly, for $p \in X$ and $u, v \in (T_pX)^{1,0}$, we have $\omega(u, \bar{v}) = ig(u, v)$. Moreover, in any local holomorphic coordinate neighborhood $(U, z = x + iy)$, we have

$$\omega = \omega(\partial/\partial z, \partial/\partial \bar{z})\, dz \wedge d\bar{z} = \left|\frac{\partial}{\partial z}\right|_g^2 i dz \wedge d\bar{z} = 2\left|\frac{\partial}{\partial z}\right|_g^2 dx \wedge dy,$$

so ω is a $\mathcal{C}^\infty$ positive real $(1, 1)$-form, that is, a Kähler form. Again, uniqueness follows. □

Definition 3.5.4 A Hermitian metric g in the holomorphic tangent bundle $(TX)^{1,0} = K_X^*$ is called a *Kähler metric* on X.

According to Lemma 3.5.3, a Kähler metric has a corresponding Kähler form and vice versa.

According to the following lemma, the proof of which is left to the reader (see Exercise 3.5.2), Hermitian metrics in holomorphic line bundles induce Hermitian metrics in dual bundles and tensor product bundles:

Lemma 3.5.5 *Let (E, h), (E', h'), and (E'', h'') be Hermitian holomorphic line bundles on X. Then*:

(a) *There exists a unique Hermitian metric h^* in the dual bundle E^* such that*

$$h^*(\alpha, \beta) = h_p^*(\alpha, \beta) \equiv \frac{\alpha(s)\overline{\beta(s)}}{|s|_h^2},$$

for each point $p \in X$, each pair $\alpha, \beta \in E_p^*$, and each element $s \in E_p \setminus \{0\}$. Equivalently, for each nonzero element $s \in E$, $|1/s|_{h^*}^2 = 1/|s|_h^2$.

(b) *There exists a unique Hermitian metric $h \otimes h'$ in the tensor product bundle $E \otimes E'$ such that*

$$(h \otimes h')(u \otimes u', v \otimes v') = h(u, v) \cdot h'(u', v')$$

for each point $p \in X$, each pair $u, v \in E_p$, and each pair $u', v' \in E'_p$.

(c) *Under the identification of $(E^*)^*$ with E, we have $(h^*)^* = h$ (where $(h^*)^*$ is the Hermitian metric obtained by applying part* (a) *to (E^*, h^*)); under the identification of $E' \otimes E$ with $E \otimes E'$, we have $h' \otimes h = h \otimes h'$ (where $h' \otimes h$ and $h \otimes h'$ are the Hermitian metrics obtained by applying part* (b)); *and under the identification of $(E \otimes E') \otimes E''$ with $E \otimes (E' \otimes E'')$, we have $(h \otimes h') \otimes h'' = h \otimes (h' \otimes h'')$.*

Definition 3.5.6 Let (E, h), (E', h'), and (E'', h'') be Hermitian holomorphic line bundles on X. The Hermitian metric h^* in E^* given by Lemma 3.5.5 is called the *dual Hermitian metric*. The Hermitian metric $h \otimes h'$ in $E \otimes E'$ given by Lemma 3.5.5 is called the *tensor product Hermitian metric*. Under the natural identification of $(E \otimes E' \otimes E'', (h \otimes h') \otimes h'')$ with $(E \otimes E' \otimes E'', h \otimes (h' \otimes h''))$, we set $h \otimes h' \otimes h'' \equiv (h \otimes h') \otimes h'' = h \otimes (h' \otimes h'')$. For any $r \in \mathbb{Z}_{>0}$, we also denote the Hermitian metric $h \otimes \cdots \otimes h$ in E^r by h^r.

If (E, h) and (E', h') are Hermitian holomorphic line bundles over a complex 1-manifold X, and s and s' are nonvanishing local holomorphic sections of E and E', respectively, then, setting $\varphi \equiv -\log |s|_h^2$ and $\varphi' \equiv -\log |s'|_{h'}^2$, we get

$$-\log |s^{-1}|_{h^*}^2 = \log |s|_h^2 = -\varphi \quad \text{and} \quad -\log |s \otimes s'|_{h\otimes h'}^2 = \varphi + \varphi'.$$

Thus h^* and $h \otimes h'$ are locally represented by the weight functions $-\varphi$ and $\varphi + \varphi'$, respectively.

Exercises for Sect. 3.5

3.5.1 Verify that the function h constructed in Example 3.5.2 is a Hermitian metric with the required properties.
3.5.2 Prove Lemma 3.5.5.

3.6 L^2 Forms with Values in a Hermitian Holomorphic Line Bundle

Throughout this section, (E, h) denotes a Hermitian holomorphic line bundle on a complex 1-manifold X. The goal of this section is the development of a suitable L^2 space of differential forms. As discussed in Sect. 2.6, since the objects that we integrate on oriented real 2-dimensional manifolds are the 2-forms and we wish to pair forms of the same degree and integrate in order to get an inner product, it is natural to consider a pointwise pairing that gives a 2-form. For this, the following notation is convenient (see [De3]):

Definition 3.6.1 Given a point $x \in X$, integers $m, n \in \mathbb{Z}_{\geq 0}$, and elements $\alpha \in [\Lambda^m(T^*X)_{\mathbb{C}} \otimes E]_x$ and $\beta \in [\Lambda^n(T^*X)_{\mathbb{C}} \otimes E]_x$, we set

$$\{\alpha, \beta\}_{(E,h)} = \{\alpha, \beta\}_h = \{\alpha, \beta\} \equiv \begin{cases} (\frac{\alpha}{s}) \wedge \overline{(\frac{\beta}{s})} \cdot |s|_h^2 & \text{if } m+n \leq 2, \\ 0 & \text{if } m+n > 2, \end{cases}$$

in $\Lambda^{m+n}(T_x^*X)_{\mathbb{C}}$ for any nonzero element $s \in E_x$.

In other words, $\{\alpha, \beta\}$ is equal to the product of the exterior product of the form parts (with the second factor conjugated) and the inner product of the line bundle parts. Observe that the above definition does not depend on the choice of s. For $m = n = 0$, we have $\{\alpha, \beta\}_h = h(\alpha, \beta)$. Moreover, if $\varphi = -\log |s|^2$ for a nonvanishing local holomorphic section s of E, and α and β are differential forms with values in E and $\alpha_s = \alpha/s$ and $\beta_s = \beta/s$, then

$$\{\alpha, \beta\}_h = \alpha_s \wedge \bar{\beta}_s \cdot e^{-\varphi},$$

which is the pointwise pairing that appears in the definition of the L^2 inner product of scalar-valued forms (see Sect. 2.6). In particular, if α and β are continuous ($\mathcal{C}^k$ with $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, measurable), then the scalar-valued differential form $\{\alpha, \beta\}_h$ is continuous (respectively, $\mathcal{C}^k$, measurable). Moreover, if $d, d' \in [1, \infty]$, $(1/d) + (1/d') = 1$, and α and β are in L^d_{loc} and $L^{d'}_{\mathrm{loc}}$, respectively, then $\{\alpha, \beta\}_h$ is in L^1_{loc}. Based on the above, we make the following definition (cf. Definition 2.6.1):

Definition 3.6.2 Let S be a measurable subset of X, and let α and β be E-valued measurable differential forms of type (p, q) that are defined on S.

(a) For $(p, q) = (1, 0)$, we define

$$\|\alpha\|_{L^2_{1,0}(S,E,h)} \equiv \left[\int_S i\{\alpha, \alpha\}_h\right]^{1/2} \in [0, \infty].$$

If $\{\alpha, \beta\}_h$ is integrable on S, then we define

$$\langle \alpha, \beta \rangle_{L^2_{1,0}(S,E,h)} \equiv \int_S i\{\alpha, \beta\}_h \in \mathbb{C}.$$

(b) For $(p, q) = (0, 1)$, we define

$$\|\alpha\|_{L^2_{0,1}(S,E,h)} \equiv \left[-\int_S i\{\alpha, \alpha\}_h\right]^{1/2} \in [0, \infty].$$

If $\{\alpha, \beta\}_h$ is integrable on S, then we define

$$\langle \alpha, \beta \rangle_{L^2_{0,1}(S,E,h)} \equiv -\int_S i\{\alpha, \beta\}_h \in \mathbb{C}.$$

(c) If $(p, q) = (0, 0)$ and ω is a *nonnegative* measurable form of type $(1, 1)$ defined on S, then we define

$$\|\alpha\|_{L^2_{0,0}(S,E,\omega,h)} \equiv \left[\int_S |\alpha|^2_h \cdot \omega\right]^{1/2} = \left[\int_S \{\alpha, \alpha\}_h \cdot \omega\right]^{1/2} \in [0, \infty].$$

If $h(\alpha, \beta) \cdot \omega = \{\alpha, \beta\}_h \cdot \omega$ is integrable on S, then we define

$$\langle \alpha, \beta \rangle_{L^2_{0,0}(S,E,\omega,h)} \equiv \int_S h(\alpha, \beta) \cdot \omega = \int_S \{\alpha, \beta\}_h \cdot \omega \in \mathbb{C}.$$

(d) If $(p, q) = (1, 1)$ and ω is a *positive* measurable form of type $(1, 1)$ defined on S, then we define

$$\|\alpha\|_{L^2_{1,1}(S,E,\omega,h)} \equiv \left[\int_S \left|\frac{\alpha}{\omega}\right|^2_h \omega\right]^{1/2} = \left[\int_S \left\{\frac{\alpha}{\omega}, \frac{\alpha}{\omega}\right\}_h \omega\right]^{1/2}$$
$$= \|\alpha/\omega\|_{L^2_{0,0}(S,E,\omega,h)} \in [0, \infty].$$

If $\{\alpha/\omega, \beta/\omega\}_h\omega$ is integrable on S, then we define

$$\langle\alpha, \beta\rangle_{L^2_{1,1}(S,E,\omega,h)} \equiv \int_S \left\{\frac{\alpha}{\omega}, \frac{\beta}{\omega}\right\}_h \omega = \left\langle\frac{\alpha}{\omega}, \frac{\beta}{\omega}\right\rangle_{L^2_{0,0}(S,E,\omega,h)} \in \mathbb{C}.$$

(e) When there is no danger of confusion (for example, when the choice of (p, q), S, E, ω, or h is understood from the context), we will suppress parts of the notation. For example, we will often write $\|\alpha\|_{L^2_{p,q}(S,E,\omega,h)}$ simply as $\|\alpha\|_{S,E,\omega,h}$, $\|\alpha\|_{S,\omega,h}$, $\|\alpha\|_{\omega,h}$, $\|\alpha\|_h$, $\|\alpha\|_E$, $\|\alpha\|_\omega$, $\|\alpha\|_{S,h}$, $\|\alpha\|_{L^2(S,h)}$, $\|\alpha\|_{S,E}$, $\|\alpha\|_{L^2(S,E,\omega,h)}$, $\|\alpha\|_{L^2(S,\omega,h)}$, $\|\alpha\|_{L^2(S,\omega,h)}$, $\|\alpha\|_{L^2(\omega,h)}$, $\|\alpha\|_{L^2(S,h)}$, $\|\alpha\|_{L^2(S,E)}$, $\|\alpha\|_{L^2(h)}$, $\|\alpha\|_{L^2(E)}$, or $\|\alpha\|$. The analogous simplified notation will be used for $\langle\alpha, \beta\rangle_{L^2_{p,q}(S,E,\omega,h)}$, for $\|\alpha\|_{L^2_{p,q}(S,E,h)}$, and for $\langle\alpha, \beta\rangle_{L^2_{p,q}(S,E,h)}$.

(f) Let γ and η be measurable E-valued 1-forms defined on S with (r, s) parts $\gamma^{r,s}$ and $\eta^{r,s}$, respectively, for each $(r, s) \in \{(1, 0), (0, 1)\}$. Then we set $\|\gamma\|^2_{L^2_1(S,E,h)} \equiv \|\gamma^{1,0}\|^2_h + \|\gamma^{0,1}\|^2_h$. We also set

$$\langle\gamma, \eta\rangle_{L^2_1(S,E,h)} \equiv \langle\gamma^{1,0}, \eta^{1,0}\rangle_h + \langle\gamma^{0,1}, \eta^{0,1}\rangle_h,$$

provided each of the summands on the right-hand side is defined. We also use the simplified notation analogous to that appearing in (e).

Definition 3.6.3 Let S be a measurable subset of X.

(a) For $(p, q) = (1, 0)$ or $(0, 1)$, the set $L^2_{p,q}(S, E, h)$ consists of all equivalence classes of measurable E-valued differential forms α of type (p, q) on S with $\|\alpha\|_{L^2(S,h)} < \infty$, where we identify any two elements that are equal almost everywhere. The set $L^2_1(S, E, h)$ consists of all equivalence classes of measurable E-valued 1-forms α on S with $\|\alpha\|_{L^2(S,h)} < \infty$, where again, we identify any two elements that are equal almost everywhere.

(b) For ω a *nonnegative* measurable differential form of type $(1, 1)$ that is defined on S, $L^2_{0,0}(S, E, \omega, h)$ consists of all equivalence classes of measurable sections α of E on S with $\|\alpha\|_{L^2(S,\omega,h)} < \infty$, where we identify any two elements that are equal almost everywhere.

(c) For ω a *positive* measurable differential form of type $(1, 1)$ that is defined on S, $L^2_{1,1}(S, E, \omega, h)$ consists of all equivalence classes of E-valued measurable differential forms α of type $(1, 1)$ on S with $\|\alpha\|_{L^2(S,\omega,h)} < \infty$, where we identify any two elements that are equal almost everywhere.

(d) Suppose $E = 1_X = X \times \mathbb{C} \to X$ is the trivial holomorphic line bundle, h_0 is the standard Hermitian metric in E, and φ is the real-valued $\mathcal{C}^\infty$ function on X for which $h = e^{-\varphi}h_0$. Then, for a suitable pair (p, q), we set $L^2_{p,q}(S, \varphi) \equiv L^2_{p,q}(S, E, h)$, and for a suitable ω, $L^2_{p,q}(S, \omega, \varphi) \equiv L^2_{p,q}(S, E, \omega, h)$ (cf. Definition 2.6.2). The analogous notation is used for L^2_1, and for $\langle\cdot, \cdot\rangle_{L^2}$ and $\|\cdot\|_{L^2}$ (cf. Definition 2.6.1).

Remarks 1. In Definitions 3.6.2 and 3.6.3, we could allow the Hermitian metric h to be only measurable (in an appropriate sense) and defined only on the given measurable set S, but this is not necessary for our purposes.

2. If ω is a positive measurable $(1,1)$-form defined on a measurable set $S \subset X$, $\alpha \in L^2_{p,q}(S, E, \omega, h)$ with $p+q=0$ or 2 or $\alpha \in L^2_{p,q}(S, E, h)$ with $p+q=1$, $\beta \in L^2_{r,s}(S, E, \omega, h)$ with $r+s=0$ or 2 or $\beta \in L^2_{r,s}(S, E, h)$ with $r+s=1$, and $(p,q) \neq (r,s)$, then we set $\langle \alpha, \beta \rangle_{L^2} = 0$.

The proof of the following proposition, which is left to the reader (see Exercise 3.6.1) is similar to that of Proposition 2.6.3:

Proposition 3.6.4 *Let S be a measurable subset of X.*

(a) *The pair $(L^2_1(S, E, h), \langle \cdot, \cdot \rangle_h)$, and the pair $(L^2_{p,q}(S, E, h), \langle \cdot, \cdot \rangle_h)$ for $(p,q) = (1,0)$ or $(0,1)$, are Hilbert spaces (where the inner product on any two equivalence classes is given by the pairing of any representatives). Moreover, we have the Hilbert space orthogonal decomposition*

$$L^2_1(S, E, h) = L^2_{1,0}(S, E, h) \oplus L^2_{0,1}(S, E, h).$$

(b) *For ω a continuous positive differential form of type $(1,1)$ that is defined on S and for $(p,q) = (0,0)$ or $(1,1)$, $(L^2_{p,q}(S, E, \omega, h), \langle \cdot, \cdot \rangle_{\omega,h})$ is a Hilbert space (where the inner product on any two equivalence classes is given by the pairing of any representatives).*

Moreover, in each of the above spaces, any sequence converging to an element α admits a subsequence that converges to α pointwise almost everywhere in S.

Exercises for Sect. 3.6

3.6.1 Prove Proposition 3.6.4.

3.6.2 Prove a version of Theorem 2.6.4 for sections of a Hermitian holomorphic line bundle. In other words, prove that if X is a Riemann surface, ω is a Kähler form on X, and (E, h) is a Hermitian holomorphic line bundle on X, then for every compact set $K \subset X$, there is a constant $C = C(X, K, \omega, h) > 0$ such that

$$\max_K |s|_h \leq C \|s\|_{L^2(X,\omega,h)} \quad \forall s \in \Gamma(X, \mathcal{O}(E)).$$

3.7 The Connection and Curvature in a Line Bundle

Throughout this section, (E, h) denotes a Hermitian holomorphic line bundle on a complex 1-manifold X. If α is a $\mathcal{C}^\infty$ differential form with values in E, then the form $\bar{\partial}\alpha$ is well defined by $\bar{\partial}(\alpha/s) \cdot s = \bar{\partial}\alpha$ for every nonvanishing local holomorphic section s of E (see Definition 3.4.1). However, in order to define an analogue of the exterior derivative d for line-bundle-valued differential forms, one must add a suitable correction term to the exterior derivative given by a local trivialization. For a Hermitian holomorphic line bundle, there is a canonical choice.

Theorem 3.7.1 *There exists a unique operator $D = D_E = D_h = D_{(E,h)}$ with the following properties*:

(i) *If α is a $\mathcal{C}^1$ r-form with values in E on an open set $\Omega \subset X$, then $D\alpha$ is a $\mathcal{C}^0$ $(r+1)$-form with values in E on Ω*;

(ii) *If α is a $\mathcal{C}^1$ E-valued differential form on an open set $\Omega \subset X$ and s is any nonvanishing holomorphic section of E on an open set $U \subset \Omega$, then on U,*

$$D\alpha = d(\alpha/s)\cdot s + |s|_h^{-2}\partial(|s|_h^2)\wedge\alpha = d(\alpha/s)\cdot s + (\partial \log|s|_h^2)\wedge\alpha$$

(that is, for $\varphi \equiv -\log|s|_h^2$, $D_h\alpha = [D_\varphi(\alpha/s)]\cdot s$);

(iii) *If ρ is a $\mathcal{C}^1$ p-form and α is a $\mathcal{C}^1$ E-valued r-form on an open set $\Omega \subset X$, then $D(\rho\wedge\alpha) = d\rho\wedge\alpha + (-1)^p\rho\wedge D\alpha$*;

(iv) *If α and β are $\mathcal{C}^1$ E-valued differential forms on an open set $\Omega \subset X$ with α of degree r, then $d\{\alpha,\beta\}_h = \{D\alpha,\beta\}_h + (-1)^r\{\alpha, D\beta\}_h$.*

Proof Let α and β be $\mathcal{C}^1$ E-valued differential forms on an open set $\Omega \subset X$, let $r \equiv \deg\alpha$, let s be a nonvanishing holomorphic section of E on an open set $U \subset \Omega$, and let $\alpha_s \equiv \alpha/s$ and $\beta_s \equiv \beta/s$ on U.

We first show that D is well defined by (ii). If t is another nonvanishing holomorphic section on U, then setting $\alpha_t \equiv \alpha/t$ and $f \equiv t/s$, we get, since $\bar\partial f = 0$,

$$\begin{aligned}
&(d\alpha_t)\cdot t + |t|_h^{-2}\partial(|t|_h^2)\wedge\alpha \\
&\quad = d(f^{-1}\alpha_s)\cdot fs + |f|^{-2}|s|_h^{-2}\partial(|f|^2|s|_h^2)\wedge\alpha \\
&\quad = -f^{-2}df\wedge(\alpha_s)\cdot fs + d(\alpha_s)\cdot s + |f|^{-2}\bar f\partial f\wedge\alpha + |s|_h^{-2}\partial(|s|_h^2)\wedge\alpha \\
&\quad = (d\alpha_s)\cdot s + |s|_h^{-2}\partial(|s|_h^2)\wedge\alpha.
\end{aligned}$$

Thus $D\alpha$ is well defined and (i) follows easily.

If ρ is a $\mathcal{C}^1$ p-form on Ω, then

$$\begin{aligned}
D(\rho\wedge\alpha) &= d(\rho\wedge\alpha_s)\cdot s + |s|_h^{-2}\partial(|s|_h^2)\wedge\rho\wedge\alpha \\
&= d\rho\wedge\alpha + (-1)^p\rho\wedge(d\alpha_s)\cdot s + (-1)^p\rho\wedge|s|_h^{-2}\partial(|s|_h^2)\wedge\alpha \\
&= d\rho\wedge\alpha + (-1)^p\rho\wedge D\alpha
\end{aligned}$$

on U. Thus (iii) holds.

For the proof of (iv), observe that

$$\begin{aligned}
d\{\alpha,\beta\}_h &= d[\alpha_s\wedge\overline{\beta_s}\cdot|s|_h^2] \\
&= (d\alpha_s)\wedge\overline{\beta_s}\cdot|s|_h^2 + (-1)^r\alpha_s\wedge\overline{d\beta_s}\cdot|s|_h^2 + (d|s|_h^2)\wedge\alpha_s\wedge\overline{\beta_s}
\end{aligned}$$

$$\begin{aligned}
&= (d\alpha_s) \wedge \overline{\beta_s} \cdot |s|_h^2 + (-1)^r \alpha_s \wedge \overline{d\beta_s} \cdot |s|_h^2 \\
&\quad + (\partial |s|_h^2) \wedge \alpha_s \wedge \overline{\beta_s} + (-1)^r \alpha_s \wedge \overline{(\partial |s|_h^2) \wedge \beta_s} \\
&= [(d\alpha_s) + |s|_h^{-2}(\partial |s|_h^2) \wedge \alpha_s] \wedge \overline{\beta_s} \cdot |s|_h^2 \\
&\quad + (-1)^r \alpha_s \wedge \overline{[d\beta_s + |s|_h^{-2}(\partial |s|_h^2) \wedge \beta_s]} \cdot |s|_h^2 \\
&= [(D\alpha)/s] \wedge \overline{\beta_s} \cdot |s|_h^2 + (-1)^r \alpha_s \wedge \overline{[(D\beta)/s]} \cdot |s|_h^2 \\
&= \{D\alpha, \beta\}_h + (-1)^r \{\alpha, D\beta\}_h
\end{aligned}$$

on U. Thus (iv) is proved. □

Definition 3.7.2 The operator D provided by Theorem 3.7.1 is called the *canonical connection* (or the *Chern connection*) in (E, h). We write $D = D' + D'' = D' + \bar{\partial}$, where for each $\mathcal{C}^1$ E-valued differential form α, $D''\alpha \equiv \bar{\partial}\alpha$ is the $(0, 1)$ part and $D'\alpha \equiv D\alpha - D''\alpha$ is the $(1, 0)$ part. We also write

$$D = D_E = D_h = D_{(E,h)}, \qquad D' = D'_E = D'_h = D'_{(E,h)}, \quad \text{and} \quad \bar{\partial} = D'' = D''_E.$$

For $E = 1_X$ and φ the weight function representing h (i.e., $h = e^{-\varphi} h_0$, where h_0 is the standard Hermitian metric), we set $D_\varphi \equiv D_h$ and $D'_\varphi \equiv D'_h$ (cf. Definition 2.7.2).

Let α be a $\mathcal{C}^\infty$ differential form with values in E, let s be a nonvanishing local holomorphic section of E, let $\alpha_s = \alpha/s$, and let $\varphi = -\log |s|_h^2$. Then we have

$$D'\alpha = \partial\alpha_s \cdot s + e^\varphi \partial(e^{-\varphi}) \wedge \alpha = e^\varphi \partial[e^{-\varphi}\alpha_s] \cdot s = [D'_\varphi \alpha_s] \cdot s$$

and

$$D''\alpha = \bar{\partial}\alpha = [\bar{\partial}\alpha_s] \cdot s.$$

Consequently, $(D')^2 = (D'')^2 = 0$ and $D^2 = D'D'' + D''D'$. Locally, since $\partial\bar{\partial} + \bar{\partial}\partial = 0$, we have

$$\begin{aligned}
D^2\alpha &= (\partial\bar{\partial}\alpha_s) \cdot s + e^\varphi \partial(e^{-\varphi}) \wedge \bar{\partial}\alpha_s \cdot s \\
&\quad + (\bar{\partial}\partial\alpha_s) \cdot s + \bar{\partial}(e^\varphi \partial(e^{-\varphi}) \wedge \alpha_s) \cdot s \\
&= \bar{\partial}(e^\varphi \partial(e^{-\varphi})) \wedge \alpha = \partial\bar{\partial}\varphi \wedge \alpha = \Theta_h \wedge \alpha,
\end{aligned}$$

where $\Theta_h = \partial\bar{\partial}\varphi$ is a well-defined scalar-valued form of type $(1, 1)$.

Definition 3.7.3 (Cf. Definition 2.8.1) The *curvature* (or *curvature form*) of (E, h) is the scalar-valued $(1, 1)$-form Θ, which we also denote by Θ_E, Θ_h, or $\Theta_{(E,h)}$, given locally by $\Theta = \partial\bar{\partial}(-\log |s|_h^2)$ for every nonvanishing local holomorphic section s of E. In other words,

$$D^2 = D'D'' + D''D' = \Theta \wedge (\cdot).$$

For $E = 1_X$ and φ the weight function representing h, we set $\Theta_\varphi \equiv \Theta_h = i\partial\bar{\partial}\varphi$ (cf. Definition 2.8.1).

Remarks 1. If g is a Kähler metric with Kähler form ω, and (U, z) is a local holomorphic coordinate neighborhood, then $\omega = G(i/2)\,dz \wedge d\bar{z}$ and $|\partial/\partial z|_g^2 = G/2$ on U for some positive $\mathcal{C}^\infty$ function G. Hence $\Theta_g = \partial\bar{\partial}(-\log G) = \Theta_\omega$ (see Definition 2.12.1).

2. In a slight abuse of language, we say that (E, h) has *nonnegative* (*positive*, *zero*, *nonpositive*, *negative*) *curvature* if $i\Theta_h \geq 0$ (respectively, $i\Theta_h > 0$, $i\Theta_h = 0$, $i\Theta_h \leq 0$, $i\Theta_h < 0$). A holomorphic line bundle E that admits a Hermitian metric of positive curvature is called *positive*.

3. For any nonvanishing local holomorphic section s of E and any Kähler form ω, we have

$$i\Theta_h = i\Theta_{-\log|s|_h^2} = [\Delta_\omega(-\log|s|_h^2)] \cdot \omega$$

(see Definitions 2.8.1 and 2.14.3). Combined with the results of Sect. 2.14, this observation will later allow us to construct Hermitian metrics of positive curvature (see Sects. 3.11 and 4.2).

4. The dual Hermitian metric h^* in E^* satisfies $i\Theta_{h^*} = -i\Theta_h$, and if (E', h') is a second Hermitian holomorphic line bundle on X, then $i\Theta_{h\otimes h'} = i\Theta_h + i\Theta_{h'}$.

Example 3.7.4 Recall that the hyperplane bundle $E \to \mathbb{P}^1$ is the holomorphic line bundle $E = [D]$ for the divisor D with $D(0) = 1$ and $D(q) = 0$ for $q \in \mathbb{P}^1 \setminus \{0\}$ (see Examples 3.1.5, 3.3.4, and 3.5.2). If s is the holomorphic section of E with $\mathrm{div}(s) = D$ considered in Example 3.1.5, and h is the Hermitian metric in E with $|s|_h^2 = |z|^2/(1 + |z|^2)$ ($|s_\infty|_h^2 = 1$) considered in Example 3.5.2, then

$$\Theta_h = -\partial\bar{\partial}\log|s|_h^2 = \frac{dz \wedge d\bar{z}}{1 + |z|^2} - \frac{\bar{z}z \cdot dz \wedge d\bar{z}}{(1 + |z|^2)^2} = \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

In particular, $i\Theta_h > 0$. In fact, $2i\Theta_h$ is the chordal Kähler form considered in Example 2.12.3.

The following will play a role analogous to that of Proposition 2.8.2 in the L^2 $\bar{\partial}$-method:

Proposition 3.7.5 (Fundamental estimate) *Let $D' = D_h$, $\Theta = \Theta_h$, and $D'' = \bar{\partial}$. Then, for every $t \in \mathcal{D}(E)(X)$, we have*

$$\|D't\|_{L^2(X,h)}^2 = \|D''t\|_{L^2(X,h)}^2 + \int_X i\Theta|t|_h^2 \geq \int_X i\Theta|t|_h^2.$$

Remarks 1. In particular, if in the above, $i\Theta \geq 0$, then

$$\|D't\|_{L^2(X,h)}^2 = \|D''t\|_{L^2(X,h)}^2 + \|t\|_{L^2(X,i\Theta,h)}^2 \geq \|t\|_{L^2(X,i\Theta,h)}^2.$$

2. The corresponding equality for inner products of $\mathcal{C}^\infty$ compactly supported sections also holds (see Exercise 3.7.2).

Proof of Proposition 3.7.5 Given $t \in \mathcal{D}(E)(X)$, we may choose a finite covering $\{U_\nu\}_{\nu=1}^m$ of $\operatorname{supp} t$ by relatively compact open subsets of X such that for each ν, there exists a nonvanishing holomorphic section s_ν of E on U_ν. We may also choose $\mathcal{C}^\infty$ functions $\{\eta_\nu\}_{\nu=1}^m$ such that $\operatorname{supp} \eta_\nu \subset U_\nu$ for each ν and $\sum \eta_\nu \equiv 1$ on $\operatorname{supp} t$. Thus, setting $\varphi \equiv -\log |s_\nu|_h^2$ and $f_\nu \equiv t/s_\nu \in \mathcal{C}^\infty(U_\nu)$ for each ν, and applying Proposition 2.8.2, we get

$$\begin{aligned}
\|D_h' t\|^2_{L^2(X,h)} &= \int_X i\{D_h' t, D_h' t\}_h = \sum_\nu \int_X i\{D_h' t, D_h'(\eta_\nu t)\}_h \\
&= \sum \int_{U_\nu} i\{[D'_{\varphi_\nu} f_\nu] \cdot s_\nu, [D'_{\varphi_\nu}(\eta_\nu f_\nu)] \cdot s_\nu\}_h \\
&= \sum \int_{U_\nu} i[D'_{\varphi_\nu} f_\nu] \wedge \overline{D'_{\varphi_\nu}(\eta_\nu f_\nu)} \cdot e^{-\varphi_\nu} \\
&= -\sum \int_{U_\nu} i(\bar{\partial} f_\nu) \wedge \overline{\bar{\partial}(\eta_\nu f_\nu)} \cdot e^{-\varphi_\nu} + \sum \int_{U_\nu} i\Theta_{\varphi_\nu} f_\nu \overline{\eta_\nu f_\nu} e^{-\varphi_\nu} \\
&= -\sum \int_{U_\nu} i\{D'' t, D''(\eta_\nu t)\}_h + \sum \int_{U_\nu} \bar{\eta}_\nu \cdot i\Theta_h \cdot |t|_h^2 \\
&= -\int_X i\{D'' t, D'' t\}_h + \int_X i\Theta_h |t|_h^2 \\
&= \|D'' t\|^2_{L^2(X,h)} + \int_X i\Theta_h |t|_h^2.
\end{aligned}$$

□

Exercises for Sect. 3.7

3.7.1 Let (E, h) be a Hermitian holomorphic line bundle on a Riemann surface X. Prove that for every point $p \in X$, there exists a nonvanishing holomorphic section t of E on a neighborhood of p such that $|t_p|_h^2 = 1$ and $(d|t|_h^2)_p = 0$. Conclude from this that for every $\mathcal{C}^1$ E-valued differential form α on a neighborhood of p, we have $(D_{(E,h)}\alpha)_p = (d(\alpha/t) \cdot t)_p$.

3.7.2 Prove that in Proposition 3.7.5, the corresponding equality for inner products of $\mathcal{C}^\infty$ compactly supported sections also holds (cf. Proposition 2.8.2). That is, prove that if (E, h) is a Hermitian holomorphic line bundle on a Riemann surface X, $D' = D_h'$, $\Theta = \Theta_h$, $D'' = \bar{\partial}$, and $s, t \in \mathcal{D}(E)(X)$, then

$$\langle D' s, D' t\rangle_{L^2(X,h)} = \langle D'' s, D'' t\rangle_{L^2(X,h)} + \int_X h(s,t) i\Theta.$$

3.7.3 Prove that if (E, h) is a Hermitian holomorphic line bundle on a compact Riemann surface X with *negative curvature* (i.e., $-i\Theta_h > 0$), then $\Gamma(X, \mathcal{O}(E)) = 0$.

3.8 The Distributional $\bar\partial$ Operator in a Holomorphic Line Bundle

Throughout this section, X denotes a complex 1-manifold, (E, h) denotes a Hermitian holomorphic line bundle on X, and $D = D' + D'' = D' + \bar\partial$ denotes the associated canonical connection. The following is a natural definition for the distributional $\bar\partial$ operator in E:

Definition 3.8.1 Let α and β be locally integrable differential forms on X with values in E. We write $D''_{\mathrm{distr}}\alpha = \bar\partial_{\mathrm{distr}}\alpha = \beta$ if for every nonvanishing local holomorphic section s of E, we have $\bar\partial_{\mathrm{distr}}(\alpha/s) = (\beta/s)$ (see Definition 2.7.1).

The following proposition gives an equivalent, and more intrinsic, version (cf. Proposition 2.7.3):

Proposition 3.8.2 *Let α and β be locally integrable differential forms with values in E on an open set $\Omega \subset X$.*

(a) *If α is of type $(1, 0)$ and β is of type $(1, 1)$, then $\bar\partial_{\mathrm{distr}}\alpha = \beta$ if and only if*

$$\int_\Omega \{\alpha, D't\}_h = \int_\Omega \{\beta, t\}_h \quad \forall t \in \mathcal{D}(E)(\Omega).$$

(b) *If α is of type $(0, 0)$ and β is of type $(0, 1)$, then $\bar\partial_{\mathrm{distr}}\alpha = \beta$ if and only if*

$$-\int_\Omega \{\alpha, D'\gamma\}_h = \int_\Omega \{\beta, \gamma\}_h \quad \forall \gamma \in \mathcal{D}^{0,1}(E)(\Omega).$$

(c) *If α is of type (p, q) with $p \geq 2$ or $q \geq 1$, then $\bar\partial_{\mathrm{distr}}\alpha = 0$.*

Remark The proof will show that if the condition in Definition 3.8.1 holds for the forms α/s and β/s for *some* nonvanishing holomorphic section s on a neighborhood of each point, then it holds for *every* choice of s.

Proof of Proposition 3.8.2 Suppose α is of type $(1, 0)$ and β is of type $(1, 1)$. If s is a nonvanishing holomorphic section of E on an open set $U \subset X$, $\alpha_s = \alpha/s$, $\beta_s = \beta/s$, and $\varphi \equiv -\log|s|_h^2$, then given a $\mathcal{C}^\infty$ section t of E with compact support in U, we get, for $f \equiv t/s \in \mathcal{D}(U)$ (which we may view as a function in $\mathcal{D}(\Omega)$),

$$\int_\Omega \{\beta, t\}_h = \int_U \beta_s \cdot \bar f \cdot e^{-\varphi}$$

and

$$\int_\Omega \{\alpha, D'_h t\}_h = \int_U \{\alpha, (D'_\varphi f) \cdot s\}_h = \int_U \alpha_s \wedge \overline{D'_\varphi f} \cdot e^{-\varphi}.$$

This observation and Proposition 2.7.3 together imply that if the above left-hand sides are always equal, then $\bar\partial_{\mathrm{distr}}\alpha = \beta$. Conversely, if $\bar\partial_{\mathrm{distr}}\alpha = \beta$ and $t \in \mathcal{D}(E)(\Omega)$, then, choosing finitely many $\mathcal{C}^\infty$ functions $\{\eta_\nu\}_{\nu=1}^m$ on Ω such that $\sum \eta_\nu \equiv 1$ on $\operatorname{supp} t$ and such that for each ν, the support of η_ν is contained in some neighborhood U_ν on which there exists a nonvanishing holomorphic section of E, we get, by the above,

$$\int_\Omega \{\alpha, D_h' t\}_h = \sum_\nu \int_{U_\nu} \{\alpha, D_h'(\eta_\nu t)\}_h = \sum_\nu \int_{U_\nu} \{\beta, \eta_\nu t\}_h = \int_\Omega \{\beta, t\}_h.$$

Thus part (a) is proved. Part (c) is obvious, and the proof of part (b) is left to the reader (see Exercise 3.8.1). $\square$

Theorem 2.7.4 immediately gives the following in this context:

Theorem 3.8.3 *If $p \in \{0, 1\}$, α is a locally integrable form of type $(p, 0)$ with values in E, and $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$ for an E-valued form β of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class $\mathcal{C}^k$. In particular, if $\bar{\partial}_{\mathrm{distr}}\alpha = 0$, then α is a holomorphic p-form with values in E.*

It is sometimes convenient to consider L^2 versions of Dolbeault cohomology (cf. Definition 3.4.3).

Definition 3.8.4 Let ω be a Kähler form on X; let $\mathcal{Z}^1 = L^2_{0,1}(X, E, h)$; and let $\mathcal{B}^1 \subset \mathcal{Z}^1$ be the subspace consisting of elements of the form $\bar{\partial}_{\mathrm{distr}}\alpha$, where $\alpha \in L^2_{0,0}(X, E, \omega, h)$ is an L^2 section for which $\bar{\partial}_{\mathrm{distr}}\alpha$ exists and is in $L^2_{0,1}(X, E, h)$. The *first L^2 Dolbeault cohomology of* (X, E, ω, h) is then the quotient vector space

$$H^1_{\mathrm{Dol}, L^2}(X, E, \omega, h) \equiv \mathcal{Z}^1/\mathcal{B}^1.$$

The *first $\mathcal{C}^\infty$ L^2 Dolbeault cohomology of* (X, E, ω, h) is the quotient vector space

$$H^1_{\mathrm{Dol}, L^2 \cap \mathcal{E}}(X, E, \omega, h) \equiv \mathcal{Z}^1_\infty/\mathcal{B}^1_\infty,$$

where $\mathcal{Z}^1_\infty = \mathcal{Z}^1 \cap \mathcal{E}^{0,1}(X, E)$ and $\mathcal{B}^1_\infty = \mathcal{Z}^1 \cap \bar{\partial}[L^2_{0,0}(X, E, \omega, h) \cap \mathcal{E}^{0,0}(X, E)]$.

Exercises for Sect. 3.8

3.8.1 Prove part (b) of Proposition 3.8.2.

3.9 The L^2 $\bar{\partial}$-Method for Line-Bundle-Valued Forms of Type $(1, 0)$

The following is a direct generalization of Theorem 2.9.1:

Theorem 3.9.1 *Let X be a Riemann surface, let (E, h) be a Hermitian holomorphic line bundle on X with $i\Theta = i\Theta_h \geq 0$, and let $Z = \{x \in X \mid \Theta_x = 0\}$.*

Then, for every measurable E-valued $(1,1)$-form β on X with $\beta = 0$ a.e. in Z and $\beta\restriction_{X\setminus Z} \in L^2_{1,1}(X\setminus Z, E, i\Theta, h)$ (in particular, β is in L^2_{loc} on X), there exists a form $\alpha \in L^2_{1,0}(X, E, h)$ such that

$$D''_{\mathrm{distr}}\alpha = \bar{\partial}_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,E,h)} \le \|\beta\|_{L^2(X\setminus Z,E,i\Theta,h)}.$$

In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0}\cup\{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.

Remark The form β is in L^2_{loc} because β vanishes on Z, and hence for any compact set K in any local holomorphic coordinate neighborhood (U, z), and for any nonvanishing holomorphic section s of E on U, we have

$$\|\beta/s\|_{L^2_{1,1}(K,(i/2)\,dz\wedge d\bar{z})} \le A\|\beta\|_{L^2_{1,1}(K\setminus Z,i\Theta,h)} < \infty,$$

where

$$A = \left(\sup_K \frac{i\Theta}{(i/2)\,dz\wedge d\bar{z}}|s|_h^{-2}\right)^{1/2} < \infty.$$

It is often more convenient to apply Theorem 3.9.1 in one of the following forms (cf. Corollary 2.9.2 and Corollary 2.9.3), the proofs of which are left to the reader (see Exercises 3.9.1 and 3.9.2):

Corollary 3.9.2 *Suppose that X is a Riemann surface, (E, h) is a Hermitian holomorphic line bundle on X, ω is a Kähler form on X, ρ is a nonnegative measurable function on X with $i\Theta_h \ge \rho\omega$, and $Z = \{x \in X \mid \rho(x) = 0\}$. Then, for every measurable E-valued $(1,1)$-form β on X with $\beta = 0$ a.e. in Z and $\beta\restriction_{X\setminus Z} \in L^2_{1,1}(X\setminus Z, E, \rho\omega, h) = L^2_{1,1}(X\setminus Z, E, \omega, e^{-\log\rho}h)$ (in particular, β is in L^2_{loc} on X), there exists a form $\alpha \in L^2_{1,0}(X, E, h)$ such that*

$$D''_{\mathrm{distr}}\alpha = \bar{\partial}_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,h)} \le \|\beta\|_{L^2(X\setminus Z,\rho\omega,h)}.$$

In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0}\cup\{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.

Corollary 3.9.3 *Suppose that X is a Riemann surface, (E, h) is a Hermitian holomorphic line bundle on X, ω is a Kähler form on X, and C is a positive constant with $i\Theta_h \ge C^2\omega$. Then, for every form $\beta \in L^2_{1,1}(X, E, \omega, h)$, there exists a form $\alpha \in L^2_{1,0}(X, E, h)$ such that*

$$D''_{\mathrm{distr}}\alpha = \bar{\partial}_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,h)} \le C^{-1}\|\beta\|_{L^2(X,\omega,h)}.$$

In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0}\cup\{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.

Proof of Theorem 3.9.1 Let $N \equiv \|\beta\|_{L^2(X\setminus Z, i\Theta, h)} = \|\beta/(i\Theta)\|_{L^2(X\setminus Z, i\Theta, h)}$, and let $\mathcal{V} \equiv \mathcal{D}(E)(X)$. For each section $t \in \mathcal{V}$, the Schwarz inequality and the fundamental estimate (Proposition 3.7.5) give

$$\left|\int_X \{t, \beta\}_h\right| = \left|\int_{X\setminus Z} h\left(t, \frac{\beta}{i\Theta}\right) \cdot i\Theta\right| = \left|\langle t, \beta/(i\Theta)\rangle_{L^2(X\setminus Z, i\Theta, h)}\right|$$
$$\leq \|t\|_{L^2(X\setminus Z, i\Theta, h)} \cdot \|\beta/(i\Theta)\|_{L^2(X\setminus Z, i\Theta, h)}$$
$$\leq N \cdot \|t\|_{L^2(X, i\Theta, h)} \leq N \cdot \|D't\|_{L^2(X,h)}.$$

It follows that the mapping $\Upsilon\colon [D't] \mapsto -i\int_X \{t, \beta\}_h$ is a well-defined bounded complex linear functional on the subspace $D'\mathcal{V}$ of $L^2_{1,0}(X, E, h)$. For by the above inequality, Υ is well defined, and for each $t \in \mathcal{V}$, we have $|\Upsilon[D't]| \leq N \cdot \|D't\|_{L^2(X,h)}$. In particular, $\|\Upsilon\| \leq N$. By the Hahn–Banach theorem (Theorem 7.5.11), there exists a bounded linear functional $\widehat{\Upsilon}$ on $L^2_{1,0}(X, E, h)$ such that $\widehat{\Upsilon}\restriction_{D'\mathcal{V}} = \Upsilon$ and $\|\widehat{\Upsilon}\| = \|\Upsilon\|$. Therefore, by Theorem 7.5.10, there exists a (unique) element $\alpha \in L^2_{1,0}(X, E, h)$ such that $\|\alpha\|_{L^2(X,h)} = \|\widehat{\Upsilon}\| \leq N$ and $\widehat{\Upsilon}(\cdot) = \langle\cdot, \alpha\rangle_{L^2(X,h)}$. Moreover, for each $t \in \mathcal{V}$, we have

$$\int_X i\{\alpha, D't\}_h = \overline{\Upsilon(D't)} = \overline{\int_X (-i)\{t, \beta\}_h} = \int_X i\{\beta, t\}_h.$$

Therefore, by Proposition 3.8.2, $D''_{\mathrm{distr}}\alpha = \beta$, as required. Finally, the regularity statement at the end follows from Theorem 3.8.3. □

Exercises for Sect. 3.9

3.9.1 Prove Corollary 3.9.2.
3.9.2 Prove Corollary 3.9.3.

3.10 The L^2 $\bar{\partial}$-Method for Line-Bundle-Valued Forms of Type (0, 0)

Throughout this section, X denotes a Riemann surface (although all of the results to be stated also hold for X a complex 1-manifold), g denotes a Kähler metric on X with Kähler form ω, and (E, h) denotes a Hermitian holomorphic line bundle on X. The goal of this section is a proof that with appropriate curvature assumptions, one may solve the inhomogeneous Cauchy–Riemann equation $\bar{\partial}\alpha = \beta$ for β a differential form of type $(0, 1)$ with values in E (cf. Sect. 2.12). We first consider the following:

Proposition 3.10.1 *For $q = 0, 1$, let $\Phi_q : \Lambda^{(1,q)}T^*X \otimes E \to \Lambda^{(0,q)}T^*X \otimes K_X \otimes E$ be the map given by*

$$\Phi_q : \alpha = \theta \wedge \gamma \cdot s = (-1)^q \gamma \wedge \theta \cdot s \mapsto (-1)^q \gamma \cdot (\theta \otimes s)$$

for each $x \in X$, $\gamma \in \Lambda^{(0,q)}T_x^*X$, $\theta \in (K_X)_x$, and $s \in E_x$. Then we have the following:

(a) *For each $q = 0, 1$, Φ_q is a well-defined mapping for which the restriction $\Lambda^{(1,q)}T_x^*X \otimes E_x \to \Lambda^{(0,q)}T_x^*X \otimes (K_X \otimes E)_x$ is a linear isomorphism for each point $x \in X$ (in fact, Φ_0 is equal to the identity).*

(b) *If α is an E-valued differential form of type $(1,q)$, then α is continuous ($\mathcal{C}^k$ with $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, measurable, in L^p_{loc} with $p \in [1,\infty]$) if and only if the $K_X \otimes E$-valued differential form $\Phi_q(\alpha)$ of type $(0,q)$ is continuous (respectively, $\mathcal{C}^k$, measurable, in L^p_{loc}). For $q = 0$, α is a local holomorphic 1-form with values in E if and only if $\Phi_0(\alpha)$ is a local holomorphic section of $K_X \otimes E$.*

(c) *For each point $x \in X$ and all $\alpha, \beta \in \Lambda^{(1,q)}T_x^*X \otimes E_x$, we have*

$$\{\Phi_0(\alpha), \Phi_0(\beta)\}_{g^*\otimes h} \cdot \omega = i\{\alpha, \beta\}_h \quad \textit{for } q = 0,$$

$$-i\{\Phi_1(\alpha), \Phi_1(\beta)\}_{g^*\otimes h} = \left\{\frac{\alpha}{\omega}, \frac{\beta}{\omega}\right\}_h \cdot \omega \quad \textit{for } q = 1.$$

(d) *For each $q = 0, 1$, Φ_q preserves L^2 inner products and L^2 norms. More precisely, suppose α and β are measurable E-valued differential forms of type $(1,q)$ on a measurable set S. Then we have*

$$\|\Phi_0(\alpha)\|_{L^2(S,\omega,g^*\otimes h)} = \|\alpha\|_{L^2(S,h)} \quad \textit{for } q = 0,$$

$$\|\Phi_1(\alpha)\|_{L^2(S,g^*\otimes h)} = \|\alpha\|_{L^2(S,\omega,h)} \quad \textit{for } q = 1.$$

Moreover, for $q = 0$,

$$\langle\Phi_0(\alpha), \Phi_0(\beta)\rangle_{L^2(S,\omega,g^*\otimes h)} = \langle\alpha, \beta\rangle_{L^2(S,h)},$$

where both of the above terms are defined if either one is defined. For $q = 1$,

$$\langle\Phi_1(\alpha), \Phi_1(\beta)\rangle_{L^2(S,g^*\otimes h)} = \langle\alpha, \beta\rangle_{L^2(S,\omega,h)},$$

where again, both of the above terms are defined if either one is defined.

(e) *For any locally integrable E-valued differential form α of type $(1,0)$ on X, $\bar\partial_{\text{distr}}\alpha$ exists if and only if $\bar\partial_{\text{distr}}[\Phi_0(\alpha)]$ exists. Moreover, if these objects do exist, then $\bar\partial_{\text{distr}}[\Phi_0(\alpha)] = \Phi_1[\bar\partial_{\text{distr}}\alpha]$.*

Proof In terms of a local holomorphic coordinate z and a nonvanishing local holomorphic section s of E, the maps are given by

$$\Phi_0 \colon a\,dz \cdot s \mapsto a\,dz \otimes s,$$

$$\Phi_1 \colon a\,dz \wedge d\bar z \cdot s = -a\,d\bar z \wedge dz \cdot s \mapsto -a\,d\bar z \cdot (dz \otimes s);$$

and parts (a) and (b) follow. Fixing a point $r \in X$ and choosing the local holomorphic coordinate z so that $|dz|^2_{g^*} = 2$ at r, we get $\omega_r = (i/2)(dz \wedge d\bar z)_r$. For

$\alpha = a\,dz \cdot s$ and $\beta = b\,dz \cdot s$ at r, we have

$$\begin{aligned}\{\Phi_0(\alpha), \Phi_0(\beta)\}_{g^*\otimes h} \cdot \omega &= a\bar{b} \cdot |dz \otimes s|^2_{g^*\otimes h} \cdot \omega = a\bar{b} \cdot |dz|^2_{g^*} \cdot |s|^2_h \cdot \omega \\ &= a\bar{b} \cdot |s|^2_h \cdot i\,dz \wedge d\bar{z} = i\{\alpha, \beta\}_h.\end{aligned}$$

For $\alpha = a\,dz \wedge d\bar{z} \cdot s$ and $\beta = b\,dz \wedge d\bar{z} \cdot s$ at r, we have

$$\begin{aligned}-i\{\Phi_1(\alpha), \Phi_1(\beta)\}_{g^*\otimes h} &= -i\{-a\,d\bar{z} \cdot (dz \otimes s), -b\,d\bar{z} \cdot (dz \otimes s)\}_{g^*\otimes h} \\ &= -ia\bar{b}(d\bar{z} \wedge dz) \cdot |dz \otimes s|^2_{g^*\otimes h} \\ &= ia\bar{b}(dz \wedge d\bar{z}) \cdot 2|s|^2_h \\ &= \left[(i/2)\frac{\alpha}{\omega \cdot s}\right] \cdot \overline{\left[(i/2)\frac{\beta}{\omega \cdot s}\right]} \cdot 4\omega \cdot |s|^2_h \\ &= \left\{\frac{\alpha}{\omega}, \frac{\beta}{\omega}\right\}_h \cdot \omega.\end{aligned}$$

Parts (c) and (d) now follow.

Finally, for the proof of (e), we may assume without loss of generality that X is an open subset of $\mathbb{C}$. If $\alpha = a\,dz \cdot s$ is a locally integrable $(1,0)$-form with values in E, then, according to Definition 2.7.1 and Definition 3.8.1, the existence of $\bar{\partial}_{\mathrm{distr}}\Phi_0(\alpha)$ and the existence of $\bar{\partial}_{\mathrm{distr}}\alpha$ are each equivalent to the existence of

$$b \equiv \left(\frac{\partial a}{\partial \bar{z}}\right)_{\mathrm{distr}} \in L^1_{\mathrm{loc}}(X).$$

Moreover, if this function does exist, then

$$\begin{aligned}\Phi_1(\bar{\partial}_{\mathrm{distr}}\alpha) &= \Phi_1(-b\,dz \wedge d\bar{z} \cdot s) = b\,d\bar{z} \cdot (dz \otimes s) = (\bar{\partial}_{\mathrm{distr}}a) \cdot (dz \otimes s) \\ &= \bar{\partial}_{\mathrm{distr}}[\Phi_0(\alpha)].\end{aligned}$$ □

Remarks 1. Φ_1 is an example of a $\mathcal{C}^\infty$ line bundle isomorphism.

2. By the above proposition, the operator $D'' = \bar{\partial}$ and its distributional version are preserved (in the appropriate sense) under the identifications given by Φ_0 and Φ_1. However, the operators D', D, and Θ (associated to the Hermitian metrics in the holomorphic line bundles E and $K_X \otimes E$) are *not* preserved under these identifications; in fact, even the resulting types do not correspond. For example, in the notation of Proposition 3.10.1, for a $\mathcal{C}^\infty$ $(1,0)$-form $\alpha \equiv a\,dz \in \mathcal{E}^{1,0}(1_{\mathbb{C}})(\mathbb{C})$ with values in the trivial line bundle over $\mathbb{C}$, we have $\partial\alpha = D'\alpha = 0$ (as a $(2,0)$-form), but $D'\Phi_0(\alpha) = \frac{\partial a}{\partial z}dz \cdot dz \in \mathcal{E}^{1,0}(K_{\mathbb{C}})(\mathbb{C})$.

3. The above proposition also gives the mapping

$$(\Lambda^{(1,q)}T^*X) \otimes K_X^* \otimes E \to (\Lambda^{(0,q)}T^*X) \otimes K_X \otimes K_X^* \otimes E = (\Lambda^{(0,q)}T^*X) \otimes E$$

for $q = 0, 1$.

We have the following facts, which are equivalent to, respectively, Theorem 3.9.1, Corollary 3.9.2, and Corollary 3.9.3:

Corollary 3.10.2 *Assume that $i\Theta \equiv i\Theta_g + i\Theta_h \geq 0$ on X, and let $\rho \equiv i\Theta/\omega$ and $Z \equiv \{x \in X \mid \rho(x) = 0\} = \{x \in X \mid \Theta_x = 0\}$. Then, for every measurable E-valued $(0,1)$-form β on X with $\beta = 0$ a.e. in Z and $\beta\restriction_{X\setminus Z} \in L^2_{0,1}(X \setminus Z, E, e^{-\log\rho}h)$, there exists a section $\alpha \in L^2_{0,0}(X, E, \omega, h)$ such that*

$$D''_{\mathrm{distr}}\alpha = \bar{\partial}_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,E,\omega,h)} \leq \|\beta\|_{L^2(X\setminus Z,E,e^{-\log\rho}h)}.$$

In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.

Corollary 3.10.3 *Suppose that ρ is a nonnegative measurable function on X with $i\Theta_g + i\Theta_h \geq \rho\omega$, and $Z = \{x \in X \mid \rho(x) = 0\}$. Then, for every measurable E-valued $(0,1)$-form β on X with $\beta = 0$ a.e. in Z and $\beta\restriction_{X\setminus Z} \in L^2_{0,1}(X \setminus Z, E, e^{-\log\rho}h)$, there exists a section $\alpha \in L^2_{0,0}(X, E, \omega, h)$ such that*

$$D''_{\mathrm{distr}}\alpha = \bar{\partial}_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,E,\omega,h)} \leq \|\beta\|_{L^2(X\setminus Z,E,e^{-\log\rho}h)}.$$

In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.

Corollary 3.10.4 *Suppose that C is a positive constant with $i\Theta_g + i\Theta_h \geq C^2\omega$ on X. Then, for every $\beta \in L^2_{0,1}(X, E, h)$, there exists a section $\alpha \in L^2_{0,0}(X, E, \omega, h)$ such that*

$$D''_{\mathrm{distr}}\alpha = \bar{\partial}_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,\omega,h)} \leq C^{-1}\|\beta\|_{L^2(X,h)}.$$

In particular, if β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.

Corollary 3.10.4 immediately gives the following example of a *vanishing theorem* (see Definitions 3.4.3 and 3.8.4):

Corollary 3.10.5 *Let X be a Riemann surface and let E be a holomorphic line bundle on X.*

(a) *If g is a Kähler metric on X with Kähler form ω and h is a Hermitian metric in E satisfying $i\Theta_h \geq C^2\omega$ on X for some positive constant C, then*

$$H^1_{\mathrm{Dol},L^2}(X, K_X \otimes E, \omega, g^* \otimes h) = H^1_{\mathrm{Dol},L^2\cap\mathcal{E}}(X, K_X \otimes E, \omega, g^* \otimes h) = 0.$$

Equivalently, if $i\Theta_g + i\Theta_h \geq C^2\omega$, then

$$H^1_{\mathrm{Dol},L^2}(X, E, \omega, h) = H^1_{\mathrm{Dol},L^2\cap\mathcal{E}}(X, E, \omega, h) = 0.$$

(b) *Assume that X is compact. If E is positive (i.e., there exists a Hermitian metric h in E with $i\Theta_h > 0$ on X), then $H^1_{\mathrm{Dol}}(X, K_X \otimes E) = 0$. Equivalently, if there exist a Kähler metric g on X and a Hermitian metric h in E with $i\Theta_g + i\Theta_h > 0$, then $H^1_{\mathrm{Dol}}(X, E) = 0$. Consequently, if E is positive, then, given a holomorphic line bundle F on X, we have $H^1_{\mathrm{Dol}}(X, F \otimes E^r) = 0$ for all $r \gg 0$.*

The proof of Corollary 3.10.2 appears below, while the proofs of Corollary 3.10.3, Corollary 3.10.4, and Corollary 3.10.5 are left to the reader (see Exercises 3.10.1–3.10.3).

Proof of Corollary 3.10.2 Let $\hat{\beta}$ be the form of type $(1, 1)$ with values in $K_X^* \otimes E$ corresponding to β (see the third remark following Proposition 3.10.1); that is, if, in terms of a local holomorphic coordinate z and a nonvanishing local holomorphic section s of E, we have $\beta = b\, d\bar{z} \cdot s$, then

$$\hat{\beta} = b\, d\bar{z} \wedge dz \cdot \left(\left(\frac{\partial}{\partial z}\right) \otimes s\right) = -b\, dz \wedge d\bar{z} \cdot \left(\left(\frac{\partial}{\partial z}\right) \otimes s\right).$$

We have

$$\begin{aligned}\|\hat{\beta}\|_{L^2(X\setminus Z,\rho\omega,g\otimes h)} &= \|\hat{\beta}\|_{L^2(X\setminus Z,\omega,e^{-\log\rho} g\otimes h)} = \|\hat{\beta}\|_{L^2(X\setminus Z,\omega,g\otimes(e^{-\log\rho} h))} \\ &= \|\beta\|_{L^2(X\setminus Z,e^{-\log\rho} h)} < \infty\end{aligned}$$

and

$$i\Theta_{(K_X^* \otimes E, g\otimes h)} = i\Theta_{(K_X^*,g)} + i\Theta_h = i\Theta = \rho\omega.$$

Therefore, by Theorem 3.9.1, there exists a form $\hat{\alpha}$ of type $(1, 0)$ with values in $K_X^* \otimes E$ (with $\hat{\alpha}$ of class $\mathcal{C}^k$ if β, and therefore $\hat{\beta}$, is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$) such that $D''_{\mathrm{distr}}\hat{\alpha} = \bar{\partial}_{\mathrm{distr}}\hat{\alpha} = \hat{\beta}$ and $\|\hat{\alpha}\|_{L^2(X,g\otimes h)} \leq \|\hat{\beta}\|_{L^2(X\setminus Z,\rho\omega,g\otimes h)}$. The identification given by (the third remark following) Proposition 3.10.1 gives the desired solution α. □

Exercises for Sect. 3.10

3.10.1 Prove Corollary 3.10.3.
3.10.2 Prove Corollary 3.10.4.
3.10.3 Prove Corollary 3.10.5 (for the proof of the last statement in part (b), the reader may wish to apply Proposition 3.11.1 from Sect. 3.11).

3.11 Positive Curvature on an Open Riemann Surface

According to Radó's theorem (Theorem 2.11.1), every Riemann surface is second countable. From second countability, one gets the following:

Proposition 3.11.1 *Every holomorphic line bundle $\Pi\colon E \to X$ on a Riemann surface X admits a Hermitian metric.*

Proof Since X is second countable, there exist a locally finite collection of local holomorphic trivializations $\{(U_\nu, (\Pi, \Phi_\nu))\}_{\nu\in N}$ and a $\mathcal{C}^\infty$ partition of unity $\{\lambda_\nu\}$ with $\operatorname{supp}\lambda_\nu \subset U_\nu$ (and $\lambda_\nu \geq 0$) for each $\nu \in N$ (see Sect. 9.3). The pairing

$$h(u, v) = h_p(u, v) \equiv \sum_{\nu\in N} \lambda_\nu(p)\Phi_\nu(u)\overline{\Phi_\nu(v)} \quad \forall p \in X,\ u, v \in E_p$$

is then a Hermitian metric in E. The verification is left to the reader (see Exercise 3.11.1). □

We now (begin to) address the natural question as to when a holomorphic line bundle on a Riemann surface admits a positive-curvature Hermitian metric. For a holomorphic line bundle on an *open* Riemann surface, the facts proved in Sect. 2.14 provide a Hermitian metric with arbitrarily large positive curvature. In other words, we have the following:

Theorem 3.11.2 *Let X be an open Riemann surface, let E be a holomorphic line bundle on X, and let θ be a continuous real $(1, 1)$-form on X. Then E admits a Hermitian metric h with $i\Theta_h \geq \theta$. In particular, if g is a Kähler metric on X with associated Kähler form ω, then E admits a Hermitian metric h with*

$$i\Theta_h \geq \omega \quad \textit{and} \quad i\Theta_g + i\Theta_h \geq \omega.$$

Proof Let us fix a Hermitian metric k in E and a Kähler metric g with Kähler form ω on X (provided by Proposition 3.11.1). Given a continuous real $(1, 1)$-form θ on X, Theorem 2.14.4 provides a $\mathcal{C}^\infty$ (exhaustion) function φ such that

$$\Delta_\omega\varphi \geq 1 + \left|\frac{\theta}{\omega}\right| + \left|\frac{i\Theta_k}{\omega}\right| + \left|\frac{i\Theta_g}{\omega}\right|.$$

Thus the Hermitian metric $h \equiv e^{-\varphi}k$ satisfies $i\Theta_h = [\Delta_\omega\varphi]\cdot\omega + i\Theta_k \geq \theta$, $i\Theta_h \geq \omega$, and $i\Theta_g + i\Theta_h \geq \omega$. □

Remarks 1. When second countability of a particular Riemann surface X is evident (for example, when X is an open subset of a compact Riemann surface), one may construct Hermitian metrics and, for X an open Riemann surface, $\mathcal{C}^\infty$ strictly subharmonic exhaustion functions (as in Sect. 2.14) and positive-curvature Hermitian metrics (as above) without any reliance on Radó's theorem, and hence with almost no reliance on Sects. 2.6, 2.7, 2.9–2.12, 3.6, and 3.8–3.10.

2. A holomorphic line bundle E on a *compact* Riemann surface need *not* admit a positive-curvature Hermitian metric. In Chap. 4, it will be shown that in fact, a holomorphic line bundle E on a compact Riemann surface admits a Hermitian metric of positive curvature if and only if E is the line bundle associated to a divisor of *positive degree* (Theorem 4.3.1).

The above considerations, together with the facts proved in Sects. 3.9 and 3.10, give the following (cf. Theorem 2.14.10):

Theorem 3.11.3 *Let X be an open Riemann surface, let E be a holomorphic line bundle on X, let $p \in \{0, 1\}$, and let β be a locally square-integrable differential form of type $(p, 1)$ with values in E on X. Then there exists a locally square-integrable differential form α of type $(p, 0)$ with values in E such that $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$. If β is of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then α is also of class $\mathcal{C}^k$ and $\bar{\partial}\alpha = \beta$.*

Proof We may fix a Kähler metric g with Kähler form ω, and a Hermitian metric k in E. Applying Theorem 2.14.4, with the compact set given by $K = \emptyset$ and the function ρ chosen (with the help of Lemma 9.7.13) so large that $\beta \in L^2_{0,1}(X, E, e^{-\rho}k)$ or $L^2_{1,1}(X, E, \omega, e^{-\rho}k)$ and $\rho > 1 + |i\Theta_k/\omega| + |i\Theta_g/\omega|$, we get a positive $\mathcal{C}^\infty$ strictly subharmonic (exhaustion) function φ on X such that the Hermitian metric $h = e^{-\varphi}k$ satisfies

$$\begin{aligned} i\Theta_g + i\Theta_h &\geq \omega \quad \text{and} \quad \|\beta\|_{L^2(X,E,h)} < \infty \qquad \text{if } p = 0, \\ i\Theta_h &\geq \omega \quad \text{and} \quad \|\beta\|_{L^2(X,E,\omega,h)} < \infty \quad \text{if } p = 1. \end{aligned}$$

Corollary 3.10.4 and Corollary 3.9.3 now provide the desired E-valued differential form α. $\square$

In particular, we have the following vanishing theorem:

Corollary 3.11.4 *If E is a holomorphic line bundle on an open Riemann surface X, then $H^1_{\mathrm{Dol}}(X, E) = 0$.*

The construction of a holomorphic section of a holomorphic line bundle with prescribed values on a given discrete set (or prescribed Laurent series about each of the points in the discrete set) is one of the main applications of the L^2 $\bar{\partial}$-method. For an open Riemann surface, the above considerations give the following generalization of the Mittag-Leffler theorem (Theorem 2.15.1):

Theorem 3.11.5 (Mittag-Leffler theorem for a holomorphic line bundle) *Let X be an open Riemann surface, let E be a holomorphic line bundle on X, let P be a discrete subset of X, and for each point $p \in P$, let U_p be a neighborhood of p with $U_p \cap P = \{p\}$, let t_p be a holomorphic section of E on $U_p \setminus \{p\}$, and let m_p be a positive integer. Then there exists a holomorphic section s of E on $X \setminus P$ such that for each point $p \in P$, $s - t_p$ extends to a holomorphic section on U_p that either vanishes on a neighborhood of p or has a zero of order at least m_p at p (in other words, if z is a local holomorphic coordinate and u is a nonvanishing holomorphic section of E on a neighborhood of p, and $s/u = \sum_{n\in\mathbb{Z}} a_n(z - z(p))^n$ and $t_p/u = \sum_{n\in\mathbb{Z}} b_n(z - z(p))^n$ are the corresponding Laurent series expansions centered at p, then $a_{m_p-n} = b_{m_p-n}$ for $n = 1, 2, 3, \ldots$).*

Remarks 1. Recall that we also denote the value of a section v of a line bundle E at a point p by v_p. The section t_p in the above theorem should not be confused with such a value.

2. It follows that one may actually choose the above section $s \in \Gamma(X \setminus P, \mathcal{O}(E))$ so that for each point $p \in P$, $s - t_p$ extends to a holomorphic section on U_p with a zero of order *equal to* m_p at p (see Exercise 3.11.3).

3. In Exercise 3.11.2, the reader is asked to give a proof that is a direct generalization of the proof of Theorem 2.15.1. The proof provided below, which uses the language of divisors, is more efficient (in particular, the proof is, in part, a demonstration of the quiet power of divisors). It should also be noted, however, that according to the Weierstrass theorem (Theorem 3.12.1), every holomorphic line bundle on an open Riemann surface is holomorphically trivial; so it turns out that the Mittag-Leffler theorem (Theorem 2.15.1) and its generalization for sections of a line bundle (Theorem 3.11.5) are actually equivalent.

The proof of the following is left to the reader (see Exercise 3.11.4):

Corollary 3.11.6 (Cf. Corollary 2.15.2) *Let X be an open Riemann surface, let E be a holomorphic line bundle on X, and let P be a discrete subset of X.*

(a) *If t_p is a meromorphic section of E on a neighborhood of p and $m_p \in \mathbb{Z}_{>0}$ for each point $p \in P$, then there exists a section $s \in \Gamma(X, \mathcal{M}(E))$ such that s is holomorphic on $X \setminus P$ and* $\operatorname{ord}_p(s - t_p) \geq m_p$ *for every $p \in P$.*
(b) *If t_p is a holomorphic section of E on a neighborhood of p and $m_p \in \mathbb{Z}_{>0}$ for each point $p \in P$, then there exists a holomorphic section $s \in \Gamma(X, \mathcal{O}(E))$ with* $\operatorname{ord}_p(s - t_p) \geq m_p$ *for every $p \in P$.*
(c) *If $\zeta_p \in E_p$ for each point $p \in P$, then there exists a section $s \in \Gamma(X, \mathcal{O}(E))$ with $s(p) = \zeta_p$ for every $p \in P$.*

Of course, for a given holomorphic line bundle on a *compact* Riemann surface X, a holomorphic section with prescribed values on a given discrete set need not exist (for example, $\Gamma(X, \mathcal{O}(1_X)) = \mathcal{O}(X) = \mathbb{C}$). Much of Chap. 4 is devoted to determining when (and how many) such sections exist.

Proof of Theorem 3.11.5 We may assume without loss of generality that $P \neq \emptyset$; we may let D be the (nontrivial) effective divisor given by $D \equiv \sum_{p \in P} m_p \cdot p$ (i.e., $D(p) = m_p$ for each point $p \in P$ and D vanishes on $X \setminus P$); we may fix a holomorphic section $u \in \Gamma(X, \mathcal{O}([D]))$ with $\operatorname{div}(u) = D$; and finally, we may fix a $\mathcal{C}^\infty$ section γ of the holomorphic line bundle $E \otimes [-D]$ on the open set $Y \equiv X \setminus P$ such that $\gamma = t_p/u$ near each point $p \in P$.

The $E \otimes [-D]$-valued $(0, 1)$-form $\bar{\partial}\gamma$ extends to a $\mathcal{C}^\infty$ $E \otimes [-D]$-valued $(0, 1)$-form β on X that vanishes on a neighborhood of P. Theorem 3.11.3 provides a $\mathcal{C}^\infty$ section α of $E \otimes [-D]$ on X with $\bar{\partial}\alpha = \beta$. In particular, α is holomorphic near P, and the section $s \equiv (\gamma - \alpha) \otimes u = \gamma \otimes u - \alpha \otimes u$ of $E \otimes [-D] \otimes [D] = E$ is holomorphic on Y. Finally, near each point $p \in P$, we have $s - t_p = -\alpha \otimes u$, and therefore, since $\alpha \otimes u$ is holomorphic near p with $\mathrm{ord}_p(\alpha \otimes u) \geq \mathrm{ord}_p u = m_p$, s has the required properties. □

Corollary 3.11.7 (Cf. Theorem 4.2.3) *Any holomorphic line bundle E on an open Riemann surface X is equal to the line bundle associated to some nontrivial effective divisor D on X; in other words, E admits a nontrivial (i.e., not everywhere zero) holomorphic section with at least one zero.*

Further generalizations of the Mittag-Leffler theorem, and also of the Runge approximation theorem, are considered in the exercises.

Exercises for Sect. 3.11

3.11.1 Verify that the function h constructed the proof of Proposition 3.11.1 is a Hermitian metric.

3.11.2 Give a proof of Theorem 3.11.5 that is analogous to the proof of Theorem 2.15.1 given in Sect. 2.15 (and that does not use divisors).

3.11.3 Prove that in Theorem 3.11.5, one may actually choose the holomorphic section s on $X \setminus P$ so that for each point $p \in P$, $s - t_p$ extends to a holomorphic section on U_p with a zero of order *equal to m_p* at p.

3.11.4 Prove Corollary 3.11.6

3.11.5 The goal of this exercise is a generalization of the Behnke–Stein theorem (Corollary 2.15.3). Let (E, h) be a Hermitian holomorphic line bundle on an open Riemann surface X. Prove that:
   (i) *Holomorphic convexity.* If S is any infinite discrete subset of X, then there exists a holomorphic section s of E on X such that $|s|_h$ is unbounded on S;
   (ii) *Separation of points.* If $p, q \in X$ and $p \neq q$, then there exists a holomorphic section s of E on X such that $s(p) = 0$ and $s(q) \neq 0$; and
   (iii) *Global sections give local coordinates.* For each point $p \in X$, there exists a holomorphic section s of E on X such that $s_p = 0$ and $(d(s/t))_p \neq 0$ for each nonvanishing holomorphic section t of E on a neighborhood of p.

3.11.6 The goal of this exercise is a version of Theorem 3.11.5 that is analogous to the version of the Mittag-Leffler theorem considered in Exercise 2.15.6. Let X be an open Riemann surface, let E be a holomorphic line bundle on X, let P be a discrete subset of X, let $\{m_p\}_{p \in P}$ be a collection of positive integers, let $\{U_i\}_{i \in I}$ be an open covering of X, and for each pair of indices $i, j \in I$, let $t_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}(E))$ with $\mathrm{ord}_p t_{ij} \geq m_p$ for each point $p \in P \cap U_i \cap U_j$. Assume that the family $\{t_{ij}\}$ satisfies the (additive) *cocycle relation*

$$t_{ik} = t_{ij} + t_{jk} \quad \text{on } U_i \cap U_j \cap U_k \quad \forall i, j, k \in I.$$

Prove that there exist sections $\{s_i\}_{i \in I}$ such that
   (i) For each index $i \in I$, $s_i \in \Gamma(U_i, \mathcal{O}(E))$ and $\mathrm{ord}_p s_i \geq m_p$ for each point $p \in P \cap U_i$; and

(ii) For each pair of indices $i, j \in I$, we have $t_{ij} = s_j - s_i$ on $U_i \cap U_j$.
Prove also that the above implies Theorem 3.11.5.

Hint. Using a $\mathcal{C}^\infty$ partition of unity $\{\lambda_\nu\}$ such that each point in P lies in $\operatorname{supp} \lambda_\nu$ for exactly one index ν and such that for each ν, $\operatorname{supp} \lambda_\nu \subset U_{k_\nu}$ for some index $k_\nu \in I$, one may form a $\mathcal{C}^\infty$ solution of the problem of the form $v_i = \sum_\nu \lambda_\nu \cdot t_{k_\nu i}$. In particular, the E-valued forms $\{\bar{\partial} v_i\}$ agree on the overlaps and therefore determine a well-defined E-valued (0, 1)-form β on X.

3.11.7 The goal of this exercise is a version of the Runge approximation theorem (Theorem 2.16.1) for sections of a holomorphic line bundle. Let X be an open Riemann surface, let (E, h) be a Hermitian holomorphic line bundle on X, and let $K \subset X$ be a compact set satisfying $\mathfrak{h}_X(K) = K$. Prove that for every holomorphic section t of E on a neighborhood of K in X and for every $\epsilon > 0$, there exists a section $s \in \Gamma(X, \mathcal{O}(E))$ such that $|s - t|_h < \epsilon$ on K.

3.11.8 The goal of this exercise is a version of Theorem 2.16.3 for sections of a holomorphic line bundle. Suppose K is a compact subset of a Riemann surface X, (E, h) is a Hermitian holomorphic line bundle on X, P is a finite subset of $X \setminus K$, $Y = X \setminus P$, $\mathfrak{h}_Y(K) = K$, t is a holomorphic section of E on a neighborhood of K in X, and $\epsilon > 0$. Prove that there exists a meromorphic section s of E on X such that s is holomorphic on Y, s has a pole at each point in P, and $|s - t|_h < \epsilon$ on K.

3.11.9 The goal of this exercise is a generalization of Exercise 2.16.7. Let X be an open Riemann surface, let (E, h) be a Hermitian holomorphic line bundle on X, let $K \subset X$ be a compact subset with $\mathfrak{h}_X(K) = K$, let $P \subset X$ be a discrete subset with $P \subset X \setminus K$, let t_0 be a holomorphic section of E on a neighborhood of K in X, and for each point $p \in P$, let t_p be a holomorphic section of E on $U_p \setminus \{p\}$ for some neighborhood U_p of p in X with $U_p \cap P = \{p\}$, and let m_p be a positive integer. Prove that for every $\epsilon > 0$, there exists a holomorphic section s of E on $X \setminus P$ such that $|s - t_0|_h < \epsilon$ on K and such that for each point $p \in P$, $s - t_p$ extends to a holomorphic section on U_p that either vanishes on a neighborhood of p or has a zero of order at least m_p at p.

3.12 The Weierstrass Theorem

The main goal of this section is the following analogue, due to Florack [Fl], of the classical theorem of Weierstrass:

Theorem 3.12.1 (Weierstrass theorem) *Every holomorphic line bundle E on an open Riemann surface X is* (*holomorphically*) *trivial*; *that is*, *equivalently*, *every divisor on X has a solution.*

Recall that a *solution* of a divisor D on a complex 1-manifold X is a meromorphic function f such that f does not vanish identically on any open set and

$\operatorname{div}(f) = D$ (see Definition 3.3.1). To prove the theorem, we must show that E admits a nonvanishing holomorphic section. For this, the main step will be to show that E is $\mathcal{C}^\infty$ *trivial*; i.e., E admits a nonvanishing $\mathcal{C}^\infty$ section. The $\bar{\partial}$-method will then provide a correction that yields a suitable holomorphic section. For a divisor D with $[D] = E$ and a meromorphic section s of E with $\operatorname{div}(s) = D$, E is $\mathcal{C}^\infty$ trivial if and only if there exists a $\mathcal{C}^\infty$ function ρ on $X \setminus D^{-1}((-\infty, 0))$ such that s is nonvanishing on $X \setminus \operatorname{supp} D$ and s/ρ extends to a nonvanishing $\mathcal{C}^\infty$ section of E on X (equivalently, ρ is nonvanishing on $X \setminus \operatorname{supp} D$ and for each point $p \in \operatorname{supp} D$, there are a local holomorphic coordinate neighborhood (U, z) with $z(p) = 0$ and a nonvanishing $\mathcal{C}^\infty$ function ψ on U such that $\rho = z^{D(p)} \cdot \psi$ on $U \setminus \{p\}$). Such a function ρ is called a *weak solution* of the divisor D. The construction of a weak solution very closely parallels that of a strictly subharmonic exhaustion function provided in Sect. 2.14. The first step is the following local construction:

Lemma 3.12.2 *Let $(V, \Phi, \Delta \equiv \Delta(0; 1))$ be a local holomorphic chart in a Riemann surface X. Then, for each pair of distinct points $p, q \in V$, there exists a weak solution ρ of the divisor $D = p - q$ with $\rho \equiv 1$ on $X \setminus V$.*

Proof Given a pair of distinct points $p, q \in V$, setting $\zeta = \Phi(p)$ and $\xi = \Phi(q)$, we may choose a holomorphic branch of the function

$$z \mapsto \log\left(\frac{z - \zeta}{z - \xi}\right)$$

on the complement $\Delta \setminus [\zeta, \xi]$ of the line segment $[\zeta, \xi]$ from ζ to ξ (see Exercise 3.12.2). In other words, there exists a holomorphic function g on $\Delta \setminus [\zeta, \xi]$ such that

$$e^{g(z)} = \frac{z - \zeta}{z - \xi} \quad \forall z \in \Delta \setminus [\zeta, \xi].$$

We may also choose R with $|\zeta|, |\xi| < R < 1$ and a $\mathcal{C}^\infty$ function λ with compact support in Δ such that $\lambda \equiv 1$ on $\Delta(0; R)$. The function ρ given by $\rho \equiv 1$ on $X \setminus V$ and

$$\rho(\Phi^{-1}(z)) \equiv \begin{cases} \exp(\lambda(z) \cdot g(z)) & \text{if } z \in \Delta(0; 1) \setminus \Delta(0; R), \\ \frac{z-\zeta}{z-\xi} & \text{if } z \in \Delta(0; R) \setminus \{\xi\}, \end{cases}$$

is then a weak solution of the divisor $D = p - q$ on X. □

Lemma 3.12.3 *Let X be an open Riemann surface, let $K \subset X$ be a compact set with $\mathfrak{h}_X(K) = K$, and let $p \in X \setminus K$. Then there exists a weak solution of the divisor $D = 1 \cdot p$ on X with $\rho \equiv 1$ on K.*

Proof By Lemma 2.13.5, there exists a locally finite (in X) sequence of relatively compact open subsets $\{V_m\}_{m=1}^\infty$ of $X \setminus K$ such that $p \in V_1$ and such that for each m, V_m is biholomorphic to a disk and $V_m \cap V_{m+1} \neq \emptyset$. Hence we may choose a sequence of points $\{p_m\}_{m=0}^\infty$ such that $p_0 = p$ and $p_m \in V_m \cap V_{m+1}$ for each $m \geq 1$.

By Lemma 3.12.3, for each $m = 1, 2, 3, \ldots$, there exists a weak solution ρ_m of the divisor $D_m = p_{m-1} - p_m$ with $\rho_m \equiv 1$ on $X \setminus V_m$. The locally finite product $\prod \rho_m$ then yields a weak solution ρ of the divisor $D = 1 \cdot p$ with $\rho \equiv 1$ on K. □

Lemma 3.12.4 *Every divisor D on an open Riemann surface X has a weak solution.*

Proof By Lemma 2.13.4, there exists a sequence of compact sets $\{K_\nu\}_{\nu=0}^\infty$ such that $K_0 = \emptyset$ and $X = \bigcup_{\nu=0}^\infty K_\nu$, and such that for each $\nu = 0, 1, 2, \ldots$, we have $K_\nu \subset \overset{\circ}{K}_{\nu+1}$ and $\mathfrak{h}_X(K_\nu) = K_\nu$. For each point $p \in \operatorname{supp} D$, there is a unique index $\nu(p)$ such that $p \in K_{\nu(p)+1} \setminus K_{\nu(p)}$. Moreover, by Lemma 3.12.3, there exists a weak solution ρ_p of the divisor $D_p = 1 \cdot p$ on X such that $\rho_p \equiv 1$ on $K_{\nu(p)}$. Since the collection of points in $\operatorname{supp} D$ and the collection of sets $\{X \setminus K_\nu\}$ are locally finite in X, the product

$$\prod_{p \in \operatorname{supp} D} \rho_p^{D(p)}$$

is locally finite and yields a weak solution of the divisor D. □

Proof of Theorem 3.12.1 By Lemma 3.12.4, E admits a nonvanishing $\mathcal{C}^\infty$ section v. The quotient $\beta = (\bar{\partial} v)/v$ is then a $\mathcal{C}^\infty$ scalar-valued form of type (0, 1) (which, locally, may be thought of as $\bar{\partial} \log v$). By Theorem 3.11.3 (or Theorem 2.14.10 or Corollary 3.11.4), there exists a $\mathcal{C}^\infty$ function α on X such that $\bar{\partial}\alpha = \beta$. The nonvanishing $\mathcal{C}^\infty$ section $s \equiv e^{-\alpha} \cdot v$ of E is then holomorphic because

$$\bar{\partial} s = -e^{-\alpha} \cdot \frac{\bar{\partial} v}{v} \cdot v + e^{-\alpha} \cdot \bar{\partial} v = 0. \qquad \square$$

Remarks 1. Since, according to the Weierstrass theorem, a holomorphic line bundle on an open Riemann surface is holomorphically trivial, one need only really consider *scalar-valued* differential forms (along with weights) as in Chap. 2 in this case.

2. Suitable higher-dimensional analogues of the Mittag-Leffler and Weierstrass theorems, due to Oka and Cartan, hold on *Stein manifolds* (see, for example, [Hö] or [KaK]). The problem of obtaining an analogue of the Mittag-Leffler theorem (the *additive Cousin problem* or the *Cousin problem* I) and the problem of obtaining an analogue of the Weierstrass theorem (the *multiplicative Cousin problem* or the *Cousin problem* II) played central roles in the development of several complex variables; and these analogues are fundamental tools in the subject.

Exercises for Sect. 3.12

3.12.1 Let D be a divisor on a Riemann surface X, let $E = [D]$, and let s be a meromorphic section of E with $\operatorname{div}(s) = D$. Verify that E is $\mathcal{C}^\infty$ trivial if and only if there exists a $\mathcal{C}^\infty$ function ρ on $X \setminus D^{-1}((-\infty, 0))$ such that s

is nonvanishing on $X \setminus \operatorname{supp} D$ and s/ρ extends to a nonvanishing $\mathcal{C}^\infty$ section of E on X. Verify also that a $\mathcal{C}^\infty$ function ρ on $X \setminus D^{-1}((-\infty, 0))$ has the above property if and only if ρ is nonvanishing on $X \setminus \operatorname{supp} D$ and for each point $p \in \operatorname{supp} D$, there are a local holomorphic coordinate neighborhood (U, z) with $z(p) = 0$ and a nonvanishing $\mathcal{C}^\infty$ function ψ on U such that $\rho = z^{D(p)} \cdot \psi$ on $U \setminus \{p\}$.

3.12.2 Verify that given a pair of distinct points $\zeta, \xi \in \mathbb{C}$, there exists a holomorphic branch of the function

$$z \mapsto \log\left(\frac{z-\zeta}{z-\xi}\right)$$

on the complement $\mathbb{C} \setminus [\zeta, \xi]$ of the line segment $[\zeta, \xi]$ from ζ to ξ. In other words, there exists a holomorphic function g on $\mathbb{C} \setminus [\zeta, \xi]$ such that

$$e^{g(z)} = \frac{z-\zeta}{z-\xi} \quad \forall z \in \mathbb{C} \setminus [\zeta, \xi]$$

(this fact was used in the proof of Lemma 3.12.2).

3.12.3 Let D be a divisor on a compact Riemann surface X. Prove that the associated holomorphic line bundle $E = [D]$ is $\mathcal{C}^\infty$ trivial (i.e., E admits a nonvanishing $\mathcal{C}^\infty$ section) if and only if $\deg D = 0$.

3.12.4 As is the case for the Mittag-Leffler theorem (see Exercises 2.15.6 and 3.11.6), the Weierstrass theorem may be stated in terms of *cocycles* (in higher dimensions, this is known as the solution of the *multiplicative Cousin problem* or the *Cousin problem* II). Let X be a Riemann surface.

(a) Suppose $\Pi: E \to X$ is a holomorphic line bundle and $\{f_{ij} = \Phi_i/\Phi_j\}_{i,j \in I}$ are the transition functions associated to some holomorphic line bundle atlas $\{(U_i, \Psi_i = (\Pi, \Phi_i))\}_{i \in I}$ for E. Prove that E is (holomorphically) trivial if and only if there exists a collection $\{f_i\}_{i \in I}$ such that f_i is a nonvanishing holomorphic function on U_i for each $i \in I$ and $f_{ij} = f_j/f_i$ for all $i, j \in I$.

(b) Suppose $\{U_i\}_{i \in I}$ is an open covering of X and f_{ij} is a nonvanishing holomorphic function on $U_i \cap U_j$ for each pair of indices $i, j \in I$. Assume that the family $\{f_{ij}\}_{i,j \in I}$ satisfies the (multiplicative) cocycle relation

$$f_{ik} = f_{ij} f_{jk} \quad \text{on } U_i \cap U_j \cap U_k \quad \forall i, j, k \in I.$$

Prove that there exists a unique holomorphic line bundle $\Pi: E \to X$ for which there exists a holomorphic line bundle atlas $\{(U_i, \Psi_i = (\Pi, \Phi_i))\}_{i \in I}$ with transition functions given by $\Phi_i/\Phi_j = f_{ij}$ for each pair $i, j \in I$. Conclude that in particular, if X is an open Riemann surface, then there exists a collection $\{f_i\}_{i \in I}$ such that f_i is a nonvanishing holomorphic function on U_i for each $i \in I$ and $f_{ij} = f_j/f_i$ for all $i, j \in I$.

Part II
Further Topics

Chapter 4
Compact Riemann Surfaces

In this chapter, we consider some facts concerning holomorphic line bundles (and their holomorphic sections) on compact Riemann surfaces. We first consider conditions that guarantee the existence of holomorphic sections with prescribed values. Unlike the open Riemann surface case (in which one has Theorem 3.11.5), a holomorphic line bundle need not have the positivity required for such a section to exist. For example, the space of holomorphic functions on a compact Riemann surface X has dimension 1, and a negative holomorphic line bundle on X has *no* nontrivial global holomorphic sections. After the above considerations, we consider the fact that a holomorphic line bundle is positive if and only if its degree is positive, and we then consider finiteness of Dolbeault cohomology. The Riemann–Roch formula is then proved (using finiteness). Finally, we consider the Serre duality theorem and the Hodge decomposition theorem, and some of their consequences. Different approaches to the above facts appear in (for example) [For] and [Ns4].

4.1 Existence of Holomorphic Sections on a Compact Riemann Surface

In this section, we consider consequences of the following:

Theorem 4.1.1 *Let E be a holomorphic line bundle on a compact Riemann surface X. If there exist a Kähler metric g on X and a Hermitian metric h in E with $i\Theta_g + i\Theta_h > 0$ on X, then for every effective divisor D on X, the linear map*

$$\begin{aligned}\Gamma(X, \mathcal{O}(E\otimes[D])) \longrightarrow\, & \Gamma(X, \mathcal{Q}_D(E\otimes[D])) \\ & = \Gamma(X, \mathcal{O}(E\otimes[D])/\mathcal{O}_{-D}(E\otimes[D]))\end{aligned}$$

T. Napier, M. Ramachandran, *An Introduction to Riemann Surfaces*, Cornerstones,
DOI 10.1007/978-0-8176-4693-6_4,

$$= \bigoplus_{p \in \operatorname{supp} D} \mathcal{O}(E \otimes [D])_p \big/ (\mathfrak{m}_p^{D(p)} \cdot \mathcal{O}(E \otimes [D])_p)$$
$$\cong \bigoplus_{p \in \operatorname{supp} D} \mathbb{C}^{D(p)} \cong \mathbb{C}^{\deg D}$$

is surjective. In other words, given a holomorphic section t_p of $E \otimes [D]$ on a neighborhood of p in X for each point $p \in \operatorname{supp} D$, there exists a holomorphic section s of $E \otimes [D]$ on X such that $\operatorname{ord}_p(s - t_p) \geq D(p)$ for each point $p \in \operatorname{supp} D$. In particular, $\dim \Gamma(X, \mathcal{O}(E \otimes [D])) \geq \deg D$.

Proof By Corollary 3.10.5, we have $H^1_{\mathrm{Dol}}(X, E) = 0$, and therefore the Dolbeault exact sequence (see Theorem 3.4.4 and Definition 3.4.5) is

$$0 \to \Gamma(X, \mathcal{O}(E)) \to \Gamma(X, \mathcal{O}(E \otimes [D])) \to \Gamma(X, \mathcal{Q}_D(E \otimes [D])) \to 0.$$

Furthermore, any choice of a holomorphic section t_p of $E \otimes [D]$ on a neighborhood of p in X for each point $p \in \operatorname{supp} D$ determines a section $\xi \in \Gamma(X, \mathcal{Q}_D(E \otimes [D]))$. Every section $s \in \Gamma(X, \mathcal{O}(E \otimes [D]))$ with image ξ in $\Gamma(X, \mathcal{Q}_D(E \otimes [D]))$ then satisfies $\operatorname{ord}_p(s - t_p) \geq D(p)$ for each point $p \in \operatorname{supp} D$. □

Remark One may also prove a stronger version (see Exercise 4.1.4) that is analogous to the Mittag-Leffler theorem (Theorem 2.15.1 and Theorem 3.11.5).

The proof of the following equivalent version of Theorem 4.1.1 is left to the reader (see Exercise 4.1.1):

Corollary 4.1.2 *Let E be a positive holomorphic line bundle on a compact Riemann surface X. Then, for every effective divisor D on X, the associated linear map*

$$\Gamma(X, \mathcal{O}(K_X \otimes E \otimes [D])) \longrightarrow \Gamma(X, \mathcal{Q}_D(K_X \otimes E \otimes [D]))$$

is surjective. In particular, $\dim \Gamma(X, \mathcal{O}(K_X \otimes E \otimes [D])) \geq \deg D$.

Corollary 4.1.3 *Let X be a compact Riemann surface, let F be a holomorphic line bundle on X, and let E be a positive holomorphic line bundle on X. Then for every integer $r \gg 0$ and for every effective divisor D on X (r does not depend on the choice of D), the linear map*

$$\Gamma(X, \mathcal{O}(F \otimes E^r \otimes [D])) \longrightarrow \Gamma(X, \mathcal{Q}_D(F \otimes E^r \otimes [D]))$$

is surjective. In particular, $\dim \Gamma(X, \mathcal{O}(F \otimes E^r \otimes [D])) \geq \deg D$.

The proof is left to the reader (see Exercise 4.1.2).

Corollary 4.1.4 *Given an effective divisor D on a compact Riemann surface X, there exists a Kähler form ω_0 on X such that for every Hermitian holomorphic line bundle (E, h) on X with $i\Theta_h \geq \omega_0$, the associated linear map*

$$\Gamma(X, \mathcal{O}(E)) \longrightarrow \Gamma(X, \mathcal{Q}_D(E)) \cong \mathbb{C}^{\deg D}$$

is surjective (in particular, $\dim \Gamma(X, \mathcal{O}(E)) \geq \deg D$*). In fact, we may choose ω_0 so that if (E, h) is a Hermitian holomorphic line bundle on X with $i\Theta_h \geq \omega_0$ and t_p is a holomorphic section of E on a neighborhood of p in X for each point $p \in \operatorname{supp} D$, then there exists a holomorphic section s of E on X such that* $\operatorname{ord}_p(s - t_p) = D(p)$ *for each point $p \in \operatorname{supp} D$.*

Proof We may fix a Kähler metric g on X with associated Kähler form ω and a Hermitian metric k in $[D]$ on X. We may now choose a Kähler form ω_0 such that

$$\omega_0 + i\Theta_g - i\Theta_k > 0$$

(for example, we may take $\omega_0 = C\omega$ for $C \gg 0$). Hence, if (E, h) is a Hermitian holomorphic line bundle on X with $i\Theta_h \geq \omega_0$, then the Hermitian holomorphic line bundle $(E \otimes [-D], h \otimes k^*)$ satisfies

$$i\Theta_g + i\Theta_{h\otimes k^*} = i\Theta_g + i\Theta_h - i\Theta_k \geq i\Theta_g + \omega_0 - i\Theta_k > 0.$$

Applying Theorem 4.1.1 to the line bundle $E \otimes [-D] \otimes [D] \cong E$, we get surjectivity of the mapping $\Gamma(X, \mathcal{O}(E)) \longrightarrow \Gamma(X, \mathcal{Q}_D(E))$.

Now, for each point $p \in \operatorname{supp} D$, we may fix a holomorphic section u_p of E on a neighborhood of p such that $\operatorname{ord}_p u_p = D(p)$. Applying the above to the divisor $2D$, we get a Kähler metric ω_0 such that if (E, h) is a Hermitian holomorphic line bundle on X with $i\Theta_h \geq \omega_0$ and t_p is a holomorphic section of E on a neighborhood of p for each point $p \in \operatorname{supp} D$, then there exists a section $s \in \Gamma(X, \mathcal{O}(E))$ with $\operatorname{ord}_p(s - t_p - u_p) \geq 2D(p) > D(p)$, and hence $\operatorname{ord}_p(s - t_p) = D(p)$, for each point $p \in \operatorname{supp} D$. $\square$

Corollary 4.1.5 *Let D be an effective divisor on a compact Riemann surface X, let F be a holomorphic line bundle on X, and let E be a positive holomorphic line bundle on X. Then, for $r \gg 0$ (depending on the choice of D, F, and E), the associated linear map*

$$\Gamma(X, \mathcal{O}(F \otimes E^r)) \longrightarrow \Gamma(X, \mathcal{Q}_D(F \otimes E^r))$$

is surjective (in particular, $\dim \Gamma(X, \mathcal{O}(F \otimes E^r)) \geq \deg D$*). In fact, if $r \gg 0$ and t_p is a holomorphic section of $F \otimes E^r$ on a neighborhood of p in X for each point $p \in \operatorname{supp} D$, then there exists a holomorphic section s of $F \otimes E^r$ on X such that* $\operatorname{ord}_p(s - t_p) = D(p)$ *for each point $p \in \operatorname{supp} D$.*

The proof is left to the reader (see Exercise 4.1.3).

Exercises for Sect. 4.1

4.1.1 Prove Corollary 4.1.2.
4.1.2 Prove Corollary 4.1.3.
4.1.3 Prove Corollary 4.1.5.
4.1.4 The goal of this problem is a generalization of Theorem 4.1.1 that is analogous to the Mittag-Leffler theorem (Theorem 2.15.1 and Theorem 3.11.5). Let E be a holomorphic line bundle on a compact Riemann surface X. Assume that there exist a Kähler metric g on X and a Hermitian metric h in E with $i\Theta_g + i\Theta_h > 0$ on X. Let D be an effective divisor on X, and for each point $p \in \operatorname{supp} D$, let t_p be a holomorphic section of $E \otimes [D]$ on $U_p \setminus \{p\}$ for some neighborhood U_p of p with $U_p \cap \operatorname{supp} D = \{p\}$. Prove that there exists a holomorphic section s of $E \otimes [D]$ on $X \setminus \operatorname{supp} D$ such that for each point $p \in \operatorname{supp} D$, $s - t_p$ extends to a holomorphic section u_p of $E \otimes [D]$ on U_p with $\operatorname{ord}_p u_p \geq D(p)$.

4.2 Positive Curvature on a Compact Riemann Surface

According to Theorem 3.11.2, every holomorphic line bundle E on an open Riemann surface is positive (i.e., E admits a positive-curvature Hermitian metric). This is not true in general for a holomorphic line bundle E on a *compact* Riemann surface. It will be shown in Sect. 4.3 (see Theorem 4.3.1) that in fact, a holomorphic line bundle E on a compact Riemann surface is positive if and only if E is the line bundle associated to a divisor of *positive degree*. For now, we consider the following weaker fact:

Theorem 4.2.1 *Let X be a compact Riemann surface. If $E = [D]$ is the holomorphic line bundle associated to a nontrivial effective divisor D on X (i.e., E is a holomorphic line bundle that admits a nontrivial global holomorphic section with at least one zero), then E is positive.*

For the proof, it will be convenient to have the following construction of a nonnegative-curvature Hermitian metric in the holomorphic line bundle associated to a divisor given by a single point:

Lemma 4.2.2 *Let X be a Riemann surface, let $p \in X$, let D be the divisor given by $D(p) = 1$ and $D(q) = 0$ for each point $q \in X \setminus \{p\}$ (i.e., $D = 1 \cdot p$), let s be a holomorphic section of the associated holomorphic line bundle $E = [D]$ on X with $\operatorname{div}(s) = D$, and let U be a neighborhood of p. Then there exists a Hermitian metric h in E on X such that*

(i) *We have $|s|_h^2 = 1$ and $i\Theta_h = 0$ at each point in $X \setminus U$, and*
(ii) *We have $i\Theta_h \geq 0$ on X and $i\Theta_h > 0$ at p.*

Proof Clearly, we may assume that there is a holomorphic coordinate z on U with $z(p) = 0$. Applying Lemma 2.10.3, we get a constant $b > 0$, a nonnegative $\mathcal{C}^\infty$ subharmonic φ on the set $Y \equiv X \setminus \{p\}$, and a relatively compact neighborhood V of p in U such that $\varphi \equiv 0$ on a neighborhood of $X \setminus U$ and $\varphi = |z|^2 - \log |z|^2 - b$ on V. We now get a Hermitian metric h in X by requiring that $|s|_h^2 = e^{-\varphi}$. More precisely, for any $\xi \in E\restriction_Y$, we set $|\xi|_h^2 = |\xi/s|^2 e^{-\varphi}$. We may extend s/z to a nonvanishing holomorphic section t of E on U, and for any $\xi \in E\restriction_V$, we set $|\xi|_h^2 = |\xi/t|^2 e^{-|z|^2+b}$. It is now easy to see that h is well defined and that h has the properties (i) and (ii). □

Proof of Theorem 4.2.1 Suppose $E = [D]$, where $D = \sum_{j=1}^m \nu_j p_j$ is a nontrivial effective divisor with $\nu_j > 0$ for each j. For each $j = 1, \dots, m$, Lemma 4.2.2 provides a Hermitian metric h_j in $[p_j]$ with $i\Theta_{h_j} \geq 0$ on X and $i\Theta_{h_j} > 0$ on some neighborhood V_j of p_j in X. The Hermitian metric $k \equiv h_1^{\nu_1} \otimes \cdots \otimes h_m^{\nu_m}$ in E then satisfies $i\Theta_k \geq 0$ on X and $i\Theta_k > 0$ on the neighborhood $V \equiv V_1 \cup \cdots \cup V_m$ of $\operatorname{supp} D$. Applying Corollary 2.14.2 in $X \setminus \operatorname{supp} D$ and cutting off, we get a $\mathcal{C}^\infty$ function α on X such that $i\Theta_\alpha > 0$ on a neighborhood of $X \setminus V$. Thus, for $\epsilon > 0$ sufficiently small, the Hermitian metric $h \equiv e^{-\epsilon\alpha} k$ satisfies $i\Theta_h = \epsilon i\Theta_\alpha + i\Theta_k > 0$. □

Remark Since the Riemann surface X in the above theorem is compact, one gets second countability without appealing to Radó's theorem. Therefore, although a fact from Sect. 2.14 (Corollary 2.14.2) was applied in the above proof, there was no real dependence on Radó's theorem and therefore almost no dependence on Sects. 2.6, 2.7, 2.9–2.12, 3.6, and 3.8–3.10.

It follows from Theorem 4.2.1 that every compact Riemann surface admits a positive holomorphic line bundle. Moreover, we have the following consequence, which, when combined with Corollary 3.11.7, implies that holomorphic line bundles and divisors on Riemann surfaces are actually equivalent:

Theorem 4.2.3 *Any holomorphic line bundle E on a compact Riemann surface X is equal to the line bundle associated to some nontrivial divisor D. In fact, given any finite set $S \subset X$ and any choice of integers $\{m_p\}_{p\in S}$, there exists a meromorphic section s of E such that* $\operatorname{ord}_p s = m_p$ *for each point $p \in S$.*

Proof Suppose $S \subset X$ is a nonempty finite set and $m_p \in \mathbb{Z}$ for each point $p \in S$. Fixing a positive holomorphic line bundle (provided by Theorem 4.2.1) and applying Corollary 4.1.5, we get a holomorphic line bundle F on X such that $E \otimes F$ and F admit global holomorphic sections u and v, respectively, with $\operatorname{ord}_p u = \max(m_p, 0)$ and $\operatorname{ord}_p v = \max(-m_p, 0)$ for each point $p \in S$. The quotient $s \equiv u/v$ is then a nontrivial meromorphic section of E, $\operatorname{ord}_p s = m_p$ for each point $p \in S$, and $E = [D]$ for the nontrivial divisor $D \equiv \operatorname{div}(s)$. □

Exercises for Sect. 4.2

4.2.1 Let X be a compact Riemann surface, let E be a holomorphic line bundle on X, let $p \in X$, and let $m \in \mathbb{Z}_{\geq 2}$. Prove that if E admits a Hermitian metric with $i\Theta_h \geq 0$, then there exists a meromorphic 1-form θ with values in E on X such that θ is holomorphic on $X \setminus \{p\}$ and θ has a pole of order m at p (cf. Theorem 2.10.1).

Hint. Apply Corollary 4.1.2 to the effective divisor $D = m \cdot p$, the positive holomorphic line bundle $E \otimes [p]$, and a local holomorphic section t_p of $K_X \otimes E \otimes [(m+1)p]$ with a zero of order 1 at p. Divide by a suitable power of a section of $[p]$ with divisor p.

4.3 Equivalence of Positive Curvature and Positive Degree

In this section we consider the more complete characterization of positivity of holomorphic line bundles on compact Riemann surfaces alluded to in Sects. 3.11 and 4.2.

Theorem 4.3.1 *Let D be a divisor on a compact Riemann surface X, and let $E = [D]$ be the associated holomorphic line bundle. Then E is positive if and only if* $\deg D > 0$.

Proof If E is positive, then by Corollary 4.1.5, for $r \gg 0$, E^r admits a nontrivial holomorphic section s that has at least one zero. Setting $D' = \operatorname{div}(s)$, we get

$$\deg D = \frac{1}{r}\deg(rD) = \frac{1}{r}\deg D' > 0.$$

Conversely, suppose $\deg D > 0$. To show that E is positive, it suffices to show that $E^r = [rD]$ admits a nontrivial holomorphic section for some positive integer r. For the divisor of the section will be nontrivial (since $\deg E = r\deg D > 0$), and hence it will follow from Theorem 4.2.1 that E^r admits a positive-curvature Hermitian metric h_r. The Hermitian metric h in E determined by $|\xi|_h^2 = |\xi^r|_{h_r}^{2/r}$ for each $\xi \in E$ (see Exercise 4.3.1) will then satisfy $i\Theta_h = r^{-1} \cdot i\Theta_{h_r} > 0$.

If $D \geq 0$, then clearly, E admits a nontrivial holomorphic section. Assuming now that D is not effective, we have $D = D_+ - D_-$, where $D_\pm$ are (unique) nontrivial (i.e., not everywhere zero) effective divisors with disjoint supports, and $E = E_+ \otimes E_-$ with $E_\pm \equiv [D_\pm]$. By Theorem 4.2.1 and Corollary 4.1.3, for m a sufficiently large positive integer and n an arbitrary positive integer, we have

$$\dim \Gamma(X, \mathcal{O}(E_+^{m+n})) = \dim \Gamma(X, \mathcal{O}(E_+^m \otimes [nD_+])) \geq \deg(nD_+) = n \deg D_+.$$

For any effective divisor G, multiplication by a holomorphic section of $[G]$ with divisor G yields the exact sequence

$$\begin{aligned} 0 &\to \Gamma(X, \mathcal{O}([(m+n)D_+ - G])) \to \Gamma(X, \mathcal{O}([(m+n)D_+])) \\ &\to \Gamma(X, \mathcal{O}([(m+n)D_+])/\mathcal{O}_{-G}([(m+n)D_+])) \cong \mathbb{C}^{\deg G}. \end{aligned}$$

Setting $G = (m+n)D_-$, we get

$$\begin{aligned}\dim \Gamma(X, \mathcal{O}(E^{m+n})) &\geq \dim \Gamma(X, \mathcal{O}(E_+^{m+n})) - (m+n)\deg D_- \\ &\geq n \deg D_+ - (m+n)\deg D_- \\ &= n \deg D - m \deg D_-.\end{aligned}$$

Fixing m and choosing $n \gg 0$, we get the claim. □

Remark One may also prove the theorem by applying the Riemann–Roch formula, which will be considered in Sect. 4.5 (see Exercise 4.5.1).

Corollary 4.3.2 *If D is a divisor of positive degree on a compact Riemann surface X and $E = [D]$, then E^r admits a nontrivial holomorphic section for $r \gg 0$ and no nontrivial holomorphic sections for $r < 0$.*

Proof By Theorem 4.3.1, $E > 0$, and hence E^r admits a nontrivial holomorphic section s for $r \gg 0$ (for example, by Corollary 4.1.5). For $r < 0$, we have $\deg E^r = r \deg D < 0$, so $\Gamma(X, \mathcal{O}(E^r)) = 0$. □

Observe also that facts considered earlier concerning positive holomorphic line bundles on compact Riemann surfaces have equivalent forms in terms of divisors of positive degree. For example, Theorem 4.3.1 and Corollary 3.10.5 together give the following vanishing theorem:

Corollary 4.3.3 *If D is a divisor of positive degree on a compact Riemann surface X, then $H^1_{\mathrm{Dol}}(X, K_X \otimes [D]) = 0$. Moreover, for any holomorphic line bundle F on X, we have $H^1_{\mathrm{Dol}}(X, F \otimes [rD]) = 0$ for $r \gg 0$.*

Some other examples, which follow from the results of Sect. 4.1, are summarized in the following corollary, the proof of which is left to the reader (see Exercise 4.3.2):

Corollary 4.3.4 *Let D_0 be a divisor of positive degree on a compact Riemann surface X.*

(a) *For every effective divisor D on X, the associated linear map*

$$\Gamma(X, \mathcal{O}(K_X \otimes [D_0 + D])) \longrightarrow \Gamma(X, \mathcal{Q}_D(K_X \otimes [D_0 + D]))$$

is surjective (in particular, $\dim \Gamma(X, \mathcal{O}(K_X \otimes [D_0 + D])) \geq \deg D$).

(b) *If F is any holomorphic line bundle on X, then for every sufficiently large positive integer r and every effective divisor D on X (r is independent of the choice of D), the linear map*

$$\Gamma(X, \mathcal{O}(F \otimes [rD_0 + D])) \to \Gamma(X, \mathcal{Q}_D(F \otimes [rD_0 + D]))$$

is surjective (in particular, $\dim \Gamma(X, \mathcal{O}(F \otimes [rD_0 + D])) \geq \deg D$).

(c) *If F is any holomorphic line bundle on X and D is any effective divisor on X, then for every sufficiently large positive integer r (depending on the choice of D_0, F, and D), the linear map*

$$\Gamma(X, \mathcal{O}(F \otimes [rD_0])) \longrightarrow \Gamma(X, \mathcal{Q}_D(F \otimes [rD_0]))$$

is surjective (in particular, $\dim \Gamma(X, \mathcal{O}(F \otimes [rD_0])) \geq \deg D$*). In fact, if $r \gg 0$ and t_p is a holomorphic section of $F \otimes [rD_0]$ on a neighborhood of p in X for each point $p \in \operatorname{supp} D$, then there exists a holomorphic section s of $F \otimes [rD_0]$ on X such that* $\operatorname{ord}_p(s - t_p) = D(p)$ *for each point* $p \in \operatorname{supp} D$.

Exercises for Sect. 4.3

4.3.1 As in the proof of Theorem 4.3.1, let E be a holomorphic line bundle, let r be a positive integer, and let h_r be a Hermitian metric in the r-fold tensor power E^r. For each element $\xi \in E$, let $\xi^r = \xi \otimes \cdots \otimes \xi \in E^r$, and let $|\xi|_h^2 = |\xi^r|_{h_r}^{2/r}$. Verify that this determines a Hermitian metric h in E with $\Theta_h = (1/r)\Theta_{h_r}$.
4.3.2 Prove Corollary 4.3.4.
4.3.3 Prove that if E is a holomorphic line bundle on a compact Riemann surface X and $\deg E - \deg K_X > 0$, then $H^1_{\mathrm{Dol}}(X, E) = 0$.

4.4 A Finiteness Theorem

The vector space of holomorphic sections of a holomorphic line bundle on a *compact* Riemann surface is always finite-dimensional. In fact, we have the following:

Theorem 4.4.1 (Finiteness theorem) *If E is a holomorphic line bundle on a compact Riemann surface X, then*

$$\dim \Gamma(X, \mathcal{O}(E)) = \dim H^0_{\mathrm{Dol}}(X, E) \leq \max(\deg E + 1, 0) < \infty$$

and

$$\dim H^1_{\mathrm{Dol}}(X, E) < \infty.$$

Proof Fix a point $p \in X$ and let G be the divisor with $\operatorname{supp} G = \{p\}$ and $G(p) = 1$ (i.e., $G = 1 \cdot p$). For each positive integer r, we have the linear map

$$\Gamma(X, \mathcal{O}(E)) \to \Gamma(X, \mathcal{Q}_{(r-1)G}(E)) \cong \mathbb{C}^{r-1} \quad (= \{0\} \quad \text{if } r = 1).$$

If $\dim \Gamma(X, \mathcal{O}(E)) \geq r$, then the kernel contains a nontrivial section. On the other hand, such a section determines an effective divisor D such that $E = [D]$ and

$$\deg E = \deg D \geq r - 1.$$

Thus we get the desired bound on $\dim \Gamma(X, \mathcal{O}(E))$.

Multiplication by a holomorphic section of $[rG]$ with divisor rG gives the exact sequence of sheaves

$$0 \to \mathcal{O}(E) \to \mathcal{O}(E \otimes [rG]) \to \mathcal{Q}_{rG}(E \otimes [rG]) \to 0$$

and the corresponding Dolbeault exact sequence

$$\begin{aligned} 0 &\to \Gamma(X, \mathcal{O}(E)) \to \Gamma(X, \mathcal{O}(E \otimes [rG])) \to \Gamma(X, \mathcal{Q}_{rG}(E \otimes [rG])) \\ &\to H^1_{\mathrm{Dol}}(X, E) \to H^1_{\mathrm{Dol}}(X, E \otimes [rG]) \to 0. \end{aligned}$$

By Theorem 4.2.1 and Corollary 3.10.5 (or by Corollary 4.3.3), for $r \gg 0$, we have $H^1_{\mathrm{Dol}}(X, E \otimes [rG]) = 0$, and hence the map

$$\mathbb{C}^r \cong \Gamma(X, \mathcal{Q}_{rG}(E \otimes [rG])) \to H^1_{\mathrm{Dol}}(X, E)$$

is surjective. Thus $\dim H^1_{\mathrm{Dol}}(X, E) < \infty$. □

Definition 4.4.2 For a compact Riemann surface X, the *genus* of X is the nonnegative integer

$$\mathrm{genus}(X) \equiv \dim H^1_{\mathrm{Dol}}(X, 1_X) = \dim H^1_{\mathrm{Dol}}(X).$$

Exercises for Sect. 4.4

4.4.1 Prove that if E is a holomorphic line bundle on a compact Riemann surface X, then $\dim H^1_{\mathrm{Dol}}(X, E) \le \max(\deg K_X - \deg E + 1, 0)$ (cf. Exercise 4.3.3).

4.5 The Riemann–Roch Formula

In this section, we consider the following formula for the dimension of the space of holomorphic sections of a holomorphic line bundle on a compact Riemann surface (this formula is much more precise than the estimates for the dimension considered in previous sections):

Theorem 4.5.1 (Riemann–Roch formula) *If E is a holomorphic line bundle on a compact Riemann surface X, then*

$$\dim \Gamma(X, \mathcal{O}(E)) - \dim H^1_{\mathrm{Dol}}(X, E) = 1 - \mathrm{genus}(X) + \deg E.$$

Remark For a divisor D on a compact Riemann surface X, we set

$$h^q(D) \equiv \dim H^q_{\mathrm{Dol}}(X, [D]) \quad \text{for } q = 0, 1.$$

Setting $g \equiv \mathrm{genus}(X)$ and $d \equiv \deg D$, we get the following equivalent form for the Riemann–Roch formula, which is easier to remember:

$$h^0(D) - h^1(D) = 1 - g + d.$$

The proof given here, which is standard, is similar to that in [For]. The main point is the following induction step:

Lemma 4.5.2 *Suppose A and B are divisors on a compact Riemann surface X such that $B = A + p$ for some point $p \in X$ and such that the Riemann–Roch formula holds for A or for B. Then the Riemann–Roch formula holds both for A and for B.*

Proof Multiplication by a holomorphic section of the line bundle $[p]$ with divisor $p = B - A$ yields the Dolbeault exact sequence

$$\begin{aligned} 0 &\to \Gamma(X, \mathcal{O}([A])) \to \Gamma(X, \mathcal{O}([B])) \to \Gamma(X, \mathcal{Q}_{B-A}([B])) \\ &\to H^1_{\mathrm{Dol}}(X, [A]) \to H^1_{\mathrm{Dol}}(X, [B]) \to 0. \end{aligned}$$

Setting

$$\begin{aligned} \mathcal{V} &\equiv \mathrm{im}[\Gamma(X, \mathcal{O}([B])) \to \Gamma(X, \mathcal{Q}_{B-A}([B]))] \\ &= \ker[\Gamma(X, \mathcal{Q}_{B-A}([B])) \to H^1_{\mathrm{Dol}}(X, [A])] \subset \Gamma(X, \mathcal{Q}_{B-A}([B])) \cong \mathbb{C} \end{aligned}$$

and

$$\begin{aligned} \mathcal{W} &\equiv \mathrm{im}[\Gamma(X, \mathcal{Q}_{B-A}([B])) \to H^1_{\mathrm{Dol}}(X, [A])] \\ &= \ker[H^1_{\mathrm{Dol}}(X, [A]) \to H^1_{\mathrm{Dol}}(X, [B])], \end{aligned}$$

we get $\dim \mathcal{V} + \dim \mathcal{W} = \dim \Gamma(X, \mathcal{Q}_{B-A}(B)) = 1$, and therefore

$$\begin{aligned} h^0(B) &- h^1(B) - \deg B \\ &= (h^0(A) + \dim \mathcal{V}) - (h^1(A) - \dim \mathcal{W}) - \deg A - 1 \\ &= h^0(A) - h^1(A) - \deg A. \end{aligned}$$

The lemma now follows. □

Proof of Theorem 4.5.1 For $D = 0$, we have

$$h^0(D) - h^1(D) = 1 - g = 1 - g + d.$$

For an arbitrary divisor D on X, we have $D = B - A$ for effective divisors

$$A = p_1 + \cdots + p_m \quad \text{and} \quad B = q_1 + \cdots + q_n.$$

Setting $B_0 = 0$ and $B_\nu = q_1 + \cdots + q_\nu$ for $\nu = 1, \ldots, n$, and applying Lemma 4.5.2 inductively on ν, we get the Riemann–Roch formula for the divisor $B = B_n$. Setting $D_0 = B$ and $D_\mu = B - p_1 - \cdots - p_\mu$ for $\mu = 1, \ldots, m$, and applying Lemma 4.5.2 inductively on μ, we get the Riemann–Roch formula for the divisor $D = D_m$. □

Exercises for Sect. 4.5

4.5.1 Using the Riemann–Roch formula in place of Theorem 4.3.1, give a different proof of Corollary 4.3.2 (i.e., a different proof that for any divisor D of positive degree on a compact Riemann surface X, $[rD]$ admits a nontrivial holomorphic section for $r \gg 0$). Use this to then give a different proof of Theorem 4.3.1.

4.6 Statement of the Serre Duality Theorem

Throughout this section, E denotes a holomorphic line bundle on a Riemann surface X. Recall that we have the natural identifications $E \otimes E^* \cong E^* \otimes E \cong 1_X$ determined by $s \otimes \xi \leftrightarrow \xi \otimes s \mapsto \xi \cdot s = \xi(s)$ for $\xi \in E^*$ and $s \in E$. Combining this with the wedge product, one gets a natural pointwise bilinear pairing $\mathfrak{s}_E(\cdot, \cdot)$ of forms with values in E and forms with values in E^* that is related to the pointwise pairing $\{\cdot, \cdot\}_h$ of forms with values in E associated to a Hermitian metric h as described in Definition 3.6.1 ($\{\cdot, \cdot\}_h$ is not, however, bilinear; it is linear in the first entry and conjugate linear in the second). One then also gets an associated integration pairing $\mathfrak{S}_E(\cdot, \cdot)$ that is related to the L^2 pairing. The precise definitions are as follows:

Definition 4.6.1 Let $m, n \in \mathbb{Z}_{\geq 0}$.

(a) For each point $x \in X$, the bilinear pairing

$$\mathfrak{s}_E(\cdot, \cdot)\colon [\Lambda^m(T^*X)_{\mathbb{C}} \otimes E]_x \times [\Lambda^n(T^*X)_{\mathbb{C}} \otimes E^*]_x \to [\Lambda^{m+n}(T^*X)_{\mathbb{C}}]_x$$

is given by

$$\mathfrak{s}_E(\alpha, \beta) \equiv \alpha_0 \wedge \beta_0 \cdot s \cdot \xi = \alpha_0 \wedge \beta_0 \cdot \xi(s)$$

for all $\alpha = \alpha_0 \otimes s \in [\Lambda^m(T^*X)_{\mathbb{C}} \otimes E]_x$ and $\beta = \beta_0 \otimes \xi \in [\Lambda^n(T^*X)_{\mathbb{C}} \otimes E^*]_x$ with $s \in E_x$ and $\xi \in E_x^*$.

(b) If $m \in \{0, 1, 2\}$ and α and β are measurable differential forms of degree m and $2 - m$, respectively, with values in E and E^*, respectively, on X, then we define

$$\mathfrak{S}_E(\alpha, \beta) \equiv \int_X \mathfrak{s}_E(\alpha, \beta),$$

provided the above integral is defined.

Remarks 1. It is easy to see that $\mathfrak{s}_E(\cdot, \cdot)$ is well defined. In fact, under the identification of $E \otimes E^* \cong 1_X$, we have $\mathfrak{s}_E(\alpha, \beta) = \alpha \wedge \beta$ as in Definition 3.1.14 (see Exercise 4.6.1).

2. The pairings $\mathfrak{s}_E(\cdot, \cdot)$ and $\mathfrak{S}_E(\cdot, \cdot)$ are *bilinear*, unlike the pairings $\{\cdot, \cdot\}_h$ and $\langle \cdot, \cdot \rangle_{L^2}$, which are linear in the first entry, but conjugate linear in the second.

3. It is easy to see that $\mathfrak{s}_E(\alpha,\beta)=(-1)^{mn}\mathfrak{s}_{E^*}(\beta,\alpha)$ and $\mathfrak{S}_E(\alpha,\beta)=(-1)^{mn}\cdot\mathfrak{S}_{E^*}(\beta,\alpha)$ for all (suitable) α and β of degree m and n, respectively. Furthermore, if α is of type (p,q) and β is of type (r,s), then $\mathfrak{s}_E(\alpha,\beta)$ is of type $(p+r,q+s)$ (the value being 0 if $p+r>1$ or $q+s>1$).

4. Suppose α and β are differential forms with values in E and E^*, respectively. It is easy to verify that if α and β are continuous ($\mathcal{C}^k$, measurable), then $\mathfrak{s}_E(\alpha,\beta)$ is continuous (respectively, $\mathcal{C}^k$, measurable).

Lemma 4.6.2 *For $\mathcal{C}^\infty$ differential forms α and β with values in E and E^*, respectively, on X:*

(a) *We have $\bar\partial(\mathfrak{s}_E(\alpha,\beta))=\mathfrak{s}_E(\bar\partial\alpha,\beta)+(-1)^{\deg\alpha}\mathfrak{s}_E(\alpha,\bar\partial\beta)$.*
(b) *If α is of type $(p,0)$ and β is of type $(1-p,0)$ with $p\in\{0,1\}$, and α or β has compact support, then $\mathfrak{S}_E(\bar\partial\alpha,\beta)+(-1)^p\mathfrak{S}_E(\alpha,\bar\partial\beta)=0$. In particular, if α is a $\mathcal{C}^\infty$ section of E with compact support and β is a holomorphic 1-form with values in E^* (i.e., $\beta\in\Gamma(X,\mathcal{O}(K_X\otimes E^*))$ under the usual identification given by Proposition 3.10.1), then $\mathfrak{S}_E(\bar\partial\alpha,\beta)=0$.*

Proof The local representation of $\mathfrak{s}_E(\cdot,\cdot)$ gives Part (a) (see Exercise 4.6.2). For the proof of (b), observe that $\mathfrak{s}_E(\alpha,\beta)$ is a form of type $(1,0)$ and therefore $d(\mathfrak{s}_E(\alpha,\beta))=\bar\partial(\mathfrak{s}_E(\alpha,\beta))$. Thus, integrating the expressions in (a) and applying Stokes' theorem, we get (b). □

The above lemma allows us to make the following definition:

Definition 4.6.3 For X a compact Riemann surface, the *Serre pairing*

$$\mathfrak{S}_E(\cdot,\cdot)\colon H^1_{\mathrm{Dol}}(X,E)\times\Gamma(X,\mathcal{O}(K_X\otimes E^*))\to\mathbb{C}$$

is given by $\mathfrak{S}_E([\theta]_{\mathrm{Dol}},s)\equiv\mathfrak{S}_E(\theta,s)$ for each form $\theta\in\mathcal{E}^{0,1}(E)(X)$ and each section $s\in\Gamma(X,\mathcal{O}(K_X\otimes E^*))$. The *Serre mapping* is the linear mapping

$$\iota^E_{\mathfrak{S}}\colon\Gamma(X,\mathcal{O}(K_X\otimes E^*))\to[H^1_{\mathrm{Dol}}(X,E)]^*$$

given by $s\mapsto\mathfrak{S}_E(\cdot,s)$.

Remarks 1. Lemma 4.6.2 implies that the Serre pairing is well defined.

2. In a slight abuse of notation, we denote both the general integration pairing in Definition 4.6.1 and the Serre pairing by $\mathfrak{S}_E(\cdot,\cdot)$.

3. Here, we are identifying each global holomorphic section $s\in\Gamma(X,\mathcal{O}(K_X\otimes E^*))$ of $K_X\otimes E^*$ with the corresponding holomorphic 1-form with values in E^*.

4. If $\alpha=a\,d\bar z\cdot t$ and $\beta=b\,dz\cdot\xi$ for a local holomorphic coordinate z and for local holomorphic sections t and ξ of E and E^*, respectively, then

$$\mathfrak{s}_E(\alpha,\beta)=ab\cdot d\bar z\wedge dz\cdot\xi(t)=-ab\cdot dz\wedge d\bar z\cdot\xi(t).$$

One goal of the remainder of this chapter is the following:

Theorem 4.6.4 (Serre duality theorem) *If E is a holomorphic line bundle on a compact Riemann surface X, then the Serre mapping $\iota^E_{\mathfrak{S}} : s \mapsto \mathfrak{S}_E(\cdot, s)$ gives an isomorphism*

$$\iota^E_{\mathfrak{S}} : \Gamma(X, \mathcal{O}(K_X \otimes E^*)) \xrightarrow{\cong} [H^1_{\mathrm{Dol}}(X, E)]^*.$$

Remark Equivalently, we have an isomorphism

$$\iota^{K_X \otimes E^*}_{\mathfrak{S}} : \Gamma(X, \mathcal{O}(E)) = \Gamma(X, \mathcal{O}(K_X \otimes (K_X \otimes E^*)^*)) \xrightarrow{\cong} [H^1_{\mathrm{Dol}}(X, K_X \otimes E^*)]^*.$$

Letting K and D be divisors on X with $K_X = [K]$ and $E = [D]$ (i.e., K is the divisor of some meromorphic 1-form η and we identify (K_X, η) with $([K], s)$, where s is the associated defining section for K, and the analogous identification holds for E and D), we also get the equivalent forms

$$\iota^{[D]}_{\mathfrak{S}} : \Gamma(X, \mathcal{O}([K - D])) \xrightarrow{\cong} [H^1_{\mathrm{Dol}}(X, [D])]^*,$$
$$\iota^{[K-D]}_{\mathfrak{S}} : \Gamma(X, \mathcal{O}([D])) \xrightarrow{\cong} [H^1_{\mathrm{Dol}}(X, [K - D])]^*.$$

Corollary 4.6.5 *For X compact, the Serre map $\iota^{1_X}_{\mathfrak{S}} : \Omega(X) \to [H^1_{\mathrm{Dol}}(X)]^*$, which is given by*

$$\iota^{1_X}_{\mathfrak{S}}(\alpha)([\theta]_{\mathrm{Dol}}) = \int_X \theta \wedge \alpha$$
$$\forall \alpha \in \Omega(X) = \Gamma(X, \mathcal{O}(K_X)),\ [\theta]_{\mathrm{Dol}} \in H^1_{\mathrm{Dol}}(X),$$

is an isomorphism. In particular, $\mathrm{genus}(X) = \dim \Omega(X)$.

The Serre duality theorem and the Riemann–Roch formula have many applications. We consider a few of the immediate applications here. We first observe that Serre duality gives the following equivalent version of the Riemann–Roch formula:

Theorem 4.6.6 *For any holomorphic line bundle E on a compact Riemann surface X, we have*

$$\dim \Gamma(X, \mathcal{O}(E)) - \dim \Gamma(X, \mathcal{O}(K_X \otimes E^*)) = 1 - \mathrm{genus}(X) + \deg E.$$

Equivalently, for any divisor D of degree d on a compact Riemann surface X of genus g and for any divisor K with $[K] = K_X$, we have

$$h^0(D) - h^0(K - D) = 1 - g + d.$$

Corollary 4.6.7 *For any compact Riemann surface X of genus g, we have*

$$\deg K_X = 2g - 2.$$

Proof The above Riemann–Roch formula (applied to $E = K_X$) and Corollary 4.6.5 together imply that

$$g = \dim \Gamma(X, \mathcal{O}(K_X)) = \dim \Gamma(X, \mathcal{O}) + 1 - g + \deg K_X = 1 + 1 - g + \deg K_X.$$

The claim now follows. □

The following application will play a role in the proof of the Abel–Jacobi embedding theorem (Theorem 5.22.2):

Theorem 4.6.8 *Up to biholomorphism, the only compact Riemann surface of genus* 0 *is* $\mathbb{P}^1$. *In fact, if X is a compact Riemann surface that is not biholomorphic to* $\mathbb{P}^1$, *then for every point $p \in X$, there is a holomorphic* 1*-form θ on X with* $\theta_p \neq 0$.

Proof The verification that genus$(\mathbb{P}^1) = 0$ (i.e., that $\mathbb{P}^1$ has no nontrivial holomorphic 1-forms) is left to the reader (cf. Exercise 2.5.4). Suppose now that X is a compact Riemann surface of genus g, and $p \in X$ is a point at which every holomorphic 1-form on X vanishes. Fixing a divisor K with $[K] = K_X$, letting $D = 1 \cdot p$, and fixing a section $s \in \Gamma(X, \mathcal{O}([D]))$ with $\operatorname{div}(s) = D$, we get an isomorphism $\Gamma(X, \mathcal{O}([K])) \xrightarrow{\cong} \Gamma(X, \mathcal{O}([K - D]))$ given by $\theta \mapsto \theta/s$. The Riemann–Roch formula (in the form of Theorem 4.6.6) then gives

$$h^0(D) - g = h^0(D) - h^0(K) = h^0(D) - h^0(K - D) = 1 - g + \deg D = 1 - g + 1,$$

and hence $h^0(D) = 2$. Thus there exists a section $t \in \Gamma(X, \mathcal{O}([D])) \setminus \mathbb{C} \cdot s$, and $f \equiv t/s \colon X \to \mathbb{P}^1$ is a nonconstant meromorphic function that is holomorphic except for a simple pole at p ($t(p) \neq 0$, since otherwise, f would be a nonconstant holomorphic function on X). Therefore, by Proposition 2.5.7, f is a biholomorphism of X onto $\mathbb{P}^1$, and the claim follows. □

Remarks 1. The canonical line bundle of a compact Riemann surface X of genus 1 is trivial (see Exercise 4.6.3).

2. It follows from the above theorem that if $\theta_1, \dots, \theta_g$ is a basis for $\Omega(X)$ for a compact Riemann surface X of genus $g > 0$, then we get a Kähler form

$$\omega \equiv \sum_{j=1}^{g} i\theta_j \wedge \bar{\theta}_j.$$

We recall that if a holomorphic line bundle E is the line bundle associated to a nontrivial effective divisor, then E is positive. However, as the following example shows, the converse is false (see [Ns4]):

Observation 4.6.9 *Any compact Riemann surface X of genus $g > 1$ admits a positive holomorphic line bundle with no nontrivial holomorphic sections*. To see this, we first observe that there exists an effective divisor D such that $\deg D = g$ and $h^0(D) = 1$. For there exists a nontrivial holomorphic 1-form on X and therefore a point $p_1 \in X$ at which the form does not vanish. Thus the vector space

$$\mathcal{V}_1 \equiv \{\theta \in \Gamma(X, \mathcal{O}(K_X)) \mid \theta_{p_1} = 0\}$$

is of dimension $g - 1 > 0$. Given points $p_1, \dots, p_{m-1} \in X$ and subspaces

$$\Gamma(X, \mathcal{O}(K_X)) = \mathcal{V}_0 \supset \mathcal{V}_1 \supset \mathcal{V}_2 \supset \cdots \supset \mathcal{V}_{m-1},$$

where $1 < m \leq g$ and $\mathcal{V}_j = \{\theta \in \Gamma(X, \mathcal{O}(K_X)) \mid \theta_{p_1} = \cdots = \theta_{p_j} = 0\}$ and $\dim \mathcal{V}_j = g - j$ for $j = 1, \dots, m-1$, we may choose a nontrivial element of $\mathcal{V}_{m-1}$ and a point $p_m \in X \setminus \{p_1, \dots, p_{m-1}\}$ at which this element does not vanish. Setting

$$\mathcal{V}_m \equiv \{\theta \in \Gamma(X, \mathcal{O}(K_X)) \mid \theta_{p_1} = \cdots = \theta_{p_m} = 0\} = \{\theta \in \mathcal{V}_{m-1} \mid \theta_{p_m} = 0\} \subset \mathcal{V}_{m-1},$$

we get $\dim \mathcal{V}_m = g - (m-1) - 1 = g - m$. Thus, by induction, we get distinct points $p_1, \dots, p_g \in X$ at which no nontrivial holomorphic 1-form can simultaneously vanish. We now let K be a divisor with $[K] = K_X$, we set $D = p_1 + \cdots + p_g$, and we fix a section $s \in \Gamma(X, \mathcal{O}([D]))$ with $\operatorname{div}(s) = D$. The Riemann–Roch formula and Serre duality give $h^0(D) = h^0(K - D) + 1$. On the other hand, if $\theta \in \Gamma(X, \mathcal{O}([K - D]))$, then $\theta \cdot s$ is a holomorphic 1-form that vanishes at the points $p_1, \dots, p_g$ and hence $\theta \equiv 0$. Thus $h^0(D) = 1$, as desired.

Fixing a point $p \in X \setminus \{p_1, \dots, p_g\}$ and setting $F = D - p$, we get $\deg F = g - 1 > 0$, so $E \equiv [F]$ is positive by Theorem 4.3.1. However, letting P be the divisor with $P(p) = 1$ and $P(x) = 0$ for $x \neq p$ (i.e., $P = 1 \cdot p$), and choosing a section $t \in \Gamma(X, \mathcal{O}([P]))$ with $\operatorname{div}(t) = P$, we get the injective linear mapping $\Gamma(X, \mathcal{O}(E)) \hookrightarrow \Gamma(X, \mathcal{O}([D]))$ given by multiplication by t. Since $s(p) \neq 0$, s cannot be in the image of this mapping. Therefore $\dim \Gamma(X, \mathcal{O}(E)) < h^0(D) = 1$, and hence E has no nontrivial holomorphic sections.

Remark The above observation is false in genus 1 (see Exercise 4.6.3).

We record here for later use the observation that Lemma 4.6.2 leads one to the following equivalent characterization of the operator $\bar{\partial}_{\text{distr}}$, which is completely intrinsic, unlike Definition 3.8.1, which involves the choice of a local holomorphic coordinate and a nonvanishing local holomorphic section, and unlike the characterization in Proposition 3.8.2, which involves the choice of a Hermitian metric:

Lemma 4.6.10 *Let $p \in \{0, 1\}$ and let α and β be locally integrable differential forms of type $(p, 0)$ and type $(p, 1)$, respectively, with values in E on X. Then $\bar{\partial}_{\text{distr}} \alpha = \beta$ if and only if*

$$\mathfrak{S}_E(\alpha, \bar{\partial}\gamma) + (-1)^p \mathfrak{S}_E(\beta, \gamma) = 0 \quad (\textit{i.e.}, \mathfrak{S}_{E^*}(\bar{\partial}\gamma, \alpha) - (-1)^p \mathfrak{S}_{E^*}(\gamma, \beta) = 0)$$

for every form $\gamma \in \mathcal{D}^{1-p,0}(E^*)(X)$. In particular, α is holomorphic if and only if

$$\mathfrak{S}_E(\alpha, \bar{\partial}\gamma) = 0 \quad (\textit{i.e.}, \mathfrak{S}_{E^*}(\bar{\partial}\gamma, \alpha) = 0) \quad \forall \gamma \in \mathcal{D}^{1-p,0}(E^*)(X).$$

Proof For $p = 1$, suppose $\alpha = a\,dz \otimes s$ and $\beta = b\,dz \wedge d\bar{z} \otimes s$ on U for some local holomorphic coordinate neighborhood (U, z) and some nonvanishing holomorphic section s of E on U. For every function $f \in \mathcal{D}(U)$, we may extend f/s by 0 to a $\mathcal{C}^\infty$ differential form γ of type $(0,0)$ with values in E^* on X and compact support in U. Thus

$$\mathfrak{S}_E(\beta, \gamma) = \int_U bf\,dz \wedge d\bar{z}$$

and

$$\mathfrak{S}_E(\alpha, \bar{\partial}\gamma) = \int_U a \cdot dz \wedge \bar{\partial} f = \int_U a \cdot \frac{\partial f}{\partial \bar{z}} \cdot dz \wedge d\bar{z}.$$

Definition 3.8.1 and the above together now imply that if $\mathfrak{S}_E(\alpha, \bar{\partial}\tau) - \mathfrak{S}_E(\beta, \tau) = 0$ for every form $\tau \in \mathcal{D}^{0,0}(E^*)(X)$, then $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$.

Conversely, suppose $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$ and γ is a $\mathcal{C}^\infty$ compactly supported differential form of type $(0,0)$ with values in E^* on X. We may form a finite collection of $\mathcal{C}^\infty$ compactly supported functions $\{\eta_\nu\}_{\nu=1}^m$ on X such that $\sum \eta_\nu \equiv 1$ on $\operatorname{supp}\gamma$ and such that for each ν, the support of η_ν is contained in some local holomorphic coordinate neighborhood (U_ν, z_ν) on which there is a nonvanishing holomorphic section s_ν of E. Setting $a_\nu \equiv \alpha/(dz_\nu \otimes s_\nu) \in \mathcal{C}^\infty(U_\nu)$, $b_\nu \equiv \beta/(dz_\nu \wedge d\bar{z}_\nu \otimes s_\nu) \in \mathcal{C}^\infty(U_\nu)$, and $f_\nu \equiv \eta_\nu\gamma \cdot s_\nu \in \mathcal{D}(U_\nu)$ for each ν, we get

$$\begin{aligned}\mathfrak{S}_E(\alpha, \bar{\partial}\gamma) &= \sum_\nu \mathfrak{S}_E(\alpha, \bar{\partial}(\eta_\nu\gamma)) = \sum_\nu \int_{U_\nu} a_\nu \cdot \frac{\partial f_\nu}{\partial \bar{z}_\nu} \cdot dz_\nu \wedge d\bar{z}_\nu \\ &= \sum_\nu \int_{U_\nu} b_\nu f_\nu \cdot dz_\nu \wedge d\bar{z}_\nu = \sum_\nu \mathfrak{S}_E(\beta, \eta_\nu\gamma) = \mathfrak{S}_E(\beta, \gamma).\end{aligned}$$

The proof of the case $p = 0$ is left to the reader (see Exercise 4.6.4). □

Remark Recall that we have the natural identification of an E-valued $(1,1)$-form with a $K_X \otimes E$-valued $(0,1)$-form provided by Proposition 3.10.1. In the present context, this identification can lead to some confusion, since the above pairings are preserved only up to sign. To see this, let (U, z) be local a holomorphic coordinate neighborhood, let s be a nonvanishing holomorphic section of E on U, and let $\alpha = dz \otimes s$ and $\beta = d\bar{z} \otimes s^{-1}$. Viewing α as a $(1,0)$-form with values in E and β as a $(0,1)$-form with values in E^*, we get $\mathfrak{s}_E(\alpha, \beta) = dz \wedge d\bar{z}$. Viewing α as a $(0,0)$-form with values in $K_X \otimes E$ and $\beta = -dz \wedge d\bar{z} \otimes ((\partial/\partial z) \otimes s^{-1})$ as a $(1,1)$-form with values in $K_X^* \otimes E^*$, we get $\mathfrak{s}_{K_X \otimes E}(\alpha, \beta) = -dz \wedge d\bar{z} = -\mathfrak{s}_E(\alpha, \beta)$. Fortunately, confusion can be avoided by specifying the reference line bundle (E or $K_X \otimes E$) in the subscript for $\mathfrak{s}(\cdot,\cdot)$ and $\mathfrak{S}(\cdot,\cdot)$. Moreover, for our purposes, the sign is not crucial. The behavior of the pairings with respect to these identifications is summarized in Proposition 4.6.11 below (cf. Proposition 3.10.1), the proof of which is left to the reader (see Exercise 4.6.5). The full proposition is not used in this book, so the reader may wish to omit it.

Proposition 4.6.11 *For $q \in \{0,1\}$ and for any holomorphic line bundle F on X, let $\Phi_q^F : \Lambda^{(1,q)}T^*X \otimes F \to \Lambda^{(0,q)}T^*X \otimes K_X \otimes F$ denote the mapping given by*

$$\Phi_q^F : \alpha = \theta \wedge \gamma \cdot s = (-1)^q \gamma \wedge \theta \cdot s \mapsto (-1)^q \gamma \cdot (\theta \otimes s)$$

for each $x \in X$, $\theta \in (K_X)_x$, $\gamma \in \Lambda^{(0,q)}T_x^*X$, and $s \in E_x$. In particular, we have $\Phi_q^{K_X^* \otimes F} : \Lambda^{(1,q)}T^*X \otimes (K_X^* \otimes F) \to \Lambda^{(0,q)}T^*X \otimes K_X \otimes K_X^* \otimes F = \Lambda^{(0,q)}T^*X \otimes F$.

(a) *For each $q = 0, 1$ and $x \in X$, we have*

$$\mathfrak{s}_E(\alpha, \beta) = (-1)^{1-q} \mathfrak{s}_{K_X \otimes E}\big(\Phi_q^E(\alpha), \big(\Phi_{1-q}^{K_X^* \otimes E^*}\big)^{-1}(\beta)\big)$$

if $\alpha \in (\Lambda^{(1,q)}T^*X \otimes E)_x$ and $\beta \in (\Lambda^{(0,1-q)}T^*X \otimes E^*)_x$; and we have

$$\mathfrak{s}_E(\alpha, \beta) = (-1)^{1-q} \mathfrak{s}_{K_X^* \otimes E}\big(\big(\Phi_q^{K_X^* \otimes E}\big)^{-1}(\alpha), \Phi_{1-q}^{E^*}(\beta)\big)$$

if $\alpha \in (\Lambda^{(0,q)}T^*X \otimes E)_x$ and $\beta \in (\Lambda^{(1,1-q)}T^*X \otimes E^*)_x$.

(b) *If $q \in \{0,1\}$ and α is a measurable differential form of type $(1,q)$ with values in E and β is a measurable differential form of type $(0, 1-q)$ with values in E^* on X, then*

$$\mathfrak{S}_E(\alpha, \beta) = (-1)^{1-q} \mathfrak{S}_{K_X \otimes E}\big(\Phi_q^E(\alpha), \big(\Phi_{1-q}^{K_X^* \otimes E^*}\big)^{-1}(\beta)\big);$$

here the left-hand side is defined if and only if the right-hand side is defined. If $q \in \{0,1\}$ and α is a measurable differential form of type $(0,q)$ with values in E and β is a measurable differential form of type $(1, 1-q)$ with values in E^* on X, then

$$\mathfrak{S}_E(\alpha, \beta) = (-1)^{1-q} \mathfrak{S}_{K_X^* \otimes E}\big(\big(\Phi_q^{K_X^* \otimes E}\big)^{-1}(\alpha), \Phi_{1-q}^{E^*}(\beta)\big);$$

here again, the left-hand side is defined if and only if the right-hand side is defined.

Exercises for Sect. 4.6 Exercises 4.6.3 and 4.6.6–4.6.10 require the Serre duality theorem (and its consequences), so the reader may wish to postpone consideration of these exercises until after consideration of the proof in Sect. 4.8.

4.6.1 Verify that in the notation of Definition 4.6.1, under the identification of $E \otimes E^* \cong 1_X$, we have $\mathfrak{s}_E(\alpha, \beta) = \alpha \wedge \beta$ as in Definition 3.1.14.

4.6.2 Prove part (a) of Lemma 4.6.2.

4.6.3 Let X be a compact Riemann surface of genus g.
  (a) Prove that if $g = 1$, then K_X is trivial and $\dim \Gamma(X, \mathcal{O}(E)) = \deg E$ for every holomorphic line bundle E of positive degree on X.
  (b) Prove that if $g > 1$, then $\dim \Gamma(X, \mathcal{O}(K_X^m)) = (2m-1)(g-1)$ for every $m \in \mathbb{Z}_{>1}$.

4.6.4 Prove Lemma 4.6.10 for $p = 0$.

4.6.5 Prove Proposition 4.6.11.

Some facts concerning *hyperelliptic Riemann surfaces* and *Weierstrass gaps* are developed in Exercises 4.6.6–4.6.10 below (see also Exercises 5.20.3–5.20.6 and 5.20.8 and, for example, [FarK] for further discussion).

4.6.6 A compact Riemann surface is called *hyperelliptic* if it admits a divisor D such that $\deg D = 2$ and $h^0(D) \geq 2$.

(a) Prove that a compact Riemann surface X is hyperelliptic if and only if there exists a meromorphic function on X with exactly two poles (counting multiplicity).

(b) Prove that every compact Riemann surface of genus $g \leq 2$ is hyperelliptic.

4.6.7 Given a point p in a compact Riemann surface X, a positive integer ν is called a *gap* (or a *Weierstrass gap*) at p if there does *not* exist a function $f \in \mathcal{M}(X)$ that is holomorphic on $X \setminus \{p\}$ and that has a pole of order ν at p. A positive integer that is not a gap is called a *nongap* (or a *Weierstrass nongap*) at p. The goal of this exercise is the following:

Theorem (Weierstrass gap theorem) *At any point p in a compact Riemann surface X of genus g,*

(i) *The sum of any two nongaps is a nongap*; *and*

(ii) *There are exactly g gaps*, *and for $g > 0$, the gaps are given by positive integers $\nu_1, \dots, \nu_g$ (the* gap sequence) *satisfying* $1 = \nu_1 < \nu_2 < \cdots < \nu_g < 2g$.

Prove the Weierstrass gap theorem following the outline below.

(a) Prove that the sum of any two nongaps at p is a nongap.

(b) Prove that for $g = 0$, there are no gaps at p.

(c) Let $g > 0$ and let $\nu \in \mathbb{Z}_{>0}$. Fixing a holomorphic section s of $[p]$ with $\operatorname{div}(s) = p$, we get the Dolbeault exact sequence (Sect. 3.4)

$$0 \to \Gamma(X, \mathcal{O}([(\nu-1)p])) \xrightarrow{\alpha} \Gamma(X, \mathcal{O}([\nu p])) \to \Gamma(X, \mathcal{Q}_p([\nu p]))$$
$$\to H^1_{\mathrm{Dol}}(X, [(\nu-1)p]) \to H^1_{\mathrm{Dol}}(X, [\nu p]) \to 0,$$

where α is the map given by $u \mapsto u \otimes s$. Prove that ν is a gap if and only if α is surjective, and conclude from this that

$$h^0(\nu p) = \begin{cases} h^0((\nu-1)p) & \text{if } \nu \text{ is a gap,} \\ h^0((\nu-1)p) + 1 & \text{if } \nu \text{ is nongap.} \end{cases}$$

(d) Let $g > 0$ and let K be a divisor with $[K] = K_X$. Use the Riemann–Roch formula in the form of Theorem 4.6.6 to prove that

$$h^0(K - \nu p + p) = \begin{cases} h^0(K - \nu p) + 1 & \text{if } \nu \text{ is a gap,} \\ h^0(K - \nu p) & \text{if } \nu \text{ is nongap.} \end{cases}$$

(e) Let $g > 0$. Prove that 1 is a gap and that each gap is at most $2g - 1$.

(f) Let $g > 0$. Prove that if ν is a gap, then there exists a holomorphic section t of $[K - (\nu - 1)p]$ with $t(p) \neq 0$. Forming such a section and multiplying by $s^{\nu-1}$ for each gap ν (for s as in (c)), show that one gets linearly independent holomorphic 1-forms $\theta_1, \dots, \theta_r$, where r is the number of gaps. Conclude that $r \leq g$.

(g) Let $g > 0$. Given a nontrivial holomorphic 1-form θ with a zero of order $\nu - 1$ at p ($\theta_p \neq 0$ if $\nu = 1$), show that ν must be a gap. Show also that by subtracting a suitable multiple of one of the holomorphic 1-forms $\theta_1, \dots, \theta_r$ produced in part (f), one gets a holomorphic 1-form that is either trivial or that has a zero of order $\geq \nu$ at p. Using this observation, complete the proof of the theorem.

4.6.8 Let $\alpha_1 < \alpha_2 < \cdots < \alpha_g = 2g$ be the nongaps in $(1, 2g]$ at a point p in a compact Riemann surface of genus $g > 1$ (as in Exercise 4.6.7).

(a) Prove that $\alpha_j + \alpha_{g-j} \geq 2g$ for $j = 1, \dots, g - 1$.

Hint. We have $j + 1$ nongaps $\alpha_{g-j} < \alpha_1 + \alpha_{g-j} < \alpha_2 + \alpha_{g-j} < \cdots < \alpha_j + \alpha_{g-j}$ for each j.

(b) Prove that $\sum_{j=1}^{g} \alpha_j \geq g(g+1)$, and prove that equality holds if and only if $\alpha_j + \alpha_{g-j} = 2g$ for $j = 1, \dots, g - 1$.

(c) Prove that $\sum_{j=1}^{g} \alpha_j = g(g+1)$ if and only if $(\alpha_1, \alpha_2, \dots, \alpha_g) = (2, 4, \dots, 2g)$.

Hint. If $\alpha_j + \alpha_{g-j} = 2g$ for $j = 1, \dots, g - 1$, then in particular, we have $\alpha_1 + \alpha_{g-1} = 2g = \alpha_g$. Hence, if $g > 2$, then the nongap $\alpha_1 + \alpha_{g-2}$ lies in (α_{g-2}, α_g). Apply this observation and induction to get $\alpha_1 + \alpha_{j-1} = \alpha_j$ for $j = 2, \dots, g$.

4.6.9 Let X be a compact Riemann surface of genus $g > 0$.

(a) For each point $p \in X$, the *weight* (or *Weierstrass weight*) at p is the integer $w(p) \equiv \sum_{j=1}^{g} (\nu_j - j)$, where $(\nu_1, \dots, \nu_g)$ (with $1 = \nu_1 < \cdots < \nu_g < 2g$) is the gap sequence (as in Exercise 4.6.7). Prove that for each point $p \in X$, we have $0 \leq w(p) \leq g(g-1)/2$. Prove also that $w(p) = 0$ if and only if the gap sequence is $(1, 2, 3, \dots, g)$ (this gap sequence is called the *generic gap sequence*) and $w(p) = g(g-1)/2$ if and only if the gap sequence is $(1, 3, 5, \dots, 2g - 1)$ (this gap sequence is called the *hyperelliptic gap sequence*, and we then say that p is a *hyperelliptic point*).

Hint. Use Exercise 4.6.8.

(b) Given a basis $\theta = (\theta_1, \dots, \theta_g)$ for $\Gamma(X, \mathcal{O}(K_X))$, the associated *Wronskian* is the section $W(\theta) \in \Gamma(X, \mathcal{O}(K_X^{g(g+1)/2}))$ given by

$$W(\theta) = \begin{vmatrix} f_1 & \cdots & f_g \\ (\partial/\partial z)(f_1) & \cdots & (\partial/\partial z)(f_g) \\ \vdots & \vdots & \vdots \\ (\partial/\partial z)^{g-1}(f_1) & \cdots & (\partial/\partial z)^{g-1}(f_g) \end{vmatrix} \cdot (dz)^{g(g+1)/2}$$

on each local holomorphic coordinate neighborhood (U, z), where $f_j \equiv \theta_j / dz$ for $j = 1, \dots, g$. Prove that the Wronskian $W(\theta)$ is well defined by the above (i.e., $W(\theta)$ does not depend on the choice of the local holomorphic coordinate z). Prove also that if $\theta' = (\theta'_1, \dots, \theta'_g)$ is another basis for $\Gamma(X, \mathcal{O}(K_X))$, then there is a constant $\zeta \in \mathbb{C}^*$ such that $W(\theta') = \zeta \cdot W(\theta)$ on X.

(c) A point $p \in X$ is called a *Weierstrass point* if $w(p) > 0$. Prove that if $\theta = (\theta_1, \dots, \theta_g)$ is a basis for $\Gamma(X, \mathcal{O}(K_X))$, then the set of Weierstrass points in X is equal to the set of zeros of the Wronskian $W(\theta)$ and the order of the zero of $W(\theta)$ at each Weierstrass point p is equal to $w(p)$.

Hint. For $g = 1$, one may prove this by using the triviality of the canonical line bundle (Exercise 4.6.3). Assume that $g > 1$, let $p \in X$, and let $(\nu_1, \dots, \nu_g)$ be the gap sequence at p. Choose the basis $\theta = (\theta_1, \dots, \theta_g)$ as in part (f) of Exercise 4.6.7. Fix a local holomorphic coordinate neighborhood (U, z) with $z(p) = 0$, and replace each θ_j with its product with a suitable constant, so that $f_j \equiv \theta_j / dz = z^{\nu_j - 1}$ + higher-order terms for each $j = 1, \dots, g$. Given a k-tuple of holomorphic functions $h = (h_1, \dots, h_k)$ on U with $k > 1$, set

$$W(h) \equiv \begin{vmatrix} h_1 & \dots & h_k \\ (\partial/\partial z)(h_1) & \dots & (\partial/\partial z)(h_k) \\ \vdots & \vdots & \vdots \\ (\partial/\partial z)^{k-1}(h_1) & \dots & (\partial/\partial z)^{k-1}(h_k) \end{vmatrix} \in \mathcal{O}(U).$$

Show that for arbitrary positive integers $\mu_1 < \dots < \mu_k$, for $h_j = z^{\mu_j - 1}$ for $j = 1, \dots, k$, and for $m = \sum_{j=1}^{k} (\mu_j - j)$, we have

$$W(h) = \prod_{1 \le i < j \le k} (\mu_j - \mu_i) z^m.$$

Conclude from this that for $f = (f_1, \dots, f_g)$ and $h = (z^{\nu_1 - 1}, \dots, z^{\nu_g - 1})$, we have $\mathrm{ord}_p(W(f) - W(h)) > w(p)$, and then that $W(f)$ has a zero of order $w(p)$ at p.

(d) Prove that $\sum_{p \in X} w(p) = (g-1)g(g+1)$ (in particular, every compact Riemann surface of genus $g > 1$ has a Weierstrass point).

(e) Assume that $g > 1$ and let m be the number of Weierstrass points. Prove the following:
  (i) We have $2g + 2 \le m \le (g-1)g(g+1)$.
  (ii) We have $m = 2g + 2$ if and only if X has at least $2g + 2$ hyperelliptic points. In particular, $m = 2g + 2$ if and only if every Weierstrass point in X is a hyperelliptic point.
  (iii) If $m = 2g + 2$, then X is a hyperelliptic Riemann surface (see Exercise 4.6.6 for the definition of a hyperelliptic Riemann surface, and see Exercise 5.20.4 for the converse).

(f) Prove that if $g = 2$, then every Weierstrass point in X is a hyperelliptic point (in particular, this gives a proof of the fact that a compact Riemann surface of genus 2 is hyperelliptic that differs slightly from the proof suggested by the hint in Exercise 4.6.6).

4.6.10 Prove that every nontrivial automorphism of a compact Riemann surface X of genus $g > 1$ has at most $2g + 2$ fixed points.

Hint. Given $\Phi \in \mathrm{Aut}(X) \setminus \{\mathrm{Id}_X\}$, fix a point $p \in X$ that is *not* a fixed point of Φ. Show that there exist a nongap α at p and a function $f \in \mathcal{M}(X)$ such that $1 < \alpha \leq g + 1$, f is holomorphic on $X \setminus \{p\}$, and f has a pole of order α at p. Show that the meromorphic function $f - f \circ \Phi$ has exactly 2α zeros (counting multiplicities).

4.7 Statement of the $\bar{\partial}$-Hodge Decomposition Theorem

Throughout this section, E denotes a holomorphic line bundle on a Riemann surface X. One may relate the pairing $\mathfrak{S}_E(\cdot,\cdot)$ and the L^2 inner product associated to a Hermitian metric h in E by considering the unique operator $\bar{*}^\flat$ satisfying $\mathfrak{S}_E(\alpha, \bar{*}^\flat\beta) = \langle\alpha,\beta\rangle_{L^2}$ for suitable L^2 E-valued 1-forms α and β. To see this, we first observe that a Hermitian metric induces a natural conjugate linear isomorphism of E and E^*.

Definition 4.7.1 For a Hermitian metric h in E with dual metric h^* in E^*, the associated *flat operator* is the $\mathcal{C}^\infty$ conjugate linear isomorphism

$$\flat = \flat_{(E,h)} = \flat_h = \flat_E : E \to E^*$$

given by

$$s \mapsto \flat(s) = s^\flat \equiv h(\cdot, s) = \begin{cases} |s|_h^2 s^{-1} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

The associated *sharp operator* is the ($\mathcal{C}^\infty$ conjugate linear) inverse mapping

$$\# = \#_{(E,h)} = \#_h = \#_E \equiv \flat_{(E,h)}^{-1} : E^* \to E,$$

for which we write $\xi \mapsto \xi^\#$.

Remarks 1. It is easy to verify that $\flat$ and # are $\mathcal{C}^\infty$ conjugate linear isomorphisms (see Exercise 4.7.1); that is, their restrictions to each fiber are conjugate linear isomorphisms, and $s^\flat$ and $\xi^\#$ are $\mathcal{C}^\infty$ sections for all local holomorphic (or $\mathcal{C}^\infty$) sections s and ξ of E and E^*, respectively. It follows that if s is a section of E which is continuous ($\mathcal{C}^k$ with $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, measurable, L^s_{loc} with $s \in [1,\infty]$), then $s^\flat$ is continuous (respectively, $\mathcal{C}^k$, measurable, L^s_{loc}).

2. One can form analogous operators for a general finite-dimensional inner product space (of arbitrary dimension).

3. This notation originated in the classical tensor calculus, in which the operators $\flat$ and # denoted, respectively, the lowering and raising of indices (in analogy with the corresponding musical notation).

Proposition 4.7.2 *For any Hermitian metric h in E with corresponding dual Hermitian metric h^* in E^*:*

(a) *Under the identification $((E^*)^*, (h^*)^*) = (E, h)$, we have $\flat_{E^*} = \#_E$.*
(b) *We have $h(s, \xi^\#) = \xi \cdot s = h^*(\xi, s^\flat)$ for all $s \in E_x$ and $\xi \in E_x^*$ with $x \in X$.*
(c) *We have $h^*(s^\flat, t^\flat) = h(t, s) = \overline{h(s, t)}$ for all $s, t \in E_x$ with $x \in X$. In particular, $|s^\flat|_{h^*} = |s|_h$ for all $s \in E$.*
(d) *We have $h(\xi^\#, \zeta^\#) = h^*(\zeta, \xi) = \overline{h^*(\xi, \zeta)}$ for all $\xi, \zeta \in E_x^*$ with $x \in X$. In particular, $|\xi^\#|_h = |\xi|_{h^*}$ for all $\xi \in E^*$.*

Proof Suppose $x \in X$, $s \in E_x$, and $\xi \in E_x^*$ with $s \neq 0$ and $\xi \neq 0$. Then

$$h(s, \xi^\#) = (\flat\#\xi) \cdot s = \xi \cdot s$$

and

$$h^*(\xi, s^\flat) = (\xi \cdot s) h^*(s^{-1}, |s|_h^2 s^{-1}) = (\xi \cdot s)|s|_h^2 \cdot |s^{-1}|_{h^*}^2 = \xi \cdot s.$$

Thus (b) is proved. Part (a) follows, since (b) gives

$$h(s, \xi^\flat) = s \cdot \xi = \xi \cdot s = h(s, \xi^\#).$$

For s and t as in (c), we have

$$h^*(s^\flat, t^\flat) = s^\flat \cdot t = h(t, \#\flat s) = h(t, s).$$

Part (d) now follows from parts (a) and (c). □

Definition 4.7.3 The *Hodge star operator* $*\colon (T^*X)_{\mathbb{C}} \to (T^*X)_{\mathbb{C}}$ is the mapping given by

$$*\colon (\alpha + \bar{\beta}) \mapsto -i\alpha + i\bar{\beta}$$

for all $\alpha, \beta \in \Lambda^{(1,0)} T_x^* X$ with $x \in X$. The *conjugate Hodge star operator* is the mapping $\bar{*}\colon (T^*X)_{\mathbb{C}} \to (T^*X)_{\mathbb{C}}$ given by

$$\bar{*}\colon \gamma \mapsto \overline{*\gamma} = *\bar{\gamma}$$

for all $\gamma \in (T^*X)_{\mathbb{C}}$.

In other words, for a local holomorphic coordinate z, we have $*dz = -i\,dz$ and $*d\bar{z} = \overline{*dz} = i\,d\bar{z}$. The proof of the following is left to the reader (see Exercise 4.7.2).

Proposition 4.7.4 *The Hodge star operator $*$ and conjugate Hodge star operator $\bar{*}$ on X have the following properties:*

(a) *For each point $r \in X$, the restrictions of $*$ and $\bar{*}$ to $(T_r^*X)_{\mathbb{C}}$ are linear and conjugate linear, respectively, isomorphisms. If α is a continuous ($\mathcal{C}^k$ with $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, measurable, L_{loc}^s with $s \in [1, \infty]$) 1-form, then $*\alpha$ and $\bar{*}\alpha$ are continuous (respectively, $\mathcal{C}^k$, measurable, L_{loc}^s).*

(b) *We have* $** = \bar{*}\bar{*} = -\mathrm{Id}$.
(c) *For* $(p,q) = (1,0)$ *or* $(0,1)$, $*$ *maps* $\Lambda^{(p,q)}T^*X$ *onto* $\Lambda^{(p,q)}T^*X$ *and* $\bar{*}$ *maps* $\Lambda^{(p,q)}T^*X$ *onto* $\Lambda^{(q,p)}T^*X$.
(d) *Each of the operators* $*$ *and* $\bar{*}$ *maps the real cotangent bundle* T^*X *onto itself and* $* = \bar{*}$ *on* T^*X.
(e) *If* $(p,q) = (1,0)$ *or* $(0,1)$, $x \in X$, *and* $\alpha, \beta \in \Lambda^{(p,q)}T_x^*X$, *then*

$$\alpha \wedge \bar{*}\beta = \alpha \wedge \overline{*\beta} = (-1)^q i\alpha \wedge \bar{\beta}.$$

In particular, $\alpha \wedge \bar{*}\alpha \geq 0$.
(f) *Let* α *and* β *be measurable forms of degree* 1 *on a measurable set* $S \subset X$, *and let* φ *be a measurable real-valued function on* S. *Then* (*see Definition* 2.6.1)

$$\|\alpha\|^2_{L^2(S,\varphi)} = \int_S \alpha \wedge \bar{*}\alpha \cdot e^{-\varphi} = \| *\alpha\|^2_{L^2(S,\varphi)} = \|\bar{*}\alpha\|^2_{L^2(S,\varphi)}.$$

Moreover, if $\alpha, \beta \in L^2_1(S,\varphi)$ (*which is equivalent to* $*\alpha, *\beta \in L^2_1(S,\varphi)$ *as well as to* $\bar{*}\alpha, \bar{*}\beta \in L^2_1(S,\varphi)$), *then*

$$\langle \alpha, \beta\rangle_{L^2(S,\varphi)} = \langle *\alpha, *\beta\rangle_{L^2(S,\varphi)} = \langle \bar{*}\beta, \bar{*}\alpha\rangle_{L^2(S,\varphi)} = \int_S \alpha \wedge \bar{*}\beta \cdot e^{-\varphi}.$$

For forms with values in a holomorphic line bundle, one may combine the conjugate Hodge star operator with the flat and sharp operators as follows:

Definition 4.7.5 Let h be a Hermitian metric in E, let $\flat = \flat_{(E,h)}$ be the associated flat operator, let $\# = \#_{(E,h)}$ be the associated sharp operator, and let $\bar{*}\colon (T^*X)_{\mathbb{C}} \to (T^*X)_{\mathbb{C}}$ be the conjugate Hodge star operator. Then the associated *conjugate Hodge star-flat operator*

$$\bar{*}^{\flat} = \bar{*}^{\flat}_{(E,h)} = \bar{*}^{\flat}_h = \bar{*}^{\flat}_E \colon \Lambda^1(T^*X)_{\mathbb{C}} \otimes E \to \Lambda^1(T^*X)_{\mathbb{C}} \otimes E^*$$

and the associated *conjugate Hodge star-sharp operator*

$$\bar{*}^{\#} = \bar{*}^{\#}_{(E,h)} = \bar{*}^{\#}_h = \bar{*}^{\#}_E \colon \Lambda^1(T^*X)_{\mathbb{C}} \otimes E^* \to \Lambda^1(T^*X)_{\mathbb{C}} \otimes E$$

are the mappings given by

$$\bar{*}^{\flat}(\alpha \otimes s) = (\bar{*}\alpha) \otimes s^{\flat} \quad \text{and} \quad \bar{*}^{\#}(\alpha \otimes \xi) = (\bar{*}\alpha) \otimes \xi^{\#}$$

for $\alpha \in \Lambda^1(T_x^*X)_{\mathbb{C}} = (T_x^*X)_{\mathbb{C}}$, $s \in E_x$, and $\xi \in E_x^*$ with $x \in X$.

Remark The conjugate linearity of $\bar{*}$, $\flat$, and $\#$ make the above operators well defined (and conjugate linear on the fibers), since for $\zeta \in \mathbb{C} \setminus \{0\}$ and for α, s, and ξ as above, we have

$$(\bar{*}(\zeta\alpha)) \otimes (s/\zeta)^{\flat} = \bar{\zeta} \cdot (1/\bar{\zeta}) \cdot (\bar{*}\alpha) \otimes s^{\flat} = (\bar{*}\alpha) \otimes s^{\flat}$$

and

$$(\bar{*}(\zeta\alpha)) \otimes (\xi/\zeta)^{\#} = \bar{\zeta} \cdot (1/\bar{\zeta}) \cdot (\bar{*}\alpha) \otimes \xi^{\#} = (\bar{*}\alpha) \otimes \xi^{\#}.$$

The basic properties of $\bar{*}_E^\flat$ and $\bar{*}_E^\#$ and the relationship between the pairings $\mathfrak{S}_E(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_{L^2(X,E,h)}$ are contained in the following proposition, the proof of which is left to the reader (see Exercise 4.7.3).

Proposition 4.7.6 *Given a Hermitian metric h in E with the associated Hermitian metric h^* in E^*, and given a pair of indices $(p,q) = (1,0)$ or $(0,1)$, the corresponding conjugate Hodge star-flat and conjugate Hodge star-sharp operators have the following properties*:

(a) *We have $\bar{*}_{E^*}^\flat = \bar{*}_E^\#$, $\bar{*}_E^\# \circ \bar{*}_E^\flat = -\mathrm{Id}$, $\bar{*}_E^\flat \circ \bar{*}_E^\# = -\mathrm{Id}$, $\bar{*}_E^\flat(\Lambda^{(p,q)}T^*X \otimes E) = \Lambda^{(q,p)}T^*X \otimes E^*$, and $\bar{*}_E^\#(\Lambda^{(p,q)}T^*X \otimes E^*) = \Lambda^{(q,p)}T^*X \otimes E$. If α is an E-valued 1-form that is continuous ($\mathcal{C}^k$ with $\mathbb{Z}_{\geq 0} \cup \{\infty\}$, measurable, L^s_{loc} with $s \in [1,\infty]$), then $\bar{*}_E^\flat\alpha$ is continuous (respectively, $\mathcal{C}^k$, measurable, L^s_{loc}).*

(b) *If $x \in X$ and $\alpha, \beta \in (\Lambda^{(p,q)}T^*X \otimes E)_x$, then*

$$\mathfrak{s}_E(\alpha, \bar{*}_E^\flat\beta) = (-1)^q i\{\alpha,\beta\}_h = (-1)^p i\{\bar{*}_E^\flat\beta, \bar{*}_E^\flat\alpha\}_{h^*}.$$

In particular, $\mathfrak{s}_E(\alpha, \bar{*}_E^\flat\alpha) \geq 0$. If $x \in X$, $\alpha \in (\Lambda^{(p,q)}T^*X \otimes E)_x$, and $\beta \in (\Lambda^{(q,p)}T^*X \otimes E^*)_x$, then

$$\mathfrak{s}_E(\alpha,\beta) = -(-1)^q i\{\alpha, \bar{*}_E^\#\beta\}_h.$$

(c) *If α is a measurable differential form of degree 1 with values in E on X, then (see Definition 3.6.2)*

$$\|\alpha\|^2_{L^2(X,E,h)} = \mathfrak{S}_E(\alpha, \bar{*}_E^\flat\alpha) = \|\bar{*}_E^\flat\alpha\|^2_{L^2(X,E^*,h^*)}.$$

(d) *The conjugate Hodge star-flat and conjugate Hodge star-sharp operators determine conjugate linear isomorphisms*

$$\bar{*}_E^\flat \colon L^2_{p,q}(X,E,h) \to L^2_{q,p}(X,E^*,h^*),$$

$$\bar{*}_E^\flat \colon L^2_1(X,E,h) \to L^2_1(X,E^*,h^*),$$

$$\bar{*}_E^\# = -(\bar{*}_E^\flat)^{-1} \colon L^2_{q,p}(X,E^*,h^*) \to L^2_{p,q}(X,E,h),$$

$$\bar{*}_E^\# = -(\bar{*}_E^\flat)^{-1} \colon L^2_1(X,E^*,h^*) \to L^2_1(X,E,h),$$

satisfying

(i) $\langle\bar{*}_E^\flat\alpha, \bar{*}_E^\flat\beta\rangle_{L^2} = \langle\beta,\alpha\rangle_{L^2}$ *, for all $\alpha,\beta \in L^2_1(X,E,h)$;*

(ii) $\langle\bar{*}_E^\#\alpha, \bar{*}_E^\#\beta\rangle_{L^2} = \langle\beta,\alpha\rangle_{L^2}$*, for all $\alpha,\beta \in L^2_1(X,E^*,h^*)$;*

(iii) $\langle\alpha,\beta\rangle_{L^2} = \mathfrak{S}_E(\alpha,\bar{*}_E^\flat\beta)$*, for all $\alpha,\beta \in L^2_1(X,E,h)$ (in particular, the right-hand side $\mathfrak{S}_E(\alpha,\bar{*}_E^\flat\beta)$ is defined);*

(iv) $\langle \alpha, \bar{*}_E^{\#}\beta\rangle_{L^2} = -\mathfrak{S}_E(\alpha, \beta)$ *for all* $\alpha \in L_1^2(X, E, h)$ *and* $\beta \in L_1^2(X, E^*, h^*)$ (*in particular,* $\mathfrak{S}_E(\alpha, \beta)$ *is defined*).

Another goal of the remainder of this chapter is the following:

Theorem 4.7.7 ($\bar{\partial}$-Hodge decomposition theorem) *For every Hermitian holomorphic line bundle* (E, h) *on a compact Riemann surface* X, *we have the orthogonal decomposition*

$$\mathcal{E}^{0,1}(E)(X) = \bar{*}_E^{\#}(\Gamma(X, \mathcal{O}(K_X \otimes E^*))) \oplus \bar{\partial}(\mathcal{E}^{0,0}(E)(X))$$

with respect to the restriction of the L^2 *inner product* $\langle \cdot, \cdot \rangle_{L^2_{0,1}(X,E,h)}$ *to* $\mathcal{E}^{0,1}(E)(X)$ (*here, we treat elements of* $\Gamma(X, \mathcal{O}(K_X \otimes E^*))$ *as* $(1, 0)$*-forms with values in* E^*).

Remark There is also a natural definition for $\bar{*}_E^{\flat}$ and $\bar{*}_E^{\#}$ on forms of degree 0 and 2, but it depends on the choice of a Kähler metric. The operators map a 0-form to a 2-form and a 2-form to a 0-form. One may also define such an operator on a complex manifold of higher dimension n. The operator then maps a (p, q)-form to an $(n-p, n-q)$-form (see, for example, [De3] or [MKo]). In our case, we have $n = 1$ and $(p, q) = (1, 0)$ or $(0, 1)$, so $(n-p, n-q) = (q, p)$. One should not view the conjugate Hodge star operator as simply switching the indices (p, q) in general, only in the special (1-dimensional) case considered in this book.

Exercises for Sect. 4.7

4.7.1 Verify that for any Hermitian holomorphic line bundle, the associated operators $\flat$ and $\#$ are $\mathcal{C}^\infty$ conjugate linear isomorphisms.

4.7.2 Prove Proposition 4.7.4.

4.7.3 Prove Proposition 4.7.6.

4.8 Proof of Serre Duality and $\bar{\partial}$-Hodge Decomposition

The Serre duality theorem and the $\bar{\partial}$-Hodge decomposition theorem will be proved simultaneously using orthogonal decomposition in a Hilbert space. Throughout this section, E denotes a holomorphic line bundle on a *compact* Riemann surface X. By Theorem 4.2.3, we may choose divisors D and K in X with $[D] = E$ and $[K] = K_X$. Finally, we may fix a Hermitian metric h in E. According to Proposition 4.7.6, we then have

$$\Gamma(X, \mathcal{O}(K_X \otimes E^*)) \hookrightarrow L^2_{1,0}(X, E^*, h^*) \xrightarrow{\bar{*}_E^{\#}} L^2_{0,1}(X, E, h),$$

where $\bar{*}_E^{\#}$ is a (surjective) conjugate linear isomorphism that satisfies

$$\langle \bar{*}_E^{\#}\alpha, \bar{*}_E^{\#}\beta\rangle_{L^2_{0,1}(X,E,h)} = \langle \beta, \alpha\rangle_{L^2_{1,0}(X,E^*,h^*)} \quad \forall \alpha, \beta \in L^2_{1,0}(X, E^*, h^*).$$

Lemma 4.8.1 *In* $L^2_{0,1}(X, E, h)$, *we have*

$$\bar{*}^\#_E(\Gamma(X, \mathcal{O}(K_X \otimes E^*))) = (\bar{\partial}(\mathcal{E}^{0,0}(E)(X)))^\perp.$$

Moreover, the conjugate linear map $\Gamma(X, \mathcal{O}(K_X \otimes E^*)) \to H^1_{\mathrm{Dol}}(X, E)$ *given by*

$$\alpha \mapsto [\bar{*}^\#_E \alpha]_{\mathrm{Dol}}$$

and the Serre mapping $\iota^E_{\mathfrak{S}} \colon \Gamma(X, \mathcal{O}(K_X \otimes E^*)) \to (H^1_{\mathrm{Dol}}(X, E))^*$ *are injective.*

Proof If $\alpha \in \Gamma(X, \mathcal{O}(K_X \otimes E^*))$ and $t \in \mathcal{E}^{0,0}(E)(X)$, then by Proposition 4.7.6 and Lemma 4.6.2 (or Lemma 4.6.10),

$$\langle \bar{\partial} t, \bar{*}^\#_E \alpha \rangle_{L^2} = -\mathfrak{S}_E(\bar{\partial} t, \alpha) = \mathfrak{S}_E(t, \bar{\partial}\alpha) = 0.$$

Conversely, suppose $\beta \in (\bar{\partial}(\mathcal{E}^{0,0}(E)(X)))^\perp$. We have $\beta = \bar{*}^\#_E \alpha$, where $\alpha \equiv -\bar{*}^\flat_E \beta \in L^2_{1,0}(X, E^*, h^*)$, and for every section $t \in \mathcal{E}^{0,0}(E)(X)$, we have

$$0 = \langle \bar{\partial} t, \bar{*}^\#_E \alpha \rangle_{L^2} = -\mathfrak{S}_E(\bar{\partial} t, \alpha).$$

Lemma 4.6.10 now implies that α is a holomorphic 1-form with values in E^*. That the map $\alpha \mapsto [\bar{*}^\#_E \alpha]_{\mathrm{Dol}}$ is injective follows easily.

Finally, if $\alpha \in \ker \iota^E_{\mathfrak{S}}$, then by Proposition 4.7.6,

$$0 = (\iota^E_{\mathfrak{S}}(\alpha)) \cdot [\bar{*}^\#_E \alpha]_{\mathrm{Dol}} = \mathfrak{S}_E(\bar{*}^\#_E \alpha, \alpha) = -\|\bar{*}^\#_E \alpha\|^2_{L^2}.$$

Thus $\bar{*}^\#_E \alpha = 0$ and hence $\alpha = 0$. Therefore $\iota^E_{\mathfrak{S}}$ is injective. □

Lemma 4.8.2 *The following are equivalent*:

(i) *The range* $\bar{\partial}(\mathcal{E}^{0,0}(E)(X))$ *is closed in the subspace* $\mathcal{E}^{0,1}(E)(X)$ *of the Hilbert space* $L^2_{0,1}(X, E, h)$.

(ii) $\mathcal{E}^{0,1}(E)(X) = \bar{*}^\#_E(\Gamma(X, \mathcal{O}(K_X \otimes E^*))) \oplus \bar{\partial}(\mathcal{E}^{0,0}(E)(X))$ (*i.e., the* $\bar{\partial}$-*Hodge decomposition theorem holds for* E).

(iii) *The conjugate linear map* $\Gamma(X, \mathcal{O}(K_X \otimes E^*)) \to H^1_{\mathrm{Dol}}(X, E)$ *given by*

$$\alpha \mapsto [\bar{*}^\#_E \alpha]_{\mathrm{Dol}}$$

is bijective.

(iv) *The Serre mapping*

$$\iota^E_{\mathfrak{S}} \colon \Gamma(X, \mathcal{O}(K_X \otimes E^*)) \to (H^1_{\mathrm{Dol}}(X, E))^*$$

is bijective (*i.e., the Serre duality theorem holds for* E).

Proof The equivalence of (i)–(iii) follows from Lemma 4.8.1 and general Hilbert space theory (see Corollary 7.5.7).

For the proof that (i)–(iii) and (iv) are equivalent, we observe that given a linear functional τ on $H^1_{\mathrm{Dol}}(X, E)$, we get the linear functional $\rho\colon t \mapsto -\overline{\tau([\bar{*}^{\#}_E t]_{\mathrm{Dol}})}$ on the *finite-dimensional* subspace $\Gamma(X, \mathcal{O}(K_X \otimes E^*)) \subset L^2_{1,0}(X, E^*, h^*)$. Thus there exists an element $s \in \Gamma(X, \mathcal{O}(K_X \otimes E^*))$ such that $\rho = \langle \cdot, s\rangle_{L^2_{1,0}(X,E^*,h^*)}$. If the orthogonal decomposition (ii) holds, then for each $\alpha \in \mathcal{E}^{0,1}(E)(X)$, we have $\alpha = \bar{*}^{\#}_E t + \bar{\partial}\beta$ for some holomorphic section $t \in \Gamma(X, \mathcal{O}(K_X \otimes E^*))$ and some $\mathcal{C}^\infty$ section $\beta \in \mathcal{E}^{0,0}(E)(X)$. Therefore, by Proposition 4.7.6,

$$\begin{aligned}
\tau([\alpha]_{\mathrm{Dol}}) &= \tau([\bar{*}^{\#}_E t]_{\mathrm{Dol}}) = -\overline{\rho(t)} = -\overline{\langle t, s\rangle_{L^2_{1,0}(X,E^*,h^*)}} = -\langle s, t\rangle_{L^2_{1,0}(X,E^*,h^*)} \\
&= -\langle \bar{*}^{\#}_E t, \bar{*}^{\#}_E s\rangle_{L^2_{0,1}(X,E,h)} = -\langle \alpha, \bar{*}^{\#}_E s\rangle_{L^2_{0,1}(X,E,h)} = \mathfrak{S}_E(\alpha, s) \\
&= \iota^E_{\mathfrak{S}}(s) \cdot [\alpha]_{\mathrm{Dol}}.
\end{aligned}$$

Thus the Serre map $\iota^E_{\mathfrak{S}}$ is surjective, and therefore, by Lemma 4.8.1, $\iota^E_{\mathfrak{S}}$ is bijective. Conversely, if there exists an element

$$\begin{aligned}
&\alpha \in \mathcal{E}^{0,1}(E)(X) \cap \mathrm{cl}(\bar{\partial}(\mathcal{E}^{0,0}(E)(X))) \setminus \bar{\partial}(\mathcal{E}^{0,0}(E)(X)) \\
&\quad \subset (\bar{*}^{\#}_E(\Gamma(X, \mathcal{O}(K_X \otimes E^*))))^{\perp},
\end{aligned}$$

then $[\alpha]_{\mathrm{Dol}} \in H^1_{\mathrm{Dol}}(X, E)$ is *not* in the image $\mathcal{V}$ of $\bar{*}^{\#}_E(\Gamma(X, \mathcal{O}(K_X \otimes E^*)))$. Hence there exists a linear functional τ on $H^1_{\mathrm{Dol}}(X, E)$ such that $\tau([\alpha]_{\mathrm{Dol}}) = 1$ and $\tau = 0$ on $\mathcal{V}$. On the other hand, for each $s \in \Gamma(X, \mathcal{O}(K_X \otimes E^*))$, we have

$$\iota^E_{\mathfrak{S}}(s) \cdot [\alpha]_{\mathrm{Dol}} = \mathfrak{S}_E(\alpha, s) = -\langle \alpha, \bar{*}^{\#}_E s\rangle_{L^2(X,E,h)} = 0,$$

so $\iota^E_{\mathfrak{S}}(s) \neq \tau$. Thus $\iota^E_{\mathfrak{S}}$ is not surjective if this is the case. □

Next, we observe that in order to obtain Serre duality for the divisor D, it suffices to obtain Serre duality for some divisor $\leq D$.

Lemma 4.8.3 *If there exists an effective divisor F on X for which the Serre map*

$$\iota^{[D-F]}_{\mathfrak{S}}\colon \Gamma(X, \mathcal{O}([K - D + F])) \to (H^1_{\mathrm{Dol}}(X, [D - F]))^*$$

is bijective, then the Serre map

$$\iota^{[D]}_{\mathfrak{S}}\colon \Gamma(X, \mathcal{O}([K - D])) \to (H^1_{\mathrm{Dol}}(X, [D]))^*$$

is also bijective.

Proof Fix a nontrivial section $u \in \Gamma(X, \mathcal{O}([F]))$. Given a nontrivial linear functional $\tau \in (H^1_{\mathrm{Dol}}(X, [D]))^*$, we may convert τ into a linear functional $\tau_u \in$

$(H^1_{\mathrm{Dol}}(X, [D - F]))^*$ by setting

$$\tau_u([\theta]_{\mathrm{Dol}}) \equiv \tau([\theta \cdot u]_{\mathrm{Dol}}) \quad \forall [\theta]_{\mathrm{Dol}} \in H^1_{\mathrm{Dol}}(X, [D - F]).$$

In other words, τ_u is the composition of τ and the surjective linear mapping

$$H^1_{\mathrm{Dol}}(X, [D - F]) = H^1_{\mathrm{Dol}}(X, [D] \otimes [F]^*) \to H^1_{\mathrm{Dol}}(X, [D])$$

induced by multiplication by u (as in the Dolbeault exact sequence). By hypothesis, there exists a section $t \in \Gamma(X, \mathcal{O}([K - D + F])) \subset \mathcal{E}^{1,0}([-D + F])(X)$ such that

$$\tau_u = \iota^{[D-F]}_{\mathfrak{S}}(t) = \mathfrak{S}_{[D-F]}(\cdot, t).$$

We will now show that the meromorphic section t/u of $[K - D]$ is actually holomorphic and that $\iota^{[D]}_{\mathfrak{S}}(v) = \tau$. Given an element $[\theta]_{\mathrm{Dol}} \in H^1_{\mathrm{Dol}}(X, [D])$, we may form local solutions of $\bar\partial\gamma = \theta$ near $\operatorname{supp}\operatorname{div}(u)$ and cut off to get a $\mathcal{C}^\infty$ section α of $[D]$ on X such that $\bar\partial\alpha = \theta$ near $\operatorname{supp}\operatorname{div}(u)$. Thus, by replacing θ with $\theta - \bar\partial\alpha$, we see that we may choose the representing class θ to vanish on a neighborhood of $\operatorname{supp}\operatorname{div}(u)$. Therefore, θ/u extends by 0 to a well-defined $\mathcal{C}^\infty$ form θ_u of type (0, 1) with values in $[D - F]$ (recall that this is how we proved surjectivity of such a map in the Dolbeault exact sequence in Sect. 3.4) and we have

$$\begin{aligned}\tau([\theta]_{\mathrm{Dol}}) &= \tau([\theta_u \cdot u]_{\mathrm{Dol}}) = \tau_u([\theta_u]_{\mathrm{Dol}}) = \mathfrak{S}_{[D-F]}(\theta_u, t)\\ &= \int_X \mathfrak{s}_{[D-F]}(\theta/u, t) = \int_X \mathfrak{s}_{[D]}(\theta, t/u).\end{aligned}$$

In fact, for $E \equiv [D]$, the meromorphic 1-form $v \equiv t/u$ with values in $E^* = [-D]$ is holomorphic. For given a pole p of v, we may choose a local holomorphic coordinate neighborhood $(U, \Phi = z)$ of p in X such that $z(p) = 0$, u and t are nonvanishing on $U \setminus \{p\}$, and there exist a nonvanishing holomorphic section s of E and a meromorphic function f with $v = f\,dz/s$ on U. We may also fix a $\mathcal{C}^\infty$ function λ with compact support in U such that $\lambda \equiv 1$ on $P \equiv \Phi^{-1}(\Delta(0; R)) \Subset U$ for some $R > 0$. Since $1/f$ has a zero at p, the section $\lambda s/(zf)$ of E on $U \setminus \{p\}$ extends to a unique $\mathcal{C}^\infty$ section β of E on X. Moreover, the form $\gamma \equiv \bar\partial\beta = (\bar\partial\lambda) \cdot s/(zf)$ vanishes outside a compact subset of U and on P. Thus

$$\begin{aligned}0 = \tau(0) = \tau([\gamma]_{\mathrm{Dol}}) &= \int_X \mathfrak{s}_E(\gamma, v) = \int_{U\setminus\overline{P}} \frac{\bar\partial\lambda}{z} \wedge dz\\ &= \int_{U\setminus\overline{P}} d\left(\lambda\frac{dz}{z}\right) = -\int_{\partial P} \frac{dz}{z} = -2\pi i \neq 0.\end{aligned}$$

We have therefore arrived at a contradiction, and hence $v = t/u$ has no poles. Therefore, for θ as above, we get

$$\tau([\theta]_{\mathrm{Dol}}) = \int_X \mathfrak{s}_E(\theta, v) = \mathfrak{S}_E([\theta]_{\mathrm{Dol}}, v).$$

Thus $\tau = \mathfrak{S}_E(\cdot, v) = \iota^E_{\mathfrak{S}}(v)$ as required. □

Proof of Theorem 4.6.4 and Theorem 4.7.7 According to the above lemmas, it suffices to show that for the divisor D on X, Serre duality holds for the line bundle $[D - F]$ for some effective divisor F. Let $E = [D]$. Fixing a point $p \in X$ and applying Theorem 4.2.1 (or Theorem 4.3.1), we see that $[-D + mp]$ will admit a positive-curvature Hermitian metric provided $m \gg 0$. Fixing a Kähler form ω and replacing D with $D - mp$ and E with $[D - mp]$, we may assume without loss of generality that E admits a Hermitian metric k with $-i\Theta_k \geq \omega$ (after choosing $m \gg 0$ or replacing ω with the product of ω and a small positive constant).

According to Lemma 4.8.2, it remains to show that the range $\bar{\partial}(\mathcal{E}^{0,0}(E)(X))$ is closed in the subspace $\mathcal{E}^{0,1}(E)(X)$ of $L^2_{0,1}(X, E, k)$. For this, suppose $\beta \in \mathcal{E}^{0,1}(E)(X)$ and $\{\alpha_\nu\}$ is a sequence in $\mathcal{E}^{0,0}(E)(X)$ with $\|\bar{\partial}\alpha_\nu - \beta\|_{L^2(X,E,k)} \to 0$. Observe that the fundamental estimate (Proposition 3.7.5) gives

$$\|\bar{\partial}\alpha\|_k^2 = \|D'_k\alpha\|_k^2 - \int_X |\alpha|_k^2 \cdot i\Theta_k \geq \|\alpha\|^2_{L^2_{0,0}(X,E,\omega,k)} \quad \forall \alpha \in \mathcal{E}^{0,0}(E)(X).$$

It follows that the sequence $\{\alpha_\nu\}$ is Cauchy, and hence that the sequence converges to some $\alpha \in L^2_{0,0}(X, E, \omega, k)$. If $\gamma \in \mathcal{E}^{0,1}(E)(X)$, then by Proposition 3.8.2,

$$\begin{aligned} -\int_X \{\alpha, D'_k\gamma\}_k = -\left\langle \alpha, \frac{D'_k\gamma}{\omega} \right\rangle_{L^2(X,E,\omega,k)} &= -\lim_{\nu\to\infty} \left\langle \alpha_\nu, \frac{D'_k\gamma}{\omega} \right\rangle_{L^2} \\ &= -\lim_{\nu\to\infty} \int_X \{\alpha_\nu, D'_k\gamma\}_k = \lim_{\nu\to\infty} \int_X \{\bar{\partial}\alpha_\nu, \gamma\}_k \\ &= i \lim_{\nu\to\infty} \langle \bar{\partial}\alpha_\nu, \gamma\rangle_{L^2} = i\langle \beta, \gamma\rangle_{L^2} = \int_X \{\beta, \gamma\}_k. \end{aligned}$$

Applying Proposition 3.8.2 (again), we get $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$. The regularity theorem (Theorem 3.8.3) now implies that α is of class $\mathcal{C}^\infty$ and $\bar{\partial}\alpha = \beta$. Thus the range is closed in $\mathcal{E}^{0,1}(E)(X)$, and the theorems follow. □

4.9 Hodge Decomposition for Scalar-Valued Forms

Applying Theorem 4.7.7 to the trivial line bundle, we get the following:

Theorem 4.9.1 (Hodge decomposition for scalar-valued forms) *If X is a compact Riemann surface, then we have the orthogonal decompositions*

$$\mathcal{E}^{0,1}(X) = \overline{\Omega}(X) \oplus \bar{\partial}(\mathcal{E}^{0,0}(X)),$$

$$\mathcal{E}^{1,0}(X) = \Omega(X) \oplus \partial(\mathcal{E}^{0,0}(X)),$$

$$\mathcal{E}^1(X) = \Omega(X) \oplus \overline{\Omega}(X) \oplus d(\mathcal{E}^0(X)) \oplus \bar{*}d(\mathcal{E}^0(X)),$$

$$\ker\bigl(\mathcal{E}^1(X) \xrightarrow{d} \mathcal{E}^2(X)\bigr) = \Omega(X) \oplus \overline{\Omega}(X) \oplus d(\mathcal{E}^0(X)),$$

where $\Omega(X) = \Gamma(X, \mathcal{O}(K_X))$ *and* $\overline{\Omega}(X) = \{\bar{\theta} \mid \theta \in \Omega(X)\}$ (*not the closure of* $\Omega(X)$).

Remark Recall that we think of a (1, 0)-form and a (0, 1)-form as orthogonal.

Proof of Theorem 4.9.1 The first equality follows from Theorem 4.7.7 and the second follows by conjugation of the spaces in the first.

For the third and fourth equalities, we first observe that $\bar{*}d(\mathcal{E}^0(X)) \perp \ker d$ in $\mathcal{E}^1(X)$, and hence $\bar{*}d(\mathcal{E}^0(X))$ is orthogonal to $\Omega(X) \oplus \overline{\Omega}(X) \oplus d(\mathcal{E}^0(X)) \subset \ker d$. For if $\alpha \in \ker d$ and $v \in \mathcal{E}^0(X)$, then Proposition 4.7.4 and Stokes' theorem give

$$\langle \alpha, \bar{*}\, dv\rangle_{L^2} = \int_X \alpha \wedge \bar{*}\bar{*}\, dv = -\int_X \alpha \wedge dv = \int_X d(\alpha v) = 0.$$

Next, we observe that the first two equalities yield the orthogonal decomposition

$$\mathcal{E}^1(X) = \Omega(X) \oplus \overline{\Omega}(X) \oplus \partial(\mathcal{E}^{0,0}(X)) \oplus \bar{\partial}(\mathcal{E}^{0,0}(X)).$$

Given $\rho \in \mathcal{E}^{0,0}(X)$, we have $d\rho = \partial\rho + \bar{\partial}\rho$ and $\bar{*}d\rho = \bar{*}\partial\rho + \bar{*}\bar{\partial}\rho = \bar{\partial}(i\bar{\rho}) + \partial(-i\bar{\rho})$. Thus

$$d(\mathcal{E}^0(X)) \oplus \bar{*}d(\mathcal{E}^0(X)) \subset \partial(\mathcal{E}^{0,0}(X)) \oplus \bar{\partial}(\mathcal{E}^{0,0}(X)).$$

Conversely, given $\alpha, \beta \in \mathcal{E}^{0,0}(X)$, by setting

$$\rho \equiv \frac{1}{2}(\alpha + \beta) \quad \text{and} \quad \tau \equiv -\frac{i}{2}(\bar{\alpha} - \bar{\beta}),$$

we get $\partial\alpha + \bar{\partial}\beta = d\rho + \bar{*}\, d\tau$. Thus the third equality holds. The fourth equality follows since the right-hand side is contained in $\ker d$ and $\bar{*}d(\mathcal{E}^0(X)) \perp \ker d$. □

For a compact Riemann surface X, the space of *harmonic* 1*-forms* is given by

$$\mathrm{Harm}^1(X) \equiv \Omega(X) \oplus \overline{\Omega}(X).$$

Clearly, the space $\mathrm{Harm}^1(X) \cap \mathcal{E}^1(X, \mathbb{R})$ of real harmonic 1-forms consists of all elements of the form $\alpha + \bar{\alpha}$ with $\alpha \in \Omega(X)$. Recall that for each $r \in \mathbb{Z}_{\geq 0}$, the rth *complex de Rham cohomology* (see Definition 10.6.1) of a $\mathcal{C}^\infty$ surface M is given by

$$H^r_{\mathrm{deR}}(M) = H^r_{\mathrm{deR}}(M, \mathbb{C}) = \ker\bigl(\mathcal{E}^r(M) \xrightarrow{d} \mathcal{E}^{r+1}(M)\bigr)/\mathrm{im}\,\bigl(\mathcal{E}^{r-1}(M) \xrightarrow{d} \mathcal{E}^r(M)\bigr),$$

and the r*th real de Rham cohomology* is given by

$$H^r_{\mathrm{deR}}(M, \mathbb{R}) = \ker\bigl(\mathcal{E}^r(M, \mathbb{R}) \xrightarrow{d} \mathcal{E}^{r+1}(M, \mathbb{R})\bigr)/\mathrm{im}\,\bigl(\mathcal{E}^{r-1}(M, \mathbb{R}) \xrightarrow{d} \mathcal{E}^r(M, \mathbb{R})\bigr)$$

(where we set $d = 0$ on $\mathcal{E}^{-1}(M) = 0$). Thus we get the following consequence of Hodge decomposition (Theorem 4.9.1), the proof of which is left to the reader (see Exercise 4.9.1):

Corollary 4.9.2 *For any compact Riemann surface X, we have the following*:

(a) *Every complex de Rham cohomology class $[\theta]_{\mathrm{deR}} \in H^1_{\mathrm{deR}}(X)$ has a unique harmonic representative*; *that is*, *the map* $\mathrm{Harm}^1(X) \to H^1_{\mathrm{deR}}(X)$ *given by* $\theta \mapsto [\theta]_{\mathrm{deR}}$ *is a* (*surjective*) *complex linear isomorphism.*
(b) *Every real de Rham cohomology class $[\theta]_{\mathrm{deR}} \in H^1_{\mathrm{deR}}(X, \mathbb{R})$ has a unique harmonic representative*; *that is*, *the map* $\mathrm{Harm}^1(X) \cap \mathcal{E}^1(X, \mathbb{R}) \to H^1_{\mathrm{deR}}(X, \mathbb{R})$ *given by $\theta \mapsto [\theta]_{\mathrm{deR}}$ is a* (*surjective*) *real linear isomorphism.*

Consequently, $\dim_{\mathbb{R}} H^1_{\mathrm{deR}}(X, \mathbb{R}) = \dim_{\mathbb{C}} H^1_{\mathrm{deR}}(X) = 2 \cdot \mathrm{genus}(X)$.

Remark In particular, for any compact Riemann surface X, we have

$$\begin{aligned} 2 \cdot \mathrm{genus}(X) &= \dim_{\mathbb{C}} H^1(X, \mathbb{C}) = \dim_{\mathbb{R}} H^1(X, \mathbb{R}) = \mathrm{rank}\, H^1(X, \mathbb{Z}) \\ &= \dim_{\mathbb{C}} H_1(X, \mathbb{C}) = \dim_{\mathbb{R}} H_1(X, \mathbb{R}) = \mathrm{rank}\, H_1(X, \mathbb{Z}) \end{aligned}$$

(see Sects. 10.6 and 10.7). Hence the (holomorphic) genus of a compact Riemann surface is actually a topological invariant; i.e., it does not depend on the holomorphic structure or even on the $\mathcal{C}^\infty$ structure. In other words, any two compact Riemann surfaces that are homeomorphic have the same genus. The converse, though true, is not proved in this book (see, for example, [T] for a proof).

Theorem 4.9.3 (Poincaré duality theorem) *Let X be a compact Riemann surface*, *and let $r \in \{0, 1, 2\}$. Then the map*

$$([\alpha]_{\mathrm{deR}}, [\beta]_{\mathrm{deR}}) \mapsto \mathfrak{P}([\alpha]_{\mathrm{deR}}, [\beta]_{\mathrm{deR}}) \equiv \int_X \alpha \wedge \beta$$

gives a well-defined bilinear pairing $\mathfrak{P}(\cdot, \cdot)\colon H^r_{\mathrm{deR}}(X) \times H^{2-r}_{\mathrm{deR}}(X) \to \mathbb{C}$. *Furthermore*, *the map* $[\beta]_{\mathrm{deR}} \mapsto \iota_{\mathfrak{P}}([\beta]_{\mathrm{deR}}) \equiv \mathfrak{P}(\cdot, [\beta]_{\mathrm{deR}})$ *gives a* (*surjective*) *linear isomorphism* $\iota_{\mathfrak{P}}\colon H^{2-r}_{\mathrm{deR}}(X) \to (H^r_{\mathrm{deR}}(X))^*$. *In particular*, $\dim_{\mathbb{C}} H^2_{\mathrm{deR}}(X) = \dim_{\mathbb{C}} H^0_{\mathrm{deR}}(X) = 1$.

Proof The pairing is well defined. For if $\alpha \in \mathcal{E}^r(X) \cap \ker d$, $\beta \in \mathcal{E}^{2-r}(X) \cap \ker d$, $\rho \in \mathcal{E}^{r-1}(X)$ (we set $\rho = 0$ if $r = 0$), and $\tau \in \mathcal{E}^{2-r-1}(X)$ (we set $\tau = 0$ if $r = 2$), then by Stokes' theorem,

$$\begin{aligned} &\int_X (\alpha + d\rho) \wedge (\beta + d\tau) \\ &\quad = \int_X \alpha \wedge \beta + \int_X \alpha \wedge d\tau + \int_X d\rho \wedge \beta + \int_X d\rho \wedge d\tau \end{aligned}$$

$$= \int_X \alpha \wedge \beta + (-1)^r \int_X d(\alpha \wedge \tau) + \int_X d(\rho \wedge \beta) + \int_X d(\rho \wedge d\tau)$$
$$= \int_X \alpha \wedge \beta.$$

Now, given $[\alpha]_{\mathrm{deR}} \in H^1_{\mathrm{deR}}(X)$, we may apply Corollary 4.9.2 to get $\alpha^{1,0} \in \Omega(X)$ and $\alpha^{0,1} \in \overline{\Omega}(X)$ with $[\alpha]_{\mathrm{deR}} = [\alpha^{1,0}]_{\mathrm{deR}} + [\alpha^{0,1}]_{\mathrm{deR}}$, and we may choose the representative α so that $\alpha = \alpha^{1,0} + \alpha^{0,1}$. For $(p,q) = (1,0)$ or $(0,1)$, we have

$$(-1)^p i\,\overline{\alpha^{p,q}} \wedge \alpha = (-1)^p i\,\overline{\alpha^{p,q}} \wedge \alpha^{p,q} \geq 0,$$

with equality at a point x if and only if $\alpha^{p,q}_x = 0$. Thus, if $\alpha^{p,q}$ is nontrivial, then

$$\mathfrak{P}([(-1)^p i\,\overline{\alpha^{p,q}}]_{\mathrm{deR}}, [\alpha]_{\mathrm{deR}}) > 0.$$

It follows that $\iota_{\mathfrak{P}}([\alpha]_{\mathrm{deR}}) = 0$ if and only if $[\alpha]_{\mathrm{deR}} = 0$, and hence that $\iota_{\mathfrak{P}}$ maps $H^1_{\mathrm{deR}}(X)$ isomorphically onto $(H^1_{\mathrm{deR}}(X))^*$.

It is easy to verify that $H^0_{\mathrm{deR}}(X) = \mathbb{C}$. To determine $H^2_{\mathrm{deR}}(X)$, let us fix a Kähler metric g with Kähler form ω on X. To a differential form $\theta \in \mathcal{E}^{1,1}(X)$, we may associate a differential form $\hat{\theta}$ of type $(0,1)$ with values in K_X. Applying the $\bar{\partial}$-Hodge decomposition theorem, we get a $\mathcal{C}^\infty$ section $\hat{\rho}$ of K_X and a holomorphic section ζ of $K_X \otimes K_X^* = 1_X$ (i.e., a complex constant) with $\hat{\theta} = \bar{*}^{\#}_{(K_X, g^*)}\zeta + \bar{\partial}\hat{\rho}$. Letting ρ be the scalar-valued $(1,0)$-form associated to $\hat{\rho}$, we get $\theta = -\bar{\zeta}\omega + \bar{\partial}\rho = -\bar{\zeta}\omega + d\rho$ (see Exercise 4.9.2). We also have $[\omega]_{\mathrm{deR}} \neq 0$, since $v \equiv \int_X \omega > 0$, so $H^2_{\mathrm{deR}}(X) = \mathbb{C}[\omega]_{\mathrm{deR}} \cong \mathbb{C}$. For any $\xi, \eta \in \mathbb{C}$, we have (here we identify ξ and η with the corresponding constant functions $x \mapsto \xi$ and $x \mapsto \eta$)

$$\mathfrak{P}([\xi]_{\mathrm{deR}}, [\eta\omega]_{\mathrm{deR}}) = \mathfrak{P}([\eta\omega]_{\mathrm{deR}}, [\xi]_{\mathrm{deR}}) = \xi\eta \cdot v,$$

which vanishes if and only if $\xi = 0$ or $\eta = 0$. It now follows that the linear mappings $\iota_{\mathfrak{P}} : H^0_{\mathrm{deR}}(X) \to (H^2_{\mathrm{deR}}(X))^*$ and $\iota_{\mathfrak{P}} : H^2_{\mathrm{deR}}(X) \to (H^0_{\mathrm{deR}}(X))^*$ are (surjective) isomorphisms. □

Exercises for Sect. 4.9

4.9.1 Prove Corollary 4.9.2.

4.9.2 Let X be a Riemann surface. According to Proposition 3.10.1, we may identify a $(1,1)$-form with a corresponding $(0,1)$-form with values in K_X, and we may identify a $(1,0)$-form with values in K_X^* with a corresponding section of $K_X \otimes K_X^* = 1_X$, i.e., with a function. Let g be a Kähler metric with associated Kähler form ω on X, and let $\zeta \in \mathbb{C}$. Viewing the constant function ζ as a holomorphic 1-form with values in K_X^*, show that the scalar-valued $(1,1)$-form associated to the K_X-valued $(0,1)$-form $\bar{*}^{\#}_{(K_X, g^*)}\zeta$ is equal to $-\bar{\zeta}\omega$ (this fact was used in the proof of the Poincaré duality theorem).

4.9.3 Let (E, h) be a Hermitian holomorphic line bundle on a compact Riemann surface X. Prove the D'_h-Hodge decomposition theorem:

$$\mathcal{E}^{1,0}(E)(X) = \Gamma(X, \mathcal{O}(K_X \otimes E)) \oplus D'_h(\mathcal{E}^{0,0}(E)(X)).$$

Hint. First prove that $\bar{*}^{\#}_{E}(\bar{\partial} u) = D'_h(\#_E(iu))$ for each section $u \in \mathcal{E}^{0,0}(E^*)(X)$. Then apply the operator $\bar{*}^{\#}_{E}$ to the summands in the $\bar{\partial}$-Hodge decomposition for $\mathcal{E}^{0,1}(E^*)(X)$.

Chapter 5
Uniformization and Embedding of Riemann Surfaces

In this chapter, we consider certain complex analytic characterizations of Riemann surfaces (some topological and $\mathcal{C}^\infty$ characterizations appear in Chap. 6). The first goal is the following Riemann surface analogue of the classical Riemann mapping theorem in the plane:

Theorem 5.0.1 (Riemann mapping theorem) *A simply connected Riemann surface is biholomorphic to the Riemann sphere* $\mathbb{P}^1$, *to the complex plane* $\mathbb{C}$, *or to the unit disk* $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$.

Exactly one of the above outcomes will hold, since no two of the Riemann surfaces $\mathbb{P}^1$, $\mathbb{C}$, and Δ are biholomorphic (see Exercise 2.2.7). Theorem 5.0.1 is also called the *uniformization theorem*. Parts of the theorem were first obtained by Riemann. The first complete proofs were given by Koebe [Koe1], [Koe2] and Poincaré [P]. The proof given in this chapter is essentially due to Simha [Sim] and Demailly [De3], although the idea of forming compactifications of exhausting subdomains is due to Koebe. The theorem allows one to classify Riemann surfaces as quotients of the Riemann sphere, the plane, or the disk (see Sect. 5.9).

The second goal of this chapter is the fact that every Riemann surface X may be obtained by holomorphic attachment of tubes at elements of a locally finite sequence of coordinate disks in a domain in $\mathbb{P}^1$. In particular, for X compact, this allows one to form a canonical homology basis.

Finally, we consider finite holomorphic branched coverings (see Sect. 5.20) as well as the following embedding theorems:

- Every open Riemann surface admits a holomorphic embedding into $\mathbb{C}^3$ (see Sect. 5.18). This fact is due to Narasimhan [Ns1] and is the 1-dimensional version of the Bishop–Narasimhan–Remmert embedding theorem for Stein manifolds.
- Every compact Riemann surface admits a holomorphic embedding into some (higher-dimensional) complex projective space (see Sect. 5.19). This fact is the 1-dimensional version of the Kodaira embedding theorem.

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- Every compact Riemann surface of positive genus admits a holomorphic embedding into some (higher-dimensional) complex torus (see Sect. 5.22). This fact is the Abel–Jacobi embedding theorem.

5.1 Holomorphic Covering Spaces

The general theory of covering spaces is reviewed in Chap. 10 (see Sects. 10.1–10.5). In this section, we consider covering spaces of Riemann surfaces.

Definition 5.1.1 A *holomorphic covering space* (or a 1-*dimensional complex covering manifold* or, for $\widehat{X}$ connected, a *covering Riemann surface*) of a complex 1-manifold X is a covering space $\Upsilon\colon \widehat{X} \to X$ in which $\widehat{X}$ is a complex 1-manifold and the mapping Υ is locally biholomorphic (see Definition 10.2.1). In other words, each point in X has a connected neighborhood U such that Υ maps each connected component of $\Upsilon^{-1}(U)$ biholomorphically onto U. The holomorphic covering space $\Upsilon\colon \widehat{X} \to X$ is *holomorphically equivalent* to a holomorphic covering space $\check{\Upsilon}\colon \check{X} \to X$ if there exists a biholomorphism $\Phi\colon \check{X} \to \widehat{X}$ that is a lifting of $\check{\Upsilon}$; i.e., Φ is a fiber-preserving biholomorphism. A simply connected holomorphic covering space $\Upsilon\colon \widetilde{X} \to X$ of a Riemann surface is also called the *holomorphic universal covering space* (or *universal covering Riemann surface*).

Remark Holomorphic equivalence of coverings is an equivalence relation (see Exercise 5.1.1).

Proposition 5.1.2 *Let X be a complex* 1*-manifold and let $\Upsilon\colon \widehat{X} \to X$ be a* (*topological*) *covering space.*

(a) *If $\widehat{X}$ is a complex* 1*-manifold and Υ is holomorphic, then $\Upsilon\colon \widehat{X} \to X$ is a holomorphic covering space.*
(b) *In general, there is a unique holomorphic structure on $\widehat{X}$ with respect to which $\Upsilon\colon \widehat{X} \to X$ is a holomorphic covering space.*

Proof Part (a) follows from the holomorphic inverse function theorem (Theorem 2.4.4). For the proof of (b), first observe that $\widehat{X}$ is Hausdorff, since X is Hausdorff (see Proposition 10.2.10). Next observe that we may form a holomorphic atlas $\mathcal{A} = \{(U_i, \Phi_i, U_i')\}_{i\in I}$ on X such that U_i is connected and evenly covered by Υ for each $i \in I$. For each $i \in I$, $\widehat{U}_i \equiv \Upsilon^{-1}(U_i)$ is the union of a collection of disjoint connected open sets $\{U_i^{(\lambda)}\}_{\lambda\in\Lambda_i}$ in $\widehat{X}$ each of which is mapped homeomorphically onto U_i. Setting $\Phi_i^{(\lambda)} = \Phi_i \circ \Upsilon\restriction_{U_i^{(\lambda)}}$, we get a complex atlas $\widehat{\mathcal{A}} \equiv \{(U_i^{(\lambda)}, \Phi_i^{(\lambda)}, U_i')\}_{\lambda\in\Lambda_i, i\in I}$ in $\widehat{X}$. Suppose $i, j \in I$, $\lambda \in \Lambda_i$, $\mu \in \Lambda_j$, and $U_i^{(\lambda)} \cap U_j^{(\mu)} \neq \emptyset$. Then the coordinate transformation

$$\Phi_i^{(\lambda)} \circ [\Phi_j^{(\mu)}]^{-1} = \Phi_i \circ \Phi_j^{-1} \upharpoonright_{\Phi_j^{(\mu)}(U_i^{(\lambda)} \cap U_j^{(\mu)})} : \Phi_j^{(\mu)}(U_i^{(\lambda)} \cap U_j^{(\mu)})$$
$$\longrightarrow \Phi_i^{(\lambda)}(U_i^{(\lambda)} \cap U_j^{(\mu)}) \subset \Phi_i(U_i \cap U_j)$$

is holomorphic, and hence the atlas determines a holomorphic structure on $\widehat{X}$. Moreover, Υ is holomorphic with respect to this holomorphic structure, since

$$\Phi_i \circ \Upsilon \circ [\Phi_i^{(\lambda)}]^{-1} = \mathrm{Id}_{U_i'} \in \mathcal{O}(U_i') \quad \forall i \in I, \lambda \in \Lambda_i.$$

Finally, given an arbitrary holomorphic structure on $\widehat{X}$ with respect to which Υ is a local biholomorphism, and given $i \in I$ and $\lambda \in \Lambda_i$, the mapping $\Phi_i^{(\lambda)} = \Phi_i \circ \Upsilon \upharpoonright_{U_i^{(\lambda)}}$ is a composition of a biholomorphism on $U_i \subset X$ and a biholomorphism on $U_i^{(\lambda)}$, and therefore $\Phi_i^{(\lambda)}$ is biholomorphic. Thus the elements of the atlas $\widehat{\mathcal{A}}$ are holomorphically compatible with the local holomorphic charts on $\widehat{X}$, and uniqueness follows. □

Remark Given a covering space $\Upsilon \colon \widehat{X} \to X$ of a complex 1-manifold X, we will assume that $\widehat{X}$ has the induced holomorphic structure unless otherwise indicated.

Theorem 5.1.3 (Holomorphic lifting theorem) *Let $\Upsilon \colon \widehat{X} \to X$ be a covering Riemann surface of a Riemann surface X, let $\Phi \colon Y \to X$ be a holomorphic mapping of a Riemann surface Y to X, let $y_0 \in Y$, let $x_0 = \Phi(y_0)$, and let $\hat{x}_0 \in \Upsilon^{-1}(x_0)$.*

(a) *If $\widehat{\Phi} \colon Y \to \widehat{X}$ is a lifting of Φ, then $\widehat{\Phi}$ is holomorphic. If Φ is a local biholomorphism* (*a holomorphic covering map*), *then $\widehat{\Phi}$ is a local biholomorphism* (*respectively, a holomorphic covering map*).
(b) *If $\Phi_*\pi_1(Y, y_0) \subset \Upsilon_*\pi_1(\widehat{X}, \hat{x}_0)$, then there is a unique lifting of Φ to a holomorphic map $\widehat{\Phi} \colon Y \to \widehat{X}$ with $\widehat{\Phi}(y_0) = \hat{x}_0$. Conversely, if such a lifting $\widehat{\Phi}$ exists, then $\Phi_*\pi_1(Y, y_0) \subset \Upsilon_*\pi_1(\widehat{X}, \hat{x}_0)$.*

Proof The proof of (a) is left to the reader (see Lemma 10.2.4 and Exercise 5.1.2). Part (b) follows from the (topological) lifting theorem (Theorem 10.2.5) together with part (a). □

Corollary 5.1.4 *Let $\Upsilon \colon \widehat{X} \to X$ and $\check{\Upsilon} \colon \check{X} \to X$ be covering Riemann surfaces of a Riemann surface X, let $x_0 \in X$, let $\hat{x}_0 \in \Upsilon^{-1}(x_0)$, and let $\check{x}_0 \in \check{\Upsilon}^{-1}(x_0)$. If $\Upsilon_*\pi_1(\widehat{X}, \hat{x}_0) \subset \check{\Upsilon}_*\pi_1(\check{X}, \check{x}_0)$, then there exists a unique commutative diagram of holomorphic covering maps*

$$\begin{tikzcd} & \check{X} \arrow[d, "\check{\Upsilon}"] \\ \widehat{X} \arrow[ur, "\widehat{\Upsilon}"] \arrow[r, "\Upsilon"'] & X \end{tikzcd}$$

with $\widehat{\Upsilon}(\hat{x}_0) = \check{x}_0$. *Moreover, if* $\Upsilon_*\pi_1(\widehat{X}, \hat{x}_0) = \check{\Upsilon}_*\pi_1(\check{X}, \check{x}_0)$, *then* $\widehat{\Upsilon}$ *is a biholomorphism*; *that is*, *the two coverings are holomorphically equivalent. In particular, up to holomorphic equivalence of holomorphic coverings*, *every Riemann surface has a unique holomorphic universal covering space.*

Proof The holomorphic lifting theorem gives the existence and uniqueness of the commutative diagram. Corollary 10.2.6 then implies that if the image groups are equal, then $\widehat{\Upsilon}$ is a biholomorphism. □

Recall the following (see Definition 2.2.1):

Definition 5.1.5 A biholomorphism of a Riemann surface X onto itself is called an *automorphism* of X. The group of automorphisms of X (with product given by composition) is denoted by $\mathrm{Aut}\,(X)$.

Recall that a *deck transformation* (or *covering transformation*) of a covering space $\Upsilon\colon \widehat{X} \to X$ is a homeomorphism $\Phi\colon \widehat{X} \to \widehat{X}$ such that $\Upsilon \circ \Phi = \Upsilon$, and the group of deck transformations is denoted by $\mathrm{Deck}(\Upsilon)$ (see Sect. 10.4). If the above is a Riemann surface covering, then the holomorphic lifting theorem (Theorem 5.1.3) implies that $\mathrm{Deck}(\Upsilon)$ is a subgroup of $\mathrm{Aut}\,(\widehat{X})$. Moreover, the holomorphic lifting theorem and Theorem 10.4.5 together give the following (the details of the proof are left to reader in Exercise 5.1.3):

Theorem 5.1.6 *Let* $\Upsilon\colon \widetilde{X} \to X$ *be the* (*holomorphic*) *universal covering space of a Riemann surface* X. *Then* $\Gamma \equiv \mathrm{Deck}(\Upsilon)$ *is a subgroup of* $\mathrm{Aut}(\widetilde{X})$ *that acts properly discontinuously and freely on* $\widetilde{X}$ (*see Definition* 10.4.3), *and there is a unique holomorphic structure on* $\Gamma\backslash\widetilde{X}$ *giving rise to a commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\Upsilon"'] \arrow[dr, "\Upsilon_\Gamma"] & \\ X & \Gamma\backslash\widetilde{X} \arrow[l, "\cong"'] \end{tikzcd}$$

in which Υ_Γ *is a holomorphic covering map and the map* $\Gamma\backslash\widetilde{X} \to X$ *is a biholomorphism.*

In other words, $\Upsilon_\Gamma\colon \widetilde{X} \to \Gamma\backslash\widetilde{X}$ is a holomorphic covering that we may identify with the original holomorphic covering $\Upsilon\colon \widetilde{X} \to X$. It follows that if $\widehat{\Upsilon}\colon \widetilde{X} \to \widehat{X}$ is the universal covering space of a Riemann surface $\widehat{X}$ and $\mathrm{Deck}(\widehat{\Upsilon}) = \mathrm{Deck}(\Upsilon)$, then $\widehat{X} \cong \Gamma\backslash\widetilde{X} \cong X$ and we may identify the holomorphic coverings $\widehat{\Upsilon}\colon \widetilde{X} \to \widehat{X}$ and $\Upsilon\colon \widetilde{X} \to X$.

Conversely, a group of automorphisms acting properly discontinuously and freely gives rise to a holomorphic universal covering space (cf. Theorem 10.4.6):

Theorem 5.1.7 *Let* $\widetilde{X}$ *be a simply connected Riemann surface*, *let* Γ *be a subgroup of* $\mathrm{Aut}\,(\widetilde{X})$ *that acts properly discontinuously and freely on* $\widetilde{X}$, *and let*

$\Upsilon = \Upsilon_\Gamma : \widetilde{X} \to X \equiv \Gamma\backslash\widetilde{X}$ *be the corresponding quotient space mapping. Then we have the following*:

(a) *There is a unique holomorphic structure on* X *with respect to which the quotient map* $\Upsilon : \widetilde{X} \to X$ *is a holomorphic universal covering space and* $\Gamma = \mathrm{Deck}(\Upsilon)$.
(b) *If* $\widehat{\Gamma}$ *is a subgroup of* Γ, *then we have a commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\Upsilon"'] \arrow[dr, "\Upsilon_{\widehat{\Gamma}}"] & \\ X & \widehat{X} \equiv \widehat{\Gamma}\backslash\widetilde{X} \arrow[l, "\widehat{\Upsilon}"'] \end{tikzcd}$$

of holomorphic covering maps. Moreover, if $\tilde{x}_0 \in \widetilde{X}$, $x_0 = \Upsilon(\tilde{x}_0)$, *and* $\hat{x}_0 = \Upsilon_{\widehat{\Gamma}}(\tilde{x}_0)$, *and*

$$\chi : \Gamma = \mathrm{Deck}(\Upsilon) \to \pi_1(X, x_0) \quad \textit{and} \quad \widehat{\chi} : \widehat{\Gamma} = \mathrm{Deck}(\Upsilon_{\widehat{\Gamma}}) \to \pi_1(\widehat{X}, \hat{x}_0)$$

are the corresponding group isomorphisms, *then*

$$\chi \circ \widehat{\chi}^{-1} = \widehat{\Upsilon}_* : \pi_1(\widehat{X}, \hat{x}_0) \to \pi_1(X, x_0).$$

In other words, *the action of* $[\hat{\gamma}] \in \pi_1(\widehat{X}, \hat{x}_0)$ *on* $\widetilde{X}$ *is the same as that of* $\widehat{\Upsilon}_*[\hat{\gamma}] = [\widehat{\Upsilon}(\hat{\gamma})] \in \pi_1(X, x_0)$.

Proof For the proof of part (a), we first observe that Theorem 10.4.6 implies that $\Upsilon : \widetilde{X} \to X$ is a covering map and $\mathrm{Deck}(\Upsilon) = \Gamma$. Let $\mathcal{A}$ be the collection of local complex charts in X of the form $\{(U, \Phi \circ (\Upsilon\restriction_{U''})^{-1}, U')\}$, where U is an evenly covered connected open subset of X and (U'', Φ, U') is a local holomorphic chart in $\widetilde{X}$ for which U'' is a connected component of $\Upsilon^{-1}(U)$. We will show that $\mathcal{A}$ is a holomorphic atlas in X. Clearly, the neighborhoods in $\mathcal{A}$ cover X. If $(U, \Phi \circ (\Upsilon\restriction_{U''})^{-1}, U')$ and $(V, \Psi \circ (\Upsilon\restriction_{V''})^{-1}, V')$ are two local complex charts in $\mathcal{A}$ and $p \in U \cap V$, then we may choose a connected evenly covered neighborhood W of p in $U \cap V$. Setting $W'' \equiv (\Upsilon\restriction_{U''})^{-1}(W)$, we get an element $\gamma \in \Gamma$ such that the connected open set $\gamma(W'') \subset \Upsilon^{-1}(V)$ meets, and hence is contained in, V''. Thus the coordinate transformation on $\Phi(W'') = \Phi \circ (\Upsilon\restriction_{U''})^{-1}(W)$ is given by

$$\Psi \circ (\Upsilon\restriction_{V''})^{-1} \circ [\Phi \circ (\Upsilon\restriction_{U''})^{-1}]^{-1} = \Psi \circ \gamma \circ (\Upsilon\restriction_{U''})^{-1} \circ \Upsilon \circ \Phi^{-1} = \Psi \circ \gamma \circ \Phi^{-1},$$

and is therefore holomorphic as a composition of holomorphic maps. Thus $\mathcal{A}$ is a holomorphic atlas. The map $\Upsilon : \widetilde{X} \to X$ is holomorphic with respect to this holomorphic structure, since given a local holomorphic chart $(U, \Phi \circ (\Upsilon\restriction_{U''})^{-1}, U') \in \mathcal{A}$, we have $\Phi \circ (\Upsilon\restriction_{U''})^{-1} \circ \Upsilon = \Phi$ on U''. Finally, if $\mathcal{A}'$ is a holomorphic atlas on X with respect to which the covering map Υ is holomorphic, $(Q, \Lambda, Q') \in \mathcal{A}'$, and $(U, \Phi \circ (\Upsilon\restriction_{U''})^{-1}, U') \in \mathcal{A}$, then the coordinate transformation on $\Phi \circ (\Upsilon\restriction_{U''})^{-1}(U \cap Q)$ is given by

$$\Lambda \circ [\Phi \circ (\Upsilon\restriction_{U''})^{-1}]^{-1} = \Lambda \circ \Upsilon \circ \Phi^{-1},$$

and is therefore holomorphic as a composition of holomorphic maps. The inverse function theorem then implies that its inverse map is also holomorphic. Thus $\mathcal{A}$ and $\mathcal{A}'$ determine the same holomorphic structure on X, and we have uniqueness.

For the proof of (b), suppose $\widehat{\Gamma}$ is a subgroup of Γ. Then Theorem 10.4.6 gives the required commutative diagram of topological covering maps, and part (a) gives the holomorphic structure on $\widehat{X}$ with respect to which $\Upsilon_{\widehat{\Gamma}}$ is a holomorphic covering map. It follows that the covering map $\widehat{\Upsilon}\colon \widehat{X} \to X$ is also holomorphic, because locally, the map may be written as the composition of the holomorphic map Υ with a local holomorphic inverse of $\Upsilon_{\widehat{\Gamma}}$. □

Corollary 5.1.8 *Let X be a Riemann surface, let $x_0 \in X$, and let Γ be a subgroup of $\pi_1(X, x_0)$. Then, up to holomorphic equivalence of holomorphic covering spaces, there are a unique covering Riemann surface $\widehat{\Upsilon}\colon \widehat{X} \to X$ and a point $\hat{x}_0 \in \widehat{\Upsilon}^{-1}(x_0)$ such that the homomorphism $\widehat{\Upsilon}_*\colon \pi_1(\widehat{X}, \hat{x}_0) \to \pi_1(X, x_0)$ maps $\pi_1(\widehat{X}, \hat{x}_0)$ isomorphically onto Γ. In fact, if $\Upsilon\colon \widetilde{X} \to X$ is the universal covering space, then we have the commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\Upsilon_\Gamma"] \arrow[d, "\Upsilon"'] & \widehat{X} = \Gamma\backslash\widetilde{X} \arrow[dl, "\widehat{\Upsilon}"] \\ X & \end{tikzcd}$$

(*here, we identify* $\pi_1(X, x_0) \supset \Gamma$ *with* $\mathrm{Deck}(\Upsilon)$).

Proof This follows immediately from Theorem 10.3.3 and Theorem 5.1.2 (which give the existence of the holomorphic universal cover), Theorem 5.1.7 (the construction of a holomorphic covering from a properly discontinuous free group action by automorphisms), and Corollary 5.1.4 (holomorphic equivalence of holomorphic coverings with the same image fundamental group). □

We now consider some fundamental examples:

Example 5.1.9 The holomorphic universal covering map of the punctured plane $\mathbb{C}^*$ is the map $\Upsilon\colon \mathbb{C} \to \mathbb{C}^*$ given by $z \mapsto \exp(2\pi i z)$. For Υ is a surjective locally biholomorphic mapping with local inverses given by holomorphic branches of the multiple-valued holomorphic function $\zeta \mapsto (\log \zeta)/(2\pi i)$ (see Example 1.6.2). Given a point $z_0 = x_0 + iy_0 \in \mathbb{C}$ with $x_0, y_0 \in \mathbb{R}$, we may form the connected neighborhood

$$\begin{aligned} U &\equiv \{\rho e^{2\pi i t} \mid 0 < \rho < \infty,\ x_0 - (1/2) < t < x_0 + (1/2)\} \\ &= \Upsilon(\{x + iy \mid y \in \mathbb{R}, x_0 - (1/2) < x < x_0 + (1/2)\}) \end{aligned}$$

of $\zeta_0 = \Upsilon(z_0)$. The inverse image $\Upsilon^{-1}(U)$ is then the union of the collection of disjoint domains $\{U_n\}_{n\in\mathbb{Z}}$, where for each n, U_n is the infinite strip

$$U_n = \{z = x + iy \mid y \in \mathbb{R},\ x_0 - (1/2) + n < x < x_0 + (1/2) + n\},$$

which is mapped biholomorphically onto U by Υ (see Fig. 5.1).

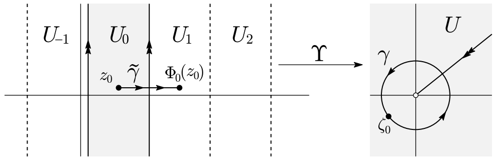

Fig. 5.1 The universal covering of the punctured plane

Fix a point $z_0 \in \mathbb{C}$, let $\zeta_0 \equiv \Upsilon(z_0) \in \mathbb{C}^*$, and for each $t \in [0, 1]$, let $\tilde{\gamma}(t) = z_0 + t$ and $\gamma(t) = \Upsilon(\tilde{\gamma}(t)) = \zeta_0 e^{2\pi i t}$. Clearly, the map $\Phi_0 \colon z \mapsto z + 1$ is a deck transformation, so we have an injective homomorphism $\mathbb{Z} \to \mathrm{Deck}(\Upsilon)$ given by $n \mapsto \Phi_0^n$ (where Φ_0^n is the translation $z \mapsto z + n$). Moreover, the fiber $\Upsilon^{-1}(\zeta_0)$ is equal to $z_0 + \mathbb{Z}$, so we also have surjectivity. Thus $\pi_1(\mathbb{C}^*) \cong \mathrm{Deck}(\Upsilon) \cong \mathbb{Z}$. In fact, since the lifting $\tilde{\gamma}$ of γ satisfies $\tilde{\gamma}(0) = z_0$ and $\tilde{\gamma}(1) = z_0 + 1 = \Phi_0(z_0)$, $\pi_1(\mathbb{C}^*, \zeta_0)$ is the infinite cyclic group generated by $[\gamma]_{\zeta_0}$. That is, $[\gamma]_{\zeta_0} \mapsto 1$ under the composition of the isomorphism $\mathrm{Deck}(\Upsilon) \to \mathbb{Z}$ (given by $\Phi \mapsto \Phi(z_0) - z_0$) and the isomorphism $\pi_1(X, \zeta_0) \mapsto \mathrm{Deck}(\Upsilon)$ (given by $[\alpha]_{\zeta_0} \mapsto \Phi$, where $\Phi(z_0) = \tilde{\alpha}(1)$ for $\tilde{\alpha}$ the lifting of α with $\tilde{\alpha}(0) = z_0$).

Any proper subgroup Γ of $\mathbb{Z} \cong \mathrm{Deck}(\Upsilon) \cong \pi_1(\mathbb{C}^*, \zeta_0)$ must be of the form $\Gamma = m\mathbb{Z}$ for some positive integer m; that is, Γ is the infinite cyclic subgroup generated by $[\gamma]_{\zeta_0}^m$. We have a holomorphic universal covering map $\Upsilon_\Gamma \colon \mathbb{C} \to \mathbb{C}^*$ given by $z \mapsto e^{2\pi i z/m}$. In fact, Υ_Γ is the composition of the holomorphic universal covering map $\mathbb{C} \to \mathbb{C}^*$ given by $z \mapsto e^{2\pi i z}$ and the biholomorphism $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z/m$. The universal covering map Υ_Γ has associated deck transformation group $\mathrm{Deck}(\Upsilon_\Gamma) = \Gamma \subset \mathrm{Deck}(\Upsilon)$, because $\Gamma \subset \mathrm{Deck}(\Upsilon_\Gamma)$, and for any point $z \in \mathbb{C}$, $\Gamma \cdot z = z + m\mathbb{Z} = \Upsilon_\Gamma^{-1}(\Upsilon_\Gamma(z))$. Therefore, $\mathbb{C}^* \cong \Gamma \backslash \mathbb{C}$ and the holomorphic covering space of $\mathbb{C}^*$ associated to the subgroup Γ as in Corollary 5.1.8 is the finite holomorphic covering $\mathbb{C}^* \to \mathbb{C}^*$ given by $\exp(2\pi i z/m) \mapsto \exp(2\pi i z)$; that is, the mapping is given by $z \mapsto z^m$.

Example 5.1.10 Let us consider complex tori from the point of view of holomorphic covering spaces (see Example 2.1.6). Recall that a *lattice* in $\mathbb{C}$ is a subgroup of the form $\Gamma = \mathbb{Z}\xi_1 + \mathbb{Z}\xi_2$, where $\xi_1, \xi_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. We have an injective homomorphism $\Gamma \hookrightarrow \mathrm{Aut}(\mathbb{C})$ under which each element $\xi \in \Gamma$ maps to the automorphism given by $z \mapsto z + \xi$. Thus we may identify Γ with a subgroup of $\mathrm{Aut}(\mathbb{C})$. Clearly, Γ acts freely. Moreover, Γ is the image of $\mathbb{Z}^2$ under the real linear isomorphism $\alpha \colon \mathbb{R}^2 \to \mathbb{C}$ given by $(t_1, t_2) \mapsto t_1\xi_1 + t_2\xi_2$, and hence Γ is a discrete subset of $\mathbb{C}$. Hence, if K is a compact subset of $\mathbb{C}$, then the set

$$\Gamma_K = \{\xi \in \Gamma \mid |\xi| \leq \operatorname{diam} K\}$$

is finite. Since $K \cap ((\Gamma \setminus \Gamma_K) \cdot K) = \emptyset$, it follows that Γ must also act properly discontinuously. Thus we have the holomorphic covering map

$$\Upsilon_\Gamma \colon \mathbb{C} \to X = \Gamma \backslash \mathbb{C},$$

where X is a complex torus.

Topologically, X is the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. In fact, we have a commutative diagram of $\mathcal{C}^\infty$ maps

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\alpha"] \arrow[d, "\Upsilon_0"'] & \mathbb{C} \arrow[d, "\Upsilon"] \\ \mathbb{T}^2 \arrow[r, "\beta"] & X \end{tikzcd}$$

where α is the real linear isomorphism (and therefore a diffeomorphism) given by $(t_1, t_2) \mapsto t_1\xi_1 + t_2\xi_2$ (we may consider $\mathbb{Z}^2$, which is mapped onto Γ, as a group of diffeomorphisms of $\mathbb{R}^2$ that is given by translations and that acts properly discontinuously and freely), Υ_0 is the $\mathcal{C}^\infty$ covering map onto the real torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ given by $(t_1, t_2) \mapsto (e^{2\pi i t_1}, e^{2\pi i t_2})$, and β is the induced diffeomorphism.

Example 5.1.11 The holomorphic universal covering of the punctured unit disk $\Delta^* = \Delta^*(0; 1) = \{\zeta \in \mathbb{C} \mid 0 < |\zeta| < 1\}$ is the holomorphic mapping $\Upsilon : \mathbb{H} \to \Delta^*$ given by $\Upsilon(z) = \exp(2\pi i z) \in \Delta^*(0; 1)$ for each point z in the *upper half-plane* $\mathbb{H} \equiv \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. For the inverse image of Δ^* under the universal covering map $\mathbb{C} \to \mathbb{C}^*$ given by $z \mapsto \exp(2\pi i z)$ (Example 5.1.9) is $\mathbb{H}$, so the restriction of the covering map to $\mathbb{H}$ gives the universal covering of Δ^*. Given a point $z_0 = x_0 + i y_0 \in \mathbb{H}$ with $x_0, y_0 \in \mathbb{R}$, the inverse image $\Upsilon^{-1}(U)$ of the connected neighborhood

$$\begin{aligned} U &\equiv \left\{\rho \exp(2\pi i t) \mid 0 < \rho < 1,\ x_0 - \frac{1}{2} < t < x_0 + \frac{1}{2}\right\} \\ &= \Upsilon\left(\left\{z = x + iy \mid y > 0,\ x_0 - \frac{1}{2} < x < x_0 + \frac{1}{2}\right\}\right) \end{aligned}$$

(an open sector) of $\zeta_0 \equiv \Upsilon(z_0)$ in $\Delta^*(0; 1)$ is equal to the union of the collection of disjoint domains $\{U_n\}_{n \in \mathbb{Z}}$, where for each $n \in \mathbb{Z}$, U_n is the infinite half-strip

$$U_n \equiv \{z = x + iy \mid y > 0, x_0 - (1/2) + n < x < x_0 + (1/2) + n\},$$

which is mapped biholomorphically onto U by Υ (see Fig. 5.2).

Fix a point $z_0 \in \mathbb{H}$, let $\zeta_0 \equiv \Upsilon(z_0) \in \Delta^*(0; 1)$, and for each $t \in [0, 1]$, let $\tilde{\gamma}(t) = z_0 + t$ and $\gamma(t) = \Upsilon(\tilde{\gamma}(t)) = \zeta_0 e^{2\pi i t}$. As in Example 5.1.9, the fundamental group $\pi_1(\Delta^*, \zeta_0) \cong \operatorname{Deck}(\Upsilon) \cong \mathbb{Z}$ is the infinite cyclic (free) group generated by $[\gamma]_{\zeta_0}$, and $\operatorname{Deck}(\Upsilon)$ is precisely the group of automorphisms given by $z \mapsto z + n$ for $n \in \mathbb{Z}$ (i.e., the restrictions to $\mathbb{H}$ of deck transformations of the universal covering $\mathbb{C} \to \mathbb{C}^*$). Any proper subgroup Γ of $\mathbb{Z} \cong \operatorname{Deck}(\Upsilon) \cong \pi_1(\mathbb{C}^*, \zeta_0)$ must be of the form $\Gamma = m\mathbb{Z}$ for some positive integer m (that is, Γ is the infinite cyclic subgroup generated by $[\gamma]_{\zeta_0}^m$), and the corresponding intermediate covering spaces are the mappings

$$\mathbb{H} \to \Delta^* = \Gamma\backslash\mathbb{H} \text{ given by } z \mapsto e^{2\pi i z/m} \quad \text{and} \quad \Delta^* \to \Delta^* \text{ given by } z \mapsto z^m.$$

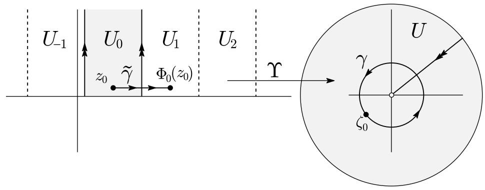

Fig. 5.2 The universal covering of the punctured unit disk

The mapping $z \mapsto 1/z$ gives a biholomorphism $\Delta^* \to \mathbb{C} \setminus \overline{\Delta(0; 1)} = \Delta(0; 1, \infty)$, so the universal covering of $\Delta(0; 1, \infty)$ is the mapping $\mathbb{H} \to \Delta(0; 1, \infty)$ given by

$$z \mapsto 1/e^{2\pi i z} = e^{-2\pi i z}.$$

Example 5.1.12 For $r \in (0, 1)$, the holomorphic universal covering of the annulus $A \equiv \Delta(0; r, 1) = \{\zeta \in \mathbb{C} \mid r < |\zeta| < 1\}$ is the holomorphic mapping $\Upsilon : \mathbb{H} \to A$ given by $z \mapsto \exp(2\pi i L(z))$, where

$$L : \mathbb{H} \to B \equiv \left\{ \zeta \in \mathbb{C} \mid 0 < \operatorname{Im} \zeta < \frac{\pi}{\log \lambda} = -\frac{\log r}{2\pi} \right\}$$

is the holomorphic branch of the logarithmic function $z \mapsto \log_\lambda z$ with base $\lambda \equiv \exp(-2\pi^2/\log r) > 1$ (see Example 1.6.2) given by

$$z \mapsto \frac{1}{\log \lambda} \cdot (\log |z| + i \operatorname{arccot}(\operatorname{Re} z / \operatorname{Im} z))$$

(and the *arccotangent function* is given by $\operatorname{arccot} \equiv (\cot \restriction_{(0,\pi)})^{-1} : \mathbb{R} \to (0, \pi)$). For B is the inverse image of A under the universal covering map $\mathbb{C} \to \mathbb{C}^*$ given by $\zeta \mapsto e^{2\pi i \zeta}$ (as in Examples 5.1.9 and 5.1.11), L is a biholomorphism of $\mathbb{H}$ onto B (with inverse map given by the exponential function $\zeta \mapsto e^{\zeta \log \lambda} = \lambda^\zeta$ with base λ), and Υ is the composition of the two mappings.

Fix $z_0 = R_0 e^{i\alpha_0} \in \mathbb{H}$ with $R_0 > 0$ and $0 < \alpha_0 < \pi$, and let $\zeta_0 \equiv \Upsilon(z_0) = \rho_0 e^{i\theta_0}$ with

$$\rho_0 = |\zeta_0| = e^{-2\pi\alpha_0/\log\lambda} = e^{(\alpha_0/\pi)\log r} \quad \text{and} \quad \theta_0 = \frac{2\pi \log R_0}{\log \lambda} = -\frac{(\log R_0)(\log r)}{\pi}.$$

For each $t \in [0, 1]$, let

$$\tilde{\gamma}(t) \equiv z_0 \cdot \lambda^t = L^{-1}(L(z_0) + t) \quad \text{and} \quad \gamma(t) \equiv \Upsilon(\tilde{\gamma}(t)) = \zeta_0 \cdot e^{2\pi i t}.$$

As in Examples 5.1.9 and 5.1.11, the deck transformations of the universal covering map $B \to A$ are precisely the translations $\zeta \mapsto \zeta + n$ for $n \in \mathbb{Z}$. Pulling back to $\mathbb{H}$, we see that the deck transformations of $\Upsilon : \mathbb{H} \to A$ are given by

$$z \mapsto L^{-1}(L(z) + n) = \exp((L(z) + n)\log\lambda) = \lambda^n z$$

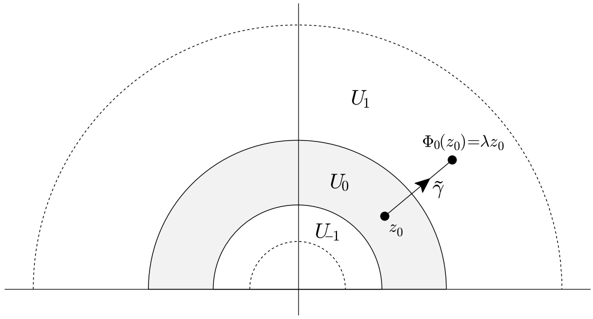

Fig. 5.3 The universal covering of an annulus

for $n \in \mathbb{Z}$. Thus $\pi_1(A) \cong \mathrm{Deck}(\Upsilon) \cong \mathbb{Z}$ with generator $\Phi_0 \colon z \mapsto \lambda z$, and in fact, $\pi_1(A, \zeta_0)$ is the infinite cyclic (free) group generated by $[\gamma]_{\zeta_0}$. Observe also that the inverse image $\Upsilon^{-1}(U)$ of the connected neighborhood

$$U \equiv \{\rho e^{i\theta} \mid r < \rho < 1, \theta_0 - \pi < \theta < \theta_0 + \pi\} \subset A$$

of ζ_0 is equal to the union of the disjoint domains $\{U_n\}_{n\in\mathbb{Z}}$, where for each $n \in \mathbb{Z}$,

$$U_n \equiv \{z \in \mathbb{C} \mid \operatorname{Im} z > 0, \lambda^{n-(1/2)} R_0 < |z| < \lambda^{n+(1/2)} R_0\}$$

and Υ maps U_n biholomorphically onto U (see Fig. 5.3).

Any proper subgroup Γ of $\mathbb{Z} \cong \mathrm{Deck}(\Upsilon) \cong \pi_1(A, \zeta_0)$ must be of the form $\Gamma = m\mathbb{Z}$ for some positive integer m; that is, Γ is the infinite cyclic subgroup generated by $[\gamma]_{\zeta_0}^m$, that is, by $\Phi_0^m \colon z \mapsto \lambda^m z$. Setting $\check{A} = \Delta(0; \sqrt[m]{r}, 1)$, we have $\exp(-2\pi^2/\log \sqrt[m]{r}) = \lambda^m$, and hence we have a holomorphic universal covering map $\Upsilon_\Gamma \colon \mathbb{H} \to \check{A}$ given by

$$z \mapsto \exp\left(\frac{2\pi i}{m} L(z)\right).$$

The universal covering map Υ_Γ has associated deck transformation group $\mathrm{Deck}(\Upsilon_\Gamma) = \Gamma \subset \mathrm{Deck}(\Upsilon)$. Therefore, $\check{A} \cong \Gamma\backslash\mathbb{H}$ and the holomorphic covering space of A associated to the subgroup Γ as in Corollary 5.1.8 is the finite holomorphic covering $\check{A} \to A$ given by $\exp(2\pi i L(z)/m) \mapsto \exp(2\pi i L(z))$; that is, the mapping is given by $z \mapsto z^m$.

For $s, t \in \mathbb{R}$ with $0 < s < t$, we have a biholomorphism $\Delta(0; s/t, 1) \to \Delta(0; s, t)$ given by $\zeta \mapsto t\zeta$. Hence, for $\lambda \equiv \exp(-2\pi^2/\log(s/t))$, the map $\mathbb{H} \to \Delta(0; s, t)$ given by $z \mapsto t \exp(2\pi i L(z)/\log \lambda)$ is the holomorphic universal covering of $\Delta(0; s, t)$. As in the case $t = 1$, every finite holomorphic covering of $\Delta(0; s, t)$ is equivalent to a covering of the form $\Delta(0; \sqrt[m]{s}, \sqrt[m]{t}) \to \Delta(0; s, t)$ given by $z \mapsto z^m$ for some $m \in \mathbb{Z}_{\geq 0}$. This characterization of finite coverings of annuli will be applied in the proof of the Riemann mapping theorem in Sect. 5.5.

Moreover, it turns out that two annuli $\Delta(0; r, 1)$ and $\Delta(0; s, 1)$ with $r, s \in (0, 1)$ are biholomorphic if and only if $r = s$. Equivalently, two annuli $\Delta(0; r, s)$ and $\Delta(0; R, S)$ with $0 < r < s < +\infty$ and $0 < R < S < +\infty$ are biholomorphic if and only if $r/s = R/S$ (see Exercise 5.8.2).

Exercises for Sect. 5.1

5.1.1 Prove that holomorphic equivalence of holomorphic covering spaces is an equivalence relation.

5.1.2 Prove part (a) of Theorem 5.1.3.

5.1.3 Prove Theorem 5.1.6.

5.1.4 Prove that for $r \in (0, 1)$, the universal covering map $\Upsilon: \mathbb{H} \to \Delta(0; r, 1)$ appearing in Example 5.1.12 extends to a universal covering map $\overline{\mathbb{H}} \setminus \{0\} \to \overline{\Delta(0; r, 1)}$ (where $\overline{\mathbb{H}}$ and $\overline{\Delta(0; r, 1)}$ are the closures of $\mathbb{H}$ and $\Delta(0; r, 1)$, respectively).

5.1.5 For each loop γ in $\mathbb{C}$ and each point $z_0 \in \mathbb{C} \setminus \gamma([0, 1])$, the *winding number of* γ *around* z_0 is given by $n(\gamma; z_0) \equiv \tilde{\gamma}(1) - \tilde{\gamma}(0)$, where $\tilde{\gamma}$ is a lifting of γ under the universal covering mapping $\mathbb{C} \to \mathbb{C} \setminus \{z_0\}$ given by $\zeta \mapsto z_0 + e^{2\pi i \zeta}$ (see Example 5.1.9).

(a) Show that the winding number is an integer that is independent of the choice of the lifting of the loop to the universal cover.

(b) Show that for distinct points $z_0, z_1 \in \mathbb{C}$, the mapping $[\gamma]_{z_1} \mapsto n(\gamma; z_0)$ determines a well-defined group isomorphism $\pi_1(\mathbb{C} \setminus \{z_0\}, z_1) \xrightarrow{\cong} \mathbb{Z}$.

(c) Prove that for each loop γ in $\mathbb{C}$ and each point $z_0 \in \mathbb{C} \setminus \gamma([0, 1])$,

$$n(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} \quad \text{(see Definition 10.5.3).}$$

5.1.6 Consideration of winding numbers (see Exercise 5.1.5) yields versions of the Cauchy integral formula and Cauchy's theorem (cf. Lemma 1.2.1 and Exercise 6.7.1), the residue theorem (cf. Exercises 2.5.9, 5.9.5, 6.7.1, and 6.7.6), and the argument principle and Rouché's theorem (cf. Exercises 2.5.8, 2.9.5, 6.7.1, and 6.7.6) for general loops. Let Ω be a domain in $\mathbb{C}$, let γ be loop in Ω that is path homotopic in Ω to a constant loop, and let $C \equiv \gamma([0, 1])$.

(a) *Cauchy's theorem*. Prove that for each function $f \in \mathcal{O}(\Omega)$,

$$\int_\gamma f(z)\, dz = 0.$$

Hint. Apply Theorem 10.5.6.

(b) *Cauchy integral formula*. Prove that for each function $f \in \mathcal{O}(\Omega)$ and each point $z_0 \in \Omega \setminus C$,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz = f(z_0) n(\gamma; z_0).$$

(c) Prove that there is a domain Ω_0 such that $C \subset \Omega_0 \Subset \Omega$ and γ is path homotopic in Ω_0 to a constant loop. Conclude that in particular,

$$n(\gamma; z_0) = 0 \quad \forall z_0 \in \mathbb{C} \setminus \Omega_0.$$

(d) *Residue theorem*. Prove that if S is a discrete subset of Ω that is contained in $\Omega \setminus C$, and θ is a holomorphic 1-form on $\Omega \setminus S$, then

$$\frac{1}{2\pi i}\int_\gamma \theta = \sum_{p\in S} n(\gamma; p)\mathrm{res}_p\theta$$

(note that the right-hand side is actually a finite sum by part (c)).

Hint. Using part (c), show that one may assume without loss of generality that S is a finite set. Writing $\theta = f(z)\,dz$, and letting g_p be the *principal part of f about p* (i.e., the sum of the negative-exponent terms of the Laurent series about p) for each $p \in S$, one gets a holomorphic 1-form $\theta - [\sum_{p\in S} g_p(z)]\,dz$ on Ω.

(e) *Argument principle*. Prove that if f is a nontrivial meromorphic function on Ω with zero set Z and pole set P both contained in $\Omega \setminus C$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{df}{f} = \sum_{z_0\in Z\cup P} n(\gamma; z_0)\,\mathrm{ord}_{z_0} f.$$

(f) *Rouché's theorem*. Suppose f and g are nontrivial meromorphic functions on Ω that do not have any zeros or poles in C, and $|g| < |f|$ on C. Let Z_f denote the zero set of f, P_f the pole set of f, Z_{f+g} the zero set of $f+g$, and P_{f+g} the pole set of $f+g$. Prove that

$$\sum_{z_0\in Z_f\cup P_f} n(\gamma; z_0)\,\mathrm{ord}_{z_0} f = \sum_{z_0\in Z_{f+g}\cup P_{f+g}} n(\gamma; z_0)\,\mathrm{ord}_{z_0}(f+g).$$

5.1.7 The goal of this problem to obtain more general (homological) versions of the facts considered in Exercise 5.1.6. Let Ω be a domain in $\mathbb{C}$, let $\gamma_1, \dots, \gamma_m$ be loops in Ω, let $C \equiv \bigcup_{j=1}^m \gamma_j([0,1])$, and let $a_1, \dots, a_m \in \mathbb{C}$. Assume that

$$\sum_{j=1}^m a_j \int_{\gamma_j} \beta = 0$$

for every $\mathcal{C}^\infty$ closed 1-form β on Ω (in the language of Sects. 10.6 and 10.7, the 1-cycle $\xi = \sum a_j\gamma_j$ is *homologous* to 0 in Ω). Note that according to Theorem 10.5.6, this is the case if, for example, γ_j is path homotopic in Ω to a constant loop for each j.

(a) *Cauchy's theorem*. Prove that for each function $f \in \mathcal{O}(\Omega)$,

$$\sum_{j=1}^m a_j \int_{\gamma_j} f(z)\,dz = 0.$$

(b) *Cauchy integral formula*. Prove that for each function $f \in \mathcal{O}(\Omega)$ and each point $z_0 \in \Omega \setminus C$,

$$\sum_{j=1}^{m} \frac{a_j}{2\pi i} \int_{\gamma_j} \frac{f(z)}{z - z_0}\, dz = f(z_0) \sum_{j=1}^{m} a_j n(\gamma_j; z_0).$$

(c) Prove that there is an open set Ω_0 such that $C \subset \Omega_0 \Subset \Omega$ and

$$\sum_{j=1}^{m} a_j n(\gamma_j; z_0) = 0 \quad \forall z_0 \in \mathbb{C} \setminus \Omega_0.$$

Hint. Observe that the left-hand side of the above equation is a continuous function of z_0.

(d) *Residue theorem*. Prove that if S is a discrete subset of Ω that is contained in $\Omega \setminus C$, and θ is a holomorphic 1-form on $\Omega \setminus S$, then

$$\sum_{j=1}^{m} \frac{a_j}{2\pi i} \int_{\gamma_j} \theta = \sum_{p \in S} \sum_{j=1}^{m} a_j n(\gamma_j; p) \mathrm{res}_p \theta$$

(note that the right-hand side is actually a finite sum by part (c)).

Hint. In the proof of part (d) of Exercise 5.1.6, one reduced to the case in which S is finite; but it is not clear that one can do so here (although it is possible to do so using, for example, the homological fact considered in Exercise 5.17.5). Assuming that S is infinite, let $\{p_n\}$ be an enumeration of S, and writing $\theta = f(z)\,dz$, let g_n be the principal part of f about p_n for each n (see part (d) of Exercise 5.1.6). Fix a sequence of compact sets $\{K_\nu\}$ such that $K_\nu \subset \mathring{K}_{\nu+1}$ and $\mathfrak{h}_\Omega(K_\nu) = K_\nu$ for each ν and $\Omega = \bigcup_{\nu=1}^{\infty} K_\nu$. For each n, apply the Runge approximation theorem to get a holomorphic function h_n on Ω such that $|g_n - h_n| < 2^{-n}$ on K_ν whenever $p_n \notin K_\nu$. Show that $\theta - [\sum_{n=1}^{\infty}(g_n - h_n)]\,dz$ is a holomorphic 1-form on Ω.

(e) *Argument principle*. Prove that if f is a nontrivial meromorphic function on Ω with zero set Z and pole set P both contained in $\Omega \setminus C$, then

$$\sum_{j=1}^{m} \frac{a_j}{2\pi i} \int_{\gamma_j} \frac{df}{f} = \sum_{z_0 \in Z \cup P} \sum_{j=1}^{m} a_j n(\gamma_j; z_0)\, \mathrm{ord}_{z_0} f.$$

(f) *Rouché's theorem*. Suppose f and g are nontrivial meromorphic functions on Ω that do not have any zeros or poles in C, and $|g| < |f|$ on C. Let Z_f denote the zero set of f, P_f the pole set of f, Z_{f+g} the zero set of $f + g$, and P_{f+g} the pole set of $f + g$. Prove that

$$\sum_{z_0 \in Z_f \cup P_f} \sum_{j=1}^{m} a_j n(\gamma_j; z_0) \operatorname{ord}_{z_0} f$$
$$= \sum_{z_0 \in Z_{f+g} \cup P_{f+g}} \sum_{j=1}^{m} a_j n(\gamma_j; z_0) \operatorname{ord}_{z_0}(f+g).$$

5.2 The Riemann Mapping Theorem in the Plane

The goal of this section is the following:

Theorem 5.2.1 (Riemann mapping theorem in the plane) *Any simply connected domain Ω in $\mathbb{C}$ with $\Omega \neq \mathbb{C}$ is biholomorphic to the unit disk $\Delta = \Delta(0; 1)$.*

The proof given here is due to Fejér and F. Riesz. We first consider the following:

Proposition 5.2.2 *Let f be a nonvanishing holomorphic function on a simply connected Riemann surface X. Then there exists a holomorphic function h such that $e^h = f$ on X (i.e., h is a single-valued holomorphic branch of the function* $\log f$*). In particular, for every positive integer q, there is a holomorphic function $k = e^{h/q}$ such that $k^q = f$ (i.e., k is a single-valued holomorphic qth root of f).*

Proof By Corollary 10.5.7, the holomorphic 1-form $\theta = df/f$ is exact, and hence there is a holomorphic function h on X such that $dh = \theta$ ($\bar{\partial} h$ is equal to the $(0,1)$ part of θ, which is 0). Fixing a point $p \in X$ and a single-valued holomorphic branch g of the function $\log f$ on a connected neighborhood U of p, we may choose h so that $h(p) = g(p)$. Since $dh = \theta = dg$ on U, it follows that the holomorphic functions e^h and f agree on U and therefore on X. □

The biholomorphism will be obtained as a limit of a sequence of holomorphic mappings, so the following observation will be required:

Lemma 5.2.3 *Let X be a Riemann surface, and let $\{f_\nu\}$ be a sequence of injective holomorphic functions on X (i.e., a sequence of biholomorphic mappings of X onto open sets in $\mathbb{C}$) that converges uniformly on compact subsets of X to a nonconstant holomorphic function f. Then f is injective.*

Proof Given a point $a \in X$, the identity theorem (Theorem 2.2.2) implies that we may choose a relatively compact connected neighborhood D of a with $f(\partial D) \subset \mathbb{C} \setminus \{f(a)\}$. Hence there exists a $\delta > 0$ such that $|f - f(a)| > \delta$ on ∂D, and for $\nu \gg 0$, we also get $|f_\nu - f_\nu(a)| > \delta$ on ∂D. For $\nu \gg 0$, we have $\delta > |f_\nu(a) - f(a)|$, and it follows that $f(a) \in f_\nu(D)$, because any point ζ in the complement of the domain $f_\nu(D)$ satisfies

$$|f_\nu(a) - \zeta| \geq \operatorname{dist}(f_\nu(a), \partial(f_\nu(D))) = \operatorname{dist}(f_\nu(a), f_\nu(\partial D)) \geq \delta.$$

Now if a_0 and a_1 are distinct points in X, then we may choose disjoint neighborhoods D_0 and D_1 of a_0 and a_1, respectively, as above with $f(a_j) \in f_\nu(D_j)$ for $j = 1, 2$ and $\nu \gg 0$. For such an index ν, we also have $f_\nu(D_0) \cap f_\nu(D_1) = \emptyset$ (since f_ν is injective), and hence $f(a_0) \neq f(a_1)$. □

The first step in the proof of Theorem 5.2.1 is the following:

Lemma 5.2.4 *Let $\Omega \subset \mathbb{C}$ be a domain such that $\Omega \neq \mathbb{C}$ and such that any nonvanishing holomorphic function on Ω has a single-valued holomorphic square root, and let $a \in \Omega$. Then there exists an injective holomorphic map $f : \Omega \to \Delta \equiv \Delta(0; 1)$ with $f(a) = 0$.*

Proof Fixing a point $z_0 \in \mathbb{C} \setminus \Omega$, we may form a function $h \in \mathcal{O}(\Omega)$ such that $[h(z)]^2 = z - z_0$ for each point $z \in \Omega$. In particular, h is injective. Moreover, no two points $w, z \in \Omega$ can satisfy $h(w) = -h(z)$. For if such a pair exists, then squaring, we get $w = z$. But then $h(w) = h(z) = 0$, and squaring again, we get $z - z_0 = 0$, which is impossible, since $z_0 \notin \Omega$. By the open mapping theorem, $h(\Omega)$ is a domain, and hence for some $\delta > 0$, we have $\Delta(h(a); \delta) \subset h(\Omega)$. It follows that $h(\Omega) \cap \Delta(-h(a); \delta) = \emptyset$. For if $\zeta \in \Delta(-h(a); \delta)$, then $-\zeta \in \Delta(h(a); \delta) \subset h(\Omega)$. Since $h(\Omega) \cap (-h(\Omega)) = \emptyset$, we must have $\zeta \notin h(\Omega)$. It is now easy to check that for any sufficiently large constant $R > 0$, the function f on Ω given by

$$f(z) = \frac{1}{R} \cdot \frac{h(z) - h(a)}{h(z) + h(a)} \quad \forall z \in \Omega$$

has the required properties (see Exercise 5.2.1). □

Theorem 5.2.1 is an immediate consequence of Proposition 5.2.2 and the following version:

Theorem 5.2.5 *Let $\Omega \subset \mathbb{C}$ be a domain such that $\Omega \neq \mathbb{C}$ and such that any nonvanishing holomorphic function on Ω has a single-valued holomorphic square root. Then Ω is biholomorphic to the unit disk $\Delta \equiv \Delta(0; 1)$.*

Proof Fixing a point $a \in \Omega$, Lemma 5.2.4 implies that

$$\mathcal{F} \equiv \{f \in \mathcal{O}(\Omega) \mid f \text{ is injective, } f(\Omega) \subset \Delta, \text{ and } f(a) = 0\} \neq \emptyset.$$

In particular, $M \equiv \sup_{f \in \mathcal{F}} |f'(a)| \in (0, \infty]$. We may choose a sequence $\{f_\nu\}$ in $\mathcal{F}$ such that $|f_\nu'(a)| \to M$ as $\nu \to \infty$, and after applying Montel's theorem (Corollary 1.2.7) and passing to a subsequence, we may assume without loss of generality that $\{f_\nu\}$ converges uniformly on compact subsets of Ω to a holomorphic function f. Corollary 1.2.6 then implies that $|f'(a)| = M$. In particular, $M \in (0, \infty)$, f is nonconstant, and therefore by Lemma 5.2.3, f is injective. We also have $|f| \leq 1$, and hence the maximum principle implies that $|f| < 1$ on Ω.

It remains to show that $f(\Omega) = \Delta$. For each point $c \in \Delta$, let Φ_c be the automorphism of the disk given by

$$\Phi_c(z) = \frac{z - c}{1 - \bar{c}z} \quad \forall z \in \Delta$$

(see Proposition 2.14.8). If there exists a point $c \in \Delta \setminus f(\Omega) \subset \mathbb{C}^*$, then the injective holomorphic function $\Phi_c \circ f : \Omega \to \Delta$ is nonvanishing, and hence there exists an injective holomorphic function $h \colon \Omega \to \Delta$ such that $h^2 = \Phi_c \circ f$. Differentiation at a then gives $2|c|^{1/2}|h'(a)| = (1 - |c|^2)M$. The injective holomorphic function $g \equiv \Phi_{h(a)} \circ h \colon \Omega \to \Delta$ satisfies $g(a) = 0$, and hence $g \in \mathcal{F}$. However, we have

$$|g'(a)| = (1 - |c|)^{-1} \cdot (1 - |c|^2)M/(2\sqrt{|c|}) = (1 + |c|)M/(2\sqrt{|c|}) > M.$$

Thus we have arrived at a contradiction, and therefore f is a biholomorphism of Ω onto the disk Δ. $\square$

Corollary 5.2.6 *If Ω is a simply connected domain in $\mathbb{P}^1$, and $\mathbb{P}^1 \setminus \Omega$ contains more than one point, then Ω is biholomorphic to $\Delta(0; 1)$.*

Proof Fixing a point $\xi \in \mathbb{C} \setminus \Omega$ and replacing Ω with its image under the automorphism $\mathbb{P}^1 \to \mathbb{P}^1$ given by $z \mapsto 1/(z - \xi)$ if necessary, we may assume that Ω is a proper domain in $\mathbb{C}$. The claim now follows from Proposition 5.2.2 and Theorem 5.2.5. $\square$

The proof of the Riemann mapping theorem gives the following:

Corollary 5.2.7 *If Ω is a domain in $\mathbb{P}^1$ and every closed $\mathcal{C}^\infty$ 1-form on Ω is exact, then Ω is biholomorphic to $\mathbb{P}^1$, to $\mathbb{C}$, or to $\Delta(0; 1)$. In particular, Ω is simply connected.*

The proof is left to the reader (see Exercise 5.2.2).

Remark According to the Osgood–Taylor–Carathéodory theorem (see [OT] and [C]), if $\Omega \subset \mathbb{P}^1$ is a simply connected region and $\partial\Omega$ is a Jordan curve (see Sect. 5.10), then any biholomorphism of Ω onto the disk Δ may be extended to a homeomorphism of $\overline{\Omega}$ onto $\overline{\Delta}$. This fact is useful (and appealing), but it is not proved (or applied) in this book. However, a related fact called *Schönflies' theorem* (Theorem 6.7.1) is considered in Chap. 6.

Exercises for Sect. 5.2

5.2.1 Verify that the function f constructed in the proof of Lemma 5.2.4 has the required properties.

5.2.2 Prove Corollary 5.2.7.

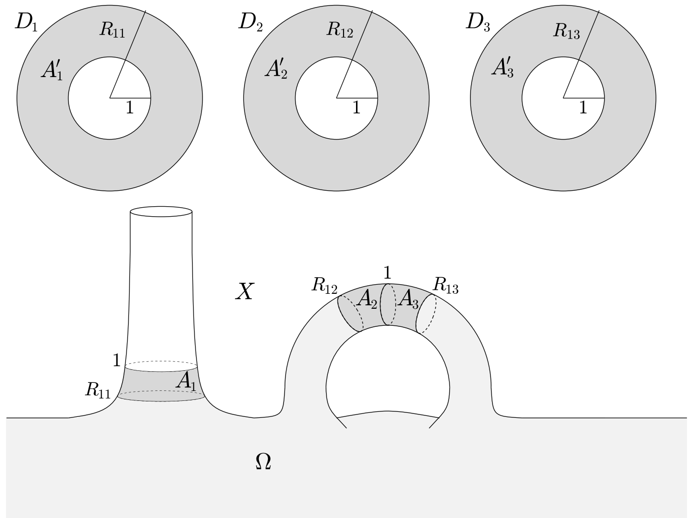

Fig. 5.4 Three disks (or caps) and a domain with three boundary coordinate annuli

5.3 Holomorphic Attachment of Caps

As considered in Sect. 2.3, one may construct examples of Riemann surfaces by holomorphic attachment. In this section, we consider holomorphic attachment of caps (i.e., disks), which in particular, may be used to compactify a suitable domain. Such compactifications will play an important role in the proof of the Riemann mapping theorem for a Riemann surface.

Let X be a complex 1-manifold; let $\Omega \subset X$ be an open set; let $\{R_{0j}\}_{j\in J}$ and $\{R_{1j}\}_{j\in J}$ be collections of numbers in $(1,\infty)$; let $\{(U_j, \Phi_j, \Delta(0; 1/R_{0j}, R_{1j}))\}_{j\in J}$ be a *locally finite* family of local holomorphic charts in X with $\partial\Omega \subset \bigcup_{j\in J} U_j$; and for each $j \in J$, let $A_j \equiv \Phi_j^{-1}(\Delta(0; 1, R_{1j}))$, $D_j \equiv \Delta(0; R_{1j})$, and $A'_j \equiv \Delta(0; 1, R_{1j}) \subset D_j$. Assume that $W \cap \Omega \subset \bigcup_{j\in J} A_j$ for some neighborhood W of $\partial\Omega$ in X, and that for each index $j \in J$, $A_j \subset \Omega$, $A_j \cap A_k = \emptyset$ for each $k \in J \setminus \{j\}$, and $U_j \cap \partial\Omega = \Phi_j^{-1}(\partial\Delta(0; 1))$ (note that the sets $\{U_j\}$ need not be disjoint; for example, we may let $\Omega \equiv \mathbb{C} \setminus \partial\Delta(0; 1) \subset X \equiv \mathbb{C}$, $U_0 = U_1 \equiv \Delta(0; 1/2, 2)$, $\Phi_0 \colon z \mapsto 1/z$, $\Phi_1 \colon z \mapsto z$, $A_0 \equiv \Delta(0; 1/2, 1)$, and $A_1 \equiv \Delta(0; 1, 2)$). Then we have the biholomorphism

$$\Psi \colon A \equiv \bigcup_{j\in J} A_j \xrightarrow{\cong} A' \equiv \bigsqcup_{j\in J} A'_j \subset D \equiv \bigsqcup_{j\in J} D_j,$$

where for every index $j \in J$ and every point $p \in A_j$, $\Psi(p)$ is the image of $\Phi_j(p)$ under the holomorphic inclusion $A'_j \hookrightarrow A'$ (see Fig. 5.4, in which $J = \{1, 2, 3\}$).

The set $\Psi^{-1}(K' \cap A')$ is closed in Ω for every compact set $K' \subset D$, and the set $\Psi(K \cap A)$ is closed in D for every compact set $K \subset \Omega$. Thus we get the holomorphic

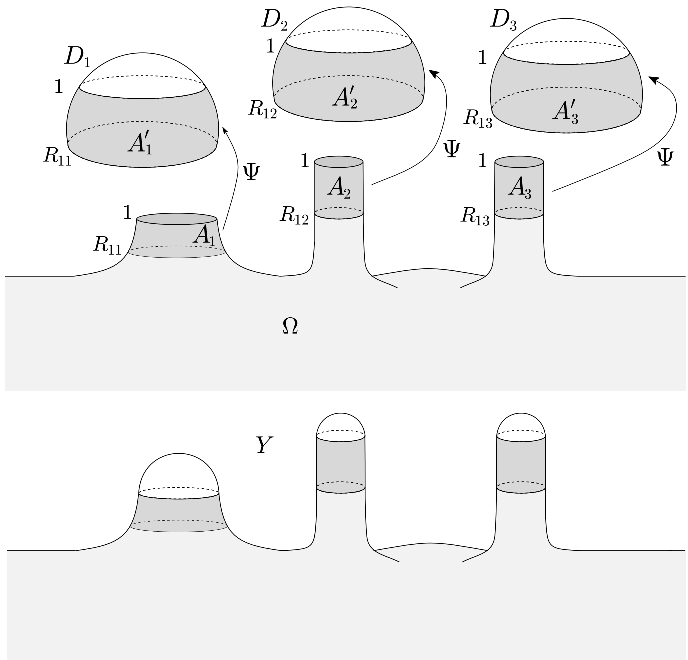

Fig. 5.5 Holomorphic attachment of the caps

attachment

$$Y \equiv D \cup_{\Psi} \Omega = D \sqcup \Omega/\sim,$$

where for $j \in J$, $p \in A_j$, and $z \in A'_j$, the images p_0 of p and z_0 of z under the inclusions $A_j, A'_j \hookrightarrow D \sqcup \Omega$ satisfy $p_0 \sim z_0$ if and only if $\Phi_j(p) = z$. In particular, for the holomorphic inclusions $\Pi_\Omega \colon \Omega \hookrightarrow Y$ and $\Pi_{D_j} \colon D_j \hookrightarrow Y$ for $j \in J$, we have

$$\Pi_\Omega(\Omega) = Y \setminus \bigcup_{j \in J} \Pi_{D_j}(\overline{\Delta(0;1)}).$$

In other words, Y is a complex 1-manifold obtained by gluing caps (i.e., disks) at the boundary of Ω (see Fig. 5.5). We say that Y is obtained by *holomorphic attachment of caps* (or *disks*) *to* Ω *at the boundary*.

Observe that Y is compact if and only if $\Omega \Subset X$. If Y is connected, then Ω is connected, but the converse is false. Observe also that if in some local holomorphic coordinate neighborhood of some boundary component, Ω is the complement of a concentric circle in an annulus, then we are separating the two pieces and gluing a cap over each piece (as pictured in Fig. 5.5, one cuts a tube into two new tubes, and then places a cap over each of the two new ends). This process of *holomorphic tube removal* will be considered in more detail in Sect. 5.12. It will be applied in the characterization of a Riemann surface by holomorphic tube attachment (see Sects. 5.13 and 5.14).

Although it is natural to picture D as being glued onto Ω (instead of Ω being glued onto D), Ψ is chosen above to map a subset of Ω into the union of disks D (instead of the reverse direction). The mapping is chosen in the above direction in order to make the notation fit with the notation of holomorphic tube attachment and removal (see Sect. 2.3 and Sect. 5.12). The points of view are, of course, equivalent, since we have $D \cup_\Psi \Omega = \Omega \cup_{\Psi^{-1}} D$.

Fixing a number $R^*_{\nu j}$ with $1 < R^*_{\nu j} \leq R_{\nu j}$ for each $\nu \in \{0, 1\}$ and $j \in J$, setting $D^*_j \equiv \Delta(0; R^*_{1j}) \subset D_j$ for each $j \in J$, and setting $D^* \equiv \bigsqcup_{j \in J} D^*_j \subset D$, we may form the holomorphic attachment $Y^* = D^* \sqcup \Omega / \sim$. The natural inclusion $D^* \sqcup \Omega \subset D \sqcup \Omega$ then descends to a natural biholomorphism $Y^* \xrightarrow{\cong} Y$ (see Exercise 5.3.1), so we may identify Y^* with Y. In particular, we may always choose $R_{0j}, R_{1j} \in (1, \infty)$ to be equal and arbitrarily close to 1 for each $j \in J$.

Exercises for Sect. 5.3

5.3.1 In the notation of this section, verify that the natural inclusion of $D^* \sqcup \Omega$ into $D \sqcup \Omega$ descends to a biholomorphism of Y^* onto Y (cf. Exercise 2.3.4).

5.4 Exhaustion by Domains with Circular Boundary Components

In this section, we see that every Riemann surface admits an exhaustion by relatively compact domains for which each boundary component is a concentric circle in a coordinate annulus, a fact that will play an important role in the proof of the Riemann mapping theorem. More precisely, we obtain the following:

Proposition 5.4.1 *Given a compact subset K of a Riemann surface X, there exist a compact Riemann surface X', a finite (possibly empty) collection of disjoint local holomorphic charts $\{(D_j, \Psi_j, \Delta(0; R_j))\}_{j=1}^n$ in X' with $R_j > 1$ for each j, and a biholomorphism of a connected relatively compact neighborhood Ω of K in X onto $X' \setminus \bigcup_{j=1}^n \Psi_j^{-1}(\overline{\Delta(0; 1)})$.*

Remark By replacing Ω with the inverse image of $X' \setminus \bigcup_{j=1}^n \Psi_j^{-1}(\overline{\Delta(0; R^*_j)})$, where $R^*_j \in (1, R_j)$ is sufficiently close to 1 for each j, we see that we may choose Ω so that there exist constants $\{R_j\}_{j=1}^n$ in $(1, \infty)$ and disjoint local holomorphic charts $\{(U_j, \Phi_j, \Delta(0; 1/R_j, R_j))\}_{j=1}^n$ in X such that $\partial\Omega \subset U_1 \cup \cdots \cup U_n$ and such that for each $j = 1, \ldots, n$, we have $U_j \cap \partial\Omega = \Phi_j^{-1}(\partial\Delta(0; 1))$ and $A_j \equiv \Phi_j^{-1}(\Delta(0; 1, R_j)) = U_j \cap \Omega$ (in particular, Ω is a smooth domain). In fact, in the proof of Proposition 5.4.1, X' is constructed by holomorphic attachment of caps to such a domain.

We first consider the following:

Lemma 5.4.2 *Suppose f is a holomorphic function on a Riemann surface X, s_0 is a positive regular value of the function $\rho \equiv |f|$, Ω is a connected component of $\{x \in X \mid \rho(x) < s_0\}$, and S is a nonempty compact connected component of $\partial\Omega$. Then there exist $m \in \mathbb{Z}_{>0}$ and $r, t \in \mathbb{R}$ with $0 < r < s \equiv \sqrt[m]{s_0} < t$, and a local holomorphic chart $(A, \Phi = z, \Delta(0; r, t))$, such that $f\restriction_A = z^m$ and*

$$A \cap \Omega = \Phi^{-1}(\Delta(0; r, s)) \quad \textit{and} \quad S = A \cap \partial\Omega = \Phi^{-1}(\partial\Delta(0; s)).$$

Proof At each point $x \in S$, we have

$$\overline{f(x)}(df)_x + f(x)(\overline{df})_x = (d\rho^2)_x = 2\rho(x)(d\rho)_x \neq 0.$$

Hence $(df)_x \neq 0$, and by the holomorphic inverse function theorem (Theorem 2.4.4), we may form a local holomorphic chart of the form $(U_x, \Psi_x = f\restriction_{U_x}, U'_x)$ such that $x \in U_x$ and the sets $U'_x \cap \Delta(0; s_0)$ and $U'_x \cap \partial\Delta(0; s_0)$ are connected. In particular, the connected set

$$\Psi_x^{-1}(U'_x \cap \Delta(0; s_0)) = U_x \cap f^{-1}(\Delta(0; s_0)) = U_x \cap \rho^{-1}((-\infty, s_0))$$

must meet, and therefore lie in, the connected component Ω of $\rho^{-1}((-\infty, s_0))$. Thus $U_x \cap \Omega = \Psi_x^{-1}(U'_x \cap \Delta(0; s_0))$, and similarly,

$$\Psi_x^{-1}(U'_x \cap \partial\Delta(0; s_0)) = U_x \cap f^{-1}(\partial\Delta(0; s_0)) = U_x \cap \rho^{-1}(s_0) = U_x \cap \partial\Omega = U_x \cap S.$$

Fixing a relatively compact connected neighborhood M of S in $\bigcup_{x \in S} U_x$, we see that $f\restriction_M$ is a local biholomorphism and

$$M \cap \Omega = M \cap f^{-1}(\Delta(0; s_0)) \quad \text{and} \quad S = M \cap \partial\Omega = M \cap f^{-1}(\partial\Delta(0; s_0)).$$

In particular, the restriction $f\restriction_S : S \to \partial\Delta(0; s_0)$ is a local diffeomorphism, and hence by part (a) of Lemma 10.2.11, a finite $\mathcal{C}^\infty$ covering map. Therefore $f(S) = \partial\Delta(0; s_0)$, and by part (b) of Lemma 10.2.11, there exist constants r_0 and t_0 and a relatively compact connected neighborhood A of S in M such that $0 < r_0 < s_0 < t_0$ and such that the restriction $f\restriction_A : A \to \Delta(0; r_0, t_0)$ is a finite holomorphic covering map. As in Example 5.1.12, for $r = \sqrt[m]{r_0}$ and $t = \sqrt[m]{t_0}$ for some choice of $m \in \mathbb{Z}_{>0}$, we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\Phi"] \arrow[d, "f\restriction_A"'] & \Delta(0; r, t) \arrow[dl] \\
\Delta(0; r_0, t_0) &
\end{tikzcd}$$

in which Φ is a biholomorphism and the map $\Delta(0; r, t) \to \Delta(0; r_0, t_0)$ is the finite holomorphic covering map given by $z \mapsto z^m$. Thus Φ has the required properties. □

Proof of Proposition 5.4.1 Clearly, we may assume that X is noncompact and that K is nonempty. According to Lemma 2.16.5, there exist a holomorphic function f on a domain $Y \supset K$, a positive regular value s_0 of the function $|f|$, and a

$\mathcal{C}^\infty$ domain Ω that is a connected component of $\{x \in Y \mid |f(x)| < s_0\}$ satisfying $K \subset \Omega \Subset Y$. In particular, $\partial\Omega$ is a compact $\mathcal{C}^\infty$ submanifold of Y.

For each connected component S of $\partial\Omega$, Lemma 5.4.2 provides a positive integer m, constants r and t with $0 < r < s \equiv \sqrt[m]{s_0} < t$, and a local holomorphic chart $(V, \widehat{\Phi}, \Delta(0; r, t))$ such that $S = \widehat{\Phi}^{-1}(\partial\Delta(0; s))$, $V \cap \Omega = \widehat{\Phi}^{-1}(\Delta(0; r, s))$, and $f\restriction_V = \widehat{\Phi}^m$. Fixing R with $s/t < 1/R < 1 < R < s/r$ and setting $\Phi \equiv (s/\widehat{\Phi})\restriction_U$, where $U \equiv \widehat{\Phi}^{-1}(\Delta(0; s/R, sR))$, we then get a local holomorphic chart $(U, \Phi, \Delta(0; 1/R, R))$ such that $S = \Phi^{-1}(\partial\Delta(0; 1))$ and $U \cap \Omega = \Phi^{-1}(\Delta(0; 1, R))$. Applying this to each boundary component, we get constants $\{R_j\}_{j=1}^n$ in $(1, \infty)$ and disjoint local holomorphic charts $\{(U_j, \Phi_j, \Delta(0; 1/R_j, R_j))\}_{j=1}^n$ in X such that $\partial\Omega \subset U_1 \cup \cdots \cup U_n$ and such that for each $j = 1, \ldots, n$, we have $U_j \cap \partial\Omega = \Phi_j^{-1}(\partial\Delta(0; 1))$ and $A_j \equiv \Phi_j^{-1}(\Delta(0; 1, R_j)) = U_j \cap \Omega$. As in Sect. 5.3, we may now let $D_j \equiv \Delta(0; R_j)$ for each $j = 1, \ldots, n$, and we may let X' be the holomorphic attachment given by

$$X' \equiv \bigsqcup_{j=1}^{n} D_j \sqcup \Omega/\sim,$$

where for each $j = 1, \ldots, n$, for $p \in A_j$, and for $z \in A'_j \equiv \Delta(0; 1, R_j) \subset D_j$, the images p_0 of p and z_0 of z under the inclusions $A_j, A'_j \hookrightarrow \bigsqcup_{j=1}^n D_j \sqcup \Omega$ satisfy $p_0 \sim z_0$ if and only if $\Phi_j(p) = z$. The natural inclusion of Ω is a biholomorphism of Ω onto $X' \setminus \bigcup_{j=1}^n \Psi_j^{-1}(\overline{\Delta(0; 1)})$, where for each $j = 1, \ldots, n$, $\Psi_j \equiv \Pi_{D_j}^{-1} \colon \Pi_{D_j}(D_j) \to D_j$ is the local holomorphic chart in X' given by the inverse of the holomorphic inclusion $\Pi_{D_j} \colon D_j \hookrightarrow X'$. □

5.5 Koebe Uniformization

We are now ready to prove the Riemann mapping theorem for a Riemann surface (Theorem 5.0.1). In fact, we will prove something stronger.

Definition 5.5.1 An orientable $\mathcal{C}^\infty$ surface M is *planar* if every closed $\mathcal{C}^\infty$ 1-form with compact support in M is exact.

Remarks 1. Of course, every exact 1-form θ of class $\mathcal{C}^\infty$ is $\mathcal{C}^\infty$-exact; in fact, every potential for θ is of class $\mathcal{C}^\infty$.

2. Planarity of second countable orientable $\mathcal{C}^\infty$ surfaces is actually a topological condition (see Exercise 5.5.2).

Lemma 5.5.2 *Let M be an orientable $\mathcal{C}^\infty$ surface.*

(a) *If M is simply connected, then M is planar.*
(b) *If M is planar, then every domain in M is planar.*
(c) *Every domain in the Riemann sphere $\mathbb{P}^1$ is planar.*

Proof Part (a) follows immediately from Corollary 10.5.7. For part (b), we simply observe that a closed $\mathcal{C}^\infty$ 1-form with compact support in a domain Ω in an orientable planar $\mathcal{C}^\infty$ surface M extends by 0 to a $\mathcal{C}^\infty$ compactly supported closed differential 1-form on M, and therefore the form is exact. Finally, part (c) follows from (a) and (b). □

In particular, the Riemann mapping theorem (Theorem 5.0.1) is an immediate consequence of the Riemann mapping theorem in the plane (Theorem 5.2.1) and the following:

Theorem 5.5.3 (Koebe uniformization theorem [Koe1], [Koe2], [Koe3], [Koe4]) *Any planar Riemann surface is biholomorphic to a domain in* $\mathbb{P}^1$.

The proof given here is essentially due to Simha [Sim] and Demailly [De3], although the idea of forming compactifications of exhausting subdomains is due to Koebe. We first consider the compact case.

Lemma 5.5.4 *Any compact planar Riemann surface is biholomorphic to* $\mathbb{P}^1$. *In fact, there exists a universal constant* $C > 0$ *such that if* X *is any compact planar Riemann surface,* p *is any point in* X, *and* $(D, \Phi = z, \Delta(0; 1))$ *is any local holomorphic chart with* $p \in D$ *and* $\Phi(p) = z(p) = 0$, *then there exists a biholomorphism* $F \colon X \to \mathbb{P}^1$ *for which*

(i) *The meromorphic function* $F - (1/z)$ *has a removable singularity at* p (*in particular,* $F(p) = \infty$); *and*
(ii) *We have* $\|dF\|_{L^2(X\setminus D)} \leq C$ *and* $\|z dF + z^{-1} dz\|_{L^2(D)} \leq C$.

Proof By Theorem 2.10.1, there exists a universal constant $C > 0$ such that if X is any compact planar Riemann surface, p is a point in X, and $(D, \Phi = z, \Delta(0; 1))$ is a local holomorphic chart with $p \in D$ and $\Phi(p) = z(p) = 0$, then there exists a meromorphic 1-form θ on X with the following properties:

(i) The meromorphic 1-form θ is holomorphic on $X \setminus \{p\}$ and has a pole of order 2 at p;
(ii) We have $\|\theta\|_{L^2(X\setminus D)} \leq C$;
(iii) The meromorphic 1-form $\theta + z^{-2}\, dz$ on D has at worst a simple pole at p; and
(iv) We have $\|z\theta + z^{-1} dz\|_{L^2(D)} \leq C$ (note that the meromorphic 1-form provided by Theorem 2.10.1 has been multiplied by -1).

For some constant $a \in \mathbb{C}$ and for some holomorphic function g on $\Delta(0; 1)$, we have

$$\theta = \left(-\frac{1}{z^2} + \frac{a}{z} + g(z)\right) \cdot dz \quad \text{on } D.$$

In particular, by Stokes' theorem,

$$2\pi i a = \int_{\partial \Phi^{-1}(\Delta(0;1/2))} \theta = -\int_{X \setminus \Phi^{-1}(\overline{\Delta(0;1/2)})} d\theta = 0.$$

The holomorphic function g has a primitive, i.e., a function $G \in \mathcal{O}(\Delta(0; 1))$ with $G' = g$ (one may see this by observing that the closed 1-form $g(z)\,dz$ must be exact, since $\Delta(0; 1)$ is simply connected; or one may apply Theorems 1.3.1 and 1.3.2). The meromorphic function h_0 on D given by $h_0 = z^{-1} + G(z)$ then satisfies $dh_0 = \partial h_0 = \theta$ on $D \setminus \{p\}$. Fixing a $\mathcal{C}^\infty$ function η with compact support in D that is equal to 1 on $\Phi^{-1}(\Delta(0; 1/2))$, we see that $\theta - d(\eta h_0)$ extends from $X \setminus \{p\}$ to a closed $\mathcal{C}^\infty$ 1-form θ_1 on X (with $\theta_1 \equiv 0$ near p). Since X is planar, there exists a $\mathcal{C}^\infty$ function h_1 on X with $dh_1 = \theta_1$. Thus $F \equiv \eta h_0 + h_1$ is a $\mathcal{C}^\infty$ function on $X \setminus \{p\}$ that satisfies $dF = \theta$. In particular, $\bar{\partial} F = 0$ (the $(0, 1)$ part of θ is 0), so F is holomorphic on $X \setminus \{p\}$. Furthermore, the holomorphic function $F - (1/z)$ on $D \setminus \{p\}$ extends to a $\mathcal{C}^\infty$, and therefore holomorphic, function on D (which is equal to $G(z) + h_1$ on $\Phi^{-1}(\Delta(0; 1/2))$). Hence F is a meromorphic function on X that is holomorphic on $X \setminus \{p\}$ and that has a simple pole at p, and Proposition 2.5.7 now implies that F is a biholomorphism of X onto $\mathbb{P}^1$. The L^2 bounds for $dF = \theta$ follow from the construction of θ. $\square$

Lemma 5.5.5 *Let M be an orientable $\mathcal{C}^\infty$ surface, let $r, R \in \mathbb{R}$ with $0 < r < R$, let $(D, \Psi, \Delta(0; R))$ be a local $\mathcal{C}^\infty$ chart in M, and let $N = M \setminus \Psi^{-1}(\overline{\Delta(0; r)})$. Then N is planar if and only if M is planar.*

Proof It is easy to see that N is connected, and Lemma 5.5.2 implies that N is planar if M is planar. For the converse, suppose N is planar and θ is a closed $\mathcal{C}^\infty$ 1-form with compact support in M. Corollary 10.5.7 provides a $\mathcal{C}^\infty$ function u on D with $du = \theta \restriction_D$. Fixing a $\mathcal{C}^\infty$ function η with compact support in D such that $\eta \equiv 1$ on $\Psi^{-1}(\Delta(0; s))$ for some $s \in (r, R)$, we see that (after extending ηu by 0 to all of M), $\theta_0 \equiv \theta - d(\eta u)$ is a closed $\mathcal{C}^\infty$ 1-form with compact support in N. Therefore, since N is planar, there exists a $\mathcal{C}^\infty$ function v on N with $dv = \theta_0 \restriction_N$. On the other hand, since $dv = \theta_0 \equiv 0$ on the coordinate annulus $\Psi^{-1}(\Delta(0; r, s))$, the function v must be equal to a constant on this set, and we may assume without loss of generality that this constant is 0. Thus v extends by 0 to a $\mathcal{C}^\infty$ function v_0 on M with $dv_0 = \theta_0$. Therefore $\theta = \theta_0 + d(\eta u) = d(v_0 + \eta u)$ is exact and M is planar. $\square$

Proof of Theorem 5.5.3 By Lemma 5.5.4, it remains to prove that any given *non-compact* planar Riemann surface X is biholomorphic to a domain in $\mathbb{P}^1$. We may apply Proposition 5.4.1 to get a sequence of nonempty domains $\{\Omega_\nu\}_{\nu=1}^\infty$ such that $X = \bigcup_{\nu=1}^\infty \Omega_\nu$ and such that for each ν, we have $\Omega_\nu \Subset \Omega_{\nu+1}$ and there exist a compact Riemann surface X_ν, a finite collection of disjoint local holomorphic charts $\{(D_j^{(\nu)}, \Psi_j^{(\nu)}, \Delta(0; R_j^{(\nu)}))\}_{j=1}^{n_\nu}$ in X_ν with $\{R_j^{(\nu)}\}_{j=1}^{n_\nu}$ in $(1, \infty)$, and a biholomorphism Φ_ν of Ω_ν onto $X_\nu \setminus \bigcup_{j=1}^{n_\nu} (\Psi_j^{(\nu)})^{-1}(\overline{\Delta(0; 1)})$.

Let us fix a point $p \in \Omega_1$ and a local holomorphic chart $(D, \Psi = z, \Delta(0; 1))$ with $p \in D \Subset \Omega_1$ and $\Psi(p) = z(p) = 0$. For each ν, Lemma 5.5.2 implies that Ω_ν is planar and therefore, by Lemma 5.5.5, X_ν is planar (by downward induction on k, one sees that the Riemann surface $X_\nu \setminus \bigcup_{j=1}^{k} (\Psi_j^{(\nu)})^{-1}(\overline{\Delta(0; 1)})$ is planar for $k = 1, \dots, n_\nu$). Therefore, by Lemma 5.5.4, for some constant $C > 0$ (independent of ν) and for each ν, there exists a biholomorphism $F_\nu \colon X_\nu \to \mathbb{P}^1$ such that for

$f_\nu \equiv F_\nu \circ \Phi_\nu \colon \Omega_\nu \to \mathbb{P}^1$, the meromorphic function $f_\nu - (1/z)$ on D has a removable singularity at p and the meromorphic 1-form $\theta_\nu \equiv df_\nu$ satisfies

$$\|\theta_\nu\|_{L^2(\Omega_\nu \setminus D)} \le C \quad \text{and} \quad \|z\theta_\nu + z^{-1}dz\|_{L^2(D)} \le C.$$

Fixing a point $q \in \Omega_1 \setminus \{p\}$, we may also assume that $f_\nu(q) = 0$ for each $\nu \in \mathbb{Z}_{>0}$ (simply by replacing F_ν with $F_\nu - F_\nu(\Phi_\nu(q))$). We will show that by passing to a subsequence, we may assume that the sequence of meromorphic functions $\{f_\nu\}$ converges to the desired biholomorphism of X onto a domain in $\mathbb{P}^1$. For this, we observe that for each compact set $K \subset X \setminus \{p\}$ and each $\nu \in \mathbb{Z}_{>0}$ with $\Omega_\nu \supset K$, we have

$$\begin{aligned}
\|\theta_\nu\|_{L^2(K)} &= \|\theta_\nu\|_{L^2(K\setminus D)} + \|\theta_\nu\|_{L^2(K\cap D)} \\
&\le \|\theta_\nu\|_{L^2(\Omega_\nu\setminus D)} + \left\|\frac{1}{z}(z\theta_\nu + z^{-1}\,dz)\right\|_{L^2(K\cap D)} + \|z^{-2}\,dz\|_{L^2(K\cap D)} \\
&\le C + \frac{C}{\min_{K\cap D}|z|} + \|z^{-2}\,dz\|_{L^2(K\cap D)} = A_K,
\end{aligned}$$

where clearly, the positive constant A_K is independent of the choice of ν. Given a point $s \in X \setminus \{p\}$, we may fix a local holomorphic chart $(U, \Lambda = \zeta, \Delta(0;2))$ with $U \Subset X \setminus \{p\}$ and $s = \Lambda^{-1}(0)$. For some $\nu_0 \in \mathbb{Z}_{>0}$, we have $U \subset \Omega_\nu$ and $\theta_\nu = (\partial f_\nu/\partial\zeta)d\zeta$ on U for all $\nu \ge \nu_0$. The above L^2-estimate and the L^∞/L^p-estimate (Theorem 1.2.4) together imply that the moduli of the holomorphic functions $\{\partial f_\nu/\partial\zeta\}_{\nu\ge\nu_0}$ are uniformly bounded by some positive constant B (depending on (U,ζ)) on the set $V \equiv \Lambda^{-1}(\Delta(0;1))$. Setting $s = q$, we see that q lies in the set Q of points $x \in X \setminus \{p\}$ for which there exist a neighborhood W and a positive integer μ such that $\{f_\nu\restriction_W\}_{\nu\ge\mu}$ is uniformly bounded. Clearly, Q is open, and given a point $x \in \overline{Q}$, we may choose the above point s and local holomorphic chart $(U, \Lambda = \zeta, \Delta(0;2))$ so that $s \in Q$ and $x \in V$. It follows that $Q = X \setminus \{p\}$, and hence $\{f_\nu\}$ is uniformly bounded on each compact subset of $X \setminus \{p\}$. Thus, after applying Montel's theorem (Corollary 2.11.4) and passing to a suitable subsequence, we may assume without loss of generality that the sequence $\{f_\nu\}$ converges uniformly on compact subsets of $X \setminus \{p\}$ to a holomorphic function f on $X \setminus \{p\}$ with $f(q) = 0$.

Furthermore, for each ν, the holomorphic function $g_\nu \equiv f_\nu - (1/z)$ on D (recall that $f_\nu - (1/z)$ has a removable singularity at p) satisfies

$$\begin{aligned}
\left\|z \cdot \frac{\partial g_\nu}{\partial z}\right\|_{L^2(D,(i/2)\,dz\wedge d\bar z)} &= \left\|z\frac{\partial f_\nu}{\partial z} + \frac{1}{z}\right\|_{L^2(D,(i/2)\,dz\wedge d\bar z)} \\
&= \frac{1}{\sqrt{2}}\left\|z\theta_\nu + \frac{1}{z}\,dz\right\|_{L^2(D)} \le C.
\end{aligned}$$

Thus, after again passing to a subsequence, we may also assume that the sequence of holomorphic functions $\{z \cdot \partial g_\nu/\partial z\}$ converges uniformly on compact subsets of

D to a holomorphic function h with $h(p) = 0$. In particular, on $D \setminus \{p\}$,

$$\frac{\partial f}{\partial z} \leftarrow \frac{\partial f_\nu}{\partial z} = \frac{\partial g_\nu}{\partial z} - \frac{1}{z^2} \to \frac{h}{z} - \frac{1}{z^2} \quad \text{as } \nu \to \infty.$$

The function h/z (with the singularity at p removed) is holomorphic on D, and therefore this function has a primitive. It follows that f is a meromorphic function on X, f is holomorphic on $X \setminus \{p\}$, f has a simple pole at p, and $f - (1/z)$ has a removable singularity at p. In particular, f is nonconstant and Lemma 5.2.3 implies that $f\restriction_{X\setminus\{p\}} \colon X \setminus \{p\} \to \mathbb{C}$ is injective. Since the holomorphic mapping $f \colon X \to \mathbb{P}^1$ satisfies $f(p) = \infty$, f is injective on all of X, and therefore f is a biholomorphism of X onto a domain in $\mathbb{P}^1$. □

The Koebe uniformization theorem and Corollary 5.2.7 together give the following:

Corollary 5.5.6 *If X is Riemann surface and every closed $\mathcal{C}^\infty$ 1-form on X is exact, then X is biholomorphic to $\mathbb{P}^1$, to $\mathbb{C}$, or to $\Delta(0; 1)$. In particular, X is simply connected.*

We also have the following equivalent version (see Sect. 10.6):

Corollary 5.5.7 *If X is a Riemann surface and $H^1_{\mathrm{deR}}(X, \mathbb{C}) = 0$, then X is biholomorphic to $\mathbb{P}^1$, to $\mathbb{C}$, or to $\Delta(0; 1)$. In particular, X is simply connected.*

Remark As shown in Sects. 10.6 and 10.7, the vanishing of the complex (de Rham) cohomology vector space $H^1_{\mathrm{deR}}(X, \mathbb{C}) \cong H^1(X, \mathbb{C})$ is equivalent to the vanishing of the real (de Rham) cohomology vector space $H^1_{\mathrm{deR}}(X, \mathbb{R}) \cong H^1(X, \mathbb{R})$, as well as to the vanishing of $H_1(X, \mathbb{R})$ (in fact, for any choice of a subring $\mathbb{A}$ of $\mathbb{C}$ containing $\mathbb{Z}$, it is equivalent to the vanishing of $H_1(X, \mathbb{A})$).

Exercises for Sect. 5.5

5.5.1 The goal of this problem is a stronger version of the Koebe uniformization theorem. Prove that there exists a universal constant $C > 0$ such that if X is any planar Riemann surface, p is a point in X, and $(D, z = \Phi, \Delta(0; 1))$ is a local holomorphic chart with $p \in D$ and $\Phi(p) = z(p) = 0$, then there exists a biholomorphism f of X onto a domain in $\mathbb{P}^1$ such that
 (i) The meromorphic function $f - (1/z)$ has a removable singularity at p (in particular, $f(p) = \infty$); and
 (ii) We have $\|df\|_{L^2(X\setminus D)} \le C$ and $\|z df + z^{-1} dz\|_{L^2(D)} \le C$.

5.5.2 This problem requires the topological characterization of cohomology in terms of *Čech 1-forms* appearing in Sect. 10.7. Let M and N be homeomorphic second countable orientable $\mathcal{C}^\infty$ surfaces. Prove that M is planar if and only if N is planar.

5.6 Automorphisms and Quotients of $\mathbb{C}$

The uniformization theorem allows one to obtain a characterization of Riemann surfaces as quotient spaces of $\mathbb{P}^1$, $\mathbb{C}$, and $\Delta = \Delta(0; 1)$. This characterization is considered in Sects. 5.6–5.9. We first determine, in this section, all of the Riemann surfaces with universal covering space $\mathbb{C}$. For this, we first observe the following:

Theorem 5.6.1 *Let $\Phi_{(a,b)}\colon \mathbb{C} \to \mathbb{C}$ be the mapping $z \mapsto az + b$ for each pair $(a, b) \in \mathbb{C}^* \times \mathbb{C}$. Then we have the following*:

(a) *For each pair $(a, b) \in \mathbb{C}^* \times \mathbb{C}$, the map $\Phi_{(a,b)}\colon \mathbb{C} \to \mathbb{C}$ is an automorphism of $\mathbb{C}$. This automorphism has a fixed point (given by $b/(1 - a)$) if and only if $a \neq 1$.*
(b) $\operatorname{Aut}(\mathbb{C}) = \{\Phi_{(a,b)} \mid (a, b) \in \mathbb{C}^* \times \mathbb{C}\}$.

Proof Part (a) is easily verified. For the proof of (b), suppose $\Phi \in \operatorname{Aut}(\mathbb{C})$. Then $\Phi(z) \to \infty$ as $z \to \infty$, so Φ extends to an automorphism of $\mathbb{P}^1$ that fixes ∞. In particular, the function $z \mapsto (\Phi(z) - \Phi(0))/z$ is holomorphic on $\mathbb{P}^1$ (since the meromorphic function $\Phi - \Phi(0)$ on $\mathbb{P}^1$ has a simple pole at ∞ and a simple zero at 0) and therefore this function is constant. $\square$

Remark The pair $(G \equiv \mathbb{C}^* \times \mathbb{C}, \cdot)$, where

$$(a, b) \cdot (c, d) = (ac, ad + b) \quad \forall (a, b), (c, d) \in G,$$

is a group, and the map $G \to \operatorname{Aut}(\mathbb{C})$ given by $(a, b) \mapsto \Phi_{(a,b)}$ (for $\Phi_{(a,b)}$ as above) is a (surjective) group isomorphism (see Exercise 5.6.1).

Examples 5.1.9 and 5.1.10, along with the following lemma, imply that any Riemann surface with holomorphic universal cover $\mathbb{C}$ must be (up to biholomorphism) $\mathbb{C}$, $\mathbb{C}^*$, or a complex torus (this will be examined further in Sect. 5.9).

Lemma 5.6.2 *For any subgroup Γ of* $\operatorname{Aut}(\mathbb{C})$ *that acts properly discontinuously and freely, and for $\Upsilon = \Upsilon_\Gamma\colon \mathbb{C} \to X = \Gamma\backslash\mathbb{C}$ the associated Riemann surface quotient, we have the following*:

(a) *Γ is a group of translations, and either $\Gamma = \{0\}$, or $\Gamma = \mathbb{Z}\xi$ for some $\xi \in \mathbb{C}^*$, or Γ is a lattice (here, we identify any $\zeta \in \mathbb{C}$ with the translation $z \mapsto z + \zeta$).*
(b) *If $\Gamma = \{0\}$, then $X = \mathbb{C}$ and $\Upsilon = \mathrm{Id}_{\mathbb{C}}$.*
(c) *If $\Gamma = \mathbb{Z}\xi$ for some $\xi \in \mathbb{C}^*$, then Υ is holomorphically equivalent to the covering $\Upsilon_0\colon \mathbb{C} \to \mathbb{C}^*$ given by $z \mapsto e^{2\pi i z}$ (as in Example 5.1.9). More precisely, there is a commutative diagram*

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\alpha"] \arrow[d, "\Upsilon_0"'] & \mathbb{C} \arrow[d, "\Upsilon"] \\ \mathbb{C}^* \arrow[r, "\beta"] & X \end{tikzcd}$$

in which α is the linear isomorphism given by $z \mapsto \xi z$ and β is a biholomorphism.

(d) *If Γ is a lattice, then there exists a complex number τ such that* $\operatorname{Im}\tau > 0$ *and such that the holomorphic covering space $\Upsilon_0 \colon \mathbb{C} \to X_0 = \Gamma_0\backslash\mathbb{C}$ corresponding to the lattice $\Gamma_0 \equiv \mathbb{Z} + \mathbb{Z}\tau$ is holomorphically equivalent to Υ. More precisely, for some (nonunique) $\tau \in \mathbb{C}$ with* $\operatorname{Im}\tau > 0$, *there is a commutative diagram*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\alpha} & \mathbb{C} \\ {\scriptstyle \Upsilon_0}\downarrow & & \downarrow{\scriptstyle \Upsilon} \\ X_0 & \xrightarrow{\beta} & X \end{array}$$

in which α is a linear isomorphism (with $\alpha(\Gamma_0) = \Gamma$) and β is a biholomorphism. Moreover, if $X' = \Gamma'\backslash\mathbb{C}$ is any complex torus that is biholomorphic to X, then $\Upsilon_0 \colon \mathbb{C} \to X_0$ is also equivalent in the above sense to the covering $\Upsilon' \colon \mathbb{C} \to X'$.

Proof By Theorem 5.6.1, every automorphism Φ of $\mathbb{C}$ must be of the form $z \mapsto az + b$ for constants $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. Moreover, Φ has a fixed point if and only if $a \neq 1$. Therefore, every element of Γ must be a translation, and hence we may identify Γ with a subgroup of $\mathbb{C}$.

Now Γ must be a discrete set in $\mathbb{C}$. For if there is a sequence $\{\xi_\nu\}$ in Γ that converges to a point ξ that is not an element of the sequence, then choosing a compact set K containing a neighborhood of $\{0, \xi\}$, we see that $0 + \xi_\nu \in K \cap (K + \xi_\nu)$ for all large ν, which is impossible since Γ acts properly discontinuously. In particular, if $\Gamma \neq \{0\}$, then we may choose an element $\xi_1 \in \Gamma \setminus \{0\}$ of minimal modulus. We then have $\Gamma \cap (\mathbb{R}\xi_1) = \mathbb{Z}\xi_1$. For if $r \in \mathbb{R}$ and $r\xi_1 \in \Gamma$, then $r\xi_1 - \lfloor r \rfloor \xi_1$ (where $\lfloor r \rfloor$ denotes the *floor* of r) is an element of Γ with modulus $(r - \lfloor r \rfloor)|\xi_1| < |\xi_1|$, and hence $r \in \mathbb{Z}$. The complement of $\mathbb{Z}\xi_1$ in Γ must be a closed set that does not meet $\mathbb{R}\xi_1$. Hence, if this complement is nonempty, then we may choose an element $\xi_2 \in \Gamma$ with $\xi_2 \notin \mathbb{Z}\xi_1$ of minimal distance from $[0, 1] \cdot \xi_1$, and an element $u\xi_1 \in [0, 1] \cdot \xi_1$ at which this minimal distance is attained. For any $\zeta \in \Gamma \setminus \mathbb{Z}\xi_1$ and $t \in \mathbb{R}$, we have

$$|\zeta - t\xi_1| = |(\zeta - \lfloor t \rfloor \xi_1) - (t - \lfloor t \rfloor)\xi_1| \geq |\xi_2 - u\xi_1| > 0,$$

so ξ_2 is actually of minimal distance from $\mathbb{R}\xi_1$, and this minimal distance is attained at $u\xi_1$. In particular, ξ_1 and ξ_2 are linearly independent over $\mathbb{R}$. It also follows that $\Gamma = \mathbb{Z}\xi_1 + \mathbb{Z}\xi_2$. For given an element $\xi \in \Gamma$, there must exist real numbers $t_1, t_2 \in \mathbb{R}$ with $\xi = t_1\xi_1 + t_2\xi_2$. In order to show that $\xi \in \mathbb{Z}\xi_1 + \mathbb{Z}\xi_2$, we may assume without loss of generality that $t_2 \in [0, 1)$, since we may replace ξ with $t_1\xi_1 + (t_2 - \lfloor t_2 \rfloor)\xi_2$. Thus ξ is an element of Γ with distance from $\mathbb{R}\xi_1$ at most

$$|\xi - (t_1 + t_2 u)\xi_1| = t_2|\xi_2 - u\xi_1| < |\xi_2 - u\xi_1|.$$

Therefore, by the choice of ξ_2, we must have $\xi \in \mathbb{Z}\xi_1$. Thus the claim (a) is proved.

The claim (b) is trivial, and the claim (c) is easy to verify (see Exercise 5.6.2).

For the proof of (d), let us assume that Γ is a lattice generated by $\xi_1, \xi_2 \in \mathbb{C}$. Since we may replace ξ_2 with $-\xi_2$ if necessary, we may choose the generators so that the number $\tau \equiv \xi_2/\xi_1$ satisfies $\operatorname{Im} \tau > 0$. The linear isomorphism $\alpha: \mathbb{C} \to \mathbb{C}$ given by $z \mapsto \xi_1 z$ then maps the lattice $\Gamma_0 \equiv \mathbb{Z} + \mathbb{Z}\tau$ onto Γ. It is now easy to verify that we get the desired commutative diagram. Finally, suppose $\Upsilon': \mathbb{C} \to X'$ is the universal covering of a complex torus $X' = \Gamma' \backslash \mathbb{C}$, and $\gamma: X \to X'$ is a biholomorphism. Then γ lifts to an automorphism $\tilde{\gamma}$ of the universal covering $\mathbb{C}$, and hence $\tilde{\gamma}$ must be of the form $z \mapsto az + b$ with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. For each element $\xi \in \Gamma$, we have

$$a\xi = \tilde{\gamma}(\xi) - \tilde{\gamma}(0) \in \Gamma'$$

and hence $a\Gamma \subset \Gamma'$. By symmetry, we then have $a^{-1}\Gamma' \subset \Gamma$ and hence $a\Gamma = \Gamma'$. Thus $\xi_1' \equiv a\xi_1, \xi_2' \equiv a\xi_2$ generate Γ', and $\xi_2'/\xi_1' = \xi_2/\xi_1 = \tau$. Thus we get the analogous commutative diagram for $\mathbb{C} \to X'$. □

Exercises for Sect. 5.6

5.6.1 In the notation of Theorem 5.6.1, show that the pair $(G \equiv \mathbb{C}^* \times \mathbb{C}, \cdot)$, where $(a, b) \cdot (c, d) = (ac, ad + b)$ for all $(a, b), (c, d) \in G$, is a group, and that the map $G \to \operatorname{Aut}(\mathbb{C})$ given by $(a, b) \mapsto \Phi_{(a,b)}$ is a (surjective) group isomorphism.

5.6.2 Prove part (c) of Lemma 5.6.2.

5.7 Aut ($\mathbb{P}^1$) and Uniqueness of the Quotient

Suppose $\Phi: \mathbb{P}^1 \to \mathbb{P}^1$ is an automorphism of $\mathbb{P}^1$. The mapping $\Psi: \mathbb{P}^1 \to \mathbb{P}^1$ given by $\Psi(0) = 0$, $\Psi(\infty) = \infty$, and, for $z \in \mathbb{C}^*$,

$$\Psi(z) = \begin{cases} \frac{\Phi(z)-\Phi(0)}{\Phi(z)-\Phi(\infty)} & \text{if } \Phi(0), \Phi(\infty) \in \mathbb{C}, \\ \Phi(z) - \Phi(0) & \text{if } \Phi(\infty) = \infty, \\ \frac{1}{\Phi(z)-\Phi(\infty)} & \text{if } \Phi(0) = \infty \end{cases}$$

(for $\Phi(0), \Phi(\infty) \in \mathbb{C}$, we set $\Psi(\Phi^{-1}(\infty)) = 1$) is then an automorphism of $\mathbb{P}^1$. For Ψ is a bijective continuous mapping that is holomorphic on $\mathbb{C} \setminus \{\Phi^{-1}(\infty)\}$, so Riemann's extension theorem (Theorem 2.2.2) implies that Ψ is a bijective holomorphic mapping. In particular, the restriction $\Psi\restriction_{\mathbb{C}}: \mathbb{C} \to \mathbb{C}$ is an automorphism of $\mathbb{C}$ with $\Psi(0) = 0$, and hence by Theorem 5.6.1, there exists a constant $\alpha \in \mathbb{C}^*$ such that $\Psi(z) = \alpha z$ for each $z \in \mathbb{P}^1$. Solving this equation for $\Phi(z)$, we see that there exist constants $a, b, c, d \in \mathbb{C}$ such that for each point $z \in \mathbb{C} \setminus \{\Phi^{-1}(\infty)\}$, we have $cz + d \in \mathbb{C}^*$ and

$$\Phi(z) = \frac{az + b}{cz + d}.$$

Given two points $z_1, z_2 \in \mathbb{C} \setminus \{\Phi^{-1}(\infty)\}$, we have

$$z_1 = z_2 \quad \Longleftrightarrow \quad \frac{az_1 + b}{cz_1 + d} = \frac{az_2 + b}{cz_2 + d} \quad \Longleftrightarrow \quad (ad - bc)(z_1 - z_2) = 0.$$

Choosing the two points to be distinct, we see that $ad - bc \neq 0$. Furthermore, for $z \in \mathbb{C} \setminus \{\Phi^{-1}(\infty)\}$, we have

$$\frac{az + b}{cz + d} \to a/c \quad (= \infty \quad \text{if } c = 0) \quad \text{as } z \to \infty$$

and (since $a(-d/c) + b \neq 0$ for $c \neq 0$ and $a(-d/c) + b = \infty$ for $c = 0$)

$$\frac{az + b}{cz + d} \to \infty \quad \text{as } z \to -d/c \quad (= \infty \quad \text{if } c = 0).$$

Therefore, $\Phi(\infty) = a/c$ and $\Phi(-d/c) = \infty$. We express this by saying that the equation

$$\Phi(z) = \frac{az + b}{cz + d}$$

holds for *all* $z \in \mathbb{P}^1$.

Definition 5.7.1 A *Möbius transformation* is a mapping $\Phi \colon \mathbb{P}^1 \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ of the form

$$\Phi \colon z \mapsto \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ (here, we set $\Phi(\infty) = a/c$ and $\Phi(-d/c) = \infty$).

Observe that the constants a, b, c, d giving a Möbius transformation Φ as above are not unique, since for any constant $\lambda \in \mathbb{C}^*$, the constants $\lambda a, \lambda b, \lambda c, \lambda d$ give the same Möbius transformation Φ. As shown above, every automorphism of $\mathbb{P}^1$ is a Möbius transformation. The converse, as well as other appealing facts, also hold. For the statements, it will be convenient to have the following terminology:

Definition 5.7.2 Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $n \in \mathbb{Z}_{>0}$.

(a) The *general linear group* $\mathrm{GL}(n, \mathbb{F})$ over $\mathbb{F}$ is the group of nonsingular $n \times n$ matrices with entries in $\mathbb{F}$ (with product given by matrix multiplication).
(b) The *special linear group* $\mathrm{SL}(n, \mathbb{F})$ over $\mathbb{F}$ is the subgroup of $\mathrm{GL}(n, \mathbb{F})$ given by the nonsingular $n \times n$ matrices with entries in $\mathbb{F}$ and determinant 1.
(c) The *projectivized special linear group* $\mathrm{PSL}(n, \mathbb{F})$ over $\mathbb{F}$ is the quotient of the group $\mathrm{SL}(n, \mathbb{F})$ by the normal subgroup $\{I, -I\}$; that is,

$$\mathrm{PSL}(n, \mathbb{F}) = \mathrm{SL}(n, \mathbb{F})/A \sim -A.$$

Remark It is easy to see that $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, and $\mathrm{PSL}(n,\mathbb{R})$ are subgroups of $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$, and $\mathrm{PSL}(n,\mathbb{C})$, respectively.

Theorem 5.7.3 *The Möbius transformations have the following properties*:

(a) *The Möbius transformations together with the product given by composition form a group that is equal to the automorphism group of* $\mathbb{P}^1$.
(b) *We have a surjective homomorphism* $\mathrm{GL}(2,\mathbb{C}) \to \mathrm{Aut}\,(\mathbb{P}^1)$ *given by*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \Phi_A, \quad \textit{where } \Phi_A(z) = \frac{az+b}{cz+d} \quad \forall z \in \mathbb{P}^1.$$

(c) *The map* $[A] \mapsto \Phi_A$ (*for* Φ_A *as in* (*b*)) *is a well-defined* (*surjective*) *isomorphism of* $\mathrm{PSL}(2,\mathbb{C})$ *onto* $\mathrm{Aut}\,(\mathbb{P}^1)$.
(d) *Every Möbius transformation that is not the identity has exactly one or two fixed points in* $\mathbb{P}^1$. *In particular, up to biholomorphism, the only Riemann surface with universal covering* $\mathbb{P}^1$ *is* $\mathbb{P}^1$ *itself.*
(e) *Given* $\zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3 \in \mathbb{P}^1$ *with* $\zeta_i \neq \zeta_j$ *and* $\xi_i \neq \xi_j$ *for all* $i, j \in \{1,2,3\}$ *with* $i \neq j$, *there exists a unique Möbius transformation* Φ *with* $\Phi(\zeta_j) = \xi_j$ *for* $j = 1, 2, 3$.
(f) *If* C *is a circle in* $\mathbb{P}^1$ (*i.e.*, C *is a circle in* $\mathbb{C}$ *or* $C = L \cup \{\infty\}$ *for some line* L *in* $\mathbb{C}$), *then the image of* C *under any Möbius transformation is also a circle in* $\mathbb{P}^1$.

The proof is left to the reader (see Exercise 5.7.1).

Exercises for Sect. 5.7

5.7.1 Prove Theorem 5.7.3 (part of it was already proved in the discussion preceding the statement of the theorem).

5.8 Automorphisms of the Disk

The following fact will allow us to characterize the automorphism group of the disk:

Theorem 5.8.1 (Schwarz's lemma) *Let* $f : \Delta \to \Delta$ *be a holomorphic mapping of the unit disk* $\Delta \equiv \Delta(0;1)$ *to itself with* $f(0) = 0$. *Then we have* $|f(z)| \leq |z|$ *for all* $z \in \Delta$ *and* $|f'(0)| \leq 1$. *Moreover, if* $|f(z)| = |z|$ *for some point* $z \in \Delta \setminus \{0\}$ *or* $|f'(0)| = 1$, *then there is a constant* $c \in \mathbb{C}$ *such that* $|c| = 1$ *and* $f(z) = cz$ *for all* $z \in \Delta$.

Proof Riemann's extension theorem (Theorem 1.2.10) implies that the (continuous) function h on Δ given by

$$h(z) = \begin{cases} f(z)/z & \text{if } z \in \Delta \setminus \{0\}, \\ f'(0) & \text{if } z = 0, \end{cases}$$

is holomorphic. Moreover, we have $\limsup_{z\to c} |h(z)| \leq 1$ for each point $c \in \partial\Delta$ and therefore, by the maximum principle, $|h| \leq 1$ on Δ. Furthermore, if $|h(z)| = 1$ for some point $z \in \Delta$, then h is equal to a constant c of modulus 1. $\square$

Theorem 5.8.2 (Cf. Proposition 2.14.8) *For each point* $c \in \Delta \equiv \Delta(0;1)$, *let* $\Phi_c : \mathbb{P}^1 \to \mathbb{P}^1$ *be the map given by*

$$\Phi_c(z) = \frac{z-c}{1-\bar{c}z} \quad \forall z \in \mathbb{P}^1.$$

Then we have the following:

(a) *For each point* $c \in \Delta$, Φ_c *is a Möbius transformation*, $\Phi_c(c) = 0$, $\Phi_c'(0) = 1 - |c|^2$, *and* $\Phi_c'(c) = 1/(1-|c|^2)$.
(b) *For each choice of* $c \in \Delta$ *and* $b \in \partial\Delta$, *the Möbius transformation* $\Phi \equiv b\Phi_c$ *has inverse* $z \mapsto \Phi^{-1}(z) = \Phi_{-c}(\bar{b}z) = \bar{b}\Phi_{-bc}(z)$, Φ *maps* Δ *onto* Δ, *and the restriction* $\Phi\restriction_\Delta : \Delta \to \Delta$ *is an automorphism of* Δ. *Furthermore, every automorphism of* Δ *is of this form.*

Proof For $c \in \Delta$, we have $1 \cdot 1 - \bar{c} \cdot c = 1 - |c|^2 > 0$, so Φ_c is a Möbius transformation. The remaining properties in (a) are left to the reader (see Exercise 2.14.5). We have

$$\begin{pmatrix} 1 & -c \\ -\bar{c} & 1 \end{pmatrix}^{-1} = \frac{1}{1-|c|^2} \cdot \begin{pmatrix} 1 & c \\ \bar{c} & 1 \end{pmatrix},$$

and hence for each point $z \in \mathbb{P}^1$,

$$\Phi_c^{-1}(z) = \frac{z+c}{1+\bar{c}z} = \Phi_{-c}(z).$$

Therefore, for $b \in \partial\Delta$ and $\Phi = b\Phi_c$, we have

$$\Phi^{-1}(z) = \Phi_{-c}(\bar{b}z) = \frac{\bar{b}z + c}{1+\bar{c}\bar{b}z} = \bar{b}\frac{z+bc}{1+\overline{bc}z} = \bar{b}\Phi_{-bc}(z) \quad \forall z \in \mathbb{P}^1.$$

It is easy to check that $\Phi(\partial\Delta) \subset \partial\Delta$, and hence by the above, we have $\Phi(\partial\Delta) = \partial\Delta$. In particular, $\Phi(\Delta)$ must be either Δ or $\mathbb{P}^1 \setminus \overline{\Delta}$, and it follows that $\Phi(\Delta) = \Delta$, since $\Phi(c) = 0 \in \Delta$. Therefore, $\Phi\restriction_\Delta : \Delta \to \Delta$ is an automorphism.

Finally, suppose Λ is an arbitrary automorphism of Δ. Then $\Lambda_0 \equiv \Phi_{\Lambda(0)} \circ \Lambda$ is an automorphism of Δ that maps 0 to 0. But then

$$(\Lambda_0^{-1})'(0) \cdot \Lambda_0'(0) = (\Lambda_0^{-1} \circ \Lambda_0)'(0) = 1.$$

Since by the Schwarz lemma, each of the factors in the first of the above expressions has modulus at most 1, each must have modulus equal to 1. Therefore, again by the Schwarz lemma, there is a constant $b \in \partial\Delta$ such that $\Lambda_0(z) = bz$ for all $z \in \Delta$. Therefore, for all $z \in \Delta$, $\Lambda(z) = \Phi_{-\Lambda(0)}(bz) = b\Phi_{-\bar{b}\Lambda(0)}(z)$. $\square$

It is often more convenient to work with the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\},$$

which is biholomorphic to the unit disk Δ.

Theorem 5.8.3 *For $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and $\Delta = \Delta(0; 1)$, we have the following*:

(a) *The mapping $\Psi\colon \mathbb{P}^1 \to \mathbb{P}^1$ given by $z \mapsto i\frac{1-z}{1+z} = \frac{-iz+i}{z+1}$ is a Möbius transformation that maps Δ onto $\mathbb{H}$ and $\partial\Delta$ onto $\mathbb{R} \cup \{\infty\}$.*

(b) *The set* $\mathrm{GL}_+(2, \mathbb{R}) \equiv \{A \in \mathrm{GL}(2, \mathbb{R}) \mid \det A > 0\}$ *is a subgroup of* $\mathrm{GL}(2, \mathbb{R})$, *and we have a surjective homomorphism* $\mathrm{GL}_+(2, \mathbb{R}) \to \mathrm{Aut}\,(\mathbb{H})$ *given by*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \Phi_A \restriction_{\mathbb{H}},$$

where Φ_A is the Möbius transformation given by

$$\Phi_A(z) = \frac{az+b}{cz+d} \quad \forall z \in \mathbb{P}^1.$$

(c) *The map $[A] \mapsto \Phi_A \restriction_{\mathbb{H}}$ (for Φ_A as in* (b) *and for $[A]$ the equivalence class represented by A) is a well-defined surjective isomorphism of* $\mathrm{PSL}(2, \mathbb{R})$ *onto* $\mathrm{Aut}\,(\mathbb{H})$.

Proof We have $(-i)(1) - (1)(i) = -2i \neq 0$, so Ψ is a Möbius transformation. Moreover, $\Psi(1) = 0$, $\Psi(i) = 1$, and $\Psi(-1) = \infty$. Therefore, by Theorem 5.7.3, Ψ maps $\partial\Delta$, which is the unique circle containing $1, i, -1$, onto the unique circle containing $0, 1, \infty$, i.e., onto $\mathbb{R} \cup \{\infty\}$. In particular, Ψ must map Δ onto exactly one of the two connected components of $\mathbb{P}^1 \setminus (\mathbb{R} \cup \{\infty\})$, and therefore, since $\Psi(0) = i \in \mathbb{H}$, we have $\Psi(\Delta) = \mathbb{H}$. Thus (a) is proved.

If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_+(2, \mathbb{R})$, then clearly, the corresponding Möbius transformation

$$\Phi_A\colon z \mapsto \frac{az+b}{cz+d}$$

maps the circle $\mathbb{R} \cup \{\infty\}$ onto the circle $\mathbb{R} \cup \{\infty\}$, and therefore Φ_A maps $\mathbb{H}$ onto one of the connected components of $\mathbb{P}^1 \setminus (\mathbb{R} \cup \{\infty\})$. On the other hand, since $ad - bc = \det A > 0$, we have

$$\Phi_A(i) = \frac{ai+b}{ci+d} = \frac{ac+bd}{c^2+d^2} + i\frac{ad-bc}{c^2+d^2} \in \mathbb{H}.$$

Thus $\Phi_A \restriction_{\mathbb{H}} \in \mathrm{Aut}\,(\mathbb{H})$.

For the converse, observe that any automorphism of $\mathbb{H}$ is of the form $\Psi \circ \Lambda \circ \Psi^{-1} \restriction_{\mathbb{H}}$ for some automorphism Λ of Δ, and hence by Theorem 5.8.2 and Theorem 5.7.3, the automorphism must be of the form $\Phi \restriction_{\mathbb{H}}$ for some Möbius transformation Φ with $\Phi(\mathbb{H}) = \mathbb{H}$ and $\Phi(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}$. Guided by the arguments in Sect. 5.7, let us set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\begin{smallmatrix} \mu & -\mu\Phi(0) \\ 1 & -\Phi(\infty) \end{smallmatrix}\right) & \text{if } \Phi(0), \Phi(\infty) \in \mathbb{R}, \\ \left(\begin{smallmatrix} \mu & -\mu\Phi(0) \\ 0 & 1 \end{smallmatrix}\right) & \text{if } \Phi(\infty) = \infty, \\ \left(\begin{smallmatrix} 0 & \mu \\ 1 & -\Phi(\infty) \end{smallmatrix}\right) & \text{if } \Phi(0) = \infty, \end{cases}$$

where $\mu = \pm 1$ is chosen so that $\det A > 0$, i.e., so that $A \in \mathrm{GL}_+(2, \mathbb{R})$. Then $\Phi_A \circ \Phi$ is a Möbius transformation that maps 0 to 0, ∞ to ∞, $\mathbb{R} \cup \{\infty\}$ to $\mathbb{R} \cup \{\infty\}$, and $\mathbb{H}$ to $\mathbb{H}$, and hence $\Phi_A \circ \Phi$ must be of the form $z \mapsto rz$ for some $r \in (0, \infty) \subset \mathbb{R}$. Therefore, for $B = A^{-1}\left(\begin{smallmatrix} r & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_+(2, \mathbb{R})$, we have $\Phi = \Phi_B$, and applying Theorem 5.7.3, we get (b) and (c). $\square$

Remark Clearly, Theorem 5.7.3 provided the recipe for the construction of the biholomorphism Ψ of Δ onto $\mathbb{H}$; we only needed to choose a Möbius transformation mapping three points arranged in counterclockwise order on $\partial\Delta$ to three corresponding points arranged in the same order on the circle $\mathbb{R} \cup \{\infty\}$ oriented with $\mathbb{H}$ on the left. On the other hand, one can also verify the claims in the theorem directly without applying Theorem 5.7.3.

Next, we see that Example 5.1.11 and Example 5.1.12 are the only examples of Riemann surfaces with universal covering $\Delta \cong \mathbb{H}$ and infinite cyclic fundamental group.

Lemma 5.8.4 *If X is a Riemann surface with universal covering $\Delta \equiv \Delta(0; 1) \cong \mathbb{H}$ and fundamental group $\pi_1(X) \cong \mathbb{Z}$, then X is biholomorphic to the punctured disk $\Delta^* \equiv \Delta^*(0; 1)$ or to the annulus $\Delta(0; r, 1)$ for some $r \in (0, 1)$.*

Proof We have the universal covering map $\Upsilon\colon \mathbb{H} \to X = \Gamma\backslash\mathbb{H}$ for some infinite cyclic subgroup Γ of $\mathrm{Aut}\,(\mathbb{H})$ that acts properly discontinuously and freely. By Theorem 5.8.3, Γ is generated by $\Phi\restriction_{\mathbb{H}}$ for some Möbius transformation Φ of the form $\Phi\colon z \mapsto (az+b)/(cz+d)$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. In particular, Φ has one or two fixed points in $\mathbb{P}^1$, and since Φ has no fixed points in $\mathbb{H}$ and $\Phi(\bar{z}) = \overline{\Phi(z)}$ for each point $z \in \mathbb{P}^1$, the fixed point or points must lie in $\mathbb{R} \cup \{\infty\}$.

Suppose Φ has exactly one fixed point $\lambda \in \mathbb{R} \cup \{\infty\}$. The Möbius transformation

$$\Psi_1 : z \mapsto \begin{cases} \frac{-1}{z-\lambda} & \text{if } \lambda \in \mathbb{R}, \\ z & \text{if } \Phi(\infty) = \infty, \end{cases}$$

restricts to an automorphism of $\mathbb{H}$ and maps λ to ∞. Hence the Möbius transformation $\Psi_1 \circ \Phi \circ \Psi_1^{-1}$ also restricts to an automorphism of $\mathbb{H}$ and has the unique fixed point ∞. It follows that $\Psi_1 \circ \Phi \circ \Psi_1^{-1}$ is a translation $z \mapsto z + \beta$ for some constant $\beta \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$. By replacing Φ with Φ^{-1} (which has the same unique fixed point λ) if necessary, we may assume that $\beta > 0$. Thus the Möbius transformation

$\Psi\colon z \mapsto \beta^{-1}\Psi_1(z)$ restricts to an automorphism of $\mathbb{H}$ and the Möbius transformation $\Phi_0 = \Psi \circ \Phi \circ \Psi^{-1}$ is given by $z \mapsto z+1$. Letting $\Gamma_0 \cong \mathbb{Z}$ be the subgroup of $\mathrm{Aut}(\mathbb{H})$ generated by $\Phi_0\restriction_{\mathbb{H}}$, we get a commutative diagram of holomorphic mappings

$$\begin{tikzcd} \mathbb{H} \arrow[r, "\Psi"] \arrow[d, "\Upsilon"'] & \mathbb{H} \arrow[d, "\Upsilon_0"] \\ X \arrow[r, "\Psi_0"] & \Delta^* \end{tikzcd}$$

where Υ_0 is the universal covering map $z \mapsto e^{2\pi i z}$ as in Example 5.1.11, and Ψ_0 is a biholomorphism.

Let us now assume that Φ has exactly two fixed points $\mu, \nu \in \mathbb{R} \cup \{\infty\}$, which we may order so that $\mu > \nu$ if $\mu, \nu \in \mathbb{R}$. The Möbius transformation

$$\Psi\colon z \mapsto \begin{cases} \frac{z-\mu}{z-\nu} & \text{if } \mu, \nu \in \mathbb{R}, \\ z-\mu & \text{if } \nu = \infty, \\ \frac{-1}{z-\nu} & \text{if } \mu = \infty, \end{cases}$$

restricts to an automorphism of $\mathbb{H}$ and maps μ to 0 and ν to ∞. Hence the Möbius transformation $\Phi_0 = \Psi \circ \Phi \circ \Psi^{-1}$ also restricts to an automorphism of $\mathbb{H}$ and has fixed points 0 and ∞. It follows that Φ_0 is a dilation $z \mapsto \lambda z$ for some constant $\lambda > 0$, and by replacing Φ with Φ^{-1} (which has the same fixed points) if necessary, we may assume that $\lambda > 1$. Letting $\Gamma_0 \cong \mathbb{Z}$ be the subgroup of $\mathrm{Aut}(\mathbb{H})$ generated by $\Phi_0\restriction_{\mathbb{H}}$ and $r \equiv \exp(-2\pi^2/\log\lambda) \in (0,1)$, we get a commutative diagram of holomorphic mappings

$$\begin{tikzcd} \mathbb{H} \arrow[r, "\Psi"] \arrow[d, "\Upsilon"'] & \mathbb{H} \arrow[d, "\Upsilon_0"] \\ X \arrow[r, "\Psi_0"] & \Delta(0; r, 1) \end{tikzcd}$$

where Υ_0 is the universal covering map $z \mapsto \exp(2\pi i L(z))$ (where L is the branch of $\log_\lambda z$ given by $z \mapsto \frac{1}{\log\lambda}(\log|z| + i \operatorname{arccot}(x/y))$ for each $z = x + iy \in \mathbb{H}$) as in Example 5.1.12, and Ψ_0 is a biholomorphism. $\square$

Exercises for Sect. 5.8

5.8.1 Prove part (a) of Theorem 5.8.2.

5.8.2 Prove that two annuli $\Delta(0; r, 1)$ and $\Delta(0; s, 1)$ with $r, s \in (0,1)$ are biholomorphic if and only if $r = s$.

5.9 Classification of Riemann Surfaces as Quotient Spaces

The results of the previous sections may be summarized as follows:

Theorem 5.9.1 *Up to biholomorphism, every Riemann surface X is biholomorphic to exactly one of the following*:

(i) $\mathbb{P}^1$;
(ii) $\mathbb{C}$;
(iii) $\mathbb{C}^*$;
(iv) $\Gamma\backslash\mathbb{C}$, *where* $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ *and* $\tau \in \mathbb{C}$ *with* $\operatorname{Im}\tau > 0$;
(v) $\Gamma\backslash\Delta$, *where* Γ *is a subgroup of*

$$\operatorname{Aut}(\Delta) = \left\{ b\Phi_c \mid b \in \mathbb{S}^1,\ c \in \Delta,\ \textit{and}\ \Phi_c \colon z \mapsto \frac{z-c}{1-\bar{c}z} \right\}$$

that acts properly discontinuously and freely (*equivalently*, $\Gamma\backslash\mathbb{H}$, *where* Γ *is a subgroup of*

$$\operatorname{Aut}(\mathbb{H}) = \left\{ \Phi \colon z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{R}\ \textit{with}\ ad-bc=1 \right\} \cong \mathrm{PSL}(2,\mathbb{R})$$

that acts properly discontinuously and freely).

The Riemann mapping theorem also allows one to characterize Riemann surfaces according to their fundamental groups. For example, the Riemann mapping theorem, Lemma 5.6.2, and Lemma 5.8.4 give the following characterization of Riemann surfaces with infinite cyclic fundamental group:

Theorem 5.9.2 *If X is a Riemann surface with fundamental group $\pi_1(X) \cong \mathbb{Z}$, then X is biholomorphic to exactly one of the following*:

(i) $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$;
(ii) $\Delta^* = \Delta^*(0;1) = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$;
(iii) *the annulus* $\Delta(0;r,1) = \{z \in \mathbb{C} \mid r < |z| < 1\}$ *for some* $r \in (0,1)$.

Remark According to Exercise 5.8.2, the constant r in (iii) is unique (for a given X).

Definition 5.9.3 Let X be a Riemann surface with universal covering $\widetilde{X}$. Then X is called

(i) *elliptic* if $\widetilde{X} \cong \mathbb{P}^1$;
(ii) *parabolic* if $\widetilde{X} \cong \mathbb{C}$;
(iii) *hyperbolic* if $\widetilde{X} \cong \Delta \cong \mathbb{H}$.

Remarks 1. By the Riemann mapping theorem, every Riemann surface is of exactly one of the above types.

2. A complex torus, that is, a compact Riemann surface with universal covering $\mathbb{C}$, is also called an *elliptic curve* (which should not be confused with *the* elliptic Riemann surface $\mathbb{P}^1$).

Exercises for Sect. 5.9

5.9.1 Let $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$ and $\Gamma' = \mathbb{Z} + \mathbb{Z}\tau'$ be lattices in $\mathbb{C}$ with $\operatorname{Im}\tau, \operatorname{Im}\tau' > 0$ (as in Lemma 5.6.2). Prove that the complex tori $X \equiv \Gamma\backslash\mathbb{C}$ and $X' \equiv \Gamma'\backslash\mathbb{C}$ are biholomorphic if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ for some matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) = \{2 \times 2 \text{ integral matrices in } \mathrm{SL}(2,\mathbb{R})\}.$$

Find an a example of a pair of complex tori that are *not* biholomorphic.

5.9.2 Prove that if a Riemann surface X has noncyclic Abelian fundamental group, then X is biholomorphic to a complex torus.

5.9.3 A Kähler form ω is said to be of *constant curvature* $\lambda \in \mathbb{R}$ if $i\Theta_\omega = \lambda\omega$ (see Definition 2.12.1).

(a) Show that the Kähler form $\omega \equiv y^{-2}\,dx \wedge dy$ on $\mathbb{H}$ is of constant curvature -1. Show also that if Γ is a group of automorphisms acting properly discontinuously and freely on $\mathbb{H}$, then $\Phi^*\omega = \omega$ for each element $\Phi \in \Gamma$.

(b) Let X be a Riemann surface. Using part (a) along with Examples 2.12.2 and 2.12.3, conclude that:

(i) If X is elliptic, then X admits a Kähler form of constant curvature 1.

(ii) If X is parabolic, then X admits a Kähler form of constant curvature 0.

(iii) If X is hyperbolic, then X admits a Kähler form of constant curvature -1.

(c) Let X be a *compact* Riemann surface. Prove that the converse of each of the statements (i)–(iii) in part (b) holds for X; that is, prove that X is elliptic (parabolic, hyperbolic) if and only if X admits a Kähler form ω of constant curvature 1 (respectively, 0, -1). Show that the converses do not hold in general for open Riemann surfaces.

(d) Let X be a compact Riemann surface of genus g. Using part (c) above and the results of Chap. 4 (other approaches will be considered in Exercise 5.16.1 and Sect. 5.22), prove that

(i) X is elliptic if and only if $g = 0$.

(ii) X is parabolic if and only if $g = 1$.

(iii) X is hyperbolic if and only if $g > 1$.

Hint. According to Exercise 4.6.3, the canonical line bundle of a compact Riemann surface of genus 1 is trivial.

5.9.4 An element g of a group is called a *torsion* element if g is of finite order (i.e., $g^m = e$ for some positive integer m). A group in which only the identity is a torsion element is said to be *torsion-free*.

(a) Prove that every torsion element of $\operatorname{Aut}(\mathbb{H})$ must have fixed point.

(b) Prove that the fundamental group of every Riemann surface is torsion-free.

5.9.5 The uniformization theorem leads to a natural generalization of the notion of the winding number of a loop around a point (see Exercises 5.1.5, 5.1.6, and 5.1.7). Let X be an open Riemann surface, let $\Upsilon\colon \widetilde{X} \to X$ be the universal covering of X, let γ be a loop in X that is path homotopic to a constant loop, let $C = \gamma([0,1])$, and let $\tilde{\gamma}$ be a lifting of γ to a loop in $\widetilde{X}$. In particular, we have $\widetilde{X} = \mathbb{H}$ or $\mathbb{C}$. Given a point $p \in X \setminus C$, we set

$$n_X(\gamma; p) \equiv \sum_{z_0 \in \Upsilon^{-1}(p)} n(\tilde{\gamma}; z_0) \in \mathbb{Z}$$

(the right-hand side is a finite sum by Exercise 5.1.6).

(a) Prove that if $p \in X \setminus C$ and θ is a holomorphic 1-form on $X \setminus \{p\}$ with $\mathrm{res}_p \theta = 1$, then

$$n_X(\gamma; p) = \frac{1}{2\pi i} \int_\gamma \theta.$$

Show that such a 1-form θ always exists (see Corollary 2.15.4 and Exercise 2.15.4), and conclude that $n_X(\gamma; p)$ is independent of the choice of the holomorphic universal covering map Υ and the lifting $\tilde{\gamma}$.

(b) Prove that there is a domain Ω such that $C \subset \Omega \Subset X$ and γ is path homotopic in Ω to a constant loop. Conclude that in particular, $n_X(\gamma; p) = 0$ for each point $p \in X \setminus \Omega$.

(c) *Residue theorem*. Prove that if S is a discrete subset of X that is contained in $X \setminus C$, and θ is a holomorphic 1-form on $X \setminus S$, then

$$\frac{1}{2\pi i} \int_\gamma \theta = \sum_{p \in S} n_X(\gamma; p)\, \mathrm{res}_p \theta$$

(note that the right-hand side is actually a finite sum by part (b)).

(d) *Argument principle*. Prove that if f is a nontrivial meromorphic function on X with zero set Z and pole set P both contained in $X \setminus C$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{df}{f} = \sum_{p \in Z \cup P} n_X(\gamma; p)\, \mathrm{ord}_p f.$$

(e) *Rouché's theorem*. Suppose f and g are nontrivial meromorphic functions on X that do not have any zeros or poles in C, and $|g| < |f|$ on C. Let Z_f denote the zero set of f, P_f the pole set of f, Z_{f+g} the zero set of $f+g$, and P_{f+g} the pole set of $f+g$. Prove that

$$\sum_{p \in Z_f \cup P_f} n_X(\gamma; p)\, \mathrm{ord}_p f = \sum_{p \in Z_{f+g} \cup P_{f+g}} n_X(\gamma; p)\, \mathrm{ord}_p (f+g).$$

(f) Suppose X is simply connected, S is a discrete subset of X, and θ is a holomorphic 1-form on $X \setminus S$. Prove that there exists a function $f \in \mathcal{O}(X \setminus S)$ with $df = \theta$ (i.e., θ is exact) if and only if $\mathrm{res}_p \theta = 0$ for each point $p \in S$.

5.10 Smooth Jordan Curves and Homology

Sections 5.10–5.17 rely on the Koebe uniformization theorem (Theorem 5.5.3) as well as elementary facts concerning the first homology and cohomology groups of a second countable $\mathcal{C}^\infty$ surface (as reviewed in Sects. 10.6 and 10.7), but we will avoid using the classification of Riemann surfaces as quotient spaces considered in Sects. 5.6–5.9. The main goal is the fact that every Riemann surface may be obtained by holomorphic attachment of tubes to a planar domain (as in Sects. 2.3 and 5.12). The idea of the proof is to find holomorphic annuli (i.e., tubes) in the Riemann surface, and then to inductively remove them. Reversing the process then gives the desired construction of the given Riemann surface. Each annulus to be removed is found in a small neighborhood of some diffeomorphic image of the circle S^1 (that is, a smooth Jordan curve) that represents a nonzero homology class (in fact, one may take the holomorphic annulus to be the small neighborhood, but it will be slightly easier to allow for other possibilities). In this section, we consider the first step, which is to show that the first homology group of a second countable $\mathcal{C}^\infty$ surface is generated by smooth Jordan curves.

Definition 5.10.1 Let M be a topological surface. By a *Jordan curve* (or a *Jordan loop* or a *simple loop* or a *simple closed curve*) in M, we will mean either a loop $\gamma\colon [a,b] \to M$ for which $\gamma\restriction_{[a,b)}$ is injective or the image $C = \gamma([a,b])$ of such a loop. We will often identify γ with the homeomorphism $\mathbb{S}^1 \to C$ given by $e^{2\pi i t} \mapsto \gamma(a + t(b-a))$ for each $t \in [0,1]$ (recall that a bijective continuous mapping of compact Hausdorff spaces is a homeomorphism).

Definition 5.10.2 Let M be a $\mathcal{C}^\infty$ surface and let $\gamma\colon [a,b] \to M$ be a path.

(a) For $c \in (a,b)$, we say that γ is *smooth at c* if on some neighborhood of c, γ is a $\mathcal{C}^\infty$ map with nonvanishing tangent vector $\dot{\gamma}$. We say that γ is *smooth from the right* at a if γ extends to a path in M that is defined on some interval $[a-\epsilon, b]$ for some $\epsilon > 0$ and that is smooth at a. Similarly, we say that γ is *smooth from the left* at b if γ extends to a path that is defined on some interval $[a, b+\epsilon]$ for some $\epsilon > 0$ and that is smooth at b. We call γ a *smooth path* (or a *smooth curve*) if γ is smooth at each point in (a,b), smooth from the right at a, and smooth from the left at b. We also call the image of a smooth path a *smooth path* (or a *smooth curve*).

(b) We say that γ is *loop-smooth at a* (or, equivalently, *at b*) if $\gamma(a) = \gamma(b)$ (i.e., γ is a loop) and on some neighborhood of $1 = e^{2\pi i 0} = e^{2\pi i(1)} \in \mathbb{C}$ in $\mathbb{S}^1$, the associated map $\hat{\gamma}\colon \mathbb{S}^1 \to M$ given by $e^{2\pi i t} \mapsto \gamma(a + t(b-a))$ for $t \in [0,1]$ is $\mathcal{C}^\infty$ with nonvanishing tangent map $\hat{\gamma}_*$. Equivalently, on some neighborhood of $a + \mathbb{Z}(b-a)$, the map $\mathbb{R} \to M$ given by $t \mapsto \gamma(t - \llcorner \frac{t-a}{b-a} \lrcorner (b-a))$ (where $\llcorner x \lrcorner$ is the floor of $x \in \mathbb{R}$) is $\mathcal{C}^\infty$ with nonvanishing tangent vector (see Exercise 5.10.1). We call γ a *smooth loop* (or a *smooth closed curve*) if γ is a smooth path that is loop-smooth at a and b (in particular, γ is a loop); that is, the associated map $\mathbb{S}^1 \to M$ is a $\mathcal{C}^\infty$ immersion (see Definition 9.9.3). We also call the image of a smooth loop a *smooth loop* (or a *smooth closed curve*).

(c) We call γ a *piecewise smooth path* (or *piecewise smooth curve*) in M if for some partition $a = t_0 < t_1 < \cdots < t_m = b$, $\gamma\restriction_{[t_{j-1},t_j]}$ is a smooth path for each $j = 1, \ldots, m$. If, in addition, $\gamma(a) = \gamma(b)$, then we also call γ a *piecewise smooth loop* (or a *piecewise smooth closed curve*). We will also call the image of a piecewise smooth path a *piecewise smooth path* (or a *piecewise smooth curve*) and the image of a piecewise smooth loop a *piecewise smooth loop* (or a *piecewise smooth closed curve*).

(d) We call γ a *smooth Jordan curve* (or a *smooth Jordan loop* or a *smooth simple loop* or a *smooth simple closed curve*) in M if γ is a smooth loop that is also a Jordan curve (that is, the associated map $\mathbb{S}^1 \to \gamma([a, b]) \subset M$ is a $\mathcal{C}^\infty$ embedding). In this case, we will also call the image $\gamma([0, 1])$ a *smooth Jordan curve* (or a *smooth Jordan loop* or a *smooth simple loop* or a *smooth simple closed curve*). We call γ a *piecewise smooth Jordan curve* (or a *piecewise smooth Jordan loop* or a *piecewise smooth simple loop* or a *piecewise smooth simple closed curve*) if γ is a piecewise smooth loop that is also a Jordan curve. In this case, we also call the image $\gamma([0, 1])$ a *piecewise smooth Jordan curve* (or a *piecewise smooth Jordan loop* or a *piecewise smooth simple loop* or a *piecewise smooth simple closed curve*).

Remarks 1. As usual, we assume that all given loops and paths have domain $[0, 1]$ or $\mathbb{S}^1$ unless otherwise noted.

2. The Jordan curves in a surface M are precisely those subsets that are homeomorphic to $\mathbb{S}^1$.

3. The smooth Jordan curves in a smooth surface M are precisely the 1-dimensional compact connected $\mathcal{C}^\infty$ submanifolds (see Theorem 9.10.1).

4. A smooth path $\gamma\colon [a, b] \to M$ may be a loop and still not be a *smooth loop*. There is the added requirement that γ meet itself smoothly at the base point $\gamma(a) = \gamma(b)$.

5. If a path $\gamma\colon [a, b] \to M$ is smooth from the right at c, then we may define the *right-hand tangent vector* $\dot{\gamma}_+(c)$ to be the tangent vector at c of some smooth extension of $\gamma\restriction_{[c,c+\epsilon)}$ for some $\epsilon > 0$ to a smooth path on a neighborhood of c. Similarly, if γ is smooth from the left at c, then we get a *left-hand tangent vector* $\dot{\gamma}_-(c)$. Actually, one only needs γ to be $\mathcal{C}^1$ from the right at c in the former case, and from the left at c in the latter case. If γ is a loop that is loop-smooth at a (and b), then we may define the *tangent vector at the base point* by $\dot{\gamma}(a) = \dot{\gamma}(b) \equiv \dot{\gamma}_+(a) = \dot{\gamma}_-(b)$.

6. The same terminology is applied to paths in 2-dimensional topological and $\mathcal{C}^\infty$ manifolds.

We will now see that one may smooth a piecewise smooth injective or Jordan curve. For the proof, it will be convenient to first make the pieces of the curve transverse at the nonsmooth points, and to then choose special coordinates in a neighborhood of a corner at which two pieces of the curve meet transversely. Two curves in a $\mathcal{C}^\infty$ surface are *transverse* at a point of intersection that is a smooth point for each of the curves if their tangent vectors at the point are linearly independent. The following two lemmas accomplish these steps.

Lemma 5.10.3 *Let $\alpha\colon [a,b]\to M$ and $\beta\colon [c,d]\to M$ be smooth paths in a $\mathcal{C}^\infty$ surface M such that $0\in(a,b)\cap(c,d)$, $\alpha(0)=\beta(0)$, and $\dot\alpha(0)$ and $\dot\beta(0)$ are linearly independent (i.e., α and β are transverse at this point). Then there exists a local $\mathcal{C}^\infty$ chart (W,Φ,W') in M such that for some $r>0$, we have $[-r,r]\subset\alpha^{-1}(W)\cap\beta^{-1}(W)$ and*

$$\Phi(\alpha(t))=(t,0)\quad\textit{and}\quad\Phi(\beta(t))=(0,t)\quad\forall t\in[-r,r].$$

Proof The claim is local, so we may assume without loss of generality that M is an open subset of $\mathbb{R}^2$ with $\alpha(0)=\beta(0)=(0,0)\in M$. Writing $\alpha=(\alpha_1,\alpha_2)$ and $\beta=(\beta_1,\beta_2)$, we see that since $\alpha'(0)\neq 0$ ($\alpha'(0)$ and $\beta'(0)$ are linearly independent), we may apply a linear change of coordinates to get $\alpha_1'(0)\neq 0$. Hence, by the 1-dimensional $\mathcal{C}^\infty$ inverse function theorem (the easy case of Theorem 9.9.1), we may assume that α_1 extends to a diffeomorphism of an open interval containing $[a,b]$ onto an open interval containing 0. The map $(x,y)\mapsto(\alpha_1^{-1}(x),y-\alpha_2(\alpha_1^{-1}(x)))$ is then a diffeomorphism of neighborhoods of $(0,0)$ in $\mathbb{R}^2$ (with $\mathcal{C}^\infty$ inverse mapping $(u,v)\mapsto(\alpha_1(u),v+\alpha_2(u))$) and hence, applying this diffeomorphism to get a change of coordinates, we may assume that $\alpha(t)=(t,0)\in\mathbb{R}^2$ for each $t\in[a,b]$. Similarly, since $\alpha'(0)$ and $\beta'(0)$ are linearly independent, we have $\beta_2'(0)\neq 0$, and hence the mapping $(x,y)\mapsto(x-\beta_1(\beta_2^{-1}(y)),\beta_2^{-1}(y))$ is a diffeomorphism of neighborhoods of $(0,0)$ (with $\mathcal{C}^\infty$ inverse mapping $(u,v)\mapsto(u+\beta_1(v),\beta_2(v))$). This diffeomorphism maps the point $(t,0)$ to $(t-\beta_1(\beta_2^{-1}(0)),\beta_2^{-1}(0))=(t,0)$. Therefore, after applying this diffeomorphism, we get $\alpha(t)=(t,0)$ and $\beta(t)=(0,t)$ for all t near 0. $\square$

Lemma 5.10.4 *Let M be a $\mathcal{C}^\infty$ surface, let $\alpha\colon I=[a,b]\to M$ be an injective piecewise smooth path for which $0\in(a,b)$ and α is smooth at each point in $(a,b)\setminus\{0\}$, let V be a neighborhood of $p=\alpha(0)$ in M, and let U be a neighborhood of 0 in $(a,b)\cap\alpha^{-1}(V)$. Then there exists an injective piecewise smooth path $\beta\colon[a,b]\to M$ such that for some partition $a=t_0<t_1=0<t_2<\cdots<t_m=b$ of I,*

(i) *The path β is smooth at each point in $(a,b)\setminus\{t_1,\dots,t_{m-1}\}$, while $\dot\beta_-(t_j)$ and $\dot\beta_+(t_j)$ are linearly independent for each $j=1,\dots,m-1$; and*
(ii) *We have $\{t_1,\dots,t_{m-1}\}\subset U$, $\beta=\alpha$ on $\{0\}\cup(I\setminus U)$, $\beta(U)\subset V$, and β is path homotopic to α.*

Remark In fact, as will be clear from the proof, we may choose β so that $m\leq 4$ and $\beta=\alpha$ on $[a,0]$ (see Fig. 5.6).

Proof of Lemma 5.10.4 Since α is injective, we may fix a constant $R>0$ and a local $\mathcal{C}^\infty$ chart $(W,\Phi=(\Phi_1,\Phi_2),(-R,R)\times(-R,R))$ in M with $p\in W\subset V$, $\Phi(p)=(0,0)$, and $\alpha^{-1}(W)\subset U$. For some $r>0$ with $[-r,r]\subset\alpha^{-1}(W)$, we have smooth paths $\alpha_1,\alpha_2\colon[-r,r]\to W$ with $\alpha_1\restriction_{[-r,0]}=\alpha\restriction_{[-r,0]}$ and $\alpha_2\restriction_{[0,r]}=\alpha\restriction_{[0,r]}$. Setting $\tau\equiv\Phi(\alpha_1)=(\tau_1,\tau_2)$ and $\rho\equiv\Phi(\alpha_2)=(\rho_1,\rho_2)$ and applying Lemma 5.10.3 (to α_1 and an arbitrary smooth path transverse to α_1 at p), we see that we may choose Φ so that $\tau(t)=(\tau_1(t),\tau_2(t))=(t,0)\in\mathbb{R}^2$ for each $t\in[-r,r]$

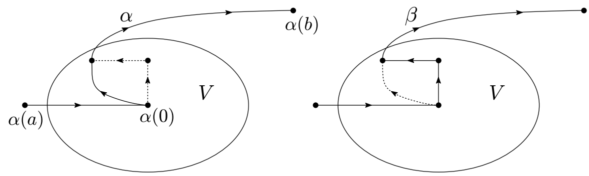

Fig. 5.6 Modifying an injective piecewise smooth path to get transversality

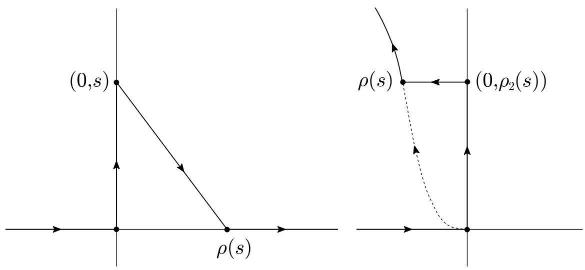

Fig. 5.7 The two possible local representations of the path and their local modifications

(in particular, $r < R$). We may also fix numbers s and S such that $0 < s < S < r$ and

$$\begin{aligned}\alpha([-s,s]) \subset \alpha_1([-s,s]) \cup \alpha_2([-s,s]) \subset \Phi^{-1}((-S,S)\times(-S,S))\\ \subset W \setminus \alpha([a,-r]\cup[r,b]).\end{aligned}$$

The lemma is trivial if $\dot\alpha_-(0)$ and $\dot\alpha_+(0)$ are linearly independent, so we may assume that $\rho'(0) = (\rho_1'(0), 0)$. Since α_2 is smooth, we then have $\rho_1'(0) \neq 0$, and hence we may assume that ρ_1' is nonvanishing on $[-r, r]$.

We now consider the two possible cases (see Fig. 5.7). If $\rho_2 = 0$ at all points in $[0, r]$ that are near 0, then, after shrinking r, we may assume that $\rho_2 \equiv 0$ on $[0, r]$, and since α is injective, we must then have $\rho_1' > 0$ on $[-r, r]$. In this case, we let $\gamma\colon [0, s] \to [0, \rho_1(s)] \times [0, s] \subset [-S, S] \times [-S, S]$ be the injective piecewise smooth path from $(0, 0)$ to $\rho(s) = (\rho_1(s), 0)$ given by the vertical line segment path in $\mathbb{R}^2$ from $(0, 0)$ to $(0, s)$, followed by the line segment path from $(0, s)$ to $\rho(s)$, where each segment is parametrized so as to be smooth. If there exist nonnegative points arbitrarily close to 0 at which ρ_2 is nonzero, then we may choose s so that $\rho_2(s) \neq 0$ and $\rho_2'(s) \neq 0$. We may then let

$$\gamma\colon [0,s] \to [-|\rho_1(s)|, |\rho_1(s)|] \times [-|\rho_2(s)|, |\rho_2(s)|] \subset [-S,S]\times[-S,S]$$

be the injective piecewise smooth path from $(0, 0)$ to $\rho(s)$ given by the vertical line segment path from $(0, 0)$ to $(0, \rho_2(s))$, followed by the horizontal line segment path from $(0, \rho_2(s))$ to $\rho(s)$, where again, each segment is parametrized so as to be smooth. In particular, in this case, since ρ_1 is either strictly increasing on $[-r, r]$ or strictly decreasing on $[-r, r]$, γ will meet $\rho([s, r])$ only at the point $\gamma(s) = \rho(s)$. In either case, it is easy to verify that the piecewise smooth path $\beta\colon [a, b] \to M$

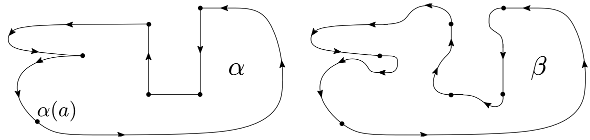

Fig. 5.8 Smoothing a piecewise smooth Jordan curve

given by $\beta \equiv \alpha$ on $[a, 0] \cup [s, b]$ and $\beta \equiv \Phi^{-1}(\gamma)$ on $[0, s]$ has the required properties. $\square$

Lemma 5.10.5 *Let M be a $\mathcal{C}^\infty$ surface, let $\alpha\colon I = [a, b] \to M$ be an injective piecewise smooth path or a piecewise smooth Jordan curve that is loop-smooth at a and b, let $a = t_0 < t_1 < \cdots < t_m = b$ be a partition of I such that α is smooth at each point in $(a, b) \setminus \{t_1, \ldots, t_{m-1}\}$, and for each $j = 1, \ldots, m-1$, let V_j be a neighborhood of $\alpha(t_j)$ in M and let U_j be a neighborhood of t_j in $(a, b) \cap \alpha^{-1}(V_j)$. Then there exists a path $\beta\colon I \to M$ such that*

(i) *The path β is an injective smooth path if α is an injective path, and β is a smooth Jordan curve if α is a Jordan curve; and*
(ii) *We have $\beta = \alpha$ on $\{t_1, \ldots, t_{m-1}\} \cup (I \setminus (U_1 \cup \cdots \cup U_{m-1}))$, $\beta(U_j) \subset V_j$ for each $j = 1, \ldots, m-1$, and β is path homotopic to α.*

In other words, an injective piecewise smooth path (or a piecewise smooth Jordan curve that is loop-smooth at the base point) is path homotopic to an injective smooth path (respectively, a smooth Jordan curve) that agrees with the original path outside an arbitrarily small neighborhood of the nonsmooth points, as well as at the nonsmooth points (see Fig. 5.8).

Proof of Lemma 5.10.5 Clearly, we may assume that the neighborhoods $\{V_j\}$ are disjoint. Next we observe that we may assume without loss of generality that $\dot\alpha_-(t_j)$ and $\dot\alpha_+(t_j)$ are linearly independent (i.e., that we have transversality) for each $j = 1, \ldots, m-1$. For we may choose a constant $u > 0$ so that $2u < \min_{1 \le j \le m}(t_j - t_{j-1})$ and $[t_j - u, t_j + u] \subset U_j$ for each $j = 1, \ldots, m-1$. For $j = 1, \ldots, m-1$, Lemma 5.10.4, when applied to the curve $t \mapsto \alpha(t_j + t)$ for $t \in [-u, u]$ along with a neighborhood V_j' of $\alpha(t_j)$ in $V_j \setminus \alpha(I \setminus (t_j - u, t_j + u))$ and a neighborhood U_j' of 0 in $(U_j + (-t_j)) \cap (-u, u)$ with $\alpha(U_j' + t_j) \subset V_j'$, yields an injective curve $\beta_j\colon [-u, u] \to M$ with the properties described in the lemma. The new curve $\widehat{\alpha}$, which is equal to α on $I \setminus \bigcup_{j=1}^{m-1}[t_j - u, t_j + u]$ and given by $t \mapsto \beta_j(t - t_j)$ on $[t_j - u, t_j + u]$ for each j, agrees with α off of U_j for each j, as well as near the endpoints and at the points $t_1, \ldots, t_{m-1}$. Moreover, $\hat\alpha$ is an injective piecewise smooth path if α is such a path, and a piecewise smooth Jordan curve that is loop-smooth at a if α is such a Jordan curve, $\hat\alpha$ is homotopic to α, $\hat\alpha$ maps U_j into V_j for each j, and $\hat\alpha$ has the desired transversality property at those interior points $\{s_1, \ldots, s_k\} \supset \{t_1, \ldots, t_{m-1}\}$ at which it is not smooth. Thus, by replacing α with $\hat\alpha$, replacing $\{t_1, \ldots, t_{m-1}\}$ with $\{s_1, \ldots, s_k\}$, and replacing $\{U_j\}$ with $\{\widehat{U}_i\}_{i=1}^k$ and $\{V_j\}$

with $\{\widehat{V}_i\}_{i=1}^k$, where if $s_i - t_j$ is one of the points in U'_j at which β_j is not smooth, then $\widehat{U}_i \subset U'_j + t_j$ and $\widehat{V}_i \subset V'_j$ are suitable sufficiently small neighborhoods of s_i and $\hat{\alpha}(s_i)$, respectively, we may assume that α has the desired transversality property.

By Lemma 5.10.3, for some $r > 0$, we may choose disjoint local $\mathcal{C}^\infty$ charts

$$\{(W_j, \Phi_j, (t_j - r, t_j + r) \times (t_j - r, t_j + r))\}_{j=1}^{m-1}$$

such that for each $j = 1, \ldots, m-1$, we have $W_j \Subset V_j$,

$$(t_j - r, t_j + r) \subset (t_{j-1}, t_{j+1}) \cap \alpha^{-1}(W_j) \cap U_j,$$

and

$$\Phi_j(\alpha(t)) = \begin{cases} (t, t_j) & \text{if } t \in (t_j - r, t_j], \\ (t_j, t) & \text{if } t \in [t_j, t_j + r) \end{cases}$$

(one obtains Φ_j by applying the lemma to the curve $t \mapsto \alpha(t + t_j)$ and then letting Φ_j be the sum of the resulting local chart with the point (t_j, t_j)). Since either $\alpha\restriction_{[a,b]}$ is injective or $\alpha\restriction_{[a,b)}$ is injective with $\alpha(a) = \alpha(b)$, we may choose a constant $s \in (0, r)$ such that for each $j = 1, \ldots, m-1$,

$$\Phi_j^{-1}([t_j - s, t_j + s] \times [t_j - s, t_j + s]) \cap \alpha([a, t_j - r] \cup [t_j + r, b]) = \emptyset.$$

Fixing a nondecreasing $\mathcal{C}^\infty$ function $\lambda\colon \mathbb{R} \to [0, \infty)$ such that $\lambda \equiv 0$ on a neighborhood of $(-\infty, 0]$ and $\lambda \equiv 1$ on a neighborhood of $[s, \infty)$ (for example, for a nonnegative $\mathcal{C}^\infty$ function χ with compact support in $(0, s)$ such that $q = \int_0^s \chi(v)\,dv > 0$, the function $\lambda\colon t \mapsto q^{-1}\int_0^t \chi(v)\,dv$ has the required properties), we see that the path $\beta\colon [a, b] \to M$ given by

$$t \mapsto \begin{cases} \alpha(t) & \text{if } t \in [a, b] \setminus \bigcup_{j=1}^{m-1} [t_j, t_j + s], \\ \Phi_j^{-1}((1 - \lambda(t - t_j)) \cdot (t, t_j) & \\ \quad + \lambda(t - t_j) \cdot (t_j, t)) & \text{if } t \in [t_j, t_j + s] \text{ for some } j, \end{cases}$$

has the required properties. The details of the verification are left to the reader (see Exercise 5.10.2). □

We record here for later use (see Sect. 5.15) the following consequence of Lemma 5.10.5:

Lemma 5.10.6 *If p and q are distinct points in a $\mathcal{C}^\infty$ surface M, then there exists an injective smooth path in M from p to q.*

Proof Fixing a point $p \in M$, we must show that the set N of points $x \in M \setminus \{p\}$ for which there is an injective smooth path in M from p to x is equal to the set $M \setminus \{p\}$. Clearly, $N \neq \emptyset$. Given a point $x_0 \in M \setminus \{p\}$, we may choose a local $\mathcal{C}^\infty$ chart $(U, \Phi, \Delta(0; 2))$ in M with $\Phi(x_0) = 0$ and $p \notin U$, and we may set $D \equiv \Phi^{-1}(\Delta(0; 1))$. If D meets N, then there exists an injective smooth path α

in M from p to a point in D, and we may set $t_0 \equiv \min \alpha^{-1}(\overline{D}) \in (0, 1)$. Hence, given a point $x \in D$, the product of suitable reparametrizations of the paths $\alpha\restriction_{[0,t_0]}$ and $t \mapsto \Phi^{-1}((1-t)\Phi(\alpha(t_0)) + t\Phi(x))$ is an injective piecewise smooth path in M from p to x. Applying Lemma 5.10.5, we get an injective smooth path in M from p to x. Thus $D \subset N$. From this, it follows that N is open with empty boundary in the connected set $M \setminus \{p\}$, and hence that $N = M \setminus \{p\}$. □

We now come to the main goal of this section:

Proposition 5.10.7 *Let M be a $\mathcal{C}^\infty$ surface. Then, for every (continuous) loop α in M, there exist smooth Jordan curves $\alpha_1, \dots, \alpha_n$ in M such that*

$$\int_\alpha \theta = \sum_{j=1}^{n} \int_{\alpha_j} \theta$$

for every closed $\mathcal{C}^\infty$ 1-form θ on M. Consequently, if M is second countable and $\mathbb{A}$ is a subring of $\mathbb{C}$ containing $\mathbb{Z}$, then every 1-cycle in M with coefficients in $\mathbb{A}$ is homologous (as a 1-cycle over $\mathbb{C}$) to an $\mathbb{A}$-linear combination of smooth Jordan curves.

Remark The second part of the above proposition follows immediately from the first part and Lemma 10.6.6. The only reason for including the second countability condition in this second part is that in this book, homology is considered only for second countable surfaces. Second countability does not play an essential role in the proof. It also follows that for M second countable, $H_1(M, \mathbb{A})$ is generated by those homology classes that are represented by smooth Jordan curves, provided M is orientable (or M admits an orientable $\mathcal{C}^\infty$ surface structure) if $\mathbb{A} \neq \mathbb{R}$ and $\mathbb{A} \neq \mathbb{C}$ (see Sects. 10.6 and 10.7).

Proof of Proposition 5.10.7 Let $\alpha\colon [0, 1] \to M$ be a loop in M. We may assume without loss of generality that M is second countable, since (for example) we may replace M with a connected relatively compact neighborhood of $\alpha([0, 1])$ (second countability will allow us to use the convenient language of homology). We may choose a partition $0 = t_0 < t_1 < \cdots < t_m = 1$, local $\mathcal{C}^\infty$ charts $\{(U_j, \Phi_j, U'_j)\}_{j=1}^m$, and open sets $\{D_j\}_{j=1}^m$ such that for each $j = 1, \dots, m$, U_j is simply connected, $\alpha([t_{j-1}, t_j]) \subset D_j \subset U_j$, $\Phi_j(D_j)$ is a disk in $\mathbb{R}^2$, and $D_j \subset U_k$ for any $k \in \{1, \dots, m\}$ with $D_j \cap D_k \neq \emptyset$ (to get the last property, one may apply Lemma 9.3.6). We may also assume that α is piecewise smooth and that $\alpha\restriction_{(t_{j-1},t_j]}$ is injective for $j = 1, \dots, m$ (for example, we may replace the segment $\alpha\restriction_{[t_{j-1},t_j]}\colon [t_{j-1}, t_j] \to D_j$ with a suitable smooth path in D_j from $\alpha(t_{j-1})$ to $\alpha(t_j)$ that is injective on $(t_{j-1}, t_j]$). We now proceed by induction on m to show that α is homologous to a sum of smooth Jordan curves. The cases $m = 1$ and $m = 2$, in which α lies in the simply connected open set U_1, are trivial. Assume now that $m > 2$ and that the claim holds for partitions of $[0, 1]$ into fewer than m intervals.

If $\alpha(r) = \alpha(s)$ for some r, s, and j with

$$t_0 \le r \le t_{j-1} < t_j \le s \le t_{m-1} \qquad \text{or} \qquad t_1 \le r \le t_{j-1} < t_j \le s \le t_m,$$

then α is homologous to the sum of the two piecewise smooth loops $\alpha_1 * \alpha_3$ and α_2, where α_1, α_2, and α_3 are suitable reparametrizations of $\alpha\restriction_{[t_0,r]}$, $\alpha\restriction_{[r,s]}$, and $\alpha\restriction_{[s,t_m]}$, respectively. For each of these loops, we get a partition as above containing fewer than m subintervals, and therefore, by the induction hypothesis, each is homologous to a sum of smooth Jordan curves. Thus we may assume that no such points r and s exist.

Given an index $j \in \{1, \dots, m-1\}$, we may set

$$r \equiv \min([t_{j-1}, t_j] \cap \alpha^{-1}(\alpha([t_j, t_{j+1}]))) \in (t_{j-1}, t_j]$$

and

$$s \equiv \max([t_j, t_{j+1}] \cap \alpha^{-1}(\alpha(r))) \in [t_j, t_{j+1}).$$

If $r < s$, then α is homologous to the sum of the (path homotopically trivial) loop β_1 in U_j given by a suitable reparametrization of $\alpha\restriction_{[r,s]}$ and the piecewise smooth loop $\beta_2 \colon [0,1] \to M$ determined by $\beta_2 \circ \varphi = \alpha\restriction_{[t_0,r]\cup[s,t_m]}$, where $\varphi \colon [t_0, r] \cup [s, t_m] \to [0,1]$ is the map given by

$$\varphi(t) = \begin{cases} \frac{1}{2} \cdot \frac{t-t_0}{r-t_0} & \text{if } t \in [t_0, r], \\ \frac{1}{2} + \frac{1}{2} \cdot \frac{t-s}{t_m-s} & \text{if } t \in [s, t_m]. \end{cases}$$

By replacing α with β_2, we get a path in which the corresponding segment satisfies $r = s$ (the partition corresponding to β_2 has the m intervals given by $\varphi([t_{k-1}, t_k]) \subset \beta_2^{-1}(D_k)$ for $k \in \{1, \dots, m\} \setminus \{j, j+1\}$, $\varphi([t_{j-1}, r]) \subset \beta_2^{-1}(D_j)$, and $\varphi([s, t_{j+1}]) \subset \beta_2^{-1}(D_{j+1})$). Performing this procedure inductively on j, we see that we may assume that $\alpha([t_{j-1}, t_j))$ and $\alpha([t_j, t_{j+1}])$ are disjoint and that $\alpha([t_{j-1}, t_j])$ and $\alpha((t_j, t_{j+1}])$ are disjoint for each $j = 1, \dots, m-1$. A similar argument shows that we may assume without loss of generality that $\alpha([t_{m-1}, t_m))$ and $\alpha([t_0, t_1])$ are disjoint and that $\alpha([t_{m-1}, t_m])$ and $\alpha((t_0, t_1])$ are disjoint. Thus we may assume that α is a piecewise smooth Jordan curve.

Finally, fixing a point $a \in (0, t_1)$ at which α is smooth, we see that α is homologous to the piecewise smooth Jordan curve γ given by $t \mapsto \alpha(t+a)$ on $[0, 1-a]$ and $t \mapsto \alpha(t-1+a)$ on $[1-a, 1]$. Since γ is also loop-smooth at 0, Lemma 5.10.5 implies that γ is homologous to a smooth Jordan curve, and the claim follows. $\square$

Exercises for Sect. 5.10

5.10.1 Let $\gamma \colon [a,b] \to M$ be a loop in a $\mathcal{C}^\infty$ surface M. Verify that γ is loop-smooth at a (i.e., on some neighborhood of $1 = e^{2\pi i \cdot 0}$, the associated map $\hat{\gamma} \colon \mathbb{S}^1 \to M$ given by $e^{2\pi i t} \mapsto \gamma(a + t(b-a))$ for $t \in [0,1]$ is $\mathcal{C}^\infty$ with nonvanishing tangent map) if and only if on some neighborhood of $a + \mathbb{Z}(b-a) \subset \mathbb{R}$, the map $\mathbb{R} \to M$ given by $t \mapsto \gamma(t - \llcorner \frac{t-a}{b-a} \lrcorner (b-a))$ is $\mathcal{C}^\infty$ with nonvanishing tangent vector.

5.10.2 Verify that the path β constructed in the proof of Lemma 5.10.5 has the required properties.

5.10.3 Prove that every Jordan curve $\alpha: [0,1] \to M$ (injective path) in a $\mathcal{C}^\infty$ surface M is path homotopic to a smooth Jordan curve (respectively, a smooth injective path) β. Prove also that if $0 = t_0 < \cdots < t_n = 1$ is a partition and U_j is a neighborhood of $\gamma([t_{j-1}, t_j])$ for $j = 1, \dots, n$, then β may be chosen so that $\beta([t_{j-1}, t_j]) \subset U_j$ for $j = 1, \dots, n$.

Hint. One may form a partition of the curve into segments, each of which lies in a coordinate disk. One may then replace a small segment of the curve about each of the partition points with a smooth segment that agrees with the original curve at the partition points. The complement of a suitable small neighborhood of the partition points is equal to a finite collection of disjoint segments, and Lemma 5.10.6 allows one to replace each of these segments with a smooth segment. One may then apply Lemma 5.10.5.

5.11 Separating Smooth Jordan Curves

In this section, we consider some properties of smooth Jordan curves that allow us to find and remove annuli. The main goals are to show that a smooth Jordan curve in an oriented smooth surface admits a neighborhood that is diffeomorphic to an annulus, and to show that the de Rham pairing of a separating smooth Jordan curve and the cohomology class of a compactly supported $\mathcal{C}^\infty$ closed 1-form is zero.

Lemma 5.11.1 *Let $\gamma: [0,1] \to M$ be an injective smooth path or a smooth Jordan curve in an oriented $\mathcal{C}^\infty$ surface M, and let $C = \gamma([0,1])$.*

(a) *If γ is a smooth Jordan curve, then for some $R > 1$, there exists an orientation-preserving diffeomorphism Φ of some neighborhood A of C in M onto the annulus $\Delta(0; 1/R, R)$ with $\Phi(\gamma(t)) = e^{2\pi i t}$ for each $t \in [0,1]$ (see Fig. 5.9). Consequently, $M \setminus C$ has either one or two connected components, and each of these connected components has boundary equal to C.*

(b) *If γ is an injective smooth path, then for some $\epsilon > 0$, there exists an orientation-preserving diffeomorphism Φ of some neighborhood R of C in M onto the open rectangle $(-\epsilon, 1+\epsilon) \times (-\epsilon, \epsilon)$ with $\Phi(\gamma(t)) = (t, 0)$ for each $t \in [0,1]$. Consequently, $M \setminus C$ is connected.*

Remarks 1. For a smooth Jordan curve in a *nonorientable* surface, one may find a neighborhood that is diffeomorphic to either an annulus or a Möbius band. This fact will be proved and applied in the discussion of $\mathcal{C}^\infty$ structures on surfaces in Chap. 6 (see Sect. 6.10).

2. As indicated in part (a), the existence of annular neighborhoods implies that the complement $M \setminus C$ of a smooth Jordan curve C in an oriented smooth surface M has exactly one or two connected components, and each of these connected components has boundary equal to C. On the other hand, one may also obtain this fact directly (see Exercise 5.11.3).

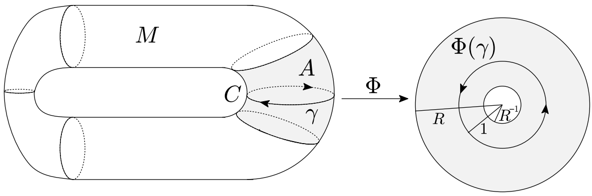

Fig. 5.9 A smooth annular neighborhood of a smooth Jordan curve

3. For our purposes, the easier fact that any smooth Jordan curve is separating in some neighborhood of the curve would suffice. However, it will be convenient to have the annular neighborhood provided by Lemma 5.11.1.

4. Given $a, b, c, d \in \mathbb{R}$ with $a < 1 < b$ and $c < 1 < d$, there exists an orientation-preserving diffeomorphism $\Lambda\colon \Delta(0; a, b) \xrightarrow{\approx} \Delta(0; c, d)$ such that $\Lambda\restriction_{\mathbb{S}^1} = \mathrm{Id}_{\mathbb{S}^1}$ (see Exercise 5.11.4). Consequently, part (a) of the lemma actually holds for *any* $R > 1$. Similarly, part (b) holds for *any* $\epsilon > 0$.

Proof of Lemma 5.11.1 We prove part (a) and leave the proof of part (b), which is similar, to the reader (see Exercise 5.11.1). Assuming that γ is a smooth Jordan curve, we may form the associated mapping $\gamma_0\colon \mathbb{S}^1 \to C$ given by $e^{2\pi i t} \mapsto \gamma(t)$ and the mapping $\alpha\colon \mathbb{R} \to C$ given by $t \mapsto \gamma_0(e^{2\pi i t}) = \gamma(t - \llcorner t \lrcorner)$ (which is actually the extension of γ to a periodic mapping with period 1). By Theorem 9.9.2, C is a compact $\mathcal{C}^\infty$ submanifold of M, γ_0 is a diffeomorphism, and α is a local diffeomorphism (in fact, α is a $\mathcal{C}^\infty$ covering map). Observe that $\mathcal{C}^\infty$ local extensions of local inverses of these maps have nonvanishing differentials at points in C. Hence we may form a finite covering of C by positively oriented local $\mathcal{C}^\infty$ charts $\{(U_j, \Lambda_j = (x_j, y_j), R_j = I_j \times (-\delta_j, \delta_j))\}_{j=1}^m$ in M such that for each $j = 1, \dots, m$, I_j is an open interval, $\delta_j > 0$, $C \cap U_j = \{p \in U_j \mid y_j(p) = 0\} = \alpha(I_j)$, and $x_j(\alpha(t)) = t$ for each point $t \in I_j$.

In particular, since α_* maps $(T\mathbb{R})_t$ bijectively onto $(TC)_{\gamma(t)}$ for each point $t \in \mathbb{R}$, we have $dx_j\restriction_{T(C\cap U_j\cap U_k)} = dx_k\restriction_{T(C\cap U_j\cap U_k)}$ for each pair of indices j, k. On the other hand, we have $dy_j\restriction_{T(C\cap U_j)} = 0$ for each j, so we have

$$(dx_j \wedge dy_j)_p = (dx_k \wedge dy_j)_p \quad \forall p \in C \cap U_j \cap U_k.$$

For we have $(dx_k)_p = a(dx_j)_p + b(dy_j)_p$ for some scalars $a, b \in \mathbb{R}$, and restricting to TC, we see that $a = 1$.

We may also form nonnegative $\mathcal{C}^\infty$ functions $\{\lambda_j\}_{j=1}^m$ such that $\operatorname{supp}\lambda_j \subset U_j$ for each j and $\sum \lambda_j \equiv 1$ on a neighborhood of C. Thus we may define a $\mathcal{C}^\infty$ mapping $\Psi\colon M \to \mathbb{C} = \mathbb{R}^2$ by

$$\Psi = \sum_{j=1}^m \lambda_j \cdot e^{2\pi i(x_j + i y_j)} = \sum_{j=1}^m \lambda_j \cdot e^{-2\pi y_j} \cdot (\cos 2\pi x_j, \sin 2\pi x_j)$$

$$= \left(\sum_{j=1}^{m} \lambda_j \cdot e^{-2\pi y_j} \cdot \cos 2\pi x_j, \sum_{j=1}^{m} \lambda_j \cdot e^{-2\pi y_j} \cdot \sin 2\pi x_j \right) = (\Psi_1, \Psi_2)$$

(as usual, we have extended $\lambda_j \cdot e^{2\pi i(x_j + iy_j)}$ by 0 to all of M for each j). For $p \in C$ and $\zeta = \gamma_0^{-1}(p) = u + iv \leftrightarrow (u, v)$, we have $\zeta = e^{2\pi i x_j(p)}$ (i.e., $(u, v) = (\cos(2\pi x_j(p)), \sin(2\pi x_j(p)))$) for *any* index j with $p \in U_j$. Hence

$$\Psi(p) = \sum_{j=1}^{m} \lambda_j(p) \cdot e^{2\pi i x_j(p)} = \sum_{j=1}^{m} \lambda_j(p) \cdot \gamma_0^{-1}(p) = \gamma_0^{-1}(p)$$

and (since $\sum d\lambda_j = d(\sum \lambda_j) = 0$ near C)

$$\begin{aligned}(d\Psi)_p &= ((d\Psi_1)_p, (d\Psi_2)_p)\\ &= \left(\sum_{j=1}^{m} (-2\pi)\lambda_j(p)(v(dx_j)_p + u(dy_j)_p), \right.\\ &\qquad \left. \sum_{j=1}^{m} 2\pi\lambda_j(p)(u(dx_j)_p - v(dy_j)_p) \right).\end{aligned}$$

Therefore, since $u^2 + v^2 = |\zeta|^2 = 1$, we have

$$\begin{aligned}(d\Psi_1 \wedge d\Psi_2)_p &= 4\pi^2 \sum_{j,k=1}^{m} \lambda_j(p)\lambda_k(p)v^2(dx_j \wedge dy_k)_p\\ &\quad + 4\pi^2 \sum_{j,k=1}^{m} \lambda_j(p)\lambda_k(p)u^2(dx_k \wedge dy_j)_p\\ &= 4\pi^2 \sum_{j,k=1}^{m} \lambda_j(p)\lambda_k(p)v^2(dx_j \wedge dy_k)_p\\ &\quad + 4\pi^2 \sum_{j,k=1}^{m} \lambda_j(p)\lambda_k(p)u^2(dx_j \wedge dy_k)_p\\ &= 4\pi^2 \sum_{j,k=1}^{m} \lambda_j(p)\lambda_k(p)(dx_j \wedge dy_k)_p.\end{aligned}$$

On the other hand, if P is the set of indices j for which $p \in U_j$, then we have $(dx_j \wedge dy_k)_p = (dx_k \wedge dy_k)_p$ for all $j, k \in P$. Therefore

$$\begin{aligned}(d\Psi_1 \wedge d\Psi_2)_p &= 4\pi^2 \sum_{j,k \in P} \lambda_j(p)\lambda_k(p)(dx_k \wedge dy_k)_p\\ &= 4\pi^2 \sum_{k \in P} \lambda_k(p)(dx_k \wedge dy_k)_p > 0.\end{aligned}$$

Therefore, by continuity, $d\Psi_1 \wedge d\Psi_2 > 0$ on some neighborhood Ω of C, and hence by the $\mathcal{C}^\infty$ inverse function theorem (Theorem 9.9.1 and Theorem 9.9.2), $\Psi\restriction_\Omega$ is an orientation-preserving local diffeomorphism. Furthermore, the intersection of $\Psi^{-1}(\mathbb{S}^1)$ with a sufficiently small neighborhood of C must be equal to C. For if this were not the case, then we could choose a sequence $\{p_\nu\}$ in $M \setminus C$ that converged to a point $p \in C$ and that satisfied $\Psi(p_\nu) \in \mathbb{S}^1$ for each ν. Setting $q_\nu \equiv \gamma_0(\Psi(p_\nu)) \in C$ for each ν, we would get

$$q_\nu = \gamma_0(\Psi(p_\nu)) \to \gamma_0(\Psi(p)) = p.$$

But then, for large ν, both p_ν and q_ν would lie in a neighborhood of p on which Ψ was injective. Since $\Psi(q_\nu) = \gamma_0^{-1}(q_\nu) = \Psi(p_\nu)$, we have arrived at a contradiction. Lemma 10.2.11 now implies that Ψ maps a neighborhood H of C diffeomorphically onto a neighborhood of $\mathbb{S}^1$. Choosing $R > 1$ sufficiently close to 1, we see that the restriction Φ of Ψ to $(\Psi\restriction_H)^{-1}(\Delta(0; 1/R, R))$ has the required properties.

Finally, setting $Q_1 \equiv \Delta(0; 1/R, 1)$ and $Q_2 \equiv \Delta(0; 1, R)$, we see that $\Phi^{-1}(Q_j)$ must lie in some connected component P_j of $M \setminus C$ for each $j = 1, 2$. Any connected component of $M \setminus C$ containing neither $\Phi^{-1}(Q_1)$ nor $\Phi^{-1}(Q_2)$ would have to be a connected component of M, since it would have no boundary points in M. Therefore, since M is connected, the set of connected components of $M \setminus C$ is $\{P_1, P_2\}$. Finally, observe that $\partial P_j = \Phi^{-1}(\partial Q_j) = C$ for $j = 1, 2$. □

Definition 5.11.2 A closed subset of a connected topological space X is called *separating* in X if $X \setminus K$ is *not* connected. If $X \setminus K$ is connected, then K is called *nonseparating* in X.

For example, a circle in $\mathbb{C}$ is separating; in fact, as will be proved in Chap. 6, *every* Jordan curve in $\mathbb{C}$ is separating (see Corollary 6.7.2). The smooth Jordan curve $C = \gamma([0, 1])$ appearing in Fig. 5.9 is nonseparating.

Lemma 5.11.3 *If C is a separating smooth Jordan curve in an oriented smooth surface M, then $\int_C \theta = 0$ for every closed $\mathcal{C}^\infty$ 1-form θ with compact support.*

Remark The converse, which is also true, will be proved in Sect. 5.15 (see Proposition 5.15.2); and the proof of the converse will be applied in the construction of a canonical homology basis for a compact Riemann surface in Sect. 5.16.

Proof of Lemma 5.11.3 By Lemma 5.11.1, $M \setminus C$ must have exactly two connected components M_1 and M_2, and we must have $\partial M_1 = \partial M_2 = C$. Orienting C positively with respect to M_1 and applying Stokes' theorem (Theorem 9.7.17), we get

$$\int_C \theta = \int_{M_1} d\theta = \int_{M_1} 0 = 0.$$

□

We close this section with the following elementary observation, which will allow us to choose *disjoint tubes* for removal from a Riemann surface, thereby reducing its homology.

Lemma 5.11.4 *Let M be a topological surface; let $K = \bigcup_{\alpha\in A} K_\alpha$, where $\{K_\alpha\}$ is a countable locally finite collection of disjoint compact sets each of which is mapped onto a closed disk in the plane in some local chart; and let $N = M \setminus K$. Then every path $\gamma\colon [0,1] \to M$ with $\gamma(0), \gamma(1) \in N$ is path homotopic (in M) to a path in N. In particular, N is connected.*

Proof Suppose first that for some constants r and R with $0 < r < R < \infty$, $(D, \Psi, \Delta(0; R))$ is a local chart in M and $K = \Psi^{-1}(\overline{\Delta(0; r)})$. Fixing $r_0 \in (r, R)$ with $\gamma(0), \gamma(1) \in M \setminus \Psi^{-1}(\overline{\Delta(0; r_0)})$, we get real numbers $\{a_j\}_{j=1}^k$ and $\{b_j\}_{j=1}^k$ such that $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k < 1$, $I_j \equiv [a_j, b_j] \subset \gamma^{-1}(\Psi^{-1}(\Delta(0; r_0)))$ for $j = 1, \dots, k$, and $\gamma([0,1] \setminus \bigcup_{j=1}^k I_j) \subset N$ (the compact set $\gamma^{-1}(\Psi^{-1}(\overline{\Delta(0; r)}))$ admits a finite covering by disjoint open intervals that are relatively compact in $\gamma^{-1}(\Psi^{-1}(\Delta(0; r_0)))$, and we may take $\{(a_j, b_j)\}_{j=1}^k$ to be these intervals). In particular, for some $r_1 \in (r, r_0)$ and for each $j = 1, \dots, k$, $\gamma(a_j)$ and $\gamma(b_j)$ lie in the topological annulus $\Psi^{-1}(\Delta(0; r_1, r_0))$, and hence we have a path homotopy $H_j\colon \gamma\restriction_{[a_j, b_j]} \sim \lambda_j$ to a path in $\lambda_j\colon [a_j, b_j] \to \Psi^{-1}(\Delta(0; r_1, r_0)) \subset N$. The map $H\colon [0,1] \times [0,1] \to M$ given by

$$H(t, s) = \begin{cases} H_j(t, s) & \text{if } (t, s) \in [a_j, b_j] \times [0,1] \text{ for some } j, \\ \gamma(t) & \text{if } (t, s) \in ([0,1] \setminus \bigcup_{j=1}^k [a_j, b_j]) \times [0,1], \end{cases}$$

is then a path homotopy from γ to a path in N.

Now for the general case, let $A_0 = \{\alpha_1, \dots, \alpha_m\} \subset A$ be the finitely many indices α for which $\gamma([0,1])$ meets K_α. By the above, we may homotope γ within the connected component of $M \setminus \bigcup_{\alpha\in A\setminus A_0} K_\alpha$ containing the connected set $\gamma([0,1]) \cup \bigcup_{\alpha\in A_0} K_\alpha$ to a path γ_1 in $M \setminus \bigcup_{\alpha\in\{\alpha_1\}\cup(A\setminus A_0)} K_\alpha$. Proceeding inductively, we get the claim. □

Exercises for Sect. 5.11

5.11.1 Prove part (b) of Lemma 5.11.1.

5.11.2 Prove that in part (a) of Lemma 5.11.1, if $\Phi_0\colon V \to W$ is an orientation-preserving diffeomorphism of a neighborhood V of $\gamma([s_0, s_1])$ onto a neighborhood W of the arc $E \equiv \{e^{2\pi i t} \mid s_0 \le t \le s_1\}$ for some pair s_0, s_1 with $0 < s_0 < s_1 < 1$, and $\Phi_0(\gamma(t)) = e^{2\pi i t}$ for each $t \in \gamma^{-1}(V)$, then we may choose R and Φ so that $S \equiv \Phi^{-1}(\{r\zeta \mid 1/R < r < R, \zeta \in E\}) \subset V$ and $\Phi = \Phi_0$ on S. Prove also that in part (b), if $\Phi_0\colon V \to W$ is an orientation-preserving diffeomorphism of a neighborhood V of $\gamma([s_0, s_1])$ onto a neighborhood W of the line segment $L = [s_0, s_1] \times \{0\}$ for some pair s_0, s_1 with $0 \le s_0 < s_1 \le 1$ and $\Phi_0(\gamma(t)) = (t, 0)$ for each $t \in \gamma^{-1}(V)$, then we may choose ϵ and Φ so that $S \equiv \Phi^{-1}([s_0, s_1] \times (-\epsilon, \epsilon)) \subset V$ and $\Phi = \Phi_0$ on S.

5.11.3 Prove directly from local descriptions (without applying the existence of annular neighborhoods as in Lemma 5.11.1) that the complement $M \setminus C$ of a smooth Jordan curve C in a smooth (not necessarily orientable) surface M has exactly one or two connected components, and each of these connected components has boundary equal to C.

5.11.4 Prove that if $a < 1 < b$ and $c < 1 < d$, then there exists an orientation-preserving diffeomorphism $\Lambda \colon \Delta(0; a, b) \xrightarrow{\approx} \Delta(0; c, d)$ such that $\Lambda\restriction_{\mathbb{S}^1} = \mathrm{Id}_{\mathbb{S}^1}$ (hence part (a) of Lemma 5.11.1 holds for *any* $R > 1$).

5.11.5 Let $\gamma \colon \mathbb{R} \to M$ be a $\mathcal{C}^\infty$ embedding of $\mathbb{R}$ into an oriented $\mathcal{C}^\infty$ surface M (i.e., a proper $\mathcal{C}^\infty$ mapping for which the tangent vector $\dot{\gamma}$ is nonvanishing), and let $C = \gamma(\mathbb{R})$. Prove that there exists an orientation-preserving diffeomorphism Φ of some neighborhood of C in M onto the open strip $\mathbb{R} \times (-1, 1)$ with $\Phi(\gamma(t)) = (t, 0)$ for each $t \in \mathbb{R}$. Conclude that in particular, $M \setminus C$ has either one or two connected components.

5.11.6 Prove that there exists a universal constant $C > 0$ such that if X is a Riemann surface, p is a point in X, and $(D, z = \Phi, \Delta(0; 1))$ is a local holomorphic chart with $p \in D$ and $\Phi(p) = z(p) = 0$, then there exists a meromorphic 1-form θ on X such that (cf. Exercise 5.5.1)

(i) The meromorphic 1-form θ is holomorphic on $X \setminus \{p\}$;

(ii) We have $\|\theta\|_{L^2(X\setminus D)} \leq C$;

(iii) The meromorphic 1-form $\theta + z^{-2}\,dz$ on D has a removable singularity at p;

(iv) We have $\|z\theta + z^{-1}\,dz\|_{L^2(D)} \leq C$; and

(v) We have $\int_\gamma \theta = 0$ for every separating smooth Jordan curve γ in $X \setminus \{p\}$.

5.12 Holomorphic Attachment and Removal of Tubes

In this section we develop notation for the holomorphic attachment and removal of a locally finite family of disjoint tubes (cf. Example 2.3.3).

5.12.1 Holomorphic Attachment of Tubes

Let Y be a complex 1-manifold; let $\{R_{0j}\}_{j\in J}$ and $\{R_{1j}\}_{j\in J}$ be numbers in the interval $(1, \infty)$; let

$$\{(D_{0j}, \Phi_{0j}, \Delta(0; R_{0j}))\}_{j\in J} \quad \text{and} \quad \{(D_{1j}, \Phi_{1j}, \Delta(0; R_{1j}))\}_{j\in J}$$

be locally finite families of disjoint local holomorphic charts in Y such that for all $j, k \in J$, $D_{0j} \cap D_{1k} = \emptyset$, and for each $j \in J$, let $A'_{\nu j} \equiv \Phi_{\nu j}^{-1}(\Delta(0; 1, R_{\nu j})) \subset D_{\nu j}$ for $\nu = 0, 1$, let $T_j \equiv \Delta(0; 1/R_{0j}, R_{1j})$, let $A_{0j} \equiv \Delta(0; 1/R_{0j}, 1) \subset T_j$, and let $A_{1j} \equiv \Delta(0; 1, R_{1j}) \subset T_j$. Setting

$$A'_\nu \equiv \bigcup_{j\in J} A'_{\nu j} \subset D_\nu \equiv \bigcup_{j\in J} D_{\nu j} \subset Y \quad \text{and} \quad A_\nu \equiv \bigsqcup_{j\in J} A_{\nu j} \subset T \equiv \bigsqcup_{j\in J} T_j$$

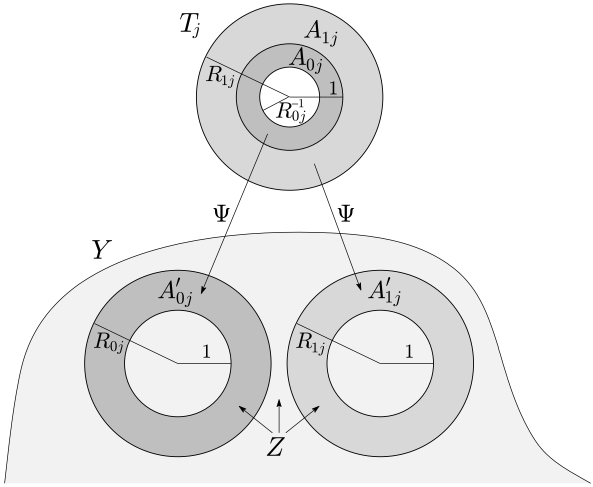

Fig. 5.10 An annulus (or tube) and two disjoint disks

for $\nu = 0, 1$, we get a biholomorphism $\Psi\colon A_0 \cup A_1 \to A'_0 \cup A'_1$, where for each index $\nu \in \{0, 1\}$, each index $j \in J$, and each point $z \in A_{\nu j}$, the image p of z under the inclusion $A_{\nu j} \hookrightarrow A_\nu$ satisfies

$$\Psi(p) \equiv \begin{cases} \Phi_{0j}^{-1}(1/z) \in A'_{0j} & \text{if } \nu = 0, \\ \Phi_{1j}^{-1}(z) \in A'_{1j} & \text{if } \nu = 1. \end{cases}$$

Setting

$$Z \equiv Y \setminus \Bigl[\bigcup_{\nu \in \{0,1\},\, j \in J} \Phi_{\nu j}^{-1}\bigl(\overline{\Delta(0; 1)}\bigr) \Bigr] \supset A'_0 \cup A'_1,$$

we see that $\Psi^{-1}(K' \cap (A'_0 \cup A'_1))$ is closed in T for every compact set $K' \subset Z$, and $\Psi(K \cap (A_0 \cup A_1))$ is closed in Z for every compact set $K \subset T$ (see Fig. 5.10).

We may therefore form the holomorphic attachment $X \equiv Z \cup_\Psi T = Z \sqcup T/\sim$, where for $\nu \in \{0, 1\}$, $j \in J$, $p \in A'_{\nu j}$, and $z \in A_{\nu j}$, the images p_0 of p and z_0 of z in $Z \sqcup T$ under the inclusions $A'_{\nu j}, A_{\nu j} \hookrightarrow Z \sqcup T$ satisfy $p_0 \sim z_0$ if and only if $z \cdot \Phi_{0j}(p) = 1$ for $\nu = 0$ and $z = \Phi_{1j}(p)$ for $\nu = 1$ (see Fig. 5.11, in which $J = \{1, 2, 3\}$). In other words, X is a complex 1-manifold obtained by removing the unit disks in each of the coordinate disks D_{0j} and D_{1j}, and gluing in a tube (i.e., an annulus) T_j (equivalently, the boundaries of the unit disks are glued together). We call X the *complex* 1-*manifold obtained by holomorphic attachment of tubes at elements of the locally finite family of disjoint coordinate disks* $\{(D_{\nu j}, \Phi_{\nu j}, \Delta(0; R_{\nu j}))\}_{\nu \in \{0,1\}, j \in J}$ (*or simply the complex* 1-*manifold obtained by holomorphic attachment of tubes at* $\{D_{\nu j}\}_{\nu \in \{0,1\}, j \in J}$). Observe that X is compact if and only if Y is compact. If Y is connected, then X is connected. However, X may be connected even if Y is not (see Exercise 2.3.2).

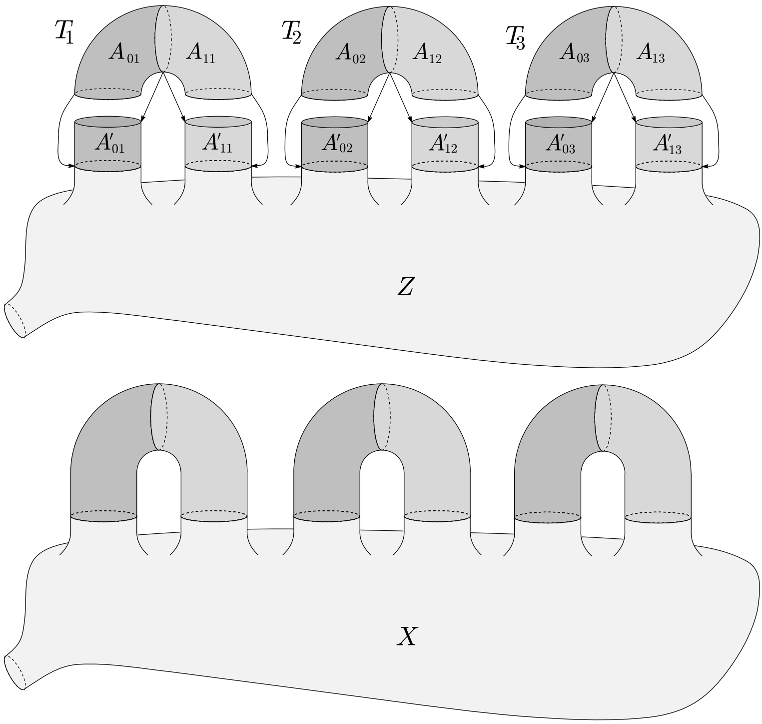

Fig. 5.11 Holomorphic attachment of tubes

Fixing a number $R^*_{\nu j}$ with $1 < R^*_{\nu j} \le R_{\nu j}$ for each $\nu \in \{0, 1\}$ and $j \in J$, setting $T^*_j \equiv \Delta(0; 1/R^*_{0j}, R^*_{1j}) \subset T_j$ for each $j \in J$, and setting $T^* \equiv \bigsqcup_{j \in J} T^*_j \subset T$, we may form the holomorphic attachment $X^* \equiv Z \sqcup T^*/\sim$ of the tubes $\{T^*_j\}_{j \in J}$ at the coordinate disks

$$\left\{\left(D^*_{\nu j} \equiv \Phi^{-1}_{\nu j}(\Delta(0; R^*_{\nu j})), \Phi_{\nu j}\restriction_{D^*_{\nu j}}, \Delta(0; R^*_{\nu j})\right)\right\}_{\nu \in \{0,1\}, j \in J}.$$

The natural inclusion $Z \sqcup T^* \subset Z \sqcup T$ then descends to a natural biholomorphism $X^* \xrightarrow{\cong} X$ (see Exercise 5.12.1), so we may identify X^* with X. In particular, for each $j \in J$, we may always choose R_{0j} and R_{1j} to be equal and arbitrarily close to 1.

5.12.2 Holomorphic Removal of Tubes

The inverse operation of tube attachment is *tube removal*. Let X be a complex 1-manifold, let $\{R_{0j}\}_{j \in J}$ and $\{R_{1j}\}_{j \in J}$ be collections of numbers in $(1, \infty)$, let $\{(T_j, \Phi_j, \Delta(0; 1/R_{0j}, R_{1j}))\}_{j \in J}$ be a locally finite collection of disjoint local holomorphic charts in X, and for each $j \in J$, let $A_{0j} \equiv \Phi^{-1}_j(\Delta(0; 1/R_{0j}, 1)) \subset T_j$

and $A_{1j} \equiv \Phi_j^{-1}(\Delta(0; 1, R_{1j})) \subset T_j$. For each $\nu = 0, 1$ and each $j \in J$, let $D_{\nu j} \equiv \Delta(0; R_{\nu j})$ and $A'_{\nu j} \equiv \Delta(0; 1, R_{\nu j}) \subset D_{\nu j}$. Setting

$$A_\nu \equiv \bigcup_{j \in J} A_{\nu j} \subset T \equiv \bigcup_{j \in J} T_j \quad \text{and} \quad A'_\nu \equiv \bigsqcup_{j \in J} A'_{\nu j} \subset D_\nu \equiv \bigsqcup_{j \in J} D_{\nu j}$$

for $\nu = 0, 1$, we get a biholomorphism $\Psi\colon A_0 \cup A_1 \to A'_0 \sqcup A'_1$, where for each index $\nu \in \{0, 1\}$, each index $j \in J$, and each point $p \in A_{\nu j}$, $\Psi(p)$ is the image in $A'_0 \sqcup A'_1$ of the point

$$\begin{cases} 1/\Phi_j(p) \in A'_{0j} & \text{for } \nu = 0, \\ \Phi_j(p) \in A'_{1j} & \text{for } \nu = 1, \end{cases}$$

under the inclusion $A'_{\nu j} \hookrightarrow A'_0 \sqcup A'_1 \subset D_0 \sqcup D_1$. Setting

$$\begin{aligned} Z &\equiv X \setminus [T \cap \partial(A_0 \cup A_1)] = X \setminus (T \cap \partial A_0) = X \setminus (T \cap \partial A_1) \\ &= X \setminus \bigcup_{j \in J} \Phi_j^{-1}(\partial\Delta(0; 1)) \supset A_0 \cup A_1, \end{aligned}$$

we see that $\Psi^{-1}(K' \cap (A'_0 \sqcup A'_1))$ is closed in Z for every compact set $K' \subset D_0 \sqcup D_1$, and $\Psi(K \cap (A_0 \cup A_1))$ is closed in $D_0 \sqcup D_1$ for every compact set $K \subset Z$. Thus we get the holomorphic attachment $Y \equiv (D_0 \sqcup D_1) \cup_\Psi Z = D_0 \sqcup D_1 \sqcup Z/{\sim}$, where for $\nu \in \{0, 1\}$, $j \in J$, $p \in A_{\nu j}$, and $z \in A'_{\nu j}$, the images p_0 of p and z_0 of z in $D_0 \sqcup D_1 \sqcup Z$ under the inclusions $A_{\nu j}, A'_{\nu j} \hookrightarrow D_0 \sqcup D_1 \sqcup Z$ satisfy $p_0 \sim z_0$ if and only if $z \cdot \Phi_j(p) = 1$ for $\nu = 0$ and $z = \Phi_j(p)$ for $\nu = 1$. In other words, Y is the complex 1-manifold obtained by removing the unit circles in each of the tubes (i.e., coordinate annuli) T_j and gluing in caps (i.e., disks) D_{0j} and D_{1j} on each end of the remaining two pieces. That is, Y is obtained by attaching caps to the open subset Z of X. We call Y the *complex* 1*-manifold obtained by holomorphic removal of the locally finite family of disjoint tubes* $\{(T_j, \Phi_j, \Delta(0; 1/R_{0j}, R_{1j}))\}_{j \in J}$ *from* X (or simply *by holomorphic removal of* $\{T_j\}_{j \in J}$). Observe that X is compact if and only if Y is compact. If Y is connected, then X is connected, but connectivity of X does not imply connectivity of Y (see Exercise 5.12.2).

As is the case for holomorphic attachment of tubes, shrinking the annuli about the unit circle does not affect the outcome. More precisely, fixing a number $R^*_{\nu j}$ with $1 < R^*_{\nu j} \leq R_{\nu j}$ for each $\nu \in \{0, 1\}$ and $j \in J$, setting $T^*_j \equiv \Phi_j^{-1}(\Delta(0; 1/R^*_{0j}, R^*_{1j})) \subset T_j$ for each $j \in J$, setting $D^*_{\nu j} \equiv \Delta(0; R^*_{\nu j})$ for each $\nu \in \{0, 1\}$ and $j \in J$, and setting $D^*_\nu \equiv \bigsqcup_{j \in J} D^*_{\nu j} \subset D_\nu$ for $\nu = 0, 1$, we may form the complex 1-manifold $Y^* \equiv D^*_0 \sqcup D^*_1 \sqcup Z/\sim$ obtained by holomorphic removal of the tubes $\{T^*_j\}_{j \in J}$. The natural inclusion $D^*_0 \sqcup D^*_1 \sqcup Z \subset D_0 \sqcup D_1 \sqcup Z$ then descends to a natural biholomorphism $Y^* \xrightarrow{\cong} Y$ (see Exercise 5.12.3), so we may identify Y^* with Y. In particular, for each $j \in J$, we may always choose R_{0j} and R_{1j} to be equal and arbitrarily close to 1.

5.12.3 Holomorphic Reattachment of Tubes

Tube attachment and tube removal are inverse operations (up to biholomorphism). In other words, if one removes tubes in a complex 1-manifold X as in Sect. 5.12.2, and then (re)attaches tubes with the disks and constants as in Sect. 5.12.1, then the resulting complex 1-manifold is biholomorphic to the original complex 1-manifold X. In fact, if one reattaches only those tubes in some subfamily, then the resulting complex 1-manifold is biholomorphic to the complex 1-manifold obtained by removal from X of those tubes in the complementary subfamily. Similarly, if one attaches tubes to a complex 1-manifold Y as in Sect. 5.12.1, and then removes tubes with the annuli and constants as in Sect. 5.12.2, then the resulting complex 1-manifold is biholomorphic to the original complex 1-manifold Y. The details of holomorphic reattachment of tubes are provided in this section. The details of the inverse operation (holomorphic attachment followed by holomorphic removal of tubes) are left to the reader (see Exercise 5.12.5).

In the notation of Sect. 5.12.2, let $Y = (D_0 \sqcup D_1) \cup_\Psi Z$, let $\Pi_Z \colon Z \hookrightarrow Y$ be the holomorphic inclusion of Z, and for each $\nu \in \{0, 1\}$ and $j \in J$, let $\Pi_{D_{\nu j}} \colon D_{\nu j} \hookrightarrow Y$ be the holomorphic inclusion of the disk $D_{\nu j} = \Delta(0; R_{\nu j})$, let $\widehat{D}_{\nu j} = \Pi_{D_{\nu j}}(D_{\nu j})$, and let $\widehat{\Phi}_{\nu j} \equiv \Pi_{D_{\nu j}}^{-1} \colon \widehat{D}_{\nu j} \xrightarrow{\cong} D_{\nu j}$. Given a set of indices $\widehat{J} \subset J$, holomorphic attachment of tubes at elements of the locally finite family of disjoint coordinate disks $\{(\widehat{D}_{\nu j}, \widehat{\Phi}_{\nu j}, \Delta(0; R_{\nu j}))\}_{\nu\in\{0,1\}, j\in\widehat{J}}$ in Y yields a complex 1-manifold $\widehat{X}$. More precisely, for each $j \in \widehat{J}$, we may set

$$\begin{aligned}\widehat{A}_{0j} &\equiv \Delta(0; 1/R_{0j}, 1),\\ \widehat{A}_{1j} &\equiv \Delta(0; 1, R_{1j}) \subset \widehat{T}_j \equiv \Delta(0; 1/R_{0j}, R_{1j}),\end{aligned}$$

and

$$\widehat{A}_{\nu j}{}' \equiv \Pi_{D_{\nu j}}(A'_{\nu j}) = \Pi_Z(A_{\nu j}) = \widehat{\Phi}_{\nu j}^{-1}(\Delta(0; 1, R_{\nu j})) \subset \widehat{D}_{\nu j} \subset Y \quad \text{for } \nu = 0, 1.$$

As in Sect. 5.12.1, setting

$$\widehat{A}'_\nu \equiv \bigcup_{j\in\widehat{J}} \widehat{A}'_{\nu j} \subset \widehat{D}_\nu \equiv \bigcup_{j\in\widehat{J}} \widehat{D}_{\nu j} \subset Y \quad \text{and} \quad \widehat{A}_\nu \equiv \bigsqcup_{j\in\widehat{J}} \widehat{A}_{\nu j} \subset \widehat{T} \equiv \bigsqcup_{j\in\widehat{J}} \widehat{T}_j$$

for $\nu = 0, 1$, we get a biholomorphism $\widehat{\Psi} \colon \widehat{A}_0 \cup \widehat{A}_1 \to \widehat{A}'_0 \cup \widehat{A}'_1$, where for each index $\nu \in \{0, 1\}$, each index $j \in \widehat{J}$, and each point $z \in \widehat{A}_{\nu j}$, the image p of z under the inclusion $\widehat{A}_{\nu j} \hookrightarrow \widehat{A}_\nu$ satisfies

$$\widehat{\Psi}(p) \equiv \begin{cases} \widehat{\Phi}_{0j}^{-1}(1/z) = \Pi_{D_{0j}}(1/z) \in \widehat{A}'_{0j} & \text{if } \nu = 0,\\ \widehat{\Phi}_{1j}^{-1}(z) = \Pi_{D_{1j}}(z) \in \widehat{A}'_{1j} & \text{if } \nu = 1.\end{cases}$$

Setting

$$\begin{aligned}\widehat{Z} &\equiv Y \setminus \left[\bigcup_{\nu\in\{0,1\},\, j\in\widehat{J}} \widehat{\Phi}_{\nu j}^{-1}\big(\overline{\Delta(0;1)}\big)\right] = Y \setminus \left[\bigcup_{\nu\in\{0,1\},\, j\in\widehat{J}} \Pi_{D_{\nu j}}\big(\overline{\Delta(0;1)}\big)\right] \\ &= \Pi_Z(Z) \cup \bigcup_{\nu\in\{0,1\},\, j\in J\setminus\widehat{J}} \widehat{D}_{\nu j} \supset \Pi_Z(Z) \cap (\widehat{D}_0 \cup \widehat{D}_1) = \widehat{A}'_0 \cup \widehat{A}'_1,\end{aligned}$$

we may form the holomorphic attachment (of tubes) $\widehat{X} \equiv \widehat{Z} \cup_{\widehat{\Psi}} \widehat{T}$, with inclusion mappings $\Pi_{\widehat{Z}} \colon \widehat{Z} \hookrightarrow \widehat{X}$ and $\Pi_{\widehat{T}_j} \colon \widehat{T}_j \hookrightarrow \widehat{X}$ for each $j \in \widehat{J}$. We call $\widehat{X}$ the *complex* 1-*manifold obtained by holomorphic reattachment of the tubes* $\{T_j\}_{j\in\widehat{J}}$ *to* Y.

Setting $\widetilde{J} \equiv J \setminus \widehat{J}$, holomorphic removal of the locally finite family of disjoint tubes $\{(T_j, \Phi_j, \Delta(0; 1/R_{0j}, R_{1j}))\}_{j\in\widetilde{J}}$ from X yields a complex 1-manifold $\widetilde{X}$. More precisely, setting

$$\widetilde{A}_\nu \equiv \bigcup_{j\in\widetilde{J}} A_{\nu j} \subset \widetilde{T} \equiv \bigcup_{j\in\widetilde{J}} T_j \subset T \quad \text{and} \quad \widetilde{A}'_\nu \equiv \bigsqcup_{j\in\widetilde{J}} A'_{\nu j} \subset \widetilde{D}_\nu \equiv \bigsqcup_{j\in\widetilde{J}} D_{\nu j}$$

for $\nu = 0, 1$, we get a biholomorphism $\widetilde{\Psi} \colon \widetilde{A}_0 \cup \widetilde{A}_1 \to \widetilde{A}'_0 \sqcup \widetilde{A}'_1$, where for each index $\nu \in \{0,1\}$, each index $j \in \widetilde{J}$, and each point $p \in A_{\nu j}$, $\widetilde{\Psi}(p)$ is the image in $\widetilde{A}'_0 \sqcup \widetilde{A}'_1$ of the point

$$\begin{cases} 1/\Phi_j(p) \in A'_{0j} & \text{for } \nu = 0, \\ \Phi_j(p) \in A'_{1j} & \text{for } \nu = 1, \end{cases}$$

under the inclusion $A'_{\nu j} \hookrightarrow \widetilde{A}'_0 \sqcup \widetilde{A}'_1 \subset \widetilde{D}_0 \sqcup \widetilde{D}_1$. Setting

$$\begin{aligned}\widetilde{Z} &\equiv X \setminus [\widetilde{T} \cap \partial(\widetilde{A}_0 \cup \widetilde{A}_1)] = X \setminus (\widetilde{T} \cap \partial\widetilde{A}_0) = X \setminus (\widetilde{T} \cap \partial\widetilde{A}_1) \\ &= X \setminus \bigcup_{j\in\widetilde{J}} \Phi_j^{-1}(\partial\Delta(0;1)) = Z \cup \bigcup_{j\in\widehat{J}} T_j \supset Z \cap \widetilde{T} = \widetilde{A}_0 \cup \widetilde{A}_1,\end{aligned}$$

we may form the holomorphic attachment $\widetilde{Y} \equiv (\widetilde{D}_0 \sqcup \widetilde{D}_1) \cup_{\widetilde{\Psi}} \widetilde{Z}$, with inclusion mappings $\Pi_{\widetilde{Z}} \colon \widetilde{Z} \hookrightarrow \widetilde{Y}$ and $\widetilde{\Pi}_{D_{\nu j}} \colon D_{\nu j} \hookrightarrow \widetilde{Y}$ for each $\nu \in \{0,1\}$ and $j \in \widetilde{J}$. One may now verify that the natural mapping $\widetilde{Y} \to \widehat{X}$ given by

$$\begin{aligned} \Pi_{\widetilde{Z}}(p) &\mapsto \Pi_{\widehat{Z}}(\Pi_Z(p)) && \forall p \in Z, \\ \Pi_{\widetilde{Z}}(p) &\mapsto \Pi_{\widehat{T}_j}(\Phi_j(p)) && \forall j \in \widehat{J}, p \in T_j, \\ \widetilde{\Pi}_{D_{\nu j}}(z) &\mapsto \Pi_{\widehat{Z}}(\Pi_{D_{\nu j}}(z)) && \forall \nu \in \{0,1\}, j \in \widetilde{J}, z \in D_{\nu j}, \end{aligned}$$

is a well-defined biholomorphism that maps $\Pi_{\widetilde{Z}}(Z)$ onto $\Pi_{\widehat{Z}}(\Pi_Z(Z))$, $\Pi_{\widetilde{Z}}(T_j)$ onto $\Pi_{\widehat{T}_j}(\widehat{T}_j)$ for each index $j \in \widehat{J}$, and $\widetilde{\Pi}_{D_{\nu j}}(D_{\nu j})$ onto $\Pi_{\widehat{Z}}(\widehat{D}_{\nu j})$ for each index $\nu \in \{0,1\}$ and each index $j \in \widetilde{J}$ (see Exercise 5.12.4). Thus we may identify $\widetilde{Y}$ with $\widehat{X}$. In other words, holomorphic reattachment of the tubes $\{T_j\}_{j\in\widehat{J}}$ to Y yields the same complex 1-manifold as holomorphic removal of the tubes $\{T_j\}_{j\in\widetilde{J}}$ from X.

5.12.4 Finite Sequential Holomorphic Removal/Attachment of Tubes

If a locally finite family of disjoint tubes in a complex 1-manifold is divided into finitely many disjoint subfamilies, and the subfamilies are then inductively removed (i.e., if the subfamilies are removed one at a time), then the resulting complex 1-manifold will be naturally biholomorphic to the complex 1-manifold obtained by removing the tubes simultaneously. More precisely, in the notation of Sect. 5.12.2, suppose $\{J_k\}_{k=1}^m$ are nonempty disjoint subsets of the index set J with union J; that is, $\{J_k\}_{k=1}^m$ is a *partition* of J. For each $k = 1, \dots, m$, let

$$A_\nu^{(k)} \equiv \bigcup_{j\in J_k} A_{\nu j} \subset T^{(k)} \equiv \bigcup_{j\in J_k} T_j \quad \text{and} \quad A_\nu^{(k)\prime} \equiv \bigsqcup_{j\in J_k} A'_{\nu j} \subset D_\nu^{(k)} \equiv \bigsqcup_{j\in J_k} D_{\nu j}$$

for $\nu = 0, 1$, and let $\Psi_k \colon A_0^{(k)} \cup A_1^{(k)} \to A_0^{(k)\prime} \sqcup A_1^{(k)\prime}$ be the associated biholomorphism as in Sect. 5.12.2. Now let

$$\begin{aligned} Z_1 &\equiv X \setminus [T^{(1)} \cap \partial(A_0^{(1)} \cup A_1^{(1)})] = X \setminus (T^{(1)} \cap \partial A_0^{(1)}) = X \setminus (T^{(1)} \cap \partial A_1^{(1)}) \\ &= X \setminus \bigcup_{j\in J_1} \Phi_j^{-1}(\partial\Delta(0;1)) \supset A_0^{(1)} \cup A_1^{(1)} \cup \bigcup_{k=2}^m T^{(k)}, \end{aligned}$$

let $Y_1 \equiv (D_0^{(1)} \sqcup D_1^{(1)}) \cup_{\Psi_1} Z_1$ be the complex 1-manifold obtained by holomorphic removal of the tubes $\{T_j\}_{j\in J_1}$, and let $\Pi_{Z_1} \colon Z_1 \hookrightarrow Y_1$, and $\Pi_{D_\nu^{(1)}} \colon D_\nu^{(1)} \hookrightarrow Y_1$ for $\nu = 0, 1$, be the corresponding inclusion mappings. We then have the locally finite family of disjoint tubes

$$\{(\Pi_{Z_1}(T_j), \Phi_j \circ \Pi_{Z_1}^{-1}, \Delta(0; 1/R_{0j}, R_{1j}))\}_{j\in J_2}$$

in Y_1, and holomorphic removal of these tubes yields the complex 1-manifold

$$Y_2 \equiv (D_0^{(2)} \sqcup D_1^{(2)}) \cup_{\Psi_2\circ\Pi_{Z_1}^{-1}} Z_2$$

and the corresponding inclusion mappings $\Pi_{Z_2} \colon Z_2 \hookrightarrow Y_2$ and $\Pi_{D_\nu^{(2)}} \colon D_\nu^{(2)} \hookrightarrow Y_2$ for $\nu = 0, 1$, where

$$Z_2 \equiv Y_1 \setminus \bigcup_{j\in J_2} \Pi_{Z_1}(\Phi_j^{-1}(\partial\Delta(0;1))) \supset \Pi_{Z_1}\left(A_0^{(2)} \cup A_1^{(2)} \cup \bigcup_{k=3}^m T^{(k)}\right).$$

Proceeding inductively, we get complex 1-manifolds $X = Y_0, Y_1, \dots, Y_m$ and, for each $k = 1, \dots, m$, holomorphic inclusion mappings $\Pi_{Z_k} \colon Z_k \hookrightarrow Y_k$ of an open set $Z_k \subset Y_{k-1}$ and $\Pi_{D_\nu^{(k)}} \colon D_\nu^{(k)} \hookrightarrow Y_k$ for $\nu = 0, 1$. Moreover, setting $Z_0 \equiv X$,

$\Pi_{Z_0} \equiv \mathrm{Id}_X$, $Z_{m+1} \equiv Y_m$, and $\Pi_{Z_{m+1}} \equiv \mathrm{Id}_{Y_m}$, we see that for each $k = 1, \dots, m$, the composition $\Pi_{Z_{k-1}} \circ \cdots \circ \Pi_{Z_0}$ is defined on the open set

$$X \setminus \bigcup_{j \in \bigcup_{i=1}^{k-1} J_i} \Phi_j^{-1}(\partial \Delta(0; 1)) \supset T^{(k)},$$

we have

$$Z_k = Y_{k-1} \setminus \bigcup_{j \in J_k} \Pi_{Z_{k-1}} \circ \cdots \circ \Pi_{Z_0}(\Phi_j^{-1}(\partial \Delta(0; 1))),$$

$\Pi_{Z_{m+1}} \circ \cdots \circ \Pi_{Z_{k+1}}$ is defined on $\Pi_{D_\nu^{(k)}}(D_\nu^{(k)})$ for $\nu = 0, 1$, and

$$Y_k = (D_0^{(k)} \sqcup D_1^{(k)}) \cup_{\Psi_k \circ (\Pi_{Z_{k-1}} \circ \cdots \circ \Pi_{Z_0})^{-1}} Z_k = D_0^{(k)} \sqcup D_1^{(k)} \sqcup Z_k / \sim$$

is the complex 1-manifold obtained by holomorphic removal of the tubes

$$\{\Pi_{Z_{k-1}} \circ \cdots \circ \Pi_{Z_0}(T_j)\}_{j \in J_k}$$

from Y_{k-1} (in other words, under the appropriate identifications, Y_k is the complex 1-manifold obtained by holomorphic removal of the tubes $\{T_j\}_{j \in J_k}$ from Y_{k-1}).

On the other hand, as in Sect. 5.12.2, holomorphic removal of the tubes $\{T_j\}_{j \in J}$ yields the complex 1-manifold $Y \equiv (D_0 \sqcup D_1) \cup_\Psi Z = D_0 \sqcup D_1 \sqcup Z/\sim$, where

$$Z \equiv X \setminus \bigcup_{j \in J} \Phi_j^{-1}(\partial \Delta(0; 1)),$$

and $\Psi \colon A_0 \cup A_1 \xrightarrow{\cong} A_0' \sqcup A_1'$ is the biholomorphism for which the restriction $\Psi \restriction_{A_0^{(k)} \cup A_1^{(k)}}$ is equal to the composition of the inclusion $A_0^{(k)\prime} \sqcup A_1^{(k)\prime} \hookrightarrow A_0' \sqcup A_1'$ and the biholomorphism $\Psi_k \colon A_0^{(k)} \cup A_1^{(k)} \xrightarrow{\cong} A_0^{(k)\prime} \sqcup A_1^{(k)\prime}$ for each $k = 1, \dots, m$. Let $\Pi_Z \colon Z \hookrightarrow Y$ and $\Pi_{D_\nu} \colon D_\nu \hookrightarrow Y$ for $\nu = 0, 1$ be the corresponding inclusion mappings.

One may now verify that the natural mapping $Y \to Y_m$, which sends the image $\Pi_Z(p)$ in Y of any point $p \in Z$ to the point $\Pi_{Z_m} \circ \cdots \circ \Pi_{Z_0}(p) \in Y_m$, and which for $\nu = 0, 1$ and $k = 1, \dots, m$, sends the image in Y of any point $q \in D_\nu^{(k)}$ under the composition of the inclusions Π_{D_ν} and $D_\nu^{(k)} \hookrightarrow D_\nu$ to the point $\Pi_{Z_{m+1}} \circ \cdots \circ \Pi_{Z_{k+1}} \circ \Pi_{D_\nu^{(k)}}(q)$ in Y_m, is a well-defined biholomorphism (see Exercise 5.12.6). Thus we may identify Y_m with Y.

Similarly, one may consider finite sequential holomorphic *attachment* of tubes, and verify that the resulting complex 1-manifold is naturally biholomorphic to the complex 1-manifold obtained by simultaneous holomorphic attachment of the tubes (see Exercise 5.12.7).

Exercises for Sect. 5.12

5.12.1 In the notation of Sect. 5.12.1, verify that the natural inclusion of $Z \sqcup T^*$ into $Z \sqcup T$ descends to a biholomorphism of X^* onto X.

5.12.2 Let X be a complex 1-manifold obtained by holomorphic attachment of tubes to a complex 1-manifold Y as in Sect. 5.12.1. Prove that X is connected if Y is connected. Give an example that shows that X may be connected even if Y is not (cf. Exercise 2.3.2).

5.12.3 In the notation of Sect. 5.12.2, verify that the natural inclusion of $D_0^* \sqcup D_1^* \sqcup Z$ into $D_0 \sqcup D_1 \sqcup Z$ descends to a biholomorphism of Y^* onto Y.

5.12.4 Verify that the mapping $\widetilde{Y} \to \widehat{X}$ in Sect. 5.12.3 is a well-defined biholomorphism.

5.12.5 Verify that *holomorphic tube removal undoes tube attachment*. That is, in analogy with Sect. 5.12.3, verify that if one holomorphically attaches tubes to a complex 1-manifold Y as in Sect. 5.12.1, and then holomorphically removes some of the tubes as in Sect. 5.12.2, then the resulting complex 1-manifold is biholomorphic to the complex 1-manifold obtained by holomorphic attachment of the remaining tubes to Y.

5.12.6 Verify that the mapping $Y \to Y_m$ in Sect. 5.12.4 is a well-defined biholomorphism.

5.12.7 In analogy with the description of finite sequential holomorphic removal of tubes in Sect. 5.12.4, provide details for the process of finite sequential holomorphic *attachment* of tubes to a complex 1-manifold. Include a verification that the resulting complex 1-manifold is (naturally) biholomorphic to the complex 1-manifold obtained by simultaneous holomorphic attachment of the tubes.

5.12.8 Describe, with proofs, $\mathcal{C}^\infty$ and $\mathcal{C}^0$ versions of attachment, cap attachment, tube attachment, and tube removal in which the biholomorphisms are replaced with diffeomorphisms in second countable 2-dimensional $\mathcal{C}^\infty$ manifolds and homeomorphisms in second countable 2-dimensional topological manifolds, respectively.

5.12.9 Let $T \cong \Delta(0; 1/R_0, R_1)$ be a holomorphic coordinate annulus (i.e., a tube) in a Riemann surface X as in Sect. 5.12.2 (with $R_0, R_1 > 1$). Assume that the complex 1-manifold obtained by holomorphic removal of T is connected, but *not* simply connected. Prove that there is an *infinite* covering Riemann surface $\Upsilon\colon \widehat{X} \to X$ (cf. Exercise 5.9.4) such that
   (i) T is evenly covered by Υ; and
   (ii) For any connected component T_0 of $\Upsilon^{-1}(T)$, we have $\mathrm{Deck}(\Upsilon) \cdot T_0 = \Upsilon^{-1}(T)$.

5.13 Tubes in a Compact Riemann Surface

Recall that we are working toward a proof that any Riemann surface may be obtained by holomorphic attachment of tubes to a planar domain. We consider the compact case in this section.

Theorem 5.13.1 *Up to biholomorphism, every compact Riemann surface may be obtained by holomorphic attachment of tubes at finitely many disjoint coordinate*

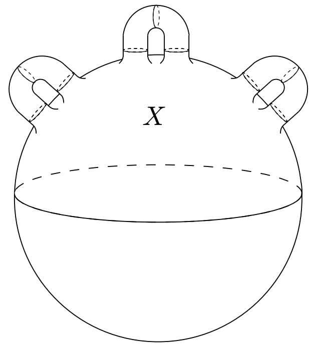

Fig. 5.12 Representation of a compact Riemann surface via holomorphic attachment of finitely many tubes to the Riemann sphere

disks in the Riemann sphere $\mathbb{P}^1$. More precisely, suppose X is a compact Riemann surface and $K \subset X$ is a union of finitely many disjoint compact sets, each of which is a closed disk in some local holomorphic chart (i.e., some local holomorphic chart maps the set onto a closed disk in $\mathbb{C}$). Then, in the notation of Sect. 5.12.1, X is biholomorphic to the compact Riemann surface $Z \cup_\Psi T$ obtained by holomorphic attachment of tubes $\{T_j\}_{j\in J}$ (with $T = \bigsqcup_{j\in J} T_j$) to the Riemann surface $Y = \mathbb{P}^1$ at elements of a finite family of disjoint coordinate disks $\{D_{\nu j}\}_{\nu\in\{0,1\},j\in J}$ (see Fig. 5.12). Moreover, the biholomorphism may be chosen so that the image of K is disjoint from the image of T in $Z \cup_\Psi T$.

According to Sect. 5.12.3, the following theorem, which is in a form that is more convenient for the proof, is equivalent to Theorem 5.13.1:

Theorem 5.13.2 *For every compact Riemann surface, one may obtain a Riemann surface that is biholomorphic to $\mathbb{P}^1$ by holomorphic removal of finitely many disjoint tubes. More precisely, suppose X is a compact Riemann surface and $K \subset X$ is a union of finitely many disjoint compact sets, each of which is a closed disk in some local holomorphic chart. Then, in the notation of Sect. 5.12.2, one may apply (finite) holomorphic tube removal to get a compact Riemann surface $Y \equiv (D_0 \sqcup D_1) \cup_\Psi Z \cong \mathbb{P}^1$, and one may choose the tubes to be disjoint from K.*

Lemma 5.13.3 *Let X be a compact Riemann surface, and let $K \subset X$ be a union of finitely many disjoint compact sets, each of which is a closed disk in some local holomorphic chart. If $H_1(X, \mathbb{R}) \neq 0$, then there exists a local holomorphic chart $(A, \Phi, \Delta(0; 1/R^*, R^*))$ in $X \setminus K$ with $R^* > 1$ and a closed $\mathcal{C}^\infty$ 1-form θ on X such that the smooth Jordan curve $\gamma : t \mapsto \Phi^{-1}(e^{2\pi i t})$ satisfies $\int_\gamma \theta = 1$. In particular, the image $C \equiv \gamma([0, 1])$ is nonseparating in X.*

Proof By hypothesis, there exists a nonzero homology class in $H_1(X, \mathbb{R})$, and since every class contains a linear combination of loops based at a point in $X \setminus K$ (Proposition 10.6.7), we may assume that this class is represented by a loop in X with base point in $X \setminus K$. By Lemma 5.11.4, we may also assume that this representing loop lies in $X \setminus K$. According to Proposition 5.10.7, this loop is homologous in $X \setminus K$, and therefore, in X, to a sum of smooth Jordan loops. It follows that there exist a smooth Jordan curve $\beta\colon [0, 1] \to X \setminus K$ and a closed $\mathcal{C}^\infty$ real 1-form τ on X with $\int_\beta \tau \neq 0$. Lemma 5.11.1 provides a diffeomorphism of a relatively compact neighborhood U of the smooth Jordan curve $B \equiv \beta([0, 1])$ in $X \setminus K$ onto an annulus that maps B onto a concentric circle. By Proposition 5.4.1, there exists a smooth relatively compact domain Ω in U such that $B \subset \Omega$, and such that each of the finitely many connected components L of $\partial\Omega$ has a neighborhood that admits a biholomorphism onto an annulus that maps L onto a concentric circle. Moreover, by Lemma 5.11.1, $\Omega \setminus B = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are disjoint smooth relatively compact domains in U, and for each $j = 1, 2$, B is a boundary component of Ω_j and $(\partial\Omega_j) \setminus B$ is a union of boundary components of $\partial\Omega$ (B is separating in U and therefore in $\Omega \Subset U$). By Stokes' theorem, $\int_{(\partial\Omega_1)\cap(\partial\Omega)} \tau = \pm \int_\beta \tau \neq 0$ (the sign depends on the choice of orientation). Hence the integral of τ along some boundary component of Ω (which is a boundary component of Ω_1) is nonzero. Choosing a suitable local holomorphic chart $(A, \Phi, \Delta(0; 1/R^*, R^*))$ in $X \setminus K$ mapping this boundary component onto $\mathbb{S}^1$, letting $\gamma\colon t \mapsto \Phi^{-1}(e^{2\pi i t})$, letting θ be a suitable rescaling of τ, and applying Lemma 5.11.3, we get the claim. □

Remark In the above proof, we used Proposition 5.4.1 in order to get a holomorphic coordinate annulus in X with nonseparating boundary components. In fact, Theorem 5.9.2 implies that a slight shrinking of the neighborhood U is itself biholomorphic to an annulus (U is itself biholomorphic to an annulus, to Δ^*, or to $\mathbb{C}^*$). A reader who has worked through the discussion in Sect. 5.9 of the classification of Riemann surfaces as quotients by groups of automorphisms may prefer to apply Theorem 5.9.2 in the above proof in place of Proposition 5.4.1 (see Exercise 5.13.1).

Proof of Theorem 5.13.2 We proceed by induction on $d \equiv \dim_{\mathbb{R}} H_1(X, \mathbb{R})$, which, according to Proposition 10.6.7, is finite, since X is compact. The case $d = 0$ follows from the Koebe uniformization theorem (Theorem 5.5.3) or just Lemma 5.5.4. Assuming that $d > 0$, Lemma 5.13.3 provides a number $R^* > 1$, a local holomorphic chart $(A, \Phi, \Delta(0; 1/R^*, R^*))$ in $X \setminus K$, and a closed $\mathcal{C}^\infty$ 1-form θ on X such that the smooth Jordan curve $\gamma\colon t \mapsto \Phi^{-1}(e^{2\pi i t})$ satisfies $\int_\gamma \theta = 1$, and (hence) the image $C \equiv \gamma([0, 1])$ is nonseparating in X.

Fixing a coordinate annulus $T = \Phi^{-1}(\Delta(0; 1/R, R)) \Subset A$ for some $R \in (1, R^*)$, holomorphic removal of the tube T as in Sect. 5.12.2 yields a compact Riemann surface Y. More precisely, setting $A'_\nu \equiv \Delta(0; 1, R) \subset D_\nu \equiv \Delta(0; R)$ for $\nu = 0, 1$, $A_0 \equiv \Phi^{-1}(\Delta(0; 1/R, 1)) \subset T$, and $A_1 \equiv \Phi^{-1}(\Delta(0; 1, R)) \subset T$, we get a biholomorphism $\Psi\colon A_0 \cup A_1 \to A'_0 \sqcup A'_1$, where for each index $\nu \in \{0, 1\}$ and each point $p \in A_\nu$, $\Psi(p)$ is the image in $A'_0 \sqcup A'_1$ of the point

$$\begin{cases} 1/\Phi(p) \in A'_0 & \text{for } \nu = 0, \\ \Phi(p) \in A'_1 & \text{for } \nu = 1, \end{cases}$$

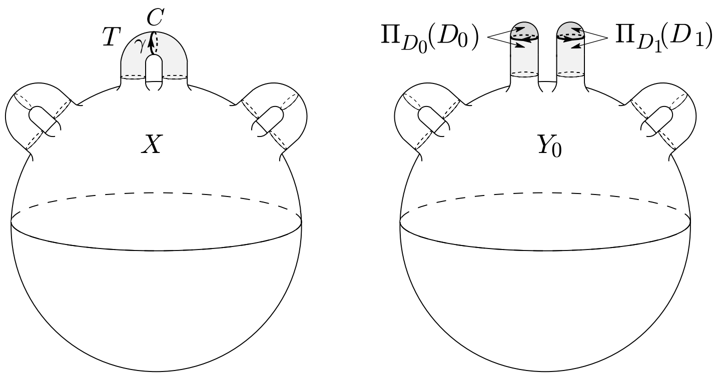

Fig. 5.13 Reduction of the dimension of the first homology via holomorphic removal of a tube

under the inclusion $A'_\nu \hookrightarrow A'_0 \sqcup A'_1 \subset D_0 \sqcup D_1$. The set $Z \equiv X \setminus C$ is connected, since C is nonseparating. Thus we get the compact Riemann surface

$$Y_0 \equiv (D_0 \sqcup D_1) \cup_\Psi Z = (D_0 \sqcup D_1) \sqcup Z/\sim,$$

where for $\nu \in \{0, 1\}$, $z \in A'_\nu$, and $p \in A_\nu$, the images z_0 of z and p_0 of p in $D_0 \sqcup D_1 \sqcup Z$ under the inclusions $A'_\nu, A_\nu \hookrightarrow D_0 \sqcup D_1 \sqcup Z$ satisfy $z_0 \sim p_0$ if and only if $z \cdot \Phi(p) = 1$ for $\nu = 0$ and $z = \Phi(p)$ for $\nu = 1$. In other words, Y_0 is a compact Riemann surface obtained by removing the unit circle C in the tube (i.e., annulus) T and gluing in caps (i.e., disks) D_0 and D_1 on each end of the remaining two pieces (see Fig. 5.13). That is, Y_0 is obtained by attaching caps to the open subset Z of X. Letting $\Pi_Z : Z \hookrightarrow Y_0$ and $\Pi_{D_\nu} : D_\nu \hookrightarrow Y_0$ for $\nu = 0, 1$ be the corresponding inclusion mappings, we get

$$\Pi_Z(T \setminus C) = \Pi_Z(A_0 \cup A_1) = \Pi_{D_0}(A'_0) \cup \Pi_{D_1}(A'_1)$$

and

$$\Pi_Z(K) \subset \Pi_Z(X \setminus \overline{T}) = Y_0 \setminus \left(\overline{\Pi_{D_0}(D_0)} \cup \overline{\Pi_{D_1}(D_1)}\right).$$

Setting $D^*_\nu \equiv \Delta(0; R^*) \supset \overline{D}_\nu$ for $\nu = 0, 1$, we also see that the inclusion $D_0 \sqcup D_1 \hookrightarrow Y_0$ extends to a biholomorphism of $D^*_0 \sqcup D^*_1$ onto a neighborhood of $\overline{\Pi_{D_0}(D_0)} \cup \overline{\Pi_{D_1}(D_1)}$ (see the remarks at the end of Sect. 5.12.2).

It now suffices to show that $\dim H_1(Y_0, \mathbb{R}) < d$. For if this is the case, then, by the induction hypothesis, one may obtain a Riemann surface $Y \cong \mathbb{P}^1$ from Y_0 by holomorphic removal of finitely many disjoint tubes that do not meet the image of K as well as the images of the caps D_0 and D_1. Thus, by simultaneously removing these tubes along with the tube removed from X above, and applying Sect. 5.12.4, we get a Riemann surface that is biholomorphic to $Y \cong \mathbb{P}^1$.

In order to show that $\dim H_1(Y_0, \mathbb{R}) < d = \dim H_1(X, \mathbb{R})$, it suffices to produce an injective linear map $H_1(Y_0, \mathbb{R}) \to H_1(X, \mathbb{R})\big/\mathcal{V}$, where $\mathcal{V} \equiv \mathbb{R} \cdot [\gamma]_{H_1}$ is the (1-dimensional) subspace spanned by $[\gamma]_{H_1}$. Let $\rho : H_1(X, \mathbb{R}) \to H_1(X, \mathbb{R})/\mathcal{V}$

be the quotient map. Given an element $[\xi]_{H_1} \in H_1(Y_0, \mathbb{R})$, Lemma 5.11.4 implies that we may choose the representative ξ to be a real linear combination of loops in

$$Y_0 \setminus \left(\overline{\Pi_{D_0}(D_0)} \cup \overline{\Pi_{D_1}(D_1)}\right) = \Pi_Z(X \setminus \overline{T}),$$

and hence we may form the mapping $[\xi]_{H_1(Y_0,\mathbb{R})} \mapsto \rho([(\Pi_Z^{-1})_*\xi]_{H_1(X,\mathbb{R})})$.

To see that this mapping is well defined, linear, and injective, let us consider a 1-cycle ξ that is a linear combination of loops in $\Pi_Z(X \setminus \overline{T})$, and let us set

$$\xi' \equiv (\Pi_Z^{-1})_*\xi = (\Pi_Z)_*^{-1}\xi \quad \text{and} \quad t \equiv \int_{\xi'} \theta \in \mathbb{R}.$$

Given a closed $\mathcal{C}^\infty$ 1-form η on X, for $s \equiv \int_\gamma \eta$, we have $\int_\gamma (\eta - s\theta) = 0$, and hence, since the path homotopy class of γ generates the fundamental group of $A \supset \overline{T}$, $(\eta - s\theta)\restriction_A$ is exact. Cutting off a potential, we get a real-valued $\mathcal{C}^\infty$ function λ with compact support in A such that $\eta - s\theta = d\lambda$ on a neighborhood of $\overline{T}$. Thus we get a well-defined closed $\mathcal{C}^\infty$ 1-form τ on Y_0 by setting $\tau = (\Pi_Z^{-1})^*(\eta - s\theta - d\lambda)$ at each point in $\Pi_Z(X \setminus \overline{T})$ and $\tau = 0$ at each point in $\overline{\Pi_{D_0}(D_0)} \cup \overline{\Pi_{D_1}(D_1)}$. We then have

$$\int_\xi \tau = \int_{\xi'} \eta - s\int_{\xi'} \theta = \int_{\xi'} \eta - st = \int_{\xi'} \eta - t\int_\gamma \eta = \int_{\xi' - t\gamma} \eta.$$

In particular, if $[\xi]_{H_1(Y_0,\mathbb{R})} = 0$, then $[\xi']_{H_1(X,\mathbb{R})} = t[\gamma]_{H_1(X,\mathbb{R})} \in \mathcal{V}$, and it follows that the mapping is well defined. Linearity is easy to verify. For injectivity, observe that if, for ξ as above, we have $[\xi']_{H_1(X,\mathbb{R})} = u[\gamma]_{H_1(X,\mathbb{R})} \in \mathcal{V}$ for some $u \in \mathbb{R}$, then we must have $u = t$. Given a closed $\mathcal{C}^\infty$ 1-form τ on Y_0, τ is exact on a neighborhood of $\overline{\Pi_{D_0}(D_0)} \cup \overline{\Pi_{D_1}(D_1)}$, and hence for some real-valued $\mathcal{C}^\infty$ function λ_0 with compact support in some small neighborhood of $\overline{\Pi_{D_0}(D_0)} \cup \overline{\Pi_{D_1}(D_1)}$, we have $\tau + d\lambda_0 \equiv 0$ on some (smaller) neighborhood of this set. Thus we may define a closed $\mathcal{C}^\infty$ 1-form η_0 on X by setting $\eta_0 = \Pi_Z^*(\tau + d\lambda_0)$ at each point in $X \setminus \overline{T}$, and $\eta_0 = 0$ at each point in $\overline{T}$. We then have

$$\int_\xi \tau = \int_\xi (\tau + d\lambda_0) = \int_{\xi'} \eta_0 = t\int_\gamma \eta_0 = 0$$

(since γ lies in T), and hence $[\xi]_{H_1(Y_0,\mathbb{R})} = 0$. □

Remarks 1. The above proof does not really use the full Koebe uniformization theorem (or the Riemann mapping theorem), just the more elementary Proposition 5.4.1 and Lemma 5.5.4. The proof of the general case in Sect. 5.14 will use the full Koebe uniformization theorem (but not the Riemann mapping theorem).

2. It turns out that the injective linear map $H_1(Y_0, \mathbb{R}) \to H_1(X, \mathbb{R})/\mathbb{R} \cdot [\gamma]_{H_1}$ in the above proof is *never* surjective, because $\dim H_1(X, \mathbb{R})$ is *even* for any compact Riemann surface X. This is a consequence of the existence of a canonical homology basis (see Sect. 5.16), as well as of the Hodge decomposition theorem (see Sect. 4.9).

Exercises for Sect. 5.13

5.13.1 Give a slightly different (and shorter) proof of Lemma 5.13.3 by applying Theorem 5.9.2 in order to show that U may be chosen to be biholomorphic to an annulus (see the remark following the proof of Lemma 5.13.3).

5.13.2 Prove that every compact oriented $\mathcal{C}^\infty$ surface M may be obtained by $\mathcal{C}^\infty$ oriented attachment (i.e., the local charts are positively oriented and the attaching maps are orientation-preserving diffeomorphisms) of a finite family of disjoint tubes (see Exercise 5.12.8) to a compact oriented $\mathcal{C}^\infty$ surface N with $H_1(N, \mathbb{R}) = 0$ (and hence $H_1(N, \mathbb{A}) = 0$ for any subring $\mathbb{A}$ of $\mathbb{C}$ containing $\mathbb{Z}$). Using this fact together with Lemma 5.5.4 and Theorem 5.9.2, give a slightly different proof of Theorem 5.13.1.

Note. It will follow from the results of Chap. 6 that in fact, N is diffeomorphic to a sphere.

5.14 Tubes in an Arbitrary Riemann Surface

The goal of this section is the following general tube-attachment theorem (see [Ri] for a more complete classification of noncompact second countable topological surfaces):

Theorem 5.14.1 *Up to biholomorphism, every Riemann surface may be obtained by holomorphic attachment of tubes at elements of a locally finite family of disjoint coordinate disks in a domain in $\mathbb{P}^1$. More precisely, suppose X is a Riemann surface, and $K \subset X$ is a union of a locally finite family of disjoint compact sets, each of which is a closed disk in some local holomorphic chart. Then, in the notation of Sect. 5.12.1, X is biholomorphic to the Riemann surface $Z \cup_\Psi T$ obtained by holomorphic attachment of tubes $\{T_j\}_{j\in J}$ (with $T = \bigsqcup_{j\in J} T_j$) to a domain Y in $\mathbb{P}^1$ at elements of a locally finite family of disjoint coordinate disks $\{D_{\nu j}\}_{\nu\in\{0,1\}, j\in J}$. Moreover, the biholomorphism may be chosen so that the image of K is disjoint from the image of T in $Z \cup_\Psi T$.*

As in the compact case, it is more convenient to prove the following version, which, according to Sect. 5.12.3, is equivalent to Theorem 5.14.1:

Theorem 5.14.2 *For every Riemann surface X, one may obtain a planar Riemann surface by holomorphic removal of a locally finite family of disjoint tubes. More precisely, suppose X is a Riemann surface, and $K \subset X$ is a union of a locally finite family of disjoint compact sets, each of which is a closed disk in some local holomorphic chart. Then, in the notation of Sect. 5.12.2, one may apply holomorphic tube removal to get a Riemann surface $Y \equiv (D_0 \sqcup D_1) \cup_\Psi Z$ that is biholomorphic to a domain in $\mathbb{P}^1$. Moreover, the tubes may be chosen to be disjoint from K.*

The idea of the proof is as follows. We first form an exhaustion of X by relatively compact smooth domains $\{\Omega_\nu\}_{\nu=1}^\infty$ such that each boundary component is a concentric circle in a coordinate annulus (i.e., in a tube). For each connected component Γ of $\Omega_{\nu+1} \setminus \overline{\Omega}_\nu$, by holomorphically removing the tubes about all but one of the connected components of $\partial\Gamma \cap \partial\Omega_\nu$, we see that we may assume that $\partial\Gamma \cap \partial\Omega_\nu$ is connected (see Lemma 5.14.4 and Fig. 5.14). Holomorphic removal of the tubes about the boundary components of the domains $\{\Omega_\nu\}$ then yields a disjoint union of compact Riemann surfaces (see Fig. 5.16). By Theorem 5.13.2, we may then get a disjoint union of copies of the Riemann sphere by holomorphic removal of tubes in each of these components (see Fig. 5.17). Finally, by holomorphically reattaching the boundary tubes about the boundary components of the domains $\{\Omega_\nu\}$, we get a planar Riemann surface (see Fig. 5.18).

Lemma 5.14.3 *Suppose X is an open Riemann surface, and $K \subset X$ is a union of a locally finite family of disjoint compact sets, each of which is a closed disk in some local holomorphic chart. Then there exists a sequence of smooth domains $\{\Omega_\nu\}_{\nu=1}^{\infty}$ in X such that*

(i) $\Omega_\nu \Subset \Omega_{\nu+1}$ *for each* $\nu \in \mathbb{Z}_{>0}$ *and* $X = \bigcup_{\nu=1}^{\infty} \Omega_\nu$; *and*
(ii) *For each* $\nu \geq 1$ *and for each connected component* C *of* $\partial\Omega_\nu$, *there is a local holomorphic chart* $(A, \Phi, \Delta(0; 1/R, R))$ *for some* $R = R(C) > 0$ *with* $C \subset A \subset X \setminus K$, $\Phi(A \cap \Omega_\nu) = \Delta(0; 1, R)$, *and* $C = \Phi^{-1}(\partial\Delta(0; 1))$.

Proof We may write $K = \bigcup_{j=1}^{\infty} K_j$ for disjoint connected compact sets $\{K_j\}$, each of which is either the empty set or a closed disk in some local holomorphic chart. We may also choose compact sets $\{L_\nu\}_{\nu=1}^{\infty}$ with $\bigcup_\nu L_\nu = X$ and $L_\nu \subset L_{\nu+1}$ for each ν, and positive integers $\{j_\nu\}_{\nu=1}^{\infty}$ such that for each ν, $j_\nu < j_{\nu+1}$ and $L_\nu \cap K_j = \emptyset$ for all $j \geq j_\nu$. By Proposition 5.4.1, we may choose a relatively compact smooth domain Ω_1 in the domain $X \setminus \bigcup_{j=j_1}^{\infty} K_j$ (this set is connected by Lemma 5.11.4) such that $L_1 \cup K_1 \cup \cdots \cup K_{j_1-1} \subset \Omega_1$ and Ω_1 has the property (ii). Given a domain $\Omega_{\nu-1} \Subset X \setminus \bigcup_{j=j_{\nu-1}}^{\infty} K_j$, we may choose a relatively compact smooth domain Ω_ν in the domain $X \setminus \bigcup_{j=j_\nu}^{\infty} K_j$ such that $\overline{\Omega}_{\nu-1} \cup L_\nu \cup K_1 \cup \cdots \cup K_{j_\nu-1} \subset \Omega_\nu$ and Ω_ν has the property (ii). Thus, by induction, we get the desired sequence of domains $\{\Omega_\nu\}$. $\square$

Lemma 5.14.4 *Suppose X is an open Riemann surface, and $K \subset X$ is a union of a locally finite family of disjoint compact sets, each of which is a closed disk in some local holomorphic chart. Then, in the notation of Sect. 5.12.2, one may apply holomorphic tube removal to get a Riemann surface $Y \equiv (D_0 \sqcup D_1) \cup_\Psi Z$ with holomorphic inclusion $\Pi_Z \colon Z \hookrightarrow Y$ and a sequence of smooth domains $\{\Omega_\nu\}_{\nu=1}^{\infty}$ in Y such that setting $\Omega_0 = \emptyset$, we have the following (see Fig. 5.14):*

(i) *The closures of the removed tubes are disjoint from* K (*in particular,* $K \subset Z$);
(ii) $\Omega_\nu \Subset \Omega_{\nu+1}$ *for each* $\nu \geq 1$ *and* $Y = \bigcup_{\nu=1}^{\infty} \Omega_\nu$;
(iii) *For each* $\nu \geq 1$ *and for each connected component* C *of* $\partial\Omega_\nu$, *there is a local holomorphic chart* $(A, \Phi, \Delta(0; 1/R, R))$ *for some* $R = R(C) > 0$ *with* $C \subset A \Subset \Pi_Z(Z \setminus K)$, $\Phi(A \cap \Omega_\nu) = \Delta(0; 1, R)$, *and* $C = \Phi^{-1}(\partial\Delta(0; 1))$;
(iv) *For each* $\nu \geq 0$, *the* (*finitely many*) *connected components of* $\Omega_{\nu+1} \setminus \overline{\Omega}_\nu$ *lie in distinct connected components of* $Y \setminus \overline{\Omega}_\nu$, *and for* $\nu \geq 1$, *if* Γ *is a connected*

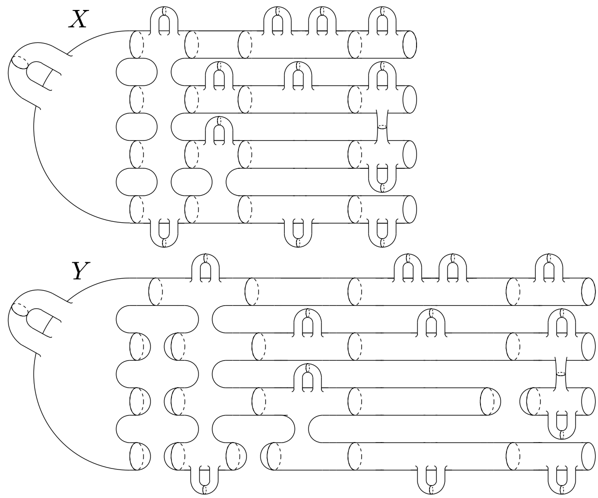

Fig. 5.14 Holomorphic removal of certain boundary tubes for an exhaustion yielding an exhaustion with separating boundary components

component of $\Omega_{\nu+1} \setminus \overline{\Omega}_\nu$ *and* V *is the connected component of* $Y \setminus \overline{\Omega}_\nu$ *containing* Γ, *then one of the boundary components of* Γ *is a boundary component* C *of* Ω_ν *satisfying* $\partial V = \partial\Gamma \cap \partial\Omega_\nu = C$; *and*

(v) *For each* $\nu \geq 1$, *each boundary component* C *of* $\partial\Omega_\nu$ *is separating in* Y.

Proof By Lemma 5.14.3, there exist a family of smooth domains $\{\Theta_\nu\}_{\nu=1}^\infty$, disjoint local holomorphic charts

$$\left\{\left(A_j^{(\nu)}, \Phi_j^{(\nu)}, \Delta(0; 1/R_\nu^*, R_\nu^*)\right)\right\}_{\nu\in\mathbb{Z}_{>0}, j\in\{1,\dots,m_\nu\}}$$

with $m_\nu \in \mathbb{Z}_{>0}$ and $R_\nu^* > 1$ for each ν, disjoint smooth Jordan curves

$$\left\{C_j^{(\nu)}\right\}_{\nu\in\mathbb{Z}_{>0}, j\in\{1,\dots,m_\nu\}},$$

and disjoint domains $\{\Gamma_j^{(\nu)}\}_{\nu\in\mathbb{Z}_{\geq 0},\, j\in\{1,\dots,k_\nu\}}$ with $k_\nu \in \{1,\dots,m_\nu\}$ (setting $m_0 = 1$) for each $\nu \geq 0$ such that setting $\Theta_0 \equiv \emptyset$, we have

(i) $\Theta_\nu \Subset \Theta_{\nu+1}$ for each $\nu \geq 0$ and $X = \bigcup_{\nu=1}^\infty \Theta_\nu$;

(ii) For each $\nu \geq 1$, $\partial\Theta_\nu \subset \bigcup_{j=1}^{m_\nu} A_j^{(\nu)} \Subset X \setminus K$, and for each $j = 1,\dots,m_\nu$, $\Phi_j^{(\nu)}(A_j^{(\nu)} \cap \Theta_\nu) = \Delta(0; 1, R_\nu^*)$ and $C_j^{(\nu)} = (\Phi_j^{(\nu)})^{-1}(\partial\Delta(0;1))$ (in particular, $\partial\Theta_\nu = \bigcup_{j=1}^{m_\nu} C_j^{(\nu)}$);

(iii) For each $\nu \geq 0$, $\Gamma_1^{(\nu)},\dots,\Gamma_{k_\nu}^{(\nu)}$ are the connected components of $\Theta_{\nu+1} \setminus \overline{\Theta}_\nu$ (in particular, $k_0 = 1$ and $\Gamma_1^{(0)} = \Theta_1$); and

(iv) For each $\nu \geq 1$ and each $j = 1,\dots,k_\nu$, $C_j^{(\nu)}$ is a connected component of $\partial\Gamma_j^{(\nu)}$ (thus we have chosen *exactly one* of the common connected components of

$\partial\Theta_\nu$ and $\Gamma_j^{(\nu)}$ for each $j = 1, \dots, k_\nu$, and holomorphic attachment of caps at the remaining boundary curves $\{C_j^{(\nu)}\}_{j=k_\nu+1}^{m_\nu}$ will yield a Riemann surface in which the images of the boundary curves $\{C_j^{(\nu)}\}_{j=1}^{k_\nu}$ will be separating).

In particular, we have $A_j^{(\nu)} \Subset \Theta_{\nu+1} \setminus \overline{\Theta}_{\nu-1}$ for each $\nu \geq 1$ and each $j = 1, \dots, m_\nu$, so the family $\{A_j^{(\nu)}\}$ is locally finite in X. Finally, choosing constants $\{R_\nu\}$ with $1 < R_\nu < R_\nu^*$ for each $\nu \geq 1$, we may set

$$T_j^{(\nu)} \equiv (\Phi_j^{(\nu)})^{-1}\big(\Delta(0; 1/R_\nu, R_\nu)\big) \Subset A_j^{(\nu)} \quad \text{and} \quad D_{0j}^{(\nu)} = D_{1j}^{(\nu)} \equiv \Delta(0; R_\nu)$$

for each $\nu \geq 1$ and $j = 1, \dots, m_\nu$,

$$C \equiv \bigcup_{\nu=1}^{\infty} \bigcup_{j=k_\nu+1}^{m_\nu} C_j^{(\nu)} \subset T \equiv \bigcup_{\nu=1}^{\infty} \bigcup_{j=k_\nu+1}^{m_\nu} T_j^{(\nu)},$$

$Z \equiv X \setminus C \supset X \setminus \overline{T} \supset K$ (i.e., the condition (i) in the statement of the lemma holds), and

$$D_\mu \equiv \bigsqcup_{\nu\geq 1, k_\nu < j \leq m_\nu} D_{\mu j}^{(\nu)} \quad \text{for } \mu = 0, 1.$$

Holomorphic removal of the tubes $\{T_j^{(\nu)}\}_{\nu \geq 1,\, k_\nu < j \leq m_\nu}$ as in Sect. 5.12.2 yields a complex 1-manifold $Y = (D_0 \sqcup D_1) \cup_\Psi Z$ (for the appropriate biholomorphism Ψ). Denoting the associated inclusions by $\Pi_Z \colon Z \hookrightarrow Y$ and $\Pi_{D_\mu} \colon D_\mu \hookrightarrow Y$ for $\mu = 0, 1$, we get $\Pi_Z(X \setminus \overline{T}) = Y \setminus (\overline{\Pi_{D_0}(D_0)} \cup \overline{\Pi_{D_1}(D_1)})$. Setting $D_{\mu j}^{(\nu)*} \equiv \Delta(0; R_\nu^*) \supset \overline{D_{\mu j}^{(\nu)}}$ for each choice of ν, j, and μ, and setting

$$D_\mu^* \equiv \bigsqcup_{\nu\geq 1, k_\nu < j \leq m_\nu} D_{\mu j}^{(\nu)*} \quad \text{for } \mu = 0, 1,$$

we also see that the inclusion $D_0 \sqcup D_1 \hookrightarrow Y$ extends to a biholomorphism of $D_0^* \sqcup D_1^*$ onto a neighborhood of $\overline{\Pi_{D_0}(D_0)} \cup \overline{\Pi_{D_1}(D_1)}$ (see the remarks at the end of Sect. 5.12.2).

Now $Z \cap \Theta_1 = \Theta_1 = \Gamma_1^{(0)}$ is connected. Assuming that $Z \cap \Theta_\nu$ is connected for some $\nu \geq 1$, we see that $Z \cap \Theta_{\nu+1}$ is equal to the union of the connected set $Z \cap \Theta_\nu$ with the connected open sets

$$\Gamma_1^{(\nu)} \cup T_1^{(\nu)}, \dots, \Gamma_{k_\nu}^{(\nu)} \cup T_{k_\nu}^{(\nu)},$$

each of which meets $Z \cap \Theta_\nu$. Thus it follows by induction that $Z \cap \Theta_\nu$ is connected for each ν, and hence that $Z = \bigcup_{\nu=1}^{\infty}(Z \cap \Theta_\nu)$ is connected. Since $Y \setminus \Pi_Z(Z)$ is covered by a union of a locally finite family of (disjoint) domains each of which meets $\Pi_Z(Z)$ (and is biholomorphic to a disk), it follows that Y is connected.

Next we observe that if $0 \leq \nu < \mu$, then $Z \cap \Theta_\mu \setminus \overline{\Theta}_\nu$ has exactly k_ν distinct connected components, each of which contains exactly one of the sets $\Gamma_1^{(\nu)}, \dots, \Gamma_{k_\nu}^{(\nu)}$. For the proof of this fact, we proceed by induction on μ, the case $\mu = \nu + 1$ being trivial. Assuming that the claim holds for $Z \cap \Theta_\mu \setminus \overline{\Theta}_\nu$, each of the disjoint connected sets $\Gamma_1^{(\mu)} \cup T_1^{(\mu)}, \dots, \Gamma_{k_\mu}^{(\mu)} \cup T_{k_\mu}^{(\mu)}$ must meet exactly one of the k_ν connected components of $Z \cap \Theta_\mu \setminus \overline{\Theta}_\nu$, because for each $j = 1, \dots, k_\mu$, $T_j^{(\mu)} \cap \Theta_\mu$ meets, and therefore lies in, exactly one such connected component. Thus the claim holds for $\mu + 1$, and therefore for all $\mu > \nu$. It also follows that for each $\nu \geq 0$, the set $Z \setminus \overline{\Theta}_\nu$ has exactly k_ν distinct connected components; and we may denote these connected components by $\{V_j^{(\nu)}\}_{j=1}^{k_\nu}$, where $\Gamma_j^{(\nu)} \subset V_j^{(\nu)}$ for each $j = 1, \dots, k_\nu$. Furthermore, for each $\nu \geq 1$ and each $j = 1, \dots, k_\nu$, $C_j^{(\nu)}$ is a separating curve in Z. In fact, $Z \setminus C_j^{(\nu)}$ has the two connected components

$$V_j^{(\nu)} \quad \text{and} \quad (Z \cap \Theta_\nu) \cup \bigcup_{1 \leq i \leq k_\nu, i \neq j} (V_i^{(\nu)} \cup T_i^{(\nu)}).$$

We now let $\Omega_0 = \emptyset$, and for each $\nu \geq 1$, we let Ω_ν be the union of $\Pi_Z(Z \cap \Theta_\nu)$ with those (finitely many) connected components of $\Pi_{D_0}(D_0) \cup \Pi_{D_1}(D_1)$ that meet this set. For each $\nu \geq 1$, Ω_ν is then a relatively compact domain in Y with distinct boundary components $\Pi_Z(C_1^{(\nu)}), \dots, \Pi_Z(C_{k_\nu}^{(\nu)})$, and since Π_Z maps Z biholomorphically onto $\Pi_Z(Z)$, Ω_ν satisfies the conditions (ii) and (iii) in the statement of the lemma. Furthermore, by construction, for each $\nu \geq 0$, $\Omega_{\nu+1} \setminus \overline{\Omega}_\nu$ is the union of the disjoint connected open sets $\{\Xi_j^{(\nu)}\}_{j=1}^{k_\nu}$, where for each ν and j, $\Xi_j^{(\nu)}$ is the union of $\Pi_Z(\Gamma_j^{(\nu)})$ with those connected components of $\Pi_{D_0}(D_0) \cup \Pi_{D_1}(D_1)$ that meet $\Pi_Z(\Gamma_j^{(\nu)})$, and $Y \setminus \overline{\Omega}_\nu$ is the union of disjoint connected open sets $\{W_j^{(\nu)}\}_{j=1}^{k_\nu}$, where for each ν and j, $W_j^{(\nu)} \supset \Xi_j^{(\nu)}$ is the union of $\Pi_Z(V_j^{(\nu)})$ with those connected components of $\Pi_{D_0}(D_0) \cup \Pi_{D_1}(D_1)$ that meet $\Pi_Z(V_j^{(\nu)})$. Finally, for each $\nu \geq 1$ and each $j \in \{1, \dots, k_\nu\}$, the boundary component $\Pi_Z(C_j^{(\nu)}) = \partial \Xi_j^{(\nu)} \cap \partial \Omega_\nu = \partial W_j^{(\nu)}$ of Ω_ν (and of $\Xi_j^{(\nu)}$) is separating in Y, since $Y \setminus \Pi_Z(C_j^{(\nu)})$ has (only) the two connected components

$$W_j^{(\nu)} \quad \text{and} \quad \Omega_\nu \cup \bigcup_{1 \leq i \leq k_\nu, i \neq j} \left(W_i^{(\nu)} \cup \Pi_Z(T_i^{(\nu)})\right).$$

□

Proof of Theorem 5.14.2 The compact case is Theorem 5.13.2, so we may assume that X is noncompact.

Step 1. Choice of a suitable exhaustion by domains. We may holomorphically remove relatively compact tubes with closures disjoint from K to get a Riemann surface X' with the properties considered in Lemma 5.14.4. Moreover, we may assume that the local holomorphic chart on each tube actually extends to a neighborhood of the closure. Thus, if we let K' be the union of the image of K with the closures of the holomorphically attached caps in X', and holomorphic removal of

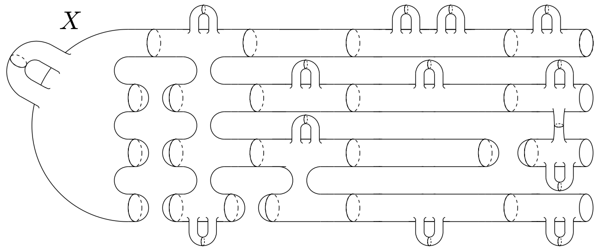

Fig. 5.15 The initial (modified) Riemann surface X

tubes in X' with closures disjoint from K' yields a planar Riemann surface Y, then, as in Sect. 5.12.4, we may identify Y with a Riemann surface obtained by holomorphic removal of tubes in X that are disjoint from K. Therefore, by replacing X with X' and K with K', we may assume without loss of generality that there exist a family of (nonempty) smooth domains $\{\Omega_\nu\}_{\nu=1}^{\infty}$, disjoint local holomorphic charts

$$\left\{\left(A_j^{(\nu)}, \Phi_j^{(\nu)}, \Delta(0; 1/R_\nu^*, R_\nu^*)\right)\right\}_{\nu\in\mathbb{Z}_{>0},\, j\in\{1,\dots,m_\nu\}}$$

with $m_\nu \in \mathbb{Z}_{>0}$ and $R_\nu^* > 1$ for each ν, disjoint smooth Jordan curves

$$\{C_j^{(\nu)}\}_{\nu\in\mathbb{Z}_{>0},\, j\in\{1,\dots,m_\nu\}},$$

disjoint domains $\{\Gamma_j^{(\nu)}\}_{\nu\in\mathbb{Z}_{\geq 0},\, j\in\{1,\dots,m_\nu\}}$ with $m_0 = 1$, and for each $\nu \geq 0$, disjoint domains $\{V_j^{(\nu)}\}_{j\in\{1,\dots,m_\nu\}}$ such that setting $\Omega_0 \equiv \emptyset$, we have (see Fig. 5.15)

(i) $\Omega_\nu \Subset \Omega_{\nu+1}$ for each $\nu \geq 0$ and $X = \bigcup_{\nu=1}^{\infty} \Omega_\nu$;

(ii) For each $\nu \geq 1$, $\partial\Omega_\nu \subset \bigcup_{j=1}^{m_\nu} A_j^{(\nu)} \Subset X \setminus K$, and for each $j = 1, \dots, m_\nu$,

$$\Phi_j^{(\nu)}(A_j^{(\nu)} \cap \Omega_\nu) = \Delta(0; 1, R_\nu^*) \quad \text{and} \quad C_j^{(\nu)} = (\Phi_j^{(\nu)})^{-1}(\partial\Delta(0; 1))$$

(in particular, $\partial\Omega_\nu = \bigcup_{j=1}^{m_\nu} C_j^{(\nu)}$);

(iii) For each $\nu \geq 0$, $\Gamma_1^{(\nu)}, \dots, \Gamma_{m_\nu}^{(\nu)}$ are the distinct connected components of $\Omega_{\nu+1} \setminus \overline{\Omega}_\nu$ (in particular, $\Gamma_1^{(0)} = \Omega_1$), $V_1^{(\nu)}, \dots, V_{m_\nu}^{(\nu)}$ are the distinct connected components of $X \setminus \overline{\Omega}_\nu$, and $\Gamma_j^{(\nu)} \subset V_j^{(\nu)}$ for each $j = 1, \dots, m_\nu$; and

(iv) For each $\nu \geq 1$ and each $j = 1, \dots, m_\nu$, the smooth Jordan curve $C_j^{(\nu)}$ is separating in X and $C_j^{(\nu)} = \partial\Gamma_j^{(\nu)} \cap \partial\Omega_\nu = \partial V_j^{(\nu)}$ is a boundary component of Ω_ν and of $\Gamma_j^{(\nu)}$ (and of $V_j^{(\nu)}$).

In particular, we have $A_j^{(\nu)} \Subset \Omega_{\nu+1} \setminus \overline{\Omega}_{\nu-1}$ for each $\nu \geq 1$ and each $j = 1, \dots, m_\nu$, so the family $\{A_j^{(\nu)}\}$ is locally finite in X. Finally, choosing constants $\{R_\nu\}$ with $1 < R_\nu < R_\nu^*$ for each $\nu \geq 1$, we may set

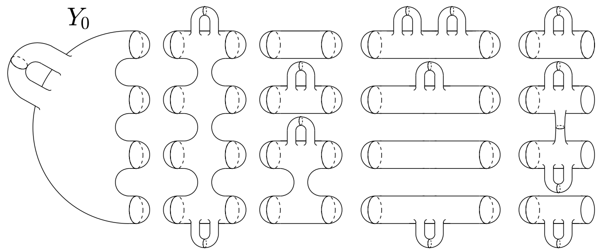

Fig. 5.16 The complex 1-manifold Y_0 obtained by holomorphic removal of the boundary tubes for the exhaustion of X

$$T_j^{(\nu)} \equiv (\Phi_j^{(\nu)})^{-1}(\Delta(0; 1/R_\nu, R_\nu)) \Subset A_j^{(\nu)} \quad \text{and} \quad D_{0j}^{(\nu)} = D_{1j}^{(\nu)} \equiv \Delta(0; R_\nu)$$

for each $\nu \geq 1$ and each $j = 1, \dots, m_\nu$, and

$$C_0 \equiv \bigcup_{\nu,j} C_j^{(\nu)} \subset T_0 \equiv \bigcup_{\nu,j} T_j^{(\nu)} \subset X \setminus K,$$

$$Z_0 \equiv X \setminus C_0 = \bigcup_{\nu,j} \Gamma_j^{(\nu)} \supset K,$$

$$D_\mu \equiv \bigsqcup_{\nu,j} D_{\mu j}^{(\nu)} \quad \text{for } \mu = 0, 1.$$

Step 2. Removal of the boundary tubes, removal of the interior tubes, and reattachment of the boundary tubes. Holomorphic removal of the tubes $\{T_j^{(\nu)}\}$ as in Sect. 5.12.2 yields a second countable complex 1-manifold $Y_0 = (D_0 \sqcup D_1) \cup_{\Psi_0} Z_0$ (for the appropriate biholomorphism Ψ_0). Denoting the associated inclusions by $\Pi_{Z_0} \colon Z_0 \hookrightarrow Y_0$ and $\Pi_{D_\mu} \colon D_\mu \hookrightarrow Y_0$ for $\mu = 0, 1$, we get $\Pi_{Z_0}(X \setminus \overline{T}_0) = Y_0 \setminus (\overline{\Pi_{D_0}(D_0)} \cup \overline{\Pi_{D_1}(D_1)})$. Setting $D_{\mu j}^{(\nu)*} \equiv \Delta(0; R_\nu^*) \supset \overline{D_{\mu j}^{(\nu)}}$ for each choice of ν, j, and μ, and setting

$$D_\mu^* \equiv \bigsqcup_{\nu,j} D_{\mu j}^{(\nu)*} \quad \text{for } \mu = 0, 1,$$

we also see that the inclusion $D_0 \sqcup D_1 \hookrightarrow Y_0$ extends to a biholomorphism of $D_0^* \sqcup D_1^*$ onto a neighborhood of $\overline{\Pi_{D_0}(D_0)} \cup \overline{\Pi_{D_1}(D_1)}$ (see the remarks at the end of Sect. 5.12.2).

The complex 1-manifold Y_0 has distinct connected components $\{Y_j^{(\nu)}\}_{\nu \geq 0, 1 \leq j \leq m_\nu}$ with $\Pi_{Z_0}(\Gamma_j^{(\nu)}) \subset Y_j^{(\nu)}$ for each ν and j; in fact, $Y_j^{(\nu)}$ may be viewed as the compact Riemann surface obtained by holomorphic attachment of caps to $\Gamma_j^{(\nu)}$ at the boundary as in Sect. 5.3 (see Fig. 5.16). Therefore, by Theorem 5.13.2, we may apply holomorphic tube removal in each of these connected components to get a Riemann surface that is biholomorphic to $\mathbb{P}^1$. Thus simultaneous holomorphic removal of these tubes from Y_0 yields a complex 1-manifold Y_1 in which each connected component is biholomorphic to $\mathbb{P}^1$ (see Fig. 5.17). Moreover, we may choose the tubes

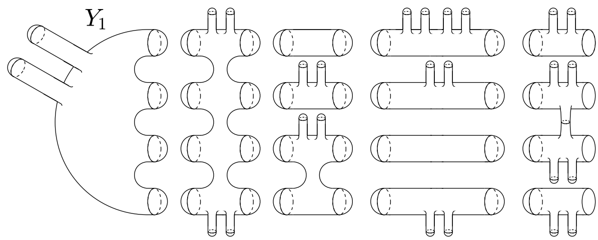

Fig. 5.17 The complex 1-manifold Y_1 obtained by holomorphic removal of tubes in each connected component of Y_0

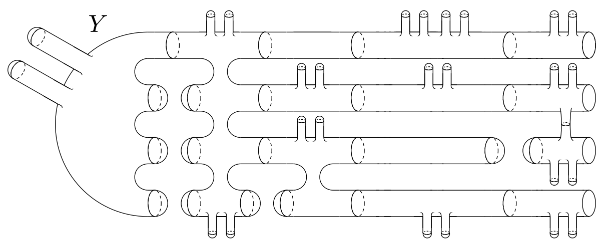

Fig. 5.18 The desired Riemann surface Y obtained by holomorphic reattachment of the boundary tubes to Y_1 (equivalently, via holomorphic removal of tubes from X)

so that their closures do not meet the set

$$\Pi_{Z_0}(K) \cup \overline{\Pi_{D_0}(D_0)} \cup \overline{\Pi_{D_1}(D_1)}.$$

As in Sect. 5.12.4, we may identify Y_1 with the Riemann surface obtained by holomorphic removal of all of the above tubes (those removed in the construction of Y_0 together with those removed in the construction of Y_1) *from* X. Hence holomorphic *reattachment* of the tubes $\{T_j^{(\nu)}\}$ to Y_1 as in Sect. 5.12.3 yields a Riemann surface Y that is biholomorphic to the Riemann surface obtained by holomorphic removal of a locally finite collection of disjoint tubes in X whose closures do not meet $\overline{T}_0 \cup K$ (see Fig. 5.18).

Step 3. Planarity of Y. By the Koebe uniformization theorem (Theorem 5.5.3), it now suffices to show that Y is planar. For this, we denote by T the union of the tubes in $X \setminus (\overline{T}_0 \cup K)$ that were removed in order to get Y, we denote by $C \subset T$ the union of the corresponding (unit) circles in each of the tubes in T (which were cut out in the tube-removal process), we set $Z = X \setminus C$, and we let $\Pi_Z : Z \hookrightarrow Y$ denote the corresponding holomorphic inclusion. The complement $Y \setminus \Pi_Z(Z)$ is a union of a locally finite family of disjoint compact sets, each of which is a closed disk in some local holomorphic chart. By construction, Y is connected, and hence $Z \cong \Pi_Z(Z)$ is connected (for example, by Lemma 5.11.4). Moreover, for each $\nu = 0, 1, 2, \ldots$, the set $Z \cap \Omega_\nu = \Omega_\nu \setminus C$, and the sets $Z \cap \Gamma_j^{(\nu)} = \Gamma_j^{(\nu)} \setminus C$ and $Z \cap V_j^{(\nu)} = V_j^{(\nu)} \setminus C$ for $j = 1, \ldots, m_\nu$, are connected (see Exercise 5.14.1).

Let $\Theta_0 = \emptyset$, and for each $\nu = 1, 2, 3, \ldots$, let $\Theta_\nu \subset Y$ be the connected component of $Y \setminus \Pi_Z(\partial\Omega_\nu)$ containing $\Pi_Z(Z \cap \Omega_\nu) = \Pi_Z(\Omega_\nu \setminus C)$ (which we may identify with the Riemann surface obtained by holomorphic removal from Ω_ν of the tubes given by those connected components of T that are contained in Ω_ν). Similarly, for each $\nu \geq 0$ and each $j = 1, \ldots, m_\nu$, let $\Xi_j^{(\nu)}$ be the connected component of $Y \setminus \Pi_Z(C_0)$ containing $\Pi_Z(Z \cap \Gamma_j^{(\nu)}) = \Pi_Z(\Gamma_j^{(\nu)} \setminus C)$ (which we may identify with the Riemann surface obtained by holomorphic removal from $\Gamma_j^{(\nu)}$ of the tubes given by those connected components of T that are contained in $\Gamma_j^{(\nu)}$), and let $W_j^{(\nu)}$ be the connected component of $Y \setminus \Pi_Z(\partial\Omega_\nu)$ containing $\Pi_Z(Z \cap V_j^{(\nu)}) = \Pi_Z(V_j^{(\nu)} \setminus C)$ (which we may identify with the Riemann surface obtained by holomorphic removal from $V_j^{(\nu)}$ of the tubes given by those connected components of T which are contained in $V_j^{(\nu)}$). Then $\{\Theta_\nu\}$ is a sequence of smooth domains with $Y = \bigcup_\nu \Theta_\nu$ and $\Theta_\nu \Subset \Theta_{\nu+1}$ for each $\nu \geq 0$. Moreover, for each $\nu \geq 0$, $\{\Xi_j^{(\nu)}\}$ and $\{W_j^{(\nu)}\}$ are the distinct connected components of $\Theta_{\nu+1} \setminus \overline{\Theta}_\nu$ and $Y \setminus \overline{\Theta}_\nu$, respectively, and, for each $j = 1, \ldots, m_\nu$, we have $\Xi_j^{(\nu)} \subset W_j^{(\nu)}$. Furthermore, for each $\nu \geq 1$, the distinct boundary components of Θ_ν are given by $\Pi_Z(C_j^{(\nu)}) = \partial\Xi_j^{(\nu)} \cap \partial\Theta_\nu = \partial W_j^{(\nu)}$ for $j = 1, \ldots, m_\nu$. Moreover, for each $\nu \geq 1$ and $j = 1, \ldots, m_\nu$, $\Pi_Z(C_j^{(\nu)})$ is separating in Y with complement equal to the union of the disjoint connected open sets

$$W_j^{(\nu)} \quad \text{and} \quad \Theta_\nu \cup \bigcup_{1 \leq i \leq m_\nu, i \neq j} W_i^{(\nu)} \cup \Pi_Z(T_i^{(\nu)}).$$

To verify that Y is planar, we must show that every closed $\mathcal{C}^\infty$ 1-form θ with compact support on Y is exact. For each $\nu \geq 1$ and each $j = 1, \ldots, m_\nu$, $\Pi_Z(C_j^{(\nu)})$ is separating in Y, so

$$\int_{\Pi_Z(C_j^{(\nu)})} \theta = 0 \quad \text{(for either orientation)}$$

by Lemma 5.11.3. Hence, since there is a biholomorphism of a neighborhood of $\Pi_Z(\overline{T_j^{(\nu)}})$ onto an annulus that maps $\Pi_Z(C_j^{(\nu)})$ onto a concentric circle, θ is exact on this neighborhood. Therefore, by replacing θ with the difference of θ and the differential of a $\mathcal{C}^\infty$ compactly supported function on Y with differential equal to θ on a neighborhood of $\Pi_Z(\overline{T_0})$ (note that we may choose the function to be compactly supported because θ is identically zero on a neighborhood of all but finitely many connected components of $\Pi_Z(\overline{T_0})$), we may assume that

$$\operatorname{supp}\theta \subset Y \setminus \Pi_Z(\overline{T_0}) \subset \bigcup_{\nu, j} \Xi_j^{(\nu)}.$$

Identifying Y_1 with the complex 1-manifold obtained by holomorphic removal of the tubes $\{\Pi_Z(T_j^{(\nu)})\}$ from Y (see Sect. 5.12.4), we get the holomorphic inclusion $\Pi_{Y \setminus \Pi_Z(C_0)} \colon Y \setminus \Pi_Z(C_0) \hookrightarrow Y_1$. Thus the restriction $\theta\restriction_{Y \setminus \Pi_Z(C_0)}$ determines a

1-form on the image $\Pi_{Y\setminus\Pi_Z(C_0)}(Y\setminus\Pi_Z(C_0))$ in Y_1 that extends by 0 to a closed compactly supported $\mathcal{C}^\infty$ 1-form τ on Y_1. On the other hand, each connected component P of Y_1 is a copy of $\mathbb{P}^1$ in which, for some unique choice of ν and j,

$$P\cap\Pi_{Y\setminus\Pi_Z(C_0)}(Y\setminus\Pi_Z(C_0))=\Pi_{Y\setminus\Pi_Z(C_0)}(\Xi_j^{(\nu)}),$$

and the complement $P\setminus\Pi_{Y\setminus\Pi_Z(C_0)}(\Xi_j^{(\nu)})$ is a union of finitely many disjoint connected compact sets, each of which is mapped onto a closed disk by some local holomorphic chart. Hence $\tau\restriction_P=d\lambda$ for some $\mathcal{C}^\infty$ function λ on P, and in particular, λ is constant on the domain $\Pi_{Y\setminus\Pi_Z(C_0)}(T_j^{(\nu)}\cap\Xi_j^{(\nu)})$. Pulling back to $\Xi_j^{(\nu)}$ and extending by the associated constant, we see that for each $\nu\geq 1$ and each $j=1,\dots,m_\nu$, there exists a $\mathcal{C}^\infty$ function $\rho_j^{(\nu)}$ on the domain $\Xi_j^{(\nu)}\cup\Pi_Z(T_j^{(\nu)})$ such that $d\rho_j^{(\nu)}=\theta\restriction_{\Xi_j^{(\nu)}\cup\Pi_Z(T_j^{(\nu)})}$. Similarly, we also have a $\mathcal{C}^\infty$ function $\rho_1^{(0)}$ on $\Xi_1^{(0)}=\Theta_1$ with $d\rho_1^{(0)}=\theta\restriction_{\Xi_1^{(0)}}$.

Now let $\eta_1=\rho_1^{(0)}$ on $\Theta_1=\Xi_1^{(0)}$. Given a function $\eta_\nu\in\mathcal{C}^\infty(\Theta_\nu)$ with $d\eta_\nu=\theta\restriction_{\Theta_\nu}$ for some $\nu\geq 1$ (in particular, η_ν is locally constant on $T_0\cap\Theta_\nu$), by replacing each function $\rho_j^{(\nu)}$ with the sum of $\rho_j^{(\nu)}$ and some constant, we may assume that $\rho_j^{(\nu)}=\eta_\nu$ on the domain

$$[\Xi_j^{(\nu)}\cup\Pi_Z(T_j^{(\nu)})]\cap\Theta_\nu=\Pi_Z(T_j^{(\nu)})\cap\Theta_\nu.$$

The function $\eta_{\nu+1}$ given by $\eta_{\nu+1}\restriction_{\Theta_\nu}=\eta_\nu$ and $\eta_{\nu+1}\restriction_{\Xi_j^{(\nu)}\cup\Pi_Z(T_j^{(\nu)})}=\rho_j^{(\nu)}$ for $j=1,\dots,m_\nu$ is then a $\mathcal{C}^\infty$ function on $\Theta_{\nu+1}$ with $d\eta_\nu=\theta\restriction_{\Theta_{\nu+1}}$. Thus, by induction, we get a sequence of $\mathcal{C}^\infty$ functions $\{\eta_\nu\}$, and we get a well-defined $\mathcal{C}^\infty$ function η on Y with $d\eta=\theta$ by setting $\eta\restriction_{\Theta_\nu}=\eta_\nu$ for each $\nu=1,2,3,\dots$. $\square$

Remark One may instead complete the proof of the above theorem by showing more directly that $\int_\gamma\theta=0$ for every loop γ in Y. For any such loop is homologous to a sum of loops, each of which lies in one of the sets $\overline{\Xi_j^{(\nu)}}$ (here, one uses the fact that the smooth Jordan curves $\{C_j^{(\nu)}\}$ are separating in Y). One may also choose the loops to avoid C_0. Integrating the form τ on Y_1 over the images of these loops in Y_1, one gets the claim.

Exercises for Sect. 5.14

5.14.1 Verify that in the proof of Theorem 5.14.2, for each $\nu=0,1,2,\dots$, the set $Z\cap\Omega_\nu=\Omega_\nu\setminus C$, and the sets $Z\cap\Gamma_j^{(\nu)}=\Gamma_j^{(\nu)}\setminus C$ and $Z\cap V_j^{(\nu)}=V_j^{(\nu)}\setminus C$ for $j=1,\dots,m_\nu$, are connected.

5.14.2 According to Sard's theorem (see, for example, [Mi]), the set of critical values of a $\mathcal{C}^\infty$ function on a second countable $\mathcal{C}^\infty$ manifold is a set of measure 0. Using Sard's theorem, Theorem 9.10.1, Lemma 5.11.1, and arguments similar to those appearing in this section, prove that every second countable oriented $\mathcal{C}^\infty$ surface may be obtained by $\mathcal{C}^\infty$ oriented attachment of a locally finite family of disjoint tubes to an oriented planar $\mathcal{C}^\infty$ surface N (cf. Exercises 5.12.8 and 5.13.2). Using this fact together with Theorem 5.5.3 and Theorem 5.9.2, construct a slightly different proof of Theorem 5.14.1.

5.15 Nonseparating Smooth Jordan Curves

The goal of this section is a proof of equivalence of the topological (as in Definition 5.11.2) and analytic (as in Lemma 5.11.3) characterizations of separating smooth Jordan curves in an oriented smooth surface M; that is, the goal is a proof that for every nonseparating smooth Jordan curve C in M, there exists a compactly supported closed $\mathcal{C}^\infty$ 1-form with nonzero integral along C. This fact will not be applied directly until Sect. 5.17, but the construction of the associated 1-form will be a key element in the construction in Sect. 5.16 of a canonical homology basis for a compact Riemann surface.

The main point of this section is the following:

Lemma 5.15.1 *For any smooth Jordan curve $\beta\colon [0,1] \to M$ with nonseparating image $B = \beta([0,1])$ in an oriented $\mathcal{C}^\infty$ surface M, we have the following*:

(a) *There exists a smooth Jordan curve $\alpha\colon [0,1] \to M$ with image $A \equiv \alpha([0,1])$ such that $A \cap B = \{\alpha(0)\} = \{\beta(0)\}$ and such that the pair of tangent vectors $\dot{\alpha}(0), \dot{\beta}(0)$ is a positively oriented basis for the tangent space $T_{\alpha(0)}M$.*

(b) *For any choice of a smooth Jordan curve α with the properties in* (a), *the sets $A \equiv \alpha([0,1])$ and $A \cup B$ are nonseparating. Moreover, for any choice of neighborhoods U and V of A and B, respectively, there exist closed compactly supported $\mathcal{C}^\infty$ real 1-forms θ and τ on M such that* $\operatorname{supp}\theta \subset V \setminus B$, $\operatorname{supp}\tau \subset U \setminus A$, $\int_\alpha \theta = 1$, $\int_\beta \theta = 0$, $\int_\alpha \tau = 0$, $\int_\beta \tau = 1$, *and for any loop γ in M, we have $\int_\gamma \theta \in \mathbb{Z}$ and $\int_\gamma \tau \in \mathbb{Z}$* (*hence, for M second countable, we have* $[\theta]_{H^1}, [\tau]_{H^1} \in H^1(M, \mathbb{Z})$).

Proof According to Lemma 5.11.1, there exists a positively oriented local $\mathcal{C}^\infty$ chart $(T, \Psi, \Delta(0; 1/R, R))$ with $R > 1$, $B \subset T$, and $\Psi(\beta(t)) = e^{2\pi i t}$ for each $t \in [0,1]$. Fixing $R_0 \in (1, R)$, we may let $\alpha_1\colon [-1,1] \to T$ be the path from $\Psi^{-1}(1/R_0)$ to $\Psi^{-1}(R_0)$ given by

$$t \mapsto \Psi^{-1}(R_0^t) = \Psi^{-1}(e^{t \log R_0}) = \Psi^{-1}(e^{t \log R_0} + 0i).$$

By Lemma 5.10.6, there is an injective smooth path α_2 in the domain $M \setminus B$ from $\Psi^{-1}(R_0)$ to $\Psi^{-1}(1/R_0)$, and we may set

$$s_0 \equiv \max \alpha_2^{-1}(\alpha_1([0,1])) \in [0,1),$$

$$s_1 \equiv \min([s_0, 1] \cap \alpha_2^{-1}(\alpha_1([-1,0]))) \in (s_0, 1],$$

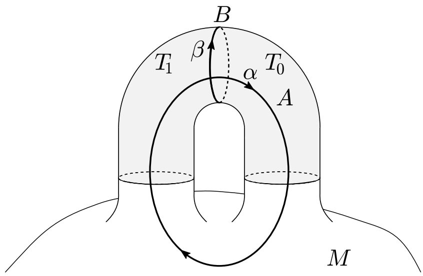

Fig. 5.19 Transverse nonseparating smooth Jordan curves and a smooth annular neighborhood

$$t_0 \equiv \alpha_1^{-1}(\alpha_2(s_0)) \in (0, 1],$$
$$t_1 \equiv \alpha_1^{-1}(\alpha_2(s_1)) \in [-1, 0).$$

We then get a piecewise smooth Jordan curve α_3 in M by setting

$$\alpha_3(t) = \begin{cases} \alpha_1(3t) & \text{if } t \in [0, t_0/3], \\ \alpha_2(s_0 + \frac{s_1 - s_0}{3 + t_1 - t_0}(3t - t_0)) & \text{if } t \in [t_0/3, 1 + (t_1/3)], \\ \alpha_1(3(t-1)) & \text{if } t \in [1 + (t_1/3), 1]. \end{cases}$$

Moreover, α_3 is loop-smooth at $\alpha_3(0) = \beta(0) = \Psi^{-1}(1)$, α_3 is smooth at each point in $(0, 1) \setminus \{t_0/3, 1 + (t_1/3)\}$, and $\alpha_3((0, 1)) \subset M \setminus B$. Applying Lemma 5.10.5, we get a smooth Jordan curve α that agrees with α_3 outside a small neighborhood of the set $\{t_0/3, 1 + (t_1/3)\}$, and that maps that small neighborhood into a small neighborhood of $\{\alpha_3(t_0/3), \alpha_3(1 + (t_1/3))\}$. In particular, we may choose α so that $\alpha([0, 1]) \cap B = \{\alpha(0)\} = \{\Phi^{-1}(1)\} = \{\beta(0)\}$, and the pair $\dot{\alpha}(0), \dot{\beta}(0)$ forms a positively oriented basis. Thus part (a) is proved.

For the proof of part (b), suppose that α is an arbitrary smooth Jordan curve for which the image $A \equiv \alpha([0, 1])$ meets B only in the point $\alpha(0) = \beta(0)$ and the pair $\dot{\alpha}(0), \dot{\beta}(0)$ forms a positively oriented basis for $T_{\alpha(0)}M$. Let $\hat{\alpha}\colon \mathbb{R} \to M$ be the 1-periodic $\mathcal{C}^\infty$ immersion given by $t \mapsto \alpha(t - \llcorner t \lrcorner)$. Given a neighborhood V of B in M, we may choose the annular neighborhood T to be relatively compact in V. Finally, let us denote the two connected components of $T \setminus B$ by $T_0 \equiv \Psi^{-1}(\Delta(0; 1, R))$ and $T_1 \equiv \Psi^{-1}(\Delta(0; 1/R, 1))$ (see Fig. 5.19).

Since polar coordinates (r, θ) (see Example 9.7.20) near $\Psi(\beta(0)) = 1$ form a positively oriented local $\mathcal{C}^\infty$ chart, we have

$$\begin{aligned} 0 &< (dr \wedge d\theta)\big(\Psi_*\dot{\alpha}(0), \Psi_*\dot{\beta}(0)\big) \\ &= (dr \wedge d\theta)\left(\Psi_*\dot{\alpha}(0), 2\pi\left(\frac{\partial}{\partial\theta}\right)_1\right) \\ &= 2\pi\, dr(\Psi_*\dot{\alpha}(0)) = 2\pi \frac{d}{dt} r(\Psi(\hat{\alpha}(t)))\bigg|_{t=0}. \end{aligned}$$

It follows that the function $t \mapsto |\Psi(\hat{\alpha}(t))|$ is strictly increasing on a neighborhood of each integer. Hence we may fix a number $s \in (0, 1/2)$ such that $\alpha([0, s]) \subset T_0$ and $\alpha([1 - s, 1]) \subset T_1$.

We may choose a real-valued $\mathcal{C}^\infty$ function φ on $M \setminus B$ such that $\operatorname{supp}\varphi \Subset V$, $\varphi \equiv 0$ on T_0, and $\varphi \equiv 1$ on T_1. Hence $d\varphi$ extends to a unique closed $\mathcal{C}^\infty$ 1-form θ on M with $\operatorname{supp}\theta \Subset V \setminus T \subset V \setminus B$, and in particular, we have $\int_\beta \theta = 0$. Since $\theta = 0$ on $T \supset \alpha([0,1] \setminus [s, 1-s])$ and $\theta = d\varphi$ on $M \setminus B \supset \alpha([s, 1-s])$, we have

$$\int_\alpha \theta = \int_{\alpha\restriction_{[s,1-s]}} \theta = \varphi(\alpha(1-s)) - \varphi(\alpha(s)) = 1.$$

It also follows that θ has an integer-valued integral along each loop γ in M. To see this, we may assume that $\gamma(0) = \gamma(1) \in T \setminus B$, and we may choose a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of $[0,1]$ such that for each $j = 1, \dots, m$, we have $\gamma([t_{j-1}, t_j]) \subset M \setminus B$ or T and $\gamma(t_j) \in T \setminus B$. In particular, we have $\varphi(\gamma(t_j)) \in \{0, 1\}$ for each j, and, letting $J \equiv \{j \in \{1, \dots, m\} \mid \gamma([t_{j-1}, t_j]) \subset M \setminus B\}$, we get

$$\int_\gamma \theta = \sum_{j\in J} \int_{\gamma\restriction_{[t_{j-1},t_j]}} \theta = \sum_{j \in J} [\varphi(\gamma(t_j)) - \varphi(\gamma(t_{j-1}))] \in \mathbb{Z}.$$

Now since $\int_\alpha \theta \neq 0$, Lemma 5.11.3 implies that A is nonseparating in M. Moreover, the reverse curve α^- is a smooth Jordan curve with nonseparating image A, and the pair $\dot\beta(0), \dot\alpha^-(0)$ (i.e., the pair $\dot\beta(0), -\dot\alpha(0)$) is positively oriented. Hence, given a neighborhood U of A, we may construct (as above) a closed $\mathcal{C}^\infty$ 1-form τ such that $\operatorname{supp}\tau \subset U \setminus A$, $\int_\alpha \tau = 0$, $\int_\beta \tau = 1$, and $\int_\gamma \tau \in \mathbb{Z}$ for each loop γ.

It remains to show that the set $A \cup B$ is nonseparating in M. For this, let Ω be a connected component of $M \setminus (A \cup B)$, and let $p = \alpha(0) = \beta(0)$. For each point $q \in (A \cup B) \setminus \{p\}$, there is a local $\mathcal{C}^\infty$ chart $(U_q, \Phi_q, U'_q = (-1,1) \times (-1,1))$ such that $q \in U_q$, U_q meets exactly one of the sets A, B, and $U_q \cap (A \cup B) = \Phi_q^{-1}((-1,1) \times \{0\})$. It follows that the set $(\partial\Omega) \setminus \{p\} \subset (A \cup B) \setminus \{p\}$ is both open and closed in $(A \cup B) \setminus \{p\}$. Since $\partial\Omega$ is not a singleton (otherwise, a connected neighborhood punctured at the boundary point would lie in Ω, and hence the connected set $A \cup B$ would be contained in the singleton $\partial\Omega$, which is impossible), $\partial\Omega$ must meet $(A \cup B) \setminus \{p\}$ and therefore contain one of the connected sets $A \setminus \{p\}$, $B \setminus \{p\}$. On the other hand, according to Lemma 5.10.3, there exists a local $\mathcal{C}^\infty$ coordinate neighborhood in which p is represented by the origin, and A and B are represented by the x-axis and y-axis, respectively, near the origin. It follows that Ω must contain the part of some open coordinate quadrant near the origin, and therefore that $A \cup B \subset \partial\Omega$. Thus $\partial\Omega = A \cup B$. Similarly, the set of points $q \in (A \cup B) \setminus \{p\}$ for which $U_q \setminus (A \cup B) \subset \Omega$ is open and closed in $(A \cup B) \setminus \{p\}$. If the intersection of this set with $A \setminus \{p\}$ is empty, then Ω is a smooth open subset of $M \setminus B$. Since θ has compact support in $M \setminus B$, Stokes' theorem gives

$$1 = \int_\alpha \theta = \pm \int_{(M\setminus B)\cap\partial\Omega} \theta = \int_\Omega d\theta = 0,$$

and we have arrived at a contradiction. Similarly, we must have $U_q \setminus (A \cup B) \subset \Omega$ for each point $q \in B \setminus \{p\}$. The local coordinate representation about p considered above implies that the complement of $A \cup B$ in some neighborhood of p also lies in Ω. Thus $M \setminus (A \cup B) = \Omega$, and $A \cup B$ is nonseparating in M. $\square$

Proposition 5.15.2 *For any orientable $\mathcal{C}^\infty$ surface M:*

(a) *A smooth Jordan curve C in M is separating if and only if $\int_C \theta = 0$ for every closed $\mathcal{C}^\infty$ 1-form θ with compact support in M.*

(b) *M is planar (i.e., every $\mathcal{C}^\infty$ compactly supported closed 1-form on M is exact) if and only if every smooth Jordan curve C in M is separating.*

Proof Part (a) follows from Lemma 5.11.3 and Lemma 5.15.1. Part (b) follows from Proposition 5.10.7, Proposition 10.5.5, and part (a). □

Exercises for Sect. 5.15

5.15.1 Let M be an oriented second countable $\mathcal{C}^\infty$ surface, let $M^* = M \cup \{\infty\}$ be the one-point compactification of M (see Definition 9.1.11), and let $\beta\colon [0,1] \to M$ be a smooth Jordan curve with image $B \equiv \beta([0,1])$.

(a) Assuming that B is separating in M but *nonseparating* in M^*, prove that:

(i) There exists a $\mathcal{C}^\infty$ embedding $\alpha\colon \mathbb{R} \to M$ with image $A \equiv \alpha(\mathbb{R})$ such that $A \cap B = \{\alpha(0)\} = \{\beta(0)\}$ and such that the pair of tangent vectors $\dot\alpha(0), \dot\beta(0)$ is a positively oriented basis for the tangent space $T_{\alpha(0)}M$; and

(ii) If $\alpha\colon \mathbb{R} \to M$ is any $\mathcal{C}^\infty$ embedding with the properties in (a), then the set $A \equiv \alpha(\mathbb{R})$ is nonseparating in M, the set $A \cup B$ is nonseparating in M^*, and for any choice of a neighborhood U of A, there exists a closed $\mathcal{C}^\infty$ real 1-form τ on M such that $\operatorname{supp}\tau \subset U \setminus A$ and $\int_\beta \tau = 1$, and such that $\int_\gamma \tau \in \mathbb{Z}$ for every loop γ in M (cf. Exercise 5.11.5).

(b) Assuming that B is separating in M^*, prove that $\int_\beta \theta = 0$ for every closed $\mathcal{C}^\infty$ 1-form θ on M (i.e., $[\beta]_{H_1} = 0$).

5.16 Canonical Homology Bases in a Compact Riemann Surface

In this section, we construct a canonical homology basis associated to holomorphically attached tubes in a compact Riemann surface. The required facts concerning homology and cohomology appear in Sects. 10.6 and 10.7.

Theorem 5.16.1 *Let $g \in \mathbb{Z}_{\geq 0}$, and let X be a compact Riemann surface obtained by holomorphic attachment of g disjoint tubes to $\mathbb{P}^1$ (as in Sect. 5.12.1 and Theorem 5.13.1). Then we have the following (the case $g = 3$ is illustrated in Fig. 5.20):*

(a) *For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we have*

$$2g = \operatorname{rank} H_1(X,\mathbb{Z}) = \operatorname{rank} H^1(X,\mathbb{Z}) = \dim_{\mathbb{F}} H_1(X,\mathbb{F}) = \dim_{\mathbb{F}} H^1(X,\mathbb{F}).$$

(b) *There exist smooth Jordan curves $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ in X with images $A_j = \alpha_j([0,1])$ and $B_j = \beta_j([0,1])$ for $j = 1, \dots, g$ such that*

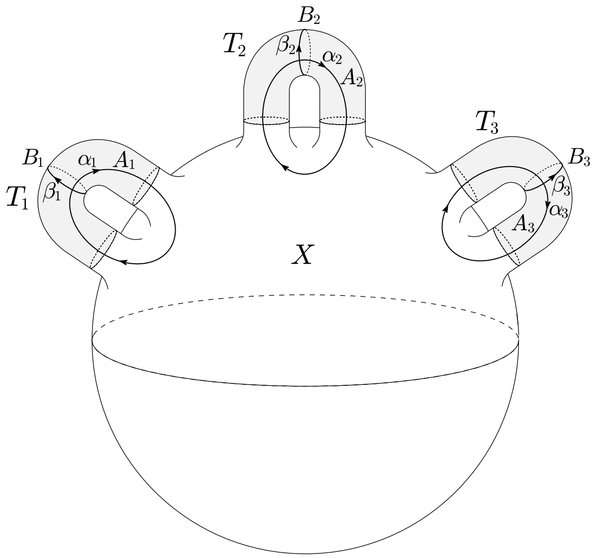

Fig. 5.20 A canonical homology basis

(i) *For all* $i, j = 1, \ldots, g$ *with* $i \neq j$, $A_i \cap A_j = A_i \cap B_j = B_i \cap B_j = \emptyset$;
(ii) *For each* $j = 1, \ldots, g$, $A_j \cap B_j = \{\alpha_j(0)\} = \{\beta_j(0)\}$; *and*
(iii) *For each* $j = 1, \ldots, g$, *the pair of tangent vectors* $\dot{\alpha}_j(0), \dot{\beta}_j(0)$ *is a positively oriented basis for the tangent space* $T_{\alpha_j(0)}X$ *(i.e.,* $\omega(\dot{\alpha}(0), \dot{\beta}(0)) > 0$ *for every positive* 2*-form* ω*).*

(c) *If* $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ *are smooth Jordan curves in* X *satisfying the conditions* (i)–(iii) *in* (b), *then the homology classes* $[\alpha_1]_{H_1}, [\beta_1]_{H_1}, \ldots, [\alpha_g]_{H_1}, [\beta_g]_{H_1}$ *form a basis for the* $\mathbb{A}$*-module* $H_1(X, \mathbb{A})$ *for any subring* $\mathbb{A}$ *of* $\mathbb{C}$ *containing* $\mathbb{Z}$. *Given neighborhoods* U_j *of* A_j *and* V_j *of* B_j *in* X *for* $j = 1, \ldots, g$, *there exist* $\mathcal{C}^\infty$ *closed* 1*-forms* $\{\theta_j\}_{j=1}^g$ *and* $\{\tau_j\}_{j=1}^g$ *such that* $\operatorname{supp}\theta_j \subset V_j \setminus B_j$ *and* $\operatorname{supp}\tau_j \subset U_j \setminus A_j$ *for* $j = 1, \ldots, g$, *and for any subring* $\mathbb{A}$ *of* $\mathbb{C}$ *containing* $\mathbb{Z}$, *the cohomology classes* $\{[\theta_j]_{H^1}\}_{j=1}^g \cup \{[\tau_j]_{H^1}\}_{j=1}^g$ *form a basis for* $H^1(X, \mathbb{A})$ *that is dual to the basis* $\{[\alpha_j]_{H_1}\}_{j=1}^g \cup \{[\beta_j]_{H_1}\}_{j=1}^g$; *that is, for* $i, j = 1, \ldots, g$,

$$([\theta_i]_{H^1}, [\beta_j]_{H_1})_{\mathrm{deR}} = ([\tau_i]_{H^1}, [\alpha_j]_{H_1})_{\mathrm{deR}} = 0$$

and

$$([\theta_i]_{H^1}, [\alpha_j]_{H_1})_{\mathrm{deR}} = ([\tau_i]_{H^1}, [\beta_j]_{H_1})_{\mathrm{deR}} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(d) *Suppose* $T_1, \ldots, T_g$ *are open subsets of* X *with disjoint closures such that* $X \setminus (\overline{T}_1 \cup \cdots \cup \overline{T}_g)$ *is connected and for each* $j = 1, \ldots, g$, T_j *is a relatively compact subset of a local holomorphic chart that maps* T_j *onto an annulus. Then holomorphic removal of the tubes* $T_1, \ldots, T_g$ *yields a compact Riemann surface that is biholomorphic to* $\mathbb{P}^1$. *Moreover, we may choose smooth Jordan* *curves* $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ *as in* (b) *so that for each* $j = 1, \ldots, g$, B_j *is a concentric circle in* T_j *with respect to the biholomorphism onto an annulus and* $A_i \cap \overline{T}_j = \emptyset$ *for* $i = 1, \ldots, \hat{j}, \ldots, g$.

(e) *Suppose* $T_1, \ldots, T_g$ *are open subsets of* X *with disjoint closures such that for each* $j = 1, \ldots, g$, T_j *is a relatively compact subset of a local holomorphic chart that maps* T_j *onto an annulus. Assume that there exist smooth Jordan curves* $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ *as in* (b) *such that for each* $j = 1, \ldots, g$, $B_j \subset T_j$. *Then* $[\beta_j]_{\beta_j(0)}$ *generates the fundamental group* $\pi_1(T_j, \beta_j(0))$ *for each* $j = 1, \ldots, g$, *and holomorphic removal of the tubes* $T_1, \ldots, T_g$ *yields a compact Riemann surface that is biholomorphic to* $\mathbb{P}^1$.

Remarks 1. Part (a) of the theorem (together with the discussion of homology in Sects. 10.6 and 10.7) implies that g, the number of tubes that one attaches to $\mathbb{P}^1$ in order to construct a given compact Riemann surface X, depends only on the topology of X, not on the holomorphic structure or even the $\mathcal{C}^\infty$ structure.

2. It will follow from the results of Chap. 6 that any compact orientable topological surface M (see Sect. 6.10) may be obtained by attaching g tubes to the sphere, where $2g = \operatorname{rank} H_1(M, \mathbb{Z})$. One may also obtain this fact directly (see, for example, [T]).

3. Actually, any two compact orientable topological surfaces are homeomorphic if and only if they have isomorphic homology groups (again, see [T]). Of course, two homeomorphic compact Riemann surfaces need not be biholomorphic. For example, two complex tori need not be biholomorphic, although all complex tori are diffeomorphic (see Exercise 5.9.1).

4. Part (e) is a partial converse of part (d), but in (e), one may not be able to choose each coordinate annulus T_j so that the given curve B_j becomes a concentric circle. However, any choice of such a concentric circle in T_j will represent $[\beta_j]_{H_1}$ or $-[\beta_j]_{H_1}$, and one may homotope α_j slightly to get transversality.

5. For $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ as in part (b), choosing a path γ_j from a fixed base point x_0 to $\alpha_j(0) = \beta_j(0)$ and letting $u_j = [\gamma_j * \alpha_j * \gamma_j^-]$, $v_j = [\gamma_j * \beta_j * \gamma_j^-] \in \pi_1(X, x_0)$ for each $j = 1, \ldots, g$, one can show that $u_1, v_1, \ldots, u_g, v_g$ generate $\pi_1(X, x_0)$ (see Exercise 5.17.3), and that in fact, $\pi_1(X, x_0)$ is equal to the free group on these generators modulo the relation $u_1 v_1 u_1^{-1} v_1^{-1} \cdots u_g v_g u_g^{-1} v_g^{-1} = e$ (see, for example, [Hat]). However, we will not use this fact.

Before addressing the proof of Theorem 5.16.1, we consider a definition and some consequences.

Definition 5.16.2 Let X be a compact Riemann surface. A homology basis $[\alpha_1]_{H_1}$, $[\beta_1]_{H_1}, \ldots, [\alpha_g]_{H_1}, [\beta_g]_{H_1}$ for $H_1(X, \mathbb{Z})$ (which will also be a basis for $H_1(X, \mathbb{A})$ for any subring $\mathbb{A}$ of $\mathbb{C}$ containing $\mathbb{Z}$) with homology classes represented by smooth Jordan curves $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ satisfying the conditions in part (b) of Theorem 5.16.1 is called a *canonical homology basis*.

Applying Theorem 5.13.1, Lemma 5.11.1, Theorem 5.9.2, and Theorem 5.16.1, we get the following:

Corollary 5.16.3 *Let X be a compact Riemann surface, let $2g = \operatorname{rank} H_1(X, \mathbb{Z})$, and let $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ be smooth Jordan curves satisfying the conditions in part* (b) *of Theorem 5.16.1* (*in particular,* $[\alpha_1]_{H_1}, [\beta_1]_{H_1}, \dots, [\alpha_g]_{H_1}, [\beta_g]_{H_1}$ *is a canonical homology basis*). *Then there exist open subsets $T_1, \dots, T_g$ of X with disjoint closures such that $X \setminus (\overline{T}_1 \cup \cdots \cup \overline{T}_g)$ is connected and such that for each $j = 1, \dots, g$, T_j is a relatively compact subset of a local holomorphic chart that maps T_j onto an annulus, $B_j \subset T_j$, $[\beta_j]_{\beta_j(0)}$ generates the fundamental group $\pi_1(T_j, \beta_j(0))$, $A_i \cap \overline{T}_j = \emptyset$ for all $i = 1, \dots, \hat{j}, \dots, g$, and holomorphic removal of the tubes $T_1, \dots, T_g$ yields a Riemann surface that is biholomorphic to $\mathbb{P}^1$.*

Theorem 5.13.1, Theorem 5.16.1, Corollary 4.9.2, and Corollary 4.6.5 together give the following:

Corollary 5.16.4 *Up to biholomorphism, every compact Riemann surface X may be obtained by holomorphic attachment of g disjoint tubes to $\mathbb{P}^1$, where*

$$g = \dim_{\mathbb{C}} H^1_{\mathrm{Dol}}(X) = \dim \Gamma(X, \mathcal{O}(K_X)) = \operatorname{genus}(X).$$

Proof of Theorem 5.16.1 Throughout this proof, $\mathbb{A}$ will denote a subring of $\mathbb{C}$ containing $\mathbb{Z}$.

Step 1. We construct smooth Jordan curves satisfying (b). By hypothesis, we may choose open subsets $\{T_j\}_{j=1}^g$ of X with disjoint closures, constants $\{R_{0j}\}_{j=1}^g$ and $\{R_{1j}\}_{j=1}^g$ in $(1, \infty)$, and a biholomorphism Φ_j of a neighborhood of $\overline{T}_j$ onto a domain in $\mathbb{C}$ with $\Phi_j(T_j) = \Delta(0; 1/R_{0j}, R_{1j})$ for each $j = 1, \dots, g$ such that if $\beta_j(t) = \Phi_j^{-1}(e^{2\pi i t})$ for each $t \in [0, 1]$, $B_j = \beta_j([0, 1]) \subset T_j$ for each $j = 1, \dots, g$, and $Z = X \setminus (B_1 \cup \cdots \cup B_g)$, then there is a biholomorphism Π_Z of Z onto a domain in $\mathbb{P}^1$ with complement $\mathbb{P}^1 \setminus \Pi_Z(Z)$ equal to the union of a finite collection of disjoint compact sets, each of which is mapped onto a closed disk by some local holomorphic chart in $\mathbb{P}^1$. Observe that $X_1 \equiv X \setminus (B_2 \cup \cdots \cup B_g) = Z \cup T_1$ is connected, since Z and T_1 are connected and $Z \cap T_1 \neq \emptyset$. Moreover, since $Z = X_1 \setminus B_1$ is connected, B_1 is nonseparating in X_1. Therefore, by Lemma 5.15.1, there exists a smooth Jordan curve α_1 with image $A_1 = \alpha_1([0, 1])$ in X_1 such that $A_1 \cap B_1 = \{\alpha_1(0)\} = \{\beta_1(0)\}$ and the pair of tangent vectors $\dot{\alpha}_1(0), \dot{\beta}_1(0)$ is a positively oriented basis for the tangent space $T_{\alpha_1(0)}X$. Moreover, the sets A_1, B_1, and $A_1 \cup B_1$ are nonseparating in X_1 (for any choice of α_1). Suppose $k \in \{2, \dots, g\}$ and we have constructed smooth Jordan curves $\alpha_1, \dots, \alpha_{k-1}$ such that for each $j = 1, \dots, k-1$, we have

$$A_j \equiv \alpha_j([0, 1]) \subset X_j \equiv X \setminus (A_1 \cup \cdots \cup A_{j-1} \cup B_1 \cup \cdots \cup \widehat{B}_j \cup \cdots \cup B_g),$$

X_j is connected, A_j, B_j, and $A_j \cup B_j$ are nonseparating in X_j, $A_j \cap B_j = \{\alpha_j(0)\} = \{\beta_j(0)\}$, and the pair of tangent vectors $\dot{\alpha}_j(0), \dot{\beta}_j(0)$ is a positively oriented basis for the tangent space $T_{\alpha_j(0)}X$. Applying Lemma 5.15.1 to the smooth Jordan curve β_k in the Riemann surface

$$X_k \equiv X \setminus (A_1 \cup \cdots \cup A_{k-1} \cup B_1 \cup \cdots \cup \widehat{B}_k \cup \cdots \cup B_g)$$

(B_k is nonseparating in X_k because $A_{k-1} \cup B_{k-1}$ is nonseparating in X_{k-1} and $X_k \setminus B_k = X_{k-1} \setminus (A_{k-1} \cup B_{k-1})$), we get a smooth Jordan curve α_k such that

$$A_k \equiv \alpha_k([0,1]) \subset X_k,$$

$A_k \cap B_k = \{\alpha_k(0)\} = \{\beta_k(0)\}$, the pair of tangent vectors $\dot{\alpha}_k(0), \dot{\beta}_k(0)$ is a positively oriented basis for the tangent space $T_{\alpha_k(0)}X$, and (for any choice of such an α_k) the sets A_k, B_k, and $A_k \cup B_k$ are nonseparating in X_k. Thus, by induction, we get smooth Jordan curves α_j and β_j for $j = 1, \dots, g$ satisfying the conditions in (b).

Step 2. We show that the homology classes $[\alpha_1]_{H_1}, [\beta_1]_{H_1}, \dots, [\alpha_g]_{H_1}, [\beta_g]_{H_1}$ *represented by the Jordan curves* $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ *constructed in Step 1 are linearly independent in* $H_1(X, \mathbb{A})$. We may fix domains $U_1, V_1, \dots, U_g, V_g$ in X such that for each $j = 1, \dots, g$, we have $A_j \subset U_j$, $B_j \subset V_j$, and for each $i = 1, \dots, \hat{j}, \dots, g$, $(U_i \cup V_i) \cap (U_j \cup V_j) = \emptyset$. By Lemma 5.15.1, for each $j = 1, \dots, g$, we may choose closed $\mathcal{C}^\infty$ 1-forms θ_j and τ_j on X such that $\operatorname{supp}\theta_j \subset V_j \setminus B_j$, $\operatorname{supp}\tau_j \subset U_j \setminus A_j$, $\int_{\alpha_j} \theta_j = \int_{\beta_j} \tau_j = 1$, $\int_{\beta_j} \theta_j = \int_{\alpha_j} \tau_j = 0$, $\int_\gamma \theta, \int_\gamma \tau \in \mathbb{Z}$ for every loop γ in X, and (clearly)

$$\int_{\alpha_i} \theta_j = \int_{\beta_i} \theta_j = \int_{\alpha_i} \tau_j = \int_{\beta_i} \tau_j = 0 \qquad \text{for } i = 1, \dots, \hat{j}, \dots, g.$$

It follows that the homology classes $[\alpha_1]_{H_1}, [\beta_1]_{H_1}, \dots, [\alpha_g]_{H_1}, [\beta_g]_{H_1}$ are linearly independent over $\mathbb{A}$. For if $\xi = \sum_{j=1}^g r_j \cdot \alpha_j + \sum_{j=1}^g s_j \cdot \beta_j \in Z_1(X, \mathbb{A})$, with $r_j, s_j \in \mathbb{A}$ for $j = 1, \dots, g$, then for each $j = 1, \dots, g$, we have

$$r_j = \int_\xi \theta_j \quad \text{and} \quad s_j = \int_\xi \tau_j.$$

So if ξ is homologous to 0, then $r_j = s_j = 0$ for each $j = 1, \dots, g$. Similarly, the cohomology classes $[\theta_1]_{H^1}, [\tau_1]_{H^1}, \dots, [\theta_g]_{H^1}, [\tau_g]_{H^1} \in H^1(X, \mathbb{A})$ are linearly independent.

Step 3. We show that the cohomology classes $[\theta_1]_{H^1}, [\tau_1]_{H^1}, \dots, [\theta_g]_{H^1}, [\tau_g]_{H^1}$ *represented by the* 1*-forms* $\theta_1, \tau_1, \dots, \theta_g, \tau_g$ *constructed in Step 2 span* $H^1(X, \mathbb{A})$. For this, we must show that each closed $\mathcal{C}^\infty$ 1-form ρ on X with integrals along loops taking values in $\mathbb{A}$ is cohomologous to a linear combination over $\mathbb{A}$ of the differential forms $\theta_1, \tau_1, \dots, \theta_g, \tau_g$. By replacing ρ with the 1-form

$$\rho - \sum_{j=1}^g \left(\int_{\alpha_j} \rho \right) \cdot \theta_j - \sum_{j=1}^g \left(\int_{\beta_j} \rho \right) \cdot \tau_j,$$

we may assume without loss of generality that

$$\int_{\alpha_j} \rho = \int_{\beta_j} \rho = 0 \quad \text{for } j = 1, \dots, g.$$

In this case, we will show that in fact, ρ is exact.

For each $j = 1, \dots, g$, we have $\int_{\beta_j} \rho = 0$, and hence, since the path homotopy class $[\beta_j]$ generates the fundamental group of a neighborhood of $\overline{T_j}$, ρ is exact on this neighborhood. Choosing a function with differential ρ on this neighborhood and subtracting from ρ the differential of a suitable cut-off of the function, we may assume that $\rho \equiv 0$ on T_j for each $j = 1, \dots, g$. It follows that there is a unique closed $\mathcal{C}^\infty$ 1-form $\hat{\rho}$ on $\mathbb{P}^1$ with $\hat{\rho} = 0$ on the open set $W \equiv \Pi_Z(\bigcup_{j=1}^g (T_j \setminus B_j)) \cup (\mathbb{P}^1 \setminus \Pi_Z(Z))$ and $\Pi_Z^* \hat{\rho} = \rho$ on Z. Since $\mathbb{P}^1$ is simply connected, there is a $\mathcal{C}^\infty$ function $\hat{\lambda}$ on $\mathbb{P}^1$ with $d\hat{\lambda} = \hat{\rho}$. In particular, $\hat{\lambda}$ is locally constant on W. On the other hand, for each $j = 1, \dots, g$, choosing $s \in (0, 1/2)$ with $\alpha_j([0, s] \cup [1 - s, 1]) \subset T_j$, we get

$$0 = \int_{\alpha_j} \rho = \int_{\alpha_j \restriction_{[s,1-s]}} \rho = \int_{\Pi_Z(\alpha_j \restriction_{[s,1-s]})} \hat{\rho} = \hat{\lambda}(\Pi_Z(\alpha_j(1 - s))) - \hat{\lambda}(\Pi_Z(\alpha_j(s))).$$

Thus $\hat{\lambda}$ has the same constant value on the two connected components

$$\Pi_Z(\Phi_j^{-1}(\Delta(0; 1/R_{0j}, 1))) \quad \text{and} \quad \Pi_Z(\Phi_j^{-1}(\Delta(0; 1, R_{1j})))$$

of the set $\Pi_Z(T_j \setminus B_j) \subset W$. It follows that the function $\hat{\lambda} \circ \Pi_Z$ extends to a unique $\mathcal{C}^\infty$ function λ on X (which is constant on T_j for each $j = 1, \dots, g$), and we have $d\lambda = \rho$, since $d\lambda = \Pi_Z^* \hat{\rho} = \rho$ on Z and $d\lambda = 0 = \rho$ on $\bigcup T_j$. Thus $[\theta_1]_{H^1}, [\tau_1]_{H^1}, \dots, [\theta_g]_{H^1}, [\tau_g]_{H^1}$ is a basis for $H^1(X, \mathbb{A})$.

Step 4. We show that the classes $[\alpha_1]_{H_1}, [\beta_1]_{H_1}, \dots, [\alpha_g]_{H_1}, [\beta_g]_{H_1}$ *form a basis for* $H_1(X, \mathbb{A})$. Given a 1-cycle $\xi \in Z_1(X, \mathbb{A})$, the 1-cycle

$$\zeta \equiv \sum_{j=1}^g \left(\int_\xi \theta_j\right) \cdot \alpha_j + \sum_{j=1}^g \left(\int_\xi \tau_j\right) \cdot \beta_j \in Z_1(X, \mathbb{A})$$

satisfies

$$\int_{\xi - \zeta} \theta_j = \int_{\xi - \zeta} \tau_j = 0 \quad \text{for } j = 1, \dots, g,$$

and it follows that ξ is homologous to ζ. Thus $[\alpha_1]_{H_1}, [\beta_1]_{H_1}, \dots, [\alpha_g]_{H_1}, [\beta_g]_{H_1}$ is a basis for $H_1(X, \mathbb{A})$. In particular, part (a) is proved.

Step 5. We prove part (c). Suppose that $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ are *arbitrary* smooth Jordan curves in X satisfying the conditions (i)–(iii) in (b), and for each $j = 1, \dots, g$, U_j is a neighborhood of A_j and V_j is a neighborhood of B_j. Then, for each $j = 1, \dots, g$, α_j and β_j are nonseparating in X (since by Lemma 5.11.1 and Lemma 5.10.3, the connected set $\beta_j((0, 1))$ meets each connected component of $X \setminus A_j$ and the connected set $\alpha_j((0, 1))$ meets each connected component of $X \setminus B_j$). By Lemma 5.15.1, there exist $\mathcal{C}^\infty$ closed 1-forms $\theta_1, \tau_1, \dots, \theta_g, \tau_g$ on X such that for $i, j = 1, \dots, g$, $\operatorname{supp} \theta_j \subset V_j \setminus B_j$, $\operatorname{supp} \tau_j \subset U_j \setminus A_j$,

$$\int_{\alpha_i} \theta_j = \int_{\beta_i} \tau_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

$\int_{\alpha_i} \tau_j = \int_{\beta_i} \theta_j = 0$, and $[\theta_j]_{H^1}, [\tau_j]_{H^1} \in H^1(X, \mathbb{Z})$. The argument in Step 2 shows that the homology classes $[\alpha_1]_{H_1}, [\beta_1]_{H_1}, \dots, [\alpha_g]_{H_1}, [\beta_g]_{H_1}$ are linearly independent in $H_1(X, \mathbb{A})$, and the cohomology classes $[\theta_1]_{H^1}, [\tau_1]_{H^1}, \dots, [\theta_g]_{H^1}, [\tau_g]_{H^1}$ are linearly independent in $H^1(X, \mathbb{A})$. In particular, $[\alpha_1]_{H_1}, [\beta_1]_{H_1}, \dots, [\alpha_g]_{H_1}$, $[\beta_g]_{H_1}$ is a basis for the $2g$-dimensional vector space $H_1(X, \mathbb{R})$ (as well as for $H_1(X, \mathbb{C})$) with dual basis $[\theta_1]_{H^1}, [\tau_1]_{H^1}, \dots, [\theta_g]_{H^1}, [\tau_g]_{H^1}$ for $H^1(X, \mathbb{R})$ (as well as for $H^1(X, \mathbb{C})$). The argument in Step 4 shows that $[\alpha_1]_{H_1}, [\beta_1]_{H_1}, \dots, [\alpha_g]_{H_1}$, $[\beta_g]_{H_1}$ must also span $H_1(X, \mathbb{A})$, and hence the dual cohomology classes $[\theta_1]_{H^1}$, $[\tau_1]_{H^1}, \dots, [\theta_g]_{H^1}, [\tau_g]_{H^1}$ span $H^1(X, \mathbb{A})$. Thus part (c) is proved.

Step 6. We prove part (d). The proof of (d) is similar to the proof of Theorem 5.13.1. Let B_j be a concentric circle in the holomorphic coordinate annulus T_j for $j = 1, \dots, g$. By hypothesis, $X \setminus \bigcup_{j=1}^{g} \overline{T}_j$ and therefore $X \setminus \bigcup_{j=1}^{g} B_j$ are connected. Holomorphic removal of the tube T_g yields a compact Riemann surface with real first homology space of dimension $< 2g$. However, since the first homology of a compact Riemann surface is of even dimension, the dimension of the homology must have actually dropped by at least 2. On the other hand, removal of T_g still leaves (the images of) the remaining tubes, each of which contains a nonseparating smooth Jordan curve (given by the image of B_j for some $j \in \{1, \dots, g-1\}$). Proceeding inductively, we will not get a planar Riemann surface before we have removed *all* of the g tubes $T_1, \dots, T_g$, at which point we get a compact Riemann surface with zero homology, that is, a Riemann surface that is biholomorphic to $\mathbb{P}^1$. Observe also that Step 1 provides the desired Jordan curves satisfying (b).

Step 7. We prove part (e). Observe that for each j, the path homotopy class $[\beta_j]$ in $\pi_1(T_j)$ cannot be the identity, because by (c), $[\beta_j]_{H_1(X,\mathbb{Z})} \neq 0$. Thus we have $[\beta_j] = [\gamma_j]^{k_j}$ for some $k_j \in \mathbb{Z} \setminus \{0\}$, where γ_j is a loop corresponding to a concentric circle in T_j. By Lemma 5.15.1, there exist $\mathcal{C}^\infty$ closed 1-forms $\eta_1, \dots, \eta_g$ on X such that for all $i, j = 1, \dots, g$,

$$\int_{\beta_i} \eta_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and such that $\int_\gamma \eta_j \in \mathbb{Z}$ for each loop γ in X. In particular, for each j, we have $1 = \int_{\beta_j} \eta_j = k_j \cdot \int_{\gamma_j} \eta_j \in k_j \mathbb{Z}$, and hence we must have $k_j = \pm 1$. Thus $[\beta_j]$ generates $\pi_1(T_j)$, and $C_j \equiv \gamma_j([0,1])$ is nonseparating. Consequently, by subtracting the differential of a suitable $\mathcal{C}^\infty$ function, we may also assume that

$$\operatorname{supp} \eta_j \subset X \setminus (\overline{T}_1 \cup \cdots \cup \widehat{\overline{T}_j} \cup \cdots \cup \overline{T}_g).$$

It follows by induction that for $j = 2, \dots, g$, $X \setminus (\overline{T}_1 \cup \cdots \cup \overline{T}_{j-1})$ is a domain in which C_j and $\overline{T}_j$ are nonseparating, and hence $X \setminus (\overline{T}_1 \cup \cdots \cup \overline{T}_g)$ is connected. Part (d) now gives the claim (e). $\square$

Remark The above proof did not use Theorems 10.7.16 and 10.7.18 in an essential way. In fact, a proof of these theorems for the special case of a compact Riemann surface is actually contained within the above.

Exercises for Sect. 5.16

5.16.1 Using Exercise 5.9.2 and the results of this section, prove that every compact Riemann surface of genus 1 is biholomorphic to a complex torus (a different approach was considered in Exercise 5.9.3, and yet another approach will be considered in Sect. 5.22).

5.16.2 Let X be a compact Riemann surface of genus 2. Prove that there is a covering Riemann surface $\Upsilon\colon \widehat{X} \to X$ such that
- (i) $\widehat{X}$ is obtained by holomorphic attachment of a locally finite infinite sequence of disjoint tubes $\{T_\nu\}_{\nu=1}^{\infty}$ to $\mathbb{C}$;
- (ii) We have $\operatorname{Deck}(\Upsilon) \cong \mathbb{Z}^2$;
- (iii) We have $T_\nu \cap \Phi(T_\nu) = \emptyset$ for each $\Phi \in \operatorname{Deck}(\Upsilon) \setminus \{\mathrm{Id}_{\widehat{X}}\}$ and each $\nu \in \mathbb{Z}_{>0}$; and
- (iv) We have $\operatorname{Deck}(\Upsilon) \cdot T_1 = \bigcup_{\nu=1}^{\infty} T_\nu$.

Hint. Apply Exercise 5.16.1 (cf. Exercise 5.12.9).

5.16.3 The goal of this exercise is a proof of a theorem of Abel (see part (d) below). Let $[\alpha_1]_{H_1}, [\beta_1]_{H_1}, \dots, [\alpha_g]_{H_1}, [\beta_g]_{H_1}$ be a canonical homology basis for a compact Riemann surface X of genus $g > 0$.

(a) Prove that if η_1 and η_2 are two closed $\mathcal{C}^\infty$ 1-forms on X, then

$$\int_X \eta_1 \wedge \eta_2 = \sum_{j=1}^{g}\left[\left(\int_{\alpha_j}\eta_1\right)\cdot\left(\int_{\beta_j}\eta_2\right) - \left(\int_{\beta_j}\eta_1\right)\cdot\left(\int_{\alpha_j}\eta_2\right)\right].$$

Hint. Choose the representatives $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ to be smooth Jordan curves as in part (b) of Theorem 5.16.1. For each $j = 1, \dots, g$, construct $\mathcal{C}^\infty$ annular neighborhoods $J_j \supset A_j$ and $T_j \supset B_j$, as in Lemma 5.11.1. Also choose the neighborhoods so that $(\overline{J}_i \cup \overline{T}_i) \cap (\overline{J}_j \cup \overline{T}_j) = \emptyset$ for $i \neq j$. Form dual closed 1-forms θ_j and τ_j with $\operatorname{supp}\theta_j \subset T_j \setminus B_j$ and $\operatorname{supp}\tau_j \subset J_j \setminus A_j$ for $j = 1, \dots, g$. Observe that for all i, j, $\int_X \theta_i \wedge \theta_j = \int_X \tau_i \wedge \tau_j = 0$, and for $i \neq j$, $\int_X \theta_i \wedge \tau_j = 0$. Show that θ_j has a potential λ_j on T_j (in particular, λ_j is locally constant near $B_j \cup \partial T_j$), and hence $\theta_j \wedge \tau_j = d(\lambda_j \tau_j)$ on T_j. Applying Stokes' theorem, prove that $\int_X \theta_j \wedge \tau_j = 1$, and then prove the claim.

(b) Prove that for every holomorphic 1-form η on X,

$$\|\eta\|_{L^2}^2 = 2\sum_{j=1}^{g}\operatorname{Im}\left[\left(\int_{\beta_j}\eta\right)\cdot\overline{\left(\int_{\alpha_j}\eta\right)}\right].$$

(c) Prove that there is a unique basis $\eta_1, \dots, \eta_g$ for $\Omega(X)$ such that for all $i, j = 1, \dots, g$,

$$\int_{\alpha_j}\eta_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(d) For the basis $\eta_1, \ldots, \eta_g$ in part (c), let $Z = (z_{ij})$ be the complex $g \times g$ matrix with entries

$$z_{ij} \equiv \int_{\beta_j} \eta_i \quad \forall i, j = 1, \ldots, g.$$

Prove that Z is symmetric (i.e., $z_{ij} = z_{ji}$ for all i and j) and $\operatorname{Im} Z = (\operatorname{Im} z_{ij})$ is positive definite (i.e., $\sum_{i,j=1}^{g} \operatorname{Im} z_{ij} u_i u_j > 0$ for all $(u_1, \ldots, u_g) \in \mathbb{R}^g \setminus \{0\}$).

Hint. For symmetry, first observe that $\eta_i \wedge \eta_j \equiv 0$ for all i and j. Then apply part (a). In order to prove that $\operatorname{Im} Z$ is positive definite, first prove that $\langle \eta_i, \eta_j \rangle_{L^2} = 2 \operatorname{Im} z_{ij}$ for all i and j.

5.16.4 Prove a $\mathcal{C}^\infty$ version of Theorem 5.16.1 for an oriented compact $\mathcal{C}^\infty$ surface M obtained by $\mathcal{C}^\infty$ oriented attachment of g tubes to a compact planar oriented $\mathcal{C}^\infty$ surface N (cf. Exercises 5.12.8 and 5.13.2).

5.17 Complements of Connected Closed Subsets of $\mathbb{P}^1$

The following fact will be applied in the construction of embeddings of open Riemann surfaces (in Sect. 5.18) and in the proof of Schönflies' theorem (in Sects. 6.7–6.9).

Lemma 5.17.1 *The complement and boundary of a simply connected domain in $\mathbb{P}^1$ are connected, and each connected component of the complement of any connected closed subset of $\mathbb{P}^1$ is simply connected.*

Remark Each connected component of the complement of a nonempty proper open subset of a topological surface must meet, and therefore contain, a boundary component of the set. Hence the complement is connected if the boundary is connected (it is easy to see that the converse is false).

Proof of Lemma 5.17.1 Let Ω be a nonempty domain in $\mathbb{P}^1$.

Suppose first that Ω is simply connected, and that A and B are disjoint (and possibly empty) subsets of $\partial\Omega$ that are open in $\partial\Omega$ and that satisfy $\partial\Omega = A \cup B$. Then $A = (\partial\Omega) \setminus B$ and $B = (\partial\Omega) \setminus A$ are also compact, so we may choose disjoint open subsets U and V of $\mathbb{P}^1$ with $A \subset U$ and $B \subset V$. In particular, $\partial U \cap \partial\Omega = \partial V \cap \partial\Omega = \emptyset$. The Riemann mapping theorem implies that Ω is equal to the union of an increasing sequence of relatively compact domains with connected boundary, so the compact set $(\partial U \cup \partial V) \cap \overline{\Omega} = (\partial U \cup \partial V) \cap \Omega$ is contained in such a domain Ω_0. Consequently, each of the sets $(\Omega \setminus \overline{\Omega_0}) \cap U = (\Omega \setminus \overline{\Omega_0}) \cap \overline{U}$ and $(\Omega \setminus \overline{\Omega_0}) \cap V = (\Omega \setminus \overline{\Omega_0}) \cap \overline{V}$ is both open and closed in the connected open set $\Omega \setminus \overline{\Omega_0}$. If $A \neq \emptyset$, then the first of the above sets is nonempty, and hence $\Omega \setminus \overline{\Omega_0} \subset U \subset \mathbb{P}^1 \setminus V$; so $B \subset (\partial\Omega) \cap V = \emptyset$. Thus $\partial\Omega$ is connected, and it follows that $\mathbb{P}^1 \setminus \Omega$ is also connected.

Assuming now that the domain Ω is a connected component of the complement $\mathbb{P}^1 \setminus K$ of a nonempty connected compact set $K \subsetneqq \mathbb{P}^1$, we will show that Ω must be simply connected. For this, suppose that θ is an arbitrary closed $\mathcal{C}^\infty$ 1-form on Ω. Given a loop γ in Ω, Proposition 5.4.1 provides a domain Ω_0 such that $\gamma([0,1]) \subset \Omega_0 \Subset \Omega$, $\partial\Omega_0$ is the union of disjoint smooth Jordan curves $C_1, \dots, C_m$, and for some $r > 1$ and for each $j = 1, \dots, m$, there is a local holomorphic chart $(U_j, \Phi_j, \Delta(0; 1/r, r))$ in Ω with $C_j = \Phi_j^{-1}(\partial\Delta(0;1))$ and $\Omega_0 \cap U_j = \Phi_j^{-1}(\Delta(0;1,r))$. Let E_j be the connected component of $\mathbb{P}^1 \setminus \Omega_0$ containing C_j for each j. Since C_j is a separating smooth Jordan curve in $\mathbb{P}^1$ (for example, by Proposition 5.15.2), it follows that $E_j \cap \overline{\Omega}_0 = C_j$ (if this were not the case, then there would exist a path in the set $(E_j \setminus C_j) \cup \Omega_0 \subset \mathbb{P}^1 \setminus C_j$ from $\Phi_j^{-1}(\Delta(0;1/r,1))$ to $\Phi_j^{-1}(\Delta(0;1,r))$). In particular, the connected components $E_1, \dots, E_m$ are distinct and therefore disjoint. Reordering so that $K \subset E_1$, we see that the set $\Omega_1 \equiv \Omega_0 \cup E_2 \cup \cdots \cup E_m$ is a smooth domain in $\mathbb{P}^1$ that lies in the connected component Ω of $\mathbb{P}^1 \setminus K$ and that satisfies $\partial\Omega_1 = C_1 \subset \Omega$ and $\Omega_1 \cap U_1 = \Phi_1^{-1}(\Delta(0;1,r))$.

Now Stokes' theorem implies that $\int_{\partial\Omega_1} \theta = \int_{\Omega_1} d\theta = 0$. Hence, since $\partial\Omega_1 = C_1 = \Phi_1^{-1}(\partial\Delta(0;1))$, the restriction of θ to U_1 must be exact. Forming a potential and cutting off, we get a $\mathcal{C}^\infty$ function ρ on Ω such that $d\rho = \theta$ on a neighborhood of $\partial\Omega_1$. Hence the 1-form τ that is equal to $\theta - d\rho$ at each point in Ω_1 and 0 at each point in $\mathbb{P}^1 \setminus \Omega_1$ is closed and of class $\mathcal{C}^\infty$, and therefore

$$\int_\gamma \theta = \int_\gamma (\theta - d\rho) = \int_\gamma \tau = 0.$$

It follows that θ is exact (on Ω), and therefore, by Corollary 5.2.7, Ω must be simply connected. $\square$

Exercises for Sect. 5.17

5.17.1 Two additional approaches to the proof of the second part of Lemma 5.17.1 (each connected component of the complement of any connected closed subset of $\mathbb{P}^1$ is simply connected) are outlined in parts (a) and (b) below. Let K be a connected closed subset of $\mathbb{P}^1$, and let Ω be nonempty connected component of $\mathbb{P}^1 \setminus K$.

(a) According to Sard's theorem (see, for example, [Mi]), the set of critical values of a $\mathcal{C}^\infty$ function on a second countable $\mathcal{C}^\infty$ manifold is a set of measure 0. Prove that Ω is simply connected by modifying the proof of Lemma 5.17.1 so that Sard's theorem, Theorem 9.10.1, and Lemma 5.11.1 are applied in place of Proposition 5.4.1.

(b) Suppose θ is a closed $\mathcal{C}^\infty$ 1-form on Ω. Applying Lemma 5.15.2, show that θ integrates to 0 along every smooth Jordan curve C in Ω, and applying Proposition 5.10.7, conclude that θ is exact. From this, conclude that Ω is simply connected.

5.17.2 A Riemann surface is said to be of *finite topological type* if its fundamental group is finitely generated. Prove that a Riemann surface X is of finite topological type if and only if X is biholomorphic to the complement $X \cong Y \setminus K$ in some compact Riemann surface Y of some (possibly empty) compact set $K \subset Y$ that is equal to a union of finitely many disjoint compact sets, each of which is either a singleton or a closed disk in some local holomorphic chart (X is said to be of *finite type* if K may be chosen to be a finite set).

Hint. Assuming that $\pi_1(X)$ is finitely generated, show that after holomorphic removal of finitely many tubes, one may assume that $X \subsetneqq \mathbb{P}^1$ (see Sect. 5.13). Suppose $\Omega \Subset X$ is a topologically Runge smooth domain, and $U_1, \dots, U_m$ are the connected components of $\mathbb{P}^1 \setminus \Omega$. Applying Proposition 5.15.2, prove that $\partial\Omega$ has exactly m connected components. Assuming that $\infty \in U_1$ and fixing a point $z_j \in U_j \setminus X$ for each $j = 2, \dots, m$, show that the closed $\mathcal{C}^\infty$ 1-forms $\theta_j = dz/(z - z_j)$ for $j = 2, \dots, m$ on X represent linearly independent cohomology classes, and from this, obtain a bound on m. Also show that $\Omega \cup U_1 \cup \cdots \cup \widehat{U_j} \cup \cdots \cup U_m$ is simply connected, and use this observation to obtain Y and K.

5.17.3 Let X be a compact Riemann surface of genus g; let $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ be smooth Jordan curves in X with the properties in part (b) of Theorem 5.16.1; let $A_j \equiv \alpha_j([0,1])$ and $B_j \equiv \beta_j([0,1])$ for $j = 1, \dots, g$; let K be a connected compact subset of $X \setminus [\bigcup_{j=1}^g (A_j \cup B_j)]$ with connected complement $X \setminus K$ in X; let $\{D_\nu\}_{\nu=1}^m$ be relatively compact smooth domains in $X \setminus [K \cup \bigcup_{j=1}^g (A_j \cup B_j)]$ with disjoint closures and connected boundaries; and for each $\nu = 1, \dots, m$, let K_ν be a connected compact subset of D_ν and γ_ν a smooth Jordan curve with $C_\nu \equiv \gamma_\nu([0,1]) = \partial D_\nu$ (cf. Exercise 5.17.2).

(a) Prove that the open set $\Omega \equiv X \setminus [K \cup \bigcup_{\nu=1}^m \overline{D}_\nu]$ is connected.

(b) Prove that if Y is the connected component of $X \setminus [K \cup \bigcup_{\nu=1}^m K_\nu]$ containing Ω, $p \in \Omega$, $\hat{\alpha}_j = \lambda_j * \alpha_j * \lambda_j^-$ and $\hat{\beta}_j = \lambda_j * \beta_j * \lambda_j^-$ for some path λ_j in Ω from p to $\alpha_j(0) = \beta_j(0)$ for $j = 1, \dots, g$, and $\hat{\gamma}_\nu = \eta_\nu * \gamma_\nu * \eta_\nu^-$ for some path η_ν in $\overline{\Omega} \setminus K$ from p to $\gamma_\nu(0)$ for $\nu = 1, \dots, m$, then the path homotopy classes of the loops $\hat{\alpha}_1, \dots, \hat{\alpha}_g, \hat{\beta}_1, \dots, \hat{\beta}_g, \hat{\gamma}_1, \dots, \hat{\gamma}_m$ in Y generate $\pi_1(Y, p)$. In particular, the path homotopy classes of the loops $\hat{\alpha}_1, \dots, \hat{\alpha}_g, \hat{\beta}_1, \dots, \hat{\beta}_g$ in X (in $X \setminus K$) generate $\pi_1(X, p)$ (respectively, generate $\pi_1(X \setminus K, p)$).

Hint. Set $K_0 \equiv K$. For the proof of (b), first consider the case $g = 0$. That is, for $X = \mathbb{P}^1$, prove that $[\hat{\gamma}_1], \dots, [\hat{\gamma}_m]$ generate $\pi_1(Y, p)$. For this, prove first that for each $\nu = 0, \dots, m$, K_ν has a small simply connected neighborhood in which the complement of K_ν is biholomorphic to a punctured disk or an annulus (the associated neighborhood of K_0 will be denoted by D_0). Deduce from this that fixing a point $z_\nu \in K_\nu$ for each ν, there is a continuous mapping of $X \setminus \{z_0, \dots, z_m\}$ into Y that fixes points in an arbitrarily large compact subset of Y. Prove also that on the other hand, every loop in Y with base point p is homotopic within $X \setminus \{z_0, \dots, z_m\}$ to a loop in $X \setminus \bigcup_\nu D_\nu$. Using these observations, reduce to the case in which $K_\nu = \{z_\nu\}$ for each ν.

5.17.4 Let X be a Riemann surface. The *commutator* of elements a and b of a group G is the element $[a, b] \equiv aba^{-1}b^{-1}$. The *commutator subgroup* of G is the subgroup generated by the set of commutators and is denoted by $[G, G]$.

(a) Prove that for any group G, $[G, G]$ is a normal subgroup of G and the quotient group $G/[G, G]$ is an Abelian group. Prove also that $[G, G]$ is the smallest normal subgroup of G with this property, that is, if H is any normal subgroup of G for which G/H is Abelian, then $H \supset [G, G]$.

(b) Prove that for any point $p \in X$, the map

$$\pi_1(X, p)\big/[\pi_1(X, p), \pi_1(X, p)] \to H_1(X, \mathbb{Z})$$

given by

$$[\gamma]_{\pi_1(X,p)} \cdot [\pi_1(X, p), \pi_1(X, p)] \mapsto [\gamma]_{H_1(X,\mathbb{Z})}$$

is a well-defined group isomorphism.

Hint. In exhausting smooth topologically Runge subdomains, construct generating loops as in Exercise 5.17.3 along with dual closed 1-forms. In order to obtain the dual closed 1-forms associated to the boundary Jordan curves, apply Lemma 5.15.1 and Exercise 5.15.1.

(c) Prove that for any subring $\mathbb{A}$ of $\mathbb{C}$ containing $\mathbb{Z}$, we have $H_1^{\Delta}(X, \mathbb{A}) \cong H_1(X, \mathbb{A})$ (see the remarks at the end of Sect. 10.7 for the definition of the singular homology group $H_1^{\Delta}(X, \mathbb{A})$, and the exercises for Sect. 10.7 for some of its properties).

(d) Prove that for any subring $\mathbb{A}$ of $\mathbb{C}$ containing $\mathbb{Z}$, the mapping

$$[\xi]_{H_1(X,\mathbb{A})} \mapsto ([\xi]_{H_1(X,\mathbb{A})}, \cdot)_{\mathrm{deR}}$$

gives an injective homomorphism $H_1(X, \mathbb{A}) \to \mathrm{Hom}(H^1(X, \mathbb{A}), \mathbb{A})$ (according to Theorem 10.7.18, this homomorphism is also surjective if $\pi_1(X)$ is finitely generated).

5.17.5 Let K be a compact subset of a Riemann surface X. Prove that there exist domains Ω and Ω' such that $K \subset \Omega \Subset \Omega' \Subset X$ and such that for every $\mathcal{C}^\infty$ closed 1-form ρ on Ω', there exist a $\mathcal{C}^\infty$ closed 1-form τ on X and a function $\lambda \in \mathcal{D}(\Omega')$ such that $\rho - \tau = d\lambda$ on Ω. Conclude from this that

$$\mathrm{im}[H^1_{\mathrm{deR}}(X) \to H^1_{\mathrm{deR}}(\Omega)] = \mathrm{im}[H^1_{\mathrm{deR}}(\Omega') \to H^1_{\mathrm{deR}}(\Omega)].$$

Hint. For a large smooth topologically Runge subdomain Ω', construct generating boundary loops along with dual closed 1-forms as in the above exercises. Also form a large subdomain Ω. For ρ as above, one can then form a $\mathcal{C}^\infty$ closed 1-form θ on X such that $\rho - \theta$ is exact near $\partial\Omega'$. Forming a potential near $\partial\Omega'$ and cutting off, one gets a function $\lambda \in \mathcal{D}(\Omega')$ such that $\rho - \theta - d\lambda$ vanishes near $\partial\Omega$. The restriction of $\rho - \theta - d\lambda$ to Ω then extends to a $\mathcal{C}^\infty$ closed 1-form τ_0 on X. The 1-form $\tau \equiv \tau_0 + \theta$ and the function λ then have the required properties.

5.18 Embedding of an Open Riemann Surface into $\mathbb{C}^3$

So far in this chapter, we have considered two characterizations of Riemann surfaces: as quotients of the planar domains $\mathbb{P}^1$, $\mathbb{C}$, and $\Delta(0; 1)$, and as planar domains to which tubes have been holomorphically attached. In the remaining sections of this chapter, we will consider other characterizations. Specifically, we will see in this section that every open Riemann surface admits a holomorphic embedding into $\mathbb{C}^3$, in Sect. 5.19 that every compact Riemann surface admits a holomorphic embedding into a higher-dimensional complex projective space, in Sect. 5.20 that every compact Riemann surface admits a finite holomorphic branched covering mapping onto $\mathbb{P}^1$, and in Sect. 5.22 that every compact Riemann surface of positive genus admits a holomorphic embedding into a higher-dimensional complex torus.

Definition 5.18.1 Let $F = (f_1, \dots, f_n)\colon X \to \mathbb{C}^n$ be a mapping of a complex 1-manifold X into some complex Euclidean space $\mathbb{C}^n$.

(a) F is *holomorphic* if f_j is a holomorphic function for each $j = 1, \dots, n$.
(b) F is a *holomorphic immersion* if for each point $p \in X$, we have $(df_j)_p \neq 0$ for some $j \in \{1, \dots, n\}$, that is, the linear map

$$(dF)_p \equiv ((df_1)_p, \dots, (df_n)_p)\colon (T_p X)^{1,0} \to \mathbb{C}^n$$

is nonsingular.
(c) F is called a *holomorphic embedding* if F is a proper injective holomorphic immersion.

The goal of this section is a proof of the following theorem of Narasimhan [Ns1]:

Theorem 5.18.2 *Every open Riemann surface X admits a* (*proper*) *holomorphic embedding into* $\mathbb{C}^3$.

Remarks 1. Given a holomorphic embedding $F\colon X \to \mathbb{C}^n$ of a Riemann surface X into $\mathbb{C}^n$, we may identify X with its image $Y \equiv F(X)$. In fact, it is a consequence of the theory of several complex variables that every holomorphic function on X is the restriction of some holomorphic function on $\mathbb{C}^n$ (see, for example, [Hö] or [KaK]).

2. The higher-dimensional analogue for Stein manifolds (and Stein spaces) was obtained independently by Bishop, Narasimhan, and Remmert. The proof given here for a Riemann surface is essentially Narasimhan's original proof (with some modifications), which may be generalized to higher-dimensional Stein manifolds [Ns2]. One may adapt Bishop's proof (see [Hö]) and Remmert's proof of the higher-dimensional case to the Riemann surface case.

3. Compact Riemann surfaces do *not* admit holomorphic embeddings into Euclidean spaces, since every holomorphic function on a compact Riemann surface is constant. However, every compact Riemann surface admits a holomorphic embedding into a *complex projective space* (see Sect. 5.19). Moreover, every compact Riemann surface of positive genus admits a holomorphic embedding into a higher-dimensional complex torus (see Sect. 5.22).

The main tool in the proof of the embedding theorem is the following *uniform* version of the Runge approximation theorem for a locally finite family of disjoint coordinate disks (see [Ns1]):

Theorem 5.18.3 *Let Ξ be an open subset of an open Riemann surface X such that Ξ is equal to the union of a locally finite family of disjoint relatively compact open subsets of X, each of which is biholomorphic to a disk. Then, for every holomorphic function g on Ξ, every closed subset K of X with $K \subset \Xi$, and every positive continuous function ρ on Ξ, there is a holomorphic function f on X such that $|f - g| < \rho$ on K.*

The idea of Narasimhan's proof of the embedding theorem is as follows. First, one forms a covering of the Riemann surface as in the following:

Proposition 5.18.4 *In any open Riemann surface X, there exist three nonempty open subsets Ξ_0, Ξ_1, and Ξ_2 such that $X = \Xi_0 \cup \Xi_1 \cup \Xi_2$ and such that for $j = 0, 1, 2$, Ξ_j is the union of a locally finite family of disjoint relatively compact open subsets of X, each of which is biholomorphic to a disk.*

By applying Theorem 5.18.3 in order to uniformly approximate a suitable locally constant function on Ξ_j for each $j = 0, 1, 2$, one gets a proper holomorphic mapping of X into $\mathbb{C}^3$. A Baire category argument then provides an approximation of this mapping that is a holomorphic embedding.

Our first goal is the proof of Theorem 5.18.3. Recall that the topological hull $\mathfrak{h}_Y(A)$ of a subset A of a Hausdorff space Y is the union of A with all of the connected components of $Y \setminus A$ that are relatively compact in Y (see Definition 2.13.1).

Lemma 5.18.5 *Let X be a connected, locally connected, locally compact Hausdorff space; let K_0 be a compact subset of X; and let $\{C_\nu\}_{\nu=1}^{\infty}$ be a locally finite family of disjoint connected compact subsets of X. Then there exists a compact subset K of X such that $\mathfrak{h}_X(K) = K$, $K_0 \subset \overset{\circ}{K}$, and for each ν, $C_\nu \subset K$ or $C_\nu \cap K = \emptyset$.*

Proof By local finiteness, there exists a positive integer μ such that $C_\nu \cap K_0 = \emptyset$ for each $\nu > \mu$. Thus we may choose a relatively compact neighborhood U of the compact set $K_0 \cup \bigcup_{\nu=1}^{\mu} C_\nu$ in the open set $X \setminus \bigcup_{\nu=\mu+1}^{\infty} C_\nu$. In particular, for each ν, C_ν is contained in U, or in a connected component of $X \setminus \overline{U}$ that is relatively compact in X, or in a connected component of $X \setminus \overline{U}$ that is *not* relatively compact in X. Lemma 2.13.2 and Lemma 2.13.3 together now imply that the set $K \equiv \mathfrak{h}_X(\overline{U})$ has the required properties. $\square$

Lemma 5.18.6 *Let $\{K_\nu\}_{\nu=0}^{m}$ be a family of disjoint compact subsets of a topological surface M such that $\mathfrak{h}_M(K_0) = K_0$ and such that for each index $\nu = 1, \dots, m$, there is a local chart on a neighborhood of K_ν that maps K_ν onto a closed disk. Then the compact set $K \equiv \bigcup_{\nu=0}^{m} K_\nu$ satisfies $\mathfrak{h}_M(K) = K$.*

Proof If U is a connected component of $M \setminus K_0$, then U is not relatively compact in M, and hence by Lemma 5.11.4, $U \setminus (K_1 \cup \cdots \cup K_m)$ is a connected open set that is not relatively compact in X. Thus each of the connected components of $M \setminus K$ is of this form, and therefore $\mathfrak{h}_M(K) = K$. □

For the proof of Theorem 5.18.3, we will apply the L^2 $\bar{\partial}$-method with a weight function provided by the following:

Lemma 5.18.7 *Let X be an open Riemann surface. Suppose that $\Xi \subset X$ is the union of a locally finite family of disjoint relatively compact open subsets of X, each of which is biholomorphic to a disk, ω is a positive continuous $(1,1)$-form on X, ρ is a positive continuous function on X, and K is a closed subset of X with $K \subset \Xi$. Then there exists a $\mathcal{C}^\infty$ strictly subharmonic function φ on X such that*

(i) *On X, $i\Theta_\varphi \geq \omega$;*
(ii) *On K, $\varphi < -\rho$; and*
(iii) *On $X \setminus \Xi$, $\varphi > \rho$.*

Proof We may assume without loss of generality that Ξ has infinitely many connected components $\{\Xi_\nu\}_{\nu=1}^\infty$. Replacing each of these connected components with a slightly smaller coordinate disk and replacing K with a suitable (larger) subset of Ξ, we may also assume that the components of Ξ have disjoint closures and that for each ν, the biholomorphism of Ξ_ν onto a disk extends to a biholomorphism on a neighborhood of $\overline{\Xi}_\nu$, and the image of $K \cap \Xi_\nu$ is a closed disk. We may also fix a neighborhood Ω of K with $\overline{\Omega} \subset \Xi$.

By applying Lemma 5.18.5 inductively (together with Lemma 9.3.6 and Radó's theorem), we get a sequence of nonempty compact sets $\{K_m\}_{m=1}^\infty$ such that $\bigcup_{m=1}^\infty K_m = X$ and $\overline{\Xi}_1 \subset K_1$, and such that for each m, we have $\mathfrak{h}_X(K_m) = K_m$, $K_m \subset \overset{\circ}{K}_{m+1}$, each connected component $\overline{\Xi}_\nu$ of $\overline{\Xi}$ lies either in K_m or in $X \setminus K_m$, and $\overline{\Xi} \cap (K_{m+1} \setminus K_m) \neq \emptyset$. Thus by reordering, we may assume that for some strictly increasing sequence of positive integers $\{\nu_m\}$, we have, for each m,

$$\overline{\Xi}_1 \cup \cdots \cup \overline{\Xi}_{\nu_m} \subset K_m \quad \text{and} \quad X \setminus K_m \supset \bigcup_{\nu=\nu_m+1}^{\infty} \overline{\Xi}_\nu.$$

Set $K_0 = \Xi_0 \equiv \emptyset$, $\nu_0 \equiv 0$, and $\Gamma_m \equiv \bigcup_{\nu=\nu_{m-1}+1}^{\nu_m} \Xi_\nu \subset K_m \setminus K_{m-1}$ for each $m = 1, 2, 3, \ldots$. Theorem 2.14.1 provides a $\mathcal{C}^\infty$ strictly subharmonic function φ_0 on X such that $i\Theta_{\varphi_0} \geq \omega$ and $\varphi_0 > \rho$. Given $\mathcal{C}^\infty$ subharmonic functions $\varphi_0, \ldots, \varphi_{m-1}$ on X, we may choose a real-valued $\mathcal{C}^\infty$ function α_m on X such that $\operatorname{supp} \alpha_m \subset \Gamma_m$ and on $\Omega \cap \Gamma_m$, $\alpha_m + \varphi_0 + \cdots + \varphi_{m-1} < -\rho$ and α_m is subharmonic. Lemma 5.18.6 and Theorem 2.14.1 provide a nonnegative $\mathcal{C}^\infty$ subharmonic function ψ_m on X such that $\psi_m \equiv 0$ on $K_{m-1} \cup (K \cap \Gamma_m)$, and $\psi_m + \alpha_m \geq 0$ and $i\Theta_{\psi_m + \alpha_m} \geq 0$ on $\Gamma_m \setminus \Omega$. Thus the $\mathcal{C}^\infty$ function $\varphi_m \equiv \psi_m + \alpha_m$ is subharmonic on X, $\varphi_m \geq 0$ on $X \setminus \Xi$, $\varphi_m \equiv 0$ on K_{m-1}, and by induction, $\sum_{j=0}^m \varphi_j < -\rho$ on $K \cap \bigcup_{\nu=1}^{\nu_m} \Xi_\nu$. Thus we get a sequence of functions $\{\varphi_m\}$, and it follows that the locally finite sum $\sum_{m=0}^{\infty}\varphi_m$ determines a $\mathcal{C}^\infty$ strictly subharmonic function φ with the required properties. $\square$

Proof of Theorem 5.18.3 We may assume without loss of generality that Ξ has infinitely many connected components $\{\Xi_\nu\}_{\nu=1}^{\infty}$. Hence we have local holomorphic charts $\{(\Xi_\nu, \Phi_\nu, \Delta(0;1))\}_{\nu=1}^{\infty}$, and we may fix nonempty neighborhoods Ω and D of K in Ξ such that $\overline{\Omega}\subset D$ and, for each ν, $\Phi_\nu(D\cap\Xi_\nu)$ is a relatively compact disk in $\Delta(0;1)$. We may also fix a $\mathcal{C}^\infty$ function τ on X such that $\operatorname{supp}\tau\subset\Xi$ and $\tau\equiv g$ on D. In particular, the support of the $\mathcal{C}^\infty$ $(0,1)$-form $\gamma\equiv\bar\partial\tau$ lies in $\Xi\setminus D$.

Fixing a Kähler form ω on X and applying Theorem 2.14.1, we get a $\mathcal{C}^\infty$ strictly subharmonic function φ_0 on X such that $i\Theta_\omega + i\Theta_{\varphi_0}\geq\omega$. By Theorem 2.6.4, for each ν, there exists a constant $\delta_\nu\in(0,1)$ such that $|h|<\rho$ on $K\cap\Xi_\nu$ for every function $h\in\mathcal{O}(\Omega\cap\Xi_\nu)$ with $\|h\|_{L^2(\Omega\cap\Xi_\nu,\omega,\varphi_0)}<\delta_\nu$. By applying Lemma 5.18.7, we may form a $\mathcal{C}^\infty$ subharmonic function φ_1 on X such that for each ν, $\varphi_1<2\log\delta_\nu$ on $\Omega\cap\Xi_\nu$, and for $\varphi\equiv\varphi_0+\varphi_1$,

$$\|\gamma\|^2_{L^2(\Xi_\nu,\varphi)} = \|\gamma\|^2_{L^2(\Xi_\nu\setminus D,\varphi)} < 2^{-\nu}.$$

In particular, $\|\gamma\|^2_{L^2(X,\varphi)} = \sum_{\nu=1}^{\infty}\|\gamma\|^2_{L^2(\Xi_\nu,\varphi)} < 1$, and by Corollary 2.12.6, there exists a $\mathcal{C}^\infty$ function α on X such that $\bar\partial\alpha=\gamma$ and $\|\alpha\|_{L^2(X,\omega,\varphi)}<1$. The function $f\equiv\tau-\alpha$ is then a holomorphic function on X. Furthermore, since $\tau=g$ on D, α must be holomorphic on $D\supset\Omega$, and for each ν,

$$\|\alpha\|_{L^2(\Omega\cap\Xi_\nu,\omega,\varphi_0)} \leq \delta_\nu\cdot\|\alpha\|_{L^2(\Omega\cap\Xi_\nu,\omega,\varphi)} \leq \delta_\nu\cdot\|\alpha\|_{L^2(X,\omega,\varphi)} < \delta_\nu.$$

It follows that $|f-g|=|\alpha|<\rho$ on K. $\square$

We now turn to the construction of the three subsets in Proposition 5.18.4. For this, we form a graph that divides X into small simply connected sets (actually, as the reader is asked to prove in Exercise 6.11.4, X admits a triangulation; but we will not use this more general fact). The three sets then consist of a union of small neighborhoods of the vertices in the graph, a union of small neighborhoods of the open edges, and finally, the complement of the graph.

For the construction of the graph, we will need some elementary facts, the first of which is the following analogue of the identity theorem (Corollary 1.3.3) for real analytic functions:

Lemma 5.18.8 *If the zero set $Z=f^{-1}(0)$ of a real-valued real analytic function f on an open interval $I\subset\mathbb{R}$ has a limit point in I, then $f\equiv 0$.*

Proof Given a point $a\in Z$, for some $\epsilon>0$, the function f may be written as a power series $x\mapsto f(x)=\sum_{k=0}^{\infty}c_k(x-a)^k$ on the interval $(a-\epsilon,a+\epsilon)\subset I$. This series converges absolutely, so we may define a holomorphic function g on the disk

$\Delta(a;\epsilon)$ in $\mathbb{C}$ by

$$g(z)=\sum_{k=0}^{\infty}c_k(z-a)^k \quad \forall z\in\Delta(a;\epsilon).$$

If a is a limit point of Z, then the identity theorem implies that g is constant. Thus the interior $\overset{\circ}{Z}$ is nonempty. The above also implies that the boundary of the nonempty open set $\overset{\circ}{Z}$ does not meet I, and hence that $Z=\overset{\circ}{Z}=I$. □

The desired graph will be constructed as the union of coordinate circles and line segments, and the following lemma will give local finiteness of intersections:

Lemma 5.18.9 *Suppose $f\colon\Omega_0\to\Omega_1$ is a biholomorphism of domains Ω_0 and Ω_1 in $\mathbb{C}$, and for each $j=0,1$, $z_j\in\mathbb{C}$, $r_j>0$, and $\Delta(z_j;r_j)\Subset\Omega_j$. Then the intersection $f(\partial\Delta(z_0;r_0))\cap\partial\Delta(z_1;r_1)$ is infinite if and only if $f(\partial\Delta(z_0;r_0))=\partial\Delta(z_1;r_1)$.*

Proof Let $\rho\colon\mathbb{R}\to\mathbb{C}$ be the periodic function $t\mapsto|f(z_0+r_0e^{2\pi it})-z_1|$. If the above intersection is infinite, then the set $\rho^{-1}(r_1)$ has a limit point $t_0\in\mathbb{R}$. On the other hand, on some neighborhood of t_0, the function $\log\rho$ is the restriction to $\mathbb{R}$ of the real part of the composition of a holomorphic branch of the logarithmic function with the holomorphic function $\zeta\mapsto f(z_0+r_0e^{2\pi i\zeta})-z_1$, and hence $\log\rho$ is real analytic (in fact, the composition of real analytic functions is real analytic, so one can show that ρ is itself real analytic near t_0). Lemma 5.18.8 then implies that ρ is constant near t_0, and hence the interior of the set $\rho^{-1}(r_1)$ in $\mathbb{R}$ is nonempty; and, moreover, this interior has no boundary points. Thus $\rho\equiv r_1$ on $\mathbb{R}$, and hence $f(\partial\Delta(z_0;r_0))\subset\partial\Delta(z_1;r_1)$. The reverse containment follows by symmetry. □

In order to get simply connected neighborhoods of the edges in a graph, we will apply the following elementary fact:

Lemma 5.18.10 *If I is an open interval in $\mathbb{R}$ and W is a neighborhood of the set $I\times\{0\}$ in $\mathbb{R}^2$, then there exists a simply connected neighborhood V of $I\times\{0\}$ in $W\cap(I\times\mathbb{R})$ such that $V\cap(\mathbb{R}\times\{0\})=I\times\{0\}$ and $V\setminus\mathbb{R}=V\setminus(I\times\{0\})$ has exactly two connected components*:

$$V_-\equiv\{(x,y)\in V\mid y<0\}\quad \textit{and}\quad V_+\equiv\{(x,y)\in V\mid y>0\},$$

both of which are nonempty. In particular, $I\times\{0\}=(I\times\mathbb{R})\cap\partial V_-\cap\partial V_+$.

Proof The sets

$$V_-\equiv\{(x,y)\in\mathbb{R}^2\mid x\in I,\, y<0,\ \text{and}\ \{x\}\times[y,0]\subset W\}$$

and

$$V_+\equiv\{(x,y)\in\mathbb{R}^2\mid x\in I,\ y>0,\ \text{and}\ \{x\}\times[0,y]\subset W\}$$

are open connected sets. For if $(x_1, y_1), (x_2, y_2) \in V_-$ with $x_1 < x_2$, then for $\epsilon > 0$ sufficiently small, we have $-\epsilon > y_j$ for $j = 1, 2$, and the connected set

$$(\{x_1\} \times [y_1, 0)) \cup ([x_1, x_2] \times \{-\epsilon\}) \cup (\{x_2\} \times [y_2, 0))$$

lies in V_-. Similar arguments show that V_+ is also connected, and that V_- and V_+ are open.

The set $V \equiv V_- \cup V_+ \cup (I \times \{0\}) \subset W \cap (I \times \mathbb{R})$ is a connected neighborhood of $I \times \{0\}$ with $V \cap \mathbb{R} = I \times \{0\}$, because each point in $I \times \{0\}$ has an open rectangular neighborhood of the form $J \times (-\delta, \delta) \subset W$ for some open interval $J \subset I$ and some $\delta > 0$, and hence this neighborhood is contained in V and meets both V_- and V_+. Finally, the set V is simply connected because if $\alpha = (u, v)$ is a loop in V with base point $(t_0, 0) \in I \times \{0\}$, then the map

$$(t, s) \mapsto (u(t), (1 - s)v(t))$$

is a path homotopy in V from α to a loop β in $I \times \{0\}$, and the map

$$(t, s) \mapsto (u(t) + s(t_0 - u(t)), 0)$$

is a path homotopy in $I \times \{0\}$ from β to the trivial loop. □

Proof of Proposition 5.18.4 We first construct a suitable graph. Since X is second countable (by Radó's theorem), Lemma 9.3.6 provides a locally finite sequence of local holomorphic charts $\{(U_\nu, \Phi_\nu, U'_\nu)\}$ such that

(i) For each $\nu \in \mathbb{Z}_{>0}$, we have $\Delta \equiv \Delta(0; 1) \Subset U'_\nu$;
(ii) The family $\{D_\nu\} \equiv \{\Phi_\nu^{-1}(\Delta)\}$ is a (locally finite) covering of X; and
(iii) If $\mu, \nu \in \mathbb{Z}_{>0}$ with $\overline{D}_\mu \cap \overline{D}_\nu \neq \emptyset$, then $D_\mu \Subset U_\nu$.

In particular, the subset $M_0 \equiv \bigcup_{\nu=1}^\infty \partial D_\nu$ of X is closed, and by Lemma 5.18.9, the set $\partial D_\mu \cap \partial D_\nu$ is finite for all $\mu, \nu \in \mathbb{Z}_{>0}$ for which $\partial D_\mu \neq \partial D_\nu$. We will now construct additional sets so that the union becomes a connected set that has connected intersection with each of the sets $\{\overline{D}_\nu\}$.

For all $\mu, \nu \in \mathbb{Z}_{>0}$, the set $D_\nu \cap \partial D_\mu$ has only finitely many connected components, because each nonempty connected component is the image under Φ_μ^{-1} of an open arc of the circle $\mathbb{S}^1 = \partial\Delta$, and any endpoint of the arc must map into the finite set $\partial D_\mu \cap \partial D_\nu$. It follows that for each ν, the set $M_0 \cap \overline{D}_\nu$ has only finitely many connected components (and each of these connected components is compact). Setting $L_0 = \emptyset$, we now construct sequences of closed sets $\{M_k\}_{k=0}^\infty$ and compact sets $\{L_k\}_{k=0}^\infty$ inductively as follows. Suppose we have constructed closed sets $M_0, \dots, M_{k-1}$ and compact sets $L_1, \dots, L_{k-1}$ so that for each $j = 1, \dots, k-1$, $M_j = M_{j-1} \cup L_j$ and $M_j \cap \overline{D}_\nu$ has only finitely many connected components for each ν. If $M_{k-1} \cap \overline{D}_\nu$ is connected for each ν, then we set $L_k = \emptyset$ and $M_k = M_{k-1}$. Otherwise, letting ν be the minimal index in $\mathbb{Z}_{>0}$ for which the set $M_{k-1} \cap \overline{D}_\nu$ is not connected, we may choose a set $L_k \subset \overline{D}_\nu$ such that $\Phi_\nu(L_k)$ is the line segment from some point in a connected component A of $\Phi_\nu(M_{k-1} \cap \overline{D}_\nu)$ to a point

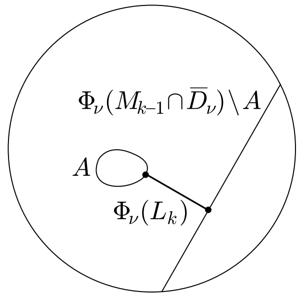

Fig. 5.21 Local construction of the graph

in $\Phi_\nu(M_{k-1} \cap \overline{D}_\nu) \setminus A$ with length $\ell(\Phi_\nu(L_k)) = \operatorname{dist}(A, \Phi_\nu(M_{k-1} \cap \overline{D}_\nu) \setminus A)$ (see Fig. 5.21). Thus the set $M_k \equiv M_{k-1} \cup L_k$ is closed, and the number of connected components of $M_k \cap \overline{D}_\nu$ is strictly less than that of $M_{k-1} \cap \overline{D}_\nu$. Observe also that for any $\mu \in \mathbb{Z}_{>0}$ with $L_k \cap \overline{D}_\mu \neq \emptyset$, either $L_k \subset D_\mu$ (and hence the endpoints of L_k lie in $M_{k-1} \cap D_\mu$) or every connected component of $L_k \cap \overline{D}_\mu$ meets ∂D_μ (and hence meets $M_0 \cap \overline{D}_\mu$). Thus, in either case, the number of connected components of $M_k \cap \overline{D}_\mu$ is not more than that of $M_{k-1} \cap \overline{D}_\mu$.

Thus we get sequences of sets $\{M_k\}$ and $\{L_k\}$, and it follows from the construction that for each ν, the set $M_k \cap \overline{D}_\nu$ is connected for each $k \gg 0$. Moreover, by local finiteness of the family $\{D_\nu\}$, the family of compact sets $\{L_k\}$ is locally finite, and hence for every compact S, $M_{k+1} \cap S = M_k \cap S$ for $k \gg 0$. Thus the set

$$X_1 \equiv \bigcup_{k=1}^{\infty} M_k = M_0 \cup \bigcup_{k=1}^{\infty} L_k$$

is a closed subset of X, and $X_1 \cap \overline{D}_\nu$ is connected for each ν. We may also choose a discrete subset X_0 of X_1 such that X_0 meets ∂D_ν for each ν, X_0 contains $\partial D_\mu \cap \partial D_\nu$ whenever $\mu, \nu \in \mathbb{Z}_{>0}$ with $\partial D_\mu \neq \partial D_\nu$, and X_0 contains the endpoints of L_k for each k. In particular, the collection of connected components of $X_1 \setminus X_0$ is locally finite in X, and each connected component is relatively compact in X. Observe also that for each k, in some local holomorphic chart, L_k is a line segment whose interior points do not meet $M_{k-1} \supset M_0$. Thus the sets $L_k \cap \partial D_\nu$ for each ν, and $L_k \cap L_j$ for each $j < k$, must consist of endpoints of L_k (or be empty), and hence these sets must be contained in X_0. Furthermore, since every disk is mapped onto the upper half-plane by some automorphism of $\mathbb{P}^1$ (Theorem 5.8.3), the closure of each connected component L of $X_1 \setminus X_0$ is contained in some local holomorphic coordinate neighborhood for which L is mapped onto an open interval in $\mathbb{R}$.

Now Lemma 9.3.6 provides a neighborhood Ξ_0 of the discrete set X_0 in X such that Ξ_0 is a union of a locally finite family of disjoint relatively compact open subsets of X, each of which is biholomorphic to a disk and contains exactly one point from X_0 (see Fig. 5.22). Lemma 9.3.6, Lemma 5.18.10, and the Riemann mapping theorem in the plane (Theorem 5.2.1) provide a neighborhood Ξ_1 of $X_1 \setminus X_0$ in $X \setminus X_0$ such that Ξ_1 is a union of a locally finite family of disjoint relatively

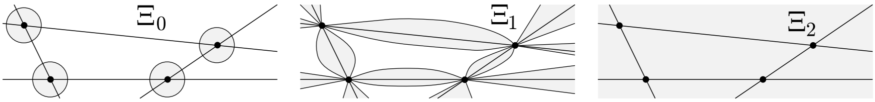

Fig. 5.22 The three desired open sets

compact connected open subsets of X, and such that if V is one of these subsets (i.e., one of the connected components of Ξ_1), then V is biholomorphic to a disk, V contains *exactly one* connected component L of $X_1 \setminus X_0$, and $V \setminus L$ has exactly two connected components (see Fig. 5.22).

We will now show that the third set may be taken to be the open set $\Xi_2 \equiv X \setminus X_1$ (see Fig. 5.22). Given a connected component Ω of Ξ_2, we have $\Omega \cap D_\nu \neq \emptyset$ for some ν. Since $\partial D_\nu \subset X_1$, we must have $\Omega \subset D_\nu \setminus X_1 \Subset X$. Thus $\Phi_\nu(\Omega) \subset \Delta$ is a connected component of the complement of the connected compact set $\Phi_\nu(X_1 \cap \overline{D}_\nu)$ in $\mathbb{P}^1$. Therefore, Lemma 5.17.1 and the Riemann mapping theorem in the plane imply that Ω is biholomorphic to a disk. Furthermore, $\partial\Omega$ is a connected subset of $X_1 \cap \overline{D}_\nu$ that cannot be a single point (for if $p \in \partial\Omega$, then $\mathbb{P}^1 \setminus \{\Phi_\nu(p)\}$ is a connected open set that meets, but is not contained in, $\Phi_\nu(\Omega) \subset \Delta$, so $\mathbb{P}^1 \setminus \{\Phi_\nu(p)\}$ must meet $\partial\Phi_\nu(\Omega) = \Phi_\nu(\partial\Omega)$). Hence $\partial\Omega$ meets some connected component L of $X_1 \setminus X_0$ with $L \subset \overline{D}_\nu$. On the other hand, if V is the connected component of Ξ_1 containing L, then $V \setminus L = V_- \cup V_+$, where V_- and V_+ are nonempty disjoint connected open subsets of $X \setminus X_1$, and hence Ω must contain V_- or V_+. It follows that the collection of connected components of Ξ_2 must be locally finite in X. $\square$

Remarks 1. As is clear from the above proof, the closure of each connected component of $X_1 \setminus X_0$ is the image of a smooth injective path with endpoints in X_0. Thus the set X_1 may be viewed as a *graph* embedded in X with vertices equal to the points of X_0.

2. The above decomposition of X into manifolds of increasing dimension X_0, $X_1 \setminus X_0$, $X \setminus X_1$ (i.e., $X_2 \setminus X_1$ for $X_2 = X$) is an example of a *stratification* of X.

3. One may also push the construction further to get a *triangulation* of X (see Exercise 6.11.4).

4. The set $X_1 \setminus X_0$ is a *real analytic submanifold* of $X \setminus X_0$. On the other hand, for our purposes, it is not really necessary that the local charts, in which the connected components of $X_1 \setminus X_0$ become intervals, be holomorphic. Continuous local charts would suffice, and in fact, the existence of suitable local $\mathcal{C}^\infty$ charts would follow from smoothness of the components, Theorem 9.10, and a natural analogue of Lemma 5.11.1. However, that one may choose the local charts to be holomorphic follows directly from the construction.

The above facts will allow us to produce a proper holomorphic mapping into $\mathbb{C}^3$, and the following lemma will allow us to approximate the mapping by an embedding:

Lemma 5.18.11 *Let X be an open Riemann surface, and let ω be a Kähler form on X. Then there exists a Kähler form ω' on X such that every $\mathcal{C}^\infty$ function φ on X with $i\Theta_\varphi \geq \omega'$ has the following properties*:

(i) *For every pair of distinct points $p, q \in X$ and every constant $\zeta \in \mathbb{C}$, there exists a function $f \in \mathcal{O}(X) \cap L^2(X, \omega, \varphi)$ with $f(p) - f(q) = \zeta$.*
(ii) *For every point $p \in X$ and every cotangent vector $\eta \in \Lambda^{1,0}T^*_p X$, there exists a function $f \in \mathcal{O}(X) \cap L^2(X, \omega, \varphi)$ with $(df)_p = \eta$.*

Proof We may fix a locally finite collection of local holomorphic charts

$$\{(D_\nu, z_\nu = \Phi_\nu, \Delta(0;3))\}_{\nu=1}^\infty$$

with $X = \bigcup_\nu \Phi_\nu^{-1}(\Delta(0;1))$, and a nonnegative function $\lambda \in \mathcal{D}(\Delta(0;3))$ with $\lambda \equiv 1$ on $\Delta(0;2)$. For each point $p \in \Phi_\nu^{-1}(\Delta(0;1))$, we have

$$\begin{aligned} &i\partial\bar{\partial}[\lambda(z_\nu)\log|z_\nu - z_\nu(p)|^2] \\ &\quad = \left(\frac{\partial^2\lambda}{\partial z\partial\bar{z}}(z_\nu)\log|z_\nu - z_\nu(p)|^2 + 2\,\mathrm{Re}\left[\frac{(\partial\lambda/\partial z)(z_\nu)}{\bar{z}_\nu - \bar{z}_\nu(p)}\right]\right) i\,dz_\nu \wedge d\bar{z}_\nu \geq -C_\nu\omega \end{aligned}$$

on $D_\nu \setminus \{p\}$, where the constant

$$C_\nu \equiv \left((\log 16)\max\left|\frac{\partial^2\lambda}{\partial z\partial\bar{z}}\right| + 2\max\left|\frac{\partial\lambda}{\partial z}\right|\right) \cdot \max_{\Phi_\nu^{-1}(\mathrm{supp}\,\lambda)} \frac{i\,dz_\nu \wedge d\bar{z}_\nu}{\omega}$$

is independent of the choice of the point p. Moreover, a standard partition of unity argument yields a $\mathcal{C}^\infty$ function ρ on X such that $\rho > 1 + 2C_\nu + |\Theta_\omega/\omega|$ on D_ν for each ν, and we may set $\omega' \equiv \rho \cdot \omega$.

Suppose now that φ is a $\mathcal{C}^\infty$ function on X with $i\Theta_\varphi \geq \omega'$. Given two (not necessarily distinct) points $p, q \in X$ and a constant $\zeta \in \mathbb{C}$, we may fix indices μ and ν with $p \in \Phi_\mu^{-1}(\Delta(0;1))$ and $q \in \Phi_\nu^{-1}(\Delta(0;1))$ (and with $\mu = \nu$ if $p = q$) and a function $\tau \in \mathcal{D}(X)$ such that $\tau \equiv \zeta$ on a neighborhood of p and $\tau \equiv 0$ on a neighborhood of q if $p \neq q$, and $\tau \equiv \zeta \cdot (z_\mu - z_\mu(p))$ on a neighborhood of p if $p = q$. The $\mathcal{C}^\infty$ $(0,1)$-form $\gamma \equiv \bar{\partial}\tau$ then has compact support in the Riemann surface $Y \equiv X \setminus \{p, q\}$, and the $\mathcal{C}^\infty$ function

$$\psi \equiv \varphi + \lambda(z_\mu)\log|z_\mu - z_\mu(p)|^2 + \lambda(z_\nu)\log|z_\nu - z_\nu(q)|^2$$

on Y satisfies

$$i\Theta_\omega + i\Theta_\psi \geq \omega.$$

Therefore, by Corollary 2.12.6, there exists a $\mathcal{C}^\infty$ function α on Y such that $\bar{\partial}\alpha = \gamma$ and $\|\alpha\|_{L^2(Y,\omega,\psi)} < \infty$. In particular, α is holomorphic near p and q. Moreover, if $p \neq q$, then for some constant $C > 0$, we have

$$\int_{\Phi_\mu^{-1}(\Delta(0;1))} |\alpha/(z_\mu - z_\mu(p))|^2 i\,dz_\mu \wedge d\bar{z}_\mu \leq C \cdot \|\alpha\|^2_{L^2(\Phi_\mu^{-1}(\Delta(0;1)),\omega,\psi)} < \infty,$$

and the analogous inequality holds near q. Therefore, by Riemann's extension theorem (Theorem 1.2.10), α extends to a $\mathcal{C}^\infty$ function on X (which we will also denote by α) that vanishes at p and q. Thus $f \equiv \tau - \alpha$ is then a holomorphic function on X, $f(p) - f(q) = \zeta$, and since $\varphi = \psi$ on $X \setminus (\Phi_\mu^{-1}(\operatorname{supp}\lambda) \cup \Phi_\nu^{-1}(\operatorname{supp}\lambda)) \subset Y$, we have $f \in L^2(X, \omega, \varphi)$. Thus the property (i) holds. If $p = q$ and $\eta = \zeta \cdot (dz_\mu)_p$, then

$$\int_{\Phi_\mu^{-1}(\Delta(0;1))} |\alpha/(z_\mu - z_\mu(p))^2|^2 i\, dz_\mu \wedge d\bar{z}_\mu \le C \cdot \|\alpha\|^2_{L^2(\Phi_\mu^{-1}(\Delta(0;1)),\omega,\psi)} < \infty$$

for some constant $C > 0$, and hence α extends to a $\mathcal{C}^\infty$ function on X (which we will also denote by α) that is holomorphic near p with a zero of order at least 2 at p. Hence $f \equiv \tau - \alpha \in L^2(X, \omega, \varphi)$ is then a holomorphic function on X satisfying $(df)_p = \eta$, and the property (ii) also holds. □

Lemma 5.18.12 *Let X be an open Riemann surface, let Ξ be an open subset that is equal to the union of a locally finite family of disjoint relatively compact open subsets of X, each of which is biholomorphic to a disk, let K be a closed subset of X with $K \subset \Xi$, let C be a countable subset of X, let $\eta_p \in \Lambda^{1,0}T_p^*X$ for each point $p \in C$, let D be a countable set of pairs of distinct points (p, q) in X, let $\zeta_{p,q} \in \mathbb{C}$ for each pair $(p, q) \in D$, and let ρ be a positive continuous function on X. Then there exists a holomorphic function f on X such that $|f| < \rho$ on K, $(df)_p \neq \eta_p$ for each point $p \in C$, and $f(p) - f(q) \neq \zeta_{p,q}$ for each pair $(p, q) \in D$.*

Proof We may assume without loss of generality that Ξ has infinitely many connected components $\{\Xi_\nu\}_{\nu=1}^\infty$, we may fix a neighborhood Ω of K with $\overline{\Omega} \subset \Xi$, and we may fix Kähler forms ω and ω' on X with the properties described in Lemma 5.18.11. Theorem 2.14.1 then provides a $\mathcal{C}^\infty$ strictly subharmonic function α on X with $i\Theta_\alpha \ge \omega'$. By Theorem 2.6.4, for each ν, there exists a constant $\delta_\nu \in (0, 1)$ such that $|h| < \rho$ on $K \cap \Xi_\nu$ for every function $h \in \mathcal{O}(\Omega \cap \Xi_\nu)$ with $\|h\|_{L^2(\Omega\cap\Xi_\nu,\omega,\alpha)} < \delta_\nu$. Moreover, Lemma 5.18.7 provides a $\mathcal{C}^\infty$ subharmonic function β on X such that $\beta < 2\log\delta_\nu$ on $\Omega \cap \Xi_\nu$ for each $\nu = 1, 2, 3, \ldots$. Setting $\varphi \equiv \alpha + \beta$, we see that if $h \in \mathcal{O}(X)$ and $\|h\|_{L^2(X,\omega,\varphi)} < 1$, then for each ν,

$$\|h\|_{L^2(\Omega\cap\Xi_\nu,\omega,\alpha)} \le \delta_\nu \cdot \|h\|_{L^2(\Omega\cap\Xi_\nu,\omega,\varphi)} < \delta_\nu,$$

and hence $|h| < \rho$ on K.

According to Theorem 2.6.4, $\mathcal{O}(X) \cap L^2(X, \omega, \varphi)$ is a closed subspace of $L^2(X, \omega, \varphi)$, and is therefore itself a Hilbert space. By construction, for each point $p \in X$, each cotangent vector $\eta \in \Lambda^{1,0}T_p^*X$, each point $q \in X \setminus \{p\}$, and each constant $\zeta \in \mathbb{C}$, the sets

$$A_{p,\eta} \equiv \{f \in \mathcal{O}(X) \cap L^2(X, \omega, \varphi) \mid (df)_p \neq \eta\},$$

$$B_{p,q,\zeta} \equiv \{f \in \mathcal{O}(X) \cap L^2(X, \omega, \varphi) \mid f(p) - f(q) \neq \zeta\},$$

are nonempty, and it follows from Theorem 2.6.4 and Theorem 1.2.4 that the above sets are also open in $\mathcal{O}(X) \cap L^2(X, \omega, \varphi)$. Moreover, these sets are dense. For if

$f \in \mathcal{O}(X) \cap L^2(X,\omega,\varphi)$ with $(df)_p = \eta$ (respectively, $f(p) - f(q) = \zeta$), then fixing $h \in A_{p,0_p}$ (respectively, $h \in B_{p,q,0}$), we get $f + \epsilon h \in A_{p,\eta}$ (respectively, $f + \epsilon h \in B_{p,q,\zeta}$) for every $\epsilon \in \mathbb{C}^*$, and $\|(f + \epsilon h) - f\| \to 0$ as $\epsilon \to 0$.

According to the Baire category theorem (see for example, [Mu] or [Rud1]), the intersection of any countable collection of dense open subsets of a Hilbert space (or any complete metric space) is dense. Hence there must exist an element f of

$$\bigcap_{p\in C} A_{p,\eta_p} \cap \bigcap_{(p,q)\in D} B_{p,q,\zeta_{p,q}}$$

with $\|f\|_{L^2(X,\omega,\varphi)} < 1$, and the claim follows. □

Proof of Theorem 5.18.2 By Proposition 5.18.4, there exists a covering of X by three open sets Ξ_0, Ξ_1, and Ξ_2 such that for each $j = 0, 1, 2$, the connected components $\{\Xi_j^{(\nu)}\}_{\nu=1}^{\infty}$ of Ξ_j form a locally finite collection of relatively compact open subsets of X and each connected component is biholomorphic to a disk. By shrinking these sets slightly and applying Theorem 5.18.3, we may also assume that there exists a holomorphic function g_j on X such that $|g_j - \nu| < 1$ on $\Xi_j^{(\nu)}$ for $\nu = 1, 2, 3, \dots$. In particular, the holomorphic mapping $g = (g_0, g_1, g_2)\colon X \to \mathbb{C}^3$ is proper. We will approximate g by a (proper) holomorphic embedding.

We may fix closed subsets K_1 and K_2 of X such that $K_1 \subset \Xi_1$, $K_2 \subset \Xi_2$, and $X = \Xi_0 \cup K_1 \cup K_2$, and we may set $f_0 \equiv g_0$. By the identity theorem, the set $C' \equiv \{p \in X \mid (df_0)_p = 0\}$ is discrete, and fixing a countable dense subset C of X containing C', we see that the set D of pairs of distinct points (p, q) in X with $p \in C$ and $f_0(p) = f_0(q)$ is countable. Applying Lemma 5.18.12, we get a holomorphic function h_1 on X such that $(dh_1)_p \neq -(dg_1)_p$ for each point $p \in C$, $h_1(p) - h_1(q) \neq g_1(q) - g_1(p)$ for each pair $(p, q) \in D$, and $|h_1| < 1$ on K_1. Thus the holomorphic function $f_1 \equiv g_1 + h_1$ satisfies $|f_1 - \nu| < 2$ on $K_1 \cap \Xi_1^{(\nu)}$ for each ν, $(df_1)_p \neq 0$ for each point $p \in C \supset C'$ (in particular, (f_0, f_1) is a holomorphic immersion into $\mathbb{C}^2$), and $f_1(p) \neq f_1(q)$ for each pair $(p, q) \in D$. It now suffices to show that the set $E \subset X \times X$ of pairs of distinct points (p, q) with $(f_0(p), f_1(p)) = (f_0(q), f_1(q))$ is countable. For the above argument will then provide a holomorphic function f_2 such that $|f_2 - \nu| < 2$ on $K_2 \cap \Xi_2^{(\nu)}$ for each ν and $f_2(p) \neq f_2(q)$ for each pair $(p, q) \in E$. The mapping $f = (f_0, f_1, f_2)\colon X \to \mathbb{C}^3$ will then be a holomorphic embedding.

Since the set of points in X at which $df_0 = 0$ or $df_1 = 0$ is countable, and $E \cap (\{p\} \times X)$ is countable for each point $p \in X$, it suffices to show that the set of points $(p, q) \in E$ at which $(df_0)_p \neq 0$ and $(df_1)_p \neq 0$ is countable. Therefore, since $X \times X$ is second countable, is suffices to show that each such point (p, q) is an isolated point in E. By the open mapping theorem, the holomorphic inverse function theorem, and the identity theorem, there exist disjoint connected neighborhoods U and V of p and q, respectively, such that for $j = 0, 1$, f_j maps U biholomorphically onto a domain $U_j \subset \mathbb{C}$, $f_j(V) \subset U_j$, and $f_j^{-1}(f_j(p)) \cap V = \{q\}$. For each point $(r, s) \in E \cap (U \times V)$, we have

$$(f_0\restriction_U)^{-1}(f_0(s)) = r = (f_1\restriction_U)^{-1}(f_1(s)).$$

Since C is dense in X, we may fix a point $r \in C \cap U \cap f_0^{-1}(f_0(V))$ and a point $s \in V \cap f_0^{-1}(f_0(r))$. We then get $(r, s) \in D$, and hence $f_1(r) \neq f_1(s)$; that is, $(f_0 \restriction_U)^{-1} \circ f_0 - (f_1 \restriction_U)^{-1} \circ f_1$ is not identically zero on the domain V. It follows that we may choose V so small that q is the only zero of the above holomorphic function, that is, so that $E \cap (U \times V) \subset U \times \{q\}$. Hence $E \cap (U \times V) = \{((f_0 \restriction_U)^{-1}(f_0(q)), q)\} = \{(p, q)\}$, and the claim follows. $\square$

Exercises for Sect. 5.18

5.18.1 Let X be an open Riemann surface. Using only Theorem 5.18.3 and Proposition 5.18.4, prove directly (without appealing to the Baire category theorem) that there exists a proper holomorphic immersion of X into $\mathbb{C}^3$.

5.18.2 Prove Theorem 5.18.3 directly using the Runge approximation theorem (Theorem 2.16.1) in place of the L^2 $\bar{\partial}$-method.

5.18.3 The proof of Theorem 5.18.2 shows that every open Riemann surface admits a holomorphic immersion into $\mathbb{C}^2$. Prove this fact directly by applying the Mittag-Leffler theorem (Theorem 2.15.1).

5.19 Embedding of a Compact Riemann Surface into $\mathbb{P}^n$

In this section, we consider an embedding theorem for a compact Riemann surface X. As mentioned in Sect. 5.18, X cannot admit a holomorphic embedding into a Euclidean space, because $\mathcal{O}(X) = \mathbb{C}$. However, X *does* admit a holomorphic embedding into a complex projective space.

Definition 5.19.1 Let $n \in \mathbb{Z}_{>0}$. The *n-dimensional complex projective space* $\mathbb{P}^n$ is the quotient space

$$\mathbb{P}^n \equiv \mathbb{C}^{n+1} \setminus \{0\}/\sim,$$

where for all points $w, z \in \mathbb{C}^{n+1} \setminus \{0\}$, we have $w \sim z$ if and only if $w = \lambda z$ for some scalar $\lambda \in \mathbb{C}^*$. We call any representative $(\zeta_0, \dots, \zeta_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ for a point $\zeta \in \mathbb{P}^n$ *homogeneous coordinates* for ζ, and we write $\zeta = [\zeta_0, \dots, \zeta_n]$.

Remarks 1. In other words, $\mathbb{P}^n$ is the space of complex lines in $\mathbb{C}^{n+1}$ that pass through the origin.

2. For each $j = 0, \dots, n$, the set $U_j \equiv \{[\zeta_0, \dots, \zeta_n] \in \mathbb{P}^n \mid \zeta_j \neq 0\}$ is well defined and open (in the quotient topology), and the mapping $U_j \to \mathbb{C}^n$ given by

$$[\zeta_0, \dots, \zeta_n] \mapsto (\zeta_0/\zeta_j, \dots, \widehat{\zeta_j/\zeta_j}, \dots, \zeta_n/\zeta_j)$$

is a well-defined homeomorphism (see Exercise 5.19.1). Moreover, $\mathbb{P}^n = \bigcup_{j=0}^n U_j$.

3. For $n = 1$, $\mathbb{P}^1$ (as defined above) is a connected compact Riemann surface with local holomorphic charts $(U_0, [\zeta_0, \zeta_1] \mapsto \zeta_1/\zeta_0, \mathbb{C})$ and $(U_1, [\zeta_0, \zeta_1] \mapsto \zeta_0/\zeta_1, \mathbb{C})$

given by the mappings in Remark 2 above, and the quotient mapping $\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ is holomorphic. Moreover, although we also denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by $\mathbb{P}^1$, there is no ambiguity because we have a biholomorphism $\mathbb{C}^2 \setminus \{0\}/\sim \to \mathbb{C} \cup \{\infty\}$ given by $[\zeta_0, \zeta_1] \mapsto \zeta_0/\zeta_1$ ($[\zeta_0, 0] \mapsto \infty$).

4. For an arbitrary $n \geq 1$, $\mathbb{P}^n$ is a connected compact complex manifold of dimension n. The local holomorphic charts are provided by the mappings in Remark 2 above, and the quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is holomorphic (see Exercise 5.19.1). We will not use this fact directly, but it justifies the use of the terminology appearing in Definition 5.19.2 below (cf. Definition 5.18.1).

Definition 5.19.2 Let X be a complex 1-manifold, and let $F\colon X \to \mathbb{P}^n$ be a continuous mapping into a complex projective space $\mathbb{P}^n$. For each $j = 0, \dots, n$, let $\Phi_j\colon U_j \to \mathbb{C}^n$ be the homeomorphism of the open set $U_j \equiv \{[\zeta_0, \dots, \zeta_n] \in \mathbb{P}^n \mid \zeta_j \neq 0\}$ onto $\mathbb{C}^n$ given by $[\zeta_0, \dots, \zeta_n] \mapsto (\zeta_0/\zeta_j, \dots, \widehat{\zeta_j/\zeta_j}, \dots, \zeta_n/\zeta_j)$.

(a) F is called *holomorphic* if the mapping $\Phi_j \circ F\colon F^{-1}(U_j) \to \mathbb{C}^n$ is holomorphic for each $j = 0, \dots, n$.
(b) F is called a *holomorphic immersion* if the mapping $\Phi_j \circ F\colon F^{-1}(U_j) \to \mathbb{C}^n$ is a holomorphic immersion for each $j = 0, \dots, n$.
(c) F is called a *holomorphic embedding* if X is compact and F is an injective holomorphic immersion (i.e., F is a proper injective holomorphic immersion).

One may obtain holomorphic mappings into complex projective spaces from holomorphic sections of a holomorphic line bundle as follows:

Lemma 5.19.3 *Let E be a holomorphic line bundle on a Riemann surface X, and let $s = (s_0, \dots, s_n)$ be an $(n + 1)$-tuple of holomorphic sections of E with no common zeros. Then the mapping $\kappa_s\colon X \to \mathbb{P}^n$ given by*

$$\kappa_s(x) \equiv \left[\frac{s_0(x)}{\xi}, \dots, \frac{s_n(x)}{\xi}\right] \quad \forall x \in X \text{ and } \xi \in E_x \setminus \{0_x\}$$

is well defined and holomorphic.

Proof For $x \in X$ and $\xi, \eta \in E_x \setminus \{0_x\}$, we have

$$\left(\frac{s_0(x)}{\xi}, \dots, \frac{s_n(x)}{\xi}\right) = \frac{\eta}{\xi} \cdot \left(\frac{s_0(x)}{\eta}, \dots, \frac{s_n(x)}{\eta}\right),$$

so the mapping is well defined. Moreover, given a point $x_0 \in X$ and an index j with $s_j(x_0) \neq 0$, then s_j is nonvanishing on some neighborhood U of x_0 in X and the composition of the mapping $[\zeta_0, \dots, \zeta_n] \mapsto (\zeta_0/\zeta_j, \dots, \widehat{\zeta_j/\zeta_j}, \dots, \zeta_n/\zeta_j)$ with the mapping $\kappa_s\restriction_U$ is the mapping

$$x \mapsto \big(s_0(x)/s_j(x), \dots, s_j\widehat{(x)/s_j}(x), \dots, s_n(x)/s_j(x)\big),$$

which is holomorphic. $\square$

Definition 5.19.4 Let E be a holomorphic line bundle on a Riemann surface X, and let $s = (s_0, \dots, s_n)$ be an $(n+1)$-tuple of holomorphic sections of E with no common zeros. Then the associated holomorphic mapping $X \to \mathbb{P}^n$ given by

$$x \mapsto \left[\frac{s_0(x)}{\xi}, \dots, \frac{s_n(x)}{\xi}\right] \quad \forall x \in X \text{ and } \xi \in E_x \setminus \{0_x\}$$

is denoted by κ_s or by $[s_0, \dots, s_n]$.

Example 5.19.5 Let $E \to \mathbb{P}^1$ be the hyperplane bundle (see Examples 3.1.5 and 3.3.4); that is, $E = [D]$, where D is the divisor with $D(0) = 1$ and $D(x) = 0$ for each point $x \in \mathbb{P}^1 \setminus \{0\}$. Let s_0 be a holomorphic section with $\operatorname{div}(s_0) = D$. We also have the holomorphic section $s_1 \equiv s_0/z$ ($s_1(\infty) = 0$), and the corresponding mapping $\kappa_{(s_0,s_1)} = [s_0, s_1]$ is equal to inverse of the biholomorphism $(\mathbb{C}^2 \setminus \{0\})/\!\sim\, \to \mathbb{C} \cup \{\infty\}$ described in the remarks following Definition 5.19.1 (see Exercise 5.19.2).

Recall that the vector space of holomorphic sections of a holomorphic line bundle on a compact Riemann surface is finite-dimensional (Theorem 4.4.1).

Lemma 5.19.6 *Let E be holomorphic line bundle on a compact Riemann surface X, and let $s = (s_0, \dots, s_n)$ for a basis $s_0, \dots, s_n$ for $\Gamma(X, \mathcal{O}(E))$.*

(a) *A point $p \in X$ satisfies $s(p) = (0, \dots, 0)$ if and only if every holomorphic section of E on X vanishes at p.*
(b) *Suppose that the sections $s_0, \dots, s_n$ have no common zeros. Then the associated map $\kappa_s = [s_0, \dots, s_n]$ is injective if and only if for each pair of distinct points $p, q \in X$, there exists a holomorphic section t of E with $t(p) = 0$ and $t(q) \neq 0$.*
(c) *Suppose (again) that the sections $s_0, \dots, s_n$ have no common zeros. Then the associated map $\kappa_s = [s_0, \dots, s_n]$ is a holomorphic immersion if and only if for each point $p \in X$, there is a holomorphic section t with a simple zero at p.*

Remark In the standard terminology (see, for example, [GriH]), the point p in part (a) is a *base point* for the holomorphic sections (or, more precisely, for the linear system of divisors associated to the nontrivial holomorphic sections of E). In (b), the holomorphic sections are said to *separate points* in X. In (c), the holomorphic sections are said to *give local coordinates* in X.

Proof of Lemma 5.19.6 If $p \in X$ with $s(p) = (0, \dots, 0)$, then every section $t \in \Gamma(X, \mathcal{O}(E))$ must vanish at p (since t is a linear combination of the sections $s_0, \dots, s_n$). Part (a) now follows. Let us assume for the rest of this proof that the sections $s_0, \dots, s_n$ have no common zeros.

Given distinct points $p, q \in X$, there exists an index $j \in \{0, \dots, n\}$ with $s_j(q) \neq 0$. If $s_j(p) = 0$, then the section s_j separates the points p and q as in (b), and we have $\kappa_s(p) \neq \kappa_s(q)$. Assuming that $s_j(p) \neq 0$ and that κ_s is injective, the corresponding points

$$\left(s_0(p)/s_j(p), \dots, s_j\widehat{(p)/s_j}(p), \dots, s_n(p)/s_j(p)\right)$$

and

$$\big(s_0(q)/s_j(q), \dots, s_j\widehat{(q)/s_j}(q), \dots, s_n(q)/s_j(q)\big)$$

in $\mathbb{C}^n$ must be distinct, and therefore

$$\frac{s_i(p)}{s_j(p)} \neq \frac{s_i(q)}{s_j(q)} \quad \text{for some } i \in \{0, \dots, \hat{j}, \dots, n\}.$$

Thus the holomorphic section

$$t \equiv s_i - \frac{s_i(p)}{s_j(p)} \cdot s_j$$

satisfies $t(p) = 0$ and $t(q) \neq 0$. For the converse, suppose that $s_j(p) \neq 0$ and $s_j(q) \neq 0$, and that there exists a holomorphic section t with $t(p) = 0$ and $t(q) \neq 0$. We then have constants $\zeta_0, \dots, \zeta_n \in \mathbb{C}$ with $t = \sum_{i=0}^n \zeta_i s_i$, and hence

$$\sum_{i=0}^n \zeta_i \frac{s_i(p)}{s_j(p)} = 0 \quad \text{and} \quad \sum_{i=0}^n \zeta_i \frac{s_i(q)}{s_j(q)} \neq 0.$$

It follows that the points

$$\big(s_0(p)/s_j(p), \dots, s_j\widehat{(p)/s_j}(p), \dots, s_n(p)/s_j(p)\big)$$

and

$$\big(s_0(q)/s_j(q), \dots, s_j\widehat{(q)/s_j}(q), \dots, s_n(q)/s_j(q)\big)$$

in $\mathbb{C}^n$ must be distinct and therefore that $\kappa_s(p) \neq \kappa_s(q)$. The claim (b) now follows.

For the proof of (c), let $p \in X$ and let $j \in \{0, \dots, n\}$ with $s_j(p) \neq 0$. If κ_s is a holomorphic immersion, then for some $i \in \{0, \dots, \hat{j}, \dots, n\}$, we have $(d(s_i/s_j))_p \neq 0$, and hence the holomorphic section

$$t \equiv s_i - \frac{s_i(p)}{s_j(p)} \cdot s_j$$

has a simple zero at p. Conversely, given a section $t \in \Gamma(X, \mathcal{O}(E))$ with a simple zero at p, we have constants $\zeta_0, \dots, \zeta_n \in \mathbb{C}$ with $t = \sum_{i=0}^n \zeta_i s_i$, and hence

$$0 \neq (d(t/s_j))_p = \sum_{i=0}^n \zeta_i \cdot (d(s_i/s_j))_p.$$

Thus $(d(s_i/s_j))_p \neq 0$ for some $i \in \{0, \dots, \hat{j}, \dots, n\}$, and (c) follows. □

Definition 5.19.7 A holomorphic line bundle E on a compact Riemann surface X is called *ample* if E is positive; that is, E admits a positive-curvature Hermitian metric or, equivalently (by Theorem 4.3.1), $\deg E > 0$. The line bundle E is called *very ample* if the map $\kappa_s \colon X \to \mathbb{P}^n$ associated to any basis s for $\Gamma(X, \mathcal{O}(E))$ exists and is a holomorphic embedding.

Remark Since the line bundle associated to any nontrivial effective divisor has positive degree, if E is very ample, then E is ample.

The results of Chap. 4 and the above observations now give us the following:

Theorem 5.19.8 *Let E be a holomorphic line bundle on a compact Riemann surface X. Then we have the following*:

(a) *If E is ample, then E^d is very ample for every $d \gg 0$.*
(b) *If* $\deg E > 2$, *then $K_X \otimes E$ is very ample.*
(c) *If* $\deg E > 2 \cdot \mathrm{genus}(X)$, *then E is very ample.*

Proof Clearly, part (a) follows from part (c). Moreover, part (c) follows from part (b) applied to the holomorphic line bundle $K_X^* \otimes E$, since this line bundle has degree $-2\,\mathrm{genus}(X) + 2 + \deg E$ by Corollary 4.6.7. Thus it suffices to prove part (b).

Assuming that $\deg E > 2$, suppose $p, q \in X$ are two (possibly equal) points and D is the divisor given by $D = p + q$. Then, applying Corollary 4.1.2 to the positive holomorphic line bundle $E \otimes [-D]$ (we have $\deg E \otimes [-D] > 2 - 2 = 0$) and the effective divisor D, we get a holomorphic section t of $K_X \otimes E \otimes [-D] \otimes [D] = K_X \otimes E$ such that $t(p) = 0$ and $t(q) \neq 0$ if $p \neq q$, and t has a simple zero at p if $p = q$. Thus, by Lemma 5.19.6, the map κ_s is defined for any basis s of $\Gamma(X, \mathcal{O}(K_X \otimes E))$, and κ_s is a holomorphic embedding. $\square$

Remark According to the higher-dimensional analogue of Theorem 5.19.8, which is called the *Kodaira embedding theorem*, if E is an ample holomorphic line bundle on a connected compact complex manifold X, then E^d is very ample for $d \gg 0$ (see, for example, [GriH], [MKo], or [Wel]).

Exercises for Sect. 5.19

5.19.1 Prove that $\mathbb{P}^n$ is a complex manifold of dimension n with local holomorphic charts given by the mappings in Definition 5.19.2, and the quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is holomorphic (see Exercise 2.2.6 for the required definitions).

5.19.2 Verify that the mapping $\kappa_{(s_0,s_1)} = [s_0, s_1]$ in Example 5.19.5 is equal to the inverse of the biholomorphism $(\mathbb{C}^2 \setminus \{0\})/\sim \, \to \mathbb{C} \cup \{\infty\}$ described in the remarks following Definition 5.19.1.

5.20 Finite Holomorphic Branched Coverings

As we will see in this section, any nonconstant proper holomorphic mapping of Riemann surfaces is actually a finite holomorphic covering map away from a discrete set of *branch points* (see Proposition 5.20.1 and Definition 5.20.2 below). We first recall some terminology and facts concerning the topology of manifolds (Sect. 9.3). A continuous mapping of Hausdorff spaces is called *proper* if the inverse image of every compact subset of the target is compact (see Definition 9.3.9). A sequence in a manifold converges to a point p if for each neighborhood U of p, all but finitely many terms of the sequence lie in U. According to Theorem 9.3.2, a compact subset K of a manifold has the Bolzano–Weierstrass property; that is, every sequence in K admits a convergent subsequence.

Proposition 5.20.1 *Let $\Phi\colon X \to Y$ be a nonconstant proper holomorphic mapping of Riemann surfaces X and Y, and let $m_q = \operatorname{mult}_q \Phi \in \mathbb{Z}_{>0}$ denote the multiplicity of Φ at q for each point $q \in X$. Then we have the following*:

(a) *Φ is surjective, and counting multiplicities, the cardinalities of the fibers of Φ are all equal to some finite constant $n \in \mathbb{Z}_{>0}$; that is,*

$$\sum_{q \in \Phi^{-1}(p)} m_q = n \quad \forall p \in Y.$$

(b) *If $p \in Y$ and $Z \equiv \Phi^{-1}(p)$, then there exist a connected neighborhood V of p and disjoint connected open sets $\{U_q\}_{q \in Z}$ such that $\Phi^{-1}(V) = \bigcup_{q \in Z} U_q$ and such that, for each $q \in Z$, we have $q \in U_q$, $\Phi\restriction_{U_q} : U_q \to V$ is a surjective proper holomorphic mapping, and, counting multiplicities, the number of points in each fiber of $\Phi\restriction_{U_q}$ is equal to m_q.*

(c) *The set of critical values*

$$C \equiv \Phi(\{q \in X \mid (\Phi_*)_q = 0\}) = \Phi(\{q \in X \mid \operatorname{mult}_q \Phi > 1\})$$

is discrete, and the mapping $\Phi\restriction_{X \setminus \Phi^{-1}(C)} : X \setminus \Phi^{-1}(C) \to Y \setminus C$ is a finite (proper) holomorphic covering map.

(d) *If $\Phi^{-1}(p)$ is a singleton for some point $p \in Y \setminus C$, then Φ is a biholomorphism of X onto Y.*

Proof By the open mapping theorem (Theorem 2.2.2), $\Omega \equiv \Phi(X)$ is an open subset of Y. If $p \in \overline{\Omega}$, then there is a sequence $\{q_\nu\}$ in X such that $\Phi(q_\nu) \to p$, and by replacing this sequence with a suitable subsequence, we may assume that $\{q_\nu\}$ converges to some point $q \in X$. But we then have $\Phi(q_\nu) \to \Phi(q)$, and hence $p = \Phi(q) \in \Omega$. Thus Ω is both open and closed in the Riemann surface Y, and hence Φ is surjective.

Once again, let us fix a point $p \in Y$ and let $Z \equiv \Phi^{-1}(p)$. Since Z is both compact (because Φ is proper) and discrete (by the identity theorem), Z must be finite. The local representation of holomorphic mappings (Lemma 2.2.3) provides a local holomorphic coordinate neighborhood $(V, \Psi = \zeta, V')$ with $p \in V$ and $\zeta(p) = 0$, disjoint local holomorphic coordinate neighborhoods $\{(U_q, \Lambda_q = z_q, U_q')\}_{q \in Z}$ in X, and positive integers $\{m_q\}_{q \in Z}$ such that for each $q \in Z$, we have $q \in U_q$, $z_q(q) = 0$, $\Phi(U_q) \subset V$, and $\zeta(\Phi) = z_q^{m_q}$ on U_q. For $\epsilon > 0$ sufficiently small, we have

$\Delta(0;\epsilon) \subset V'$ and $\Phi^{-1}(\Psi^{-1}(\Delta(0;\epsilon))) \subset U \equiv \bigcup_{q\in Z} U_q$. For if this were not the case, then there would exist a sequence $\{q_\nu\}$ in $X \setminus U$ such that $\Phi(q_\nu) \to p$ and $q_\nu \to q \in X \setminus U$ with $\Phi(q) = p$. But we would then have $q \in Z \setminus U$, which is impossible. We may also choose ϵ so small that $\Delta(0;\epsilon^{1/m_q}) \subset U'_q$ for each $q \in Z$. Therefore, by replacing V' and V with $\Delta(0;\epsilon)$ and $\Psi^{-1}(\Delta(0;\epsilon))$, respectively, and replacing U'_q and U_q with $\Delta(0;\epsilon^{1/m_q})$ and $\Lambda_q^{-1}(\Delta(0;\epsilon^{1/m_q}))$, respectively, for each $q \in Z$, we may assume that the local holomorphic charts are given by

$$(V, \Psi = \zeta, V' = \Delta(0;\epsilon)) \quad \text{and} \quad \{(U_q, z_q, U'_q = \Delta(0;\epsilon^{1/m_q}))\}_{q\in Z}.$$

In particular, for each point $q \in Z$, Φ is represented on U_q by the holomorphic map $\Delta(0;\epsilon^{1/m_q}) \to \Delta(0;\epsilon)$ given by $z \mapsto z^{m_q}$, and hence, counting multiplicities, the cardinality of the fiber of $\Phi\restriction_{U_q}$ over y is equal to m_q for each point $y \in V$, and part (b) follows. Furthermore, counting multiplicities, the number of points in each fiber over each point in V is equal to the constant $n = \sum_{q\in Z} m_q$. It follows that the function on Y given by the cardinality of each fiber, again counting multiplicities, is locally constant and therefore constant.

The claim (c) now follows from Lemma 10.2.11. The claim (d) follows from the above (or from part (a) and Theorem 2.4.4). $\square$

Definition 5.20.2 A proper holomorphic mapping $\Phi\colon X \to Y$ of a complex 1-manifold X onto a Riemann surface Y for which the restriction to each connected component is surjective and for which almost every fiber consists of n points (i.e., every fiber has n points counting multiplicities) is called a *finite holomorphic branched covering map* (or a *finite holomorphic ramified covering map*) with *n sheets*. Each critical point p of Φ is called a *branch point* (or *ramification point*) of *order $m - 1$*, where $m = \mathrm{mult}_p\Phi > 1$. The sum of all of the orders of the branch points of Φ is called the *branching order* of Φ.

According to Corollary 2.10.4, every Riemann surface admits a nonconstant meromorphic function. Thus we have the following:

Theorem 5.20.3 *Every compact Riemann surface admits a finite holomorphic branched covering map onto $\mathbb{P}^1$.*

We now record the following fact for use in the proof of Abel's theorem (see Sect. 5.21):

Proposition 5.20.4 *Let X be a complex 1-manifold, let Y be a Riemann surface, let $\Phi\colon X \to Y$ be a finite holomorphic branched covering map with n sheets, and let $C \subset Y$ be the set of critical values.*

(a) *If $\gamma\colon [0,1] \to Y$ is an injective path with $\gamma((0,1)) \subset Y \setminus C$, then there exist exactly n distinct liftings $\gamma_1, \dots, \gamma_n$ of γ to paths in X. Moreover, the sets $\{\gamma_j((0,1))\}_{j=1}^n$ are disjoint, and if $p \in \Phi^{-1}(\gamma(0))$ and $m = \mathrm{mult}_p\Phi$, then exactly m of the liftings have initial point p.*

(b) *If f is a holomorphic function on X, then the function $Y \setminus C \to \mathbb{C}$ given by*

$$y \mapsto \sum_{x \in \Phi^{-1}(y)} f(x)$$

has a unique extension to a holomorphic function on Y.

(c) *Given a holomorphic 1-form α on X, there is a unique holomorphic 1-form β on Y such that if γ is any injective path in Y with $\gamma((0,1)) \subset Y \setminus C$, and $\gamma_1, \dots, \gamma_n$ are the distinct liftings of γ to paths in X (as given by part* (a)), *then*

$$\int_\gamma \beta = \sum_{j=1}^n \int_{\gamma_j} \alpha.$$

Proof If γ is an injective path in Y with $\gamma((0,1)) \subset Y \setminus C$, then, since the restriction $X \setminus \Phi^{-1}(C) \to Y \setminus C$ is an (unbranched) covering map with exactly n points in each fiber, there exist exactly n distinct liftings $\delta_1, \dots, \delta_n$ of $\gamma\restriction_{(0,1)}$ to paths in $X \setminus \Phi^{-1}(C)$. Moreover, the images of these liftings are disjoint. For if $\delta_i(t_0) = \delta_j(t_1)$ for some pair of indices $i, j \in \{1, \dots, n\}$ and some pair of numbers $t_0, t_1 \in (0,1)$, then

$$\gamma(t_0) = \Phi(\delta_i(t_0)) = \Phi(\delta_j(t_1)) = \gamma(t_1),$$

and the injectivity of γ implies that $t_0 = t_1$. Uniqueness of liftings in a covering space then implies that $\delta_i = \delta_j$, and hence $i = j$. Now let $p_1, \dots, p_k$ be the distinct points in $\Phi^{-1}(\gamma(0))$, and let $m_i \equiv \operatorname{mult}_{p_i}\Phi$ for each $i = 1, \dots, k$ (in particular, $n = \sum_{i=1}^k m_i$). Then, by Proposition 5.20.1, there exist a connected neighborhood V of $\gamma(0)$ in Y and disjoint connected open sets $U_1, \dots, U_k$ such that $\Phi^{-1}(V) = \bigcup_{i=1}^k U_i$ and such that for each $i = 1, \dots, k$, we have $p_i \in U_i$, $U_i \cap \Phi^{-1}(C) \subset \{p_i\}$, and $\Phi\restriction_{U_i} : U_i \to V$ is a finite branched holomorphic covering map with m_i sheets. Let J be the connected component of $\gamma^{-1}(V)$ containing 0. Then, for each $j = 1, \dots, n$, $\delta_j(J \cap (0,1))$ lies in U_i for some unique index i, and properness implies that $\delta_j(t) \to p_i$ as $t \to 0^+$. A similar argument applies at $t = 1$. Thus δ_j extends to a unique lifting $\gamma_j : [0,1] \to X$ of γ, and this lifting satisfies $\gamma_j(0) = p_i$. Furthermore, fixing a point $t_0 \in J \cap (0,1)$, we see that for each $i = 1, \dots, k$, each of the m_i points in $\Phi^{-1}(\gamma(t_0)) \cap U_i$ will be equal to the value at $t = t_0$ for *exactly one* of the paths $\delta_1, \dots, \delta_n$. Thus exactly m_i of the paths $\gamma_1, \dots, \gamma_n$ will have initial value p_i, and part (a) is proved.

For the proof of (b), suppose $f \in \mathcal{O}(X)$ and $h : Y \setminus C \to \mathbb{C}$ is the function given by $y \mapsto \sum_{x \in \Phi^{-1}(y)} f(x)$. If $U \subset Y \setminus C$ is a domain that is evenly covered by the holomorphic covering map $X \setminus \Phi^{-1}(C) \to Y \setminus C$, and $U_1, \dots, U_n$ are the distinct connected components of the inverse image $\Phi^{-1}(U)$, then we have $h\restriction_U = \sum_{j=1}^n f_j$, where for each $j = 1, \dots, n$, $f_j \equiv f \circ (\Phi\restriction_{U_j})^{-1} : U \to \mathbb{C}$. It follows that h is a holomorphic function on $Y \setminus C$. Moreover, by properness, h must also be bounded on $K \setminus C$ for each compact set $K \subset Y$, and hence by Riemann's extension theorem (Theorem 1.2.10), h extends to a unique holomorphic function on Y.

Finally, for the proof of (c), suppose α is a holomorphic 1-form on X. In analogy with part (b), the corresponding holomorphic 1-form on Y is obtained by averaging over the fibers. More precisely, if $U \subset Y \setminus C$ is a domain that is evenly covered by the holomorphic covering map $X \setminus \Phi^{-1}(C) \to Y \setminus C$, and $U_1, \dots, U_n$ are the distinct connected components of $\Phi^{-1}(U)$, then we let η_U be the holomorphic 1-form on U given by $\eta_U \equiv \sum_{j=1}^n \alpha_j$, where for each $j = 1, \dots, n$, α_j is the unique holomorphic 1-form on U satisfying $(\Phi\restriction_{U_j})^*\alpha_j = \alpha\restriction_{U_j}$. These holomorphic 1-forms agree on the overlaps, and therefore they determine a well-defined holomorphic 1-form η on $Y \setminus C$. Given a point $q \in C$, we may form neighborhoods U and V of $\Phi^{-1}(q)$ and q, respectively, such that $\Phi\restriction_U : U \to V$ is a finite holomorphic branched covering map and such that for some function $f \in \mathcal{O}(U)$, we have $\alpha\restriction_U = df$ (for we may choose V so that the connected components of $\Phi^{-1}(V)$ lie in a union of disjoint simply connected open sets). By part (b), the function $y \mapsto \sum_{x\in\Phi^{-1}(y)} f(y)$ on $V \setminus C$ extends to a unique holomorphic function g on V, and as is clear from the construction of η, we have $dg = \eta$ on $V \setminus C$. It follows that η extends to a holomorphic 1-form β on Y.

Suppose γ is an injective path in Y with $\gamma((0,1)) \subset Y \setminus C$, and $\gamma_1, \dots, \gamma_n$ are the distinct liftings of γ. For each $\epsilon \in (0, 1/2)$, there is a partition $\epsilon = t_0 < \cdots < t_k = 1 - \epsilon$ such that for each $\nu = 1, \dots, k$, $\gamma([t_{\nu-1}, t_\nu])$ is contained in an evenly covered domain in $Y \setminus C$. The paths $\{\gamma_j\restriction_{[t_{\nu-1},t_\nu]}\}_{j=1}^n$ are then the distinct liftings of $\gamma\restriction_{[t_{\nu-1},t_\nu]}$, and hence

$$\int_{\gamma\restriction_{[\epsilon,1-\epsilon]}} \beta = \sum_{\nu=1}^k \int_{\gamma\restriction_{[t_{\nu-1},t_\nu]}} \beta = \sum_{\nu=1}^k \sum_{j=1}^n \int_{\gamma_j\restriction_{[t_{\nu-1},t_\nu]}} \alpha = \sum_{j=1}^n \int_{\gamma_j\restriction_{[\epsilon,1-\epsilon]}} \alpha.$$

Letting $\epsilon \to 0^+$, we get

$$\int_\gamma \beta = \sum_{j=1}^n \int_{\gamma_j} \alpha.$$

Finally, suppose β' is another holomorphic 1-form with the above property. Each point $p \in Y \setminus C$ has a connected neighborhood U in $Y \setminus C$ on which $\beta = df$ for some function $f \in \mathcal{O}(U)$. Since f may be obtained by integration of β along injective paths in U with initial point p, and each of these integrals must agree with the corresponding integral of β', we have $\beta' = df = \beta$ on U. Thus $\beta' = \beta$ on $Y \setminus C$ and therefore on Y. □

Exercises for Sect. 5.20

5.20.1 Suppose X is a complex 1-manifold, Y is a Riemann surface, $\Phi : X \to Y$ is a finite holomorphic branched covering map with n sheets, $C \subset Y$ is the set of critical values, $f \in \mathcal{O}(X)$, and $P(z_1, \dots, z_n)$ is a complex polynomial in n variables that is symmetric (that is, $P(z_{\sigma(1)}, \dots, z_{\sigma(n)}) = P(z_1, \dots, z_n)$ for every point $(z_1, \dots, z_n) \in \mathbb{C}^n$ and every permutation σ of $\{1, \dots, n\}$).

For each point $y \in Y \setminus C$, let $g(y) = P(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are the distinct points in $\Phi^{-1}(y)$. Prove that there is a unique holomorphic extension of g to a holomorphic function on Y.

Exercises 5.20.2–5.20.9 below require facts stated in Sect. 4.6 (cf. [FarK]).

5.20.2 *Riemann–Hurwitz formula*. Let $\Phi\colon X \to Y$ be a finite holomorphic branched covering mapping of compact Riemann surfaces X and Y, let b be the branching order of Φ, and let n be the number of sheets. Prove that

$$\operatorname{genus}(X) = \frac{b}{2} + n \cdot (\operatorname{genus}(Y) - 1) + 1.$$

Hint. We may fix a nontrivial meromorphic 1-form β on Y (Theorem 2.10.1), and we may let α be the nontrivial meromorphic 1-form on X given by $\alpha \equiv \Phi^*\beta$. Set $D = \operatorname{div}(\alpha)$ and $E = \operatorname{div}(\beta)$. Given a point $q \in Y$ and a point $p \in \Phi^{-1}(q)$, apply the local representation of holomorphic maps to see that

$$D(p) = m - 1 + mE(q),$$

where $m \equiv \operatorname{mult}_p \Phi$. Now sum over all of the critical values of Φ and nonzero points of E, and apply Corollary 4.6.7.

5.20.3 Prove that a compact Riemann surface X is hyperelliptic (see Exercise 4.6.6) if and only if X admits a 2-sheeted holomorphic branched covering mapping onto $\mathbb{P}^1$.

5.20.4 Let X be a hyperelliptic Riemann surface of genus g, and let $\Phi\colon X \to \mathbb{P}^1$ be a 2-sheeted holomorphic branched covering map (see Exercises 4.6.6 and 5.20.3).

(a) Prove that Φ has exactly $2g + 2$ branch points, and that each branch point is of order 1.

Hint. Apply the Riemann–Hurwitz formula (Exercise 5.20.2).

(b) Prove that if $g > 1$, then every Weierstrass point in X is a hyperelliptic point (see Exercise 4.6.9) and the set of Weierstrass points is precisely the set of branch points of Φ (in particular, X has exactly $2g + 2$ Weierstrass points, and combining this with Exercise 4.6.9, we see that a compact Riemann surface of genus $g > 1$ is hyperelliptic if and only if it has exactly $2g + 2$ Weierstrass points).

Hint. Show that the branch points are hyperelliptic (Weierstrass) points and apply Exercise 4.6.9.

(c) Prove that if $g > 1$ and $\Psi\colon X \to \mathbb{P}^1$ is any 2-sheeted holomorphic branched covering map, then $\Psi = \Lambda \circ \Phi$ for some automorphism $\Lambda \in \operatorname{Aut}(\mathbb{P}^1)$.

Hint. Show that after composing with automorphisms of $\mathbb{P}^1$ (i.e., with Möbius transformations), we may assume that there are Weierstrass points $p, q, r \in X$ with $\Psi(p) = \Phi(p) = 0$, $\Psi(q) = \Phi(q) = 1$, and $\Psi(r) = \Phi(r) = \infty$. Then consider the meromorphic function Ψ/Φ.

5.20.5 Let X be a hyperelliptic Riemann surface of genus $g > 1$, and let $\Phi\colon X \to \mathbb{P}^1$ be a 2-sheeted holomorphic branched covering map (see Exercise 5.20.3).

(a) Prove that there exists a unique nontrivial automorphism (i.e., an automorphism that is not the identity) J of X such that $\Phi \circ J = \Phi$. Prove also that $J^2 = \mathrm{Id}$, that the fixed points of J are precisely the $2g+2$ branch points of Φ (i.e., the $2g+2$ Weierstrass points in X), and that J is independent of the choice of Φ (see Exercise 5.20.4). The associated automorphism J is called the *hyperelliptic involution* or the *sheet interchange*.

(b) Let J be the hyperelliptic involution on X. Prove that if $\Psi \in \mathrm{Aut}(X) \setminus \{\mathrm{Id}, J\}$ (i.e., Ψ is not in the subgroup generated by J), then Ψ has at most four fixed points (in particular, J is the unique nontrivial automorphism of X with at least $2g+2$ fixed points).

Hint. Applying Exercise 5.20.4, one sees that $\Phi \circ \Psi = \Lambda \circ \Phi$ for some Möbius transformation Λ. Show that if m is the number of fixed points of Ψ, then the number of fixed points of Λ is at least $m/2$.

(c) Prove that the hyperelliptic involution J commutes with every automorphism of X (i.e., J is in the center of $\mathrm{Aut}(X)$).

5.20.6 Let X be a compact Riemann surface of genus $g > 1$. In this exercise, the theorem of Schwarz on finiteness of the automorphism group $\mathrm{Aut}(X)$ (with a bound on the order of the group) is obtained as an application of Exercises 4.6.6–4.6.10 and Exercises 5.20.2–5.20.5 (a more precise bound on the order of $\mathrm{Aut}(X)$ is obtained in Exercise 5.20.9 below).

(a) Let S be the set of Weierstrass points in X. Prove that $\Phi(S) = S$ for each $\Phi \in \mathrm{Aut}(X)$. Conclude from this that the map $\Phi \mapsto \Phi\restriction_S$ is a group homomorphism of $\mathrm{Aut}(X)$ into the permutation group of S.

(b) Prove that if X is *not* hyperelliptic, then the homomorphism considered in (a) is injective, and the order of $\mathrm{Aut}(X)$ is at most $[(g-1)g(g+1)]!$.

Hint. Apply Exercises 4.6.9 and 4.6.10.

(c) Prove that if X is hyperelliptic with hyperelliptic involution J (see Exercise 5.20.5), then the kernel of the homomorphism considered in (a) is equal to $\{1, J\}$, and the order of $\mathrm{Aut}(X)$ is at most $2 \cdot [(2g+2)!]$.

5.20.7 One way to obtain examples of finite holomorphic branched coverings is to form quotients of Riemann surfaces by finite groups of automorphisms. Throughout this exercise, X will denote a Riemann surface, and Γ will denote a (finite) subgroup of order n in $\mathrm{Aut}(X)$. The quotient space by Γ (see Definition 10.4.3),

$$Y \equiv \Gamma \backslash X = X/[p \sim q \iff \Psi(p) = q \text{ for some } \Psi \in \Gamma],$$

with quotient map $\Upsilon \colon X \to Y$, will be assumed to be equipped with the quotient topology. For each point $p \in X$, Γ_p will denote the subgroup of Γ consisting of those elements that fix p (i.e., Γ_p is the *isotropy subgroup* at p).

(a) Let $p \in X$ and let m be the order of Γ_p. Prove that there exist a local holomorphic chart $(D, \Phi, \Delta(0;1))$ and a primitive mth root of unity ω (i.e., $\omega^m = 1$ but $\omega^k \neq 1$ if $1 \leq k < m$) such that $\Psi(D) = D$ for each $\Psi \in \Gamma_p$, $p = \Phi^{-1}(0)$, and the group $\{\Phi \circ \Psi \circ \Phi^{-1} \mid \Psi \in \Gamma_p\}$ is precisely the (rotation) group of automorphisms of $\Delta(0; 1)$ given by $z \mapsto \omega^k z$ for $k = 0, 1, 2, \ldots, m-1$. Prove also that Γ_p is a cyclic group.

Hint. Fix a local holomorphic chart (U, Λ, U') with $p = \Lambda^{-1}(0)$. Applying Exercise 1.5.2, show that for every sufficiently small $\epsilon > 0$, $\Lambda \circ \Psi \circ \Lambda^{-1}$ maps the disk $\Delta(0; \epsilon) \Subset U'$ biholomorphically onto a relatively compact convex neighborhood of 0 in U' for each $\Psi \in \Gamma_p$, and $\bigcap_{\Psi \in \Gamma_p} \Lambda \circ \Psi \circ \Lambda^{-1}(\Delta(0; \epsilon))$ is a convex neighborhood of 0 on which the restriction of $\Lambda \circ \Psi \circ \Lambda^{-1}$ is an automorphism for each $\Psi \in \Gamma_p$. Show that any convex domain is simply connected, apply the Riemann mapping theorem (Theorem 5.2.1) in order to obtain the desired local holomorphic chart, and finally, apply Theorem 5.8.2 in order to obtain the desired primitive mth root.

(b) Prove that $\Upsilon \colon X \to Y$ is an open mapping and that Y is a Hausdorff space.

(c) Prove that there is a unique holomorphic structure on Y (i.e., a unique structure of a Riemann surface) for which $\Upsilon \colon X \to Y$ is a finite holomorphic branched covering map with n sheets. Prove also that a point $p \in X$ is a branch point of order $m-1$ if and only if Γ_p is of order $m > 1$.

Hint. Given $p \in X$, fix a local holomorphic chart $(D, \Phi = z, \Delta(0; 1))$ as in (a) in which $p = \Phi^{-1}(0)$ and the restrictions of the distinct elements of Γ_p are represented by the rotations $z \mapsto \omega^k z$ for $k = 0, \ldots, m-1$. Show that D may be chosen so that $D \cap [(\Gamma \setminus \Gamma_p) \cdot D] = \emptyset$. Then show that this yields a local complex chart $(\widehat{D} = \Upsilon(D), \widehat{\Phi} = \zeta, \Delta(0; 1))$ in Y determined by $\widehat{\Phi}(\Upsilon(q)) = (\Phi(q))^m$ for $q \in D$ (in other words, the restriction of Υ to D is represented by the finite holomorphic branched covering map $\Delta(0; 1) \to \Delta(0; 1)$ given by $z \mapsto \zeta = z^m$). Show that the corresponding atlas is a holomorphic atlas in Y and that Υ is an n-sheeted finite holomorphic branched covering map with the stated properties.

5.20.8 Let X be a compact Riemann surface of genus $g > 1$. Prove that X is hyperelliptic (see Exercises 4.6.6 and 5.20.3–5.20.5) if and only if X admits a holomorphic involution Ψ (i.e., $\Psi \in \operatorname{Aut}(X)$ and $\Psi^2 = \mathrm{Id}$) with exactly $2g+2$ fixed points.

Hint. Given a nontrivial holomorphic involution Ψ of X, Exercise 5.20.7 provides a 2-sheeted branched holomorphic covering map $X \to Y \equiv \Gamma \backslash X$, where $\Gamma = \{1, \Psi\}$ is the group generated by Ψ. Now apply Exercises 5.20.2 and 5.20.3 and Theorem 4.6.8.

5.20.9 According to Exercise 5.20.6, every compact Riemann surface of genus $g > 1$ has finite automorphism group. In this exercise, the following bound on the order of the automorphism group is obtained as an application of Exercises 5.20.2–5.20.8:

Theorem (Hurwitz) *Let X be a compact Riemann surface of genus $g > 1$, and let n be the order of the automorphism group $\Gamma \equiv \operatorname{Aut}(X)$. Then $n \leq 84(g-1)$.*

For the proof, let $\Upsilon\colon X \to Y \equiv \Gamma\backslash X$ be the n-sheeted finite holomorphic branched covering provided by Exercise 5.20.7, let $h \equiv \operatorname{genus}(Y)$, and let $q_1,\dots,q_k$ be the distinct critical values of Υ.

(a) Prove that for each $j = 1,\dots,k$, the groups $\Gamma_p \equiv \{\Phi \in \Gamma \mid \Phi(p) = p\}$ for $p \in \Upsilon^{-1}(q_j)$ have the same order $m_j > 1$ and that m_j divides n.

(b) Prove that

$$2g-2 = n\cdot(2h-2) + n\cdot\sum_{j=1}^{k}\left(1-\frac{1}{m_j}\right)$$

(which is interpreted as $2g-2 = n\cdot(2h-2)$ if Υ has no critical values).

(c) Prove that if $h \geq 2$, then $n \leq g-1$.

(d) Prove that if $h = 1$, then $n \leq 4(g-1)$.

(e) Prove that if $h = 0$ and $k \geq 5$, then $n \leq 4(g-1)$.

(f) Prove that if $h = 0$ and $k = 4$, then $n \leq 12(g-1)$.

(g) Prove that if $h = 0$ and $k = 3$, then $n \leq 84(g-1)$.

(h) Prove that if $h = 0$, then $k > 2$.

5.21 Abel's Theorem

According to the Weierstrass theorem (Theorem 3.12.1), every holomorphic line bundle on an open Riemann surface is holomorphically trivial. Equivalently, every divisor on an open Riemann surface has a solution; that is, there exists a meromorphic function with arbitrary prescribed discrete zeros and poles. This is not the case for a compact Riemann surface. For example, any divisor that is linearly equivalent to the zero divisor must have degree 0. In fact, most divisors of degree zero on a compact Riemann surface of positive genus do *not* have a solution. The goal of this section is Abel's theorem (Theorem 5.21.2), which provides a necessary and sufficient condition for a divisor of degree 0 on a compact Riemann surface to have a solution. Abel's theorem will also allow us to form a holomorphic embedding into a higher-dimensional complex torus (see Sect. 5.22). The approach taken here is similar to the approach in, for example, [For].

Observe that for a Riemann surface X, we may identify the group of divisors with finite support in X with the group of integral 0-chains $C_0(X,\mathbb{Z})$ (see Definition 10.6.3) under the isomorphism

$$D \mapsto \sum_{p\in\operatorname{supp} D} D(p)\cdot p.$$

Thus we have the homomorphism $\partial\colon C_1(X,\mathbb{Z}) \to \operatorname{Div}(X)$ (the boundary operator) and the group of integral 1-cycles $Z_1(X,\mathbb{Z}) = \ker\partial$. Moreover, we have the following fact, the proof of which is left to the reader (see Exercise 5.21.1).

Lemma 5.21.1 *For any divisor D on a compact Riemann surface X, we have* $\deg D = 0$ *if and only if* $D \in \operatorname{im}(\partial\colon C_1(X,\mathbb{Z}) \to \operatorname{Div}(X))$.

Recall that every holomorphic line bundle on a Riemann surface is equal to the line bundle associated to some nontrivial divisor (Corollary 3.11.7 and Theorem 4.2.3). Moreover, a divisor D has a solution (that is, D is the divisor of some meromorphic function) if and only if the associated holomorphic line bundle is holomorphically trivial (Proposition 3.3.2).

The goal of this section is the following:

Theorem 5.21.2 (Abel's theorem) *Suppose D is a divisor of degree 0 on a compact Riemann surface X. Then D has a solution (i.e., the associated holomorphic line bundle $E=[D]$ is holomorphically trivial) if and only if there exists an integral 1-chain $\xi \in C_1(X,\mathbb{Z})$ in X such that $\partial\xi = D$ and $\int_\xi \theta = 0$ for every $\theta \in \Omega(X)$.*

We first recall the notion of a *weak solution* of a divisor D on a Riemann surface X (see Sect. 3.12). Let $E=[D]$ and fix a meromorphic section s of E with $\operatorname{div}(s)=D$ (in particular, s is nontrivial). Then E is $\mathcal{C}^\infty$ trivial if and only if there is a $\mathcal{C}^\infty$ function ρ on $X \setminus D^{-1}((-\infty,0))$ such that ρ is nonvanishing on $X\setminus \operatorname{supp} D$ and s/ρ extends to a nonvanishing $\mathcal{C}^\infty$ section v of E on X. Such a function ρ is called a *weak solution* of the divisor D. Equivalently, a $\mathcal{C}^\infty$ function ρ on $X\setminus D^{-1}((-\infty,0))$ is a weak solution if ρ is nonvanishing on $X\setminus\operatorname{supp} D$ and for each point $p\in\operatorname{supp} D$, there are a local holomorphic coordinate neighborhood (U,z) with $z(p)=0$ and a nonvanishing $\mathcal{C}^\infty$ function ψ on U such that $\rho = z^{D(p)}\cdot\psi$ on $U\setminus\{p\}$. Observe that the *logarithmic differential $d\rho/\rho$* is locally integrable on X, since writing $\rho = z^{D(p)}\cdot\psi$ on a coordinate neighborhood (U,z) with $z(p)=0$ as above, we get, near p,

$$\frac{d\rho}{\rho} = D(p)\cdot\frac{dz}{z} + \frac{d\psi}{\psi}.$$

Observe also that for $v=s/\rho$ as above, we have

$$\frac{\bar\partial v}{v} = -\frac{\bar\partial\rho}{\rho} \quad \text{on } X\setminus\operatorname{supp} D,$$

and for (U,z) as above, we have

$$\frac{\bar\partial v}{v} = -\frac{\bar\partial\psi}{\psi} \quad \text{on } U.$$

Lemma 5.21.3 *If U is a disk in $\mathbb{C}$, $z_0, z_1\in U$, τ is a weak solution of the divisor $F=z_1-z_0$ with $\tau\equiv 1$ on $\mathbb{C}\setminus U$, and f is a holomorphic function on a neighborhood of $\overline{U}$, then*

$$\frac{1}{2\pi i}\int_U \frac{d\tau}{\tau}\wedge df = f(z_1)-f(z_0).$$

Proof The function $z\mapsto \tau(z)\cdot(z-z_0)/(z-z_1)$ extends to a nonvanishing $\mathcal{C}^\infty$ function ψ on $\mathbb{C}$. For $\epsilon>0$ sufficiently small, Stokes' theorem (Theorem 9.7.17) and the Cauchy integral formula and Cauchy's theorem (Lemma 1.2.1) together give

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{U\setminus(\Delta(z_0;\epsilon)\cup\Delta(z_1;\epsilon))}\frac{d\tau}{\tau}\wedge df\\
&\quad=\frac{1}{2\pi i}\int_{\partial\Delta(z_1;\epsilon)} f\cdot\frac{d\tau}{\tau}+\frac{1}{2\pi i}\int_{\partial\Delta(z_0;\epsilon)} f\cdot\frac{d\tau}{\tau}\\
&\quad=\frac{1}{2\pi i}\int_{\partial\Delta(z_1;\epsilon)}\frac{f(z)}{z-z_1}\,dz-\frac{1}{2\pi i}\int_{\partial\Delta(z_0;\epsilon)}\frac{f(z)}{z-z_0}\,dz\\
&\qquad+\frac{1}{2\pi i}\int_{\partial\Delta(z_0;\epsilon)\cup\partial\Delta(z_1;\epsilon)} f\cdot\frac{d\psi}{\psi}\\
&\quad=f(z_1)-f(z_0)+\frac{1}{2\pi i}\int_{\Delta(z_0;\epsilon)\cup\Delta(z_1;\epsilon)} df\wedge\frac{d\psi}{\psi}.
\end{aligned}$$

Letting $\epsilon\to 0^+$, we get the claim. □

Lemma 5.21.4 *Let W be a relatively compact neighborhood of the image $C = \gamma([0,1])$ of a path γ in a Riemann surface X. Then there exists a weak solution ρ of the divisor $D = \partial\gamma = \gamma(1)-\gamma(0)$ such that $\rho\equiv 1$ on $X\setminus W$ and*

$$\int_\gamma\theta=\frac{1}{2\pi i}\int_X\frac{d\rho}{\rho}\wedge\theta\quad\forall\theta\in\Omega(X).$$

Proof By Lemma 3.12.2, there exist a partition $0=t_0<t_1<t_2<\cdots<t_m=1$, local holomorphic charts $\{(U_j,\Phi_j,\Delta(0;1))\}_{j=1}^m$ in W, connected open sets $\{V_j\}_{j=1}^m$ with $\gamma([t_{j-1},t_j])\subset V_j\Subset U_j$ for each $j=1,\dots,m$, and for each $j=1,\dots,m$, a weak solution ρ_j of the divisor $\gamma(t_j)-\gamma(t_{j-1})$ with $\rho_j\equiv 1$ on $X\setminus V_j$. The product $\rho_1\cdots\rho_m$ then extends to a weak solution ρ of D that satisfies $\rho\equiv 1$ on $X\setminus\bigcup_j V_j\supset X\setminus W$. Furthermore, if θ is a holomorphic 1-form on X, then for each $j=1,\dots,m$, we have $\theta\restriction_{U_j}=df_j$ for some function $f_j\in\mathcal{O}(U_j)$ (by Corollary 10.5.7). Therefore, by Lemma 5.21.3,

$$\begin{aligned}
\frac{1}{2\pi i}\int_X\frac{d\rho}{\rho}\wedge\theta&=\sum_{j=1}^m\frac{1}{2\pi i}\int_{V_j}\frac{d\rho_j}{\rho_j}\wedge df_j\\
&=\sum_{j=1}^m\bigl(f_j(\gamma(t_j))-f_j(\gamma(t_{j-1}))\bigr)=\int_\gamma\theta.
\end{aligned}$$

□

Proof of Theorem 5.21.2 For the proof, we let D be a divisor of degree 0, which we may assume to be nontrivial. Suppose D has a solution, that is, there exists a nonconstant meromorphic function f with $\operatorname{div}(f)=D$. Thus $f\colon X\to\mathbb{P}^1$ is an n-sheeted finite holomorphic branched covering map with finite set of critical values C, and we may choose an injective path $\gamma\colon[0,1]\to\mathbb{P}^1$ with $\gamma(0)=\infty$, $\gamma(1)=0$, and $\gamma((0,1))\subset\mathbb{P}^1\setminus(C\cup\{0,\infty\})$. According to Proposition 5.20.4, γ has exactly n

distinct liftings $\gamma_1, \dots, \gamma_n$ to paths in X. Moreover, for each point $p \in f^{-1}(\infty)$, exactly $-D(p)$ of these liftings have initial point p, and for each point $q \in f^{-1}(0)$, exactly $D(q)$ of these liftings have terminal point q. It follows that the integral 1-chain $\xi \equiv \sum_{j=1}^{n} \gamma_j$ satisfies $\partial\xi = D$. Furthermore, if θ is a holomorphic 1-form on X, then Proposition 5.20.4 provides a holomorphic 1-form β on $\mathbb{P}^1$ such that

$$\int_{\xi} \theta = \int_{\gamma} \beta.$$

However, $\mathbb{P}^1$ has no *nontrivial* holomorphic 1-forms (see, for example, Exercise 2.5.4 and Theorem 4.6.8), so $\int_{\xi} \theta = 0$.

Conversely, suppose we have an integral 1-chain ξ in X such that $\partial\xi = D$ and such that every holomorphic 1-form on X integrates to 0 along ξ. Applying Lemma 5.21.4, we get a weak solution ρ of D such that

$$\int_X \frac{\bar{\partial}\rho}{\rho} \wedge \theta = \int_X \frac{d\rho}{\rho} \wedge \theta = 0 \quad \forall \theta \in \Omega(X)$$

(see Exercise 5.21.2). Fixing a meromorphic section s of $E = [D]$ with $\operatorname{div}(s) = D$, the section s/ρ extends to a nonvanishing $\mathcal{C}^\infty$ section v of E. Hence the $\mathcal{C}^\infty$ differential form $\eta \equiv \bar{\partial}v/v$ of type (0, 1), which agrees with $-\bar{\partial}\rho/\rho$ on $X \setminus \operatorname{supp} D$, satisfies

$$\langle \eta, \tau \rangle_{L^2_{0,1}(X)} = \int_X (-i)\eta \wedge \bar{\tau} = 0 \quad \forall \tau \in \overline{\Omega}(X).$$

Applying the Hodge decomposition theorem for scalar-valued forms (Theorem 4.9.1), we get a $\mathcal{C}^\infty$ function φ with $\eta = \bar{\partial}\varphi$ on X. The nonvanishing $\mathcal{C}^\infty$ section $t \equiv e^{-\varphi}v$ of E then satisfies

$$\bar{\partial}t = -e^{-\varphi}\eta \cdot v + e^{-\varphi}\bar{\partial}v = 0,$$

and therefore t is holomorphic. Thus $E = [D]$ is holomorphically trivial and D has a solution. □

Exercises for Sect. 5.21

5.21.1 Prove Lemma 5.21.1.

5.21.2 Verify the claim in the proof of Abel's theorem that for any integral 1-chain ξ in a compact Riemann surface X such that $\partial\xi = D$ and such that every holomorphic 1-form on X integrates to 0 along ξ, there exists a weak solution ρ of D such that

$$\int_X \frac{\bar{\partial}\rho}{\rho} \wedge \theta = \int_X \frac{d\rho}{\rho} \wedge \theta = 0 \quad \forall \theta \in \Omega(X).$$

5.22 The Abel–Jacobi Embedding

The goal of this section is the fact that every compact Riemann surface of positive genus admits a holomorphic embedding into a higher-dimensional complex torus. We first consider the basic facts and definitions concerning such higher-dimensional complex tori, which are analogous to those corresponding to 1-dimensional complex tori considered in Example 2.1.6 (see also Example 5.1.10). The verifications of the basic facts are left to the reader.

For $n \in \mathbb{Z}_{>0}$, a *lattice* in $\mathbb{C}^n$ is a subgroup of the form $\Lambda = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_{2n}$, where $\lambda_1, \ldots, \lambda_{2n} \in \mathbb{C}^n$ are elements that are linearly independent over $\mathbb{R}$. We have an isomorphism of $\mathbb{C}^n$ onto the group of translations in $\mathbb{C}^n$ (which is, of course, a subgroup of the group of self-diffeomorphisms of $\mathbb{C}^n$) under which each element $\zeta \in \mathbb{C}^n$ is identified with the translation given by $z \mapsto z + \zeta$. Thus we may identify Λ with a subgroup of the group of translations in $\mathbb{C}^n$. Clearly, Λ acts freely. Moreover, Λ is the image of $\mathbb{Z}^{2n}$ under the real linear isomorphism $\alpha\colon \mathbb{R}^{2n} \to \mathbb{C}^n$ given by $(t_1, \ldots, t_{2n}) \mapsto t_1\xi_1 + \cdots + t_{2n}\xi_{2n}$, and hence Λ is a discrete subset of $\mathbb{C}^n$. We have the quotient map

$$\Upsilon\colon \mathbb{C}^n \to \mathfrak{T} \equiv \Lambda \backslash \mathbb{C}^n,$$

and the quotient space (which is a quotient group) $\mathfrak{T}$ is called a *complex torus of* (*complex*) *dimension* n.

As in Example 5.1.10, Theorem 10.4.6 implies that $\mathfrak{T}$ admits a unique structure of a $2n$-dimensional compact smooth manifold for which Υ is the $\mathcal{C}^\infty$ universal covering map. In fact, one may show that the local charts given by local inverses of Υ form a holomorphic atlas, and hence that $\mathfrak{T}$ is a (fundamental) example of a complex manifold of complex dimension n and that Υ is a holomorphic covering map (see Exercise 5.22.1). We will not use this fact directly, but it justifies the use of the terminology appearing in Definition 5.22.1 below. As in the 1-dimensional case, we also have a commutative diagram of $\mathcal{C}^\infty$ maps

$$\begin{tikzcd}
\mathbb{R}^{2n} \arrow[r, "\alpha"] \arrow[d, "\Upsilon_0"'] & \mathbb{C}^n \arrow[d, "\Upsilon"] \\
\mathbb{T}^{2n} \arrow[r, "\beta"] & \mathfrak{T}
\end{tikzcd}$$

where α is the real linear isomorphism given by $(t_1, \ldots, t_{2n}) \mapsto t_1\lambda_1 + \cdots + t_{2n}\lambda_{2n}$, Υ_0 is the $\mathcal{C}^\infty$ covering map onto the real torus $\mathbb{T}^{2n} = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ given by $(t_1, \ldots, t_{2n}) \mapsto (e^{2\pi i t_1}, \ldots, e^{2\pi i t_{2n}})$, and β is the induced diffeomorphism.

Definition 5.22.1 Let $n \in \mathbb{Z}_{>0}$, let Λ be a lattice in $\mathbb{C}^n$, let $\Upsilon\colon \mathbb{C}^n \to \mathfrak{T}$ be the associated quotient map onto the complex torus $\mathfrak{T} \equiv \Lambda \backslash \mathbb{C}^n$, and let $\Phi\colon X \to \mathfrak{T}$ be a continuous mapping of a complex 1-manifold X into $\mathfrak{T}$.

(a) Φ is called *holomorphic* if the mapping $\Psi \circ \Phi$ is holomorphic for each local inverse Ψ of Υ.

(b) Φ is called a *holomorphic immersion* if the mapping $\Psi \circ \Phi$ is a holomorphic immersion for each local inverse Ψ of Υ.
(c) Φ is called a *holomorphic embedding* if X is compact and Φ is an injective holomorphic immersion (i.e., Φ is a proper injective holomorphic immersion).

The main goal of this section is the following:

Theorem 5.22.2 (Abel–Jacobi embedding theorem) *Let X be a compact Riemann surface of genus $g > 0$, and let $\theta = (\theta_1, \dots, \theta_g)$ be a* (*complex*) *basis for $\Omega(X)$. Then we have the following*:

(a) *The mapping*

$$[\xi]_{H_1(X,\mathbb{Z})} \mapsto \int_\xi \theta \equiv \left(\int_\xi \theta_1, \dots, \int_\xi \theta_g\right)$$

determines a well-defined injective group homomorphism $H_1(X,\mathbb{Z}) \to \mathbb{C}^g$, and the image Λ_θ of this homomorphism is a lattice in $\mathbb{C}^g$. In fact, the image of any integral basis for $H_1(X,\mathbb{Z})$ is a real basis for $\mathbb{C}^g$.

(b) *Let $\Upsilon_\theta \colon \mathbb{C}^g \to \mathfrak{T}_\theta$ be the covering map associated to the complex torus $\mathfrak{T}_\theta \equiv \Lambda_\theta \backslash \mathbb{C}^g$. Given a point $p_0 \in X$, the mapping*

$$p \mapsto \Upsilon_\theta\left(\int_\gamma \theta\right) = \Upsilon_\theta\left(\int_\gamma \theta_1, \dots, \int_\gamma \theta_g\right),$$

where γ is an arbitrary path in X from p_0 to p, determines a well-defined holomorphic embedding $\mathfrak{J}_{\theta,p_0} \colon X \to \mathfrak{T}_\theta$.

Definition 5.22.3 Given a compact Riemann surface X of genus $g > 0$ and a basis $\theta = (\theta_1, \dots, \theta_g)$ for $\Omega(X)$, the lattice Λ_θ in $\mathbb{C}^g$ provided by the Abel–Jacobi embedding theorem is called the *period lattice* associated to θ. The corresponding complex torus $\Lambda_\theta \backslash \mathbb{C}^g$ is called the *Jacobi variety* of X and is denoted by $\mathrm{Jac}(X)$ or $\mathrm{Jac}(X,\theta)$. Given a point $p_0 \in X$, the holomorphic embedding $\mathfrak{J}_{\theta,p_0} \colon X \to \mathrm{Jac}(X)$ provided by the Abel–Jacobi embedding theorem is called the *Abel–Jacobi embedding* associated to the basis θ and the point p_0.

Remarks 1. As defined above, the period lattice, Jacobi variety, and Abel–Jacobi embedding depend on the choice of a basis for $\Omega(X)$ (and the Abel–Jacobi embedding depends on the choice of an initial point $p_0 \in X$). A more intrinsic, but equivalent, point of view also exists (see, for example, [GriH]).

2. For a higher-dimensional smooth complex projective variety (or even for a connected compact complex manifold) with nontrivial holomorphic 1-forms, one may form the analogous holomorphic mapping, which is called the *Albanese map*. The target complex torus is called the *Albanese variety*. This mapping is important in algebraic geometry (see, for example, [GriH]). In higher dimensions, the Albanese map need not be an embedding (for example, for Y a compact Riemann surface of positive genus, $X \equiv Y \times \mathbb{P}^1$ is a smooth projective variety that admits

nontrivial holomorphic 1-forms, but the corresponding Albanese map is constant on $\{y\} \times \mathbb{P}^1$ for each point $y \in Y$).

For genus 1, the Abel–Jacobi embedding theorem immediately gives the following (cf. Exercises 5.9.3 and 5.16.1):

Corollary 5.22.4 *Every compact Riemann surface of genus* 1 *is biholomorphic to a complex torus* (*of dimension* 1).

Proof of Theorem 5.22.2 It follows from the definition of integral homology (Definition 10.7.9) that the mapping $H_1(X, \mathbb{Z}) \to \mathbb{C}^g$ in (a) is a well-defined group homomorphism. Moreover, since

$$(\theta, \bar{\theta}) \equiv (\theta_1, \dots, \theta_g, \bar{\theta}_1, \dots, \bar{\theta}_g)$$

is a basis for $\Omega(X) \oplus \overline{\Omega}(X) \cong H^1_{\mathrm{deR}}(X, \mathbb{C})$ (Corollary 4.9.2), the (well-defined) linear map $H_1(X, \mathbb{C}) \to \mathbb{C}^{2g}$ given by

$$[\xi]_{H_1} \mapsto \left(\int_\xi \theta, \int_\xi \bar{\theta}\right) = \left(\int_\xi \theta_1, \dots, \int_\xi \theta_g, \int_\xi \bar{\theta}_1, \dots, \int_\xi \bar{\theta}_g\right) = \left(\int_\xi \theta, \overline{\int_\xi \theta}\right)$$

is injective, and by Theorem 10.7.18, the images under this map of the elements of any integral basis $\xi_1, \dots, \xi_{2g}$ for $H_1(X, \mathbb{Z}) \subset H_1(X, \mathbb{C})$ form a complex basis for $\mathbb{C}^{2g}$. Setting $\lambda_j \equiv \int_{\xi_j} \theta \in \mathbb{C}^g$ for each $j = 1, \dots, 2g$, it follows that $\lambda_1, \dots, \lambda_{2g}$ is a real basis for $\mathbb{C}^g$. For if $a_j \in \mathbb{R}$ for $j = 1, \dots, 2g$ and $\sum_{j=1}^{2g} a_j \lambda_j = 0$, then we have

$$\sum_{j=1}^{2g} a_j \left(\int_{\xi_j} \theta, \overline{\int_{\xi_j} \theta}\right) = \left(\sum_{j=1}^{2g} a_j \lambda_j, \overline{\sum_{j=1}^{2g} a_j \lambda_j}\right) = (0, 0),$$

and therefore $a_1 = \cdots = a_{2g} = 0$. Part (a) now follows.

For the proof of (b), let us fix a point $p_0 \in X$. We will let p and q denote points in X, and we will let α and β denote two paths in X with $\alpha(0) = \beta(0) = p_0$, $\alpha(1) = p$, and $\beta(1) = q$.

If $p = q$, then $\int_\alpha \theta - \int_\beta \theta = \int_{\alpha * \beta^-} \theta \in \Lambda_\theta$, and hence $\Upsilon_\theta(\int_\alpha \theta) = \Upsilon_\theta(\int_\beta \theta)$. Thus the associated map $\mathfrak{J}_{\theta, p_0} \colon X \to \mathfrak{T}_\theta$ is well defined. Conversely, if p and q satisfy $\mathfrak{J}_{\theta, p_0}(p) = \mathfrak{J}_{\theta, p_0}(q)$, then

$$\int_\alpha \theta - \int_\beta \theta = \lambda = \int_\xi \theta \in \Lambda_\theta$$

for some integral 1-cycle $\xi \in Z_1(X, \mathbb{Z})$. Hence the 1-chain $\zeta \equiv \alpha - \beta - \xi \in C_1(X, \mathbb{Z})$ has boundary divisor $D = \partial\zeta = p - q$ and satisfies

$$\int_\zeta \eta = 0 \quad \forall \eta \in \Omega(X).$$

Abel's theorem (Theorem 5.21.2) now implies that $p = q$. For if D were nontrivial (i.e., if p and q were distinct points), then Abel's theorem would provide a nontrivial meromorphic function $f: X \to \mathbb{P}^1$ with $\mathrm{div}(f) = D$, and by Proposition 2.5.7 (or Proposition 5.20.1), f would be a biholomorphism. Thus $\mathfrak{J}_{\theta, p_0}$ is injective.

We may fix a holomorphic mapping $F = (f_1, \dots, f_g): U \to \mathbb{C}^g$ on a connected neighborhood U of p in X such that $dF = (df_1, \dots, df_g) = \theta$ on U, $F(p) = \int_\alpha \theta$, and $F(U)$ is contained in a connected neighborhood V of $F(p)$ that Υ_θ maps homeomorphically onto its image $\Upsilon_\theta(V)$. Therefore, if W is a connected neighborhood of $\mathfrak{J}_{\theta, p_0}(p) = \Upsilon_\theta(F(p))$ in $\Upsilon_\theta(V)$, $\Phi: W \to \Phi(W) \subset \mathbb{C}^g$ is a local inverse of Υ_θ, and $Q \equiv \Upsilon_\theta^{-1}(W) \cap V$, then the homeomorphism $\Phi \circ \Upsilon_\theta \restriction_Q : Q \to \Phi(W)$ is given by $z \mapsto z + \lambda$ for some (constant) element $\lambda \in \Lambda_\theta$. Thus, on the neighborhood $F^{-1}(Q)$ of p, we have

$$G \equiv \Phi \circ \mathfrak{J}_{\theta, p_0} = \Phi \circ \Upsilon_\theta \circ F = F + \lambda.$$

Thus G is holomorphic and $(dG)_p = ((\theta_1)_p, \dots, (\theta_g)_p)$. In particular, since the nontrivial holomorphic 1-forms $\theta_1, \dots, \theta_g$ have no common zeros (by Theorem 4.6.8), we have $(dG)_p \neq 0$. Thus the mapping $\mathfrak{J}_{\theta, p_0}: X \to \mathfrak{T}_\theta$ is a holomorphic embedding. □

Exercises for Sect. 5.22

5.22.1 Verify that an n-dimensional complex torus $X = \Lambda \backslash \mathbb{C}^n$ admits the structure of an n-dimensional complex manifold (see Exercise 2.2.6) for which the quotient mapping $\mathbb{C}^n \to X$ is holomorphic with local holomorphic inverses.

Chapter 6
Holomorphic Structures on Topological Surfaces

In this chapter, we address the natural problem of determining conditions for a topological surface to admit a holomorphic structure. One necessary condition is, of course, that the surface be orientable. According to Radó's theorem (Theorem 2.11.1), another necessary condition is that the surface be second countable. It turns out that these two conditions are also sufficient.

Sections 6.1–6.6 consist of a proof of the theorem of Korn and Lichtenstein, according to which every almost complex structure on a smooth surface is *integrable*; that is, it is induced by a holomorphic structure. It then follows that every orientable second countable smooth surface admits a holomorphic structure. The proof of integrability appearing in this chapter is based on the proofs of the higher-dimensional analogue appearing in [Hö] and [De3].

Sections 6.7–6.11 consist of a proof that every second countable topological surface admits a smooth structure. The first part of the chapter then implies that any second countable orientable topological surface admits a holomorphic structure. One of the main tools in the proof of the existence of smooth structures is Schönflies' theorem, and a proof (due to Kneser and Radó) of this fact appears in Sects. 6.7–6.9.

The proof of integrability and the proof of the existence of smooth structures may be read independently.

6.1 Almost Complex Structures on Smooth Surfaces

The main goal of Sects. 6.1–6.6 is the existence of holomorphic structures on second countable oriented smooth surfaces. One obtains a holomorphic structure by first producing an *almost complex structure*.

Definition 6.1.1 Let X be a 2-dimensional smooth manifold.

(a) An *almost complex structure* on X is a choice of subsets

$$(TX)^{1,0} = \bigcup_{p\in X} (T_pX)^{1,0} \quad \text{and} \quad (TX)^{0,1} = \bigcup_{p\in X} (T_pX)^{0,1}$$

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of $(TX)_{\mathbb{C}}$ such that

(i) For each point $p \in X$, $(T_pX)^{1,0}$ and $(T_pX)^{0,1}$ are complex subspaces of $(T_pX)_{\mathbb{C}}$ with $(T_pX)_{\mathbb{C}} = (T_pX)^{1,0} \oplus (T_pX)^{0,1}$ and $(T_pX)^{1,0} = \overline{(T_pX)^{0,1}}$; and

(ii) For each point in X, there is a nonvanishing $\mathcal{C}^\infty$ vector field $\mathcal{Z}$ on a neighborhood U such that $(T_pX)^{1,0} = \mathbb{C} \cdot \mathcal{Z}_p$ (hence $(T_pX)^{0,1} = \mathbb{C} \cdot \overline{\mathcal{Z}_p}$) for each point $p \in U$.

We write $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$. The manifold X together with an almost complex structure is called an *almost complex manifold of dimension* 2 (or an *almost complex surface* if X is connected).

(b) For any almost complex structure decomposition $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$, we have the associated decomposition $(T^*X)_{\mathbb{C}} = (T^*X)^{1,0} \oplus (T^*X)^{0,1}$ of the cotangent bundle with summands

$$(T^*X)^{1,0} = \bigcup_{p \in X} (T_p^*X)^{1,0} \quad \text{and} \quad (T^*X)^{0,1} = \bigcup_{p \in X} (T_p^*X)^{0,1},$$

where for each point $p \in X$,

$$(T_p^*X)^{1,0} \equiv \{\alpha \in (T_p^*X)_{\mathbb{C}} \mid \alpha(v) = 0 \; \forall v \in (T_pX)^{0,1}\} \cong [(T_pX)^{1,0}]^*$$

and

$$\begin{aligned}(T_p^*X)^{0,1} &\equiv \overline{(T_p^*X)^{1,0}} = \{\alpha \in (T_p^*X)_{\mathbb{C}} \mid \alpha(v) = 0 \; \forall v \in (T_pX)^{1,0}\} \\ &\cong [(T_pX)^{0,1}]^*\end{aligned}$$

(in particular, $(T_p^*X)_{\mathbb{C}} = (T_p^*X)^{1,0} \oplus (T_p^*X)^{0,1}$).

(c) Let $\mathcal{Z}$ be a nonvanishing $\mathcal{C}^\infty$ vector field with values in $(TX)^{1,0}$ on an open set $U \subset X$, and let θ be the 1-form on U determined by $\theta(\mathcal{Z}) \equiv 1$ and $\theta(\overline{\mathcal{Z}}) \equiv 0$. For any vector field v, we call the functions $a \equiv \theta(v)$ and $b \equiv \bar{\theta}(v)$ (i.e., $v = a\mathcal{Z} + b\overline{\mathcal{Z}}$) the *coefficients of* v with respect to $\mathcal{Z}$ and $\overline{\mathcal{Z}}$. For any 1-form α, we call the functions $A \equiv \alpha(\mathcal{Z})$ and $B \equiv \alpha(\overline{\mathcal{Z}})$ (i.e., $\alpha = A\theta + B\bar{\theta}$) the *coefficients of* α with respect to θ and $\bar{\theta}$.

(d) An almost complex structure $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ on X is *integrable* if this decomposition is equal to the decomposition of $(TX)_{\mathbb{C}}$ into $(1,0)$ and $(0,1)$ parts associated to some holomorphic structure on X; that is, the almost complex structure is *induced* by the holomorphic structure.

Remark For an almost complex structure $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$, we use the notation and terminology analogous to that used in the integrable case. For example, an element of $(TX)^{1,0}$ is of *type* $(1,0)$.

Recall that by definition, a vector field or differential form on a smooth manifold is continuous ($\mathcal{C}^k$ with $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, measurable, in L^d_{loc} with $d \in [1,\infty]$) if its coefficients in local $\mathcal{C}^\infty$ charts are continuous (respectively, $\mathcal{C}^k$, measurable, in L^d_{loc}). We have the following analogous characterization on an almost complex surface:

Proposition 6.1.2 *Let $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ be an almost complex structure on a smooth surface X, let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, and let $d \in [1, \infty]$.*

(a) *If v is a vector field on a set $S \subset X$, then the following are equivalent:*
  (i) *The vector field v is continuous ($\mathcal{C}^k$, measurable, in L^d_{loc}).*
  (ii) *The coefficients of v with respect to $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ are continuous (respectively, $\mathcal{C}^k$, measurable, in L^d_{loc}) for every nonvanishing $\mathcal{C}^\infty$ local vector field $\mathcal{Z}$ of type $(1, 0)$.*
  (iii) *For every point $p \in S$, the coefficients of v with respect to $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ are continuous (respectively, $\mathcal{C}^k$, measurable, in L^d_{loc}) for some nonvanishing $\mathcal{C}^\infty$ vector field $\mathcal{Z}$ of type $(1, 0)$ on a neighborhood of p.*
  (iv) *The $(1, 0)$ and $(0, 1)$ parts of v are continuous (respectively, $\mathcal{C}^k$, measurable, in L^d_{loc}).*

(b) *If α is a 1-form on a set $S \subset X$, then the following are equivalent:*
  (i) *The 1-form α is continuous ($\mathcal{C}^k$, measurable, in L^d_{loc}).*
  (ii) *For every nonvanishing $\mathcal{C}^\infty$ local vector field $\mathcal{Z}$ of type $(1, 0)$, the coefficients of α with respect to the corresponding dual 1-forms θ and $\bar{\theta}$ (determined by $\theta(\mathcal{Z}) \equiv 1$ and $\theta(\overline{\mathcal{Z}}) \equiv 0$) are continuous (respectively, $\mathcal{C}^k$, measurable, in L^d_{loc}).*
  (iii) *For every point $p \in S$, there exists a nonvanishing $\mathcal{C}^\infty$ local vector field $\mathcal{Z}$ of type $(1, 0)$ on a neighborhood of p such that the coefficients of α with respect to the corresponding dual 1-forms θ and $\bar{\theta}$ are continuous (respectively, $\mathcal{C}^k$, measurable, in L^d_{loc}).*
  (iv) *The $(1, 0)$ and $(0, 1)$ parts of α are continuous (respectively, $\mathcal{C}^k$, measurable, in L^d_{loc}).*

(c) *A nonvanishing local vector field $\mathcal{Z}$ of type $(1, 0)$ is continuous ($\mathcal{C}^k$, measurable) if and only if its dual 1-form θ (of type $(1, 0)$) is continuous (respectively, $\mathcal{C}^k$, measurable). In particular, for each point in $p \in X$, there exists a nonvanishing $\mathcal{C}^\infty$ differential form of type $(1, 0)$ on a neighborhood of p.*

Proof Let $\mathcal{Z}$ be a nonvanishing $\mathcal{C}^\infty$ vector field of type $(1, 0)$ on an open set $U \subset X$, and let θ and $\bar{\theta}$ be the corresponding dual 1-forms. Suppose also that we have a local $\mathcal{C}^\infty$ chart $(U, \Phi = (x, y), \Phi(U))$. Then the corresponding coefficients $a = dx(\mathcal{Z})$ and $b = dy(\mathcal{Z})$ of $\mathcal{Z}$ are of class $\mathcal{C}^\infty$, and we have

$$\mathcal{Z} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \quad \text{and} \quad \overline{\mathcal{Z}} = \bar{a}\frac{\partial}{\partial x} + \bar{b}\frac{\partial}{\partial y}.$$

Hence the matrix $C \equiv \left(\begin{smallmatrix} a & b \\ \bar{a} & \bar{b} \end{smallmatrix}\right)$ is nonsingular at each point and

$$C^{-1} = \frac{1}{a\bar{b} - \bar{a}b}\begin{pmatrix} \bar{b} & -b \\ -\bar{a} & a \end{pmatrix}$$

has $\mathcal{C}^\infty$ entries. Setting $A \equiv \theta(\partial/\partial x)$ and $B \equiv \theta(\partial/\partial y)$, we get $\theta = A\,dx + B\,dy$ and

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \theta(\mathcal{Z}) \\ \theta(\overline{\mathcal{Z}}) \end{pmatrix} = C \begin{pmatrix} A \\ B \end{pmatrix}.$$

Hence the column vector

$$\begin{pmatrix} A \\ B \end{pmatrix} = C^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

has $\mathcal{C}^\infty$ entries, and therefore θ is a $\mathcal{C}^\infty$ form of type $(1, 0)$. Armed with this fact, it is now easy to verify (a)–(c) (see Exercise 6.1.1). □

Proposition 6.1.3 *Let X be a smooth surface.*

(a) *If X admits an almost complex structure $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$, then X is orientable. In fact, there is a unique orientation for which $i\alpha \wedge \bar{\alpha} > 0$ for each $\alpha \in (T^*X)^{1,0} \setminus \{0\}$.*
(b) *If X is oriented and second countable, then X admits an almost complex structure that induces (as in (a)) the given orientation.*

Proof For the proof of (a), suppose $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ is an almost complex structure. If $(U, (x, y))$ is a connected local $\mathcal{C}^\infty$ coordinate neighborhood, then either

$$\frac{i\alpha \wedge \bar{\alpha}}{(dx \wedge dy)_p} > 0 \quad \forall p \in U, \alpha \in (T_p^*X)^{1,0} \setminus \{0\},$$

or

$$\frac{i\alpha \wedge \bar{\alpha}}{(dx \wedge dy)_p} < 0 \quad \forall p \in U, \alpha \in (T_p^*X)^{1,0} \setminus \{0\}.$$

For if θ is a nonvanishing $\mathcal{C}^\infty$ differential form of type $(1, 0)$ on an open set $V \subset U$, then

$$\frac{i\alpha \wedge \bar{\alpha}}{(dx \wedge dy)_p} = |\alpha/\theta_p|^2 \frac{i(\theta \wedge \bar{\theta})_p}{(dx \wedge dy)_p} \quad \forall p \in V, \alpha \in (T_p^*X)^{1,0} \setminus \{0\},$$

and the real-valued $\mathcal{C}^\infty$ function $i(\theta \wedge \bar{\theta})/(dx \wedge dy)$ is nonvanishing. So if, for some choice of θ and V, $i(\theta \wedge \bar{\theta})/(dx \wedge dy) > 0$ at some point in V, then the set of points $p \in U$ at which the inequality $i(\alpha \wedge \bar{\alpha})/(dx \wedge dy)_p > 0$ holds for every $\alpha \in (T_p^*X)^{1,0} \setminus \{0\}$ is nonempty with empty boundary in U, and is therefore equal to U. Similarly, if (x, y) and (x', y') are two choices of local $\mathcal{C}^\infty$ coordinates for which the above quotients are positive, then we have $(dx \wedge dy)/(dx' \wedge dy') > 0$, and hence they have compatible orientations. The claim (a) now follows.

As motivation for the proof of (b), we first examine how the orientation in $\mathbb{C} = \mathbb{R}^2$ relates to the standard (integrable) almost complex structure. For the standard coordinate $z = x + iy \leftrightarrow (x, y)$, we have, at each point p,

$$(T_p\mathbb{R}^2)^{1,0} = (T_p\mathbb{C})^{1,0} = \mathbb{C} \cdot \left(\frac{\partial}{\partial z}\right)_p = \mathbb{C} \cdot \frac{1}{2}\left[\left(\frac{\partial}{\partial x}\right)_p - i\left(\frac{\partial}{\partial y}\right)_p\right].$$

Under the identification of $T_p\mathbb{R}^2$ with $\{p\} \times \mathbb{R}^2$, we see that $(\partial/\partial y)_p$ is obtained by multiplication of $(\partial/\partial x)_p$ by i, that is, by rotation of the vector counterclockwise by 90°. On a general surface, one may define a counterclockwise 90° rotation if one has a Riemannian metric (to determine the rotational angle measure) and an orientation (to determine the counterclockwise sense).

Suppose now that X is a second countable oriented smooth surface. By Proposition 9.11.2, X admits a Riemannian metric g. For each point $p \in X$, we may let $(T_pX)^{1,0}$ be the collection of all vectors of the form $w = \frac{1}{2}(u - iv)$, where $u, v \in T_pX$ have the following properties:

(i) We have $g(u, v) = 0$ (i.e., $u \perp v$) and $|u|_g = |v|_g$; and
(ii) We have $\alpha(u, v) \geq 0$ for every positive $\alpha \in \Lambda^2 T_p^* X$.

Setting $(T_pX)^{0,1} \equiv \overline{(T_pX)^{1,0}}$ for each point $p \in X$, we now show that the spaces

$$(TX)^{1,0} \equiv \bigcup_{p\in X} (T_pX)^{1,0} \quad \text{and} \quad (TX)^{0,1} \equiv \bigcup_{p\in X} (T_pX)^{0,1}$$

determine an almost complex structure $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ on X. For this, let us fix a positively oriented local $\mathcal{C}^\infty$ coordinate neighborhood $(U, (x, y))$ in X. Then we have $\mathcal{C}^\infty$ real vector fields

$$u \equiv \frac{\partial/\partial x}{|\partial/\partial x|_g} \quad \text{and} \quad v \equiv \frac{(\partial/\partial y) - g(\partial/\partial y, u)u}{|(\partial/\partial y) - g(\partial/\partial y, u)u|_g}$$

that satisfy $g(u, v) \equiv 0$ and $|u|_g = |v|_g \equiv 1$ on U. Moreover, for every point $p \in U$ and every positive $\alpha \in \Lambda^2 T_p^* X$, we have

$$\begin{aligned}\alpha(u_p, v_p) &= \frac{\alpha}{(dx \wedge dy)_p} \cdot (dx \wedge dy)(u_p, v_p)\\ &= \frac{\alpha}{(dx \wedge dy)_p} \cdot \frac{1}{|(\partial/\partial x)_p|_g} \cdot \frac{1}{|(\partial/\partial y)_p - g((\partial/\partial y)_p, u_p)u_p|_g} > 0.\end{aligned}$$

Thus the values of the nonvanishing $\mathcal{C}^\infty$ vector field $\mathcal{Z} \equiv (1/2)(u - iv)$ on U lie in $(TX)^{1,0}$. For each point $p \in U$ and each complex number $\zeta = a + ib$ with $a, b \in \mathbb{R}$, we have

$$\zeta \cdot \mathcal{Z}_p = \frac{1}{2}((au_p + bv_p) - i(-bu_p + av_p)),$$

$g(au_p + bv_p, -bu_p + av_p) = -ab + ba = 0$, $|au_p + bv_p|_g^2 = |\zeta|^2 = |-bu_p + av_p|_g^2$, and for each positive $\alpha \in \Lambda^2(T_p^* X)$,

$$\alpha(au_p + bv_p, -bu_p + av_p) = |\zeta|^2 \alpha(u_p, v_p) \geq 0.$$

Thus $\mathbb{C}\cdot\mathcal{Z}_p\subset(T_pX)^{1,0}$. Conversely, given an element $w=(1/2)(r-is)\in(T_pX)^{1,0}$, we have $r=au_p+bv_p$ with $a,b\in\mathbb{R}$, $s\perp r$, and $|s|_g^2=|r|_g^2=a^2+b^2$. Thus $s=\pm(-bu_p+av_p)$. On the other hand, we also have $(dx\wedge dy)(r,s)\geq 0$ and

$$(dx\wedge dy)(r,-bu_p+av_p)=(a^2+b^2)(dx\wedge dy)(u_p,v_p)\geq 0$$

(with equality if and only if $r=s=0$). Thus $s=-bu_p+av_p$, and hence for $\zeta=a+ib$, we have $w=\zeta\cdot\mathcal{Z}_p$. Therefore, $(T_p^{1,0}X)^{1,0}=\mathbb{C}\cdot\mathcal{Z}_p$. It also follows that $\mathcal{Z}_p$ and $\overline{\mathcal{Z}_p}$ are linearly independent over $\mathbb{C}$, since $(dx\wedge dy)(u_p,-v_p)<0$. Thus we have produced an almost complex structure $(TX)_{\mathbb{C}}=(TX)^{1,0}\oplus(TX)^{0,1}$. The verification that this almost complex structure induces the given orientation is left to the reader (see Exercise 6.1.2). □

A holomorphic structure induces an almost complex structure. The main goal of Sects. 6.1–6.6 is the following:

Theorem 6.1.4 (Integrability theorem) *Every almost complex structure on a smooth surface is integrable. In fact, every almost complex structure is induced by a unique* (1*-dimensional*) *holomorphic structure.*

Radó's theorem (Theorem 2.11.1) and the above integrability theorem together give the following:

Corollary 6.1.5 *Any smooth surface that admits an almost complex structure must be second countable.*

Proposition 6.1.3 and the integrability theorem together give the following:

Corollary 6.1.6 *Every oriented second countable smooth surface admits a holomorphic structure that induces the given orientation.*

Definition 6.1.7 Let $(TX)_{\mathbb{C}}=(TX)^{1,0}\oplus(TX)^{0,1}$ be an almost complex structure on a 2-dimensional smooth manifold X, let α be a $\mathcal{C}^1$ differential form of degree r on an open set $\Omega\subset X$, and let $\mathcal{P}^{p,q}\colon\Lambda^1(T^*X)_{\mathbb{C}}\to\Lambda^{p,q}T^*X$ be the associated projection for each pair $(p,q)\in\{(1,0),(0,1)\}$. Then we define

$$\partial\alpha\equiv\begin{cases}\mathcal{P}^{1,0}(d\alpha) & \text{if } r=0,\\ d(\mathcal{P}^{0,1}\alpha) & \text{if } r=1,\\ 0 & \text{if } r\geq 2,\end{cases}\quad\text{and}\quad\bar\partial\alpha\equiv\begin{cases}\mathcal{P}^{0,1}(d\alpha) & \text{if } r=0,\\ d(\mathcal{P}^{1,0}\alpha) & \text{if } r=1,\\ 0 & \text{if } r\geq 2.\end{cases}$$

Remarks Proposition 6.1.2 allows us to make the above definition because according to this proposition, the (r,s) part of a $\mathcal{C}^k$ differential form is of class $\mathcal{C}^k$. We also recall that we use the notation and terminology analogous to that used in the integrable case. For example, a differential form α is $\bar\partial$*-closed* if $\bar\partial\alpha=0$. According to Proposition 6.1.8 below, the operators ∂ and $\bar\partial$ retain at least some of the properties that these operators have in the integrable case (cf. Proposition 2.5.5).

Proposition 6.1.8 *Let $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ be an almost complex structure on a smooth surface X. Then*

(a) *The operators d, ∂, and $\bar{\partial}$ satisfy $d = \partial + \bar{\partial}$ on $\mathcal{C}^1$ forms and $d^2 = \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$ on $\mathcal{C}^2$ forms.*
(b) *If α is a $\mathcal{C}^1$ differential form of type (p, q), then $\partial\alpha$ is of type $(p+1, q)$ and $\bar{\partial}\alpha$ is of type $(p, q+1)$. In particular, $\partial\alpha = 0$ if $p \geq 1$ and $\bar{\partial}\alpha = 0$ if $q \geq 1$.*
(c) *If α and β are $\mathcal{C}^1$ differential forms, then*

$$\partial(\alpha \wedge \beta) = (\partial\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \partial\beta$$

and

$$\bar{\partial}(\alpha \wedge \beta) = (\bar{\partial}\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \bar{\partial}\beta.$$

The proof is left to the reader (see Exercise 6.1.3). Note that we have not included an analogue of the property (d) of Proposition 2.5.5 (as well as (c), (f), and (g)). The formation of such an analogue is at the heart of the proof of the integrability theorem.

As a first step toward the proof of the integrability theorem, we show that the integrability condition is completely local. For this, we will need the following:

Lemma 6.1.9 *Let X be an almost complex surface, let $f = u + iv \colon X \to \mathbb{C}$ be a $\mathcal{C}^1$ function with $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ (which we may also view as a $\mathcal{C}^1$ mapping $f = (u, v) \colon X \to \mathbb{R}^2$), and let $p \in X$. Assume that $(\bar{\partial} f)_p = 0$. Then the complex linear extension $(f_*)_p \colon (T_p X)_{\mathbb{C}} \to (T_{f(p)}\mathbb{C})_{\mathbb{C}}$ of the real linear tangent map $(f_*)_p \colon T_p X \to T_{f(p)}\mathbb{C} = T_{f(p)}\mathbb{R}^2$ preserves the almost complex structures; that is,*

$$(f_*)_p (T_p X)^{r,s} \subset (T_{f(p)}\mathbb{C})^{r,s} \quad \forall (r, s) \in \{(1, 0), (0, 1)\}.$$

Moreover, if, in addition, $(df)_p \neq 0$ (i.e., $(\partial f)_p \neq 0$), then the real linear maps

$$((du)_p, (dv)_p) \colon T_p X \to \mathbb{R}^2 \quad \text{and} \quad (f_*)_p \colon T_p X \to T_p\mathbb{R}^2 = T_p\mathbb{C},$$

and the complex linear maps

$$\begin{aligned} &(f_*)_p \colon (T_p X)_{\mathbb{C}} \to (T_p\mathbb{C})_{\mathbb{C}}, \\ &(df)_p\restriction_{(T_p X)^{1,0}} = dz \circ (f_*)_p\restriction_{(T_p X)^{1,0}} \colon (T_p X)^{1,0} \to \mathbb{C}, \\ &(d\bar{f})_p\restriction_{(T_p X)^{0,1}} = d\bar{z} \circ (f_*)_p\restriction_{(T_p X)^{0,1}} \colon (T_p X)^{0,1} \to \mathbb{C}, \end{aligned}$$

are bijective.

Proof To avoid confusion, when thinking of the complex-valued function f as a mapping of X into the Riemann surface $\mathbb{C}$, let us denote this mapping by F. Letting $z = x + iy$ denote the complex coordinate in $\mathbb{C} = \mathbb{R}^2$ with $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$, we then have $f = z \circ F$ and $\bar{f} = \bar{z} \circ F$. Note also that for each point $a \in \mathbb{C}$, $(dz)_a$

maps $(T_a\mathbb{C})^{1,0}$ isomorphically onto $\mathbb{C}$ and vanishes on $(T_a\mathbb{C})^{0,1}$, while $(d\bar{z})_a$ maps $(T_a\mathbb{C})^{0,1}$ isomorphically onto $\mathbb{C}$ and vanishes on $(T_a\mathbb{C})^{1,0}$. Given a tangent vector $v \in (T_pX)^{1,0}$, we have

$$0 = \bar{\partial} f(\bar{v}) = df(\bar{v}) = dz(F_*\bar{v}) \quad \text{and} \quad 0 = \overline{df(\bar{v})} = d\bar{f}(v) = d\bar{z}(F_*v).$$

It follows that $(F_*)_p(T_pX)^{r,s} \subset (T_{F(p)}\mathbb{C})^{r,s}$ for $(r,s) \in \{(1,0),(0,1)\}$. Furthermore, if $(df)_p \neq 0$, then we may choose a tangent vector $w \in (T_pX)_{\mathbb{C}}$ with

$$0 \neq df(w) = \partial f(w) = df(\mathcal{P}^{1,0}w).$$

Thus $(df)_p\colon (T_pX)^{1,0} \to \mathbb{C}$ is a nontrivial linear mapping of 1-dimensional complex vector spaces, and is therefore an isomorphism. It follows that the complex linear maps

$$(F_*)_p\restriction_{(T_pX)^{1,0}} = \left[dz\restriction_{(T_{F(p)}\mathbb{C})^{1,0}}\right]^{-1} \circ (df)_p\restriction_{(T_pX)^{1,0}}\colon (T_pX)^{1,0} \to (T_{F(p)}\mathbb{C})^{1,0}$$

and

$$(F_*)_p\restriction_{(T_pX)^{0,1}} = \left[d\bar{z}\restriction_{(T_{F(p)}\mathbb{C})^{0,1}}\right]^{-1} \circ (d\bar{f})_p\restriction_{(T_pX)^{0,1}}\colon (T_pX)^{0,1} \to (T_{F(p)}\mathbb{C})^{0,1}$$

are bijective. In particular, the complexified tangent map $(F_*)_p\colon (T_pX)_{\mathbb{C}} \to (T_{F(p)}\mathbb{C})_{\mathbb{C}}$ is bijective, and hence, by Proposition 8.1.3, the corresponding real tangent map $(F_*)_p\colon T_pX \to T_{F(p)}\mathbb{C}$ must also be bijective, as must be the map

$$((du)_p, (dv)_p) = ((dx)_p, (dy)_p) \circ F_*\colon T_pX \to \mathbb{R}^2. \qquad \square$$

We also have the following uniqueness property:

Lemma 6.1.10 *Let X be a Riemann surface with induced almost complex structure $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ and corresponding $(0,1)$ part of the d operator denoted by $\bar{\partial}$.*

(a) *Let $(TX)_{\mathbb{C}} = \widehat{(TX)^{1,0}} \oplus \widehat{(TX)^{0,1}}$ be a second almost complex structure on X with corresponding $(0,1)$ part of the d operator denoted by $\widehat{\bar{\partial}}$. If $p \in X$ and there exists a local holomorphic chart (U, Φ, U') in X with $p \in U$ and $(\widehat{\bar{\partial}}\Phi)_p = 0$, then the two almost complex structures agree at p; that is,*

$$\widehat{(T_pX)^{1,0}} = (T_pX)^{1,0} \quad \text{and} \quad \widehat{(T_pX)^{0,1}} = (T_pX)^{0,1}.$$

(b) *Any holomorphic structure on the smooth surface underlying X that induces the same given almost complex structure is equal to the given holomorphic structure on X.*

Proof In the situation of part (a), by Lemma 6.1.9, $(\Phi_*)_p$ maps $(T_pX)_{\mathbb{C}}$ isomorphically onto $(T_{\Phi(p)}\mathbb{C})_{\mathbb{C}}$, and both of the subspaces $\widehat{(T_pX)^{1,0}}$ and $(T_pX)^{1,0}$ isomorphically onto $T^{1,0}_{\Phi(p)}\mathbb{C} \subset (T_{\Phi(p)}\mathbb{C})_{\mathbb{C}}$. Part (a) now follows.

For proof of (b), we observe that on a smooth surface, any two 1-dimensional holomorphic structures with the same underlying smooth structures and induced (almost) complex structures have the same $\bar{\partial}$ operator. Hence, if (U, z) is a local holomorphic coordinate neighborhood with respect to one of the holomorphic structures, then $\bar{\partial} z = 0$, and hence z is a local holomorphic coordinate in the other holomorphic structure. Thus, holomorphic atlases for the two holomorphic structures are holomorphically equivalent; i.e., the holomorphic structures are the same. □

Lemma 6.1.11 *Let X be an almost complex surface. If for each point $p \in X$, there exists a nonvanishing $\mathcal{C}^\infty$ $\bar{\partial}$-closed differential form of type $(1,0)$ on a neighborhood of p, then the almost complex structure is integrable.*

Proof By hypothesis, there exists a covering $\{U_j\}$ of X by open sets such that for each j, there is a nonvanishing $\mathcal{C}^\infty$ form α_j of type $(1,0)$ on U_j with $d\alpha_j = \bar{\partial}\alpha_j = 0$. By the Poincaré lemma (Lemma 9.5.7), we may also assume that for each j, there exists a $\mathcal{C}^\infty$ function Φ_j on U_j with $d\Phi_j = \alpha_j$, and therefore $\bar{\partial}\Phi_j = 0$. According to Lemma 6.1.9, for $u_j \equiv \operatorname{Re}\Phi_j$ and $v_j \equiv \operatorname{Im}\Phi_j$, $du_j \wedge dv_j$ is nonvanishing, and hence by the $\mathcal{C}^\infty$ inverse function theorem (Theorem 9.9.1 and Theorem 9.9.2), we may also assume that Φ_j is a $\mathcal{C}^\infty$ diffeomorphism of U_j onto an open subset U'_j of the plane. For each pair of indices j, k, applying Lemma 6.1.9, we see that for the $\mathcal{C}^\infty$ coordinate transformation $f_{jk} \equiv \Phi_j \circ \Phi_k^{-1}$ on $\Phi_k(U_j \cap U_k)$, we have

$$\begin{aligned}\bar{\partial} f_{jk} &= (d\Phi_j) \circ (\Phi_k^{-1})_* \circ \mathcal{P}^{0,1} = (\partial\Phi_j) \circ ((\Phi_k)_*)^{-1} \circ \mathcal{P}^{0,1} \\ &= (\partial\Phi_j) \circ \mathcal{P}^{0,1} \circ ((\Phi_k)_*)^{-1} \equiv 0,\end{aligned}$$

where $\mathcal{P}^{r,s}$ denotes the projection of tangent vectors onto their (r, s) parts for $(r, s) \in \{(1,0), (0,1)\}$; that is, f_{jk} is holomorphic. Thus the atlas $\{(U_j, \Phi_j, U'_j)\}$ determines a holomorphic structure on X, and by Lemma 6.1.10, the induced complex structure is equal to the given almost complex structure. □

Remark An almost complex structure on a higher-dimensional $\mathcal{C}^\infty$ manifold is induced by a holomorphic structure if and only if for every $\mathcal{C}^\infty$ differential form β of type $(0,1)$, the 2-form $d\beta$ has no $(2,0)$ part (see, for example, [Hö]). Of course, on a Riemann surface, this condition holds automatically.

Exercises for Sect. 6.1

6.1.1 Complete the proof of Proposition 6.1.2.

6.1.2 Verify that the almost complex structure constructed in the proof of part (b) of Proposition 6.1.3 induces the given orientation.

6.1.3 Prove Proposition 6.1.8.

6.1.4 Let g_1 and g_2 be two Riemannian metrics on an oriented smooth surface X. Prove that g_1 and g_2 induce the same almost complex structure (as in the proof of Proposition 6.1.3) if and only if there exists a $\mathcal{C}^\infty$ real-valued function φ on X such that $g_2 = e^\varphi g_1$.

6.2 Construction of a Special Local Coordinate

According to Lemma 6.1.11, to obtain integrability in an almost complex surface, it suffices to obtain a local $\bar{\partial}$-closed $\mathcal{C}^\infty$ differential form of type $(1, 0)$ that is nonzero at a given point p. On a Riemann surface, the L^2 $\bar{\partial}$-method provides a technique for producing nonconstant holomorphic (or meromorphic) functions and forms (and sections of holomorphic line bundles). Most of the L^2 $\bar{\partial}$-method on a Riemann surface as considered in Chap. 2 works without change on an almost complex surface, but there are three difficulties. The first is that on a Riemann surface, the technique relied on the (obvious) existence of nonconstant *local* holomorphic functions and forms. But on an almost complex surface, it is the existence of such local functions and forms that is the problem to be solved. The second difficulty is that the construction of a positive-curvature weight function (or Hermitian metric) in Sect. 2.14 used a local holomorphic coordinate. Again, in the present context, the existence of a local holomorphic coordinate is the problem to be solved. The third difficulty is that the proof of regularity for $\bar{\partial}$ on a Riemann surface (Theorem 1.2.8 and Theorem 2.7.4) relied on normal families of holomorphic functions, which again, are not a priori available on an almost complex surface. The proof of regularity provided in this chapter, which is similar to the proofs in [De3] and [Hö], requires elements of Sobolev space theory (see Chap. 11). For the first two difficulties, following Demailly [De3], the idea is to work with a local $\mathcal{C}^\infty$ complex coordinate z such that although $\bar{\partial} z$ does not vanish identically, $\bar{\partial} z$ does vanish to high order *at* the point p. The existence of such a coordinate is given by the following lemma, which is the main goal of this section:

Lemma 6.2.1 *Let X be an almost complex surface, let $p \in X$, and let $m \in \mathbb{Z}_{\geq 0}$. Then there exists a $\mathcal{C}^\infty$ function f on a neighborhood of p in X such that $(\partial f)_p = (df)_p \neq 0$ and $\bar{\partial} f$ has vanishing derivatives of order $\leq m$ at p (see Definition* 9.5.8).

Proof Since the claim is local, we may assume that X is an open subset of $\mathbb{C}$ to which an almost complex structure $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ has been assigned. This almost complex structure may be different from that induced by $\mathbb{C}$. In particular, ∂ and $\bar{\partial}$ will be the operators associated to the almost complex structure on X, not the standard holomorphic structure on $\mathbb{C}$. We may also assume that $p = 0 \in \mathbb{C}$. By replacing z with $\bar{z}$ and X with a small neighborhood of p if necessary, we may assume without loss of generality that the coordinate function z in $\mathbb{C}$ satisfies $\partial z \neq 0$ on X. Thus we may form the $\mathcal{C}^\infty$ function

$$\mu \equiv \frac{(\bar{\partial} z)}{(\overline{\partial z})} = \frac{(\bar{\partial} z)}{(\bar{\partial} \bar{z})} \quad \text{on } X.$$

Note that the above denotes a quotient of $(0, 1)$-forms, not a partial derivative. In other words, μ is the $\mathcal{C}^\infty$ function defined by $\bar{\partial} z = \mu \cdot \overline{\partial z} = \mu \cdot \bar{\partial} \bar{z}$.

We proceed by induction on m. For the case $m = 0$, we simply set $f(z) = z - \mu(0)\bar{z}$. We then get

$$(\bar{\partial} f)_0 = (\bar{\partial} z)_0 - \mu(0)(\bar{\partial}\bar{z})_0 = 0,$$

and hence

$$(\partial f)_0 = (df)_0 = (dz)_0 - \mu(0)(d\bar{z})_0 \neq 0$$

(since dz and $d\bar{z}$ are pointwise linearly independent). Thus the claim is proved for $m = 0$.

Assume now that $m > 0$ and that the desired function exists for lower orders. In other words, there exists a $\mathcal{C}^\infty$ function h on a neighborhood of 0 such that $\bar{\partial} h$ has vanishing derivatives of order $\leq m - 1$ at 0 and $(dh)_0 \neq 0$. In particular, Lemma 6.1.9 and the $\mathcal{C}^\infty$ inverse function theorem imply that $(\operatorname{Re} h, \operatorname{Im} h)$ gives local $\mathcal{C}^\infty$ coordinates in a neighborhood of 0. Therefore, after replacing the coordinate z with the local coordinate $h - h(0)$, we may assume that the $\mathcal{C}^\infty$ function μ has vanishing derivatives of order $\leq m-1$ at 0 (see Proposition 9.5.9). Equivalently, the degree-m Taylor polynomial P for μ centered at 0 is either homogeneous of degree m or trivial. Thus, expressing P as a polynomial in z and $\bar{z}$ (which we may do, since $x = \operatorname{Re} z = 2^{-1}(z + \bar{z})$ and $y = \operatorname{Im} z = (2i)^{-1}(z - \bar{z})$), we get

$$P(z) = A_0 z^m + A_1 z^{m-1}\bar{z} + \cdots + A_m \bar{z}^m,$$

for some complex constants $A_0, \ldots, A_m$. Setting

$$Q(z) \equiv A_0 z^m \bar{z} + A_1 z^{m-1}\frac{1}{2}\bar{z}^2 + \cdots + A_m \frac{1}{m+1}\bar{z}^{m+1},$$

we get

$$\frac{\partial Q}{\partial \bar{z}} = P \quad \text{and} \quad \frac{\partial Q}{\partial z}(0) = \frac{\partial Q}{\partial \bar{z}}(0) = 0.$$

We will show that the function $f(z) \equiv z - Q(z)$ has the required properties.

We have

$$\begin{aligned}
\bar{\partial} f = df \circ \mathcal{P}^{0,1} &= \left[dz - \frac{\partial Q}{\partial z} dz - \frac{\partial Q}{\partial \bar{z}} d\bar{z}\right] \circ \mathcal{P}^{0,1} \\
&= \bar{\partial} z - \frac{\partial Q}{\partial z} \cdot \bar{\partial} z - P \cdot \bar{\partial}\bar{z} \\
&= \left(\mu - P - \frac{\partial Q}{\partial z}\mu\right) \cdot \overline{\partial z}.
\end{aligned}$$

Since $\mu - P$ has vanishing derivatives of order $\leq m$ at 0, $\partial Q/\partial z = 0$ at 0, and μ has vanishing derivatives of order $\leq m - 1$ at 0, we see that $\bar{\partial} f$ has vanishing derivatives of order $\leq m$ at 0. Similarly, we have

$$(df)_0 = (dz)_0 - \frac{\partial Q}{\partial z}(0) \cdot (dz)_0 - \frac{\partial Q}{\partial \bar{z}}(0) \cdot (d\bar{z})_0 = (dz)_0 \neq 0.$$

Thus f has the required properties and the lemma follows. □

6.3 Regularity of Solutions on an Almost Complex Surface

For regularity, we will use Theorem 11.0.1, which is a general regularity theorem for first-order differential operators satisfying a certain estimate. In order to obtain the required $(1,0)$-form in Lemma 6.1.11 on a neighborhood of a given point in an almost complex surface, it suffices to work locally. Thus it suffices to consider a nonempty relatively compact domain X in $\mathbb{R}^2$ (obtained by replacing the given surface with a connected local $\mathcal{C}^\infty$ coordinate neighborhood) on which we have a given almost complex structure $TX_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ and in which $p = (0,0)$. Letting (x, y) denote the real coordinates in $\mathbb{R}^2$, we get a second almost complex structure under the identification $\mathbb{R}^2 = \mathbb{C}$ given by $(x, y) \leftrightarrow z = x + iy$. To avoid confusion, we will use the notation $\mathcal{P}^{r,s}$, ∂, $\bar\partial$, $\Lambda^{r,s}TX$, and $\Lambda^{r,s}T^*X$ only for the operators and spaces associated to the given almost complex structure on X, not the standard holomorphic structure on $\mathbb{C}$. According to Lemma 6.2.1, Lemma 6.1.9, and the $\mathcal{C}^\infty$ inverse function theorem (Theorem 9.9.1 and Theorem 9.9.2), we may also assume that the $(1,0)$-form $\bar\partial z$ has vanishing derivatives of order ≤ 2 at $p = 0$ (actually, for the results considered in this section, order 0 would suffice, but we will need vanishing derivatives of order ≤ 2 in later sections). In particular, we have $(\partial z)_0 = (dz)_0 \neq 0$, so we may also assume that $\partial z \neq 0$ at every point in X (replacing X with a small neighborhood of 0 if necessary).

We now derive local formulas for ∂ and $\bar\partial$. Given a local $\mathcal{C}^1$ function f, we have

$$\begin{aligned}\bar\partial f = df \circ \mathcal{P}^{0,1} &= \left[\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z\right] \circ \mathcal{P}^{0,1} \\ &= \frac{\partial f}{\partial z}\bar\partial z + \frac{\partial f}{\partial \bar z}\bar\partial \bar z = \frac{\partial f}{\partial z}\bar\partial z + \frac{\partial f}{\partial \bar z}\overline{\partial z} \\ &= \left[\frac{\partial f}{\partial \bar z} + \mu \cdot \frac{\partial f}{\partial z}\right] \cdot \overline{\partial z} = \overline{\mathcal{Z}} f \cdot \overline{\partial z},\end{aligned}$$

where μ is the $\mathcal{C}^\infty$ function on X given by

$$\mu \equiv \frac{(\bar\partial z)}{(\overline{\partial z})} = \frac{(\bar\partial z)}{(\bar\partial \bar z)} \quad \text{(i.e., } \bar\partial z = \mu \cdot \bar\partial \bar z\text{)},$$

and $\overline{\mathcal{Z}}$ is the $\mathcal{C}^\infty$ vector field (and therefore linear differential operator of order 1 with $\mathcal{C}^\infty$ coefficients) on X given by

$$\overline{\mathcal{Z}} \equiv \frac{\partial}{\partial \bar z} + \mu \cdot \frac{\partial}{\partial z}.$$

Furthermore, by the choice of the local representation of X, the function μ has vanishing derivatives of order ≤ 2 at $p = 0$. In particular, we may assume that $|\mu|^2 < 3/16$ on X. We also have

$$\partial f = \overline{(\bar\partial \bar f)} = \overline{\overline{\mathcal{Z}} \bar f \cdot \overline{\partial z}} = \mathcal{Z} f \cdot \partial z, \quad \text{where } \mathcal{Z} \equiv \overline{\overline{\mathcal{Z}}} = \frac{\partial}{\partial z} + \bar\mu \cdot \frac{\partial}{\partial \bar z}.$$

Given a local $\mathcal{C}^1$ differential form α of type $(1,0)$, we have $\alpha = f\partial z$ for the $\mathcal{C}^1$ function $f \equiv \alpha/(\partial z)$. Thus

$$\begin{aligned}\bar\partial\alpha &= (\bar\partial f)\wedge\partial z + f\bar\partial\partial z = \left[\frac{\partial f}{\partial\bar z} + \mu\cdot\frac{\partial f}{\partial z}\right]\cdot\overline{\partial z}\wedge\partial z + f\bar\partial\partial z \\ &= 2\sqrt{-1}\left[\frac{\partial f}{\partial\bar z} + \mu\cdot\frac{\partial f}{\partial z}\right]\cdot\frac{\sqrt{-1}}{2}\partial z\wedge\overline{\partial z} + 2\sqrt{-1}f\frac{\sqrt{-1}}{2}\partial\bar\partial z \\ &= 2\sqrt{-1}\left[\frac{\partial f}{\partial\bar z} + \mu\cdot\frac{\partial f}{\partial z} + \tau\cdot f\right]\omega = \bar A f\cdot\omega,\end{aligned}$$

where ω is the nonvanishing real $\mathcal{C}^\infty$ differential form of type $(1,1)$ given by

$$\omega \equiv \frac{\sqrt{-1}}{2}\partial z\wedge\overline{\partial z},$$

τ is the $\mathcal{C}^\infty$ function on X given by

$$\tau \equiv \frac{\sqrt{-1}}{2}\cdot\frac{\partial\bar\partial z}{\omega} = \frac{\partial\bar\partial z}{\partial z\wedge\overline{\partial z}}$$

(see Exercise 6.3.1 for a more explicit expression for τ in terms of μ), and A and $\bar A$ are the linear differential operators of order 1 with $\mathcal{C}^\infty$ coefficients on X given by

$$A \equiv -2\sqrt{-1}\left[\frac{\partial}{\partial z} + \bar\mu\cdot\frac{\partial}{\partial\bar z} + \bar\tau\right] = -2\sqrt{-1}(\mathcal{Z} + \bar\tau)$$

and

$$\bar A \equiv 2\sqrt{-1}\left[\frac{\partial}{\partial\bar z} + \mu\cdot\frac{\partial}{\partial z} + \tau\right] = 2\sqrt{-1}(\overline{\mathcal{Z}} + \tau).$$

In particular, since $\bar\partial z$ has vanishing derivatives of order ≤ 2 at 0, τ has vanishing derivatives of order ≤ 1 at 0. For $\beta = f\overline{\partial z}$ a $\mathcal{C}^1$ differential form of type $(0,1)$, we have

$$\partial\beta = \overline{\bar\partial\bar\beta} = \overline{\bar\partial(\bar f\partial z)} = \overline{(\bar A\bar f\cdot\omega)} = Af\cdot\omega.$$

We may summarize the above as follows:

Lemma 6.3.1 *For any given point p in an almost complex surface X, in order to show that there exists a nonvanishing $\bar\partial$-closed $\mathcal{C}^\infty$ differential form of type $(1,0)$ on a neighborhood of p, it suffices to consider the case in which*:

(i) *X is a relatively compact domain in $\mathbb{R}^2$ together with a given almost complex structure $TX_{\mathbb{C}} = (TX)^{1,0}\oplus(TX)^{0,1}$, as well as the holomorphic structure associated to the identification $\mathbb{R}^2 = \mathbb{C}$ given by $(x,y)\leftrightarrow z = x+iy$, and $p = 0$*;

(ii) *The $(1,0)$-form ∂z is nonvanishing on X*;

(iii) *We have $\mathcal{C}^\infty$ functions μ and τ and a nonvanishing $\mathcal{C}^\infty$ real differential form ω of type* $(1,1)$ *on* X *such that* μ *and* τ *have vanishing derivatives of order* ≤ 2 *and of order* ≤ 1*, respectively, at* 0, $|\mu|^2 < 3/16$ *on* X*, and*

$$\bar{\partial} z = \mu \cdot \overline{\partial z}, \qquad \omega = \frac{\sqrt{-1}}{2} \partial z \wedge \overline{\partial z}, \quad \textit{and} \quad \frac{\sqrt{-1}}{2} \partial \bar{\partial} z = \tau \cdot \omega;$$

and

(iv) *For every local $\mathcal{C}^1$ function f, we have*

$$\bar{\partial} f = \overline{\mathcal{Z}} f \cdot \overline{\partial z} \quad \textit{and} \quad \partial f = \mathcal{Z} f \cdot \partial z,$$

where

$$\overline{\mathcal{Z}} = \frac{\partial}{\partial \bar{z}} + \mu \cdot \frac{\partial}{\partial z} \quad \textit{and} \quad \mathcal{Z} \equiv \frac{\partial}{\partial z} + \bar{\mu} \cdot \frac{\partial}{\partial \bar{z}},$$

and we have

$$\bar{\partial}(f \partial z) = \bar{A} f \cdot \omega \quad \textit{and} \quad \partial(f \overline{\partial z}) = A f \cdot \omega,$$

where $\bar{A}$ and A are the linear differential operators of order 1 *with $\mathcal{C}^\infty$ coefficients on X given by*

$$\bar{A} \equiv 2\sqrt{-1}\left[\frac{\partial}{\partial \bar{z}} + \mu \cdot \frac{\partial}{\partial z} + \tau\right] = 2\sqrt{-1}(\overline{\mathcal{Z}} + \tau)$$

and

$$A \equiv -2\sqrt{-1}\left[\frac{\partial}{\partial z} + \bar{\mu} \cdot \frac{\partial}{\partial \bar{z}} + \bar{\tau}\right] = -2\sqrt{-1}(\mathcal{Z} + \bar{\tau}).$$

Fixing the choices and notation in the above lemma for the rest of this section, the main goal is the following regularity property for the operator $\bar{A}$:

Proposition 6.3.2 (Regularity) *If Ω is a relatively compact open subset of X, and $f \in L^2_{\mathrm{loc}}(\Omega)$ with $\bar{A}_{\mathrm{distr}} f \in \mathcal{C}^\infty(\Omega)$, then $f \in \mathcal{C}^\infty(\Omega)$.*

Observe that for a local $\mathcal{C}^1$ function f on X, we have

$$\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 = 2\left|\frac{\partial f}{\partial z}\right|^2 + 2\left|\frac{\partial f}{\partial \bar{z}}\right|^2.$$

Furthermore, if $f \in \mathcal{D}(X)$, then (denoting Lebesgue measure by λ) we get

$$\begin{aligned} \left\|\frac{\partial f}{\partial z}\right\|^2_{L^2(X)} &= \int_X \left|\frac{\partial f}{\partial z}\right|^2 d\lambda = \int_X \frac{\partial f}{\partial z} \overline{\left(\frac{\partial f}{\partial z}\right)} d\lambda \\ &= \int_X \frac{\partial f}{\partial z} \cdot \frac{\partial \bar{f}}{\partial \bar{z}} d\lambda = -\int_X f \cdot \frac{\partial^2 \bar{f}}{\partial z \partial \bar{z}} d\lambda \end{aligned}$$

$$= \int_X \frac{\partial f}{\partial \bar{z}} \cdot \frac{\partial \bar{f}}{\partial z}\, d\lambda = \int_X \frac{\partial f}{\partial \bar{z}} \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)}\, d\lambda = \left\| \frac{\partial f}{\partial \bar{z}} \right\|^2_{L^2(X)}.$$

Thus Proposition 6.3.2 will follow immediately from the general first-order regularity theorem (Theorem 11.0.1) together with the following:

Lemma 6.3.3 *For every open set $\Omega \Subset X$, there is a constant $C > 0$ such that*

$$\left\| \frac{\partial f}{\partial \bar{z}} \right\|^2_{L^2(\Omega)} \leq C \cdot \left(\|f\|^2_{L^2(\Omega)} + \|\bar{A} f\|^2_{L^2(\Omega)} \right) \quad \forall f \in \mathcal{D}(\Omega).$$

Proof Given an open set $\Omega \Subset X$ and a function $f \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned} 4\left\| \frac{\partial f}{\partial \bar{z}} \right\|^2_{L^2(\Omega)} &= \left\| \bar{A} f - 2i \cdot \mu \cdot \frac{\partial f}{\partial z} - 2i \cdot \tau \cdot f \right\|^2_{L^2(\Omega)} \\ &\leq 2\|\bar{A} f\|^2_{L^2(\Omega)} + 16 \cdot \sup_{\Omega} |\mu|^2 \cdot \left\| \frac{\partial f}{\partial z} \right\|^2_{L^2(\Omega)} \\ &\quad + 16 \cdot \sup_{\Omega} |\tau|^2 \cdot \|f\|^2_{L^2(\Omega)}. \end{aligned}$$

Thus, since $|\mu|^2 < 3/16$ on X (by construction) and τ is bounded on Ω, there is a constant $C > 0$ independent of the choice of f such that

$$\begin{aligned} 4\left\| \frac{\partial f}{\partial \bar{z}} \right\|^2_{L^2(\Omega)} &\leq C \cdot \left[\|f\|^2_{L^2(\Omega)} + \|\bar{A} f\|^2_{L^2(\Omega)} \right] + 3 \cdot \left\| \frac{\partial f}{\partial z} \right\|^2_{L^2(\Omega)} \\ &= C\left[\|f\|^2_{L^2(\Omega)} + \|\bar{A} f\|^2_{L^2(\Omega)} \right] + 3\left\| \frac{\partial f}{\partial \bar{z}} \right\|^2_{L^2(\Omega)}. \end{aligned}$$

The desired inequality now follows. □

Finally, we note that the real volume forms ω and $dx \wedge dy = (i/2)\, dz \wedge d\bar{z}$ induce the same orientation in X, since X is connected and the quotient

$$\eta \equiv \frac{\omega}{dx \wedge dy}$$

is a nonvanishing $\mathcal{C}^\infty$ function that is equal to 1 at 0 (in fact, as the reader is asked to verify in Exercise 6.3.1, $\eta = 1/(1 - |\mu|^2)$). We will use this orientation when integrating 2-forms.

Exercises for Sect. 6.3

6.3.1 Show that $\eta = 1/(1 - |\mu|^2)$ and

$$\tau = \frac{1}{1 - |\mu|^2} \left[\frac{\partial |\mu|^2}{\partial \bar{z}} + \frac{\partial \mu}{\partial z} - \frac{\partial \mu}{\partial z} \cdot |\mu|^2 + \mu \cdot \frac{\partial |\mu|^2}{\partial z} \right].$$

6.4 The Distributional $\bar{\partial}$, Connection, and Curvature

We continue in this section with the notation of Sect. 6.3. In particular, we have the positive $(1,1)$-form $\omega = (i/2)\partial z \wedge \overline{\partial z}$ on X. We also fix a real-valued $\mathcal{C}^\infty$ function φ on X. For every measurable set $S \subset X$ and every continuous real $(1,1)$-form θ defined at points of S, we may define $\langle\cdot,\cdot\rangle_{L^2(S,\varphi)}$, $\langle\cdot,\cdot\rangle_{L^2(S,\theta,\varphi)}$, $\|\cdot\|_{L^2(S,\varphi)}$, and $\|\cdot\|_{L^2(S,\theta,\varphi)}$ on scalar-valued differential forms and the associated L^2 (Hilbert) spaces $L^2_{p,q}(S,\varphi)$ and $L^2_{p,q}(S,\theta,\varphi)$ (with $\theta \geq 0$ or $\theta > 0$ wherever appropriate) with respect to the almost complex structure $(TX)_{\mathbb{C}} = (TX)^{1,0} \oplus (TX)^{0,1}$ exactly as in Sect. 2.6. We may also define the corresponding *canonical connection* (or *Chern connection*)

$$D = D_\varphi = D' + D'' = D'_\varphi + D''_\varphi \equiv e^\varphi \partial[e^{-\varphi}(\cdot)] + \bar{\partial}$$

on scalar-valued differential forms exactly as in Sect. 2.7, and the corresponding *curvature*

$$\Theta = \Theta_\varphi \equiv \partial\bar{\partial}\varphi$$

as in Sect. 2.8. In particular, we have $D^2 = D'D'' + D''D' = \Theta \wedge (\cdot)$.

Definition 6.4.1 Let α and β be locally integrable differential forms on an open subset U of X. We write $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$ if one of the following holds:

(i) On U, α is a 0-form, $\beta = b\overline{\partial z}$, and $\overline{\mathcal{Z}}_{\mathrm{distr}}\alpha = b$;
(ii) On U, $\alpha = a_1\partial z + a_2\overline{\partial z}$, $\beta = b\omega$, and $\bar{A}_{\mathrm{distr}}(a_1) = b$;
(iii) The form α is of degree > 1 and $\beta \equiv 0$.

In analogy with Proposition 2.7.3, we have the following equivalent form for the definition of $\bar{\partial}_{\mathrm{distr}}$:

Proposition 6.4.2 *Let α and β be locally integrable differential forms on an open set $U \subset X$.*

(a) *If α is of type $(1,0)$ and β is of type $(1,1)$, then $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$ if and only if*

$$\int_U \alpha \wedge \overline{D'f}e^{-\varphi} = \int_U \beta \cdot \bar{f} \cdot e^{-\varphi} \quad \forall f \in \mathcal{D}(U).$$

(b) *If α is of type $(0,0)$ and β is of type $(0,1)$, then $\bar{\partial}_{\mathrm{distr}}\alpha = \beta$ if and only if*

$$\int_U \alpha \cdot \overline{(-D'\gamma)} \cdot e^{-\varphi} = \int_U \beta \wedge \overline{\gamma} \cdot e^{-\varphi} \quad \forall \gamma \in \mathcal{D}^{0,1}(U).$$

(c) *If α is of type (p,q) with $p \geq 2$ or $q \geq 1$, then $\bar{\partial}_{\mathrm{distr}}\alpha = 0$.*

Proof Part (c) is trivial and the proof of (b) is left to the reader (see Exercise 6.4.1). For the proof of (a), suppose $\alpha = a\partial z$ and $\beta = b\omega$ are locally integrable forms of type $(1,0)$ and $(1,1)$, respectively, on an open set $U \subset X$. By the (weak) Friedrichs lemma (Lemma 7.3.1), we may choose a sequence of $\mathcal{C}^\infty$ functions $\{a_\nu\}$ converging to a in L^1_{loc}. Setting $\alpha_\nu = a_\nu \partial z$ for each ν, we get $\alpha_\nu \to \alpha$ in L^1_{loc}. Recall also that we have $\omega = \eta\, dx \wedge dy$.

If $\bar\partial_{\mathrm{distr}}\alpha = \beta$, then $\bar A_{\mathrm{distr}} a = b$. Hence, for every function $f \in \mathcal{D}(U)$, we have

$$\begin{aligned}
\int_U \alpha \wedge \overline{D'f} \cdot e^{-\varphi} &= \lim_{\nu\to\infty} \int_U \alpha_\nu \wedge \overline{D'f} \cdot e^{-\varphi} = \lim_{\nu\to\infty} \int_U \alpha_\nu \wedge \overline{\partial(e^{-\varphi} f)} \\
&= \lim_{\nu\to\infty} \int_U \alpha_\nu \wedge \bar\partial(\overline{e^{-\varphi} f}) = \lim_{\nu\to\infty} \int_U (\bar\partial \alpha_\nu) \cdot (\overline{e^{-\varphi} f}) \\
&= \lim_{\nu\to\infty} \int_U \bar A(a_\nu) \cdot \overline{e^{-\varphi} f} \cdot \omega = \lim_{\nu\to\infty} \int_U \bar A(a_\nu) \cdot \overline{e^{-\varphi} f \eta} \cdot dx \wedge dy \\
&= \lim_{\nu\to\infty} \int_U a_\nu \cdot \overline{\bar A^*(e^{-\varphi} f \eta)} \cdot dx \wedge dy \\
&= \int_U a \cdot \overline{\bar A^*(e^{-\varphi} f \eta)} \cdot dx \wedge dy \\
&= \int_U b \cdot \overline{e^{-\varphi} f \eta} \cdot dx \wedge dy = \int_U \beta \bar f e^{-\varphi}.
\end{aligned}$$

Conversely, suppose

$$\int_U \alpha \wedge \overline{D'f} \cdot e^{-\varphi} = \int_U \beta \bar f e^{-\varphi} \quad \forall f \in \mathcal{D}(U).$$

Given $u \in \mathcal{D}(U)$, setting $f = e^{\varphi} u/\eta$, we get

$$\begin{aligned}
\int_U a \cdot \overline{\bar A^* u} \cdot dx \wedge dy &= \int_U a \cdot \overline{\bar A^*(e^{-\varphi} f \eta)} \cdot dx \wedge dy \\
&= \lim_{\nu\to\infty} \int_U a_\nu \cdot \overline{\bar A^*(e^{-\varphi} f \eta)} \cdot dx \wedge dy \\
&= \lim_{\nu\to\infty} \int_U \alpha_\nu \wedge \overline{D'f} \cdot e^{-\varphi} = \int_U \alpha \wedge \overline{D'f} \cdot e^{-\varphi} = \int_U \beta \bar f e^{-\varphi} \\
&= \int_U b \cdot \overline{e^{-\varphi} f \eta} \cdot dx \wedge dy = \int_U b \cdot \bar u \cdot dx \wedge dy.
\end{aligned}$$

Hence $\bar A_{\mathrm{distr}} a = b$ and therefore $\bar\partial_{\mathrm{distr}}\alpha = \beta$. Thus (a) is proved. □

Remark The main point of the above proof is that the characterization holds for $\mathcal{C}^\infty$ forms, and the $\mathcal{C}^\infty$ forms are dense in L^1_{loc}. One may also prove Proposition 6.4.2 more directly without density of the $\mathcal{C}^\infty$ forms (see Exercise 6.4.2).

The proof of Proposition 2.8.2 (with Proposition 6.4.2 in place of Proposition 2.7.3) gives the following identical fundamental estimate:

Proposition 6.4.3 (Fundamental estimate) *For all functions $u, v \in \mathcal{D}(X)$, we have*

$$\langle D'u, D'v\rangle_{L^2(X,\varphi)} = \langle D''u, D''v\rangle_{L^2(X,\varphi)} + \int_X i\Theta_\varphi u\bar{v}e^{-\varphi}.$$

In particular,

$$\|D'u\|^2_{L^2(X,\varphi)} = \|D''u\|^2_{L^2(X,\varphi)} + \int_X i\Theta_\varphi |u|^2 e^{-\varphi} \geq \int_X i\Theta_\varphi |u|^2 e^{-\varphi}.$$

Remark If $i\Theta_\varphi \geq 0$ in the above, then we get

$$\langle D'u, D'v\rangle_{L^2(X,\varphi)} = \langle D''u, D''v\rangle_{L^2(X,\varphi)} + \langle u, v\rangle_{L^2(X,i\Theta_\varphi,\varphi)}$$

and

$$\|D'u\|^2_{L^2(X,\varphi)} = \|D''u\|^2_{L^2(X,\varphi)} + \|u\|^2_{L^2(X,i\Theta_\varphi,\varphi)} \geq \|u\|^2_{L^2(X,i\Theta_\varphi,\varphi)}.$$

Exercises for Sect. 6.4

6.4.1 Prove part (b) of Proposition 6.4.2.

6.4.2 Prove part (a) of Proposition 6.4.2 by direct computation without using the density of the space of $\mathcal{C}^\infty$ forms.

Hint. Apply the formulas for η and τ appearing in Exercise 6.3.1.

6.5 L^2 Solutions on an Almost Complex Surface

Carrying over the notation of Sects. 6.3 and 6.4, we get the following analogue of Theorem 2.9.1, which gives local existence of solutions of the Cauchy–Riemann equation on an almost complex surface along with L^2 estimates. The proof, which is left to the reader (see Exercise 6.5.1), is identical to the proof of Theorem 2.9.1, except that one must use Proposition 6.4.2 in place of Proposition 2.7.3, Proposition 6.4.3 in place of Proposition 2.8.2, and Proposition 6.3.2 in place of Theorem 2.7.4.

Theorem 6.5.1 *Assume that $i\Theta = i\Theta_\varphi \geq 0$, and let $Z = \{x \in X \mid \Theta_x = 0\}$. Then, for every measurable $(1,1)$-form β on X with $\beta = 0$ a.e. in Z and $\beta\restriction_{X\setminus Z} \in L^2_{1,1}(X \setminus Z, i\Theta, \varphi)$, there exists a form $\alpha \in L^2_{1,0}(X, \varphi)$ such that*

$$D''_{\mathrm{distr}}\alpha = \bar\partial_{\mathrm{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,\varphi)} \leq \|\beta\|_{L^2(X\setminus Z,i\Theta,\varphi)}.$$

In particular, if β is of class $\mathcal{C}^\infty$, then α is also of class $\mathcal{C}^\infty$ and $\bar\partial\alpha = \beta$.

The proof of the following corollary is also left to the reader (see Exercise 6.5.2):

Corollary 6.5.2 (Cf. Corollary 2.9.3) *Suppose that C is a positive constant with $i\Theta_{\varphi} \geq C^2\omega$ on X. Then, for every $\beta \in L^2_{1,1}(X, \omega, \varphi)$, there exists a form $\alpha \in L^2_{1,0}(X, \varphi)$ such that*

$$D''_{\text{distr}}\alpha = \bar{\partial}_{\text{distr}}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_{L^2(X,\varphi)} \leq C^{-1}\|\beta\|_{L^2(X,\omega,\varphi)}.$$

In particular, if β is of class $\mathcal{C}^\infty$, then α is also of class $\mathcal{C}^\infty$ and $\bar{\partial}\alpha = \beta$.

Exercises for Sect. 6.5

6.5.1 Prove Theorem 6.5.1.
6.5.2 Prove Corollary 6.5.2.

6.6 Proof of Integrability

The integrability theorem (Theorem 6.1.4) follows from Lemma 6.1.11 together with the following:

Lemma 6.6.1 *In the notation of Sects. 6.3–6.5, there exists a nonvanishing $\mathcal{C}^\infty$ $\bar{\partial}$-closed differential form of type $(1, 0)$ on a neighborhood of $p = 0$ in $X \subset \mathbb{C}$.*

Proof Let $\varphi(z) = |z|^2 + \log|z|^2$ for each point $z \in \mathbb{C}^*$, and for each $\epsilon > 0$, let

$$\varphi_\epsilon(z) = |z|^2 + \log(|z|^2 + \epsilon) \quad \forall z \in \mathbb{C}.$$

Then $\varphi_\epsilon \geq \varphi$ on $\mathbb{C}^*$ for each $\epsilon > 0$. On X (with the choices and notation given in Lemma 6.3.1), we have

$$\bar{\partial}|z|^2 = z\overline{\partial z} + \bar{z}\bar{\partial}z = (z + \bar{z}\mu) \cdot \overline{\partial z},$$

and hence

$$\frac{i}{2}\partial|z|^2 \wedge \bar{\partial}|z|^2 = \frac{i}{2}|z + \bar{z}\mu|^2 \partial z \wedge \overline{\partial z} = |z + \bar{z}\mu|^2 \cdot \omega$$

and

$$\begin{aligned}\frac{i}{2}\partial\bar{\partial}|z|^2 &= \frac{i}{2}\partial z \wedge \overline{\partial z} + \frac{i}{2}z\partial\overline{\partial z} + \frac{i}{2}\partial\bar{z} \wedge \bar{\partial}z + \frac{i}{2}\bar{z}\partial\bar{\partial}z \\ &= (1 + z\bar{\tau} + |\mu|^2 + \bar{z}\tau) \cdot \omega.\end{aligned}$$

Thus, for each $\epsilon > 0$, we have

$$\begin{aligned}i\Theta_{\varphi_\epsilon} = i\partial\bar{\partial}\varphi_\epsilon &= i\partial\bar{\partial}|z|^2 + i\frac{\partial\bar{\partial}|z|^2}{(|z|^2 + \epsilon)} - i\frac{\partial|z|^2 \wedge \bar{\partial}|z|^2}{(|z|^2 + \epsilon)^2} \\ &= 2 \cdot \left[(1 + 2\operatorname{Re}(z\bar{\tau}) + |\mu|^2) + \frac{1 + 2\operatorname{Re}(z\bar{\tau}) + |\mu|^2}{|z|^2 + \epsilon} - \frac{|z + \bar{z}\mu|^2}{(|z|^2 + \epsilon)^2}\right] \cdot \omega \\ &= 2 \cdot (1 + \rho_\epsilon) \cdot \omega,\end{aligned}$$

where (using $|z+\bar{z}\mu|^2 = |z|^2(1+|\mu|^2) + 2\operatorname{Re}(z^2\bar{\mu})$)

$$\begin{aligned}\rho_\epsilon &\equiv 2\operatorname{Re}(z\bar{\tau}) + |\mu|^2 + \frac{(|z|^2+\epsilon)(1+2\operatorname{Re}(z\bar{\tau})+|\mu|^2) - |z+\bar{z}\mu|^2}{(|z|^2+\epsilon)^2} \\ &= 2\operatorname{Re}(z\bar{\tau}) + |\mu|^2 + \frac{\epsilon(1+|\mu|^2) + (|z|^2+\epsilon)2\operatorname{Re}(z\bar{\tau}) - 2\operatorname{Re}(z^2\bar{\mu})}{(|z|^2+\epsilon)^2}.\end{aligned}$$

Since μ and τ have vanishing derivatives of orders ≤ 2 and ≤ 1, respectively, at 0, after replacing X with a relatively compact neighborhood of 0, we may assume that there exists a constant $C > 0$ such that $|\mu| \leq C|z|^3$ and $|\tau| \leq C|z|^2$ on X (see Exercise 9.5.10). Therefore

$$\rho_\epsilon \geq -2C|z|^3 - 2C\frac{|z|^3}{|z|^2+\epsilon} - 2C\frac{|z|^5}{(|z|^2+\epsilon)^2} \geq -2C(|z|^3 + 2|z|).$$

Replacing X again with a sufficiently small neighborhood of 0, we see that we may assume without loss of generality that $\rho_\epsilon \geq -1/2$, and hence $i\Theta_{\varphi_\epsilon} \geq \omega$ on X, for every $\epsilon > 0$. Finally, recall that we have the positive $\mathcal{C}^\infty$ function $\eta = \omega/(dx \wedge dy)$, which we may assume to be bounded above and below by positive constants.

Now let β be the $\mathcal{C}^\infty$ form of type $(1,1)$ given by

$$\beta \equiv \bar{\partial}\partial z = 2i\tau \cdot \omega = 2i\tau \cdot \eta \cdot dx \wedge dy.$$

For every $\epsilon > 0$, we have

$$\|\beta\|^2_{L^2(X,\omega,\varphi_\epsilon)} = \int_X 4|\tau|^2 e^{-\varphi_\epsilon}\omega \leq \int_X 4|\tau|^2 e^{-\varphi}\eta\, d\lambda \leq \int_X 4C^2|z|^2 e^{-|z|^2}\eta\, d\lambda.$$

Replacing X again with a sufficiently small neighborhood of 0, we may assume without loss of generality that $\|\beta\|_{L^2(X,\omega,\varphi_\epsilon)} \leq 1$ for every $\epsilon > 0$.

According to Corollary 6.5.2, for every $\epsilon > 0$, there exists a $\mathcal{C}^\infty$ form α_ϵ of type $(1,0)$ on X such that

$$\bar{\partial}\alpha_\epsilon = \beta \quad \text{and} \quad \|\alpha_\epsilon\|_{L^2(X,\varphi_\delta)} \leq \|\alpha_\epsilon\|_{L^2(X,\varphi_\epsilon)} \leq 1 \quad \forall \delta > \epsilon.$$

Fix a sequence of positive numbers $\{\delta_\nu\}$ with $\delta_\nu \searrow 0$. Applying weak sequential compactness in a Hilbert space (see Theorem 7.6.1) and Cantor's diagonal process, we get a sequence of positive numbers $\{\epsilon_\nu\}$ such that $\epsilon_\nu \searrow 0$ and such that for each $\mu \in \mathbb{Z}_{>0}$, there is a form $\theta_\mu \in L^2_{1,0}(X, \varphi_{\delta_\mu})$ with $\|\theta_\mu\|_{L^2(X,\varphi_{\delta_\mu})} \leq 1$ and

$$\langle \gamma, \alpha_{\epsilon_\nu}\rangle_{L^2(X,\varphi_{\delta_\mu})} \to \langle \gamma, \theta_\mu\rangle_{L^2(X,\varphi_{\delta_\mu})} \quad \text{as } \nu \to \infty \quad \forall \gamma \in L^2_{1,0}(X, \varphi_{\delta_\mu})$$

(for each μ, we have, for $\nu \gg 0$, $\epsilon_\nu < \delta_\mu$ and hence $\|\alpha_{\epsilon_\nu}\|_{L^2(X,\varphi_{\delta_\mu})} \leq 1$). Since $X \Subset \mathbb{R}^2$, the set of (equivalence classes of) forms in $L^2_{1,0}(X, \varphi_\delta)$ is actually equal to $L^2_{1,0}(X)$ for each $\delta > 0$ (although the inner products differ). In particular, given a form $\gamma_0 \in L^2_{1,0}(X)$, the above (with $\gamma = e^{\varphi_{\delta_\mu}} \cdot \gamma_0$) shows that for each $\mu \in \mathbb{Z}_{>0}$, we have $\langle \gamma_0, \alpha_{\epsilon_\nu} \rangle_{L^2(X)} \to \langle \gamma_0, \theta_\mu \rangle_{L^2(X)}$. Thus in fact, we have a single form α such that $\alpha = \theta_\mu$ for *every* μ. In particular, we have $\bar{\partial}_{\text{distr}}\alpha = D''_{\text{distr}}\theta_\mu = \beta$ (for example, by Proposition 6.4.2), and by Fatou's lemma, we have

$$\int_X i\alpha \wedge \bar{\alpha}\, e^{-|z|^2} \frac{1}{|z|^2} = \int_X i\alpha \wedge \bar{\alpha}\, e^{-\varphi} \leq 1.$$

The regularity property (Proposition 6.3.2) implies that α is of class $\mathcal{C}^\infty$, and finiteness of the above integral implies that $\alpha = 0$ at the point $p = 0$. Thus $\gamma \equiv \partial z - \alpha$ is a $\bar{\partial}$-closed $\mathcal{C}^\infty$ form of type $(1, 0)$ on X, and at 0, we have $\gamma = dz - 0 \neq 0$. Thus the lemma is proved (and integrability follows). $\square$

Exercises for Sect. 6.6

6.6.1 The integrability theorem (together with Sard's theorem) may be used to construct a proof of the Koebe uniformization theorem (Theorem 5.5.3) that differs slightly from the proof appearing in Chap. 5.

(a) Let $\Omega \neq \emptyset$ be a $\mathcal{C}^\infty$ relatively compact domain in an oriented smooth surface M. By applying Theorem 9.10.1 and Lemma 5.11.1, prove (directly, without using Proposition 5.4.1) that there are an oriented compact $\mathcal{C}^\infty$ surface M', a finite (possibly empty) collection of disjoint positively oriented local $\mathcal{C}^\infty$ charts $\{(D_j, \Psi_j, \Delta(0; R_j))\}_{j=1}^n$ in M' with $R_j > 1$ for each j, and an orientation-preserving diffeomorphism $\Lambda\colon \Omega' \to \Lambda(\Omega')$ of a connected relatively compact neighborhood Ω' of $\overline{\Omega}$ in M onto a domain $\Lambda(\Omega')$ in M' such that $\Lambda(\Omega) = M' \setminus \bigcup_{j=1}^n \Psi_j^{-1}(\overline{\Delta(0; 1)})$. Prove also that M' is planar if and only if Ω is planar.

(b) For the objects in part (a), prove that for any almost complex structure $(TM)_{\mathbb{C}} = (TM)^{1,0} \oplus (TM)^{0,1}$ inducing the given orientation in M, there exists an almost complex structure $(TM')_{\mathbb{C}} = (TM')^{1,0} \oplus (TM')^{0,1}$ in M' such that

$$\Lambda_*(T_pM)^{1,0} = (T_{\Lambda(p)}M')^{1,0} \quad \text{and} \quad \Lambda_*(T_pM)^{0,1} = (T_{\Lambda(p)}M')^{0,1}$$

for every point p in some neighborhood of $\overline{\Omega}$ in Ω'.

(c) According to Sard's theorem (see, for example, [Mi]), the set of critical values of a $\mathcal{C}^\infty$ function on a second countable smooth manifold is a set of measure zero. Using this fact together with Proposition 9.3.11 and Theorem 9.9.2, prove that each compact subset of a second countable $\mathcal{C}^\infty$ surface M lies in some $\mathcal{C}^\infty$ relatively compact domain in M.

(d) Let K be a compact subset of a Riemann surface X. By applying parts (a)–(c) above together with Theorem 6.1.4, prove that there exist a $\mathcal{C}^\infty$ domain Ω with $K \subset \Omega \Subset X$ and a biholomorphism $\Lambda\colon \Omega' \to \Lambda(\Omega')$ of a connected relatively compact neighborhood Ω' of $\overline{\Omega}$ in X onto a domain $\Lambda(\Omega')$ in a compact Riemann surface X'. Prove also that if X is planar, then one may choose X' to be planar.

(e) Give a proof of the Koebe uniformization theorem that is analogous to the proof appearing in Sect. 5.5 but that uses parts (a)–(d) above in place of Proposition 5.4.1.

Exercises 6.6.2–6.6.4 below require the discussion of homology and cohomology appearing in Sects. 10.6 and 10.7, as well as some of the facts and exercises from Chap. 5.

6.6.2 Prove that if M is a compact oriented $\mathcal{C}^\infty$ surface with $H_1(M,\mathbb{R})=0$, then M is diffeomorphic to a sphere.

6.6.3 Prove that if M is a second countable noncompact oriented $\mathcal{C}^\infty$ surface with $H_1(M,\mathbb{R})=0$, then M is diffeomorphic to $\mathbb{R}^2$.

6.6.4 Let M be a second countable $\mathcal{C}^\infty$ surface.

(a) Prove that the universal cover of M is diffeomorphic to the plane $\mathbb{R}^2$ or the sphere $\mathbb{S}^2$.

Hint. Apply integrability, uniformization, and Exercise 6.6.8.

(b) Prove that if M is orientable, then $\pi_1(M)$ is torsion-free and

$$H_1(M,\mathbb{Z})\cong \pi_1(M)\big/[\pi_1(M),\pi_1(M)],$$

while if M is nonorientable, then every torsion element of $\pi_1(M)$ has order 1 or 2 and $\pi_1(M)/\Gamma\cong\mathbb{Z}/2\mathbb{Z}$ for some torsion-free normal subgroup Γ of $\pi_1(M)$.

Hint. Apply Exercise 5.9.4 and Exercise 5.17.4.

(c) Let $\mathbb{A}$ be a subfield of $\mathbb{C}$ containing $\mathbb{Z}$, and let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Prove that if M is orientable, then $H_1^\Delta(M,\mathbb{A})\cong H_1(M,\mathbb{A})$, while if M is nonorientable, then $H_1^\Delta(M,\mathbb{F})\cong H_1(M,\mathbb{F})$.

(d) Prove that if M is orientable, then for any subring $\mathbb{A}$ of $\mathbb{C}$ containing $\mathbb{Z}$, the mapping

$$[\xi]_{H_1(M,\mathbb{A})}\mapsto \big([\xi]_{H_1(M,\mathbb{A})},\cdot\big)_{\mathrm{deR}}$$

gives an injective homomorphism $H_1(M,\mathbb{A})\to \mathrm{Hom}(H^1(M,\mathbb{A}),\mathbb{A})$ (according to Theorem 10.7.18, this homomorphism is also surjective if $\pi_1(X)$ is finitely generated).

(e) Assume that M is orientable, and let $K\subset M$ be a compact set. Prove that there exist domains Ω and Ω' such that $K\subset\Omega\Subset\Omega'\Subset M$ and such that for every $\mathcal{C}^\infty$ closed 1-form ρ on Ω', there exist a $\mathcal{C}^\infty$ closed 1-form τ on M and a function $\lambda\in\mathcal{D}(\Omega')$ such that $\rho-\tau=d\lambda$ on Ω. Conclude from this that

$$\mathrm{im}[H^1_{\mathrm{deR}}(M)\to H^1_{\mathrm{deR}}(\Omega)]=\mathrm{im}[H^1_{\mathrm{deR}}(\Omega')\to H^1_{\mathrm{deR}}(\Omega)].$$

Hint. See Exercise 5.17.5.

6.7 Statement of Schönflies' Theorem

The main goal of the rest of this chapter is the fact that every second countable topological surface admits a $\mathcal{C}^\infty$ structure. For the proof, one forms a $\mathcal{C}^0$ atlas and

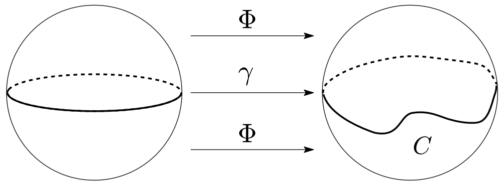

Fig. 6.1 The homeomorphism provided by Schönflies' theorem

then inductively modifies the local charts on the overlaps to get $\mathcal{C}^\infty$ compatibility. For the modification, one applies the following (see Fig. 6.1):

Theorem 6.7.1 (Schönflies' theorem) *Let $\gamma\colon \mathbb{S}^1 \to \mathbb{P}^1$ be a Jordan curve with image $C \equiv \gamma(\mathbb{S}^1)$ in the Riemann sphere $\mathbb{P}^1$. Then there exists a homeomorphism $\Phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $\Phi\restriction_{\mathbb{S}^1} = \gamma$ and such that $\Phi\restriction_{\mathbb{P}^1\setminus\mathbb{S}^1}\colon \mathbb{P}^1 \setminus \mathbb{S}^1 \to \mathbb{P}^1 \setminus C$ is a diffeomorphism.*

In particular, we have the following:

Corollary 6.7.2 (Jordan curve theorem) *If $\gamma\colon \mathbb{S}^1 \to \mathbb{P}^1$ is a Jordan curve with image $C \equiv \gamma(\mathbb{S}^1)$, then $\mathbb{P}^1 \setminus C$ has exactly two connected components. Moreover, the boundary of each of these connected components is equal to C.*

Remark One may choose the homeomorphism Φ in Schönflies' theorem to map the unit disk $\Delta(0; 1)$ onto either of the connected components of $\mathbb{P}^1 \setminus C$ by replacing Φ with the homeomorphism $z \mapsto \Phi(z/|z|^2) = \Phi(1/\bar{z})$ $(0 \mapsto \Phi(\infty), \infty \mapsto \Phi(0))$ if necessary.

A proof of Schönflies' theorem, which is due to Kneser and Radó [KnR], appears in Sect. 6.9 (a different approach, as well as a more complete treatment of triangulations and the classification of surfaces, can be found in [T]). In fact, their proof allows one to choose the diffeomorphism Φ so that the restriction of Φ^{-1} to one of the connected components of $\mathbb{P}^1 \setminus C$, and the restriction of $1/\overline{\Phi^{-1}}$ to the other connected component, are harmonic. For parts of the proof, we will apply the Riemann mapping theorem in the plane (Theorem 5.2.1), although it is not really necessary. For now, we consider a preliminary observation, namely, that a Jordan curve in $\mathbb{P}^1$ that is piecewise smooth away from some point on the curve is separating.

Lemma 6.7.3 *Let $\alpha\colon [a, b] \to \mathbb{P}^1$ be a Jordan curve in $\mathbb{P}^1$ such that for some point $c \in (a, b)$, $\alpha\restriction_{[a,r]}$ and $\alpha\restriction_{[s,b]}$ are piecewise smooth paths for all $r \in (a, c)$ and $s \in (c, b)$. Then the image $C \equiv \alpha([a, b])$ is separating in $\mathbb{P}^1$.*

Proof We have $C \neq \mathbb{P}^1$ (for example, C is not simply connected), so by applying a suitable Möbius transformation, we may assume without loss of generality that $\infty \notin C$. By choosing a smooth point $u \in (a, b)$ and replacing α with the path given by $t \mapsto \alpha(t - a + u)$ for $t \in [a, b + a - u]$ and $t \mapsto \alpha(t - b + u)$ for $t \in [b + a - u, b]$

(and modifying c accordingly), we may assume that α is loop-smooth at a (see Definition 5.10.2). We may also fix $r_0, r_1 \in \mathbb{R}$ such that $a < r_0 < r_1 < c$ and such that $\alpha\restriction_{[a,r_1]}$ is a smooth path. Theorem 9.9.2 then implies that there is a connected neighborhood U of $\alpha(r_0)$ such that $\overline{U} \cap C \subset \alpha((a, r_1))$ and $U \setminus C$ has exactly two connected components U_0 and U_1. Moreover, if $\mathbb{P}^1 \setminus C$ is connected, then there exists a loop $\beta\colon [0, 1] \to \mathbb{P}^1$ such that $\beta(0) = \beta(1) = \alpha(r_0)$, $\beta((0, 1)) \subset \mathbb{P}^1 \setminus C$, and for some $\epsilon > 0$, $\beta((0, \epsilon)) \subset U_0$ and $\beta((1-\epsilon, 1)) \subset U_1$. Consequently, we may choose $R > 0$ so small that for $D \equiv \Delta(\alpha(c); R)$, we have $D \Subset \mathbb{P}^1 \setminus (\overline{U} \cup \beta([0, 1]) \cup \alpha([a, r_1]))$, and setting

$$r \equiv \min\{t \in [a, c] \mid \alpha(t) \in \overline{D}\} \quad \text{and} \quad s \equiv \max\{t \in [c, b] \mid \alpha(t) \in \overline{D}\},$$

we get $r_1 < r < c < s < b$ and $\alpha([a, b] \setminus (r, s)) \subset \mathbb{P}^1 \setminus D$. Thus the loop $\gamma\colon [a, b] \to \mathbb{P}^1$ given by

$$\gamma(t) = \begin{cases} \alpha(t) & \text{if } t \in [a, b] \setminus (r, s), \\ \alpha(r) + \frac{t-r}{s-r} \cdot (\alpha(s) - \alpha(r)) & \text{if } t \in [r, s], \end{cases}$$

is a piecewise smooth Jordan curve that is loop-smooth at a. Lemma 5.10.5 provides a smooth Jordan curve $\eta\colon [a, b] \to \mathbb{P}^1$ such that the image $A \equiv \eta([a, b])$ satisfies $A \cap U = C \cap U$ and $\beta((0, 1)) \subset \mathbb{P}^1 \setminus A$. But β meets both connected components U_0 and U_1 of $U \setminus A = U \setminus C$, and therefore, since A is separating in $\mathbb{P}^1$ (by Proposition 5.15.2) and each connected component of $\mathbb{P}^1 \setminus A$ has boundary equal to A (by Lemma 5.11.1), we have arrived at a contradiction. Thus C is separating. $\square$

Exercises for Sect. 6.7 The exercises for this section require Schönflies' theorem (and its consequences), so the reader may wish to postpone consideration of these exercises until after consideration of the proof in Sect. 6.9.

6.7.1 Let $\gamma\colon [0, 1] \to \mathbb{C}$ be a (continuous) Jordan curve in $\mathbb{C}$ with interior Ω (i.e., Ω is the unique bounded connected component of $\mathbb{C} \setminus C$ provided by Corollary 6.7.2), and let $C = \gamma([0, 1])$. Assume that γ is oriented so that Ω is *on the left*; that is, for the corresponding homeomorphism $\gamma_0\colon \mathbb{S}^1 \to C$ given by $e^{2\pi i t} \mapsto \gamma(t)$, after choosing the homeomorphism $\Phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ with $\Phi\restriction_{\mathbb{S}^1} = \gamma_0$ provided by Schönflies' theorem (Theorem 6.7.1) to map the unit disk $\Delta \equiv \Delta(0; 1)$ diffeomorphically onto Ω (see the remark following the statement of Corollary 6.7.2), γ and γ_0 are directed so that this diffeomorphism is orientation-preserving (to ensure this, one simply replaces γ with the reverse curve and Φ with the mapping $z \mapsto \Phi(\bar{z})$ if necessary).

(a) Prove that γ is homotopic in $\overline{\Omega}$ to a trivial loop.

(b) Prove the following version of *Cauchy's theorem* (cf. Lemma 1.2.1 and Exercises 5.1.6 and 5.1.7): If f is a holomorphic function on a neighborhood of $\overline{\Omega}$, then $\int_\gamma f(z)\,dz = 0$.

(c) Prove the following version of the *Cauchy integral formula* (cf. Lemma 1.2.1 and Exercises 5.1.6 and 5.1.7): If f is a holomorphic function on a neighborhood of $\overline{\Omega}$ and $z_0 \in \Omega$, then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$

(d) Prove the following version of the *residue theorem* (cf. Exercises 2.5.9, 5.1.6, 5.1.7, 5.9.5, and 6.7.6): If S is a finite subset of Ω and θ is a holomorphic 1-form on $\Xi \setminus S$ for some neighborhood Ξ of $\overline{\Omega}$, then

$$\frac{1}{2\pi i} \int_\gamma \theta = \sum_{p \in S} \operatorname{res}_p \theta.$$

(e) Prove the following version of the *argument principle* (cf. Exercises 2.5.8, 5.1.6, 5.1.7, 5.9.5, and 6.7.6): If f is a nontrivial meromorphic function on a neighborhood of $\overline{\Omega}$ with no zeros or poles in $\partial\Omega$, and counting multiplicities, μ is the number of zeros of f in Ω and ν is the number of poles in Ω, then

$$\mu - \nu = \frac{1}{2\pi i} \int_\gamma \frac{df}{f}.$$

(f) Prove the following version of *Rouché's theorem* (cf. Exercises 2.5.8, 5.1.6, 5.1.7, 5.9.5, and 6.7.6): If f and g are nontrivial meromorphic functions on a neighborhood of $\overline{\Omega}$ that do not have any zeros or poles in $\partial\Omega$, and $|g| < |f|$ on $\partial\Omega$, then $\mu_f - \nu_f = \mu_{f+g} - \nu_{f+g}$, where counting multiplicities, μ_f is the number of zeros of f in Ω, ν_f is the number of poles of f in Ω, μ_{f+g} is the number of zeros of $f + g$ in Ω, and ν_{f+g} is the number of poles of $f + g$ in Ω.

6.7.2 Let M be a topological surface.

(a) Let $\alpha\colon [0, 1] \to M$ be an injective path with image $A \equiv \alpha([0, 1])$. Prove that for every $t_0 \in (0, 1)$, there exist numbers a, b and a local $\mathcal{C}^0$ chart $(U, \Phi, U' = (-1, 1) \times (a, b))$ in M such that $0 \leq a < t_0 < b \leq 1$, $A \cap U = \gamma((a, b))$, and $\Phi(\alpha(t)) = (0, t)$ for every $t \in (a, b)$. Prove also that if M is a $\mathcal{C}^\infty$ surface, then the local chart may be chosen to map $U \setminus A$ diffeomorphically onto $U' \setminus (\{0\} \times \mathbb{R})$.

(b) Prove that if B is a Jordan curve in M, then $M \setminus B$ has exactly one or two connected components. Prove also that for each connected component U of $M \setminus B$, we have $\partial U = B$.

(c) Suppose M is an oriented smooth surface, $\gamma\colon [0, 1] \to M$ is a Jordan curve, and $C = \gamma([0, 1])$. Prove that C is separating in M if and only if $\int_\gamma \theta = 0$ for every closed compactly supported $\mathcal{C}^\infty$ 1-form θ on M. Prove also that M is planar if and only if every Jordan curve in M is separating.

Hint. Apply Exercise 5.10.3 and Proposition 5.15.2.

6.7.3 Let $\alpha\colon [0, 1] \to M$ be an injective path in a topological surface M. Prove that there exists a Jordan curve $\beta\colon [0, 2] \to M$ with $\beta\restriction_{[0,1]} = \alpha$.

Hint. Apply Exercise 6.7.2.

6.7.4 Prove that for any nonempty relatively compact domain $\Omega \subsetneqq M$ in a topological surface M, the following are equivalent:

(i) Each point in $\partial\Omega$ admits a local coordinate neighborhood $(U, \Phi = (x, y))$ in M such that $\Omega \cap U = \{q \in U \mid x(q) < 0\}$.

(ii) There exist disjoint Jordan curves $C_1, \dots, C_n$ such that $\partial\Omega = C_1 \cup \cdots \cup C_n$ and such that for each $j = 1, \dots, n$, C_j is separating in the connected component of $M \setminus (C_1 \cup \cdots \cup \widehat{C}_j \cup \cdots C_n)$ containing C_j.

(iii) There exist Jordan curves $\gamma_j \colon [0, 1] \to M$, with corresponding continuous 1-periodic extensions $\tilde{\gamma}_j \colon \mathbb{R} \to M$, for $j = 1, \dots, n$, such that the images $C_j \equiv \gamma_j([0, 1])$ for $j = 1, \dots, n$ are disjoint; $\partial\Omega = C_1 \cup \cdots \cup C_n$; and for each index $j = 1, \dots, n$ and for each point $t_0 \in \mathbb{R}$, there is a local coordinate neighborhood $(U, \Phi = (x, y), U' = (-1, 1) \times (a, b))$ in M with $\Omega \cap U = \{q \in U \mid x(q) < 0\}$, $t_0 \in (a, b) \subset \tilde{\gamma}_j^{-1}(U)$, and $\Phi(\tilde{\gamma}_j(t)) = (0, t)$ for every $t \in (a, b)$.

Prove also that if M is a smooth surface, then each of the local charts (U, Φ) in (i) and (iii) may be chosen so that the restriction $\Phi\restriction_{U \setminus \partial\Omega}$ is a diffeomorphism of $U \setminus \partial\Omega$ onto its image.

Hint. Apply Exercise 6.7.2 and Theorem 9.10.1.

6.7.5 In this exercise, we consider some facts concerning orientation preservation.

(a) Let M be a $\mathcal{C}^\infty$ surface, let $\Omega \subset M$ be a $\mathcal{C}^\infty$ open set, let $p \in \partial\Omega$, and let $(U, \Phi = (x, y), U')$ and $(V, \Psi = (u, v), V')$ be local $\mathcal{C}^\infty$ charts in M with $p \in U \cap V \cap \partial\Omega$ and

$$\Omega \cap U = \{p \in U \mid x(p) < 0\} \quad \text{and} \quad \Omega \cap V = \{p \in V \mid u(p) < 0\}.$$

Assume that Φ and Ψ induce the same orientation in $\partial\Omega$ at p; that is, for the inclusion mapping $\iota \colon U \cap V \cap \partial\Omega \hookrightarrow U \cap V$, we have $(\iota^* dv)_p / (\iota^* dy)_p > 0$. Prove that Φ and Ψ induce the same orientation in some neighborhood of p in $U \cap V$ (and hence in the connected component of $U \cap V$ containing p).

(b) Let $\Phi = (u, v) \colon U \to V$ be a homeomorphism of neighborhoods U and V of $(0, 0)$ in $\mathbb{R}^2$ such that

(i) $\Phi(0, 0) = (0, 0)$;

(ii) $U \cap ((-\infty, 0) \times \mathbb{R}) = \{(x, y) \in U \mid u(x, y) < 0\}$;

(iii) $\Phi\restriction_{U \setminus (\{0\} \times \mathbb{R})}$ maps $U \setminus (\{0\} \times \mathbb{R})$ diffeomorphically onto $V \setminus (\{0\} \times \mathbb{R})$; and

(iv) The function $t \mapsto v(0, t)$ is increasing.

Prove that there exists a neighborhood W of $(0, 0)$ in U for which the restriction $\Phi\restriction_{W \setminus (\{0\} \times \mathbb{R})}$ is orientation-preserving.

Hint. Produce a smooth Jordan curve C bounding a smooth domain $\Omega \Subset U \cap ((-\infty, 0) \times \mathbb{R})$ such that part of C is a directed line segment with Ω lying to the left, and another part is the inverse image under Φ of a directed line segment with $\Phi(\Omega)$ lying to the left. Then apply part (a).

(c) Let M be an oriented smooth surface, and let $(U, \Phi = (x, y), U' = (-1, 1) \times (a, b))$ be a local $\mathcal{C}^0$ chart in M for which the restriction

$$\Phi_0 \equiv \Phi\restriction_{\Phi^{-1}(U' \setminus (\{0\} \times (a,b)))} \colon \Phi^{-1}(U' \setminus (\{0\} \times (a, b))) \to U' \setminus (\{0\} \times (a, b))$$

is a diffeomorphism. Prove that Φ_0 must be either orientation-preserving or orientation-reversing.

Hint. Show that it suffices to consider the case $M \Subset \mathbb{C} \subset \mathbb{P}^1$. Fix a suitable disk $D \Subset U'$ that meets the y-axis. Applying Schönflies' theorem, one may form a homeomorphism $\Psi\colon \mathbb{P}^1 \to \mathbb{P}^1$ that agrees with Φ^{-1} on ∂D and that maps D diffeomorphically onto its image $\Psi(D) = \Phi^{-1}(D) \Subset U$. In particular, $\Psi\restriction_D$ is orientation-preserving or -reversing. Now apply part (b) to the homeomorphism $\Phi_0 \circ \Psi$ near points in $(\partial D) \setminus (\{0\} \times \mathbb{R})$.

(d) Prove that for any Jordan curve $\gamma\colon \mathbb{S}^1 \to \mathbb{P}^1$ with image C, the restriction $\Phi\restriction_{\mathbb{P}^1 \setminus \mathbb{S}^1}\colon \mathbb{P}^1 \setminus \mathbb{S}^1 \to \mathbb{P}^1 \setminus C$ of the homeomorphism provided by Schönflies' theorem must be either an orientation-preserving or an orientation-reversing diffeomorphism (in particular, by replacing Φ with the homeomorphism $z \mapsto \Phi(z/|z|^2) = \Phi(1/\bar{z})$, one may always choose the homeomorphism so that this restriction is orientation-preserving).

(e) Let M be an oriented smooth surface, and let $\alpha\colon [0, 1] \to M$ be an injective path with image $A \equiv \alpha([0, 1])$. Prove that for every $t_0 \in (0, 1)$, there exist numbers a, b and a local $\mathcal{C}^0$ chart $(U, \Phi, U' = (-1, 1) \times (a, b))$ in M such that $0 \le a < t_0 < b \le 1$, $A \cap U = \gamma((a, b))$, $\Phi(\alpha(t)) = (0, t)$ for every $t \in (a, b)$, and the restriction $\Phi\restriction_{U \setminus A}\colon U \setminus A \to U' \setminus (\{0\} \times \mathbb{R})$ is an orientation-preserving diffeomorphism.

(f) Let M be an oriented smooth surface, and let $\Omega \Subset M$ be a domain with the properties (i)–(iii) listed in Exercise 6.7.4. Prove that each of the boundary curves γ_j and the local charts (U, Φ) in (iii) may be chosen so that the restriction $\Phi\restriction_{U \setminus \partial\Omega}$ is an *orientation-preserving* diffeomorphism of $U \setminus \partial\Omega$ onto its image. Prove also that (with the above choice of orientation), if θ is a $\mathcal{C}^1$ closed 1-form on M, then (see Definition 10.5.3) $\sum_{j=1}^{n} \int_{\gamma_j} \theta = 0$.

6.7.6 One may apply Exercise 6.7.5 to obtain versions of the residue theorem (cf. Exercises 2.5.9, 5.1.6, 5.1.7, 5.9.5, and 6.7.1), and the argument principle and Rouché's theorem (cf. Exercises 2.5.8, 5.1.6, 5.1.7, 5.9.5, and 6.7.1), for a suitable nonsmooth domain in a Riemann surface. Let X be a Riemann surface, let Ω be a relatively compact domain in X, and let $\gamma_j\colon [0, 1] \to X$ be a Jordan curve with 1-periodic extension $\tilde{\gamma}_j\colon \mathbb{R} \to X$ for $j = 1, \dots, n$. Assume that:

(i) The images $C_j \equiv \gamma_j([0, 1])$ for $j = 1, \dots, n$ are disjoint;

(ii) We have $\partial\Omega = C_1 \cup \cdots \cup C_n$; and

(iii) For each index $j = 1, \dots, n$ and for each point $t_0 \in \mathbb{R}$, there is a local $\mathcal{C}^0$ coordinate neighborhood $(U, \Phi = (x, y), U' = (-1, 1) \times (a, b))$ in X such that $\Omega \cap U = \{q \in U \mid x(q) < 0\}$, $t_0 \in (a, b) \subset \tilde{\gamma}_j^{-1}(U)$, $\Phi(\tilde{\gamma}_j(t)) = (0, t)$ for every $t \in (a, b)$, and $\Phi\restriction_{U \setminus C_j}\colon U \setminus C_j \to U' \setminus (\{0\} \times (a, b))$ is an orientation-preserving diffeomorphism.

(a) *Residue theorem.* Prove that if S is a finite subset of Ω and θ is a holomorphic 1-form on X, then

$$\sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \theta = \sum_{p \in S} \mathrm{res}_p \theta.$$

(b) *Argument principle.* Prove that if f is a nontrivial meromorphic function on X with no zeros or poles in $\partial\Omega$, and counting multiplicities, μ is the number of zeros of f in Ω and ν is the number of poles in Ω, then

$$\mu - \nu = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{df}{f}.$$

(c) *Rouché's theorem.* Prove that if f and g are nontrivial meromorphic functions on X that do not have any zeros or poles in $\partial\Omega$, and $|g| < |f|$ on $\partial\Omega$, then $\mu_f - \nu_f = \mu_{f+g} - \nu_{f+g}$, where counting multiplicities, μ_f is the number of zeros of f in Ω, ν_f is the number of poles of f in Ω, μ_{f+g} is the number of zeros of $f + g$ in Ω, and ν_{f+g} is the number of poles of $f + g$ in Ω.

6.7.7 Prove that if M is an orientable $\mathcal{C}^\infty$ surface and $\gamma\colon [0,1] \to M$ is a Jordan curve with image $C = \gamma([0,1])$, then for some $R > 1$, there exists a neighborhood Ω of C in M and a homeomorphism $\Phi\colon \Omega \to \Delta(0; 1/R, R)$ with $\Phi(\gamma(t)) = e^{2\pi i t}$ for each $t \in [0,1]$.

Hint. Apply Exercises 6.7.2 and 6.7.5.

6.7.8 State and prove an analogue of Lemma 5.15.1 for a continuous Jordan curve with nonseparating image in an orientable smooth surface M (see Exercise 6.7.7). Using this fact (and Exercise 6.7.5), prove that the image C of a Jordan curve γ in M is separating if and only if $\int_\gamma \theta = 0$ for every $\mathcal{C}^\infty$ closed 1-form θ with compact support in M (a different proof of this analogue of Proposition 5.15.2 was suggested in Exercise 6.7.2).

6.8 Harmonic Functions and the Dirichlet Problem

Kneser and Radó's proof of Schönflies' theorem relies on the solution of the Dirichlet problem, but only for harmonic functions on simply connected domains in $\mathbb{P}^1$ with complement containing at least two points. In this section, we consider an approach to the Dirichlet problem that is an ad hoc version of the important *Perron method*. In particular, by restricting our attention to simply connected domains, we are able to avoid some work by applying the Riemann mapping theorem in the plane. However, it should be noted that the general Perron method can be developed entirely from elementary results, and it leads to a more complete solution. Moreover, most of the results concerning open Riemann surfaces appearing in this book can be obtained by using the Perron method in place of the L^2 $\bar{\partial}$-method (see, for example, [AhS] or [For]).

A complex-valued $\mathcal{C}^2$ function u on a complex 1-manifold X is *harmonic* if $\partial\bar{\partial}u = 0$ (cf. Definition 2.8.1). A real-valued $\mathcal{C}^2$ function ρ on X is *subharmonic* (*strictly subharmonic*) if $i\partial\bar{\partial}\rho \geq 0$ (respectively, $i\partial\bar{\partial}\rho > 0$).

Lemma 6.8.1 *Let X be a Riemann surface.*

(a) *If u is a real-valued harmonic function on X and X is simply connected, then $u = \operatorname{Re} f$ for some holomorphic function f on X. In particular, the real-valued harmonic functions on a Riemann surface are precisely the functions that are locally equal to the real part of a local holomorphic function.*
(b) Strong maximum principle for harmonic functions. *If u is a real-valued harmonic function on X that attains a local maximum or local minimum value, then u is constant.*
(c) *If $\{u_n\}$ is a sequence of real-valued harmonic functions on X that is uniformly bounded on each compact subset of X, then some subsequence $\{u_{n_k}\}$ converges uniformly on compact subsets of X to a harmonic function on X.*
(d) Weak maximum principle for subharmonic functions. *If $\Omega \subsetneqq X$ is a relatively compact domain in X and φ is a continuous real-valued function on $\overline{\Omega}$ that is $\mathcal{C}^2$ and subharmonic on Ω, then $\max_{\overline{\Omega}} \varphi = \max_{\partial\Omega} \varphi$.*

Proof If u is a real-valued harmonic function on X, then $\theta \equiv \partial u$ is a (closed) holomorphic 1-form, since $\bar{\partial}\theta = -\partial\bar{\partial}u = 0$. Therefore, if X is simply connected, then by Corollary 10.5.7, $2\theta = df$ for some $\mathcal{C}^\infty$ function f on X. Since θ is of type $(1,0)$, we have $\bar{\partial}f = 0$, so f is holomorphic. On the other hand,

$$\partial(u - \operatorname{Re} f) = \theta - \frac{1}{2}df = 0 \quad \text{and} \quad \bar{\partial}(u - \operatorname{Re} f) = \overline{\partial(u - \operatorname{Re} f)} = 0,$$

so $u - \operatorname{Re} f$ is constant. Choosing f to agree with u at some point, we get $u = \operatorname{Re} f$ on X in this case, and (a) follows.

For a real-valued harmonic function u on X, part (a) implies that e^u is locally the modulus of a holomorphic function (for $u = \operatorname{Re} f$, we have $e^u = |e^f|$). Therefore, if u attains a local maximum value of c at some point, then the maximum principle for holomorphic functions (Theorem 1.3.4) implies that $u \equiv c$ in a neighborhood of the point. Hence the interior Ω of $u^{-1}(c)$ is nonempty. Corollary 1.3.3 implies that $u \equiv c$ near any point in $\overline{\Omega}$. Hence Ω is both open and closed in X, and therefore $\Omega = X$.

Suppose $\{u_n\}$ is a sequence of real-valued harmonic functions on X that is uniformly bounded on each compact subset of X. Each point has a neighborhood $D \Subset X$ such that D is biholomorphic to a disk and for some constant $R = R(D) > 0$, $|u_n| \leq R$ on D for each $n \in \mathbb{Z}_{>0}$. In particular, by part (a), for each n, we have $u_n{\restriction}_D = \operatorname{Re} f_n$ for some function $f_n \in \mathcal{O}(D)$. The holomorphic function $F_n \equiv e^{f_n}$ then satisfies $0 < e^{-R} \leq |F_n| = e^{u_n} \leq e^R$. Montel's theorem (Corollary 2.11.4) implies that some subsequence of $\{F_n\}$ converges uniformly on compact subsets of D to a holomorphic function F. Uniform continuity of the logarithmic function on the interval $[e^{-R}, e^R] \subset (0, \infty)$ then implies that some subsequence of $\{u_n{\restriction}_D\} = \{\log|F_n|\}$ converges uniformly on compact subsets of D to the har-

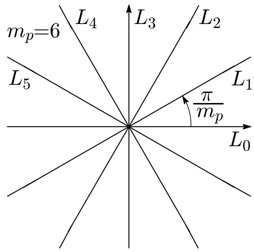

Fig. 6.2 Local representation of the zero set of a harmonic function

monic function $\log|F|$. Covering X by a locally finite collection of such coordinate disks D and applying Cantor's diagonal process, we get the claim (c).

Finally, for the proof of (d), assuming that $\max_{\overline{\Omega}}\varphi > \max_{\partial\Omega}\varphi$, we reason to a contradiction. By replacing Ω with a sufficiently large relatively compact open subset of Ω, we may assume without loss of generality that $\overline{\Omega} \neq X$. Fixing a point $p \in X \setminus \overline{\Omega}$ and applying Corollary 2.14.2, we get a positive $\mathcal{C}^\infty$ strictly subharmonic function ρ on $X \setminus \{p\}$. For $\epsilon > 0$ sufficiently small and for $\psi \equiv \varphi + \epsilon\rho$, we have $\max_{\overline{\Omega}}\psi > \max_{\partial\Omega}\psi$; and hence ψ attains its maximum value at some interior point $q \in \Omega$. On the other hand, for any local holomorphic coordinate neighborhood $(U, z = x + iy)$ of q in Ω, we have

$$\frac{\partial^2\psi}{\partial x^2}(q) + \frac{\partial^2\psi}{\partial y^2}(q) > 0;$$

so we have arrived at a contradiction. □

The following local description of the zero set of a harmonic function is an easy consequence of Lemma 6.8.1 and the local description of holomorphic mappings given by Lemma 2.2.3. The proof is left to the reader (see Exercise 6.8.2).

Lemma 6.8.2 *Let $Z = \{p \in X \mid u(p) = 0\}$ be the zero set of a nonconstant real-valued harmonic function u on a Riemann surface X. Then, for each point $p \in Z$, there exist a constant $R_p > 0$, a positive integer m_p, and a local holomorphic chart $(U_p, \Phi_p = z_p, \Delta(0; R_p))$ such that $p = \Phi_p^{-1}(0)$ and $u = \operatorname{Im}(z_p^{m_p})$ on U_p. In particular,*

$$\Phi(Z \cap U_p) = (L_0 \cup \cdots \cup L_{m_p-1}) \cap \Delta(0; R_p),$$

where $L_j \equiv \{re^{ij\pi/m_p} \mid r \in \mathbb{R}\}$ for each $j = 0, \ldots, m_p - 1$. Moreover, $m_p = 1$ if and only if $(du)_p \neq 0$, and the set of points $p \in Z$ at which $m_p > 1$ is discrete in X.

Remark The case $m_p = 6$ is pictured in Fig. 6.2. Clearly, the zero set Z in the above lemma is closed. The above characterization implies that in particular, Z is a locally finite graph.

For a nonempty open subset Ω of a Riemann surface X and a continuous function $\rho\colon \partial\Omega \to \mathbb{C}$, the associated (*classical*) *Dirichlet problem* is that of finding a continuous function $u\colon \overline{\Omega} \to \mathbb{C}$ that is harmonic on Ω and that satisfies $u\restriction_{\partial\Omega} = \rho$. This problem is *not* always solvable. For example, let $\Delta \equiv \Delta(0;1)$, let $\Omega \equiv \Delta \setminus \{0\}$, and suppose u is a continuous function on $\overline{\Delta} = \overline{\Omega}$ that is harmonic on Ω and that satisfies $u(0) = 1$ and $u \equiv 0$ on $\partial\Delta$. By replacing u with $\operatorname{Re} u$, we may assume that u is real-valued. The maximum principle (Lemma 6.8.1) implies that $0 < u < 1$ on Ω. But then, for $\epsilon > 0$ sufficiently small, the harmonic function $v\colon z \mapsto u(z) + \epsilon \log|z|^2$ on Ω, which approaches $-\infty$ at 0 and which vanishes at $\partial\Delta$, will have some positive values and therefore will attain a local maximum at some point in Ω. This contradicts the maximum principle, so no such solution u of the given Dirichlet problem can exist.

For a nonempty relatively compact domain Ω in a Riemann surface and a continuous function on $\partial\Omega$, the maximum principle implies that the associated Dirichlet problem has *at most* one solution. For if u and v are solutions, then $w \equiv u - v$ is a continuous function on $\overline{\Omega}$ that vanishes on $\partial\Omega$ and that is harmonic on Ω. Hence, by the maximum principle, the maximum and minimum values of $\operatorname{Re} w$ and $\operatorname{Im} w$ must be 0, and therefore $w \equiv 0$.

By applying the Perron method, one can prove that the Dirichlet problem is solvable on any relatively compact domain in a Riemann surface for which each boundary component contains more than one point (see, for example, [AhS]). For our purposes, the following will suffice:

Proposition 6.8.3 *The Dirichlet problem is solvable on each connected component Ω of the complement $\mathbb{P}^1 \setminus K$ of any connected closed set $K \subsetneqq \mathbb{P}^1$.*

We now work toward the proof. We first consider the (*Poisson*) formula for the solution on a disk. As motivation, we consider the following consequence of the Cauchy integral formula:

Lemma 6.8.4 (Mean value property) *If $R > 0$ and $u\colon \overline{\Delta(0;R)} \to \mathbb{R}$ is a continuous function that is harmonic on $\Delta(0;R)$, then*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\, d\theta = \operatorname{Re}\left[\frac{1}{2\pi i}\int_{\partial\Delta(0;R)} u(\zeta) \cdot \frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}\right]$$

for each point $z \in \Delta(0;R)$. In particular, we have the mean value property

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\, d\theta = \frac{1}{2\pi i}\int_{\partial\Delta(0;R)} \frac{u(\zeta)}{\zeta}\, d\zeta.$$

Proof By Lemma 6.8.1, $u\restriction_{\Delta(0;R)} = \operatorname{Re} f$ for some function $f \in \mathcal{O}(\Delta(0;R))$. Let $v \equiv \operatorname{Im} f$. For each $r \in (0,R)$, the Cauchy integral formula (Lemma 1.2.1) gives

$$\begin{aligned} f(0) &= \frac{1}{2\pi i}\int_{\partial\Delta(0;r)} \frac{f(\zeta)}{\zeta}\, d\zeta = \frac{1}{2\pi i}\int_{\partial\Delta(0;r)} \frac{u(\zeta)}{\zeta}\, d\zeta + \frac{1}{2\pi}\int_{\partial\Delta(0;r)} \frac{v(\zeta)}{\zeta}\, d\zeta \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta + \frac{i}{2\pi}\int_0^{2\pi} v(re^{i\theta})\, d\theta. \end{aligned}$$

Comparing real parts, we see that

$$u(0) = \frac{1}{2\pi i} \int_{\partial\Delta(0;r)} \frac{u(\zeta)}{\zeta} \, d\zeta = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \, d\theta,$$

and letting $r \to R^-$ (and observing that $u(re^{i\theta}) \to u(Re^{i\theta})$ uniformly), we get

$$u(0) = \frac{1}{2\pi i} \int_{\partial\Delta(0;R)} \frac{u(\zeta)}{\zeta} \, d\zeta = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \, d\theta.$$

Given a point $z \in \Delta(0; R)$, Proposition 2.14.8 provides the automorphism (i.e., the Möbius transformation)

$$\Phi_0 \colon \zeta \mapsto \frac{\zeta - (z/R)}{1 - (\bar{z}/R)\zeta}$$

of $\mathbb{P}^1$, which maps $\overline{\Delta(0; 1)}$ onto itself. Setting

$$\Phi(\zeta) \equiv R\Phi_0(\zeta/R) = \frac{R^2\zeta - R^2 z}{R^2 - \bar{z}\zeta} \quad \forall \zeta \in \mathbb{P}^1,$$

we get an automorphism $\Phi \in \operatorname{Aut}(\mathbb{P}^1)$ that maps $\overline{\Delta(0; R)}$ onto itself. Applying the above mean value property at 0 to the harmonic function $u(\Phi^{-1})$, we get

$$\begin{aligned} u(z) = u \circ \Phi^{-1}(0) &= \frac{1}{2\pi i} \int_{\partial\Delta(0;R)} \frac{u(\Phi^{-1}(\xi))}{\xi} \, d\xi \\ &= \frac{1}{2\pi i} \int_{\partial\Delta(0;R)} \frac{u(\zeta)}{\Phi(\zeta)} \cdot \Phi'(\zeta) \, d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial\Delta(0;R)} u(\zeta) \cdot \frac{R^2 - |z|^2}{(\zeta - z)(R^2 - \bar{z}\zeta)} \, d\zeta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - |z|^2}{(Re^{i\theta} - z)(R^2 - \bar{z}Re^{i\theta})} Re^{i\theta} \, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - |z|^2}{(Re^{i\theta} - z)(Re^{-i\theta} - \bar{z})} \, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \, d\theta. \end{aligned}$$

Combining the above with the equality

$$\frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2} = \operatorname{Re}\left[\frac{\zeta + z}{\zeta - z}\right],$$

we get the desired formula. □

Lemma 6.8.5 (Poisson formula) *Given a continuous function* $\rho\colon \partial\Delta(0;R) \to \mathbb{C}$ *for some* $R > 0$, *the function* $u\colon \overline{\Delta(0;R)} \to \mathbb{C}$ *given by*

$$u(z) \equiv \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \rho(Re^{i\theta}) \cdot \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\, d\theta & \text{if } z \in \Delta(0;R), \\ \rho(z) & \text{if } z \in \partial\Delta(0;R), \end{cases}$$

is the unique solution of the associated Dirichlet problem (*that is*, u *is continuous on* $\overline{\Delta(0;R)}$, u *is harmonic on* $\Delta(0;R)$, *and* $u\restriction_{\partial\Delta(0;R)} = \rho$).

Remark The function $K\colon (\zeta, z) \mapsto (R^2 - |z|^2)/(2\pi R|\zeta - z|^2)$ is called the *Poisson kernel*. We have $u(z) = \int_{\partial\Delta(0;R)} \rho(\zeta)K(\zeta,z)\,ds(\zeta)$ for $z \in \Delta(0;R)$.

Proof of Lemma 6.8.5 Clearly, we may assume without loss of generality that ρ is real-valued. As in the proof of Lemma 6.8.4, the equality $\frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2} = \mathrm{Re}[\frac{\zeta+z}{\zeta-z}]$ then implies that $u\restriction_{\Delta(0;R)}$ is equal to the real part of the function

$$z \mapsto \frac{1}{2\pi i}\int_{\partial\Delta(0;R)} \rho(\zeta) \cdot \frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}$$
$$= \frac{1}{2\pi i}\int_{\partial\Delta(0;R)} \frac{\rho(\zeta)}{\zeta - z}\, d\zeta + \frac{z}{2\pi i}\int_{\partial\Delta(0;R)} \frac{\rho(\zeta)/\zeta}{\zeta - z}\, d\zeta.$$

Moreover, by Lemma 1.2.2 (or by differentiation past the integral), the above function is holomorphic, and therefore u is harmonic on $\Delta(0;R)$.

It remains to show that u is continuous at each boundary point $z_0 \in \partial\Delta(0;R)$. Let λ denote the Lebesgue measure on $\mathbb{R}$, and for each $\delta > 0$, let

$$N_\delta \equiv \{\theta \in [0, 2\pi] \mid |Re^{i\theta} - z_0| < 2\delta\}.$$

For each $\delta > 0$ and each $z \in \Delta(0;R) \cap \Delta(z_0;\delta)$, by applying Lemma 6.8.4 to the constant (and therefore harmonic) function $\zeta \mapsto 1$, we get

$$\begin{aligned} |u(z) - u(z_0)| &= \left| \frac{1}{2\pi}\int_0^{2\pi} \rho(Re^{i\theta}) \cdot \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\, d\theta \right. \\ &\qquad \left. - \frac{1}{2\pi}\int_0^{2\pi} \rho(z_0) \cdot \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\, d\theta \right| \\ &\le \frac{1}{2\pi}\int_{N_\delta} |\rho(Re^{i\theta}) - \rho(z_0)| \cdot \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\, d\lambda(\theta) \\ &\qquad + \frac{1}{2\pi}\int_{[0,2\pi]\setminus N_\delta} |\rho(Re^{i\theta}) - \rho(z_0)| \cdot \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\, d\lambda(\theta) \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{\tau \in N_\delta} |\rho(Re^{i\tau}) - \rho(z_0)| \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \, d\theta \\
&\quad + \delta^{-2} \cdot 2 \sup_{\partial\Delta(0;R)} |\rho| \cdot \frac{1}{2\pi} \int_0^{2\pi} (R^2 - |z|^2) \, d\theta \\
&= \sup_{\Delta(z_0;2\delta)\cap\partial\Delta(0;R)} |\rho - \rho(z_0)| + \delta^{-2} \cdot 2 \sup_{\partial\Delta(0;R)} |\rho| \cdot (R^2 - |z|^2).
\end{aligned}$$

Now, given $\epsilon > 0$, the continuity of ρ implies that we may choose $\delta > 0$ so small that the first term on the right-hand side is less than $\epsilon/2$. For $z \in \Delta(0; R) \cap \Delta(z_0; \delta)$ sufficiently close to z_0, the second term will then also be less than $\epsilon/2$. Thus u is continuous at z_0. □

The solution of the Dirichlet problem on more general domains requires the following fundamental tool from the Perron method:

Definition 6.8.6 Let Ω be an open subset of a complex 1-manifold X, and let $p \in \partial\Omega$. A real-valued function β on Ω is called a $\mathcal{C}^\infty$ *barrier at p on* Ω if β is a $\mathcal{C}^\infty$ subharmonic function and

$$\lim_{z\to p} \beta(z) = 0 \quad \text{but} \quad \limsup_{z\to q} \beta(z) < 0 \quad \forall q \in (\partial\Omega) \setminus \{p\}.$$

Remark One may also define a notion of a *continuous* subharmonic function and consider *continuous* barriers.

For the construction of a barrier in our case, we will need some elementary geometric facts. We first recall that any holomorphic function f on a neighborhood of a point $z_0 \in \mathbb{C}$ with $f'(z_0) \neq 0$ is *conformal* at z_0. That is, if α and β are two smooth paths in $\mathbb{C}$ with $\alpha(s_0) = \beta(t_0) = z_0$, then f preserves the angle measure

$$\arccos\left[\frac{\mathrm{Re}(\alpha'(s_0)\overline{\beta'(t_0)})}{|\alpha'(s_0)| \cdot |\beta'(t_0)|}\right]$$

between the (tangent vectors of) the paths, because

$$(f \circ \alpha)'(s_0)\overline{(f \circ \beta)'(t_0)} = \alpha'(s_0)\overline{\beta'(t_0)}|f'(z_0)|^2 \quad \text{and} \quad |f'(z_0)|^2 > 0.$$

We also have the following fact, the proof of which is left to the reader (see Exercise 6.8.3):

Lemma 6.8.7 *Let C be a circle of radius R in the plane, let M be a line in the plane that meets C in two points, let l be the length of the line segment given by the intersection of M with the closed disk bounded by C, and let d be the distance from the center of the circle to the line M. Then, at each of the two points of intersection, the two angles between M and C are given by*

$$\theta_0 = 2 \cdot \arctan\left(\frac{l/2}{R+d}\right) \in (0, \pi/2] \quad \textit{and} \quad \theta_1 = \pi - \theta_0 = 2 \cdot \arctan\left(\frac{l/2}{R-d}\right).$$

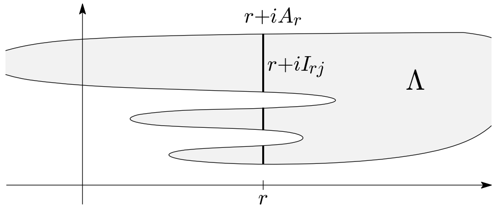

Fig. 6.3 The image Λ of the domain Ω under a branch of the logarithmic function

Lemma 6.8.8 *If K is a connected closed subset of $\mathbb{P}^1$ containing more than one point, Ω is a connected component of $\mathbb{P}^1 \setminus K$, and $p \in \partial\Omega$, then there exists a $\mathcal{C}^\infty$ barrier β at p on Ω.*

Proof This proof is based on the arguments in [AhS] (see also [BerG]). By applying an automorphism of $\mathbb{P}^1$, we may assume that $p = \infty$ and that $0 \in K$. By Lemma 5.17.1, Ω is simply connected, and hence by Proposition 5.2.2, there exists a (single-valued) holomorphic function L on Ω with $e^{L(z)} = z$ for each point $z \in \Omega \subset \mathbb{C}^*$ (i.e., L is a single-valued branch of the logarithmic function). Setting $u \equiv \operatorname{Re} L$ and $v \equiv \operatorname{Im} L$, we have $u(z) = \log|z|$ and $v(z)$ is an argument of z for each point $z \in \Omega$. The function L maps Ω biholomorphically onto a domain $\Lambda \subset \mathbb{C}$. Since $u \to -\infty$ at 0 and $+\infty$ at ∞, the intermediate value theorem implies that for each $r \in \mathbb{R}$, we have $\Lambda \cap (r + i\mathbb{R}) = r + iA_r$, where A_r is a nonempty open subset of $\mathbb{R}$ in which no two distinct points differ by a multiple of 2π. In particular, A_r is equal to the union of a (nonempty) countable collection of disjoint open intervals

$$\{I_{rj}\}_{j\in J_r} = \{(a_{rj}, b_{rj})\}_{j\in J_r}$$

(the connected components) of total length (i.e., Lebesgue measure)

$$\lambda(A_r) = \sum_{j\in J_r} (b_{rj} - a_{rj}) \leq 2\pi,$$

with $J_r = \mathbb{Z}_{>0}$ or $J_r = \{1, \dots, m_r\}$ for some $m_r \in \mathbb{Z}_{>0}$ (see Fig. 6.3).

We will construct a subharmonic function γ on Λ such that $\gamma(\zeta)$ is bounded above by a negative constant for ζ in the intersection with any half-plane $\{\operatorname{Re}\xi < r\}$, but $\gamma(\zeta) \to 0$ as $\operatorname{Re}\zeta \to \infty$. The function $\beta = \gamma(L)$ will then be a barrier at $p = \infty$ on Ω. For the construction of γ, we will first construct, for each $r \in \mathbb{R}$, a negative subharmonic function γ_r such that $\gamma_r = -1$ on $\{\zeta \in \Lambda \mid \operatorname{Re}\zeta \leq r\}$ and $\gamma_r(\zeta) \to 0$ as $\operatorname{Re}\zeta \to +\infty$. For this, we will construct a harmonic function on $\{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta > r\}$ for which the value at any point ζ will be the product of $-2/\pi$ and a sum of angle measures. Each of these angle measures will be for an angle formed by the vertical line $\operatorname{Re}\xi = r$ and, for some $j \in J_r$, the circle passing through ζ and the endpoints of the line segment $r + i\overline{I}_{rj}$. As ζ approaches any point in $r + iI_{rj}$, the associated circle will approach the line $r + i\mathbb{R}$ and the angle measure will approach π. Thus the limit of the function at points in $r + iA_r$ will be at most -2. We will get the function γ_r on Λ as an extension of the composition of a suitable nondecreasing convex function and the above function. As $\operatorname{Re}\zeta \to \infty$, the above circles will grow large and the above angle measures will shrink to 0; and it will follow that $\gamma_r(\zeta) \to 0$. To get a function γ that is bounded away from 0 on the intersection of Λ with *any* left half-plane, we will then let $\gamma = \sum 2^{-\nu}\gamma_{r_\nu}$ for some sequence $r_\nu \to \infty$.

We now consider the details of the construction. For each number $r \in \mathbb{R}$ and each index $j \in J_r$, the Möbius transformation $\Psi_{rj} \colon \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$\zeta \mapsto \frac{r + ia_{rj} - \zeta}{r + ib_{rj} - \zeta}$$

($r + ia_{rj} \mapsto 0$, $r + ib_{rj} \mapsto \infty$, $\infty \mapsto 1$) is an automorphism of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ that maps the segment $r + i\overline{I}_{rj}$ onto $\{\infty\} \cup (-\infty, 0]$, the segment $\{\infty\} \cup (r + i(\mathbb{R} \setminus I_{rj}))$ onto $[0, \infty) \cup \{\infty\}$ (here, of course, for $a \in \mathbb{R}$, $(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$ and $[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$, while $\{\infty\}$ denotes the singleton consisting of the point at infinity in $\mathbb{P}^1$), and the half-plane $H_r = \{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta > r\}$ onto the upper half-plane $\mathbb{H} = \{\zeta \in \mathbb{C} \mid \operatorname{Im}\zeta > 0\}$. We may also form a single-valued argument function that is the unique harmonic function $\alpha \colon \mathbb{C} \setminus i(-\infty, 0] \to (-\pi/2, 3\pi/2)$ satisfying $e^{i\alpha(\xi)} = \xi/|\xi|$ for each $\xi \in \mathbb{C} \setminus i(-\infty, 0]$ (i.e., α is the imaginary part of a branch of the logarithmic function on the simply connected domain $\mathbb{C} \setminus i(-\infty, 0]$). Thus

$$\alpha_{rj} \equiv -\frac{2}{\pi}\alpha \circ \Psi_{rj} \colon \Psi_{rj}^{-1}(\mathbb{C} \setminus i(-\infty, 0]) \to (-3, 1)$$

is a harmonic function, $-2 < \alpha_{rj} < 0$ on H_r, $\alpha_{rj} \equiv -2$ on $r + iI_{rj}$, and $\alpha_{rj} \equiv 0$ on $r + i(\mathbb{R} \setminus \overline{I}_{rj})$. Moreover, the fact that a Möbius transformation maps circles in $\mathbb{P}^1$ to circles (Theorem 5.7.3), the fact that (local) biholomorphisms are conformal, Lemma 6.8.7, and the inequality $0 < \arctan x < x$ for $x > 0$ together give the following estimate:

$$-\frac{2}{\pi} \cdot \frac{b_{rj} - a_{rj}}{\operatorname{Re}\zeta - r} < -\frac{4}{\pi}\arctan\left[\frac{b_{rj} - a_{rj}}{2(\operatorname{Re}\zeta - r)}\right] \leq \alpha_{rj}(\zeta) < 0 \quad \forall \zeta \in H_r$$

(as pictured in Fig. 6.4, for $\zeta \in H_r$, the inverse image under Ψ_{rj} of the ray $M \equiv [0, \infty) \cdot \Psi_{rj}(\zeta)$ is an arc of a circle that meets the ray $r + i(-\infty, a_{rj}]$ at the vertex $r + ia_{rj} = \Psi_{rj}^{-1}(0)$, thus determining an angle of the same measure as the angle formed by M and the nonnegative x-axis).

The Weierstrass M-test and Lemma 6.8.1 together now imply that the (possibly finite) series $\sum_{j \in J_r} \alpha_{rj}$ converges uniformly on compact subsets of H_r to a harmonic function α_r satisfying

$$0 > \alpha_r(\zeta) > -\sum_{j \in J_r} \frac{2}{\pi} \cdot \frac{b_{rj} - a_{rj}}{\operatorname{Re}\zeta - r} = -\frac{2}{\pi} \cdot \frac{\lambda(A_r)}{\operatorname{Re}\zeta - r} \geq -\frac{4}{\operatorname{Re}\zeta - r} \quad \forall \zeta \in H_r.$$

Now, according to Lemma 2.10.2, we may fix a $\mathcal{C}^\infty$ function $\chi : \mathbb{R} \to [-1, \infty)$ such that $\chi', \chi'' \geq 0$ on $\mathbb{R}$, $\chi \equiv -1$ on $(-\infty, -3/2]$, and $\chi(t) = t$ for each $t \geq -1/2$ (simply form the function as in the lemma with $a = -3/2$, $b = -1$, and $c = -1/2$,

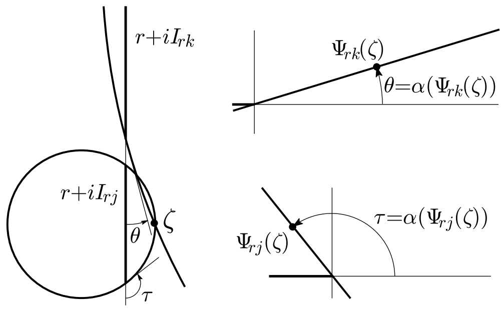

Fig. 6.4 Harmonic functions associated to two components of $\Lambda \cup (r + i\mathbb{R})$

and then subtract 1 to get χ). On the other hand, for each $r \in \mathbb{R}$, each index $j \in J_r$, and each point $\zeta \in r + iI_{rj} \subset r + iA_r = \Lambda \cap \partial H_r$, we have $\alpha_{rj}(\zeta) = -2$ and hence

$$\alpha_r \le \alpha_{rj} < -3/2$$

at all points in H_r near ζ. It follows that the function $\gamma_r\colon \Lambda \to [-1, 0)$ given by

$$\gamma_r(\zeta) = \begin{cases} -1 & \text{if } \zeta \in \Lambda \setminus H_r, \\ \chi(\alpha_r(\zeta)) & \text{if } \zeta \in \Lambda \cap H_r, \end{cases}$$

is a $\mathcal{C}^\infty$ subharmonic function. Choosing a strictly increasing sequence of real numbers $\{r_\nu\}$ with $r_\nu \to \infty$, we see that the function $\gamma \equiv \sum_{\nu=1}^{\infty} 2^{-\nu}\gamma_{r_\nu}\colon \Lambda \to [-1, 0)$ is equal to the *finite* sum

$$-2^{-\mu+1} + \sum_{\nu=1}^{\mu-1} 2^{-\nu}\gamma_{r_\nu} \le -2^{-\mu+1}$$

on the open set $\Lambda \setminus \overline{H_{r_\mu}}$ for each μ. Since these open sets form an increasing sequence with union Λ, it follows that γ is a $\mathcal{C}^\infty$ subharmonic function on Λ. Furthermore, γ is bounded above by a negative constant on the intersection of Λ with any left half-plane $\{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta < r\}$ for $r \in \mathbb{R}$. On the other hand, $\gamma(\zeta) \to 0$ as $\operatorname{Re}\zeta \to +\infty$. For given $\epsilon > 0$, we may choose $\mu \in \mathbb{Z}_{>0}$ so large that $\sum_{\nu=\mu+1}^{\infty} 2^{-\nu} = 2^{-\mu} < \epsilon/2$. Hence, if $\zeta \in \Lambda$ with $\operatorname{Re}\zeta > r_\mu$, then

$$0 > \gamma(\zeta) > -\frac{\epsilon}{2} + \sum_{\nu=1}^{\mu} 2^{-\nu}\gamma_{r_\nu}(\zeta) \ge -\frac{\epsilon}{2} + \sum_{\nu=1}^{\mu} 2^{-\nu}\chi\left[-\frac{4}{\operatorname{Re}\zeta - r_\nu}\right].$$

Since $\chi(0) = 0$, we see that for each $\zeta \in \Lambda$ with $\operatorname{Re}\zeta \gg r_\mu$, we have $0 > \gamma(\zeta) > -\epsilon$. It follows that the function $\beta \equiv \gamma(L)\colon \Omega \to [-1, 0)$ is a $\mathcal{C}^\infty$ barrier at $p = \infty$ on Ω (note that the function $z \mapsto \log|z| = \operatorname{Re} L(z)$ is bounded above on each compact subset of $\mathbb{C}$). $\square$

The last fact we will need for the proof of Proposition 6.8.3 is the following (for a much more general version, see, for example, [Mu]):

Theorem 6.8.9 (Tietze extension theorem in $\mathbb{C}$) *If* $\rho\colon K \to \mathbb{R}$ *is a continuous function on a nonempty compact subset* K *of* $\mathbb{C}$, *then there exists a continuous function* $\rho_0\colon \mathbb{C} \to \mathbb{R}$ *such that* $\rho_0{\restriction}_K = \rho$, $\inf_{\mathbb{C}} \rho_0 = \min_K \rho$, *and* $\sup_{\mathbb{C}} \rho_0 = \max_K \rho$.

Proof We may fix a number $R > 0$ so that $K \subset \Delta(0; R)$; a sequence $\{D_\nu\}$, where $D_\nu = \Delta(\zeta_\nu; r_\nu) \Subset \Delta(0; 2R) \setminus K$ for each ν, the collection $\{D_\nu\}$ is locally finite in $\mathbb{C} \setminus K$, and $\Delta(0; R) \setminus K \subset \bigcup D_\nu$; a sequence of nonnegative continuous functions $\{\lambda_\nu\}$ with $\operatorname{supp} \lambda_\nu \subset D_\nu$ for each ν, $\sum \lambda_\nu \le 1$ on $\mathbb{C}$, and $\sum \lambda_\nu \equiv 1$ on $\Delta(0; R) \setminus K$; and for each ν, a point $z_\nu \in K$ with $\operatorname{dist}(z_\nu, \zeta_\nu) = \operatorname{dist}(K, \zeta_\nu) > r_\nu$. Clearly, the function $\alpha\colon \mathbb{C} \to \mathbb{R}$ given by

$$\alpha \equiv \begin{cases} \rho & \text{on } K, \\ \sum \rho(z_\nu) \cdot \lambda_\nu & \text{on } \mathbb{C} \setminus K, \end{cases}$$

is continuous on $\mathbb{C} \setminus K$ and satisfies $\alpha{\restriction}_K = \rho$. We now show that α is continuous at each point $z_0 \in K$. Given $\epsilon > 0$, we may choose $\delta_1 > 0$ so that

$$|\alpha(z) - \alpha(z_0)| = |\rho(z) - \rho(z_0)| < \epsilon \quad \forall z \in \Delta(z_0; 3\delta_1) \cap K.$$

By local finiteness, there is an $N \in \mathbb{Z}_{>0}$ such that D_ν lies in the δ_1-neighborhood of K each $\nu > N$. In particular, for each $\nu > N$, we have

$$r_\nu < \operatorname{dist}(K, \zeta_\nu) = |z_\nu - \zeta_\nu| < \delta_1.$$

We may also choose a number δ such that $0 < \delta < \delta_1$, $\Delta(z_0; \delta) \subset \Delta(0; R)$, and $\Delta(z_0; \delta) \cap D_\nu = \emptyset$ for each $\nu = 1, \dots, N$. Therefore, if $z \in \mathbb{C} \setminus K$ with $|z - z_0| < \delta$, then we have $z \in \Delta(0; R)$, $z \notin D_\nu$ for each $\nu = 1, \dots, N$, and for each $\nu > N$ with $z \in D_\nu$, we have

$$|z_\nu - z_0| \le |z_\nu - \zeta_\nu| + |\zeta_\nu - z| + |z - z_0| < 3\delta_1.$$

Therefore

$$|\alpha(z) - \alpha(z_0)| = \left| \sum \lambda_\nu(z) \cdot (\rho(z_\nu) - \rho(z_0)) \right| \le \sum \lambda_\nu(z) \cdot |\rho(z_\nu) - \rho(z_0)| < \epsilon.$$

Thus the extension α is continuous at z_0. Hence the function $\rho_0\colon \mathbb{C} \to \mathbb{R}$ given by

$$z \mapsto \begin{cases} \alpha(z) & \text{if } \min_K \rho \le \alpha(z) \le \max_K \rho, \\ \min_K \rho & \text{if } \alpha(z) < \min_K \rho, \\ \max_K \rho & \text{if } \alpha(z) > \max_K \rho, \end{cases}$$

has the required properties. $\square$

Proof of Proposition 6.8.3 Given a connected closed set $K \subsetneqq \mathbb{P}^1$, a connected component Ω of $\mathbb{P}^1 \setminus K$, and a continuous function $\rho\colon \partial\Omega \to \mathbb{C}$, we show that the associated Dirichlet problem is solvable. By considering $\operatorname{Re} \rho$ and $\operatorname{Im} \rho$ and applying a suitable automorphism of $\mathbb{P}^1$, we may assume without loss of generality that ρ is real-valued and $\infty \in \Omega$. Any Dirichlet problem on the complement of a singleton has the constant solution, so we may also assume that K contains more than one point. By the Tietze extension theorem (Theorem 6.8.9), there exists a continuous function $\rho_0\colon \mathbb{C} \to \mathbb{R}$ with $|\rho_0| \leq M \equiv \max |\rho|$ on $\mathbb{C} \supset \partial\Omega$ and $\rho_0 = \rho$ on $\partial\Omega$. We will produce solutions on elements of an exhausting sequence of coordinate disks for Ω (with boundary values given by ρ_0). We will then pass to a convergent subsequence; and finally, we will use barriers to show that the limit is a solution of the Dirichlet problem.

By Lemma 5.17.1 and the Riemann mapping theorem in the plane (see Corollary 5.2.6), there exists a biholomorphism $\Phi\colon \Omega \to \Delta$ of Ω onto the unit disk $\Delta \equiv \Delta(0; 1)$. We may choose a sequence of numbers $\{R_\nu\}$ in $(0, 1)$ converging to 1 such that

$$\infty \in \Omega_\nu \equiv \Phi^{-1}(\Delta(0; R_\nu)) \Subset \Omega \quad \forall \nu = 1, 2, 3, \dots.$$

For each ν, Lemma 6.8.5 provides a (unique) real-valued continuous function u_ν on $\overline{\Omega}_\nu$ such that $u_\nu \restriction_{\Omega_\nu}$ is harmonic and $u_\nu \restriction_{\partial\Omega_\nu} = \rho_0 \restriction_{\partial\Omega_\nu}$. In particular, $|u_\nu| \leq M$ on $\overline{\Omega}_\nu$ by the maximum principle (part (b) of Lemma 6.8.1). Applying part (c) of Lemma 6.8.1 and passing to a subsequence, we may assume that the sequence $\{u_\nu\}$ converges uniformly on compact subsets of Ω to a real-valued harmonic function on Ω (more precisely, setting $\hat{u}_\nu = u_\nu$ on $\overline{\Omega}_\nu$ and $\hat{u}_\nu = 0$ on $\Omega \setminus \overline{\Omega}_\nu$, we see that for each μ, a subsequence of $\{\hat{u}_\nu\}$ will converge uniformly on Ω_μ, and hence Cantor's diagonal process provides a subsequence converging uniformly on compact subsets of Ω).

It now suffices to show that the function $u\colon \overline{\Omega} \to [-M, M] \subset \mathbb{R}$ given by

$$u \equiv \begin{cases} \lim_{\nu\to\infty} u_\nu & \text{on } \Omega, \\ \rho & \text{on } \partial\Omega, \end{cases}$$

is continuous at each point $z_0 \in \partial\Omega$. Given $\epsilon > 0$, there is a $\delta_1 > 0$ such that $|\rho_0(z) - \rho_0(z_0)| < \epsilon$ for each $z \in \mathbb{C}$ with $|z - z_0| < \delta_1$. By Lemma 6.8.8, there exists a barrier β at z_0 on Ω. Since $\beta < 0$ on Ω and $\limsup_{z\to\zeta} \beta(z) < 0$ at each point $\zeta \in (\partial\Omega) \setminus \{z_0\}$, we have $\sup_{\Omega\setminus\Delta(z_0;\delta_1)} \beta < 0$; and hence, by replacing β with the product of β and a sufficiently large positive constant, we may assume that $\beta < -2M$ on $\Omega \setminus \Delta(z_0; \delta_1)$. For each ν, we have

$$u_\nu + \beta \leq -M \leq -|\rho(z_0)| < \rho(z_0) + \epsilon \quad \text{on } (\partial\Omega_\nu) \setminus \Delta(z_0; \delta_1)$$

and

$$u_\nu + \beta \leq \rho_0 < \rho(z_0) + \epsilon \quad \text{on } (\partial\Omega_\nu) \cap \Delta(z_0; \delta_1).$$

Therefore, by the weak maximum principle for subharmonic functions (part (d) of Lemma 6.8.1), we have $u_\nu + \beta < \rho(z_0) + \epsilon$ on Ω_ν for each ν, and passing to the limit, we get $u+\beta \le \rho(z_0)+\epsilon$ on Ω. Therefore, since $\beta \to 0$ at z_0 and ρ is continuous, we must have

$$\limsup_{z\to z_0} u(z) \le \rho(z_0)+\epsilon \quad \forall \epsilon > 0;$$

and hence $\limsup_{z\to z_0} u(z) \le \rho(z_0) = u(z_0)$. The same argument applied to $-\rho_0$, $-\rho$, and $-u_\nu \to -u$ gives

$$\limsup_{z\to z_0}(-u(z)) \le -u(z_0),$$

and hence $\liminf_{z\to z_0} u(z) \ge u(z_0)$. Thus $\lim_{z\to z_0} u(z) = u(z_0)$, and therefore u is continuous. $\square$

Exercises for Sect. 6.8

6.8.1 State and prove a (Poisson) formula for the solution of the Dirichlet problem on a disk $\Delta(z_0; R)$ with center z_0.
6.8.2 Prove Lemma 6.8.2.
6.8.3 Prove Lemma 6.8.7.

6.9 Proof of Schönflies' Theorem

We are now ready to prove Schönflies' theorem (following [KnR]). Throughout this section, Δ denotes the unit disk $\Delta(0; 1)$. We begin with some preliminary observations.

Lemma 6.9.1 *Let $\gamma : \mathbb{S}^1 \to \mathbb{P}^1$ be a Jordan curve with image $C \equiv \gamma(\mathbb{S}^1)$, and let Ω be a connected component of $\mathbb{P}^1 \setminus C$. Then $\partial\Omega$ is an infinite connected subset of C (that is, $\gamma^{-1}(\partial\Omega)$ is either $\mathbb{S}^1$ or a closed arc in $\mathbb{S}^1$), the solution $\Psi : \overline{\Omega} \to \mathbb{C}$ of the Dirichlet problem in Ω with boundary values given by $\Psi\restriction_{\partial\Omega} = (\gamma^{-1})\restriction_{\partial\Omega} : \partial\Omega \to \mathbb{S}^1$ exists, and $\Psi\restriction_{\Omega}$ is a proper $\mathcal{C}^\infty$ mapping of Ω into Δ.*

Proof Observe that $C \neq \mathbb{P}^1$, since (for example) C is not simply connected. According to Lemma 5.17.1, $\partial\Omega$ is a nonempty connected compact subset of C. Moreover, if $\partial\Omega$ were a singleton $\{q\}$, then we would have $\Omega = \mathbb{P}^1 \setminus \{q\}$ (the connected set $\mathbb{P}^1 \setminus \{q\}$ would meet Ω, but not $\partial\Omega$), which is clearly impossible because $C \subset \mathbb{P}^1 \setminus \Omega$. Thus $\gamma^{-1}(\partial\Omega)$ is either $\mathbb{S}^1$ or a closed arc in $\mathbb{S}^1$. According to Proposition 6.8.3, the solution $\Psi : \overline{\Omega} \to \mathbb{C}$ of the Dirichlet problem in Ω with boundary values given by $\Psi\restriction_{\partial\Omega} = (\gamma^{-1})\restriction_{\partial\Omega} : \partial\Omega \to \mathbb{S}^1$ exists. Let $u \equiv \operatorname{Re}\Psi$ and $v \equiv \operatorname{Im}\Psi$. We may choose a point $p \in \overline{\Omega}$ with $|\Psi(p)| = R \equiv \max_{\overline{\Omega}} |\Psi|$, a line L in $\mathbb{C}$ with $L \cap \overline{\Delta(0; R)} = \{\Psi(p)\}$, and constants $a, b, c \in \mathbb{R}$ such that the function $\lambda : (x+iy) \mapsto ax+by+c$ for $x, y \in \mathbb{R}$ satisfies $L = \lambda^{-1}(0)$ and $\lambda < 0$ on $\Delta(0; R)$. Thus the continuous function $\lambda(\Psi) \equiv au + bv + c : \overline{\Omega} \to \mathbb{R}$, which is harmonic on Ω, attains its maximum at p. On the other hand, $\lambda(\Psi)$ is nonconstant because if $\lambda(\Psi)$ were constant, then we would have $\Psi(\partial\Omega) \subset \Psi(\overline{\Omega}) = \{\Psi(p)\}$, which is impossible since $\Psi\restriction_{\partial\Omega} = (\gamma^{-1})\restriction_{\partial\Omega}: \partial\Omega \to \mathbb{S}^1$ is injective and $\partial\Omega$ is not a singleton. Therefore, by the maximum principle (Lemma 6.8.1), $p \in \partial\Omega$, and it follows that $R = 1$ (one may also see that $R = 1$ by applying Lemma 6.8.1 to the $\mathcal{C}^\infty$ subharmonic function $|\Psi|^2$ on Ω) and $\Psi(\Omega) \subset \Delta$. Finally, since $\Psi(\partial\Omega) \subset \partial\Delta$, the restriction $\Psi\restriction_\Omega: \Omega \to \Delta$ must be a proper $\mathcal{C}^\infty$ mapping; that is, the inverse image of every compact subset of Δ is compact. □

Lemma 6.9.2 *Suppose that $\gamma: \mathbb{S}^1 \to \mathbb{P}^1$ is a Jordan curve, $C \equiv \gamma(\mathbb{S}^1)$, and for each connected component Ω of $\mathbb{P}^1 \setminus C$, there exists a continuous mapping $\Psi_\Omega: \overline{\Omega} \to \overline{\Delta}$ such that $(\Psi_\Omega)\restriction_{\partial\Omega} = (\gamma^{-1})\restriction_{\partial\Omega}: \partial\Omega \to \mathbb{S}^1$, $\Psi_\Omega(\Omega) \subset \Delta$, and $(\Psi_\Omega)\restriction_\Omega: \Omega \to \Delta$ is a proper local diffeomorphism. Then $\mathbb{P}^1 \setminus C$ has exactly two connected components Ω_0 and Ω_1 (in particular, the Jordan curve theorem holds for γ), $\partial\Omega_0 = \partial\Omega_1 = C$, and for each $\nu = 0, 1$, $\Psi_{\Omega_\nu}: \overline{\Omega}_\nu \to \overline{\Delta}$ is a homeomorphism for which the restriction $(\Psi_{\Omega_\nu})\restriction_{\Omega_\nu}: \Omega_\nu \to \Delta$ is a diffeomorphism. Consequently, the mapping $\Phi: \mathbb{P}^1 \to \mathbb{P}^1$ given by*

$$\Phi(z) \equiv \begin{cases} \Psi_{\Omega_0}^{-1}(z) & \text{if } z \in \overline{\Delta}, \\ \Psi_{\Omega_1}^{-1}(1/\bar{z}) & \text{if } z \in \mathbb{P}^1 \setminus \Delta, \end{cases}$$

is a homeomorphism for which $\Phi\restriction_{\mathbb{S}^1} = \gamma$ and $\Phi\restriction_{\mathbb{P}^1\setminus\mathbb{S}^1}: \mathbb{P}^1 \setminus \mathbb{S}^1 \to \mathbb{P}^1 \setminus \mathbb{S}^1$ is a diffeomorphism (in particular, Schönflies' theorem holds for γ).

Proof Given a connected component Ω_0 of $\mathbb{P}^1 \setminus C$, Lemma 10.2.11 implies that the proper local diffeomorphism $(\Psi_{\Omega_0})\restriction_{\Omega_0}: \Omega_0 \to \Delta$ is a (finite) $\mathcal{C}^\infty$ covering map and therefore a diffeomorphism (since Δ is simply connected). Thus

$$\overline{\Delta} = \overline{\Psi_{\Omega_0}(\Omega_0)} \subset \Psi_{\Omega_0}(\overline{\Omega}_0) \subset \overline{\Delta},$$

and hence $\Psi_{\Omega_0}: \overline{\Omega}_0 \to \overline{\Delta}$ is a continuous bijection, and therefore by compactness a homeomorphism. In particular, $\gamma^{-1}(\partial\Omega_0) = \Psi_{\Omega_0}(\partial\Omega_0) = \partial\Delta = \mathbb{S}^1$, and hence $\partial\Omega_0 = C$. Furthermore, $\mathbb{P}^1 \setminus C$ is *not* connected. For if $\Omega_0 = \mathbb{P}^1 \setminus C$, then Ψ would be a homeomorphism of $\mathbb{P}^1$ onto $\overline{\Delta}$. But these two spaces are *not* homeomorphic. For example, if we remove two distinct points from $\mathbb{P}^1$, then we get a domain that is biholomorphic to $\mathbb{C}^*$ and therefore is not simply connected. However, if we remove any two distinct points in $\partial\Delta$ from $\overline{\Delta}$, the resulting space is still simply connected.

Now, letting Ω_1 be a connected component of $\mathbb{P}^1 \setminus C$ with $\Omega_1 \neq \Omega_0$, we get a homeomorphism $\Phi: \overline{\Omega}_0 \cup \overline{\Omega}_1 = \Omega_0 \cup C \cup \Omega_1 \to \mathbb{P}^1$ by setting $\Phi = \Psi_{\Omega_0}$ on $\overline{\Omega}_0$ and $\Phi = 1/\overline{\Psi}_{\Omega_1}$ on $\overline{\Omega}_1$ (observe that the homeomorphism $\overline{\Delta} \to \mathbb{P}^1 \setminus \Delta$ given by $z \mapsto 1/\bar{z}$ is equal to the identity on $\partial\Delta$); and Φ maps $\Omega_0 \cup \Omega_1$ diffeomorphically onto $\mathbb{P}^1 \setminus C$. Suppose there exists a third connected component Ω of $\mathbb{P}^1 \setminus C$. By applying a Möbius transformation, we may assume that $0 \in \Omega_0$ and $\infty \in \Omega_1$, and we may fix $r > 0$ with $\Delta(0; r) \Subset \Omega_0$. Letting $\gamma_0(t) = \gamma(e^{2\pi i t})$ for each $t \in [0, 1]$ and fixing a path α in $\overline{\Omega}_0 \setminus \{0\} \approx \overline{\Delta} \setminus \{\Psi_{\Omega_0}(0)\}$ from $\gamma_0(0) = \gamma_0(1)$ to r, we see that the loop $\beta = \alpha^- * \gamma_0 * \alpha$ generates the fundamental group $\pi_1(\overline{\Omega}_0 \setminus \{0\}, r) \cong$

$\pi_1(\Omega_0 \setminus \{0\}, r)$ (the verification is left to the reader). Thus the loop $\sigma: t \mapsto re^{2\pi it}$ represents the element $[\beta]_r^m \in \pi_1(\overline{\Omega_0} \setminus \{0\}, r)$ for some $m \in \mathbb{Z}$, and hence

$$2\pi i = \int_\sigma \frac{1}{z}\, dz = m \int_\beta \frac{1}{z}\, dz = m \int_{\gamma_0} \frac{1}{z}\, dz.$$

However, γ_0 is path homotopic to the constant loop in $\overline{\Omega} \approx \overline{\Delta}$ and therefore in $\mathbb{C}^* \supset \overline{\Omega}$, so $\int_{\gamma_0}(1/z)\, dz = 0$. Thus we have arrived at a contradiction, and hence $\Omega_0 \cup C \cup \Omega_1 = \mathbb{P}^1$. □

Proof of Theorem 6.7.1 Let $\gamma: \mathbb{S}^1 \to \mathbb{P}^1$ be a Jordan curve, let $C \equiv \gamma(\mathbb{S}^1)$, and let Ω be a connected component of $\mathbb{P}^1 \setminus C$. According to Lemma 6.9.1, $\partial\Omega$ is an infinite connected subset of C, the solution $\Psi: \overline{\Omega} \to \mathbb{C}$ of the Dirichlet problem in Ω with boundary values given by $\Psi\restriction_{\partial\Omega} = (\gamma^{-1})\restriction_{\partial\Omega}: \partial\Omega \to \mathbb{S}^1$ exists, and the restriction $\Psi\restriction_\Omega: \Omega \to \Delta$ is a proper $\mathcal{C}^\infty$ mapping. Let $u \equiv \operatorname{Re}\Psi$ and $v \equiv \operatorname{Im}\Psi$. According to Lemma 6.9.2, it suffices to show that $\Psi\restriction_\Omega$ is a local diffeomorphism. By the $\mathcal{C}^\infty$ inverse function theorem (Theorem 9.9.1 and Theorem 9.9.2), this is the case if and only if $du \wedge dv$ is nowhere 0 in Ω. So, for the rest of this proof, we will assume that there exists a point $p \in \Omega$ at which $(du \wedge dv)_p = 0$, and we will reason to a contradiction. This condition implies that there exist constants $a, b \in \mathbb{R}$ that are not both 0, but that satisfy $a(du)_p + b(dv)_p = 0$. The continuous function

$$\rho \equiv au + bv - au(p) - bv(p): \overline{\Omega} \to \mathbb{R}$$

is then harmonic on Ω and satisfies $\rho(p) = 0$ and $(d\rho)_p = 0$. The argument will proceed (in several steps) as follows (see Fig. 6.5). We will first show that ρ is nonconstant. The local description of zero sets of nonconstant harmonic functions in Lemma 6.8.2 will then imply that for $Z \equiv \rho^{-1}(0)$, $Z \cap \Omega$ is a graph with at least four edges emanating from p. These edges cannot be rejoined by a sequence of edges in $Z \cap \Omega \setminus \{p\}$, since if this were possible, then we would get a Jordan curve in Z enclosing a region in Ω. The maximum principle would then imply that ρ vanishes on this region. Similar arguments will then imply that the (at least four) connected components of $Z \cap \Omega \setminus \{p\}$ approach at least four distinct points in $\partial\Omega$. But these *four* boundary points must then map into the intersection of the line $L \equiv \{ax + by - au(p) - bv(p) = 0\}$ with $\partial\Delta$, which consists of only *two* points. Since $\Psi\restriction_{\partial\Omega}$ is injective, this will yield the desired contradiction. The steps below contain the details of the above argument.

Step 1. Proof that ρ is nonconstant. The line L in $\mathbb{C}$ given by

$$L \equiv \{z \in \mathbb{C} \mid a \cdot \operatorname{Re} z + b \cdot \operatorname{Im} z - au(p) - bv(p) = 0\}$$

contains the point $\Psi(p) \in \Delta$, and hence $L \cap \partial\Delta$ contains exactly two points, which we will denote by ξ_1 and ξ_2. If ρ is constant, then $\Psi(\overline{\Omega}) \subset L$, and hence $\Psi(\partial\Omega) \subset \{\xi_1, \xi_2\}$, which, according to Lemma 6.9.1, is impossible. Thus ρ is nonconstant, and in particular, the set $S \equiv \{q \in \Omega \mid \rho(q) = 0 \text{ and } (d\rho)_q = 0\}$ is a discrete subset of Ω.

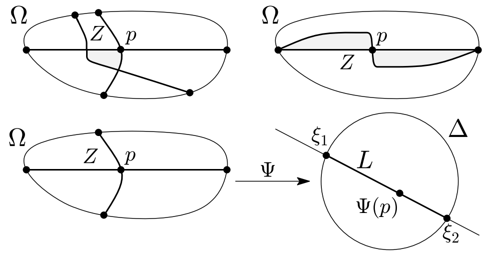

Fig. 6.5 Three possibilities for the zero set Z, the first two of which are ruled out by the maximum principle, the third by the injectivity of Ψ on the boundary

Step 2. Proof of the nonexistence of a separating Jordan curve that lies in the zero set of ρ and that does not meet $\partial\Omega$ in more than one point. Let

$$Z \equiv \Psi^{-1}(L \cap \overline{\Delta}) = \{z \in \overline{\Omega} \mid \rho(z) = 0\}.$$

Suppose there exists a separating Jordan curve A in $\mathbb{P}^1$ such that $A \subset Z$ and $A \cap \partial\Omega$ contains at most one point. Then $\mathbb{P}^1 \setminus A$ has a connected component Θ that does *not* meet the connected subset $C \setminus (A \cap \partial\Omega)$ of $\mathbb{P}^1 \setminus A$. In particular, $\Theta \cap \partial\Omega = \emptyset$ (otherwise, since $\partial\Omega$ is connected and not a singleton, Θ would contain points in $(\partial\Omega) \setminus (A \cap \partial\Omega) \subset C \setminus (A \cap \partial\Omega)$). On the other hand, $\Theta \not\subset \mathbb{P}^1 \setminus \overline{\Omega}$ (otherwise, the set $\partial\Theta$, which, according to Lemma 6.9.1, is infinite, would be contained in the set $A \setminus \Omega = A \cap \partial\Omega$, which contains at most one point), so we must have $\Theta \subset \Omega$. But then, since $\partial\Theta \subset A \subset Z$, the maximum principle implies that ρ is constant on the open set Θ and therefore on $\overline{\Omega}$. This contradicts Step 1, so no such Jordan curve A exists.

Step 3. Local description of certain injective paths near a point of $Z \cap \Omega$. Given a point $q \in Z \cap \Omega$, Lemma 6.8.2 implies that for some $R > 0$ and some $m \in \mathbb{Z}_{>0}$, there is a local holomorphic chart $(D, \Lambda = \zeta, \Delta(0; R))$ such that $D \subset \Omega$, $q = \Lambda^{-1}(0)$, and $\Lambda(Z \cap D) = (L_0 \cup \cdots \cup L_{m-1}) \cap \Delta(0; R)$ for m distinct lines $L_0, \ldots, L_{m-1}$ through 0. Suppose $\alpha : [a, b] \to Z \setminus \{q\}$ is an injective path, $\alpha([a, b]) \cap \partial\Omega$ contains at most one point, and for some point $c \in (a, b)$, $\alpha\restriction_{[a,r]}$ and $\alpha\restriction_{[s,b]}$ are piecewise smooth paths (in $\mathbb{P}^1$) for all $r \in (a, c)$ and $s \in (c, b)$. Then $\alpha([a, b])$ meets at most one of the $2m$ connected components of $Z \cap D \setminus \{q\}$. For if A and B are two distinct connected components of $Z \cap D \setminus \{q\}$ meeting $\alpha([a, b])$, then since α is injective and the functions $|\zeta|\restriction_A$ and $|\zeta|\restriction_B$ are homeomorphisms onto the interval $(0, R)$, there are unique numbers $r, s \in [a, b]$ such that $r \in \alpha^{-1}(A)$, $|\alpha(r)| = \min_{\alpha^{-1}(A)} |\zeta(\alpha)|$, $s \in \alpha^{-1}(B)$, and $|\alpha(s)| = \min_{\alpha^{-1}(B)} |\zeta(\alpha)|$. By exchanging A with B and r with s if necessary, we may assume that $r < s$. Therefore, the path β given by $\alpha\restriction_{[r,s]}$, followed by a suitable parametrization of the inverse image under Λ of the line segment from $\zeta(\alpha(s))$ to 0, and then by a suitable parametrization of the inverse image under Λ of the line segment from 0 to $\zeta(\alpha(r))$, is a Jordan curve in Ω that satisfies the conditions in Lemma 6.7.3. Thus β's image is a separating Jordan curve in $\mathbb{P}^1$ that satisfies the conditions of the Jordan curve in Step 2, and hence we have arrived at a contradiction. *Thus the image of α cannot meet more than one connected component of $Z \cap D \setminus \{q\}$.*

Step 4. Proof that the closure of each connected component of $Z \cap \Omega \setminus S$ is the image of an injective path that is smooth at each point in Ω. The (closed) zero set Z of ρ satisfies $Z \cap \partial\Omega \subset \Psi^{-1}(\{\xi_1, \xi_2\})$ (a set with at most two points), $Z \cap \Omega$ is the zero set of the nonconstant harmonic function $\rho\restriction_\Omega$, and p is a point in the discrete set $S = \{q \in Z \cap \Omega \mid (d\rho)_q = 0\}$. By Lemma 6.8.2 (or Theorem 9.9.2), $Z \cap \Omega \setminus S$ is a (properly embedded) 1-dimensional smooth submanifold of the open set $\Omega \setminus S$. Thus, by Step 2, Lemma 6.7.3 (or Proposition 5.15.2), and Theorem 9.10.1, each connected component A of $Z \cap \Omega \setminus S$ is a noncompact connected 1-dimensional smooth submanifold of $\Omega \setminus S$ that is the image of a diffeomorphism $\alpha\colon (0, 1) \to A$. We may choose a sequence $\{s_\nu\}$ in $(0, 1)$ such that $s_\nu \to 0$ and such that the sequence $\{\alpha(s_\nu)\}$ converges to a point $q_0 \in \overline{A} \subset \overline{\Omega}$. Since $\alpha\colon (0, 1) \to \Omega \setminus S$ is a proper mapping, we have $q_0 \notin A$ and therefore $q_0 \in S \cup (Z \cap \partial\Omega)$. Similarly, we may choose a sequence $\{t_\nu\}$ in $(0, 1)$ such that $t_\nu \to 1$ and such that the sequence $\{\alpha(t_\nu)\}$ converges to a point $q_1 \in S \cup (Z \cap \partial\Omega)$.

If $q_0 \in S$, then by Step 3 and Lemma 6.8.2, there is a local holomorphic chart $(D_0, \Lambda_0 = \zeta_0, \Delta(0; R_0))$ such that $D_0 \Subset \Omega$, $\overline{D}_0 \cap S = \{q_0\} = \Lambda_0^{-1}(0)$,

$$\Lambda_0(Z \cap D_0) = (L_0 \cup \cdots \cup L_{m-1}) \cap \Delta(0; R_0)$$

for $m > 1$ distinct lines $L_0, \dots, L_{m-1}$ through 0, and A meets (and hence contains) *exactly* one of the $2m$ connected components of $Z \cap D_0 \setminus \{q_0\} = Z \cap D_0 \setminus S$. After applying a rotation, we may assume that $A \cap D_0 = \Lambda_0^{-1}((0, R_0))$. It follows that for $w_0 \equiv \Lambda_0^{-1}(-R_0/2)$, the set $A_0 \equiv A \cup \Lambda_0^{-1}((-R_0/2, R_0))$ is a connected noncompact 1-dimensional smooth submanifold of the open subset $\Omega_0 \equiv \Omega \setminus [(S \setminus \{q_0\}) \cup \{w_0\}]$ of $\mathbb{P}^1$, and that the connected set A is open in A_0. Moreover, since $\Lambda_0(\alpha(s_\nu)) \to 0$, it follows that the connected open set $\alpha^{-1}(D_0) = \alpha^{-1}(\Lambda_0^{-1}((0, R_0)))$ must be equal to an interval of the form $(0, r_0)$ for some point $r_0 \in (0, 1)$. In particular, $\alpha(t) \to q_0$ as $t \to 0^+$ ($\Lambda_0(\alpha)$ is a strictly increasing real-valued $\mathcal{C}^\infty$ function on $(0, r_0)$) and $q_1 \in \overline{\alpha([r_0, 1))} \setminus \overline{D}_0$.

As above, if $q_1 \in S$, then there exist a connected neighborhood $D_1 \Subset \Omega$ with $\overline{D}_1 \cap S = \{q_1\}$, a point $w_1 \in D_1 \setminus \{q_1\}$, and a connected noncompact 1-dimensional smooth submanifold A_1 of the open subset $\Omega_1 \equiv \Omega \setminus [(S \setminus \{q_1\}) \cup \{w_1\}]$ of $\mathbb{P}^1$ such that $q_1 \in A_1 \subset A \cup D_1$ and A is a connected open set in A_1. We then have $\alpha^{-1}(D_1) = (r_1, 1)$ for some point $r_1 \in (0, 1)$ and $\alpha(t) \to q_1$ as $t \to 1^-$. Furthermore, we may choose the neighborhood D_1 so that $\overline{D}_1 \cap \overline{D}_0 = \emptyset$ if $q_0 \in S$ (and hence $0 < r_0 < r_1 < 1$ in this case).

If $q_0 \in \partial\Omega$, then $\alpha(t) \to q_0$ as $t \to 0^+$. For if this is not the case, then after fixing a sufficiently small neighborhood D of q_0 with $\overline{D} \cap Z \cap \partial\Omega = \{q_0\}$ (as we may, since $Z \cap \partial\Omega$ contains at most two points), we may form a sequence $\{a_\nu\}$ in $(0, 1)$ such that $a_\nu \to 0$ but $\alpha(a_\nu) \in \Omega \setminus D$ for each ν. In fact, since the connected image under α of the interval with endpoints a_ν and s_ν must meet ∂D for $\nu \gg 0$, we may assume that $\alpha(a_\nu) \in \partial D$ for each ν. Moreover, by passing to a subsequence, we may assume that the sequence $\{\alpha(a_\nu)\}$ converges to a point $q \in [(Z \cap \partial\Omega) \cup S] \cap \partial D$. We then cannot have $q \in S$ (otherwise, by the above arguments for the case $q_0 \in S$, we would have $\alpha(t) \to q$ as $t \to 0^+$), so $q \in Z \cap \partial\Omega \cap \partial D = \emptyset$, which is clearly impossible. Thus $\alpha(t) \to q_0$ as $t \to 0^+$. A similar argument shows that if $q_1 \in \partial\Omega$, then $\alpha(t) \to q_1$ as $t \to 1^-$. Observe also that we cannot have $q_0 = q_1 \in \partial\Omega$, because if this were the case, then α would extend to a Jordan curve $[0,1] \to \overline{\Omega}$ with separating image (by Lemma 6.7.3) and Step 2 would be violated.

Now, by the above, if both q_0 and q_1 lie in S, then A is a connected open relatively compact set in the connected noncompact 1-dimensional smooth submanifold $A_0 \cup A_1$ of the open subset $(\Omega_0 \cup \Omega_1) \setminus \{w_0, w_1\} = \Omega \setminus [(S \setminus \{q_0, q_1\}) \cup \{w_0, w_1\}]$ of $\mathbb{P}^1$, $\alpha(t) \to q_0$ as $t \to 0^+$, and $\alpha(t) \to q_1$ as $t \to 1^-$. Since $A_0 \cup A_1$ is diffeomorphic to $\mathbb{R}$ (by Theorem 9.10.1), it follows that there is a smooth injective path $\beta\colon [0,1] \to \Omega$ with $A = \beta((0,1))$, $\beta(0) = q_0$, and $\beta(1) = q_1$. If $q_0 \in S$ but $q_1 \in \partial\Omega$, then A is a connected open set in the manifold A_0, and the restriction of a suitable diffeomorphism of the interval $(-\infty, 1)$ onto A_0 then determines an injective path $\beta\colon [0,1] \to \overline{\Omega}$ such that $\beta((0,1)) = A$, β is smooth from the right at 0, β is smooth at each point in $(0,1)$, $\beta(0) = q_0$, and $\beta(1) = q_1$. A similar injective path β exists if $q_0 \in \partial\Omega$ and $q_1 \in S$. If $q_0, q_1 \in \partial\Omega$, then α extends to an injective path $\beta\colon [0,1] \to \overline{\Omega}$ with $\beta(0) = q_0$ and $\beta(1) = q_1$.

To summarize, if A is any connected component of $Z \cap \Omega \setminus S$, then there exists an injective path $\beta\colon [0,1] \to Z$ such that $\beta\restriction_{(0,1)}$ is a diffeomorphism of $(0,1)$ onto A, $\beta(0), \beta(1) \in S \cup (Z \cap \partial\Omega)$, β is smooth from the right at 0 if $\beta(0) \in S$, and β is smooth from the left at 1 if $\beta(1) \in S$.

Step 5. Formation of a piecewise smooth path to the boundary. For our point $p \in S$, we have a local holomorphic chart $(D, \Lambda = \zeta, \Delta(0;R))$ such that $D \subset \Omega$, $p = \Lambda^{-1}(0)$, and $\Lambda(Z \cap D) = (L_0 \cup \cdots \cup L_{m-1}) \cap \Delta(0;R)$ for $m \geq 2$ distinct lines $L_0, \dots, L_{m-1}$ through 0. Given a connected component A of $Z \cap D \setminus \{p\}$, we may form an injective path $\alpha\colon [0,1] \to Z$ from p to $\partial\Omega$ as follows. If B_1 is the connected component of $Z \cap \Omega \setminus S$ containing A, then by Step 6, there exists an injective path $\beta_j\colon [0,1] \to Z$ such that $\beta_1(0) = p$, $\beta_1((0,1)) = B_1$, $\beta_1(1) \in S \cup (Z \cap \partial\Omega)$, β_1 is smooth from the right at 0 and smooth at each point in $(0,1)$, and β_1 is smooth from the left at 1 if $\beta_1(1) \in S$. If $\beta_1(1) \in \partial\Omega$, then we set $\alpha = \beta_1$. If $\beta_1(1) \notin \partial\Omega$, then we proceed by induction as follows. Suppose we have constructed smooth injective paths $\beta_1, \dots, \beta_{k-1}$ in Ω such that $\beta_1(0) = p$, $B_j \equiv \beta_j((0,1))$ is a connected component of $Z \cap \Omega \setminus S$, for each $j = 1, \dots, k-1$, the connected components $B_1, \dots, B_{k-1}$ are distinct, $\beta_{j-1}(1) = \beta_j(0) \in S$ for $j = 2, \dots, k-1$, and $\beta_{k-1}(1) \in S$. By Step 3, there is a connected component B_k of $Z \cap \Omega \setminus S$ with $\beta_{k-1}(1) \in \overline{B}_k$ and $B_k \neq B_{k-1}$. In fact, Step 3 and Step 4 imply that $\overline{B}_k \cap \overline{B}_{k-1} = \{\beta_{k-1}(1)\}$ and $\overline{B}_k \cap \overline{B}_j = \emptyset$ for $j = 1, \dots, k-2$. As above, we may form an injective path $\beta_k\colon [0,1] \to Z$ such that $\beta_k(0) = \beta_{k-1}(1)$, $\beta_k((0,1)) = B_k$, $\beta_k(1) \in S \cup (Z \cap \partial\Omega)$, β_k is smooth from the right at 0 and smooth at each point in $(0,1)$, and β_k is smooth from the left at 1 if $\beta_k(1) \in S$. If $\beta_k(1) \in \partial\Omega$, then we terminate the process and we set $\alpha = \beta_1 * \cdots * \beta_k$. If $\beta_k(1) \in S$, we continue to form the next path. Thus, if the process eventually terminates, then we get an injective path α from $p = \alpha(0)$ to a point $\alpha(1) \in \partial\Omega$ such that α is piecewise smooth (with values in $Z \cap \Omega$) on each compact subinterval of $[0,1)$. If the process never terminates, then we get a sequence of smooth paths $\{\beta_k\}$, and we may define a continuous injective map $\beta\colon [0,1) \to Z \cap \Omega$ that is piecewise smooth on each compact subinterval by setting

$$\beta(t) = \beta_k\left[\left(t - 1 + \frac{1}{k}\right)k(k+1)\right]$$

for each point $t \in [1 - k^{-1}, 1 - (k+1)^{-1})]$ and each index $k = 1, 2, 3, \dots$. We may fix a sequence $\{t_\nu\}$ in $(0, 1)$ for which $t_\nu \to 1$ and $\{\beta(t_\nu)\}$ converges to a point $q \in \overline{\Omega}$. Since the collection of connected components of $Z \cap \Omega \setminus S$ is locally finite in Ω (by the local characterization of $Z \cap \Omega$), q must lie in $Z \cap \partial\Omega$, a set of at most two points. Given a neighborhood D of q with $\overline{D} \cap Z \cap \partial\Omega = \{q\}$, we see that for $\delta \in (0, 1)$ sufficiently small, we have $\beta((1 - \delta, 1)) \cap \partial D = \emptyset$ (otherwise, there would exist a sequence in $(0, 1)$ converging to 1 with image converging to a point in the set $Z \cap \partial D \cap \partial\Omega$, which is empty). Thus $\beta((1 - \delta, 1)) \subset D$, and it follows that $\beta(t) \to q$ as $t \to 1^-$. Thus we may extend β to a path $\alpha : [0, 1] \to Z$ with $\alpha(1) = q \in Z \cap \partial\Omega$.

Step 6. Completion of the proof. Continuing with the notation of Step 5, and letting $A_1, \dots, A_{2m}$ be the distinct connected components of $Z \cap D \setminus \{p\}$, we see that for each $j = 1, \dots, 2m$, we may form an injective path $\alpha_j : [0, 1] \to Z$ such that $A_j \subset \alpha_j((0, 1)) \subset \Omega \setminus \{p\}$, $\alpha_j(0) = p$, $\alpha_j(1) \in Z \cap \partial\Omega$, and the restriction of α_j to each compact subinterval of $[0, 1)$ is piecewise smooth. Step 3 implies that the sets $\alpha_1((0, 1]), \dots, \alpha_{2m}((0, 1])$ are disjoint. In particular, the $2m \geq 4$ points $\alpha_1(1), \dots, \alpha_{2m}(1) \in Z \cap \partial\Omega$ must be distinct. Thus we have arrived at a contradiction, and the theorem follows. □

The above proof also gives the following:

Theorem 6.9.3 (Kneser–Radó [KnR]) *If $\gamma : \mathbb{S}^1 \to \mathbb{P}^1$ is a Jordan curve, $C \equiv \gamma(\mathbb{S}^1)$, and Ω is a connected component of $\mathbb{P}^1 \setminus C$, then the solution Ψ of the Dirichlet problem in Ω with boundary values given by $\Psi\restriction_{\partial\Omega} = \gamma^{-1} : C \to \mathbb{S}^1$ is a homeomorphism of $\overline{\Omega}$ onto $\overline{\Delta}$ that maps Ω diffeomorphically onto Δ.*

Exercises for Sect. 6.9

6.9.1 Show that in the proof of Schönflies' theorem, we could have formed a smooth path to the boundary. In other words, prove that if ρ is a nonconstant real-valued harmonic function on a simply connected domain $\Omega \Subset \mathbb{C}$ and $p \in Z \equiv \rho^{-1}(0)$, then there exists a proper $\mathcal{C}^\infty$ embedding $\alpha : \mathbb{R} \to \Omega$ such that $\alpha(0) = p$ and $\alpha(\mathbb{R}) \subset Z$.

6.10 Orientable Topological Surfaces

In this section we consider a natural notion of orientability for a second countable topological surface. The simplest example of a nonorientable $\mathcal{C}^\infty$ surface is the Möbius band (Example 9.7.1). So one natural way in which to define orientability of a (second countable) topological surface is to require that the surface *not* contain a Möbius band. In order to verify that the definition is consistent with the definition of orientability of a $\mathcal{C}^\infty$ surface (Definition 9.7.2), we will need some preliminary facts. The first is a version of Lemma 5.11.1 for a nonorientable $\mathcal{C}^\infty$ surface:

Lemma 6.10.1 *Let $\gamma: [0, 1] \to M$ be a smooth Jordan curve in a $\mathcal{C}^\infty$ surface M, and let $C = \gamma([0, 1])$. If C does not admit an orientable neighborhood in M, then there exists a diffeomorphism $\Phi: \Omega \to B$ of some neighborhood Ω of C in M onto the Möbius band*

$$B \equiv [0, 1] \times (0, 1)/(0, t) \sim (1, 1-t) \quad \forall t \in (0, 1),$$

with the quotient map $\Pi: [0, 1] \times (0, 1) \to B$, such that $\Phi(\gamma(t)) = \Pi(t, 1/2)$ for each point $t \in [0, 1]$ (here, $(t, 1/2)$ denotes an ordered pair, not an interval).

Proof We proceed as in the proof of Lemma 5.11.1, but here, we must put a twist in the constructed band. We have the associated diffeomorphism $\gamma_0: \mathbb{S}^1 \to C$ given by $e^{2\pi i t} \mapsto \gamma(t)$, and the $\mathcal{C}^\infty$ covering map $\alpha: \mathbb{R} \to C$ given by

$$t \mapsto \gamma_0(e^{2\pi i t}) = \gamma(t - \llcorner t \lrcorner)$$

(where $\llcorner t \lrcorner$ denotes the *floor* of t). We may form local $\mathcal{C}^\infty$ charts

$$\{(U_j, \Lambda_j = (x_j, y_j), R_j = I_j \times (-\delta_j, \delta_j))\}_{j=1}^m$$

in M and a partition $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ such that

$$(U_m, \Lambda_m = (x_m, y_m), R_m) = (U_1, (1 + x_1, -y_1), (1 + I_1) \times (-\delta_1, \delta_1))$$

and such that for each $j = 1, \dots, m$, I_j is an open interval containing $[t_{j-1}, t_j]$, $0 < \delta_j < 1/2$,

$$C \cap U_j = \{p \in U_j \mid y_j(p) = 0\} = \alpha(I_j),$$

and $x_j(\alpha(t)) = t$ for each point $t \in I_j$. We may also choose the local charts so that $\ell(I_1) = \ell(I_m) < 1/2$ and $I_j \subset (0, 1)$ for $j = 2, \dots, m-1$. We may fix disjoint connected open sets $\{W_j\}_{j=1}^m$ and connected open sets $\{V_j\}_{j=1}^m$ such that

(i) We have $\gamma([t_{j-1}, t_j]) \subset V_j \subset U_j$ for each $j = 1, \dots, m$;
(ii) We have $V_m \cap V_1 \subset W_m \subset U_m = U_1$, and for each $j = 1, \dots, m-1$, we have $V_j \cap V_{j+1} \subset W_j \subset U_j \cap U_{j+1}$;
(iii) We have $V_i \cap V_j = \emptyset$ whenever $\gamma([t_{i-1}, t_i]) \cap \gamma([t_{j-1}, t_j]) = \emptyset$ (that is, whenever i and j are indices with $1 \leq i < j-1 < m-1$ or $1 < i < j-1 = m-1$); and
(iv) $V_j \cap \Lambda_1^{-1}(\{0\} \times (-\delta_1, \delta_1)) = V_j \cap \Lambda_m^{-1}(\{1\} \times (-\delta_m, \delta_m)) = V_j \cap W_m = \emptyset$ for each $j = 2, \dots, m-1$ (for this, we must first shrink $\delta_1, \dots, \delta_m$ slightly so that $\Lambda_1^{-1}(\{0\} \times (-\delta_1, \delta_1)) \cap C = \{\gamma(0)\} = \{\gamma(1)\}$).

Let $V \equiv V_1 \cup \cdots \cup V_m$. We now alter the induced orientations inductively as follows. Suppose that $1 \leq k \leq m-2$ and that for $j = 2, \dots, k$, the local coordinates (x_{j-1}, y_{j-1}) and (x_j, y_j) induce the same orientation on the connected open subset W_{j-1} of $U_{j-1} \cap U_j$; that is,

$$\frac{dx_j \wedge dy_j}{dx_{j-1} \wedge dy_{j-1}} > 0 \quad \text{on } W_{j-1}$$

(we assume that this holds vacuously for $k = 1$). Then, by replacing y_{k+1} with $-y_{k+1}$ if necessary, we may assume that the above holds for $j = k + 1$ as well. Proceeding inductively, we see that we may assume that

$$\frac{dx_j \wedge dy_j}{dx_{j-1} \wedge dy_{j-1}} > 0 \quad \text{on } W_{j-1} \supset V_{j-1} \cap V_j \quad \forall j = 2, \ldots, m-1.$$

In fact, the above condition holds for $j = m$ as well. For if this were not the case, then the local coordinates $(x_m, -y_m) = (1 + x_1, y_1)$ (on $U_m = U_1$) would induce the same orientation as the local coordinates (x_{m-1}, y_{m-1}) on the set $W_{m-1} \supset V_{m-1} \cap V_m$ and the same orientation as the local coordinates (x_1, y_1) on $W_m \supset V_m \cap V_1$. But then V would be a connected *orientable* neighborhood of C, so we would arrive at a contradiction. Thus the condition holds for $j = m$.

Let us now fix $\mathcal{C}^\infty$ functions λ and $\{\lambda_j\}_{j=2}^{m-1}$ such that $\lambda + \sum_{j=2}^{m-1} \lambda_j \equiv 1$ on a neighborhood Ω_0 of C, $0 \leq \lambda \leq 1$, $\operatorname{supp} \lambda \subset V_1 \cup V_m$, and for each $j = 2, \ldots, m-1$, $0 \leq \lambda_j \leq 1$ and $\operatorname{supp} \lambda_j \subset V_j$. We may define the characteristic functions

$$\chi_0 \equiv \begin{cases} 1 & \text{on } V \cap \Lambda_1^{-1}([0, 1/2) \times (-\delta_1, \delta_1)), \\ 0 & \text{on } V \setminus \Lambda_1^{-1}([0, 1/2) \times (-\delta_1, \delta_1)), \end{cases}$$

and

$$\chi_1 \equiv \begin{cases} 1 & \text{on } V \cap \Lambda_m^{-1}((1/2, 1) \times (-\delta_m, \delta_m)), \\ 0 & \text{on } V \setminus \Lambda_m^{-1}((1/2, 1) \times (-\delta_m, \delta_m)) \end{cases}$$

(observe that $\Lambda_1^{-1}((-1/2, 0) \times (-\delta_1, \delta_1)) = \Lambda_m^{-1}((1/2, 1) \times (-\delta_m, \delta_m))$, so $\chi_0 \cdot \chi_1 \equiv 0$ but $\chi_0 + \chi_1 \equiv 1$ on $V \cap U_1 = V \cap U_m$), and the mapping $\Psi \colon \Omega_0 \to B$ given by

$$\begin{aligned} \Psi \equiv \Pi \Bigg[& \chi_0 \cdot \lambda \cdot (x_1, (1/2) + y_1) + \chi_1 \cdot \lambda \cdot (x_m, (1/2) + y_m) \\ & + \sum_{j=2}^{m-1} \lambda_j \cdot (x_j, (1/2) + y_j) \Bigg]. \end{aligned}$$

The Möbius band B has the $\mathcal{C}^\infty$ atlas $\{(Q_j, \Phi_j, R = (0, 1) \times (0, 1))\}_{j=1}^2$, where $Q_1 \equiv \Pi(R)$, $\Phi_1 \equiv (\Pi\restriction_R)^{-1}$, $Q_2 \equiv \Pi(([0, 1] \setminus \{1/2\}) \times (0, 1))$, and

$$\Phi_2(\Pi(s, t)) \equiv \begin{cases} (s + (1/2), t) & \text{if } (s, t) \in [0, 1/2) \times (0, 1), \\ (s - (1/2), 1 - t) & \text{if } (s, t) \in (1/2, 1] \times (0, 1). \end{cases}$$

If Ω is a sufficiently small connected neighborhood of C in Ω_0, then

$$W_m \cap \Omega \supset \Omega \cap \Lambda_1^{-1}(\{0\} \times (-\delta_1, \delta_1)) = \Omega \cap \Lambda_m^{-1}(\{1\} \times (-\delta_m, \delta_m)).$$

Moreover, we have $\lambda \equiv 1$ and $\lambda_j \equiv 0$ on $W_m \cap \Omega$ for $j = 2, \dots, m-1$, $\Psi(W_m \cap \Omega) \subset Q_2$, and on $W_m \cap \Omega$, we have $\Phi_2 \circ \Psi = (\frac{1}{2} + x_1, \frac{1}{2} + y_1)$. It follows that Ψ is a $\mathcal{C}^\infty$ mapping on Ω, and in fact, $\Psi\restriction_{W_m \cap \Omega}$ is a diffeomorphism. We also have $\Psi(\Omega \setminus W_m) \subset Q_1$, and on a neighborhood of the set $\Omega \setminus W_m$, we have

$$\begin{aligned}(\rho_1, \rho_2) &\equiv \Phi_1 \circ \Psi \\ &= \Bigg(\lambda \cdot (\chi_0 x_1 + \chi_1 x_m) + \sum_{j=2}^{m-1} \lambda_j \cdot x_j, \\ &\qquad \lambda \cdot (\chi_0 \cdot (2^{-1} + y_1) + \chi_1 \cdot (2^{-1} + y_m)) + \sum_{j=2}^{m-1} \lambda_j \cdot (2^{-1} + y_j)\Bigg),\end{aligned}$$

which is of class $\mathcal{C}^\infty$. Given a point $p \in C \setminus W_m$, we have

$$\begin{aligned}(d\rho_1 \wedge d\rho_2)_p &= \lambda^2(p) \cdot ((\chi_0(p)\, dx_1 + \chi_1(p)\, dx_m) \wedge (\chi_0(p)\, dy_1 + \chi_1(p)\, dy_m))_p \\ &\quad + \sum_{j=2}^{m-1} \lambda(p)\lambda_j(p)((\chi_0(p)\, dx_1 + \chi_1(p)\, dx_m) \wedge dy_j)_p \\ &\quad + \sum_{j=2}^{m-1} \lambda(p)\lambda_j(p)(dx_j \wedge (\chi_0(p)\, dy_1 + \chi_1(p)\, dy_m))_p \\ &\quad + \sum_{i,j=2}^{m-1} \lambda_i(p)\lambda_j(p)(dx_i \wedge dy_j)_p\end{aligned}$$

(here we have used the fact that for $p = \gamma(t)$, we have $(x_j(p), y_j(p)) = (t, 0)$ and $d(\lambda + \sum_j d\lambda_j)_p = 0$). Therefore, since $(dx_j \wedge dy_j)_p = (dx_k \wedge dy_j)_p$ for all j and k with $p \in U_j \cap U_k$, we have

$$\begin{aligned}(d\rho_1 \wedge d\rho_2)_p &= \lambda^2(p)\chi_0(p)(dx_1 \wedge dy_1)_p + \lambda^2(p)\chi_1(p)(dx_m \wedge dy_m)_p \\ &\quad + \sum_{j=2}^{m-1} \lambda(p)\lambda_j(p)(dx_j \wedge dy_j)_p \\ &\quad + \sum_{j=2}^{m-1} \lambda(p)\lambda_j(p)\chi_0(p)(dx_1 \wedge dy_1)_p \\ &\quad + \sum_{j=2}^{m-1} \lambda(p)\lambda_j(p)\chi_1(p)(dx_m \wedge dy_m)_p \\ &\quad + \sum_{i,j=2}^{m-1} \lambda_i(p)\lambda_j(p)(dx_j \wedge dy_j)_p.\end{aligned}$$

Since the local charts induce the same orientations on the sets $V_i \cap V_j$ for all $i = 1, \dots, m$ and $j = 2, \dots, m-1$, the above 1-form is nonzero. Thus, by continuity, we may choose Ω so that $d\rho_1 \wedge d\rho_2$ is nonvanishing on a neighborhood of $\Omega \setminus W_m$, and therefore, by the $\mathcal{C}^\infty$ inverse function theorem (Theorem 9.9.1 and Theorem 9.9.2), $\Psi\restriction_\Omega$ is a local diffeomorphism of Ω onto a neighborhood of $\Pi([0,1] \times \{1/2\})$ in B with $\Psi(\gamma(t)) = \Pi((t, 1/2))$ for each point $t \in [0,1]$.

As in the proof of Lemma 5.11.1, we may choose Ω so small that $\Psi\restriction_\Omega$ is a diffeomorphism onto $\Psi(\Omega)$. Moreover, we may assume that for some $\epsilon \in (0, 1/2)$, we have $\Psi(\Omega) = \Pi([0,1] \times ((1/2) - \epsilon, (1/2) + \epsilon))$. On the other hand, the mapping $\Xi \colon B \to \Psi(\Omega)$ given by

$$\Pi(s, (1/2) + t) \mapsto \Pi(s, (1/2) + 2\epsilon t) \quad \forall t \in (-1/2, 1/2)$$

is a well-defined diffeomorphism that is equal to the identity on $\Pi([0,1] \times \{1/2\})$. Thus the mapping $\Phi \equiv \Xi^{-1} \circ \Psi \colon \Omega \to B$ is a diffeomorphism with the required properties. $\square$

Lemma 6.10.2 *Let M be a $\mathcal{C}^\infty$ surface. If every smooth Jordan curve in M admits an orientable neighborhood, then M is orientable.*

Proof We first show that if $\alpha \colon [a, b] \to M$ is a path for which the inverse image $\alpha^{-1}(x)$ of each point $x \in M$ contains at most two points, then the image $\alpha([0,1])$ admits an orientable neighborhood in M. For this, we let

$$d \equiv \sup\{t \in (a, b] \mid \alpha([a, t]) \text{ admits an orientable neighborhood}\} \in (a, b],$$

$$c \equiv \inf\{t \in [a, d) \mid \alpha([t, d]) \text{ admits an orientable neighborhood}\} \in [a, d).$$

If $\alpha(c) \neq \alpha(d)$, then we have $\alpha([c, c+\epsilon]) \cap \alpha([d-\epsilon, d]) = \emptyset$ for some (sufficiently small) $\epsilon \in (0, (d-c)/2)$; and we may choose connected open sets Θ_0, Θ_1, and Θ such that $\Theta_0 \cap \Theta_1 = \emptyset$, $\alpha([c, c+\epsilon]) \subset \Theta_0$, $\alpha([d-\epsilon, d]) \subset \Theta_1$, $\alpha([c+\epsilon, d-\epsilon]) \subset \Theta$, and the connected open sets $\Theta_0 \cup \Theta$ and $\Theta \cup \Theta_1$ are orientable. We may choose orientations on each of these sets that agree on their (connected) intersection Θ, and hence $\Theta_0 \cup \Theta \cup \Theta_1$ is an orientable neighborhood of $\alpha([c, d])$, and in particular, $[c, d] = [a, b]$. Thus we may assume without loss of generality that $\alpha(c) = \alpha(d)$, and we will denote this point by p.

Next, we observe that the connected set $\alpha((c, d))$ admits an orientable neighborhood in M. For $\alpha^{-1}(p) = \{c, d\}$, so there exists a sequence of open intervals $\{I_\nu\}$ such that $(c, d) = \bigcup_\nu I_\nu$ and such that for each $\nu = 1, 2, 3, \dots$, we have

$$I_\nu \Subset I_{\nu+1} \Subset (c, d) \quad \text{and} \quad \alpha^{-1}(\alpha(\overline{I}_\nu)) \cap [c, d] \subset I_{\nu+1}.$$

By the choice of c and d, for each ν there exists a nonvanishing $\mathcal{C}^\infty$ real 2-form ω_ν on a relatively compact connected neighborhood U_ν of $\alpha(I_{\nu+1})$ in M. Moreover, by inductively replacing ω_ν with $-\omega_\nu$ whenever necessary, we may assume that

$$(\omega_{\nu+1}/\omega_\nu)_q > 0 \quad \forall q \in \alpha(I_{\nu+1}) \text{ (a connected set)} \quad \forall \nu = 1, 2, 3, \dots.$$

For each $\nu = 1, 2, 3, \ldots$, we may also choose a nonnegative $\mathcal{C}^\infty$ function λ_ν on M such that $\operatorname{supp} \lambda_\nu \subset U_\nu$, $\lambda_\nu \equiv 1$ on $\alpha(I_\nu)$, and $\lambda_\nu \equiv 0$ on $\alpha([c, d] \setminus I_{\nu+1})$. If $\{\epsilon_\nu\}$ is a sequence of positive numbers converging sufficiently fast to 0, then the series

$$\sum_{\nu=1}^{\infty} \epsilon_\nu \lambda_\nu \omega_\nu$$

converges uniformly on compact subsets of M (that is, for every nonvanishing $\mathcal{C}^\infty$ real 2-form ω on an open set U, the series $\sum \epsilon_\nu \lambda_\nu \omega_\nu / \omega$ converges uniformly on compact subsets of U) to a continuous real 2-form ω. Moreover, for each index $\mu = 1, 2, 3, \ldots$ and each point $q \in \alpha(I_\mu)$, we have

$$(\omega/\omega_\mu)_q = \sum_{\nu=1}^{\infty} \epsilon_\nu \lambda_\nu(q)(\omega_\nu/\omega_\mu)_q \geq \epsilon_\mu > 0.$$

Thus ω is nonzero at each point in $\alpha((c, d))$ and therefore at each point in some connected neighborhood Ω of $\alpha((c, d))$ in $M \setminus \{p\}$. Hence this neighborhood is orientable by Proposition 9.7.4.

We may now fix a local $\mathcal{C}^\infty$ chart $(U, \Lambda, \Delta(0; 1))$ with $p = \Lambda^{-1}(0)$ and $\alpha([c, d]) \not\subset U$, a constant $\delta \in (0, (d-c)/2)$ with $\alpha([c, c+\delta] \cup [d-\delta, d]) \subset U$, and a constant $r \in (0, 1)$ so small that for $D \equiv \Lambda^{-1}(\Delta(0; r))$, $\overline{D} \cap \alpha([c+\delta, d-\delta]) = \emptyset$. Let Ω_0 and Ω_1 be the connected components of $\Omega \cap U$ containing $\alpha((c, c+\delta])$ and $\alpha([d-\delta, d))$, respectively. By replacing the neighborhood Ω with the connected component of the neighborhood $(\Omega \setminus \overline{D}) \cup \Omega_0 \cup \Omega_1$ containing $\alpha((c, d))$, we may assume that $\Omega \cap D = (\Omega_0 \cup \Omega_1) \cap D$ and $\Omega \cap \overline{D} = (\Omega_0 \cup \Omega_1) \cap \overline{D}$. We may also choose the orientation in Ω so that it agrees with the orientation induced by Λ in the connected open set $\Omega_0 \subset U \cap \Omega$. For the proof of the existence of an orientable neighborhood of $\alpha([a, b])$, it suffices to show that the two orientations also agree in Ω_1. For if they do agree, then $\Omega \cup D$ is an orientable neighborhood of $\alpha([c, d])$, and the claim follows. In particular, we may assume without loss of generality that $\Omega_0 \neq \Omega_1$ (i.e., that $\Omega_0 \cap \Omega_1 = \emptyset$).

Setting $P \equiv \Lambda^{-1}(\Delta(0; r/2))$ and applying Lemma 5.10.6, we get an injective smooth path $\beta\colon [0, 1] \to \Omega$ with $\beta(0) \in \alpha((c, d)) \cap \Omega_0 \cap P$ and $\beta(1) \in \alpha((c, d)) \cap \Omega_1 \cap P$. Setting

$$s \equiv \max\{t \in [0, 1] \mid \beta(t) \in \Omega_0 \cap \overline{P}\} \in (0, 1),$$

$$u \equiv \min\{t \in [s, 1] \mid \beta(t) \in \Omega_1 \cap \overline{P}\} \in (s, 1),$$

we get the piecewise smooth Jordan curve γ obtained by joining the path $\beta\restriction_{[s,u]}$ (with $\beta((s, u)) \subset \Omega \setminus \overline{P}$) and the path in $\overline{P} \subset D$ with image under Λ equal to the line segment from $\Lambda(\beta(u))$ to $\Lambda(\beta(s))$. Applying Lemma 5.10.5, we get a smooth Jordan curve C in $\Omega \cup D$ that is the union of two arcs A and B (i.e., A and B are each the homeomorphic image of a compact interval) such that $A \subset \Omega$, $B \subset D$, and $A \cap B = \{p_0, p_1\}$ with $p_i \in \Omega_i \cap D$ for $i = 0, 1$ (see Fig. 6.6). In particular, since

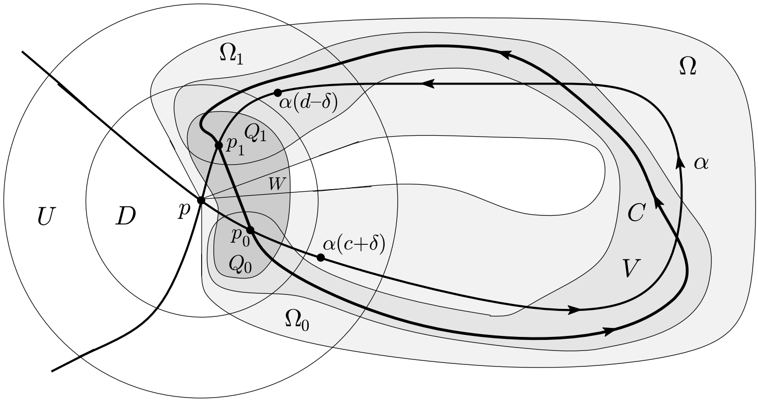

Fig. 6.6 Construction of an orientable neighborhood of the curve

by hypothesis, every smooth Jordan curve admits an orientable neighborhood, there exist connected open sets V and W such that $A \subset V \subset \Omega$, $B \subset W \subset D$, $V \cap W$ is equal to the union of two disjoint connected open sets Q_0 and Q_1 with $p_i \in Q_i \subset \Omega_i \cap D$ for $i = 0, 1$, and $V \cup W$ is orientable. Fixing the orientation in $V \cup W$ so that the orientation induced in V agrees with that induced by Ω, we see that it induces the same orientation in $Q_0 \subset V \cap \Omega_0 \cap D$ as Λ. But then $V \cup W$ and Λ must induce the same orientation in the connected open set $W \subset D$ and therefore in $Q_1 \subset V \cap \Omega_1 \cap D$. Similarly, since $V \cup W$ and Ω induce the same orientation in $V \supset Q_1$, Λ and Ω must induce the same orientation in $Q_1 \subset \Omega_1 \cap D$ and therefore in Ω_1. Therefore, since $\Omega \cap D = (\Omega_0 \cup \Omega_1) \cap D$, the neighborhood $\Omega \cup D$ of $\alpha([c, d])$ is orientable, and the claim concerning α follows.

We now get an orientation in M as follows. Let us fix a point $p_0 \in M$ and a local $\mathcal{C}^\infty$ chart $(U_0, \Phi_0 = (x_0, y_0), \Delta(0; 1))$ with $p_0 = \Phi_0^{-1}(0)$. Given a point $p_1 \in M \setminus \{p_0\}$, we may choose an injective smooth path α_1 in M from p_0 to p_1 and, by the above, a nonvanishing $\mathcal{C}^\infty$ real 2-form ω_1 on a neighborhood of $\alpha_1([0, 1])$ such that $(\omega_1)_{p_0}/(dx_0 \wedge dy_0)_{p_0} > 0$. By Theorem 9.9.2, we may now fix a local $\mathcal{C}^\infty$ chart $(U_1, \Phi_1 = (x_1, y_1), \Delta(0; 1))$ such that $p_1 = \Phi_1^{-1}(0)$, $(dx_1 \wedge dy_1)_{p_1}/(\omega_1)_{p_1} > 0$, and $\alpha_1([0, 1]) \cap U_1$ is connected. Consequently, $(dx_1 \wedge dy_1)/\omega_1 > 0$ at each point in $\alpha_1([0, 1]) \cap U_1$. We may cover M by such local $\mathcal{C}^\infty$ charts (note that even $p_0 \in U_1$ for a suitable choice of $p_1 \in M \setminus \{p_0\}$).

In order to check compatibility, let us suppose that $p_2 \in M \setminus \{p_0\}$ with $p_2 \neq p_1$ (the proof for $p_2 = p_1$ is similar) and that we have chosen an associated path α_2, form ω_2, and local chart $(U_2, \Phi_2 = (x_2, y_2), \Delta(0; 1))$. If $p_2 \in U_1 \setminus \{p_0\}$, then we may form an injective smooth path $\beta\colon [0, 1] \to U_1 \setminus \{p_0\}$ from p_1 to p_2, and we may set

$$r \equiv \max \beta^{-1}(\alpha_1([0, 1])) \in [0, 1] \quad \text{and} \quad s \equiv \alpha_1^{-1}(\beta(r)).$$

The product of suitable reparametrizations of $\alpha_1\restriction_{[0,s]}$, $\beta\restriction_{[r,1]}$, and α_2^- (omitting β if $r = 1$) is then a piecewise smooth loop γ. Moreover, γ takes any given value at most twice, so by the above, there is a nonvanishing real $\mathcal{C}^\infty$ 2-form ω on a neighborhood of $\gamma([0,1])$ with $\omega_{p_0}/(dx_0 \wedge dy_0)_{p_0} > 0$. Consequently, for $q = \alpha_1(s)$, we have

$$\frac{(dx_1 \wedge dy_1)_q}{\omega_q} = \frac{(dx_1 \wedge dy_1)_q}{(\omega_1)_q} \cdot \frac{(\omega_1)_q}{\omega_q} > 0.$$

Similarly, we have $(dx_2 \wedge dy_2)_{p_2}/\omega_{p_2} > 0$, and hence

$$\frac{(dx_1 \wedge dy_1)_{p_2}}{(dx_2 \wedge dy_2)_{p_2}} = \frac{(dx_1 \wedge dy_1)_{p_2}}{\omega_{p_2}} \cdot \frac{\omega_{p_2}}{(dx_2 \wedge dy_2)_{p_2}} > 0.$$

So the orientations agree in a neighborhood of $p_2 \in U_1 \cap U_2$. If $p_2 \in M \setminus \{p_0\}$ is arbitrary and $p \in U_1 \cap U_2$, then we may form a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, \Phi = (x, y), \Delta(0; 1))$ of p as above but with $U \subset U_1 \cap U_2$. By the above, each of the local charts Φ_1 and Φ_2 induces the same orientation in U as Φ and therefore as the other. Thus the orientations are compatible on the overlap, and hence M is orientable. □

Proposition 6.10.3 *Let M be a $\mathcal{C}^\infty$ surface. Then M is nonorientable if and only if some open subset of M is homeomorphic to the Möbius band.*

Proof If M is nonorientable, then by Lemma 6.10.2, there exists a smooth Jordan curve C in M that does *not* admit an orientable neighborhood. Lemma 6.10.1 then provides a homeomorphism (in fact, a diffeomorphism) of some neighborhood onto the Möbius band.

Conversely, suppose M is orientable and $\Psi\colon \Omega \to B$ is a homeomorphism of some open subset Ω of M onto the Möbius band

$$B \equiv [0,1] \times (0,1)/(0,t) \sim (1, 1-t) \quad \forall t \in (0,1)$$

with quotient map $\Pi\colon [0,1] \times (0,1) \to B$ (in particular, Ω is second countable). Let $\gamma\colon [0,1] \to B$ be the Jordan curve given by $s \mapsto \Pi(s, 1/2)$, and let $C \equiv \gamma([0,1])$. For the continuous mapping $R\colon B \times [0,1] \to B$ given by

$$R(\Pi(s,t), u) = \Pi(s, (1/2) + (1-u)(t - 1/2)) \quad \forall (s,t,u) \in [0,1] \times (0,1) \times [0,1],$$

we have $R(p,u) = p$ for each point $(p,u) \in (C \times [0,1]) \cup (B \times \{0\})$, and $R(p,1) \in C$ for each point $p \in B$; that is, R is a *strong deformation retraction of B onto C*. Hence, since $[\gamma]$ generates $\pi_1(C)$, $[\gamma]$ must also generate $\pi_1(B)$ (given a loop τ in B with base point $\tau(0) = \gamma(0)$, the mapping $(t,u) \mapsto R(\tau(t), u)$ is a path homotopy from τ to a loop in C). Moreover, the differential of the $\mathcal{C}^\infty$ function given by $\Pi(s,t) \mapsto s$ for all $(s,t) \in (0,1) \times (0,1)$ extends to a unique closed $\mathcal{C}^\infty$ 1-form θ on B with $\int_\gamma \theta = 1$. It follows that $H_1(B, \mathbb{R}) = \mathbb{R} \cdot [\gamma]_{H_1(B,\mathbb{R})} \cong \mathbb{R}$, and hence the Jordan curve $\beta \equiv \Psi^{-1}(\gamma)$ satisfies $H_1(\Omega, \mathbb{R}) = \mathbb{R} \cdot [\beta]_{H_1(\Omega,\mathbb{R})} \cong \mathbb{R}$ (see Sect. 10.7).

Now β is homologous in Ω to a sum of smooth Jordan curves (by Proposition 5.10.7), so there exists a smooth Jordan curve $\alpha\colon [0,1] \to \Omega$ with nonzero homology class $[\alpha]_{H_1(\Omega,\mathbb{R})}$. On the other hand, the image $A \equiv \alpha([0,1])$ is nonseparating in Ω. For Stokes' theorem implies that any separating smooth Jordan curve for which the complement has a relatively compact connected component in Ω must be homologous to 0, while the complement of any compact subset of B has exactly one connected component that is not relatively compact in B (any connected component that is not relatively compact must contain the connected open set $\Pi([0,1]\times((0,\epsilon)\cup(1-\epsilon,1)))$ for some sufficiently small $\epsilon \in (0,1/2)$). Therefore, by Lemma 5.15.1, there exists a second smooth Jordan curve σ in Ω for which the homology classes $[\alpha]_{H_1(\Omega,\mathbb{R})}$ and $[\sigma]_{H_1(\Omega,\mathbb{R})}$ are linearly independent, which is impossible because $[\alpha]_{H_1}$ spans $H_1(\Omega,\mathbb{R})$. Thus we have arrived at a contradiction, and the proposition follows. □

The above proposition implies that the following definition is not in conflict with the definitions of orientability and nonorientability for a smooth surface given in Sect. 9.7:

Definition 6.10.4 A second countable topological surface M is called *nonorientable* if some open subset of M is homeomorphic to the Möbius band. The surface M is called *orientable* if no open subset of M is homeomorphic to the Möbius band.

Exercises for Sect. 6.10

6.10.1 Let $\gamma\colon [0,1]\to M$ be a (continuous) Jordan curve in a $\mathcal{C}^\infty$ surface M, and let $C \equiv \gamma([0,1])$. Assume that C does not admit an orientable neighborhood in M. Prove that there exists a homeomorphism $\Phi\colon \Omega \to B$ of some neighborhood Ω of C in M onto the Möbius band

$$B \equiv [0,1]\times(0,1)/(0,t)\sim(1,1-t) \quad \forall t \in (0,1)$$

with quotient map $\Pi\colon [0,1]\times(0,1)\to B$ such that $\Phi(\gamma(t)) = \Pi(t,1/2)$ for each point $t \in [0,1]$ (cf. Exercise 6.7.7).

6.10.2 Let B be the Möbius band with associated $\mathcal{C}^\infty$ quotient map $\Pi\colon [0,1]\times(0,1)\to B$ (see Exercise 6.10.1), let $\gamma(t)\equiv\Pi(t,1/2)$ for each point $t\in[0,1]$, and let $C\equiv\gamma([0,1])$. Prove that C is nonseparating in B, but $\int_\gamma \theta = 0$ for every closed $\mathcal{C}^\infty$ 1-form θ with compact support in B (cf. Proposition 5.15.2).

6.11 Smooth Structures on Second Countable Topological Surfaces

The main goal of this section is the following:

Theorem 6.11.1 *Every second countable topological surface admits a smooth* (*surface*) *structure.*

Note that it is redundant to refer to a smooth *surface* structure, since no topological surface is homeomorphic to a manifold of dimension $n \neq 2$ (see Exercise 6.11.1). It follows from the above theorem and the results of Sect. 6.10 that the definition of the homology group $H_1(M, \mathbb{A})$ provided in Sect. 10.7 (Definition 10.7.9) applies to any orientable second countable topological surface M and any subring $\mathbb{A}$ of $\mathbb{C}$ containing $\mathbb{Z}$. That this group is isomorphic to the singular homology group is considered in Exercise 6.11.9.

Theorem 6.11.1, Proposition 6.10.3, and Corollary 6.1.6 together give the following:

Corollary 6.11.2 *Every second countable orientable topological surface admits a* (1*-dimensional*) *holomorphic structure.*

We now address the proof of the theorem. The idea of the proof is to inductively form smooth structures on successively larger open sets. In the inductive step, one attaches a topological disk $D \approx \Delta(0; r)$ to an open set Ω with a smooth structure $\mathcal{S}$, after first using Schönflies' theorem to modify the homeomorphism on the overlap so that it becomes smooth with respect to $\mathcal{S}$. We first consider the following case, in which one attaches an inner disk to a topological annulus Ω.

Lemma 6.11.3 *Let K be a compact subset of $\Delta \equiv \Delta(0; 1)$, and let $\mathcal{S}_0$ be a* (2*-dimensional*) *smooth structure on $\Delta \setminus K$. Then there exists a smooth structure $\mathcal{S}$ on Δ such that $\mathcal{S} = \mathcal{S}_0$ near $\partial\Delta$; that is, $\mathcal{S}$ and $\mathcal{S}_0$ induce the same restricted smooth structure on the complement $\Delta \setminus K'$ of some compact subset $K' \supset K$ of Δ.*

Proof We may assume without loss of generality that $K = \overline{\Delta(0; r)}$ for some $r \in (0, 1)$. A generator for the fundamental group of the annulus $\Delta \setminus K = \Delta(0; r, 1)$ is homologous to a sum of Jordan loops that are smooth with respect to the smooth structure $\mathcal{S}_0$ (by Proposition 5.10.7). Hence there is a Jordan curve $\gamma : [0, 1] \to \Delta \setminus K$ that is *not* homologous to 0 and that is smooth with respect to $\mathcal{S}_0$. The image $C \equiv \gamma([0, 1])$ is separating in $\mathbb{P}^1$ (by the Jordan curve theorem) and therefore in $\Delta \setminus K$. Moreover, $\Delta \setminus K$ is orientable (as shown in Sect. 6.10, orientability of surfaces is a topological property), so Stokes' theorem implies that no connected component of $(\Delta \setminus K) \setminus C$ is relatively compact in $\Delta \setminus K$ (one may also see this directly). Therefore, choosing $\epsilon \in (0, (1-r)/2)$ so small that $C \subset \Delta(0; r+\epsilon, 1-\epsilon)$, we see that the set $(\Delta \setminus K) \setminus C$ has exactly two connected components U_0 and U_1, where $U_0 \supset \Delta(0; r, r+\epsilon)$ and $U_1 \supset \Delta(0; 1-\epsilon, 1)$. Hence the domain $\Omega_0 \equiv \Delta(0; r+\epsilon) \cup U_0 \Subset \Delta$ satisfies $\partial\Omega_0 = C$.

By Lemma 5.11.1, for some $R > 1$, there are a relatively compact connected neighborhood A of C in $\Delta \setminus K$ and a homeomorphism $\Psi_0 : A \to \Delta(0; 1/R, R)$ such that $\Psi_0(C) = \mathbb{S}^1$, Ψ_0 is a diffeomorphism with respect to the smooth structure on the domain A induced by $\mathcal{S}_0$ and the standard smooth structure on the target $\Delta(0; 1/R, R)$, and $A \cap \Omega_0 = A \cap U_0 = \Psi_0^{-1}(\Delta(0; 1/R, 1))$. By Schönflies' theorem, there exists a homeomorphism $\Psi_1 : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\Psi_1 \restriction_C = \Psi_0 \restriction_C : C \to \mathbb{S}^1$ and $\Psi_1 \restriction_{\Omega_0}$ maps Ω_0 diffeomorphically (with respect to the standard smooth structure

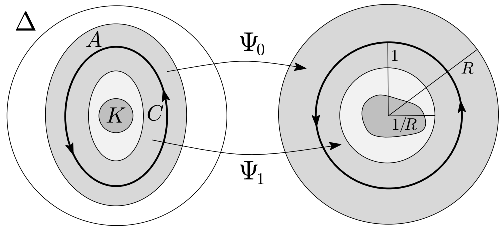

Fig. 6.7 Construction of the local chart $(\Omega, \Phi, \Delta(0; R))$

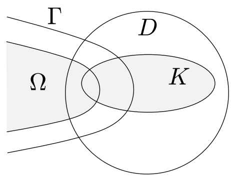

Fig. 6.8 Extension of (a modification of) the smooth structure $\mathcal{S}_0$ on Γ to a smooth structure $\mathcal{S}$ on a neighborhood of $\overline{\Omega} \cup K$

on both the domain and the target) onto Δ (and $\Psi_1{\restriction}_{\mathbb{P}^1 \setminus \overline{\Omega}_0}$ maps $\mathbb{P}^1 \setminus \overline{\Omega}_0$ diffeomorphically onto $\mathbb{P}^1 \setminus \overline{\Delta}$). Setting $\Omega \equiv \Omega_0 \cup A \Supset \Omega_0$ and

$$\Phi(z) \equiv \begin{cases} \Psi_1(z) & \text{for } z \in \Omega_0, \\ \Psi_0(z) & \text{for } z \in A \setminus \Omega_0 = \Omega \setminus \Omega_0 = \Psi_0^{-1}(\{\zeta \in \mathbb{C} \mid 1 \leq |\zeta| < R\}), \end{cases}$$

we get a homeomorphism $\Phi\colon \Omega \to \Delta(0; R)$ such that

$$\Phi{\restriction}_{\Omega \setminus \overline{\Omega}_0} = \Psi_0{\restriction}_{\Omega \setminus \overline{\Omega}_0} : \Omega \setminus \overline{\Omega}_0 \to \Delta(0; 1, R)$$

is a diffeomorphism with respect to the smooth structure induced by $\mathcal{S}_0$ on the domain and the standard smooth structure on the target (see Fig. 6.7).

Now let $\mathcal{A}$ be the collection of local charts that are equal either to $(\Omega, \Phi, \Delta(0; R))$ or to $(U \setminus \overline{\Omega}_0, \Psi{\restriction}_{U \setminus \overline{\Omega}_0}, \Psi(U \setminus \overline{\Omega}_0))$ for some local $\mathcal{C}^\infty$ chart $(U, \Psi, \Psi(U))$ in the smooth surface $(\Delta \setminus K, \mathcal{S}_0)$. Then $\mathcal{A}$ is a $\mathcal{C}^\infty$ atlas on Δ, and the corresponding $\mathcal{C}^\infty$ structure $\mathcal{S}$ induces the same $\mathcal{C}^\infty$ structure as $\mathcal{S}_0$ on the open subset $\Delta \setminus \overline{\Omega}_0 \subset \Delta \setminus K$. □

The following lemma is the main step in the proof of Theorem 6.11.1:

Lemma 6.11.4 *Let M be a second countable topological surface, let Ω and Γ be open subsets with $\overline{\Omega} \subset \Gamma$, let $\mathcal{S}_0$ be a (2-dimensional) smooth structure on Γ, let $(D, \Phi, \Delta \equiv \Delta(0; 1))$ be a local $\mathcal{C}^0$ chart in M, and let K be a compact subset of D. Then there exists a smooth structure $\mathcal{S}$ on a neighborhood of $\overline{\Omega} \cup K$ such that $\mathcal{S}$ and $\mathcal{S}_0$ induce the same smooth structure on $\Omega \setminus \overline{D}$ (see Fig. 6.8).*

Proof We may fix an open set Ω_0 with $\overline{\Omega} \subset \Omega_0 \subset \overline{\Omega}_0 \subset \Gamma$ (by Lemma 9.3.6) and a constant $r \in (0,1)$ with $K \subset D_0 \equiv \Phi^{-1}(\Delta(0;r)) \Subset D$. Given distinct points $\zeta_1, \dots, \zeta_m \in \partial\Delta(0;r)$ arranged in counterclockwise order, we let $\zeta_0 \equiv \zeta_m$, and for each $j = 1, \dots, m$, we let $\alpha_j : [0,1] \to \overline{\Delta(0;r)}$ be an injective parametrization of the line segment from ζ_{j-1} to ζ_j, we let $\beta_j : [0,1] \to \partial\Delta(0;r)$ be an injective parametrization of the counterclockwise arc of the circle $\partial\Delta(0;r)$ from ζ_{j-1} to ζ_j, we let $\gamma_j \equiv \alpha_j^- * \beta_j$, we let $C_j \equiv \gamma_j([0,1])$, and we let $\eta_j : \mathbb{S}^1 \to C_j$ be the associated homeomorphism given by $e^{2\pi i t} \mapsto \gamma_j(t)$. Exactly one of the two connected components of $\mathbb{P}^1 \setminus C_j$, which we will denote by V_j, is contained in $\Delta(0;r)$ and satisfies $\partial V_j = C_j$ (this follows from Schönflies' theorem, but it is also easy to verify directly in this case). We also set $U_j \equiv \Phi^{-1}(V_j) \subset D_0$ for $j = 1, \dots, m$ and $p_j \equiv \Phi^{-1}(\zeta_j)$ for $j = 0, \dots, m$.

We may choose such boundary points $\zeta_0, \zeta_1, \dots, \zeta_m = \zeta_0$ so close together that if $j \in \{1, \dots, m\}$ with $\overline{U}_j \cap \overline{\Omega}_0 \neq \emptyset$, then $\overline{U}_j$ is contained in some local $\mathcal{C}^\infty$ chart $(W_j, \Lambda_j, \Delta(0;1))$ in the $\mathcal{C}^\infty$ surface $(\Gamma, \mathcal{S}_0)$. Schönflies' theorem (Theorem 6.7.1) then allows us to modify Φ into a diffeomorphism with respect to $\mathcal{S}_0$ on each such set U_j, while leaving the values elsewhere unchanged. More precisely, if $j \in \{1, \dots, m\}$ with $\overline{U}_j \cap \overline{\Omega}_0 \neq \emptyset$, then Schönflies' theorem provides a homeomorphism $\overline{\Delta} \to \overline{V}_j$ that maps Δ diffeomorphically onto V_j and that is equal to the homeomorphism $\eta_j : \mathbb{S}^1 \to C_j$ on $\mathbb{S}^1 = \partial\Delta$, as well as a homeomorphism $\Lambda_j(\overline{U}_j) = \overline{\Lambda_j(U_j)} \to \overline{\Delta}$ that maps $\Lambda_j(U_j)$ diffeomorphically onto Δ and that is equal to the homeomorphism $\eta_j^{-1} \circ \Phi \circ \Lambda_j^{-1} : \Lambda_j(\partial U_j) \to \mathbb{S}^1$ on $\Lambda_j(\partial U_j) = \partial\Lambda_j(U_j)$. The associated composition $\overline{U}_j \to \Lambda_j(\overline{U}_j) \to \overline{\Delta} \to \overline{V}_j$ then yields a homeomorphism $\Psi_j : \overline{U}_j \to \overline{V}_j$ such that $\Psi_j \restriction_{\partial U_j} = \Phi \restriction_{\partial U_j} : \partial U_j \to \partial V_j$ and $\Psi_j \restriction_{U_j} : U_j \to V_j$ is a diffeomorphism with respect to the smooth structure on U_j induced by $\mathcal{S}_0$ and the standard smooth structure on $V_j \subset \mathbb{C}$ (see Fig. 6.9). Thus the mapping $\Psi : \overline{D}_0 \to \overline{\Delta(0;r)}$ given by

$$\Psi(x) \equiv \begin{cases} \Phi(x) & \text{if } x \in \overline{D}_0 \setminus (U_1 \cup \cdots \cup U_m), \\ \Phi(x) & \text{if } x \in \overline{U}_j \text{ with } j \in \{1, \dots, m\} \text{ and } \overline{U}_j \cap \overline{\Omega}_0 = \emptyset, \\ \Psi_j(x) & \text{if } x \in \overline{U}_j \text{ with } j \in \{1, \dots, m\} \text{ and } \overline{U}_j \cap \overline{\Omega}_0 \neq \emptyset, \end{cases}$$

is a well-defined homeomorphism that maps each of the sets $U_j \subset \Omega_0$ with closure meeting $\overline{\Omega}_0$ diffeomorphically, with respect to $\mathcal{S}_0$, onto its image $V_j \subset \mathbb{C}$ (which has the standard $\mathcal{C}^\infty$ structure).

Setting

$$Q \equiv \Phi^{-1}\left(\Delta(0;r) \Big\backslash \bigcup_{j=1}^m \overline{V}_j\right) = \Psi^{-1}\left(\Delta(0;r) \Big\backslash \bigcup_{j=1}^m \overline{V}_j\right) = D_0 \Big\backslash \bigcup_{j=1}^m \overline{U}_j,$$

we may now let $\mathcal{A}_1$ be the collection of local $\mathcal{C}^0$ charts in the open set

$$\Omega_1 \equiv (\Omega_0 \setminus \{p_1, \dots, p_m\}) \cup D_0 = (\Omega_0 \setminus \overline{Q}) \cup D_0,$$

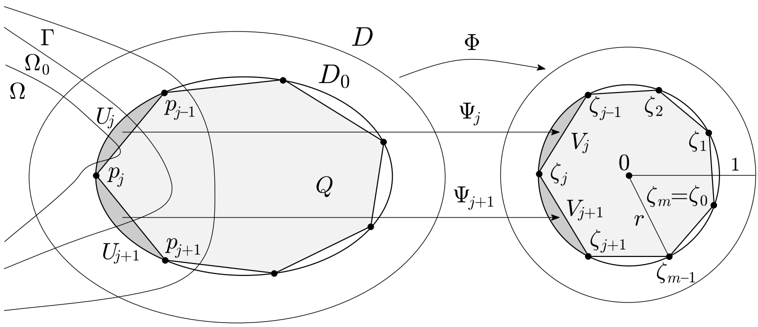

Fig. 6.9 Construction of a smooth structure outside the vertices of the polygonal region Q

which are equal either to $(D_0, \Psi\restriction_{D_0}, \Delta(0; r))$ or to

$$\left(W \cap \Omega_0 \setminus \overline{Q}, \Lambda\restriction_{(W\cap\Omega_0\setminus\overline{Q})}, \Lambda(W \cap \Omega_0 \setminus \overline{Q})\right)$$

for some local $\mathcal{C}^\infty$ chart $(W, \Lambda, \Lambda(W))$ in $(\Gamma, \mathcal{S}_0)$ with $W \cap \Omega_0 \neq \emptyset$. This collection $\mathcal{A}_1$ is then a $\mathcal{C}^\infty$ atlas, because for any local $\mathcal{C}^\infty$ chart $(W, \Lambda, \Lambda(W))$ in $(\Gamma, \mathcal{S}_0)$, we have

$$\begin{aligned}(W \cap \Omega_0 \setminus \overline{Q}) \cap D_0 &= W \cap \Omega_0 \cap \Phi^{-1}\left(\bigcup_{j=1}^{m} V_j\right) \\ &= W \cap \Omega_0 \cap \left(\bigcup_{1\le j\le m, \overline{U}_j\cap\overline{\Omega}_0\neq\emptyset} U_j\right),\end{aligned}$$

and with respect to the $\mathcal{C}^\infty$ structure $\mathcal{S}_0$, Ψ maps this set diffeomorphically onto its image. Clearly, the $\mathcal{C}^\infty$ structure $\mathcal{S}_1$ in Ω_1 determined by $\mathcal{A}_1$ agrees with $\mathcal{S}_0$ in the open subset $\Omega_0 \setminus \overline{Q} \supset \Omega_1 \setminus \overline{D} \supset \Omega \setminus \overline{D}$.

So far, we have obtained a smooth structure on the set $\Omega_1 \supset (\overline{\Omega} \setminus \{p_1, \dots, p_m\}) \cup K$. Letting $J \subset \{1, \dots, m\}$ be the set of indices j for which $p_j \in \Omega_0$, we will expand the set of definition for our smooth structure so as to include these points. We may choose a collection of disjoint local $\mathcal{C}^0$ charts $\{(E_j, \Xi_j, \Delta)\}_{j\in J}$ with $p_j = \Xi_j^{-1}(0)$ and $E_j \subset \Omega_0 \cap D$ for each index $j \in J$. In particular, for each $j \in J$, we have $E_j \setminus \{p_j\} \subset \Omega_1$. Hence the local charts in $\Delta^* = \Delta \setminus \{0\}$ given by the compositions of local $\mathcal{C}^\infty$ charts in $(\Omega_1, \mathcal{S}_1)$ meeting E_j with $\Xi_j^{-1}\restriction_{\Delta^*}$ determine a smooth structure $\mathcal{R}_j$ on Δ^* for which the mapping $\Xi_j^{-1}\restriction_{\Delta^*} : (\Delta^*, \mathcal{R}_j) \to (E_j \setminus \{p_j\}, \mathcal{S}_1)$ is a diffeomorphism. Therefore, by Lemma 6.11.3, there is a smooth structure $\mathcal{Q}_j$ on Δ that induces the same smooth structure as $\mathcal{R}_j$ on the annulus $\Delta(0; u_j, 1)$ for some $u_j \in (0, 1)$.

Setting $N \equiv \bigcup_{j\in J} \Xi_j^{-1}(\Delta(0; u_j))$, we may now let $\mathcal{A}_2$ be the collection of local $\mathcal{C}^0$ charts in the open set

$$\Omega_2 \equiv \Omega_0 \cup D_0 = \Omega_1 \cup \{p_j \mid j \in J\} \supset \overline{\Omega} \cup K$$

that are equal either to $(\Xi_j^{-1}(W), \Lambda \circ \Xi_j, P)$ for some local $\mathcal{C}^\infty$ chart (W, Λ, P) in $(\Delta, \mathcal{Q}_j)$, or to $(W \setminus \overline{N}, \Lambda\restriction_{W\setminus\overline{N}}, \Lambda(W \setminus \overline{N}))$ for some local $\mathcal{C}^\infty$ chart $(W, \Lambda, \Lambda(W))$ in $(\Omega_1, \mathcal{S}_1)$ with $W \not\subset N$. The collection $\mathcal{A}_2$ is then a $\mathcal{C}^\infty$ atlas on Ω_2 that induces the same $\mathcal{C}^\infty$ structure as $\mathcal{S}_1$ on the open set

$$\Omega_2 \setminus \overline{N} = \Omega_1 \setminus \overline{N} \supset \Omega_1 \setminus \overline{D} \supset \Omega \setminus \overline{D}.$$

Since $\mathcal{S}_1$ induces the same $\mathcal{C}^\infty$ structure as $\mathcal{S}_0$ on $\Omega \setminus \overline{D}$, the claim follows. □

We are now ready for the proof of Theorem 6.11.1. In fact, we will prove the following slightly stronger version:

Theorem 6.11.5 *Let M be a second countable topological surface, let $\mathcal{S}_0$ be a* (2*-dimensional*) *smooth structure on some* (*nonempty*) *open subset Θ of M, and let K be a closed subset of M with $K \subset \Theta$. Then there exists a smooth structure $\mathcal{S}$ on M that induces the same smooth structure as $\mathcal{S}_0$ on some neighborhood of K.*

Proof We will assume that $M \setminus \Theta$ is noncompact, the proof for M compact being similar. We may fix an open set Θ_0 with $K \subset \Theta_0 \subset \overline{\Theta_0} \subset \Theta$, a locally finite collection of local $\mathcal{C}^0$ charts $\{(U_\nu, \Phi_\nu, \Delta \equiv \Delta(0; 1))\}_{\nu=1}^\infty$, and a sequence $\{r_\nu\}_{\nu=1}^\infty$ in $(0, 1)$ such that $D_\nu \equiv \Phi_\nu^{-1}(\Delta(0; r_\nu)) \Subset U_\nu \Subset M \setminus K$ for each ν and

$$M \setminus \Theta_0 \subset \bigcup_{\nu=1}^\infty D_\nu.$$

We now set $\Gamma_0 \equiv \Theta$, $\Omega_0 \equiv \Theta_0$, and $\Omega_\nu \equiv \Theta_0 \cup D_1 \cup \cdots \cup D_\nu$ for each $\nu = 1, 2, 3, \dots$, and we proceed inductively. Suppose that for each $\mu = 0, \dots, \nu - 1$, we have constructed a (2-dimensional) $\mathcal{C}^\infty$ structure $\mathcal{S}_\mu$ on an open set $\Gamma_\mu \supset \overline{\Omega}_\mu$. Assume also that for each $\mu = 1, \dots, \nu - 1$, $\mathcal{S}_\mu$ and $\mathcal{S}_{\mu-1}$ induce the same $\mathcal{C}^\infty$ structure on the set $\Omega_{\mu-1} \setminus \overline{U}_\mu$. Then, by Lemma 6.11.4, there exists a smooth structure $\mathcal{S}_\nu$ on an open set $\Gamma_\nu \supset \overline{\Omega}_{\nu-1} \cup \overline{D}_\nu = \overline{\Omega}_\nu$ that induces the same smooth structure as $\mathcal{S}_{\nu-1}$ on the set $\Omega_{\nu-1} \setminus \overline{U}_\nu$. Thus we get a sequence of sets with smooth structures $\{(\Gamma_\nu, \mathcal{S}_\nu)\}$. These smooth structures eventually stabilize on any given compact set to give a well-defined global smooth structure. More precisely, by local finiteness, given a point $p \in M$, there are an index ν_0 and a neighborhood W of p such that $W \cap U_\nu = \emptyset$ for all $\nu > \nu_0$. In particular, we have $W \subset \Omega_{\nu_0}$, and we may choose W so that there is a local $\mathcal{C}^\infty$ chart (W, Φ, V) for $(\Omega_{\nu_0}, \mathcal{S}_{\nu_0})$. By construction, this local chart is a local $\mathcal{C}^\infty$ chart with respect to $\mathcal{S}_\nu$ for every $\nu \geq \nu_0$. It is now easy to see that the collection $\mathcal{A}$ of all such local charts is a smooth surface atlas on M, and that the associated smooth structure $\mathcal{S}$ induces the same smooth structure on the neighborhood $M \setminus \bigcup_\nu \overline{U}_\nu$ of K as $\mathcal{S}_0$. □

Exercises for Sect. 6.11

6.11.1 Let Ω be a nonempty open subset of $\mathbb{R}^2$. Prove that Ω is *not* homeomorphic to an open subset of $\mathbb{R}^n$ for $n \neq 2$.

Hint. The complement of a point in any connected neighborhood in $\mathbb{R}^2$ is *not* simply connected.

6.11.2 Prove that if M is a compact orientable topological surface with $H_1(M, \mathbb{R}) = 0$, then M is homeomorphic to a sphere.

6.11.3 Prove that if M is a second countable noncompact orientable topological surface with $H_1(M, \mathbb{R}) = 0$, then M is homeomorphic to $\mathbb{R}^2$.

6.11.4 Let M be a second countable topological surface. The goal of this exercise is a proof of a theorem of Radó, according to which M admits a triangulation. A *triangle* in M is a homeomorphism $\Phi\colon T' \to T$ of a geometric closed triangle $T' \subset \mathbb{R}^2$ onto a compact set $T \subset M$ (in a slight abuse of notation, we also denote Φ by T). The image of each vertex of T' is called a *vertex of* Φ (or *a vertex of* T), and the image of each edge of T' is called an *edge of* Φ (or *an edge of* T). A *triangulation of* M is a locally finite covering $\{T_i\}_{i\in I}$ of M by triangles such that for each pair of distinct indices $i, j \in I$, $T_i \cap T_j$ is either empty, a common vertex of T_i and T_j, or a common edge of T_i and T_j.

(a) Prove that if $\Phi\colon T' \to T$ is a triangle in M, then $\Phi(\mathring{T}') = \mathring{T}$, $\Phi(\partial T') = \partial T$, ∂T is a separating Jordan curve in M, and $\mathring{T}$ is a connected component of $M \setminus \partial T$.

(b) Suppose $\{T_i\}_{i\in I}$ is a triangulation of M, $i \in I$, and E is an edge of T_i. Prove that there is a unique index $j \in I \setminus \{i\}$ such that E is an edge of T_j. Prove also that T_i and T_j have exactly two vertices in common, namely, the endpoints of E.

(c) Prove, in the following way, that if M is orientable, then M admits a triangulation (this is a special case of the above theorem of Radó). By the results of this chapter, M admits the structure of a Riemann surface. The proof of Proposition 5.18.4 gives a locally finite covering of M by triangles with disjoint interiors. Add more edges so that no vertex will lie in the interior of an edge, and no two distinct edges will meet in more than one point.

(d) Prove, in the following way, that M admits a triangulation even if M is nonorientable (that is, prove the above theorem of Radó). By Theorem 6.11.1, M admits the structure of a $\mathcal{C}^\infty$ surface. Apply the 1-dimensional $\mathcal{C}^\infty$ version of Sard's theorem (see Exercise 9.6.5) in order to construct a suitable graph in M as in the proof of Proposition 5.18.4, and then proceed as in the proof of part (c).

6.11.5 Let $\gamma\colon [0, 1] \to M$ be a Jordan curve with image C in a topological surface M.

(a) Prove that if C admits a (topologically) orientable neighborhood, then for some $R > 1$, there exist a neighborhood Ω of C in M and a homeomorphism $\Phi\colon \Omega \to \Delta(0; 1/R, R)$ with $\Phi(\gamma(t)) = e^{2\pi i t}$ for each $t \in [0, 1]$ (cf. Exercise 6.7.7).

(b) Prove that if C does *not* admit a (topologically) orientable neighborhood, then there exists a homeomorphism $\Phi\colon \Omega \to B$ of some neighborhood Ω of C in M onto the Möbius band

$$B \equiv [0,1] \times (0,1)/(0,t) \sim (1, 1-t) \quad \forall t \in (0,1),$$

with quotient map $\Pi\colon [0,1] \times (0,1) \to B$, such that $\Phi(\gamma(t)) = \Pi(t, 1/2)$ for each point $t \in [0,1]$ (cf. Exercise 6.10.1).

6.11.6 Let M be a second countable orientable topological surface, let $\gamma\colon [0,1] \to M$ be a Jordan curve, and let $C \equiv \gamma([0,1])$. Prove that C is separating in M if and only if $\int_\gamma \theta = 0$ for every Čech 1-form θ with compact support in M. Prove also that every Jordan curve in M is separating if and only if every Čech 1-form θ with compact support in M is exact.

Hint. Apply Exercise 6.7.2.

6.11.7 Prove that every compact orientable topological surface M may be obtained by $\mathcal{C}^0$ attachment of a locally finite family of disjoint tubes (see Exercise 5.12.8) to the unit sphere $\mathbb{S}^2$.

6.11.8 Let M be a *nonorientable* second countable topological surface. Prove that there exists a connected orientable covering space $\Upsilon\colon \widehat{M} \to M$ for which each fiber contains exactly two elements (as in Exercise 9.7.8, $\Upsilon\colon \widehat{M} \to M$ is called the *orientable double cover of* M).

6.11.9 This exercise requires facts considered in Exercises 6.6.2–6.6.4. Let M be a second countable topological surface.

(a) Prove that the universal cover of M is homeomorphic to the plane $\mathbb{R}^2$ or the sphere $\mathbb{S}^2$.

(b) Prove that if M is orientable, then $\pi_1(M)$ is torsion-free and

$$H_1(M,\mathbb{Z}) \cong \pi_1(M)/[\pi_1(M), \pi_1(M)],$$

while if M is nonorientable, then every torsion element of $\pi_1(M)$ has order 1 or 2 and $\pi_1(M)/\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ for some torsion-free normal subgroup Γ of $\pi_1(M)$.

(c) Let $\mathbb{A}$ be a subfield of $\mathbb{C}$ containing $\mathbb{Z}$, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Prove that if M is orientable, then $H_1^\Delta(M,\mathbb{A}) \cong H_1(M,\mathbb{A})$, while if M is nonorientable, then $H_1^\Delta(M,\mathbb{F}) \cong H_1(M,\mathbb{F})$.

(d) Prove that if M is orientable, then for any subring $\mathbb{A}$ of $\mathbb{C}$ containing $\mathbb{Z}$, the mapping

$$[\xi]_{H_1(M,\mathbb{A})} \mapsto ([\xi]_{H_1(M,\mathbb{A})}, \cdot)_{\mathrm{deR}}$$

gives an injective homomorphism $H_1(M,\mathbb{A}) \to \mathrm{Hom}(H^1(M,\mathbb{A}),\mathbb{A})$ (according to Theorem 10.7.18, this homomorphism is also surjective if $\pi_1(X)$ is finitely generated).

(e) Assume that M is orientable, and let $K \subset M$ be a compact set. Prove that there exist domains Ω and Ω' such that $K \subset \Omega \Subset \Omega' \Subset M$ and such that

$$\operatorname{im}[H^1(M,\mathbb{C}) \to H^1(\Omega,\mathbb{C})] = \operatorname{im}[H^1(\Omega',\mathbb{C}) \to H^1(\Omega,\mathbb{C})].$$

Part III
Background Material

Chapter 7
Background Material on Analysis in $\mathbb{R}^n$ and Hilbert Space Theory

In this chapter, we recall some basic definitions and facts concerning integration and Hilbert spaces.

7.1 Measures and Integration

In this section, we recall some basic facts from measure theory. The proofs, which are omitted, can be found in, for example, [Fol], [Rud1], or [WZ].

Definition 7.1.1 Let X be a set.

(a) A *σ-algebra* in X is a collection $\mathfrak{M}$ of subsets of X such that
  (i) We have $\emptyset \in \mathfrak{M}$;
  (ii) $X \setminus S \in M$ for each set $S \in \mathfrak{M}$; and
  (iii) $\bigcup_{j\in J} S_j \in \mathfrak{M}$ for each countable family of sets $\{S_j\}_{j\in J}$ in $\mathfrak{M}$.
  The pair $(X, \mathfrak{M})$, which is often denoted simply by X, is then called a *measurable space*, and any element of $\mathfrak{M}$ is called a *measurable set*. A mapping $f: S \to Y$ of a measurable set $S \subset X$ into a topological space Y (see Sect. 9.1) is called a *measurable mapping* if $f^{-1}(U)$ is measurable for each open set $U \subset Y$. For $Y = [-\infty, \infty]$ or $\mathbb{C}$, we also call f a *measurable function*.

(b) A *positive measure* on X (or on $(X, \mathfrak{M})$) is a function $\mu: \mathfrak{M} \to [0, \infty]$ on a σ-algebra $\mathfrak{M}$ in X such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{j=1}^{\infty} S_j) = \sum_{j=1}^{\infty} \mu(S_j)$ for every sequence of disjoint measurable sets $\{S_j\}$. For each measurable set S, $\mu(S)$ is called the *measure* of S. The pair (X, μ) (or triple $(X, \mathfrak{M}, \mu)$) is called a *measure space*. The measure μ is *complete* if every subset of a set of measure 0 is measurable (and hence of measure 0). We say that a property of points holds *almost everywhere* (abbreviated a.e.) in a set $S \subset X$ if the property holds for all points in the complement $S \setminus A$ of some set $A \in \mathfrak{M}$ of measure 0.

Remarks 1. If f is a sum or product of measurable functions, a composition of a continuous function with a measurable mapping, or a supremum, infimum, or limit of a sequence of measurable functions, then f is measurable.

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2. We will occasionally consider measure spaces $(X, \mathfrak{M}, \mu)$ that are not complete (see Sect. 9.7). However, letting $\widehat{\mathfrak{M}}$ denote the collection of sets of the form $A = E \cup B$, where $E \in \mathfrak{M}$ and B is a subset of some set of measure 0, and letting $\hat{\mu}(A) = \mu(E)$ for each such set A, one gets a well-defined complete measure space $(X, \widehat{\mathfrak{M}}, \hat{\mu})$ called the *completion* of $(X, \mathfrak{M}, \mu)$ (see, for example, [Rud1]).

Definition 7.1.2 Let (X, μ) be a measure space, and let f be a measurable function on a measurable set $S \subset X$.

(a) If $f \geq 0$ a.e. in S, then the *integral of f over S* is given by

$$\int_S f(x)\, d\mu(x) \equiv \int_S f\, d\mu \equiv \sup \sum_{j=1}^{m} r_j \mu(S_j) \in [0, \infty],$$

where the supremum is taken over all choices of disjoint measurable subsets $S_1, \ldots, S_m$ of S and constants $r_1, \ldots, r_m \in [0, \infty]$ with $f \geq r_j$ a.e. in S_j for each $j = 1, \ldots, m$.

(b) We say that f is *integrable* on S if the nonnegative extended real numbers

$$R_+ \equiv \int_S [\operatorname{Re} f]^+\, d\mu, \qquad R_- \equiv \int_S [\operatorname{Re} f]^-\, d\mu,$$

$$I_+ \equiv \int_S [\operatorname{Im} f]^+\, d\mu, \qquad I_- \equiv \int_S [\operatorname{Im} f]^-\, d\mu,$$

are finite. If this is the case, then the *integral of f over S* is given by

$$\int_S f\, d\mu \equiv R_+ - R_- + iI_+ - iI_-.$$

Remarks 1. Every measurable subset Y of a measure space (X, μ) inherits a positive measure given by $S \mapsto \mu(S)$ for every measurable set S with $S \subset Y$. Integrals of functions over measurable subsets of S with respect to this induced measure then agree with the corresponding integrals with respect to μ. In an abuse of notation, we also denote the induced measure by μ.

2. A *simple function* φ on a measurable space X is a measurable complex-valued function with finite range. Letting $r_1, \ldots, r_n$ be the distinct values of φ, we have $\varphi = \sum_{j=1}^{n} r_j \chi_{S_j}$, where $S_j = \varphi^{-1}(r_j)$ for $j = 1, \ldots, n$, and for any set $E \subset X$, χ_E denotes the corresponding *characteristic function* (i.e., $\chi_E \equiv 1$ on E and $\chi_E \equiv 0$ on $X \setminus E$). If μ is a positive measure on X, $S \subset X$ is measurable, and $\varphi \geq 0$, then we have $\int_S \varphi\, d\mu = \sum_{j=1}^{n} r_j \mu(E_j \cap S)$. Consequently, if f is a measurable function on S and $f \geq 0$ a.e., then

$$\int_S f\, d\mu = \sup \left\{ \int_S \varphi\, d\mu \mid \varphi \text{ is a nonnegative simple function and } \varphi \leq f \text{ a.e. in } S \right\}.$$

Example 7.1.3 (Counting measure) Let X be a set, let $\mathfrak{M}$ be the collection of all subsets of X (which is clearly a σ-algebra), and let $\mu(S) \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \subset [0, \infty]$ be the number of elements of S for each set $S \subset X$. Then μ is a positive measure that is called the *counting measure* on X. For any function $f\colon X \to [0,\infty]$ and any set $S \subset X$, we have $\int_S f\,d\mu = \sum_{x\in S} f(x)$, the (unordered) sum of f over S (recall that $\sum_{x\in S} f(x) \equiv \sup_{F\in\mathcal{F}} \sum_{x\in F} f(x)$, where $\mathcal{F}$ is the set of all finite subsets of S). In particular, this sum is infinite if $f^{-1}((0,\infty])$ is uncountable. A function $f\colon X \to \mathbb{C}$ is integrable if and only if $\sum_{x\in S} f(x)$ is absolutely summable, and if this is the case, then the integral of f over S is equal to the sum.

Example 7.1.4 (Lebesgue measure) For $n \in \mathbb{Z}_{>0}$, an *n-cell* is a Cartesian product $Q = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ of n intervals in $\mathbb{R}$. For any n-cell Q as above, the *volume* of Q is given by $\operatorname{vol}(Q) \equiv \prod_{i=1}^n \ell(I_i) \in (0,\infty]$. For any set $S \subset X$, the *Lebesgue outer measure* of S is given by

$$\lambda^*(S) \equiv \inf\left\{ \sum_{j=1}^{\infty} \operatorname{vol}(Q_j) \mid \{Q_j\} \text{ is a sequence of open } n\text{-cells that covers } S \right\}.$$

This function is *countably subadditive*; that is, $\lambda^*(\bigcup_{j=1}^\infty S_j) \le \sum_{j=1}^\infty \lambda^*(S_j)$ for any sequence of subsets $\{S_j\}$ of $\mathbb{R}^n$. A set $S \subset \mathbb{R}^n$ is *Lebesgue measurable* if it satisfies the *Carathéodory criterion*:

$$\lambda^*(A) = \lambda^*(A \cap S) + \lambda^*(A \setminus S) \qquad \forall A \subset \mathbb{R}^n.$$

The collection $\mathfrak{M}$ of Lebesgue measurable sets is a σ-algebra, and the function

$$\lambda \equiv \lambda^*\restriction_{\mathfrak{M}} : \mathfrak{M} \to [0,\infty],$$

which is called *Lebesgue measure*, is a positive measure with the following properties:

(i) *Completeness*. Every subset of a set of Lebesgue measure 0 is measurable (in fact, any set of Lebesgue outer measure 0 is Lebesgue measurable).
(ii) *Regularity*. Every open set is Lebesgue measurable (hence $\mathfrak{M}$ contains the *Borel σ-algebra*, i.e., the smallest σ-algebra containing the open sets). Moreover, if a set $S \subset \mathbb{R}^n$ is Lebesgue measurable, then for every $\epsilon > 0$, there exist a closed set A and an open set B such that $A \subset S \subset B$ and $\lambda(B \setminus A) < \epsilon$. Consequently, there exist sets F and G such that F is a countable union of closed sets (i.e., F is an F_σ), G is a countable intersection of open sets (i.e., G is a G_δ), $F \subset S \subset G$, and $\lambda(G \setminus F) = 0$ (the completeness property (i) and the measurability of Borel sets imply that the converse is also true).
(iii) *Normalization*. For each n-cell Q, $\lambda(Q) = \operatorname{vol}(Q)$.
(iv) *Translation invariance*. For each Lebesgue measurable set $S \subset \mathbb{R}^n$ and each point $x \in \mathbb{R}^n$, the set $S + x$ is Lebesgue measurable and $\lambda(S + x) = \lambda(S)$.

For $n = 1$ and for I an interval with endpoints $a < b$, we often denote the integral of a (nonnegative measurable or integrable) function over I by $\int_a^b f$ or $\int_a^b f(x)\,dx$.

Theorem 7.1.5 (Convergence theorems) *Let* (X, μ) *be a measure space, and let* $\{f_\nu\}$ *be a sequence of measurable functions on a measurable set* $S \subset X$.

(a) Monotone convergence theorem. *If* $0 \leq f_1 \leq f_2 \leq \cdots$, *then*

$$\int_S \lim_{\nu\to\infty} f_\nu = \lim_{\nu\to\infty} \int_S f_\nu.$$

(b) Fatou's lemma. *If* $f_\nu \geq 0$ *for each* ν, *then*

$$\int_S \liminf_{\nu\to\infty} f_\nu \leq \liminf_{\nu\to\infty} \int_S f_\nu.$$

(c) Lebesgue's dominated convergence theorem. *If* $f_\nu \to f$ *and there is a nonnegative integrable function* g *such that* $|f_\nu| \leq g$ *on* S *for each* $\nu \in \mathbb{Z}_{>0}$, *then* f *and* f_ν *for each* ν *are integrable and*

$$\int_S f = \lim_{\nu\to\infty} \int_S f_\nu.$$

Definition 7.1.6 Let $\mathcal{V}$ be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

(a) A function $\|\cdot\| \colon \mathcal{V} \to [0, \infty)$ is called a *norm* if
   (i) For each vector $v \in \mathcal{V} \setminus \{0\}$, we have $\|v\| > 0$;
   (ii) For each vector $v \in \mathcal{V}$ and each scalar $a \in \mathbb{F}$, we have $\|av\| = |a|\|v\|$; and
   (iii) *Triangle inequality*. We have $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in \mathcal{V}$.
   The pair $(\mathcal{V}, \|\cdot\|)$ (or simply $\mathcal{V}$) is called a *normed vector space*.
(b) A sequence $\{v_j\}$ in $\mathcal{V}$ *converges* to v in $(\mathcal{V}, \|\cdot\|)$ if $\|v_j - v\| \to 0$ as $j \to \infty$. We also call v the *limit* of $\{v_j\}$, and we write

$$v = \lim_{j\to\infty} v_j \quad \text{or} \quad v_j \to v \quad \text{as } j \to \infty.$$

   A sequence that is not convergent is said to be *divergent*.
(c) A sequence $\{v_j\}$ in $(\mathcal{V}, \|\cdot\|)$ is a *Cauchy sequence* if for every $\epsilon > 0$, there exists an $N \in \mathbb{Z}_{>0}$ such that $\|v_i - v_j\| < \epsilon$ for all $i, j > N$. If every Cauchy sequence in $\mathcal{V}$ converges, then $(\mathcal{V}, \|\cdot\|)$ is called a *complete* normed vector space or a *Banach space*.
(d) A set $U \subset \mathcal{V}$ is *open* in $(\mathcal{V}, \|\cdot\|)$ if for each vector $v \in U$, the *ball of radius* r *centered at* v, $B(v; r) \equiv \{u \in \mathcal{V} \mid \|u - v\| < r\}$, is contained in U for some $r > 0$. A set $D \subset \mathcal{V}$ is *closed* if its complement $\mathcal{V} \setminus D$ is open. Equivalently, D is closed if the limit of every convergent sequence of vectors $\{v_j\}$ in $\mathcal{V}$ with $v_j \in D$ for each j lies in D. The *interior* $\overset{\circ}{S}$ of a set $S \subset \mathcal{V}$ is the union of all open sets that are contained in S. The *closure* $\overline{S}$ (or $\mathrm{cl}(S)$) of S is the intersection of all closed sets containing S; that is, $\overline{S}$ is the set of limits of sequences in S that converge in $\mathcal{V}$.

Remarks 1. If $\{u_j\}$ and $\{v_j\}$ are sequences in a normed vector space $(\mathcal{V}, \|\cdot\|)$, and $u_j \to u$ and $v_j \to v$, then $u_j + v_j \to u + v$, and for each scalar c, $cv_j \to cv$.

2. For vectors u and v in $(\mathcal{V}, \|\cdot\|)$, we also have $|\|u\| - \|v\|| \leq \|u+v\|$. In particular, if $v_j \to v$, then $\|v_j\| \to \|v\|$.

3. The closure $\overline{\mathcal{W}}$ of any (vector) subspace $\mathcal{W}$ of $(\mathcal{V}, \|\cdot\|)$ is a subspace.

4. Any closed subspace of a Banach space is a Banach space with respect to the restriction of the given norm.

5. If $\mathcal{V}$ is a finite-dimensional vector space, then $\mathcal{V}$ is complete with respect to *any* norm. Moreover, any two norms in $\mathcal{V}$ are equivalent, so convergence of sequences, openness of sets, etc., are independent of the choice of norm. Moreover, a sequence $\{v_j\}$ in $\mathcal{V}$ converges to a vector v if and only if $\alpha(v_j) \to \alpha(v)$ for every linear functional α on $\mathcal{V}$ (equivalently, for some basis $\alpha_1, \ldots, \alpha_n$ of the dual space $\mathcal{V}^*$, $\alpha_k(v_j) \to \alpha_k(v)$ for $k = 1, \ldots, n$).

Example 7.1.7 For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $\mathbb{F}^n$ together with the *Euclidean norm*

$$\|\zeta\| = \left[\sum_{j=1}^{n} |\zeta_j|^2\right]^{1/2} \qquad \forall \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{F}^n$$

(in fact, with any norm) is a Banach space (we set $\mathbb{F}^0 = \{0\}$).

Definition 7.1.8 Let (X, μ) be a measure space, and let $p \in [1, \infty]$.

(a) Let f be a measurable function on X. If $p < \infty$, then for each measurable function f, the L^p *norm* of F is given by

$$\|f\|_{L^p(X,\mu)} \equiv \left[\int_X |f|^p \, d\mu\right]^{1/p}.$$

The L^∞ *norm* of f is given by

$$\|f\|_{L^\infty(X,\mu)} \equiv \inf\{R \in [0, \infty] \mid |f| \leq R \text{ a.e.}\}.$$

We also denote the L^p norm by $\|\cdot\|_{L^p(\mu)}$ or $\|\cdot\|_{L^p(X)}$ or $\|\cdot\|_{L^p}$ when the measure or space is clear from the context.

(b) The L^p *space* is given by

$$L^p(X, \mu) \equiv \{f \mid f \text{ is a measurable function with } \|f\|_{L^p} < \infty\}/\sim,$$

where $f \sim g$ if and only if $f = g$ a.e. We usually denote each equivalence class by a representative f. We also denote the L^p space by $L^p(\mu)$ or $L^p(X)$ or L^p when the measure or the space are clear from the context.

(c) For Lebesgue measure λ on $\mathbb{R}^n$, we often denote $L^p(Y, \lambda)$ simply by $L^p(Y)$ for any Lebesgue measurable set $Y \subset \mathbb{R}^n$. We let $L^p_{\mathrm{loc}}(Y)$ denote the set of measurable functions f on Y such that each point in Y admits a neighborhood U in $\mathbb{R}^n$ such that $f \restriction_{U \cap Y} \in L^p(U \cap Y)$ (in particular, for Y an open set, $L^p_{\mathrm{loc}}(Y)$ is the set of functions for which the restriction to each compact subset of Y is in L^p), where as in (b), we identify any two measurable functions that are equal almost everywhere. Elements of L^1_{loc} are called *locally integrable*.

Theorem 7.1.9 *If (X, μ) is a complete measure space and $p \in [1, \infty]$, then $(L^p(X, \mu), \|\cdot\|_{L^p})$ is a* Banach space. *Moreover, if $\{f_\nu\}$ is a sequence that converges in $L^p(X, \mu)$ to a function $f \in L^p(X, \mu)$, then some subsequence of $\{f_\nu\}$ converges pointwise almost everywhere to f.*

Remarks 1. In this book, the fact that $\|\cdot\|_{L^p}$ is a norm is used, in an essential way, only for $p \in \{1, 2, \infty\}$. For $p = 1$ this follows easily from the inequality $|\int f| \leq \int |f|$; for $p = \infty$, this follows easily from the triangle inequality for $|\cdot|$; and for $p = 2$, this follows from the observation that $\|\cdot\|_{L^2}$ is the norm associated to an inner product (see Sect. 7.5). Completeness of L^p is used only for $p = 2$ or ∞. For the measure spaces of interest in this book, completeness of L^2 follows from Proposition 2.6.3. The proof of completeness of L^p for $p \in [1, \infty)$ is similar. The proof of completeness of L^∞ is relatively easy (see, for example, [Rud1]). The claim concerning pointwise convergence of a subsequence may be verified directly as follows. We may choose a subsequence $\{f_{\nu_k}\}$ such that

$$\infty > \sum_{k=1}^{\infty} \|f_{\nu_k} - f\|_{L^p}^p = \int \sum_{k=1}^{\infty} |f_{\nu_k} - f|^p$$

(for example, by the monotone convergence theorem). Hence the integrand is finite almost everywhere, and the claim follows.

2. For any compact set $K \subset \mathbb{R}^n$, the vector space $\mathcal{C}^0(K)$ of continuous functions on K is a closed vector subspace of $L^\infty(K)$, since uniform convergence preserves continuity.

3. We also recall that for $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for measurable functions f, g on (X, μ), we have the *Hölder inequality* $\|fg\|_{L^1(X,\mu)} \leq \|f\|_{L^p(X,\mu)} \cdot \|g\|_{L^q(X,\mu)}$. In this book, this inequality is used, in an essential way, only for the (relatively easy) cases $p = 1$ and $p = 2$.

4. For a measurable set $Y \subset \mathbb{R}^n$, we say that a sequence of measurable functions $\{f_\nu\}$ on Y *converges in $L^p_{\text{loc}}(Y)$* to a function f on Y if each point in Y admits a neighborhood U such that $f\restriction_{U\cap Y}$ and $f_\nu\restriction_{U\cap Y}$ for each ν are in $L^p(U \cap Y)$, and $\|f_\nu - f\|_{L^p(U\cap Y)} \to 0$.

Theorem 7.1.10 (Fubini's theorem) *Let $m, n \in \mathbb{Z}_{>0}$, and let f be a (Lebesgue) measurable function on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. Then*

(a) *For almost every point $x \in \mathbb{R}^m$, the function $f(x, \cdot)$ is measurable on $\mathbb{R}^n$.*
(b) *The function $x \mapsto \int_{\mathbb{R}^n} |f(x, y)|\, d\lambda(y)$ is measurable on $\mathbb{R}^m$. Moreover, f is integrable on $\mathbb{R}^{m+n}$ if and only if this function is integrable on $\mathbb{R}^m$.*
(c) *If f is nonnegative or integrable on $\mathbb{R}^{m+n}$, then*

$$\begin{aligned}\int_{\mathbb{R}^{m+n}} f\, d\lambda &= \int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^n} f(x, y)\, d\lambda(y) \right] d\lambda(x) \\ &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^m} f(x, y)\, d\lambda(x) \right] d\lambda(y).\end{aligned}$$

A proof is outlined in Exercise 7.1.3 (a proof for a general σ-finite product measure space can be found in, for example, [Fol] or [Rud1]).

Exercises for Sect. 7.1

7.1.1 Let λ^* and λ denote Lebesgue outer measure and Lebesgue measure, respectively, on $\mathbb{R}^n$ (see Example 7.1.4).

(a) Verify that λ^* is countably subadditive.

(b) Verify that λ is a positive measure and that λ has the properties (i)–(iv) in Example 7.1.4.

7.1.2 Let f be a nonnegative measurable function on a measure space (X, μ). Prove that there is a sequence of simple functions $\{\varphi_\nu\}_{\nu=1}^\infty$ on X such that $0 \le \varphi_\nu \le \varphi_{\nu+1}$ for each $\nu \in \mathbb{Z}_{>0}$, $\varphi_\nu \to f$ as $\nu \to \infty$, and $\varphi_\nu\restriction_A \to f\restriction_A$ uniformly on any set $A \subset X$ on which f is bounded.

Hint. For example, consider

$$\varphi_\nu \equiv \sum_{j=1}^{\nu 2^\nu} \frac{j-1}{2^\nu} \cdot \chi_{f^{-1}([(j-1)/2^\nu, j/2^\nu))} + \nu \cdot \chi_{f^{-1}([\nu,\infty])}.$$

7.1.3 The goal of this exercise is a proof of Fubini's theorem (Theorem 7.1.10). Let $m, n \in \mathbb{Z}_{>0}$, and for every set $S \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ and every point $x \in \mathbb{R}^m$, let $S_x \equiv \{y \in \mathbb{R}^n \mid (x, y) \in S\}$. Also fix an open $(m+n)$-cell $R = A \times B$ with $A \Subset \mathbb{R}^m$ and $B \Subset \mathbb{R}^n$; let $\mathcal{S}_0$ be the collection of all measurable sets $S \subset R$ for which the set $S_x \subset B \subset \mathbb{R}^n$ is measurable for almost every point $x \in A$; and let $\mathcal{S} \subset \mathcal{S}_0$ be the subcollection consisting of all sets $S \in \mathcal{S}_0$ for which the function on A given (almost everywhere) by $x \mapsto \lambda(S_x)$ is measurable and

$$\lambda(S) = \int_A \lambda(S_x)\, d\lambda(x)$$

(here, $\lambda(S)$ and $\lambda(S_x)$ denote the Lebesgue measure in $\mathbb{R}^{m+n}$ and $\mathbb{R}^n$, respectively).

(a) Prove that $\mathcal{S}_0$ is a σ-algebra in R, and that $\mathcal{S}_0$ contains the collection of Borel subsets of R.

(b) Prove that $\emptyset \in \mathcal{S}$ and that $R \setminus S \in \mathcal{S}$ for each set $S \in \mathcal{S}$.

(c) Let $\{S_\nu\}$ be a sequence of sets in $\mathcal{S}$. Prove that if $S_\nu \subset S_{\nu+1}$ for each ν or the sets $\{S_\nu\}$ are disjoint, then $\bigcup_\nu S_\nu \in \mathcal{S}$. Prove also that if $S_\nu \supset S_{\nu+1}$ for each ν, then $\bigcap_\nu S_\nu \in \mathcal{S}$.

(d) Prove that every G_δ and every F_σ in R are in $\mathcal{S}$.

Hint. First show that every open subset of $\mathbb{R}^d$ is a countable union of disjoint d-cells.

(e) Prove that every set $S \subset R$ of measure 0 is in $\mathcal{S}$.

Hint. S is contained in a G_δ of measure 0, and Lebesgue measure is complete.

(f) Prove that $\mathcal{S}$ is precisely the collection of Lebesgue measurable subsets of R.

(g) Let E be a measurable subset of $\mathbb{R}^{m+n}$. Prove that the set $E_x \subset \mathbb{R}^n$ is measurable for almost every point $x \in \mathbb{R}^m$, the function $x \mapsto \lambda(E_x)$ is measurable, and

$$\lambda(E) = \int_{\mathbb{R}^m} \lambda(E_x)\, d\lambda(x).$$

(h) Prove Fubini's theorem (Theorem 7.1.10).

Hint. First prove that the theorem holds for the characteristic function of a measurable set, and then that the theorem holds for a nonnegative simple function. Obtain the theorem for a nonnegative measurable function by applying Exercise 7.1.2. Finally, prove that the theorem holds for a measurable complex-valued function.

7.1.4 Let Ω be an open subset of $\mathbb{R}^n$ and let $p \in [1, \infty)$. The goal of this exercise is a proof that the vector space of continuous functions with compact support in Ω is dense in $L^p_{\text{loc}}(\Omega)$. This fact will be applied in the proof of the Friedrichs lemma (Lemma 7.3.1). Let K be compact subset of Ω.

(a) Prove that if f is a nonnegative locally integrable function on Ω, then for every $\epsilon > 0$, there exists a nonnegative simple function φ on $\mathbb{R}^n$ such that $\|\varphi - f\|_{L^p(K)} < \epsilon$.

Hint. Apply Exercise 7.1.2.

(b) Prove that if C is a compact subset of an open set $U \subset \mathbb{R}^n$, then there exists a nonnegative continuous function $\eta\colon \mathbb{R}^n \to [0, 1]$ such that $\eta \equiv 1$ on C and $\operatorname{supp} \eta \subset U$.

(c) Prove that if E is a measurable subset of $\mathbb{R}^n$ with characteristic function χ_E, then for every $\epsilon > 0$, there exists a nonnegative continuous function $\eta\colon \mathbb{R}^n \to [0, 1]$ such that $\|\eta - \chi_E\|_{L^p(K)} < \epsilon$.

Hint. For $\delta > 0$ sufficiently small, fix an open set U and a compact set C such that $C \subset E \cap K \subset U$ and $\lambda(U \setminus C) < \delta$ (see Example 7.1.4). Now apply part (b) above.

(d) Prove that if $f \in L^p_{\text{loc}}(\Omega)$, then for every $\epsilon > 0$, there exists a continuous function g with compact support in Ω such that $\|g - f\|_{L^p(K)} < \epsilon$.

7.1.5 Prove that if Ω is an open subset of $\mathbb{R}^n$ and $p \in [1, \infty)$, then the vector space of continuous functions with compact support in Ω is dense in $L^p(\Omega)$.

Hint. Apply Exercise 7.1.4.

7.2 Differentiation and Integration in $\mathbb{R}^n$

In this section, we recall some facts concerning differentiation and integration on $\mathbb{R}^n$ that will allow us to consider differentiation and integration on manifolds (see Chap. 9).

Definition 7.2.1 Let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let Ω be an open subset of $\mathbb{R}^n$.

(a) A function $f : \Omega \to \mathbb{F}$ is of *class* C^k if one of the following holds:

(i) We have $k = 0$ and f is continuous;
(ii) We have $k \in \mathbb{Z}_{>0}$ and all of the partial derivatives of f of order $\leq k$ exist and are continuous on Ω;
(iii) We have $k = \infty$ and f has continuous partial derivatives of *all* orders;
(iv) We have $k = \omega$ and f is real analytic (that is, in a neighborhood of every point $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, we may write f as a power series $f(x_1, \ldots, x_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} c_{m_1,\ldots,m_n}(x_1 - a_1)^{m_1} \cdots (x_n - a_n)^{m_n}$ for some choice of constants $\{c_{m_1,\ldots,m_n}\}$ in $\mathbb{F}$).

It is convenient to consider any function on $\mathbb{R}^0 = \{0\}$ to be of class $\mathcal{C}^k$.

(b) The vector space of $\mathcal{C}^k$ functions on Ω with values in $\mathbb{F}$ is denoted by $\mathcal{C}^k(\Omega, \mathbb{F})$, or simply by $\mathcal{C}^k(\Omega)$. We also denote $\mathcal{C}^\infty(\Omega, \mathbb{F})$ by $\mathcal{E}(\Omega, \mathbb{F})$ or by $\mathcal{E}(\Omega)$. The vector space of $\mathcal{C}^\infty$ $\mathbb{F}$-valued functions with compact support in Ω is denoted by $\mathcal{D}(\Omega, \mathbb{F})$ or by $\mathcal{D}(\Omega)$. For an arbitrary set $S \subset \mathbb{R}^n$, the vector space of continuous $\mathbb{F}$-valued functions on S is denoted by $\mathcal{C}^0(S, \mathbb{F})$ or by $\mathcal{C}^0(S)$.

(c) For $f \in \mathcal{C}^1(\Omega, \mathbb{F})$ and $p \in \Omega$, the *differential of f at p* is the $\mathbb{F}$-linear map $(df)_p \colon \mathbb{F}^n \to \mathbb{F}$ given by

$$(df)_p(v) = \sum_{j=1}^{n} v_j \cdot \frac{\partial f}{\partial x_j}(p) \quad \forall v = (v_1, \ldots, v_n) \in \mathbb{F}^n.$$

We also denote by df the mappings $p \mapsto (df)_p$ and $(p, v) \mapsto (df)_p(v)$ (the latter mapping $df \colon \Omega \times \mathbb{F}^n \to \mathbb{F}$ being of class $\mathcal{C}^{k-1}$ if $f \in \mathcal{C}^k(\Omega)$ with $k \in \mathbb{Z}_{>0} \cup \{\infty\}$).

Definition 7.2.2 Let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let Ω be an open subset of $\mathbb{R}^n$, and let $F = (f_1, \ldots, f_m) \colon \Omega \to \mathbb{F}^m$ be a mapping.

(a) F is of *class* $\mathcal{C}^k$ if $f_j \in \mathcal{C}^k(\Omega)$ for each $j = 1, \ldots, m$.
(b) If F is of class $\mathcal{C}^1$, then for each point $p \in \Omega$, the *differential of F at p* is the $\mathbb{F}$-linear map $(dF)_p = ((df_1)_p, \ldots, (df_m)_p) \colon \mathbb{F}^n \to \mathbb{F}^m$ given by

$$(dF)_p(v) = \sum_{j=1}^{n} v_j \cdot \frac{\partial F}{\partial x_j}(p) \quad \forall v = (v_1, \ldots, v_n) \in \mathbb{R}^n.$$

We also denote by dF the mapping $p \mapsto (dF)_p \in \mathrm{Hom}(\mathbb{F}^n, \mathbb{F}^m)$ and the mapping $(p, v) \mapsto (dF)_p(v)$ (the latter mapping $dF \colon \Omega \times \mathbb{F}^n \to \mathbb{F}^m$ being of class $\mathcal{C}^{k-1}$ if F is of class $\mathcal{C}^k$ with $k \in \mathbb{Z}_{>0} \cup \{\infty\}$).

(c) For F of class $\mathcal{C}^1$ and $m = n$, the *Jacobian determinant* is the function

$$\mathcal{J}_F = \det(dF) = \det\left(\frac{\partial f_i}{\partial x_j}\right) \colon \Omega \to \mathbb{F}.$$

(d) If F maps Ω bijectively onto an open set $\Omega' \subset \mathbb{R}^m$ and both F and F^{-1} are of class $\mathcal{C}^\infty$, then we call F a *diffeomorphism* of Ω onto Ω' (in particular, $m = n$).

Remarks 1. If $F\colon \Omega \to \mathbb{R}^m$, then under the inclusion $\mathbb{R}^m \subset \mathbb{C}^m$, we may consider, for each point $p \in \Omega$, the differential $(dF)_p\colon \mathbb{C}^n \to \mathbb{C}^m$, which is the complex linear extension of the real linear map $(dF)_p\colon \mathbb{R}^n \to \mathbb{R}^m$.

2. Although we often identify $\mathbb{C}$ with $\mathbb{R}^2$, one must proceed with caution when considering differentials. For if F is a $\mathcal{C}^1$ mapping of an open set $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^{2m}$ and $p \in \Omega$, then the complex linear extension $(dF)_p\colon \mathbb{C}^n \to \mathbb{C}^{2m}$ of the real linear differential $(dF)_p\colon \mathbb{R}^n \to \mathbb{R}^{2m}$ is clearly not the same as the differential mapping $\mathbb{C}^n \to \mathbb{C}^m$ that one obtains by considering F as a mapping of Ω into $\mathbb{C}^m$. In this book, the context will determine which differential mapping is being considered.

3. According to the *chain rule*, if $F\colon \Omega \to \mathbb{R}^m$ and $G\colon \Omega' \to \mathbb{F}^l$ are $\mathcal{C}^k$ mappings with $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^m$, and $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then $G \circ F$ is of class $\mathcal{C}^k$ and

$$d(G \circ F)_p = (dG)_{F(p)} \circ (dF)_p\colon \mathbb{C}^n \to \mathbb{C}^l$$

for each point $p \in F^{-1}(\Omega')$ (where for $\mathbb{F} = \mathbb{C}$, the above mapping $(dF)_p$ is the complex linear extension). In particular, if $F\colon \Omega \to \Omega'$ is a diffeomorphism, then for each point $p \in \Omega$, $(dF)_p^{-1} = d(F^{-1})_{F(p)}$, and hence $m = n$ in this case.

4. By the intermediate value theorem, a continuous injective real-valued function f on an interval $I \subset \mathbb{R}$ is either strictly increasing or strictly decreasing, and f maps I homeomorphically onto an interval J; that is, $f^{-1}\colon J \to I$ is continuous (in particular, the image of any endpoint of I is an endpoint of J). It follows easily that if I is open and f is of class $\mathcal{C}^\infty$ with nonvanishing derivative, then $f\colon I \to J$ is a diffeomorphism. The higher-dimensional analogue is called the $\mathcal{C}^\infty$ *inverse function theorem* (Theorem 9.9.1). This (nontrivial) theorem and its consequences are applied in the study of the topological and holomorphic structure of Riemann surfaces in Sects. 5.10–5.17, and in the study of holomorphic structures on smooth surfaces in Chap. 6.

Definition 7.2.3 A function f on a set $S \subset \mathbb{R}^n$ is said to be *locally Lipschitz* on S if for each point in S, there are a neighborhood U and a constant $C = C(U) > 0$ such that $|f(y) - f(x)| \leq C\|y - x\|$ for all $x, y \in S \cap U$. We say that f is *uniformly Lipschitz* on S if there is a constant $C > 0$ such that $|f(y) - f(x)| \leq C\|y - x\|$ for all $x, y \in S$.

Lemma 7.2.4 *If f is a $\mathcal{C}^1$ function on an open set $\Omega \subset \mathbb{R}^n$, then f is uniformly Lipschitz on every compact subset of Ω.*

The proof is left to the reader (see Exercise 7.2.1).

Proposition 7.2.5 (Differentiation past the integral) *Let $\Omega \subset \mathbb{R}^n$, let (X, μ) be a measure space, and let u be a complex-valued function on Ω. Assume that the func-*

tion $u(t,\cdot)$ is integrable on X for each fixed $t \in \Omega$, and let

$$v(t) = \int_X u(t,x)\,d\mu(x) \quad \forall t \in \Omega.$$

(a) *Suppose that $t_0 \in \Omega$, $u(\cdot,x)$ is continuous at t_0 for each fixed $x \in X$, and there exists a nonnegative integrable function g on X with $|u(t,x)| \le g(x)$ for each point $(t,x) \in \Omega \times X$. Then v is continuous at t_0.*

(b) *Suppose that Ω is open, $j \in \{1,\dots,n\}$, $\partial u/\partial t_j$ is defined on $\Omega \times X$, and there exists a nonnegative integrable function g on X with $|(\partial u/\partial t_j)(t,x)| \le g(x)$ for each point $(t,x) \in \Omega \times X$. Then $\partial v/\partial t_j$ is defined on Ω and*

$$\frac{\partial v}{\partial t_j}(t) = \int_X \frac{\partial u}{\partial t_j}(t,x)\,d\mu(x) \quad \forall t = (t_1,\dots,t_n) \in \Omega.$$

Proof Part (a) follows from the dominated convergence theorem. Clearly, for the proof of (b), we may assume that $n = 1$, that Ω is an open interval, and that u is real-valued. As a limit of a sequence of measurable functions, $\partial u/\partial t$ is measurable in the variable $x \in X$. By hypothesis, $|(\partial u/\partial t)(t,x)| \le g(x)$ for each point $(t,x) \in \Omega \times X$, and hence $\partial u/\partial t$ is integrable in $x \in X$. For distinct points $t, t_0 \in \Omega$ and for $x \in X$, we have, by the mean value theorem,

$$\left| \frac{u(t,x) - u(t_0,x)}{t - t_0} - \frac{\partial u}{\partial t}(t_0,x) \right| \le 2g(x).$$

Thus, once again, the dominated convergence theorem gives the claim. □

Lemma 7.2.6 (Taylor's formula to order one) *If $a = (a_1,\dots,a_n) \in \mathbb{R}^n$ and f is a $\mathcal{C}^2$ function on a neighborhood Ω of a with $a + t(x-a) \in \Omega$ for each point $x \in \Omega$ and each $t \in [0,1]$ (that is, Ω is starlike about a), then for each point $x = (x_1,\dots,x_n) \in \Omega$, we have*

$$f(x) = f(a) + \sum_{j=1}^{n} b_j \cdot (x_j - a_j) + \sum_{i,j=1}^{n} c_{ij}(x)(x_i - a_i)(x_j - a_j),$$

where for all indices $i, j = 1,\dots,n$ (and for coordinate functions $u = (u_1,\dots,u_n)$),

$$b_j \equiv \frac{\partial}{\partial u_j}[f(u)]\bigg|_{u=a}$$

and

$$c_{ij}(x) \equiv \int_0^1 (1-t)\frac{\partial^2}{\partial u_i \partial u_j}[f(u)]\bigg|_{u=a+t(x-a)}\,dt.$$

Proof For each point $x = (x_1, \ldots, x_n) \in \Omega$, the fundamental theorem of calculus and integration by parts give

$$
\begin{aligned}
f(x) - f(a) &= \int_0^1 \frac{d}{dt}[f(a + t(x-a))]\, dt \\
&= -(1-t)\frac{d}{dt}[f(a+t(x-a))]\Big|_{t=1} \\
&\quad + (1-t)\frac{d}{dt}[f(a+t(x-a))]\Big|_{t=0} \\
&\quad + \int_0^1 (1-t)\frac{d^2}{dt^2}[f(a+t(x-a))]\, dt.
\end{aligned}
$$

The chain rule now gives the desired expression for $f(x)$. $\square$

Remark Differentiation past the integral (Proposition 7.2.5) implies that if $f \in \mathcal{C}^k(\Omega)$ for some $k \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$, then each of the functions $x \mapsto c_{ij}(x)$ in the lemma is of class $\mathcal{C}^{k-2}$.

We close this section with a consideration of the behavior of integrals under diffeomorphisms. We first recall that the image $P = A(Q)$ of the compact n-cell $Q = [0, s_1] \times \cdots \times [0, s_n] \subset \mathbb{R}^n$ (with $s_i > 0$ for $i = 1, \ldots, n$) under a linear map $A \colon \mathbb{R}^n \to \mathbb{R}^n$ is a compact parallelepiped of measure

$$\lambda(P) = |\det A| \cdot \operatorname{vol}(Q) = |\det A| \cdot s_1 \cdots s_n.$$

This formula is trivial for $n = 1$, and relatively easy to verify for $n = 2$ (see Exercise 7.2.2), and ultimately, these are the only cases required in this book. For a general diffeomorphism, we have the following:

Theorem 7.2.7 (Change of variables formula) *Let $F \colon \Omega \to \Omega'$ be a diffeomorphism of an open set $\Omega \subset \mathbb{R}^n$ onto an open set $\Omega' \subset \mathbb{R}^n$. Then the composition $u \circ F$ of any measurable function u on Ω' with F is a measurable function on Ω. Moreover, if u is a nonnegative measurable function on Ω', then*

$$\int_{\Omega'} u\, d\lambda = \int_{\Omega} (u \circ F) \cdot |\mathcal{J}_F|\, d\lambda,$$

where $\mathcal{J}_F = \det(dF)$ is the Jacobian determinant of F. If u is a (*complex-valued*) *integrable function on Ω', then the function $(u \circ F)|\mathcal{J}_F|$ is integrable on Ω and the above equality holds. In particular, the image $F(E) \subset \Omega'$ of a set $E \subset \Omega$ is measurable if E is measurable*, *and $F(E)$ is of measure* 0 *if E is of measure* 0.

A proof is outlined in Exercise 7.2.3. The proof may be modified to give the change of variables formula for a $\mathcal{C}^1$ diffeomorphism (i.e., a bijective $\mathcal{C}^1$ mapping with $\mathcal{C}^1$ inverse), but only the $\mathcal{C}^\infty$ case is applied in this book.

Example 7.2.8 Polar coordinates. Polar coordinates play an important role in the study of complex analysis in the plane. We first recall one of the many equivalent constructions of the *real trigonometric functions*. The *real arcsine function* is the strictly increasing homeomorphism $x \mapsto \arcsin x$ of $[-1, 1]$ onto $[-\pi/2, \pi/2]$ given by $x \mapsto \int_0^x (1-t^2)^{-1/2}\,dt$, where $\pi \equiv \int_{-1}^1 (1-t^2)^{-1/2}\,dt$. The restriction to $(-1, 1)$ is a diffeomorphism onto $(-\pi/2, \pi/2)$; that is, the function and its inverse function $(-\pi/2, \pi/2) \to (-1, 1)$ are of class $\mathcal{C}^\infty$. The *real sine function* is the unique extension $\theta \mapsto \sin\theta$ of the inverse function of the arcsine function to a 2π-periodic function that satisfies $\sin(\theta+\pi) = -\sin\theta$ for each $\theta \in \mathbb{R}$. The *real cosine function* is the unique 2π-periodic function $\theta \mapsto \cos\theta$ determined by

$$\cos\theta = \sqrt{1-\sin^2\theta} \quad \forall \theta \in [-\pi/2, \pi/2]$$

and

$$\cos(\theta+\pi) = -\cos\theta \quad \forall \theta \in \mathbb{R}.$$

The *real tangent function*, the *real cotangent function*, the *real secant function*, and the *real cosecant function* are then given by

$$\begin{aligned}
\tan\theta &\equiv \frac{\sin\theta}{\cos\theta} \quad \forall \theta \in \mathbb{R} \Big\backslash \left(\frac{\pi}{2} + \mathbb{Z}\cdot\pi\right), \\
\cot\theta &\equiv \frac{\cos\theta}{\sin\theta} \quad \forall \theta \in \mathbb{R} \setminus (\mathbb{Z}\cdot\pi), \\
\sec\theta &\equiv \frac{1}{\cos\theta} \quad \forall \theta \in \mathbb{R} \Big\backslash \left(\frac{\pi}{2} + \mathbb{Z}\cdot\pi\right), \\
\csc\theta &\equiv \frac{1}{\sin\theta} \quad \forall \theta \in \mathbb{R} \setminus (\mathbb{Z}\cdot\pi).
\end{aligned}$$

The verification that these real trigonometric functions are of class $\mathcal{C}^\infty$ with the familiar derivative functions is straightforward (see Exercise 7.2.4). We also get the remaining *real inverse trigonometric functions* $\arccos \equiv (\cos\restriction_{[0,\pi]})^{-1}$, $\arctan \equiv (\tan\restriction_{[-\pi/2,\pi/2]})^{-1}$ $(\arctan(\pm\infty) \equiv \pm\pi/2)$, etc., and their derivative functions. The familiar formulas for the sine and cosine of a sum will follow from consideration of the complex exponential function in Chap. 1 (see Example 1.6.2 and Exercise 1.6.2).

For any fixed $\theta_0 \in \mathbb{R}$, the mapping $[0, \infty) \times [\theta_0, \theta_0 + 2\pi) \to \mathbb{R}^2$ given by

$$(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$$

is a continuous surjection. Moreover, the corresponding restrictions yield a bijection of the domain $(0, \infty) \times [\theta_0, \theta_0 + 2\pi)$ onto the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$, and a diffeomorphism of the domain $(0, \infty) \times (\theta_0, \theta_0 + 2\pi)$ onto the complement $\mathbb{R}^2 \setminus P$ of the ray $P \equiv \{(r\cos\theta_0, r\sin\theta_0) \mid r \geq 0\}$ in $\mathbb{R}^2$ (see Exercise 7.2.5). For any point $(x, y) \in \mathbb{R}^2$ and any pair $(r, \theta) \in [0, \infty) \times \mathbb{R}$ with $(r\cos\theta, r\sin\theta) = (x, y)$, we call (r, θ) *polar coordinates for* (x, y). Under the identification of $\mathbb{R}^2$ with $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, we also write

$$e^{i\theta} \equiv \cos\theta + i\sin\theta = (\cos\theta, \sin\theta) \quad \forall\theta \in \mathbb{R}.$$

The Jacobian determinant of the diffeomorphism $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$ described above is given by

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Hence, if f is an integrable function or a nonnegative measurable function on $\mathbb{R}^2$, then for any choice of $\theta_0 \in \mathbb{R}$, Fubini's theorem (Theorem 7.1.10) and the change of variables formula (Theorem 7.2.7) give

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy &= \int_{\mathbb{R}^2} f(x,y)\,d\lambda((x,y)) \\ &= \int_{(0,\infty)\times(\theta_0,\theta_0+2\pi)} f(r\cos\theta, r\sin\theta)r\,d\lambda((r,\theta)) \\ &= \int_0^{\infty}\int_{\theta_0}^{\theta_0+2\pi} f(r\cos\theta, r\sin\theta)r\,d\theta\,dr.\end{aligned}$$

In particular, the function $(x,y) \mapsto r^p = (x^2+y^2)^{p/2}$ is locally integrable on $\mathbb{R}^2$ for $p > -2$ (and integrable on the complement of any neighborhood of $(0,0)$ for $p < -2$).

Remark Recall that, similarly, one may define the *real logarithmic function* to be the diffeomorphism $(0,\infty) \to \mathbb{R}$ given by $x \mapsto \log x \equiv \int_1^x t^{-1}\,dt$, and one may define the *real exponential function* $x \mapsto \exp(x) = e^x$ to be the corresponding $\mathcal{C}^\infty$ inverse function. Clearly, these functions have derivative functions $x \mapsto 1/x$ and $x \mapsto 1/(1/\exp(x)) = \exp(x)$, respectively. Moreover, the verifications of the basic algebraic properties of these functions are straightforward (see Exercise 7.2.6).

Exercises for Sect. 7.2

7.2.1 Prove Lemma 7.2.4.

7.2.2 Prove that the image $P = A(Q)$ of the compact 2-cell $Q = [0,s_1] \times [0,s_2] \subset \mathbb{R}^2$ (with $s_i > 0$ for $i = 1,2$) under a linear map $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$\lambda(P) = |\det A| \cdot \mathrm{vol}(Q) = |\det A| \cdot s_1 \cdot s_2.$$

7.2.3 The goal of this exercise is a proof of the change of variables formula (Theorem 7.2.7). Let $F = (f_1,\dots,f_n)\colon \Omega \to \Omega'$ be a diffeomorphism of open sets $\Omega, \Omega' \subset \mathbb{R}^n$, and for each point $p \in \Omega$, let $G_p = (g_p^1,\dots,g_p^n)\colon \Omega \to G_p(\Omega) \subset \mathbb{R}^n$ be the diffeomorphism given by

$$x \mapsto (dF)_p^{-1}(F(x) - F(p)) = (dF^{-1})_{F(p)}(F(x) - F(p)).$$

Also, fix a relatively compact open subset U of Ω.

(a) Prove that there exists a positive constant C such that for each point $a = (a_1, \dots, a_n) \in U$, each positive number s for which the compact cubic n-cell $R = [a_1, a_1 + s] \times \cdots \times [a_n, a_n + s]$ is contained in U, and each point $x = (x_1, \dots, x_n) \in R$, we have

$$-Cs^2 \leq g_a^i(x) \leq s + Cs^2 \quad \forall i = 1, \dots, n.$$

Hint. Apply Lemma 7.2.6.

(b) Prove that for $C > 0$ and $R = [a_1, a_1 + s] \times \cdots \times [a_n, a_n + s] \subset U$ as in (a), we have

$$\begin{aligned} F(R) \subset & F(a) - (dF)_a(Cs^2, \dots, Cs^2) \\ & + (dF)_a([0, s + 2Cs^2] \times \cdots \times [0, s + 2Cs^2]), \end{aligned}$$

and conclude from this that the compact set $F(R)$ satisfies

$$\lambda(F(R)) \leq |\mathcal{J}_F(a)| \cdot \mathrm{vol}(R) \cdot (1 + 2Cs)^n.$$

(c) Prove that the open set $F(U) \subset \Omega'$ satisfies $\lambda(F(U)) \leq \int_U |\mathcal{J}_F| \, d\lambda$.

Hint. First prove that for each $\nu_0 \in \mathbb{Z}_{>0}$, U is equal to a countable union of disjoint n-cells of the form

$$[i_1 2^{-\nu}, (i_1 + 1)2^{-\nu}) \times \cdots \times [i_n 2^{-\nu}, (i_n + 1)2^{-\nu})$$

with $i_1, \dots, i_n \in \mathbb{Z}$ and $\nu \in \mathbb{Z}_{>\nu_0}$.

(d) Prove that $\lambda^*(F(E)) \leq \int_E |\mathcal{J}_F| \, d\lambda$ for every measurable set $E \subset \Omega$. Conclude that in particular, the image of every set of measure 0 under F is a set of measure 0. Finally, prove that the image of every measurable set under F is measurable.

(e) Prove the change of variables formula (Theorem 7.2.7).

Hint. For u a nonnegative measurable function on Ω', show that $u \circ F$ is measurable. Given disjoint measurable sets $E_1, \dots, E_m \subset \Omega'$ and constants $r_1, \dots, r_m \in [0, \infty]$ with $u \geq r_j$ on E_j for each $j = 1, \dots, m$, prove that

$$\sum_{j=1}^m r_j \lambda(E_j) \leq \int_\Omega (u \circ F)|\mathcal{J}_F| \, d\lambda.$$

Pass to the supremum to get $\int_{\Omega'} u \, d\lambda \leq \int_\Omega (u \circ F)|\mathcal{J}_F| \, d\lambda$. Apply this to the diffeomorphism F^{-1} and the function $(u \circ F) \cdot |\mathcal{J}_F|$ on Ω to get the reverse inequality.

7.2.4 Verify that the trigonometric functions (as defined in Example 7.2.8) are of class $\mathcal{C}^\infty$ with derivative functions

$$\frac{d}{d\theta}\sin\theta = \cos\theta, \qquad \frac{d}{d\theta}\cos\theta = -\sin\theta,$$

$$\frac{d}{d\theta}\tan\theta = \sec^2\theta, \qquad \frac{d}{d\theta}\cot\theta = -\csc^2\theta,$$

$$\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta, \qquad \frac{d}{d\theta}\csc\theta = -\csc\theta\cot\theta.$$

7.2.5 For a fixed number $\theta_0 \in \mathbb{R}$, let $\Phi\colon [0,\infty) \times [\theta_0, \theta_0 + 2\pi) \to \mathbb{R}^2$ be the continuous mapping given by $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$.

(a) Prove that Φ maps the set $(0,\infty) \times [\theta_0, \theta_0 + 2\pi)$ bijectively onto the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ (as claimed in Example 7.2.8).

(b) Prove that Φ maps the domain $(0,\infty) \times (\theta_0, \theta_0 + 2\pi)$ diffeomorphically onto the complement $\mathbb{R}^2 \setminus P$ of the ray $P = \{(r\cos\theta_0, r\sin\theta_0) \mid r \geq 0\}$ in $\mathbb{R}^2$.

7.2.6 Verify that the real logarithmic and exponential functions (as defined in the remark following Example 7.2.8) satisfy

$$\log(xy) = \log x + \log y \quad \forall x, y \in \mathbb{R}_{>0} \quad \text{and} \quad e^{x+y} = e^x \cdot e^y \quad \forall x, y \in \mathbb{R}.$$

7.3 $\mathcal{C}^\infty$ Approximation

A convenient way in which to obtain a $\mathcal{C}^\infty$ approximation of a locally integrable function is to form a *mollifier* as follows:

Lemma 7.3.1 (Friedrichs) *Let κ be a nonnegative $\mathcal{C}^\infty$ function with compact support in the unit ball $B(0;1)$ in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \kappa \, d\lambda = 1$, and let Ω be an open subset of $\mathbb{R}^n$. For each function $u \in L^1_{\mathrm{loc}}(\Omega)$ and for each $\delta > 0$, let $\kappa^\delta(x) \equiv \delta^{-n}\kappa(x/\delta)$ for each point $x \in \mathbb{R}^n$, let $\Omega_\delta \equiv \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega) > \delta\}$, and let*

$$\begin{aligned} u_\delta(x) &\equiv \int_\Omega u(y)\kappa^\delta(x-y)\, d\lambda(y) \\ &= \int_{B(0;\delta)} u(x-y)\kappa^\delta(y)\, d\lambda(y) \\ &= \int_{B(0;1)} u(x-\delta y)\kappa(y)\, d\lambda(y) \end{aligned}$$

for every $x \in \Omega_\delta$. Then, for every function $u \in L^1_{\mathrm{loc}}(\Omega)$, we have the following:

(a) *The function u_δ belongs to $\mathcal{C}^\infty(\Omega_\delta)$ for every $\delta > 0$.*

(b) *For every compact set $K \subset \Omega$, $\|u_\delta - u\|_{L^1(K)} \to 0$ as $\delta \to 0^+$ (i.e., $u_\delta \to u$ in $L^1_{\mathrm{loc}}(\Omega)$). Moreover, if $u \in \mathcal{C}^0(\Omega)$, then $u_\delta \to u$ uniformly on K.*

Remarks 1. A stronger version of the lemma (which is applied in Chap. 6) appears in Sect. 11.3 (see Lemma 11.3.1).

2. An example of such a function κ is given by

$$\kappa(x) = \begin{cases} C^{-1}\exp(-1/(1-4\|x\|^2)) & \text{if } \|x\| < 1/2, \\ 0 & \text{if } \|x\| \geq 1/2, \end{cases}$$

where

$$C = \int_{B(0;1/2)} e^{-1/(1-4\|x\|^2)}\, d\lambda(x).$$

3. The family of functions $\{\kappa^\delta\}$ is called a (*positive*) *mollifier* (or an *approximation of the identity*), and $\{u_\delta\}$ is called a *mollification* (or a *regularization*) of u. The *convolution* of two (suitable) functions f and g on $\mathbb{R}^n$ is the function

$$f * g\colon x \mapsto \int_{\mathbb{R}^n} f(x-y)g(y)\, d\lambda(y).$$

Thus, in the above lemma, u_δ is the convolution of κ^δ and an extension of u to $\mathbb{R}^n$.

Proof of Lemma 7.3.1 Part (a) follows from differentiation past the integral (Proposition 7.2.5). For the proof of (b), we fix a compact set $K \subset \Omega$ and we set

$$K^\delta \equiv \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, K) \leq \delta\}$$

for each $\delta > 0$. Fixing δ_0 with $0 < \delta_0 < \operatorname{dist}(K, \mathbb{R}^n \setminus \Omega)$, we get $K^{\delta_0} \subset \Omega$. Suppose first that u is continuous. If for $0 < \delta \leq \delta_0$, we set

$$M_\delta \equiv \sup\{|u(x) - u(y)| \mid x, y \in K^{\delta_0},\ |x-y| < \delta\},$$

then by uniform continuity, we have $M_\delta \to 0$ as $\delta \to 0^+$. For each $x \in K$, we have

$$\begin{aligned} |u_\delta(x) - u(x)| &= \left| \int_{B(0;1)} u(x-\delta y)\kappa(y)\, d\lambda(y) - \int_{B(0;1)} u(x)\kappa(y)\, d\lambda(y) \right| \\ &\leq \int_{B(0;1)} |u(x-\delta y) - u(x)| \cdot \kappa(y)\, d\lambda(y) \leq M_\delta. \end{aligned}$$

Thus $u_\delta \to u$ uniformly on K as $\delta \to 0^+$, and hence in particular, $\|u_\delta - u\|_{L^1(K)} \to 0$ as $\delta \to 0^+$.

For a general $u \in L^1_{\text{loc}}(\Omega)$, given $\epsilon > 0$, we may choose a continuous function v on Ω such that $\|v - u\|_{L^1(K^{\delta_0})} < \epsilon/3$ (see Exercise 7.1.4). For each $\delta \in (0, \delta_0)$, Fubini's theorem (see Theorem 7.1.10 and Exercise 7.3.1) then implies that

$$\|v_\delta - u_\delta\|_{L^1(K)} = \int_K \left| \int_{B(0;1)} (v(x-\delta y) - u(x-\delta y))\kappa(y)\, d\lambda(y) \right| d\lambda(x)$$

$$
\begin{aligned}
&\leq \int_K \int_{B(0;1)} |(v(x-\delta y) - u(x-\delta y))\kappa(y)|\, d\lambda(y)\, d\lambda(x) \\
&= \int_{B(0;1)} \left[\int_K |v(x-\delta y) - u(x-\delta y)| d\lambda(x) \right] \kappa(y)\, d\lambda(y) \\
&= \int_{B(0;1)} \left[\int_{K+(-\delta y)} |v(x) - u(x)|\, d\lambda(x) \right] \kappa(y)\, d\lambda(y) \\
&\leq \int_{B(0;1)} \left[\int_{K^{\delta_0}} |v(x) - u(x)|\, d\lambda(x) \right] \kappa(y)\, d\lambda(y) \\
&= \|v - u\|_{L^1(K^{\delta_0})} < \epsilon/3.
\end{aligned}
$$

On the other hand, for $\delta > 0$ sufficiently small, we have $\|v_\delta - v\|_{L^1(K)} < \epsilon/3$. Hence

$$\|u_\delta - u\|_{L^1(K)} \leq \|u_\delta - v_\delta\|_{L^1(K)} + \|v_\delta - v\|_{L^1(K)} + \|v - u\|_{L^1(K)} < \epsilon.$$

Thus (b) is proved. □

One consequence of the Friedrichs lemma is the following uniqueness property:

Lemma 7.3.2 *If Ω is an open subset of $\mathbb{R}^n$, $u, v \in L^1_{\text{loc}}(\Omega)$, and*

$$\int_\Omega u\varphi\, d\lambda = \int_\Omega v\varphi\, d\lambda \quad \forall \varphi \in \mathcal{D}(\Omega),$$

then $u = v$ almost everywhere in Ω.

Proof We may assume without loss of generality that $v \equiv 0$. In the notation of Lemma 7.3.1, we have $u_\delta \equiv 0$ on Ω_δ for every $\delta > 0$. On the other hand, for every compact set $K \subset \Omega$, we have $\|u_\delta - u\|_{L^1(K)} \to 0$, and therefore, for some sequence $\{\delta_\nu\}$, $0 = u_{\delta_\nu} \to u$ pointwise almost everywhere in K. Thus $u = 0$ almost everywhere in Ω. □

Exercises for Sect. 7.3

7.3.1 Verify that the function $(x, y) \mapsto |v(x - \delta y) - u(x - \delta y)|$ for $(x, y) \in K \times B(0; 1)$ in the proof of Lemma 7.3.1 is measurable (hence Fubini's theorem applies).

7.4 Differential Operators and Formal Adjoints

Definition 7.4.1 For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

(in particular, for $\alpha = (0, \ldots, 0)$, $(\partial/\partial x)^\alpha u = u$ for every function u). A *linear differential operator of order* $k \in \mathbb{Z}_{\geq 0}$ with coefficients $\{a_\alpha\}$ on an open set $\Omega \subset \mathbb{R}^n$ is a linear map A from $\mathcal{C}^k(\Omega)$ to the space of complex-valued functions on Ω given by

$$Au = \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n, |\alpha| \leq k} a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha u \quad \forall u \in \mathcal{C}^k(\Omega),$$

where a_α is a complex-valued function on Ω for each $\alpha \in (\mathbb{Z}_{\geq 0})^n$ with $|\alpha| \leq k$. For any open set $U \subset \Omega$, we also denote the linear differential operator

$$\sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n, |\alpha| \leq k} a_\alpha \restriction_U \left(\frac{\partial}{\partial x}\right)^\alpha : \mathcal{C}^k(U) \to \mathcal{C}^0(U)$$

by A, unless there is danger of confusion. We define the conjugate operator $\bar{A}$ by

$$\bar{A}u = \overline{A\bar{u}} = \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n,\, |\alpha| \leq k} \bar{a}_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha u \quad \forall u \in \mathcal{C}^k(\Omega).$$

Remark In practice, we usually work with simpler expressions obtained by allowing the multi-indices to be in $(\mathbb{Z}_{\geq 0})^l$ for $1 \leq l \leq n$. For example, if A is a second-order linear differential operator, then we may write

$$A = \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + c,$$

and refer to $\{a_{ij}\}$, $\{b_i\}$, and c as the coefficients of A. Observe also that by replacing $\{a_{ij}\}$ with $\{\frac{1}{2}a_{ij}\}$, we may assume that $a_{ij} = a_{ji}$ for all i and j (i.e., that the matrix (a_{ij}) is symmetric).

One may consider an extension of a differential operator to a space of functions that are not necessarily differentiable in the following way:

Definition 7.4.2 Let A be a linear differential operator of order k with (complex) coefficients $\{a_\alpha\}$ on an open set $\Omega \subset \mathbb{R}^n$ such that $a_\alpha \in \mathcal{C}^{|\alpha|}(\Omega)$ for each multi-index α.

(a) The *formal transpose of* A is the linear differential operator ${}^tA$ of order k given by

$${}^tAu \equiv \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n,\, |\alpha| \leq k} (-1)^{|\alpha|} \left(\frac{\partial}{\partial x}\right)^\alpha [a_\alpha \cdot u] \quad \forall u \in \mathcal{C}^k(\Omega).$$

(b) The *formal adjoint of* A is the linear differential operator A^* of order k given by

$$A^*u \equiv {}^t\bar{A}u = \sum_{\alpha\in(\mathbb{Z}_{\geq 0})^n,\ |\alpha|\leq k} (-1)^{|\alpha|}\left(\frac{\partial}{\partial x}\right)^{\alpha}[\bar{a}_\alpha \cdot u] \quad \forall u \in \mathcal{C}^k(\Omega).$$

(c) For $u, v \in L^1_{\mathrm{loc}}(\Omega)$, we say that Au is equal to v in the *distributional* (or *weak*) *sense*, and we write $A_{\mathrm{distr}}u = v$, if

$$\int_\Omega u \cdot \overline{A^*\varphi}\, d\lambda = \int_\Omega v \cdot \bar{\varphi}\, d\lambda \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Equivalently,

$$\int_\Omega u \cdot {}^tA\varphi\, d\lambda = \int_\Omega v \cdot \varphi\, d\lambda \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Remarks 1. The function $A_{\mathrm{distr}}u$ is unique (if it exists) by Lemma 7.3.2.

2. For any function $u \in L^1_{\mathrm{loc}}(\Omega)$, $A_{\mathrm{distr}}u$ *always* exists in a more general sense, namely, as the linear functional $\varphi \mapsto \int_\Omega u \cdot {}^tA\varphi\, d\lambda$ on $\mathcal{D}(\Omega)$ (see, for example, [Rud2]). For this reason, to say $A_{\mathrm{distr}}u \in L^1_{\mathrm{loc}}(\Omega)$ will mean that $A_{\mathrm{distr}}u$ exists as a function in $L^1_{\mathrm{loc}}(\Omega)$.

Lemma 7.4.3 *Let A and B be linear differential operators with $\mathcal{C}^\infty$ coefficients on an open set $\Omega \subset \mathbb{R}^n$, and let k be the order of A.*

(a) *If $u \in \mathcal{C}^k(\Omega)$ (or $k = 0$ and $u \in L^1_{\mathrm{loc}}(\Omega)$), then $A_{\mathrm{distr}}u = Au$.*
(b) *For $u, v \in L^1_{\mathrm{loc}}(\Omega)$, we have $A_{\mathrm{distr}}u = v$ in Ω if and only if each point $p \in \Omega$ admits a neighborhood $U \subset \Omega$ with $A_{\mathrm{distr}}[u\restriction_U] = v\restriction_U$.*
(c) *We have ${}^t({}^tA) = A$, $(A^*)^* = A$, ${}^t(\zeta A + B) = \zeta\, {}^tA + {}^tB$, and $(\zeta A + B)^* = \bar{\zeta} A^* + B^*$ for each $\zeta \in \mathbb{C}$, ${}^t(AB) = {}^tB\, {}^tA$, and $(AB)^* = B^*A^*$.*
(d) *Let $u, v \in L^1_{\mathrm{loc}}(\Omega)$ and let $\zeta \in \mathbb{C}$. Then $(\zeta A + B)_{\mathrm{distr}}u = \zeta A_{\mathrm{distr}}u + B_{\mathrm{distr}}u$, provided $A_{\mathrm{distr}}u, B_{\mathrm{distr}}u \in L^1_{\mathrm{loc}}(\Omega)$; and $A_{\mathrm{distr}}(\zeta u + v) = \zeta A_{\mathrm{distr}}u + A_{\mathrm{distr}}v$ provided $A_{\mathrm{distr}}u, A_{\mathrm{distr}}v \in L^1_{\mathrm{loc}}(\Omega)$.*
(e) *Suppose $u, B_{\mathrm{distr}}u, v \in L^1_{\mathrm{loc}}(\Omega)$. Then we have $(AB)_{\mathrm{distr}}u = v$ if and only if $A_{\mathrm{distr}}[B_{\mathrm{distr}}u] = v$.*
(f) *Suppose $u, A_{\mathrm{distr}}u \in L^1_{\mathrm{loc}}(\Omega)$, $k = 1$, the 0th-order term of A vanishes, and $\rho \in \mathcal{C}^\infty(\Omega)$. Then $A_{\mathrm{distr}}[\rho u] = \rho A_{\mathrm{distr}}u + uA\rho$.*

Proof Part (a) is trivial for $k = 0$, and it follows easily from integration by parts in each variable for $k > 0$. The details are left to the reader (see Exercise 7.4.1).

For the proof of (b), suppose that $u, v \in L^1_{\mathrm{loc}}(\Omega)$ and each point admits a neighborhood in which Au is equal to v in the distributional sense. Then, given a function $\varphi \in \mathcal{D}(\Omega)$, we may choose finitely many $\mathcal{C}^\infty$ functions $\{\eta_\nu\}_{\nu=1}^m$ such that $\sum \eta_\nu \equiv 1$ on $\operatorname{supp}\varphi$ and such that for each ν, $\operatorname{supp}\eta_\nu$ lies in an open subset U_ν of Ω on which Au is equal to v in the distributional sense (see, for example, Sect. 9.3). Hence

$$\int_\Omega u \cdot \overline{A^*\varphi}\, d\lambda = \sum_\nu \int_{U_\nu} u \cdot \overline{A^*[\eta_\nu \varphi]}\, d\lambda = \sum_\nu \int_{U_\nu} v \cdot \overline{\eta_\nu \varphi}\, d\lambda = \int_\Omega v\bar{\varphi}\, d\lambda.$$

The converse is trivial.

For (c), we observe that if $\varphi, \psi \in \mathcal{D}(\Omega)$, then

$$\int_\Omega \psi \cdot [A\varphi]\, d\lambda = \int_\Omega [{}^tA\psi] \cdot \varphi\, d\lambda = \int_\Omega \psi \cdot {}^t({}^tA)\varphi\, d\lambda,$$

and

$$\begin{aligned}\int_\Omega \psi \cdot [{}^t(AB)\varphi]\, d\lambda &= \int_\Omega [(AB)\psi] \cdot \varphi\, d\lambda \\ &= \int_\Omega [B\psi] \cdot [{}^tA\varphi]\, d\lambda \\ &= \int_\Omega \psi \cdot [{}^tB\, {}^tA\varphi]\, d\lambda.\end{aligned}$$

Lemma 7.3.2 now implies that ${}^t({}^tA)\varphi = A\varphi$ and ${}^t(AB)\varphi = {}^tB\, {}^tA\varphi$. For any function $u \in \mathcal{C}^\infty(\Omega)$ and any point $p \in \Omega$, there exists a function $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi \equiv u$ near p, so it follows that ${}^t({}^tA) = A$ and ${}^t(AB) = {}^tB\, {}^tA$. The remaining claims in (c) are easy to verify directly.

The proof of (d) is left to the reader (see Exercise 7.4.1).

For the proof of part (e), let $u \in L^1_{\mathrm{loc}}(\Omega)$ with $B_{\mathrm{distr}}u \in L^1_{\mathrm{loc}}(\Omega)$, and let $\varphi \in \mathcal{D}(\Omega)$. Then

$$\int_\Omega u \cdot [{}^t(AB)\varphi]\, d\lambda = \int_\Omega u \cdot [{}^tB\, {}^tA\varphi]\, d\lambda = \int_\Omega [B_{\mathrm{distr}}u] \cdot {}^tA\varphi\, d\lambda.$$

If $(AB)_{\mathrm{distr}}u \in L^1_{\mathrm{loc}}(\Omega)$, then the first expression is equal to

$$\int_\Omega [(AB)_{\mathrm{distr}}u] \cdot \varphi\, d\lambda,$$

and comparison with the last expression gives $A_{\mathrm{distr}}B_{\mathrm{distr}}u = (AB)_{\mathrm{distr}}u$. Similarly, if $A_{\mathrm{distr}}B_{\mathrm{distr}}u \in L^1_{\mathrm{loc}}(\Omega)$, then the last expression is equal to

$$\int_\Omega [A_{\mathrm{distr}}B_{\mathrm{distr}}u] \cdot \varphi\, d\lambda,$$

and comparison with the first expression gives $(AB)_{\mathrm{distr}}u = A_{\mathrm{distr}}B_{\mathrm{distr}}u$.

For the proof of (f), we assume that $k = 1$, $A_{\mathrm{distr}}u = v \in L^1_{\mathrm{loc}}(\Omega)$, the 0th-order term of A vanishes, and $\rho \in \mathcal{C}^\infty(\Omega)$. An easy computation shows that for $\varphi \in \mathcal{D}(\Omega)$, we have ${}^t A(\rho\varphi) = \rho \cdot ({}^t A\varphi) - (A\rho) \cdot \varphi$ (thus $A^*(\rho\varphi) = \rho \cdot (A^*\varphi) - (\bar{A}\rho) \cdot \varphi$). Hence

$$\int_\Omega \rho u \cdot ({}^t A\varphi)\, d\lambda = \int_\Omega u \cdot {}^t A(\rho\varphi)\, d\lambda + \int_\Omega u \cdot (A\rho) \cdot \varphi\, d\lambda = \int_\Omega [\rho v + u \cdot A\rho] \cdot \varphi\, d\lambda,$$

and (f) follows. □

Remarks 1. The terminology in Definition 7.4.2 is motivated by the notion of the adjoint of a linear operator on a Hilbert space (see, for example, [Rud2]) and the equality

$$\langle \varphi, A^*\psi \rangle_{L^2(\Omega)} = \langle A\varphi, \psi \rangle_{L^2(\Omega)} \quad \forall \varphi, \psi \in \mathcal{D}(\Omega)$$

(see Sect. 7.5) provided by Lemma 7.4.3.

2. Parts (a) and (b) of Lemma 9.8.3 hold even if A has $\mathcal{C}^k$ coefficients.

Lemma 7.4.4 *Let κ be a nonnegative $\mathcal{C}^\infty$ function with compact support in the unit ball $B(0;1)$ in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \kappa\, d\lambda = 1$, and let Ω be an open subset of $\mathbb{R}^n$. For each $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\delta > 0$, let $\kappa^\delta(x) \equiv \delta^{-n}\kappa(x/\delta)$ for each point $x \in \mathbb{R}^n$, let $\Omega_\delta \equiv \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega) > \delta\}$, and let*

$$\begin{aligned} u_\delta(x) &\equiv \int_\Omega u(y)\kappa^\delta(x-y)\, d\lambda(y) = \int_{B(0;\delta)} u(x-y)\kappa^\delta(y)\, d\lambda(y) \\ &= \int_{B(0;1)} u(x-\delta y)\kappa(y)\, d\lambda(y) \end{aligned}$$

for every $x \in \Omega_\delta$. If A is a constant-coefficient linear differential operator on $\mathbb{R}^n$, and $u \in L^1_{\mathrm{loc}}(\Omega)$ with $A_{\mathrm{distr}}u \in L^1_{\mathrm{loc}}(\Omega)$, then $A[u_\delta] = [A_{\mathrm{distr}}u]_\delta$ in Ω_δ for every $\delta > 0$. In particular, for every compact set $K \subset \Omega$, we have

$$\|A(u_\delta) - A_{\mathrm{distr}}u\|_{L^1(K)} \to 0 \quad \text{as } \delta \to 0^+.$$

Proof Given $\delta > 0$ and a function $\varphi \in \mathcal{D}(\Omega_\delta)$, we have, by Fubini's theorem (see Theorem 7.1.10 and Exercise 7.4.2),

$$\begin{aligned} &\int_{\Omega_\delta} u_\delta(x)[{}^t A\varphi](x)\, d\lambda(x) \\ &\quad = \int_{\Omega_\delta} \left[\int_{B(0;\delta)} u(x-y)\kappa^\delta(y)\, d\lambda(y) \right] \cdot [{}^t A\varphi](x)\, d\lambda(x) \\ &\quad = \int_{B(0;\delta)} \left[\int_{\Omega_\delta} u(x-y) \cdot [{}^t A\varphi](x)\, d\lambda(x) \right] \kappa^\delta(y)\, d\lambda(y) \\ &\quad = \int_{B(0;\delta)} \left[\int_{\Omega_\delta + (-y)} u(x) \cdot [{}^t A\varphi](x+y)\, d\lambda(x) \right] \kappa^\delta(y)\, d\lambda(y) \\ &\quad = \int_{B(0;\delta)} \left[\int_{\Omega_\delta + (-y)} u(x) \cdot [{}^t A(\varphi(\cdot + y))](x)\, d\lambda(x) \right] \kappa^\delta(y)\, d\lambda(y) \end{aligned}$$

$$= \int_{B(0;\delta)} \left[\int_{\Omega_\delta + (-y)} [A_{\text{distr}} u](x) \varphi(x+y) \, d\lambda(x) \right] \kappa^\delta(y) \, d\lambda(y)$$

$$= \int_{\Omega_\delta} [A_{\text{distr}} u]_\delta(x) \varphi(x) \, d\lambda(x)$$

(that A has constant coefficients gives the fourth equality). Therefore, $A(u_\delta) = A_{\text{distr}}(u_\delta) = [A_{\text{distr}} u]_\delta$ (by the uniqueness property provided by Lemma 7.3.2). □

Exercises for Sect. 7.4

7.4.1 Prove parts (a) and (d) of Lemma 7.4.3.

7.4.2 Verify that the function $(x, y) \mapsto u(x-y)\kappa^\delta(y)[{}^t A\varphi](x)$ for $(x, y) \in \Omega_\delta \times B(0;\delta)$ in the proof of Lemma 7.4.4 is integrable (hence Fubini's theorem applies).

7.5 Hilbert Spaces

Hilbert spaces (or *complete Hermitian inner product spaces*) are (possibly infinite-dimensional) vector spaces in which one has many of the useful geometric tools (for example, orthogonal projection) that one has in $\mathbb{R}^n$. They play a crucial role in this book. In this section, we recall some of the basic elements of Hilbert space theory. Proofs of most of the relevant facts required in this book are either provided or outlined in the exercises. Further facts concerning weak sequential compactness, which are applied only in Chap. 6, are considered in Sect. 7.6.

We first recall the following:

Definition 7.5.1 Let $\mathcal{V}$ be a complex vector space.

(a) A *Hermitian inner product* (or simply an *inner product*) on $\mathcal{V}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ such that for all $u, v, w \in \mathcal{V}$ and $\zeta \in \mathbb{C}$, we have
  (i) $\langle \zeta u + v, w \rangle = \zeta \langle u, w \rangle + \langle v, w \rangle$,
  (ii) $\langle u, v \rangle = \overline{\langle v, u \rangle}$, and
  (iii) $\langle v, v \rangle > 0$ if $v \neq 0$.
  The pair $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ is called a *Hermitian inner product space* (or simply an *inner product space*).
(b) The *norm* associated to $\langle \cdot, \cdot \rangle$ is the function $v \mapsto \|v\| \equiv \sqrt{\langle v, v \rangle}$.
(c) Two vectors u and v are *orthogonal* with respect to $\langle \cdot, \cdot \rangle$ if $\langle u, v \rangle = 0$ (equivalently, $\langle v, u \rangle = 0$). We write $u \perp v$. If $u \perp v$ and $\|u\| = \|v\| = 1$, then u and v are *orthonormal*. A nonempty set $S \subset \mathcal{V}$ is an *orthonormal set* if each pair of distinct vectors in S are orthonormal.
(d) For every nonempty set $S \subset \mathcal{V}$, we define $S^\perp \equiv \{v \in \mathcal{V} \mid v \perp u \ \forall u \in S\}$.

Remark Similarly, a *real inner product* (or simply an *inner product*) on a real vector space $\mathcal{V}$ is a real symmetric bilinear form g on $\mathcal{V}$ such that $g(v, v) > 0$ for each $v \in \mathcal{V} \setminus \{0\}$.

Theorem 7.5.2 *For any Hermitian inner product space* $(\mathcal{V}, \langle\cdot,\cdot\rangle)$, *we have the following*:

(a) Law of cosines. *We have* $\|u-v\|^2 = \|u\|^2 - 2\,\mathrm{Re}\langle u, v\rangle + \|v\|^2$ *for all* $u, v \in \mathcal{V}$.
(b) Pythagorean theorem. *We have* $\|u-v\|^2 = \|u+v\|^2 = \|u\|^2 + \|v\|^2$ *for every pair of orthogonal vectors* $u, v \in \mathcal{V}$.
(c) Parallelogram law. *We have* $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$ *for all* $u, v \in \mathcal{V}$.
(d) Schwarz inequality. *We have* $|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$ *for all* $u, v \in \mathcal{V}$.
(e) $\|\cdot\|$ *is a norm on* $\mathcal{V}$.
(f) *The function* $\langle\cdot,\cdot\rangle$ *is continuous*; *that is, if* $\{u_j\}$ *and* $\{v_j\}$ *are sequences converging to* u *and* v, *respectively, in* $(\mathcal{V}, \|\cdot\|)$, *then* $\langle u_j, v_j\rangle \to \langle u, v\rangle$.
(g) *For any set* $S \subset \mathcal{V}$, $S^\perp$ *is a closed subspace of* $(\mathcal{V}, \|\cdot\|)$, *and* $\overline{S}^\perp = S^\perp$, *where* $\overline{S} = \mathrm{cl}(S)$ *is the closure of* S. *Moreover, if* S *is a subspace of* $\mathcal{V}$, *then* $S \cap S^\perp = \{0\}$.

Proof The proofs of (a)–(c) are left to the reader (see Exercise 7.5.1). It is easy to see that $\|0\| = 0$, that $\|\zeta v\| = |\zeta| \cdot \|v\|$ for all $v \in \mathcal{V}$ and $\zeta \in \mathbb{C}$, and that $\|v\| > 0$ if $v \in \mathcal{V} \setminus \{0\}$. For the proof of (d), suppose $u, v \in \mathcal{V} \setminus \{0\}$. Setting $u' = u/\|u\|$ and $v' = v/\|v\|$, and fixing $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ and $\zeta\langle u', v'\rangle = |\langle u', v'\rangle|$, we get, for each $t \in \mathbb{R}$,

$$0 \leq \|\zeta u' - tv'\|^2 = 1 - 2t|\langle u', v'\rangle| + t^2.$$

The right-hand side is a quadratic polynomial in t that attains its minimum value at $t = |\langle u', v'\rangle|$. Substituting this value for t, we get $0 \leq 1 - 2|\langle u', v'\rangle|^2 + |\langle u', v'\rangle|^2$, and hence

$$\frac{|\langle u, v\rangle|}{\|u\| \cdot \|v\|} = |\langle u', v'\rangle| \leq 1.$$

Part (d) now follows.

For the proof of (e), it remains to verify the triangle inequality. Given $u, v \in \mathcal{V}$, we have

$$\begin{aligned}\|u+v\|^2 &= \|u\|^2 + 2\,\mathrm{Re}\langle u, v\rangle + \|v\|^2 \leq \|u\|^2 + 2|\langle u, v\rangle| + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2,\end{aligned}$$

and the triangle inequality follows.

For the proof of (f), suppose $u_j \to u$ and $v_j \to v$ in $\mathcal{V}$. Then

$$\begin{aligned}|\langle u_j, v_j\rangle - \langle u, v\rangle| &\leq |\langle u_j, v_j\rangle - \langle u_j, v\rangle| + |\langle u_j, v\rangle - \langle u, v\rangle| \\ &= |\langle u_j, v_j - v\rangle| + |\langle u_j - u, v\rangle| \\ &\leq \|u_j\| \cdot \|v_j - v\| + \|u_j - u\| \cdot \|v\| \to \|u\| \cdot 0 + 0 \cdot \|v\| = 0,\end{aligned}$$

and hence $\langle u_j, v_j\rangle \to \langle u, v\rangle$.

For the proof of (g), suppose $\{u_j\}$ is a sequence in $S^\perp$ converging to $u \in \mathcal{V}$. Then, for every vector $v \in S$, we have $0 = \langle u_j, v\rangle \to \langle u, v\rangle$. Thus $\langle u, v\rangle = 0$, and hence $u \in S^\perp$. Therefore, $S^\perp$ is closed. Since $S \subset \overline{S}$, we also have $S^\perp \supset \overline{S}^\perp$. Conversely, if $u \in S^\perp$ and $v \in \overline{S}$, then we may choose a sequence $\{v_j\}$ in S converging to v. Hence $0 = \langle u, v_j\rangle \to \langle u, v\rangle$, and it follows that $u \in \overline{S}^\perp$. Thus $S^\perp = \overline{S}^\perp$. It is easy to verify that $S^\perp$ is a subspace. Finally, if S is a (vector) subspace of $\mathcal{V}$ and $v \in S \cap S^\perp$, then $\langle v, v\rangle = 0$, so $v = 0$. Thus (g) is proved. $\square$

Definition 7.5.3 A Hilbert space is a Hermitian inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ that is complete with respect to the corresponding norm $\|\cdot\|$.

Example 7.5.4 Given a measure space (X, μ), we get the Hilbert space $L^2(X, \mu)$ with inner product and norm

$$\langle u, v\rangle_{L^2(X,\mu)} \equiv \int_X u\bar{v}\,d\mu \quad \text{and} \quad \|u\|_{L^2(X,\mu)} = \sqrt{\langle u, u\rangle_{L^2(X,\mu)}} \quad \forall u, v \in L^2(X, \mu).$$

For the inequality $|uv| \le (1/2)(|u|^2 + |v|^2)$ implies that a product of two L^2 functions is integrable (alternatively, one may use Hölder's inequality). It is then easy to check that the above is an inner product. Completeness is the case $p = 2$ of Theorem 7.1.9. On the other hand, for the related Hilbert spaces considered in this book, completeness will be proved directly (see Proposition 2.6.3 and Proposition 3.6.4).

Example 7.5.5 For any set X, $\ell^2(X)$ denotes the L^2 space associated to X together with the counting measure. The inner product and norm are then given by

$$\langle u, v\rangle_{\ell^2(X)} = \sum_{x\in X} u(x)\overline{v(x)} \quad \text{and} \quad \|u\|_{\ell^2(X)} = \sqrt{\sum_{x\in X} |u(x)|^2} \quad \forall u, v \in \ell^2(X).$$

In particular, taking $X = \mathbb{Z}_{>0}$, we get

$$\langle u, v\rangle_{\ell^2(\mathbb{Z}_{>0})} = \sum_{n=1}^{\infty} u(n)\overline{v(n)} \quad \text{and} \quad \|u\|_{\ell^2(n)} = \sqrt{\sum_{n=1}^{\infty} |u(n)|^2} \quad \forall u, v \in \ell^2(\mathbb{Z}_{>0}).$$

Theorem 7.5.6 *If $\mathcal{V}$ is a closed subspace of a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, then we have the direct sum decomposition $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^\perp$. That is, for each vector $u \in \mathcal{H}$, there are unique vectors $v \in \mathcal{V}$ and $w \in \mathcal{V}^\perp$ such that $u = v + w$. Moreover, we have $\|u - s\| > \|w\|$ for all $s \in \mathcal{V} \setminus \{v\}$.*

The above decomposition $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^\perp$ is called the associated *orthogonal decomposition*. The surjective linear mapping $u \mapsto v \in \mathcal{V}$ (with $w = u - v \in \mathcal{V}^\perp$) is called the *orthogonal projection* of $\mathcal{H}$ onto $\mathcal{V}$.

Proof of Theorem 7.5.6 Given a vector $u \in \mathcal{H}$, we may choose a sequence $\{v_j\}$ in $\mathcal{V}$ such that

$$\|u - v_j\| \to r \equiv \inf_{v \in \mathcal{V}} \|u - v\| \in [0, \infty).$$

According to the parallelogram law (see Theorem 7.5.2),

$$\begin{aligned}
\|v_i - v_j\|^2 &= \|(u - v_i) - (u - v_j)\|^2 \\
&= 2\|u - v_i\|^2 + 2\|u - v_j\|^2 - \|(u - v_i) + (u - v_j)\|^2 \\
&= 2\|u - v_i\|^2 + 2\|u - v_j\|^2 - 4\|u - (1/2)(v_i + v_j)\|^2 \\
&\le 2\|u - v_i\|^2 + 2\|u - v_j\|^2 - 4r^2.
\end{aligned}$$

Since we can make the last expression arbitrarily small by requiring that i and j be sufficiently large, $\{v_j\}$ is a Cauchy sequence, and therefore, $\{v_j\}$ converges to some vector $v \in \mathcal{V}$.

Setting $w \equiv u - v$, we see that for each vector $s \in \mathcal{V}$, $\|w\| = r \le \|u - s\|$. Fixing $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ and $\zeta \langle w, s \rangle = |\langle w, s \rangle|$, we get, for each $t \in \mathbb{R}$,

$$\|u - (v + t\zeta^{-1}s)\|^2 = \|\zeta w - ts\|^2 = \|w\|^2 - 2t|\langle w, s \rangle| + t^2\|s\|^2.$$

The quadratic polynomial on the right-hand side attains its minimum value at $t = 0$, so we must have $\langle w, s \rangle = 0$. Thus $w \in \mathcal{V}^\perp$. The decomposition $u = v + w$ is unique, since $\mathcal{V} \cap \mathcal{V}^\perp = \{0\}$. In particular, if $s \in \mathcal{V}$ with $\|u - s\| = r$, then the minimizing sequence given by $v'_j = s$ for all j must converge to v; that is, $s = v$. □

Corollary 7.5.7 *For any subspace $\mathcal{W}$ of a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, we have the following*:

(a) $(\mathcal{W}^\perp)^\perp = \overline{\mathcal{W}}$.
(b) *If $\mathcal{V}$ is a subspace of $\mathcal{H}$ with $\mathcal{W} \cup \mathcal{W}^\perp \subset \mathcal{V}$, then $\mathcal{W}$ is closed in $\mathcal{V}$ (i.e., $\overline{\mathcal{W}} \cap \mathcal{V} = \mathcal{W}$) if and only if $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^\perp$.*

Proof We have $\overline{\mathcal{W}} \perp \overline{\mathcal{W}}^\perp$ and $\overline{\mathcal{W}}^\perp = \mathcal{W}^\perp$, so $\overline{\mathcal{W}} \subset (\mathcal{W}^\perp)^\perp$. Conversely, if $u \in (\mathcal{W}^\perp)^\perp$, then we have $u = w + v$ with $w \in \overline{\mathcal{W}}$ and $v \in \mathcal{W}^\perp$. Subtracting, we get $v = u - w \in (\mathcal{W}^\perp)^\perp \cap \mathcal{W}^\perp = \{0\}$, so $u \in \overline{\mathcal{W}}$. Thus (a) is proved.

Suppose $\mathcal{W}, \mathcal{W}^\perp \subset \mathcal{V}$ as in (b). Given $u \in \mathcal{V}$, we have $u = w + v$, where $w \in \overline{\mathcal{W}}$ and $v \in \overline{\mathcal{W}}^\perp = \mathcal{W}^\perp \subset \mathcal{V}$. In particular, $w = u - v \in \overline{\mathcal{W}} \cap \mathcal{V}$. Thus, if $\mathcal{W}$ is closed in $\mathcal{V}$, then $w \in \mathcal{W}$. Conversely, if $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^\perp$ and $u \in \overline{\mathcal{W}} \cap \mathcal{V}$, then writing $u = w + v$ with $w \in \mathcal{W}$ and $v \in \mathcal{W}^\perp = \overline{\mathcal{W}}^\perp$, we get $v = u - w \in \overline{\mathcal{W}} \cap \overline{\mathcal{W}}^\perp = \{0\}$. Thus $u = w \in \mathcal{W}$ and hence $\overline{\mathcal{W}} \cap \mathcal{V} = \mathcal{W}$. □

Remark Theorem 7.5.6 and Corollary 7.5.7 do not hold in general for an inner product space (see Exercise 7.5.2).

Definition 7.5.8 Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A *bounded linear functional* α on $\mathcal{V}$ is a linear functional $\alpha\colon \mathcal{V} \to \mathbb{F}$ (i.e., an $\mathbb{F}$-valued linear map) for which the *norm*

$$\|\alpha\| \equiv \inf\{R \in [0, \infty] \mid |\alpha(v)| \leq R\|v\| \quad \forall v \in \mathcal{V}\}$$

is finite. The set of all bounded linear functionals on $\mathcal{V}$ is called the (*norm*) *dual space* of $\mathcal{V}$ and is denoted by $\mathcal{V}^*$.

Remark On a *finite-dimensional* normed vector space $\mathcal{V}$, *every* linear functional is bounded (see, for example, [Fol] or [Rud1]). Thus the *algebraic* dual space (see Sect. 8.1) is equal to the norm dual space of $\mathcal{V}$ if $\dim \mathcal{V} < \infty$.

Proposition 7.5.9 *For any normed vector space* $(\mathcal{V}, \|\cdot\|)$ *with* $\mathcal{V} \neq \{0\}$, *we have the following*:

(a) *For each* $\alpha \in \mathcal{V}^*$, $\|\alpha\| = \sup_{v \in \mathcal{V}\setminus\{0\}} |\alpha(v)|/\|v\| = \sup_{v\in\mathcal{V};\, \|v\|=1} |\alpha(v)|$.
(b) *The dual space* $(\mathcal{V}^*, \|\cdot\|)$ *is a Banach space*.

The proof is left to the reader (see Exercise 7.5.3).

The following important theorem allows one to identify a Hilbert space with its dual space up to conjugate linear isomorphism.

Theorem 7.5.10 *Let* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *be a Hilbert space. Then, for each vector* $v \in \mathcal{H}$, *the mapping* $\mathcal{H} \to \mathbb{C}$ *given by* $u \mapsto \langle u, v\rangle$ *is a bounded linear functional with norm equal to* $\|v\|$. *Conversely, given a bounded linear functional* α *on* $\mathcal{H}$, *there is a unique vector* $v \in \mathcal{H}$ *such that* $\alpha(u) = \langle u, v\rangle$ *for all* $u \in \mathcal{H}$ (*in particular*, $\|\alpha\| = \|v\|$).

Proof It follows immediately from the definition of an inner product that for each $v \in \mathcal{H}$, the mapping $\alpha\colon \mathcal{H} \to \mathbb{C}$ given by $u \mapsto \langle u, v\rangle$ is a linear functional. The Schwarz inequality implies that $|\alpha(u)| \leq \|v\|\|u\|$ for each vector $u \in \mathcal{H}$, so α is a bounded linear functional with $\|\alpha\| \leq \|v\|$. Setting $u = v$, we get $\|\alpha\| \geq \|v\|$, and hence we have equality. Observe also that if $w \in \mathcal{H}$ with $\langle\cdot, w\rangle = \alpha = \langle\cdot, v\rangle$, then setting $u = v - w$, we get

$$\|u\|^2 = \langle u, u\rangle = \langle u, v\rangle - \langle u, w\rangle = 0,$$

and hence $v = w$. In particular, this observation gives the uniqueness claim in the theorem.

Conversely, given an element $\alpha \in \mathcal{H}^*$, the kernel $\mathcal{V} \equiv \ker\alpha = \alpha^{-1}(0)$ is a *closed* subspace of $\mathcal{H}$ (see Exercise 7.5.4). If $\alpha \equiv 0$, then $\alpha = \langle\cdot, 0\rangle$. If $\alpha \neq 0$, then we may fix an element $w \in \mathcal{V}^\perp \setminus \{0\}$ and we may set $\zeta \equiv \overline{\alpha(w)}/\|w\|^2 \in \mathbb{C}\setminus\{0\}$. The vector $v \equiv \zeta w \in \mathcal{V}^\perp$ then satisfies

$$\alpha(v) = \zeta\alpha(w) = \frac{|\alpha(w)|^2}{\|w\|^2} = \langle v, v\rangle.$$

Given a vector $u \in \mathcal{H} = \mathcal{V} \oplus \mathcal{V}^\perp$, we have $u = r + s$ with $r \in \mathcal{V}$ and $s \in \mathcal{V}^\perp$. In particular,

$$s - \frac{\alpha(s)}{\|v\|^2} v \in \mathcal{V} \cap \mathcal{V}^\perp = \{0\},$$

and hence

$$\alpha(u) = \alpha(s) = \alpha(s) \frac{\langle v, v\rangle}{\|v\|^2} = \langle s, v\rangle = \langle u, v\rangle. \qquad \square$$

Theorem 7.5.11 (Hahn–Banach theorem) *If α is a bounded linear functional on a subspace $\mathcal{V}$ of a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ (that is, α is bounded with respect to the norm obtained by restriction to $\mathcal{V}$ of the norm on $\mathcal{H}$), then there exists a bounded linear functional β on $\mathcal{H}$ such that $\beta\restriction_{\mathcal{V}} = \alpha$ and $\|\beta\| = \|\alpha\|$.*

Proof We first observe that there exists a unique extension of α to a bounded linear functional on the closure $\overline{\mathcal{V}}$. For if $\{v_n\}$ is a sequence in $\mathcal{V}$ that converges to a vector v, then for all $m, n \in \mathbb{Z}_{>0}$,

$$|\alpha(v_m) - \alpha(v_n)| = |\alpha(v_m - v_n)| \le \|\alpha\| \|v_m - v_n\|.$$

It follows that the sequence $\{\alpha(v_n)\}$ in $\mathbb{C}$ is Cauchy and therefore convergent. Moreover, if $\{u_n\}$ is another sequence in $\mathcal{V}$ converging to v, then

$$|\alpha(u_n) - \alpha(v_n)| \le \|\alpha\| \|u_n - v_n\| \to 0.$$

Thus we get a well-defined mapping $\gamma : \overline{\mathcal{V}} \to \mathbb{C}$ by setting

$$\gamma(v) = \lim_{n\to\infty} \alpha(v_n),$$

for any $v \in \overline{\mathcal{V}}$ and any sequence $\{v_n\}$ in $\mathcal{V}$ converging to v. In particular, $\gamma\restriction_{\mathcal{V}} = \alpha$. The linearity properties of limits of sequences imply that γ is a linear functional, and since γ is an extension of α, we have $\|\gamma\| \ge \|\alpha\|$. On the other hand, for $v \in \overline{\mathcal{V}}$ and $\{v_n\}$ as above, we have

$$|\gamma(v)| = \lim_{n\to\infty} |\alpha(v_n)| \le \lim_{n\to\infty} \|\alpha\| \|v_n\| = \|\alpha\| \|v\|,$$

and hence $\|\gamma\| \le \|\alpha\|$. Thus γ is a bounded linear functional on $\overline{\mathcal{V}}$ and $\|\gamma\| = \|\alpha\|$.

Letting $\mathcal{P}$ be the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{V}}$, we get a linear functional $\beta \equiv \gamma \circ \mathcal{P} : \mathcal{H} \to \mathbb{C}$ with $\beta\restriction_{\overline{\mathcal{V}}} = \gamma$. In particular, $\|\beta\| \ge \|\gamma\| = \|\alpha\|$. Conversely, the Pythagorean theorem (see Theorem 7.5.2) gives

$$|\beta(v)| = |\gamma \circ \mathcal{P}(v)| \le \|\alpha\| \|\mathcal{P}v\| \le \|\alpha\| \|v\| \quad \forall v \in \mathcal{H},$$

and hence $\|\beta\| \le \|\alpha\|$. Thus $\beta \in \mathcal{H}^*$ and $\|\beta\| = \|\alpha\|$. $\square$

Remark More general versions of the Hahn–Banach theorem can be found in, for example, [Rud1] and [Rud2].

Exercises for Sect. 7.5

7.5.1 Prove parts (a)–(c) of Theorem 7.5.2.
7.5.2 Give examples that show that Theorem 7.5.6 and Corollary 7.5.7 do not hold in general for an inner product space.
7.5.3 Prove Proposition 7.5.9.
7.5.4 Show that the kernel of any element of the dual space of an inner product space is a closed subspace.
7.5.5 Suppose $\{v_n\}$ is a sequence in an inner product space $(\mathcal{V}, \langle\cdot,\cdot\rangle)$, $v \in \mathcal{V}$, and $\langle u, v_n\rangle \to \langle u, v\rangle$ for every $u \in \mathcal{V}$ (i.e., $\{v_n\}$ *converges weakly* to v). Prove that $\{v_n\}$ converges (strongly) to v (i.e., $\|v_n - v\| \to 0$) if and only $\|v_n\| \to \|v\|$.

7.6 Weak Sequential Compactness

This section is required for the discussion of integrability of almost complex structures in Chap. 6. Throughout this section, $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ denotes a Hilbert space. The main goal is the following:

Theorem 7.6.1 (Weak sequential compactness) *The closed unit ball in a Hilbert space* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *is weakly sequentially compact*; *that is*, *for every bounded sequence* $\{u_\nu\}$ *in* $\mathcal{H}$, *there is a subsequence* $\{u_{\nu_k}\}$ *and a vector* $u \in \mathcal{H}$ *such that* $\|u\| \le \sup_\nu \|u_\nu\|$ *and*

$$\langle u_{\nu_k}, v\rangle \to \langle u, v\rangle \quad \text{as } k \to \infty \quad \forall v \in \mathcal{H}.$$

Remark Conversely, by the uniform boundedness principle (see, for example, [Fol], [Rud1], or [Rud2]), a weakly convergent sequence in $\mathcal{H}$ (see Exercise 7.5.5) is bounded (we will not need this converse in this book).

For the proof, we use *complete orthonormal sets*, a fundamental object in Hilbert space theory.

Definition 7.6.2 An orthonormal set with dense span in $\mathcal{H}$ is called a *complete orthonormal set* (or a *complete orthonormal basis*).

Theorem 7.6.3 *If* $\{u_\nu\}$ *is a sequence in* $\mathcal{H} \setminus \{0\}$, *then* $\mathcal{V} \equiv \operatorname{Span}\{u_\nu\}$ *has a countable orthonormal basis*. *Consequently*, *if* $\mathcal{H}$ *has a countable dense subset* (*i.e.*, $\mathcal{H}$ *is separable*), *then* $\mathcal{H}$ *has a countable complete orthonormal set*.

Proof Let us assume that $\dim \mathcal{V} = \infty$, since the finite-dimensional case is similar, but easier. By passing to a subsequence (constructed inductively), we may assume that the vectors $\{u_\nu\}$ are linearly independent. The *Gram–Schmidt orthonormalization process* then yields an orthonormal basis $\{e_\nu\}$ for $\mathcal{V}$ determined inductively by

$$e_1 \equiv \frac{u_1}{\|u_1\|} \quad \text{and} \quad e_\nu \equiv \frac{u_\nu - \sum_{\mu=1}^{\nu-1}\langle u_\nu, e_\mu\rangle e_\mu}{\|u_\nu - \sum_{\mu=1}^{\nu-1}\langle u_\nu, e_\mu\rangle e_\mu\|} \quad \text{for } \nu > 1.$$

□

Remark A Zorn's lemma argument shows that in fact, *every* Hilbert space admits a complete orthonormal set.

Lemma 7.6.4 (Bessel's inequality) *Let $\{e_i\}_{i\in I}$ be an orthonormal set in $\mathcal{H}$, and for each vector $v \in \mathcal{H}$, let $\hat{v}(i) = \langle v, e_i\rangle$ for each index $i \in I$. Then*

$$\|v\|^2 \geq \sum_{i\in I} |\hat{v}(i)|^2.$$

In particular, at most countably many of the complex numbers $\{\hat{v}(i)\}_{i\in I}$ are nonzero.

Remarks 1. $\{\hat{v}(i)\}_{i\in I}$ is the collection of *Fourier coefficients* of v with respect to the orthonormal set.

2. One can show that in fact, equality in Bessel's inequality holds if $\{e_i\}_{i\in I}$ is a complete orthonormal set. In fact, $v \mapsto \hat{v}$ is an isometric isomorphism onto $\ell^2(I)$ in this case (see, for example, [Rud1]).

Proof of Lemma 7.6.4 Given a vector $v \in \mathcal{H}$ and a finite set of indices $J \subset I$, the Pythagorean theorem (Theorem 7.5.2) implies that

$$\|v\|^2 = \left\|\sum_{j\in J}\langle v, e_j\rangle \cdot e_j\right\|^2 + \left\|v - \sum_{j\in J}\langle v, e_j\rangle \cdot e_j\right\|^2 \geq \sum_{j\in J} |\hat{v}(j)|^2.$$

The claim now follows. □

Proof of Theorem 7.6.1 Given a sequence $\{u_\nu\}$ in $\mathcal{H}$ with $\|u_\nu\| \leq 1$ for each ν, we may assume without loss of generality that $\sup_\nu \|u_\nu\| = 1$. We may also assume that $\mathcal{V} \equiv \operatorname{Span}\{u_\nu\}$ is dense in $\mathcal{H}$. For if there exists a subsequence $\{u_{\nu_k}\}$ converging weakly to a vector u with $\|u\| \leq 1$ in the Hilbert space $\overline{\mathcal{V}}$, then letting $\mathcal{P} \colon \mathcal{H} \to \overline{\mathcal{V}}$ be the orthogonal projection, we get

$$\langle u_{\nu_k}, v\rangle = \langle u_{\nu_k}, \mathcal{P}v\rangle \to \langle u, \mathcal{P}v\rangle = \langle u, v\rangle \quad \forall v \in \mathcal{H}.$$

Theorem 7.6.3 provides a countable orthonormal basis $\{e_i\}_{i\in I}$ for $\mathcal{V}$, which is then a complete orthonormal set in $\mathcal{H}$, and we may let $\{\hat{v}(i)\}_{i\in I} = \{\langle v, e_i\rangle\}_{i\in I}$ be the Fourier coefficients of each $v \in \mathcal{H}$. Applying Bessel's inequality and Cantor's diagonal process, we see that we may assume that the sequence $\{\hat{u}_\nu(i)\}_{\nu=1}^\infty$ converges to some complex number ζ_i with $|\zeta_i| \leq 1$ for each $i \in I$. For each finite set $J \subset I$, Bessel's inequality gives

$$1 \geq \|u_\nu\|^2 \geq \sum_{j\in J} |\hat{u}_\nu(j)|^2 \to \sum_{j\in J} |\zeta_j|^2 \quad \text{as } \nu \to \infty.$$

Thus $\sum_{i\in I} |\zeta_i|^2 \leq 1$.

We may define a unique linear functional τ on $\mathcal{V}$ by setting $\tau(e_i) = \bar{\zeta}_i$ for each $i \in I$. For each vector $v \in \mathcal{V}$, we then have

$$|\tau(v)|^2 = \left|\sum_{i\in I} \hat{v}(i)\bar{\zeta}_i\right|^2 \leq \left[\sum_{i\in I} |\hat{v}(i)|^2\right] \cdot \left[\sum_{i\in I} |\zeta_i|^2\right] \leq \|v\|^2$$

(note that the sum $\sum_{i\in I} \hat{v}(i)\zeta_i$ is actually a finite sum, since $\{\hat{v}(i)\}$ has at most finitely many nonzero terms). Thus τ is a bounded linear functional of norm at most 1 on $\mathcal{V}$. Applying the Hahn–Banach theorem (or the density of $\mathcal{V}$) and Theorem 7.5.10, we get a (unique) vector $u \in \mathcal{H}$ such that $\|u\| \leq 1$ and $\langle v, u\rangle = \tau(v)$ for each $v \in \mathcal{V}$.

Now, given a vector $v \in \mathcal{H}$ and a number $\epsilon > 0$, we may choose a vector $w \in \mathcal{V}$ with $\|v - w\| < \epsilon/3$. For some finite set $J \subset I$, we have $\hat{w} \equiv 0$ on $I \setminus J$, so

$$\langle w, u_\nu\rangle = \sum_{j\in J} \hat{w}(j) \cdot \overline{\hat{u}_\nu(j)} \to \sum_{j\in J} \hat{w}(j) \cdot \bar{\zeta}_j = \tau(w) = \langle w, u\rangle \quad \text{as } \nu \to \infty.$$

Thus, for $\nu \gg 0$, we have $|\langle w, u_\nu\rangle - \langle w, u\rangle| < \epsilon/3$, and hence

$$|\langle v, u_\nu\rangle - \langle v, u\rangle| \leq |\langle v - w, u_\nu\rangle| + |\langle w, u_\nu - u\rangle| + |\langle w - v, u\rangle| < \epsilon.$$

The claim now follows. $\square$

Chapter 8
Background Material on Linear Algebra

In this chapter, we recall some basic definitions and facts concerning exterior products (which are essential in the discussion of differential forms in Sect. 9.5) and tensor products (which are essential in the discussion of holomorphic line bundles in Chap. 3). In this book, we mostly consider exterior and tensor products in vector spaces of dimension 1 or 2.

8.1 Linear Maps, Linear Functionals, and Complexifications

The vector space of linear mappings $\mathcal{V} \to \mathcal{W}$ of vector spaces $\mathcal{V}$ and $\mathcal{W}$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is denoted by $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$, which is itself a vector space over $\mathbb{F}$ (in general, for modules $\mathcal{M}$ and $\mathcal{N}$ over a ring R, the set of module homomorphisms from $\mathcal{M}$ to $\mathcal{N}$ is an R-module that is denoted by $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$). The (*algebraic*) *dual space* of $\mathcal{V}$ is the vector space $\mathcal{V}^* \equiv \mathrm{Hom}(\mathcal{V}, \mathbb{F})$. If $\dim \mathcal{V} = n < \infty$ and $e_1, \ldots, e_n$ is a basis, then there exist unique linear functionals $\lambda_1, \ldots, \lambda_n$ with $\lambda_i(e_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$. These linear functionals form a basis for $\mathcal{V}^*$ that is called the associated *dual basis*. We also have the canonical isomorphism $\mathcal{V} \cong (\mathcal{V}^*)^*$ given by $v(\alpha) = \alpha(v)$ for all $v \in \mathcal{V}$ and $\alpha \in \mathcal{V}^*$. If $\dim \mathcal{V} = 1$, then for every pair of vectors $u, v \in \mathcal{V}$ with $v \neq 0$, we denote the unique scalar $c \in \mathbb{F}$ with $u = cv$ by u/v. We also denote the linear functional $u \mapsto u/v$ by v^{-1}.

Remarks 1. If $\mathcal{V}$ is infinite-dimensional and $\{e_\alpha\}_{\alpha \in A}$ is a basis for $\mathcal{V}$, then we get a collection of linearly independent linear functionals $\{\lambda_\alpha\}_{\alpha \in A}$ determined by $\lambda_\alpha(e_\beta) = 1$ if $\alpha = \beta$, 0 if $\alpha \neq \beta$. However, these linear functionals will *not* span the dual space, since the unique linear functional λ determined by $\lambda(e_\alpha) = 1$ for all $\alpha \in A$ will not be in the span. Consequently, $\mathcal{V} \subsetneqq (\mathcal{V}^*)^*$, since there exists a nontrivial linear functional τ on $\mathcal{V}^*$ with $\tau(\lambda_\alpha) = 0$ for all $\alpha \in A$ (here, we have used the fact that every linearly independent subset of a vector space is contained in a basis).

2. For a normed vector space $(\mathcal{V}, \|\cdot\|)$, the associated norm dual space (see Definition 7.5.8) is equal to the algebraic dual space if and only if $\dim \mathcal{V} < \infty$.

T. Napier, M. Ramachandran, *An Introduction to Riemann Surfaces*, Cornerstones,
DOI 10.1007/978-0-8176-4693-6_8,

For if $\{e_\nu\}$ is a sequence of linearly independent vectors, then there exists a linear functional λ with $\lambda(e_\nu) = \nu\|e_\nu\|$ for each $\nu \in \mathbb{Z}_{>0}$.

One obtains a real vector space from a complex vector space simply by restricting scalar multiplication to real scalars. More precisely, we have the following:

Proposition 8.1.1 *Let $\mathcal{V}$ be a complex vector space with vector addition $+$ and scalar multiplication $\cdot$. Then the set $\mathcal{V}$ together with vector addition given by $+$ and scalar multiplication given by $\cdot\restriction_{\mathbb{R}\times\mathcal{V}}$ determines a real vector space $\mathcal{V}_{\mathbb{R}}$ of dimension $\dim_{\mathbb{R}} \mathcal{V}_{\mathbb{R}} = 2\dim_{\mathbb{C}} \mathcal{V}$. Moreover, for any complex basis $\{e_\nu\}$ for $\mathcal{V}_{\mathbb{C}}$, the collection $\{e_\nu\} \cup \{\sqrt{-1}e_\nu\}$ is a real basis for $\mathcal{V}_{\mathbb{R}}$.*

The proof is left to the reader (see Exercise 8.1.1).

Definition 8.1.2 For a complex vector space $\mathcal{V}$, the associated real vector space given by Proposition 8.1.1 is called the *realification* (or the *underlying real vector space*) of $\mathcal{V}$ and is denoted by $\mathcal{V}_{\mathbb{R}}$ or (in a slight abuse of notation) simply by $\mathcal{V}$.

One obtains a complex vector space $\mathcal{V}_{\mathbb{C}}$ from a real vector space $\mathcal{V}$ by forming all of the formal sums $u + \sqrt{-1}v$ for $u, v \in \mathcal{V}$. More precisely, we have the following:

Proposition 8.1.3 *Let $\mathcal{V}$ and $\mathcal{W}$ be real vector spaces.*

(a) *The set $\mathcal{V} \oplus \mathcal{V}$, together with the standard direct sum vector addition and with scalar multiplication given by $z \cdot (u, v) = (xu - yv, yu + xv)$ for all $(u, v) \in \mathcal{V} \oplus \mathcal{V}$ and $z = x + iy$ with $x, y \in \mathbb{R}$, is a complex vector space $\mathcal{V}_{\mathbb{C}}$ of dimension $\dim_{\mathbb{C}} \mathcal{V} = \dim_{\mathbb{R}} \mathcal{V}$. Denoting the element $(u, v) \in \mathcal{V}_{\mathbb{C}}$ by $u + iv$ for all $u, v \in \mathcal{V}$, we have $z \cdot (u + iv) = (xu - yv) + i(yu + xv)$ for all $z = x + iy$ with $x, y \in \mathbb{R}$. We also have a real linear inclusion $\mathcal{V} \hookrightarrow \mathcal{V}_{\mathbb{C}}$ given by $v \mapsto v + i0 \leftrightarrow (v, 0)$.*

(b) *The map $\mathcal{V}_{\mathbb{C}} \to \mathcal{V}_{\mathbb{C}}$ given by $u + iv \mapsto \overline{u + iv} \equiv u - iv$ (i.e., $(u, v) \mapsto (u, -v)$) is a conjugate linear isomorphism.*

(c) *Any real basis for $\mathcal{V}$ is a complex basis for $\mathcal{V}_{\mathbb{C}}$.*

(d) *If $\alpha, \beta \in Hom(\mathcal{V}, \mathcal{W})$ are two real linear maps, then the complex linear extension of the map $\lambda = \alpha + i\beta$, i.e., the map $\lambda\colon \mathcal{V}_{\mathbb{C}} \to \mathcal{W}_{\mathbb{C}}$ (which, in an abuse of notation, we give the same name) given by $\lambda(u + iv) = \alpha(u) - \beta(v) + i(\beta(u) + \alpha(v)) \in \mathcal{W}_{\mathbb{C}}$ for all $u, v \in \mathcal{V}$, is a complex linear mapping. Moreover, setting $\bar{\lambda} \equiv \alpha - i\beta = \alpha + i(-\beta)$, we get $\overline{\lambda(w)} = \bar{\lambda}(\bar{w})$ for each $w \in \mathcal{V}_{\mathbb{C}}$. The above correspondence gives a (canonical) isomorphism $[\mathrm{Hom}(\mathcal{V}, \mathcal{W})]_{\mathbb{C}} \cong \mathrm{Hom}(\mathcal{V}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}})$. In particular, we have $(\mathcal{V}^*)_{\mathbb{C}} \cong (\mathcal{V}_{\mathbb{C}})^*$. Furthermore, $\alpha \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is injective (surjective) if and only if the associated complex linear extension is injective (respectively, surjective).*

The proof is left to the reader (see Exercise 8.1.2).

Definition 8.1.4 For a real vector space $\mathcal{V}$, the associated complex vector space $\mathcal{V}_{\mathbb{C}}$ given by Proposition 8.1.3 is called the *complexification* of $\mathcal{V}$. We denote each element $(u, v) \in \mathcal{V}_{\mathbb{C}}$ by $w = u + iv = u + \sqrt{-1}v$, we call $\operatorname{Re} w = u$ its *real part*, we call $\operatorname{Im} w = v$ its *imaginary part*, and we call $\bar{w} = u - iv$ its *conjugate*. We identify $\mathcal{V}$ with $\mathcal{V} + i0 \subset \mathcal{V}_{\mathbb{C}}$. Given a real vector space $\mathcal{W}$, we identify each element $\lambda \in [\operatorname{Hom}(\mathcal{V}, \mathcal{W})]_{\mathbb{C}} \supset \operatorname{Hom}(\mathcal{V}, \mathcal{W})$ with its *complex linear extension* $\lambda \in \operatorname{Hom}(\mathcal{V}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}})$ (for each $\lambda \in \operatorname{Hom}(\mathcal{V}_{\mathbb{C}}, \mathcal{W}_{\mathbb{C}})$, we have $\lambda = \operatorname{Re}\lambda + i \operatorname{Im}\lambda$, where $(\operatorname{Re}\lambda)(v) = \operatorname{Re}(\lambda(v))$ and $(\operatorname{Im}\lambda)(v) = \operatorname{Im}(\lambda(v))$ for each $v \in \mathcal{V}$). We also use the identification and notation $\mathcal{V}^*_{\mathbb{C}} \equiv (\mathcal{V}^*)_{\mathbb{C}} = (\mathcal{V}_{\mathbb{C}})^*$.

Example 8.1.5 $(\mathbb{R}^n)_{\mathbb{C}} = \mathbb{C}^n$ under the identification

$$x + iy \leftrightarrow (x_1 + iy_1, \dots, x_n + iy_n) \quad \forall x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

Similarly, we have $(\mathbb{C}^n)_{\mathbb{R}} = \mathbb{R}^{2n}$.

Exercises for Sect. 8.1

8.1.1 Prove Proposition 8.1.1.
8.1.2 Prove Proposition 8.1.3.
8.1.3 Prove that the dual space of any infinite-dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ cannot have a countable basis.

8.2 Exterior Products

Throughout this section, $\mathcal{V}$ denotes a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. In this book, we require exterior products only of degree ≤ 2, so we will restrict our attention to this case.

Definition 8.2.1 For the vector space $\mathcal{V}$:

(a) A *bilinear function* on $\mathcal{V}$ (or $\mathcal{V} \times \mathcal{V}$) is a map $\alpha\colon \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ that is linear in each entry; that is, for all $(u_1, v_1), (u_2, v_2) \in \mathcal{V} \times \mathcal{V}$ and for each $\zeta \in \mathbb{F}$, we have

$$\alpha(\zeta u_1, v_1) = \alpha(u_1, \zeta v_1) = \zeta\alpha(u_1, v_1),$$
$$\alpha(u_1 + u_2, v_1) = \alpha(u_1, v_1) + \alpha(u_2, v_1),$$
$$\alpha(u_1, v_1 + v_2) = \alpha(u_1, v_1) + \alpha(u_1, v_2).$$

The vector space of bilinear functions on $\mathcal{V} \times \mathcal{V}$ (a vector subspace of the space of $\mathbb{F}$-valued functions) is denoted by $\mathcal{V}^* \otimes \mathcal{V}^*$ and is called the *tensor product* of $\mathcal{V}^*$ with itself.

(b) An element $\theta \in \mathcal{V}^* \otimes \mathcal{V}^*$ is *skew-symmetric* (*symmetric*) if $\theta(u, v) = -\theta(v, u)$ (respectively, $\theta(u, v) = \theta(v, u)$) for all $u, v \in \mathcal{V}$. For $\alpha, \beta \in \mathcal{V}^*$, the *exterior* (or *wedge*) *product* is the skew-symmetric bilinear function $\alpha \wedge \beta$ given by $(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \beta(u)\alpha(v)$ for all $u, v \in \mathcal{V}$. The vector (sub)space of skew-symmetric bilinear functions is denoted by $\Lambda^2\mathcal{V}^*$.

Remarks 1. For vector spaces (or modules) $\mathcal{U}, \mathcal{V}, \mathcal{W}$, a map $\alpha : \mathcal{U} \times \mathcal{V} \to \mathcal{W}$ that is linear in each factor is called a *bilinear pairing*.

2. Another standard definition for $\wedge$ includes a factor of $1/2$.

Proposition 8.2.2 *For the vector space $\mathcal{V}$, we have the following*:

(a) *For all $\alpha, \beta, \gamma \in \mathcal{V}^*$ and $\zeta \in \mathbb{F}$,*

$$(\zeta\alpha) \wedge \beta = \alpha \wedge (\zeta\beta) = \zeta \cdot (\alpha \wedge \beta),$$
$$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma,$$
$$\alpha \wedge \beta = -\beta \wedge \alpha.$$

(b) *If* $\dim \mathcal{V} = 1$, *then* $\Lambda^2\mathcal{V}^* = \{0\}$. *If* $\dim \mathcal{V} = 2$ *and λ_1, λ_2 is a basis for $\mathcal{V}^*$, then $\lambda_1 \wedge \lambda_2$ is a basis for $\Lambda^2\mathcal{V}^*$ (in particular,* $\dim \Lambda^2\mathcal{V}^* = 1$). *In fact, if e_1, e_2 is a basis for $\mathcal{V}$ with dual basis λ_1, λ_2, then $\alpha = \alpha(e_1, e_2)\lambda_1 \wedge \lambda_2$ for every $\alpha \in \Lambda^2\mathcal{V}^*$.*

(c) *If $\mathbb{F} = \mathbb{R}$ and $\alpha, \beta \in \mathcal{V}^* \otimes \mathcal{V}^*$ are two (real) bilinear functions, then the complex bilinear extension of the function $\lambda = \alpha + i\beta$, i.e., the map $\lambda \colon \mathcal{V}_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}} \to \mathbb{C}$ (which we give the same name) given by*

$$\begin{aligned}\lambda(\zeta, \xi) =& \alpha(u, r) - \alpha(v, s) - \beta(u, s) - \beta(v, r) \\ &+ i(\alpha(u, s) + \alpha(v, r) + \beta(u, r) - \beta(v, s))\end{aligned}$$

for $\zeta = u + iv$ and $\xi = r + is$ with $u, v, r, s \in \mathcal{V}$, is a (complex) bilinear function on $\mathcal{V}_{\mathbb{C}}$. Setting $\bar{\lambda} = \alpha - i\beta = \alpha + i(-\beta)$, we get $\overline{\lambda(\zeta, \xi)} = \bar{\lambda}(\bar{\zeta}, \bar{\xi})$ for all $\zeta, \xi \in \mathcal{V}_{\mathbb{C}}$. Moreover, if α and β are skew-symmetric, then λ is skew-symmetric. The above correspondence gives (canonical) isomorphisms

$$[\mathcal{V}^* \otimes \mathcal{V}^*]_{\mathbb{C}} \cong \mathcal{V}_{\mathbb{C}}^* \otimes \mathcal{V}_{\mathbb{C}}^* \quad \text{and} \quad [\Lambda^2\mathcal{V}^*]_{\mathbb{C}} \cong \Lambda^2(\mathcal{V}_{\mathbb{C}}^*).$$

Proof The proofs of parts (a) and (c) are left to the reader (see Exercise 8.2.1). For the proof of (b), suppose $\dim \mathcal{V} = 2$, let λ_1, λ_2 be a basis for $\mathcal{V}^*$, and let e_1, e_2 be the dual basis for $\mathcal{V}$. For each $\tau \in \Lambda^2\mathcal{V}^*$, we have, for all $u, v \in \mathcal{V}$,

$$\begin{aligned}\tau(u, v) &= \tau(\lambda_1(u)e_1 + \lambda_2(u)e_2, \lambda_1(v)e_1 + \lambda_2(v)e_2) \\ &= (\lambda_1(u)\lambda_2(v) - \lambda_2(u)\lambda_1(v)) \cdot \tau(e_1, e_2) \\ &= \tau(e_1, e_2) \cdot (\lambda_1 \wedge \lambda_2)(u, v).\end{aligned}$$

Thus $\Lambda^2\mathcal{V}^* = \mathbb{F} \cdot \lambda_1 \wedge \lambda_2$. On the other hand, $(\lambda_1 \wedge \lambda_2)(e_1, e_2) = 1$, so $\lambda_1 \wedge \lambda_2$ is a basis. It is clear that if $\dim \mathcal{V} = 1$, then $\Lambda^2\mathcal{V}^* = \{0\}$. $\square$

Remarks 1. We set $\Lambda^0\mathcal{V}^* = \mathbb{F}$, $\Lambda^1\mathcal{V}^* = \mathcal{V}^*$, and $\zeta \wedge \alpha = \zeta \cdot \alpha \in \Lambda^p\mathcal{V}^*$ if $\zeta \in \Lambda^0\mathcal{V}^*$ and $\alpha \in \Lambda^p\mathcal{V}^*$ with $p \in \{0, 1, 2\}$. If $\dim \mathcal{V} \leq 2$, then we set $\Lambda^p\mathcal{V}^* = \{0\}$ for all $p > 2$, and we set $\alpha \wedge \beta = 0$ for all $\alpha \in \Lambda^p\mathcal{V}^*$, $\beta \in \Lambda^q\mathcal{V}^*$ with $p + q > 2$. It follows that if $\dim \mathcal{V} \leq 2$, $p, q, r \in \mathbb{Z}_{\geq 0}$, $\alpha \in \Lambda^p \mathcal{V}^*$, $\beta \in \Lambda^q \mathcal{V}^*$, and $\gamma \in \Lambda^r \mathcal{V}^*$, then $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$ and $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ (we denote the latter simply by $\alpha \wedge \beta \wedge \gamma$). If $\dim \mathcal{V} > 2$, then we leave $\Lambda^p \mathcal{V}^*$ undefined for $p > 2$, since such spaces are not required in this book.

2. For $\mathbb{F} = \mathbb{R}$ and $p \in \{0, 1, 2\}$ or $p > 2 \geq \dim \mathcal{V}$, Proposition 8.2.2 allows us to use the identification and notation $\Lambda^p \mathcal{V}^*_{\mathbb{C}} \equiv [\Lambda^p \mathcal{V}^*]_{\mathbb{C}} = \Lambda^p (\mathcal{V}^*_{\mathbb{C}})$.

Definition 8.2.3 Assume that $\mathbb{F} = \mathbb{R}$ and that $n \equiv \dim \mathcal{V} = 1$ or 2.

(a) We say that two nonzero elements $\lambda_1, \lambda_2 \in \Lambda^n \mathcal{V}^*$ have *equivalent orientations* if $\lambda_1 / \lambda_2 > 0$ (this is an equivalence relation with exactly two equivalence classes).

(b) An equivalence class as in (a) is called an *orientation* in $\mathcal{V}^*$ (or in $\mathcal{V}$) and the other equivalence class is called the *opposite orientation*. If an element $\alpha \in \Lambda^2 \mathcal{V}^*$ is in the equivalence class determined by a given orientation, then α is said to be *positive* (or *positively oriented*) with respect to the orientation, we write $\alpha > 0$, and we say α *induces the orientation*. If α induces the opposite orientation, then α is said to be *negative* (or *negatively oriented*) with respect to the given orientation, and we write $\alpha < 0$. If $\alpha > 0$ (< 0) or $\alpha = 0$, then we say that α is *nonnegative* (respectively, *nonpositive*) and we write $\alpha \geq 0$ (respectively, $\alpha \leq 0$). An ordered basis $(v_1, \ldots, v_n)$ ($= v_1$ or (v_1, v_2)) for $\mathcal{V}$ is *positively oriented* if each positive element α of $\Lambda^n \mathcal{V}^*$ is positive on the ordered basis (i.e., $\alpha(v_1) > 0$ if $n = 1$, $\alpha(v_1, v_2) > 0$ if $n = 2$).

(c) Given an orientation in $\mathcal{V}^*$ and given two elements $\alpha, \beta \in \Lambda^n \mathcal{V}^*$, we write $\alpha > \beta$ ($\alpha \geq \beta$) if $\alpha - \beta > 0$ (respectively, $\alpha - \beta \geq 0$). In this context, for a sequence $\{\alpha_\nu\}$ in $\Lambda^n \mathcal{V}^*$, we may define

$$\inf_\nu \alpha_\nu \equiv \left[\inf_\nu \frac{\alpha_\nu}{\theta} \right] \cdot \theta, \qquad \sup_\nu \alpha_\nu \equiv \left[\sup_\nu \frac{\alpha_\nu}{\theta} \right] \cdot \theta,$$

$$\liminf_{\nu \to \infty} \alpha_\nu \equiv \left[\liminf_{\nu \to \infty} \frac{\alpha_\nu}{\theta} \right] \cdot \theta, \qquad \limsup_{\nu \to \infty} \alpha_\nu \equiv \left[\limsup_{\nu \to \infty} \frac{\alpha_\nu}{\theta} \right] \cdot \theta,$$

for an arbitrary positive element $\theta \in \Lambda^2 \mathcal{V}^*$, provided the above defined coefficients exist in $\mathbb{R}$.

Remark It is easy see that for a sequence $\{\alpha_\nu\}$ in $\Lambda^n \mathcal{V}^*$ as in (c) above, we have $\lim \alpha_\nu = \alpha$ if and only if $\liminf \alpha_\nu = \limsup \alpha_\nu = \alpha$.

Example 8.2.4 For the standard dual basis λ_1, λ_2 in $(\mathbb{R}^2)^*$, the orientation induced by $\lambda_1 \wedge \lambda_2$ is the right-handed orientation, while that represented by $-\lambda_1 \wedge \lambda_2 = \lambda_2 \wedge \lambda_1$ is the left-handed orientation.

Exercises for Sect. 8.2

8.2.1 Prove parts (a) and (c) of Proposition 8.2.2.

8.3 Tensor Products

Tensor products are required for the study of holomorphic line bundles in Chaps. 3 and 4 (as well as in the last part of Chap. 5). Throughout this section, $\mathbb{F}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$. We consider tensor products from the point of view of bilinear functions (cf. Definition 8.2.1).

Definition 8.3.1 Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over $\mathbb{F}$.

(a) A *bilinear function* on $\mathcal{U} \times \mathcal{V}$ is a map $\alpha\colon \mathcal{U} \times \mathcal{V} \to \mathbb{F}$ that is linear in each entry; that is, for all $(u_1, v_1), (u_2, v_2) \in \mathcal{U} \times \mathcal{V}$ and for each $\zeta \in \mathbb{F}$, we have

$$\alpha(\zeta u_1, v_1) = \alpha(u_1, \zeta v_1) = \zeta\alpha(u_1, v_1),$$
$$\alpha(u_1 + u_2, v_1) = \alpha(u_1, v_1) + \alpha(u_2, v_1),$$
$$\alpha(u_1, v_1 + v_2) = \alpha(u_1, v_1) + \alpha(u_1, v_2).$$

We call the vector space of bilinear functions on $\mathcal{U} \times \mathcal{V}$ (a vector subspace of the space of $\mathbb{F}$-valued functions) the *tensor product* of $\mathcal{U}^*$ and $\mathcal{V}^*$ and we denote it by $\mathcal{U}^* \otimes \mathcal{V}^*$. For $\dim\mathcal{U}, \dim\mathcal{V} < \infty$, the *tensor product* of $\mathcal{U}$ and $\mathcal{V}$ is given by $\mathcal{U} \otimes \mathcal{V} \equiv (\mathcal{U}^*)^* \otimes (\mathcal{V}^*)^*$, under the identification of $\mathcal{U}$ with $(\mathcal{U}^*)^*$ and $\mathcal{V}$ with $(\mathcal{V}^*)^*$.

(b) The *tensor product* of $\alpha \in \mathcal{U}^*$ and $\beta \in \mathcal{V}^*$ is the element $\alpha \otimes \beta \in \mathcal{U}^* \otimes \mathcal{V}^*$ defined by $[\alpha \otimes \beta](u, v) \equiv \alpha(u) \cdot \beta(v)$ for each pair $(u, v) \in \mathcal{U} \times \mathcal{V}$.

Proposition 8.3.2 *Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be finite-dimensional vector spaces over $\mathbb{F}$. Then we have the following*:

(a) *For all $\alpha, \beta \in \mathcal{U}^*$, $\gamma, \delta \in \mathcal{V}^*$, and $\zeta \in \mathbb{F}$, we have*

$$(\zeta\alpha) \otimes \gamma = \alpha \otimes (\zeta\gamma) = \zeta \cdot (\alpha \otimes \gamma),$$
$$(\alpha + \beta) \otimes \gamma = \alpha \otimes \gamma + \beta \otimes \gamma,$$
$$\alpha \otimes (\gamma + \delta) = \alpha \otimes \gamma + \alpha \otimes \delta.$$

(b) *If $\{\theta_i\}_{i=1}^m$ and $\{\lambda_j\}_{j=1}^n$ are bases for $\mathcal{U}^*$ and $\mathcal{V}^*$, respectively, then the tensor products $\{\theta_i \otimes \lambda_j\}_{1\leq i\leq m,\, 1\leq j\leq n}$ form a basis for $\mathcal{U}^* \otimes \mathcal{V}^*$. In particular,* $\dim \mathcal{U}^* \otimes \mathcal{V}^* = mn$.

(c) *There exist unique* (*surjective*) *isomorphisms*

  (i) $\mathcal{U}^* \otimes \mathcal{V}^* \xrightarrow{\cong} \mathcal{V}^* \otimes \mathcal{U}^*$ *satisfying* $\alpha \otimes \beta \mapsto \beta \otimes \alpha$ *for all* $\alpha \in \mathcal{U}^*$ *and* $\beta \in \mathcal{V}^*$;

  (ii) $(\mathcal{U}^* \otimes \mathcal{V}^*) \otimes \mathcal{W}^* \xrightarrow{\cong} \mathcal{U}^* \otimes (\mathcal{V}^* \otimes \mathcal{W}^*)$ *satisfying* $(\alpha \otimes \beta) \otimes \gamma \mapsto \alpha \otimes (\beta \otimes \gamma)$ *for all* $\alpha \in \mathcal{U}^*$, $\beta \in \mathcal{V}^*$, *and* $\gamma \in \mathcal{W}^*$;

  (iii) $\mathcal{V}^* \otimes \mathcal{V} \xrightarrow{\cong} \mathbb{F}$ *satisfying* $\alpha \otimes v \mapsto \alpha(v)$ *for all* $\alpha \in \mathcal{V}^*$ *and* $v \in \mathcal{V}$;

  (iv) $\mathbb{F} \otimes \mathcal{V}^* \xrightarrow{\cong} \mathcal{V}^*$ *satisfying* $\zeta \otimes \alpha \mapsto \zeta \cdot v$ *for all* $\zeta \in \mathbb{F}$ *and* $\alpha \in \mathcal{V}^*$;

  (v) $\Phi\colon \mathcal{U}^* \otimes \mathcal{V} \xrightarrow{\cong} \mathrm{Hom}(\mathcal{U}, \mathcal{V})$ *satisfying* $\Phi(\alpha \otimes v)(u) = \alpha(u) \cdot v$ *for all* $\alpha \in \mathcal{U}^*$, $v \in \mathcal{V}$, *and* $u \in \mathcal{U}$; *and*

(vi) $\Psi: \mathcal{U}^* \otimes \mathcal{V}^* \xrightarrow{\cong} (\mathcal{U} \otimes \mathcal{V})^*$ *satisfying* $\Psi(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \cdot \beta(v)$ *for all* $\alpha \in \mathcal{U}^*$, $\beta \in \mathcal{V}^*$, $u \in \mathcal{U}$, *and* $v \in \mathcal{V}$.

Proof The proof of (a) is left to the reader (see Exercise 8.3.1). For the proof of (b), let $\{e_i\}$ be the dual basis of $\mathcal{U}$ associated to $\{\theta_i\}$, and let $\{f_j\}$ be the dual basis of $\mathcal{V}$ associated to $\{\lambda_j\}$. For each $\tau \in \mathcal{U}^* \otimes \mathcal{V}^*$ and each pair $(u, v) \in \mathcal{U} \times \mathcal{V}$, we have $u = \sum_{i=1}^m \theta_i(u)e_i$ and $v = \sum_{j=1}^n \lambda_j(v) f_j$, and hence

$$\tau(u, v) = \sum_{i=1}^m \sum_{j=1}^n \theta_i(u)\lambda_j(v)\tau(e_i, f_j) = \sum_{i=1}^m \sum_{j=1}^n \tau(e_i, f_j) \cdot (\theta_i \otimes \lambda_j)(u, v).$$

Thus $\tau = \sum_{i=1}^m \sum_{j=1}^n \tau(e_i, f_j) \cdot \theta_i \otimes \lambda_j$, and it follows that the tensor products $\{\theta_i \otimes \lambda_j\}_{1\le i\le m, 1\le j\le n}$ span $\mathcal{U}^* \otimes \mathcal{V}^*$. Moreover, if $\{\zeta_{ij}\}_{1\le i\le m,\, 1\le j\le n}$ are elements of $\mathbb{F}$ and $\tau = \sum_{i=1}^m \sum_{j=1}^n \zeta_{ij} \cdot \theta_i \otimes \lambda_j$, then for each choice of indices i and j, evaluation of both of the above expressions for τ on (e_i, f_j) gives $\zeta_{ij} = \tau(e_i, f_j)$. Hence, if $\tau = 0$, then we get $\zeta_{ij} = 0$ for all i and j. Thus we have linear independence.

The proof of (c), which relies on (a) and (b), is left to the reader (see Exercise 8.3.1). □

Remarks 1. The above proposition is stated mostly with respect to dual spaces, since this is a more convenient setup for the proofs. However, for finite-dimensional vector spaces, the analogous statements also hold with $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ in place of $\mathcal{U}^*$, $\mathcal{V}^*$, and $\mathcal{W}^*$, respectively, since $\mathcal{U} \cong (\mathcal{U}^*)^*$, etc.

2. According to part (c-ii), the tensor product operation on finite-dimensional vector spaces is associative up to a canonical isomorphism. For this reason, we denote the k-fold tensor product of finite-dimensional vector spaces $\mathcal{V}_1, \dots, \mathcal{V}_k$ by $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_k$. We also identify the pairs of spaces that are canonically isomorphic as in (c).

3. For any finite-dimensional vector space $\mathcal{V}$ over $\mathbb{F}$, we denote the k-fold tensor product of $\mathcal{V}$ by $\bigotimes^k \mathcal{V}$. We also set $\bigotimes^0 \mathcal{V} = \mathbb{F}$, and we set $\zeta \otimes \lambda = \zeta \cdot \lambda$ for all $\zeta \in \mathbb{F}$ and $v \in \mathcal{V}$.

4. If $\mathcal{U}$ and $\mathcal{V}$ are finite-dimensional vector spaces over $\mathbb{F}$, $\dim \mathcal{V} = 1$, and $v \in \mathcal{V} \setminus \{0\}$, then for each $\xi \in \mathcal{U} \otimes \mathcal{V} \cong \mathcal{V} \otimes \mathcal{U}$, there is a unique vector $u \in \mathcal{U}$ with $\xi = u \otimes v$. If $\xi = \sum_{j=1}^k u_j \otimes v_j$ with $u_j \in \mathcal{U}$ and $v_j \in \mathcal{V}$ for each $j = 1, \dots, k$, then

$$u = \sum_{j=1}^k \frac{v_j}{v} \cdot u_j.$$

Equivalently, $u = \xi \otimes v^{-1} \in \mathcal{U} \otimes \mathcal{V} \otimes \mathcal{V}^*$ under the identification $\mathcal{U} \cong \mathcal{U} \otimes \mathcal{V} \otimes \mathcal{V}^*$ provided by Proposition 8.3.2. We also denote ξ by $u \cdot v$ or $v \cdot u$, and u by ξ/v. The verification that ξ/v is well defined by the above is left to the reader (see Exercise 8.3.3). Given a vector $w \in \mathcal{U}$, we may form the quotient $w/v = w \otimes v^{-1} \in$

$\mathcal{U} \otimes \mathcal{V}^*$. We may also view this as the quotient of $w \in U \otimes \mathcal{V}^* \otimes \mathcal{V} \cong \mathcal{U}$ and v. Finally, for any integer m, we set

$$\mathcal{V}^m \equiv \begin{cases} \overbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}^{m \text{ factors}} & \text{if } m > 0, \\ \mathbb{F} & \text{if } m = 0, \\ [\mathcal{V}^*]^{-m} \cong [\mathcal{V}^{-m}]^* & \text{if } m < 0; \end{cases}$$

and for any $t \in \mathcal{V}$, we set

$$t^m \equiv \begin{cases} \overbrace{t \otimes \cdots \otimes t}^{m \text{ factors}} \in \mathcal{V}^m & \text{if } m > 0, \\ 1 \in \mathcal{V}^0 = \mathbb{F} & \text{if } m = 0 \text{ and } t \neq 0, \\ (t^{-1})^{-m} = (t^{-m})^{-1} \in \mathcal{V}^m & \text{if } m < 0 \text{and } t \neq 0. \end{cases}$$

Exercises for Sect. 8.3

8.3.1 Prove parts (a) and (c) of Proposition 8.3.2.

8.3.2 Let $\mathcal{U}$ and $\mathcal{V}$ be two finite-dimensional real vector spaces. Prove that if $\alpha, \beta \in \mathcal{U}^* \otimes \mathcal{V}^*$ are two (real) bilinear functions, then the complex bilinear extension of the function $\lambda = \alpha + i\beta$, i.e., the map $\lambda : \mathcal{U}_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}} \to \mathbb{C}$ (which we give the same name) given by

$$\begin{aligned} \lambda(\zeta, \xi) =& \alpha(u, r) - \alpha(v, s) - \beta(u, s) - \beta(v, r) \\ & + i(\alpha(u, s) + \alpha(v, r) + \beta(u, r) - \beta(v, s)) \end{aligned}$$

for $\zeta = u + iv$ and $\xi = r + is$ with $u, v \in \mathcal{U}$ and $r, s \in \mathcal{V}$, is a complex bilinear function on $\mathcal{U}_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}}$. Setting $\bar{\lambda} = \alpha - i\beta = \alpha + i(-\beta)$, prove that $\overline{\lambda(\zeta, \xi)} = \bar{\lambda}(\bar{\zeta}, \bar{\xi})$ for all $\zeta \in \mathcal{U}_{\mathbb{C}}$ and $\xi \in \mathcal{V}_{\mathbb{C}}$. Finally, prove that the above correspondence gives a (canonical) isomorphism $[\mathcal{U}^* \otimes \mathcal{V}^*]_{\mathbb{C}} \cong \mathcal{U}^*_{\mathbb{C}} \otimes \mathcal{V}^*_{\mathbb{C}}$.

8.3.3 Verify that quotients ξ / v, as defined in Remark 4 following the proof of Proposition 8.3.2, are well-defined.

Chapter 9
Background Material on Manifolds

In this chapter, we recall some basic definitions and facts concerning analysis on manifolds (mainly of dimension 1 or 2).

9.1 Topological Spaces

In this section, we recall some terminology and facts from point-set topology (see, for example, [Mu]).

Definition 9.1.1 A *topology* on a set X is a collection of subsets $\mathcal{T}$ such that

(i) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$;
(ii) For every subcollection $\mathcal{U}$ of $\mathcal{T}$, we have $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$; and
(iii) We have $U \cap V \in \mathcal{T}$ for all $U, V \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ (usually denoted simply by X) is called a *topological space*. Each element of $\mathcal{T}$ is called an *open subset* of X. For each point $x \in X$, any open set $U \subset X$ containing x is called a *neighborhood* of x.

Definition 9.1.2 Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$.

(a) A is a *closed subset* of X if $X \setminus A$ is open.
(b) The *interior* $\overset{\circ}{A}$ of A is the union of all open subsets of X contained in A, the *closure* $\overline{A}$ (also denoted by $\mathrm{cl}(A)$) is the intersection of all closed sets containing A, and the *boundary* ∂A is the set $\bar{A} \setminus \overset{\circ}{A}$.
(c) A *limit point* (or *accumulation point*) of A in X is a point $p \in X$ such that $p \in \overline{A \setminus \{p\}}$. A point in A that is not a limit point is called an *isolated point* of A. If A has no limit points in X, then A is called a *discrete subset*.
(d) The collection $\mathcal{T}_A \equiv \{U \cap A \mid U \in \mathcal{T}\}$ is called the *subspace topology* on A.
(e) A mapping $\Phi \colon X \to Y$ of X to a topological space Y is *continuous* if $\Phi^{-1}(U) \subset X$ is open for each open set $U \subset Y$. If Φ is bijective with continuous

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inverse, then Φ is called a *homeomorphism* and we say that X and Y are *homeomorphic*. If Φ maps a neighborhood of each point in X homeomorphically onto an open subset of Y, then Φ is called a *local homeomorphism*. Depending on the context, $\mathcal{C}^0(X)$ will denote either the vector space of real-valued continuous functions on X or the vector space of complex-valued continuous functions on X.

(f) The set A is *compact* if every open covering of A admits a finite subcovering; that is, for every family $\mathcal{U} = \{U_i\}_{i \in I}$ of open subsets of X that satisfies $A \subset \bigcup_{i \in I} U_i$, we have $A \subset \bigcup_{i \in J} U_i$ for some finite set $J \subset I$. The set A is *relatively compact* in X if the closure $\overline{A}$ is compact, and if this is the case, then we write $A \Subset X$. If every open covering of A admits a countable subcovering, then we say that A is *σ-compact*.

(g) The set A is *connected* if for every pair of open sets U and V with $A \subset U \cup V$, we have $A \cap U \cap V \neq \emptyset$ or $A \cap U = \emptyset$ or $A \cap V = \emptyset$. For the equivalence relation $\sim$ on A determined by $x \sim y$ if and only if x and y lie in a connected subset of A, each equivalence class is (a connected set) called a *connected component* (or simply a *component*) of A. The space X is *locally connected* if for every point $p \in X$ and every neighborhood U of p, there is a connected neighborhood V of p that is contained in U (equivalently, every connected component of every open subset of X is open in X).

(h) X is called a *Hausdorff space* if for each pair of distinct points $x, y \in X$, there exist open sets U and V with $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Remarks 1. An arbitrary intersection of closed sets is closed.

2. The interior of a set is the largest open set contained in the set, and the closure of a set is the smallest closed set containing the set.

3. The subspace topology on a subset is a topology on the subset.

4. Let $A \subset X$. Then A is compact (connected) if and only if A is compact (respectively, connected) with respect to the subspace topology on A. The restriction of any continuous mapping $\Phi\colon X \to Y$ to A is continuous. Moreover, if A is compact (connected), then $\Phi(A)$ is also compact (respectively, connected).

5. A compact subset of a Hausdorff space is closed. Consequently, a bijective continuous mapping of compact Hausdorff spaces is a homeomorphism.

6. For any set $A \subset X$, $\overline{A} = \{x \in X \mid x \text{ is an isolated point or limit point of } A\}$. In particular, A is discrete if and only if A is closed and each point in A is an isolated point of A (see Exercise 9.1.1).

Definition 9.1.3 A collection $\mathcal{B}$ of subsets of a set X is a *basis* for a topology on X if $\mathcal{B}$ covers X and for each pair of sets $B_1, B_2 \in \mathcal{B}$ and each point $x \in B_1 \cap B_2$, there is an element $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$. The collection

$$\mathcal{T} = \left\{ \bigcup_{B \in \mathcal{U}} B \,\middle|\, \mathcal{U} \subset \mathcal{B} \right\}$$

is called the *topology generated by* $\mathcal{B}$ in X, and $\mathcal{B}$ is called a basis for $\mathcal{T}$. A topological space with a countable basis is called *second countable*.

Remark The topology generated by a basis $\mathcal{B}$ as above is a topology on X. Moreover, this topology is equal to the intersection of all topologies on X containing the collection.

Example 9.1.4 A normed vector space $(\mathcal{V}, \|\cdot\|)$ together with the associated open subsets is a topological space (see Definition 7.1.6).

Example 9.1.5 Given topological spaces $(X_i, \mathcal{T}_i)$ for $i = 1, \dots, n$, the collection $\mathcal{B} = \{U_1 \times \cdots \times U_n \mid U_i \in \mathcal{T}_i \text{ for } i = 1, \dots, n\}$ is a basis for a topology in the Cartesian product $X = X_1 \times \cdots \times X_n$. The topology $\mathcal{T}$ in X generated by $\mathcal{B}$ is called the *product topology*. A mapping $\Phi = (\Phi_1, \dots, \Phi_n)\colon Y \to X$ of a topological space Y into X is continuous if and only if Φ_i is continuous for each $i = 1, \dots, n$. The details are left to the reader (see Exercise 9.1.2).

Example 9.1.6 The *disjoint union* of a family of sets $\{A_\lambda\}_{\lambda\in\Lambda}$ is the set

$$A \equiv \bigsqcup_{\lambda\in\Lambda} A_\lambda \equiv \{(x, \lambda) \mid \lambda \in \Lambda, x \in A_\lambda\}.$$

In other words, each element $x \in \bigcup_{\lambda\in\Lambda} A_\lambda$ determines *distinct* copies $\{(x, \lambda)\}_{\lambda\in\Lambda, x\in A_\lambda}$ of this element in $\bigsqcup_{\lambda\in\Lambda} A_\lambda$. For each $\lambda \in \Lambda$, we identify A_λ with its image in A under the natural inclusion map $\iota_\lambda\colon x \mapsto (x, \lambda)$, provided there is no danger of confusion. In fact, given a set of indices $\Gamma \subset \Lambda$ and a subset $B_\lambda \subset A_\lambda$ for each $\lambda \in \Gamma$, we identify $\bigsqcup_{\lambda\in\Gamma} B_\lambda$ with its image in A under the natural inclusion.

For a family of topological spaces $\{(X_\lambda, \mathcal{T}_\lambda)\}_{\lambda\in\Lambda}$, the disjoint union $X \equiv \bigsqcup_{\lambda\in\Lambda} X_\lambda$ inherits the (natural) topology $\mathcal{T} \equiv \{\bigsqcup_{\lambda\in\Lambda} U_\lambda \mid U_\lambda \in \mathcal{T}_\lambda \ \forall\lambda \in \Lambda\}$. In other words, $\mathcal{T}$ is the unique topology in X for which for each $\lambda \in \Lambda$, the inclusion $\iota_\lambda\colon A_\lambda \hookrightarrow X$ maps A_λ homeomorphically onto an open set $\iota_\lambda(A_\lambda)$.

Example 9.1.7 Let $\sim$ be an equivalence relation in a topological space X. Then the *quotient topology* in the corresponding *quotient space* $Y \equiv X/{\sim}$ with *quotient map* $\Phi\colon X \to Y \equiv X/{\sim}$ is the collection of all sets $U \subset Y$ for which $\Phi^{-1}(U)$ is open in X (see Exercise 9.1.3). Observe that Φ is a surjective continuous mapping with respect to these topologies. Moreover, for an arbitrary surjective mapping $\Psi\colon X \to Z$ of X onto a set Z, the fibers are equivalence classes for the equivalence relation given by $x \sim y \iff \Psi(x) = \Psi(y)$. Hence we get an induced quotient topology in Z. If Z has a given topology $\mathcal{T}_Z$, then the quotient topology may differ from $\mathcal{T}_Z$. However, if Ψ is continuous with respect to $\mathcal{T}_Z$, then $\mathcal{T}_Z$ is contained in the quotient topology.

Definition 9.1.8 Let X be a topological space.

(a) For $p, q \in X$, a *path* (or *parametrized path* or *curve* or *parametrized curve*) from p to q in X is a continuous mapping $\gamma\colon [a, b] \to X$ for some numbers $a, b \in \mathbb{R}$ with $a < b$ with $\gamma(a) = p$ and $\gamma(b) = q$. We call p the *initial point* and q the *terminal point*. We call γ a *loop based at p* (or a *closed curve based at p*) if $p = q$.

(b) X is *path connected* if there is a path from p to q for each pair of points $p, q \in X$. The space X is *locally path connected* if for every point $p \in X$ and every neighborhood U of p, there is a path connected neighborhood V of p that is contained in U.

Remarks 1. We take the domain of a path to be the interval [0, 1] unless otherwise specified.

2. In some contexts, we also call the image of a parametrized path (with certain properties) a *path* (or *curve*). A 1-dimensional manifold (see Sect. 9.2) is also sometimes called a *curve*.

3. A path connected space is connected, and a connected locally path connected space is path connected (see Exercise 9.1.4).

Example 9.1.9 The set $(\{0\} \times \mathbb{R}) \cup \{(x, \sin(1/x)) \mid x > 0\} \subset \mathbb{R}^2$ is connected, but not path connected.

Definition 9.1.10 A Hausdorff space X is *locally compact* if for every point $p \in X$ and every neighborhood U of p, there is a relatively compact neighborhood V of p in U.

Remarks 1. A general topological space is called *locally compact* if each point has a neighborhood that lies in a compact set. For Hausdorff spaces, this condition is equivalent to the above (see, for example, [Mu]).

2. It is easy to see that if U is a neighborhood of a compact subset K in a locally compact Hausdorff space, then there exists a *relatively compact* neighborhood V of K in U (that is, $\overline{V}$ is compact and $\overline{V} \subset U$, which we indicate by writing $V \Subset U$).

Definition 9.1.11 The *one-point compactification* of a locally compact Hausdorff space $(X, \mathcal{T}_X)$ is the topological space with underlying set $X \cup \{\infty\}$ obtained by adjoining a point ∞ (not in X) to X and with topology given by

$$\mathcal{T}_X \cup \{(X \setminus K) \cup \{\infty\} \mid K \text{ is a compact subset of } X\}.$$

The element ∞ is called the *point at infinity*.

The one-point compactification $X \cup \{\infty\}$ of a locally compact Hausdorff space X is a compact Hausdorff space, and if X is noncompact and connected, then $X \cup \{\infty\}$ is connected (see Exercise 9.1.5).

Exercises for Sect. 9.1

9.1.1 Prove that for any subset A of a topological space X, we have

$$\overline{A} = \{x \in X \mid x \text{ is an isolated point or limit point of } A\}.$$

Also show that A is discrete if and only if A is closed and each point in A is an isolated point of A.

9.1.2 Verify that the product topology on the Cartesian product $X = X_1 \times \cdots \times X_n$ of (finitely many) topological spaces $X_1, \ldots, X_n$ is a topological space (see Example 9.1.5). Also verify that a mapping $\Phi = (\Phi_1, \ldots, \Phi_n)\colon Y \to X$ of a topological space Y into X is continuous if and only if Φ_i is continuous for each $i = 1, \ldots, n$.

9.1.3 Verify that the quotient topology in a quotient space of a topological space is a topology, and that the corresponding quotient map is continuous (see Example 9.1.7).

9.1.4 Prove that any path connected topological space is connected, and that any connected locally path connected topological space is path connected.

9.1.5 Verify that the one-point compactification $X \cup \{\infty\}$ of a locally compact Hausdorff space X is a topological space and that $X \cup \{\infty\}$ is compact Hausdorff. Prove also that if X is noncompact and connected, then $X \cup \{\infty\}$ is connected.

9.2 The Definition of a Manifold

Definition 9.2.1 Let M be a Hausdorff space, and let $n \in \mathbb{Z}_{\geq 0}$.

(a) A homeomorphism $\Phi : U \to U'$ of an open set $U \subset M$ onto an open set $U' \subset \mathbb{R}^n$ is called a *local chart* of *dimension* n (or an n*-dimensional local chart*) in M. We also denote this local chart by (U, Φ, U').

(b) For any two n-dimensional local charts (U_1, Φ_1, U_1') and (U_2, Φ_2, U_2') in M, the mapping

$$\Phi_1 \circ \Phi_2^{-1}\colon \Phi_2(U_1 \cap U_2) \to \Phi_1(U_1 \cap U_2)$$

is called a *coordinate transformation* (see Fig. 9.1).

(c) A collection of n-dimensional local charts $\mathcal{A} = \{(U_i, \Phi_i, U_i')\}_{i \in I}$ that covers M (i.e., for which $M = \bigcup_{i \in I} U_i$) is called an *atlas of dimension* n on M.

(d) If M admits an n-dimensional atlas, then M is called a *manifold* (or a *topological manifold* or a $\mathcal{C}^0$ *manifold*) *of dimension* n. M is also called a (*topological* or $\mathcal{C}^0$) n*-manifold*.

(e) A connected 1-dimensional manifold is also called a *curve* (or a *topological curve*). A connected 2-dimensional manifold is also called a *surface* (or a *topological surface* or a $\mathcal{C}^0$ *surface*).

Remarks 1. In this book, although we do not assume that a given manifold is second countable, we mostly consider second countable manifolds. The reader should also note that it is common to take second countability as a part of the definition of a manifold (without any specific comment).

2. Because we also call a path in a topological space a curve, we will mostly avoid referring to a connected 1-dimensional manifold as a curve unless there is no danger of confusion.

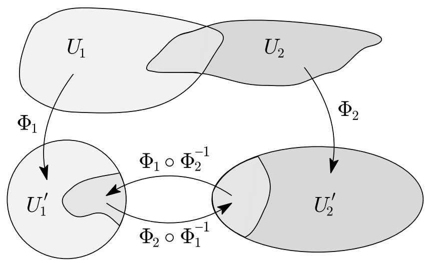

Fig. 9.1 Coordinate transformations

Definition 9.2.2 Let M be Hausdorff space, let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, and let $n \in \mathbb{Z}_{\geq 0}$.

(a) Two n-dimensional local charts (U_1, Φ_1, U_1') and (U_2, Φ_2, U_2') in M are $\mathcal{C}^k$ *compatible* if the coordinate transformations

$$\Phi_1 \circ \Phi_2^{-1} \colon \Phi_2(U_1 \cap U_2) \to \Phi_1(U_1 \cap U_2)$$

and

$$[\Phi_1 \circ \Phi_2^{-1}]^{-1} = \Phi_2 \circ \Phi_1^{-1} \colon \Phi_1(U_1 \cap U_2) \to \Phi_2(U_1 \cap U_2)$$

are of class $\mathcal{C}^k$.

(b) An atlas consisting of $\mathcal{C}^k$ compatible local charts in M is called a $\mathcal{C}^k$ *atlas* on M.

(c) Two n-dimensional $\mathcal{C}^k$ atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on M are $\mathcal{C}^k$ *equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a $\mathcal{C}^k$ atlas (this is an equivalence relation).

(d) An equivalence class $\mathcal{S}$ of n-dimensional $\mathcal{C}^k$ atlases on M is called a $\mathcal{C}^k$ *structure of dimension n* on M. The pair $(M, \mathcal{S})$ (usually denoted simply by M) is called a $\mathcal{C}^k$ *manifold of dimension n* (or a $\mathcal{C}^k$ *n-manifold*).

(e) Any local chart (U, Φ, U') in any $\mathcal{C}^k$ atlas in the $\mathcal{C}^k$ structure on a $\mathcal{C}^k$ n-manifold M is called a *local* $\mathcal{C}^k$ *chart in* M. Setting $x = (x_1, \ldots, x_n) = \Phi$, we call x *local* $\mathcal{C}^k$ *coordinates*, and for each point $p \in U$, we call (U, x) a *local* $\mathcal{C}^k$ *coordinate neighborhood of* p.

(f) A $\mathcal{C}^\infty$ manifold is also called a *smooth manifold*.

(g) A connected 2-dimensional $\mathcal{C}^k$ manifold (smooth manifold) is also called a $\mathcal{C}^k$ *surface* (respectively, *smooth surface*). A connected 1-dimensional $\mathcal{C}^k$ manifold (smooth manifold) is also called a $\mathcal{C}^k$ *curve* (respectively, *smooth curve*).

Remark A *real analytic manifold* is defined analogously (i.e., the coordinate transformations are required to be real analytic).

Examples 9.2.3 For $n \in \mathbb{Z}_{>0}$, $\mathbb{R}^n$ is a smooth (in fact, real analytic) manifold of dimension n with $\mathcal{C}^\infty$ atlas $\{(\mathbb{R}^n, x \mapsto x, \mathbb{R}^n)\}$. The unit sphere $\mathbb{S}^n$ is a $\mathcal{C}^\infty$ (in fact, real analytic) manifold of dimension n with $\mathcal{C}^\infty$ atlas $\mathcal{A} = \{(U_i^+, \Phi_i, V_i)\}_{i=1}^{n+1} \cup \{(U_i^-, \Phi_i, V_i)\}_{i=1}^{n+1}$, where for each $i = 1, \ldots, n+1$,

$$\Phi_i \colon U_i^\pm \equiv \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \mid \pm x_i > 0\} \to V_i \equiv \{t \in \mathbb{R}^n \mid \|t\| < 1\}$$

is the mapping given by $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, \widehat{x_i}, \dots, x_{n+1})$. Any open subset Ω of a $\mathcal{C}^k$ manifold M is a $\mathcal{C}^k$ manifold with $\mathcal{C}^k$ atlas consisting of the restrictions to Ω of local $\mathcal{C}^k$ coordinates in M.

The Cartesian product $M_1 \times M_2$ of two $\mathcal{C}^k$ manifolds M_1 and M_2 is a $\mathcal{C}^k$ manifold, called the *product manifold*, with $\mathcal{C}^k$ atlas consisting of local $\mathcal{C}^k$ charts given by

$$(U_1 \times U_2, (x, y) \mapsto (\Phi_1(x), \Phi_2(y)), V_1 \times V_2)$$

for local $\mathcal{C}^k$ charts (U_1, Φ_1, V_1) and (U_2, Φ_2, V_2) in M_1 and M_2, respectively.

For any family of $\mathcal{C}^k$ manifolds $\{M_\lambda\}_{\lambda\in\Lambda}$ of dimension n with corresponding atlases $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$, the disjoint union $M \equiv \bigsqcup_{\lambda\in\Lambda} M_\lambda$, with inclusion mappings $\iota_\lambda \colon M_\lambda \hookrightarrow M$ for $\lambda \in \Lambda$, inherits the structure of a $\mathcal{C}^k$ manifold of dimension n with atlas given by

$$\mathcal{A} \equiv \{(\iota_\lambda(U), \Phi \circ \iota_\lambda^{-1}, U') \mid (U, \Phi, U') \in \mathcal{A}_\lambda \text{ for some } \lambda \in \Lambda\}.$$

The verifications of the above are left to the reader (see Exercise 9.2.1).

Definition 9.2.4 For $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, a continuous mapping $\Psi \colon M \to N$ of $\mathcal{C}^k$ manifolds M and N is of *class* $\mathcal{C}^k$ if for every choice of local $\mathcal{C}^k$ charts (U_1, Φ_1, V_1) and (U_2, Φ_2, V_2) in M and N, respectively, the mapping

$$\Phi_2 \circ \Psi \circ \Phi_1^{-1} \colon \Phi_1(U_1 \cap \Psi^{-1}(U_2)) \to V_2$$

is of class $\mathcal{C}^k$. The vector space of $\mathcal{C}^k$ functions on M with values in $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is denoted by $\mathcal{C}^k(M, \mathbb{F})$. Depending on the context, $\mathcal{C}^k(M)$ will denote either $\mathcal{C}^k(M, \mathbb{R})$ or $\mathcal{C}^k(M, \mathbb{C})$. We also denote $\mathcal{C}^\infty(M, \mathbb{F})$ by $\mathcal{E}(M, \mathbb{F})$ or by $\mathcal{E}(M)$. The vector space of $\mathcal{C}^\infty$ $\mathbb{F}$-valued functions with compact support in M is denoted by $\mathcal{D}(M, \mathbb{F})$ or by $\mathcal{D}(M)$.

A $\mathcal{C}^\infty$ bijective mapping with $\mathcal{C}^\infty$ inverse is called a *diffeomorphism*. A $\mathcal{C}^\infty$ map Φ for which the restriction $\Phi\restriction_U$ to some neighborhood U of each point maps U diffeomorphically onto an open set $\Phi(U)$ is called a *local diffeomorphism*.

Remarks 1. The chain rule implies that if $\Psi \colon M \to N$ is a continuous mapping of $\mathcal{C}^k$ manifolds with $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, then Ψ is of class $\mathcal{C}^k$ if and only if for each point $p \in M$, there exist local $\mathcal{C}^k$ charts (U_1, Φ_1, V_1) and (U_2, Φ_2, V_2) on M and N, respectively, such that $p \in U_1 \cap \Psi^{-1}(U_2)$ and the mapping

$$\Phi_2 \circ \Psi \circ \Phi_1^{-1} \colon \Phi_1(U_1 \cap \Psi^{-1}(U_2)) \to V_2$$

is of class $\mathcal{C}^k$ (see Exercise 9.2.2). In other words, one need only check that the condition holds for elements of some $\mathcal{C}^k$ atlas in the $\mathcal{C}^k$ structure (not for every local $\mathcal{C}^k$ chart).

2. For real analytic manifolds, one uses the analogous terminology. Moreover, the natural analogue of the above property, as well as analogues of many other properties of $\mathcal{C}^k$ manifolds, also apply to real analytic manifolds. However, since

a proof that compositions of real analytic mappings are real analytic requires some work (for example, one may form local extensions to complex manifolds and use the fact that compositions of holomorphic maps are holomorphic), and since general facts concerning real analytic manifolds are not required in this book, we will avoid consideration of such facts.

3. It is easy to verify that sums and products of $\mathcal{C}^k$ functions are of class $\mathcal{C}^k$, so $\mathcal{C}^k(M)$ is an algebra.

Definition 9.2.5 For $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, a path $\gamma\colon [a, b] \to M$ is a $\mathcal{C}^k$ *path* (or a $\mathcal{C}^k$ *curve*) if γ extends to a $\mathcal{C}^k$ mapping of a neighborhood of $[a, b]$ into M. The path is *piecewise* $\mathcal{C}^k$ if there is a partition $a = t_0 < t_1 < \cdots < t_m = b$ such that for each $i = 1, \dots, m$, $\gamma\restriction_{[t_{i-1}, t_i]}$ is a $\mathcal{C}^k$ path.

We close this section with a consideration of subsets that inherit a $\mathcal{C}^\infty$ structure.

Definition 9.2.6 Let M be a smooth manifold of dimension n, and let $r \in \{0, \dots, n\}$. A nonempty closed set $N \subset M$ is a *smooth* (or $\mathcal{C}^\infty$) *submanifold of dimension r in M* if for each point $a \in N$, there exists a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, (x_1, \dots, x_n))$ of a in M such that

$$N \cap U = \{p \in U \mid x_{r+1}(p) = x_{r+2}(p) = \cdots = x_n(p) = 0\}.$$

Lemma 9.2.7 *A smooth submanifold N of dimension r in a smooth n-manifold M is itself a smooth manifold of dimension r with* $\mathcal{C}^\infty$ *atlas consisting of all local charts of the form* $(N \cap U, (x_1, \dots, x_r))$, *where* $(U, (x_1, \dots, x_n))$ *is a local* $\mathcal{C}^\infty$ *chart in M with* $N \cap U = \{p \in U \mid x_{r+1}(p) = x_{r+2}(p) = \cdots = x_n(p) = 0\}$.

Proof Clearly, N is Hausdorff. If $(U, \Phi = (x_1, \dots, x_n))$ and $(V, \Psi = (y_1, \dots, y_n))$ are two local $\mathcal{C}^\infty$ charts in M with

$$N \cap U = \{x_{r+1} = \cdots = x_n = 0\} \quad \text{and} \quad N \cap V = \{y_{r+1} = \cdots = y_n = 0\},$$

then each of the $\mathcal{C}^\infty$ maps $\Phi_0 \equiv (x_1, \dots, x_r)$ and $\Psi_0 \equiv (y_1, \dots, y_r)$ maps $N \cap U$ and $N \cap V$, respectively, homeomorphically onto an open subset of $\mathbb{R}^r$. Moreover,

$$\Psi_0 \circ \Phi_0^{-1}(t_1, \dots, t_r) = \Psi_0(\Phi^{-1}(t_1, \dots, t_r, 0, \dots, 0))$$

for every point $(t_1, \dots, t_r) \in \Phi_0(N \cap U \cap V)$, and hence the coordinate transformation $\Psi_0 \circ \Phi_0^{-1}\colon \Phi_0(N \cap U \cap V) \to \Psi_0(N \cap U \cap V)$ is of class $\mathcal{C}^\infty$. $\square$

Remark We assume that a given smooth submanifold has the $\mathcal{C}^\infty$ structure given by the above lemma unless otherwise indicated.

Example 9.2.8 For each $R > 0$, the circle $\partial\Delta(0; R)$ in $\mathbb{R}^2$ is a smooth submanifold of $\mathbb{R}^2$. For given a point $p \in \partial\Delta(0; R)$, polar coordinates (r, θ) (see Example 7.2.8) provide local $\mathcal{C}^\infty$ coordinates in some neighborhood U of p in $\mathbb{R}^2$. Setting $\rho = r - R$, we get the local $\mathcal{C}^\infty$ coordinate neighborhood $(U, (\theta, \rho))$ in which $U \cap \partial\Delta(0; R) = \{q \in U \mid \rho(q) = 0\}$.

Exercises for Sect. 9.2

9.2.1 Verify the claims in Examples 9.2.3.

9.2.2 Verify that if $\Psi\colon M \to N$ is a continuous mapping of $\mathcal{C}^k$ manifolds with $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, then Ψ is of class $\mathcal{C}^k$ if and only if for each point $p \in M$, there exist local $\mathcal{C}^k$ charts (U_1, Φ_1, V_1) and (U_2, Φ_2, V_2) on M and N, respectively, such that $p \in U_1 \cap \Psi^{-1}(U_2)$ and the mapping $\Phi_2 \circ \Psi \circ \Phi_1^{-1}\colon \Phi_1(U_1 \cap \Psi^{-1}(U_2)) \to V_2$ is of class $\mathcal{C}^k$.

9.3 The Topology of Manifolds

Manifolds have many nice topological properties, especially *second countable* manifolds. When working in $\mathbb{R}^n$, it is often convenient to phrase topological arguments in terms of sequences. Fortunately, in a manifold, it also often suffices to consider sequences (in a general topological space, the analogous arguments may be phrased in terms of more general objects called *nets*, as in, for example, [Ke]).

Definition 9.3.1 Let $\{a_\nu\}$ be a sequence in a topological space X. We say that $\{a_\nu\}$ *converges* (in X) to a point $a \in X$ if for every neighborhood U of a in X, there exists an integer $\mu > 0$ such that $a_\nu \in U$ for all $\nu \geq \mu$. We also say that $\{a_\nu\}$ is convergent, we call a the *limit* of $\{a_\nu\}$, and we write

$$\lim_{\nu\to\infty} a_\nu = a \quad \text{or} \quad a_\nu \to a \text{ as } \nu \to \infty.$$

If $\{a_\nu\}$ does not have a limit, then we say that the sequence *diverges* (or the sequence is *divergent*).

The following theorem, the proof of which is left to the reader (see Exercise 9.3.1), contains analogues of standard facts concerning topology in $\mathbb{R}^n$:

Theorem 9.3.2 *Let D be a subset of a manifold M. Then*

(a) Sequentially closed sets. *D is closed if and only if the limit of every convergent sequence $\{a_\nu\}$ in M with $a_\nu \in D$ for all ν lies in D. A point $p \in M$ is a limit point of D if and only if p is the limit of a sequence of points in $D \setminus \{p\}$. The closure of D is equal to the limits of all sequences in D that converge in M.*

(b) Bolzano–Weierstrass property. *If D is compact, then every sequence in D admits a subsequence that converges in M to a point in D.*

(c) Sequential continuity. *A mapping $\Phi\colon D \to N$ of D into a manifold N is continuous at a point $a \in D$ if and only if $\Phi(a_\nu) \to \Phi(a)$ for every sequence $\{a_\nu\}$ in D with $a_\nu \to a$.*

We recall that a topological space is *second countable* if it has a countable basis.

Lemma 9.3.3 *If $(X, \mathcal{T})$ is a second countable topological space and $\mathcal{B}$ is any basis for $\mathcal{T}$, then there is a countable basis $\mathcal{B}' \subset \mathcal{B}$.*

Proof Fix a set $B_0 \in \mathcal{B}$. By hypothesis, there exists a countable basis $\mathcal{A}$ for $\mathcal{T}$. For each pair $a = (A_1, A_2) \in \mathcal{A} \times \mathcal{A}$, we may fix a set $B_a \in \mathcal{B}$ with $A_1 \subset B_a \subset A_2$ if such a set exists; and we may set $B_a = B_0$ if no such set exists. Then the collection $\mathcal{B}' = \{B_a \mid a \in \mathcal{A} \times \mathcal{A}\} \subset \mathcal{B}$ is countable. Given an open set $U \subset X$ and a point $p \in U$, there exist an element $A_2 \in \mathcal{A}$ with $p \in A_2 \subset U$ (since $\mathcal{A}$ is a basis for $\mathcal{T}$), an element $B \in \mathcal{B}$ with $p \in B \subset A_2$ (since $\mathcal{B}$ is a basis for $\mathcal{T}$), and an element $A_1 \in \mathcal{A}$ with $p \in A_1 \subset B \subset A_2$. In particular, for $a \equiv (A_1, A_2) \in \mathcal{A} \times \mathcal{A}$, we have $p \in A_1 \subset B_a \subset A_2 \subset U$. It follows that $\mathcal{B}'$ is a countable basis for $\mathcal{T}$. □

Definition 9.3.4 Let X be a topological space.

(a) A family of subsets $\mathcal{A} = \{A_i\}_{i \in I}$ of X is *locally finite* if each point in X has a neighborhood that meets A_i for at most finitely many indices $i \in I$.
(b) We call X *paracompact* if every open cover of X has a locally finite refinement. That is, for every family of open sets $\{U_i\}_{i \in I}$ with $X = \bigcup_{i \in I} U_i$, there is a locally finite family of open sets $\{V_j\}_{j \in J}$ such that $X = \bigcup_{j \in J} V_j$ and such that for each $j \in J$, we have $V_j \subset U_i$ for some $i \in I$.

The proof of the following useful observation is left to the reader (see Exercise 9.3.2):

Lemma 9.3.5 *If $\{A_i\}_{i \in I}$ is a locally finite family of sets in a topological space X, then*

$$\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}.$$

The following lemma contains some useful facts concerning the topology of second countable locally compact Hausdorff spaces (for example, second countable manifolds):

Lemma 9.3.6 *For any second countable locally compact Hausdorff space X, we have the following*:

(a) *There exists a sequence of open sets $\{\Omega_\nu\}_{\nu=0}^{\infty}$ such that $\Omega_0 = \emptyset$, $X = \bigcup_{\nu=1}^{\infty} \Omega_\nu$, and $\Omega_{\nu-1} \Subset \Omega_\nu$ for each $\nu = 1, 2, 3, \dots$. In particular, X is σ-compact.*
(b) *If X is locally connected and connected, then we may choose the sequence of open sets $\{\Omega_\nu\}$ in* (a) *so that Ω_ν is connected for each ν.*
(c) *If $\{U_\alpha\}_{\alpha \in A}$ is a covering of a closed set $K \subset X$ by open subsets of X and $\mathcal{B}$ is a basis for the topology in X, then there exists a countable locally finite* (*in X*) *covering $\{B_i\}_{i \in I}$ of K by elements of the basis $\mathcal{B}$ such that for each $i \in I$, we have $B_i \Subset U_\alpha$ for some $\alpha \in A$. In particular, X is paracompact.*
(d) *If $\{U_\alpha\}_{\alpha \in A}$ is a covering of a closed set $K \subset X$ by open subsets of X and for each $\alpha \in A$, $\mathcal{B}_\alpha$ and $\mathcal{B}'_\alpha$ are bases for the topology in U_α, then there exist countable locally finite* (*in X*) *coverings $\{B_i\}_{i \in I}$ and $\{B'_i\}_{i \in I}$ of K such that*
   (i) *For each $i \in I$, we have $B_i \Subset B'_i \Subset U_\alpha$ and $B_i \in \mathcal{B}_\alpha$, $B'_i \in \mathcal{B}'_\alpha$, for some $\alpha \in A$; and*

(ii) *If* $i, j \in I$ *and* $\overline{B}_i \cap \overline{B}_j \neq \emptyset$, *then* $B_i \Subset B'_j$.

(e) *If* $\{V_i\}_{i\in I}$ *is a locally finite* (*in* X) *covering of a closed set* $K \subset X$ *by relatively compact open subsets of* X, *then there exists a covering* $\{K_i\}_{i\in I}$ *of* K *by compact subsets of* X *with* $K_i \subset V_i$ *for each* $i \in I$.

(f) *If* $\{K_i\}_{i\in I}$ *is a locally finite family of closed subsets of* X, *then there exists a locally finite family of open sets* $\{V_i\}_{i\in I}$ *such that* $K_i \subset V_i$ *for each index* $i \in I$. *Moreover, if the sets* $\{K_i\}_{i\in I}$ *are disjoint, then the sets* $\{V_i\}_{i\in I}$ *may also be chosen to be disjoint.*

Proof For the proof of (a), let us fix a countable basis $\mathcal{B}_0$ for the topology in X. We may assume that $X \neq \emptyset$, $\emptyset \notin \mathcal{B}_0$. Since X is locally compact, the collection of relatively compact elements of $\mathcal{B}_0$ is itself a basis (as one may easily check), so we may also assume that each element of $\mathcal{B}_0$ is relatively compact in X. Hence we may choose a covering of X by relatively compact basis elements $\{V_j\}_{j=1}^{\infty}$, and for each j, we may let $\Gamma_j \equiv V_1 \cup \cdots \cup V_j \Subset X$. In particular, $X = \bigcup_{j=1}^{\infty} V_j = \bigcup_{j=1}^{\infty} \Gamma_j$. Let $j_0 = 0$ and $\Gamma_0 = \emptyset$. Given indices $j_0 < j_1 < j_2 < \cdots < j_{\nu-1}$, we may choose an index $j_\nu > j_{\nu-1}$ so that the compact set $\overline{\Gamma}_{j_{\nu-1}}$ is contained in Γ_{j_ν}. Thus we get a sequence of indices $\{j_\nu\}$, and setting $\Omega_\nu \equiv \Gamma_{j_\nu}$ for $\nu = 0, 1, 2, \dots$, we get a sequence of open sets $\{\Omega_\nu\}$ with the properties required in (a).

If X is also locally connected and connected, then the collection of connected components of elements of $\mathcal{B}_0$ is countable, and hence by replacing $\mathcal{B}_0$ with this collection, we may assume that each element of the basis $\mathcal{B}_0$ is connected. For $\{V_j\}$ as above, we may instead let Γ_j be the *connected component* of $V_1 \cup \cdots \cup V_j$ containing V_1 for each $j = 1, 2, 3, \dots$. The union $\Gamma \equiv \bigcup \Gamma_j$ is then equal to X. For if $p \in \overline{\Gamma}$, then $p \in V_j$ for some j and V_j must meet Γ_k for some k. Therefore, for $l \equiv \max(j, k)$, $V_j \cup \Gamma_k$ is a connected subset of $V_1 \cup \cdots \cup V_l$ containing V_1, and hence $p \in V_j \subset \Gamma_l \subset \Gamma$. Thus Γ is both open and closed in X and is therefore equal to X. A suitable subsequence of $\{\Gamma_j\}$ (chosen inductively) will then have each term relatively compact in the next term, as required in (b).

For the proof of (c), suppose $\mathcal{B}$ is an arbitrary basis for the topology in X. We may choose a sequence of open sets $\{\Omega_\nu\}_{\nu=0}^{\infty}$ with the properties in (a). For each point $p \in K$, there is a basis element $C_p \in \mathcal{B}$ such that $p \in C_p \Subset U_\alpha$ for some $\alpha \in A$ and such that for each index $\nu = 1, 2, 3, \dots$, we have $C_p \Subset \Omega_\nu$ if and only if $p \in \Omega_\nu$, and we have $\overline{C}_p \cap \overline{\Omega}_\nu \neq \emptyset$ if and only if $p \in \overline{\Omega}_\nu$. For each $\nu = 1, 2, 3, \dots$, there exists a (possibly empty) finite collection of points $\{p_i\}_{i\in I_\nu}$ in the compact set $K \cap \overline{\Omega}_\nu \setminus \Omega_{\nu-1}$ such that $K \cap \overline{\Omega}_\nu \setminus \Omega_{\nu-1} \subset \bigcup_{i\in I_\nu} C_{p_i}$. Letting I be the disjoint union $\bigsqcup_{\nu\in\mathbb{Z}_{>0}} I_\nu$ and letting $B_i = C_{p_i}$ for each $i \in I$ (here, we identify I_ν with its image in I for each ν), we get a countable covering $\{B_i\}_{i\in I}$ of K by basis elements each of which is relatively compact in U_α for some $\alpha \in A$. Moreover, for each ν, we have $B_i \cap \Omega_\nu = \emptyset$ if $i \in I_\mu$ with $\mu > \nu + 1$, so the family $\{B_i\}$ is locally finite.

For the proof of (d), suppose we have two bases $\mathcal{B}_\alpha$ and $\mathcal{B}'_\alpha$ for the topology in U_α for each $\alpha \in A$. Again, we may assume that the empty set is not an element of any of these bases. Clearly, we may also assume that $K \neq \emptyset$. By applying part (c) to the covering $\{U_\alpha\}$ and the basis for the topology in X consisting of all open sets B such that either $B \cap K = \emptyset$ or $B \in \mathcal{B}'_\alpha$ and $B \Subset U_\alpha$ for some $\alpha \in A$, we get a countable locally finite (in X) covering $\{C_m\}_{m\in M}$ of K such that for each $m \in M$, we have $C_m \Subset U_{\alpha(m)}$ and $C_m \in \mathcal{B}'_{\alpha(m)}$ for some $\alpha(m) \in A$. Applying part (c) to the covering $\{C_m\}$, we get a countable locally finite covering $\{V_l\}_{l\in L}$ of K by nonempty open sets such that for each $l \in L$, we have $V_l \Subset C_{m(l)}$ for some $m(l) \in M$. Finally, we apply part (c) (one last time) to the covering $\{V_l\}$ and the basis for the topology in X consisting of all open sets B such that either $B \cap K = \emptyset$ or for some $l \in L$, we have $B \subset V_l$, $B \in \mathcal{B}_{\alpha(m(l))}$, and $B \Subset C_{m(k)}$ whenever $k \in L$ with $\overline{B} \cap \overline{V}_k \neq \emptyset$. Thus we get a countable locally finite (in X) open covering $\{B_i\}_{i\in I}$ of K such that for each $i \in I$, there is an index $l(i) \in L$ with $B_i \Subset V_{l(i)}$ and $B_i \in \mathcal{B}_{\alpha(m(l(i)))}$, and $B_i \Subset C_{m(k)}$ whenever $k \in L$ with $\overline{B}_i \cap \overline{V}_k \neq \emptyset$. Setting $B'_i \equiv C_{m(l(i))}$ for each $i \in I$, we get a second covering $\{B'_i\}_{i\in I}$ of K such that for each $i \in I$, we have $B'_i \in \mathcal{B}'_{\alpha(m(l(i)))}$, $B'_i \neq \emptyset$, $B_i \Subset B'_i \Subset U_{\alpha(m(l(i)))}$, and $B_i \Subset C_{m(l(j))} = B'_j$ whenever $j \in I$ with $\overline{B}_i \cap \overline{V}_{l(j)} \supset \overline{B}_i \cap \overline{B}_j \neq \emptyset$. Finally, since $\{B_i\}$ is a locally finite family of *nonempty* sets, each of the relatively compact subsets C_m of X will be equal to $C_{m(l(i))}$ (and therefore contain B_i) for only finitely many indices $i \in I$. Therefore, since the family $\{C_m\}$ is locally finite, it follows that the family $\{B'_i\} = \{C_{m(l(i))}\}$ is also locally finite.

The proofs of (e) and (f) are left to the reader (see Exercise 9.3.3). □

Theorem 9.3.7 *Let M be a second countable topological manifold.*

(a) *If K is a closed subset of M and $\mathcal{U} = \{U_i\}_{i\in I}$ is a covering of K by open subsets of M, then there exists a countable family of continuous functions $\{\lambda_j\}_{j\in J}$ on M with values in* $[0, 1]$ *such that the family* $\{\operatorname{supp}\lambda_j\}_{j\in J}$ *is locally finite in M, $\sum_{j\in J}\lambda_j \equiv 1$ on a neighborhood of K, and for each $j \in J$, we have* $\operatorname{supp}\lambda_j \subset U_i$ *for some $i \in I$. Moreover, if M is a $\mathcal{C}^\infty$ manifold, then we may choose the functions $\{\lambda_j\}$ to be of class $\mathcal{C}^\infty$. If $\mathcal{U}$ is locally finite and countable, then we may choose the family of functions so that $J = I$ and* $\operatorname{supp}\lambda_i \subset U_i$ *for each index i.*

(b) *If K_0 and K_1 are disjoint closed subsets of M, then there exists a continuous function $\lambda\colon M \to$* $[0, 1]$ *such that $\lambda \equiv 0$ on a neighborhood of K_0 and $\lambda \equiv 1$ on a neighborhood of K_1. Moreover, if M is a $\mathcal{C}^\infty$ manifold, then we may choose λ to be of class $\mathcal{C}^\infty$.*

Remark Recall that the *support* of a function is the closure of the complement of its zero set. The theorem holds for M a second countable locally compact Hausdorff space (see, for example, [Mu]), but the proof for M a manifold is easier.

Proof of Theorem 9.3.7 Let $n = \dim M$. For the proof of (a), we fix a closed set $K \subset M$ and a covering $\mathcal{U} = \{U_i\}_{i\in I}$ of K by open subsets of M. Applying Lemma 9.3.6, we see that we may assume without loss of generality that $\mathcal{U}$ is countable and locally finite. Applying Lemma 9.3.6 again, we get a countable locally finite family of local charts $\{(B'_\nu, \Phi_\nu, \Phi_\nu(B'_\nu))\}_{\nu\in N}$ in M and a covering $\{B_\nu\}_{\nu\in N}$ of M such that for each $\nu \in N$, $B_\nu \Subset B'_\nu$, $\Phi_\nu(B_\nu) = B_{\mathbb{R}^n}(0; 1) \Subset \Phi_\nu(B'_\nu)$, and either $B'_\nu \subset M \setminus K$ or $B'_\nu \subset U_{i(\nu)}$ for some index $i(\nu) \in I$. If M is a $\mathcal{C}^\infty$ manifold, then we may also assume that Φ_ν is a diffeomorphism for each ν. It is easy to verify that the function $\rho\colon \mathbb{R} \to [0,\infty)$ given by

$$t \mapsto \begin{cases} e^{1/(t-1)} & \text{if } t < 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

is of class $\mathcal{C}^\infty$ on $\mathbb{R}$. Hence, for each $\nu \in N$, $\eta_\nu \equiv \rho(|\Phi_\nu|^2)$ is a nonnegative continuous function that has compact support contained in B'_ν and that is positive on B_ν. Moreover, η_ν is of class $\mathcal{C}^\infty$ if M is a $\mathcal{C}^\infty$ manifold. The functions $\{\lambda_i\}_{i\in I}$ given by

$$\lambda_i \equiv \frac{\sum_{\nu\in N, i=i(\nu)} \eta_\nu}{\sum_{\nu\in N} \eta_\nu} \quad \forall i \in I$$

(note that in each of the above sums, locally, all but finitely many terms vanish) then have the properties required in (a). The verification of this and the proof of (b) are left to the reader (see Exercise 9.3.4). □

Definition 9.3.8 If $\mathcal{U} = \{U_i\}_{i\in I}$ is an open covering of a topological space M, then any family of continuous functions $\{\lambda_j\}_{j\in J}$ on M with values in $[0,1]$ such that the family $\{\operatorname{supp}\lambda_j\}_{j\in J}$ is locally finite in M, $\sum_{j\in J}\lambda_j \equiv 1$ on M, and for each $j \in J$, $\operatorname{supp}\lambda_j \subset U_i$ for some $i \in I$, is called a *partition of unity subordinate to the covering* $\mathcal{U}$. If M is a $\mathcal{C}^\infty$ manifold and λ_j is of class $\mathcal{C}^\infty$ for each $j \in J$, then $\{\lambda_j\}_{j\in J}$ is called a $\mathcal{C}^\infty$ *partition of unity*.

Definition 9.3.9 A continuous mapping $\Phi\colon X \to Y$ of Hausdorff spaces is called *proper* if the inverse image of every compact subset of Y is compact.

Definition 9.3.10 A real-valued function ρ on a Hausdorff space X is an *exhaustion function* if $\{x \in X \mid \rho(x) < a\} \Subset X$ for every $a \in \mathbb{R}$. A sequence of sets $\{D_\nu\}$ in X such that $D_\nu \Subset \overset{\circ}{D}_{\nu+1}$ for each ν and $\bigcup_\nu D_\nu = X$ is called an *exhaustion of* X *by the sets* $\{D_\nu\}$.

Remarks 1. If ρ is an exhaustion function, then any function $\tau \geq \rho$ is also an exhaustion function.

2. A continuous function $\rho\colon X \to \mathbb{R}$ is an exhaustion function if and only if ρ is bounded below and ρ is a proper mapping.

Proposition 9.3.11 *Every second countable $\mathcal{C}^\infty$ manifold M admits a positive $\mathcal{C}^\infty$ exhaustion function. In fact, for every continuous real-valued function τ on M, there exists a $\mathcal{C}^\infty$ exhaustion function ρ such that $\rho > \tau$ on M.*

Proof We may assume that M is noncompact, and we may choose a countable locally finite covering $\{U_\nu\}$ by relatively compact open subsets and a $\mathcal{C}^\infty$ partition of unity $\{\lambda_\nu\}$ with $\operatorname{supp}\lambda_\nu \subset U_\nu$ for each ν. Given a continuous function $\tau\colon M \to \mathbb{R}$, the locally finite sum $\sum_{\nu=1}^\infty (\nu + \max_{\operatorname{supp}\lambda_\nu}|\tau|)\cdot\lambda_\nu$ then gives a positive $\mathcal{C}^\infty$ exhaustion function that is greater than τ. □

Exercises for Sect. 9.3

9.3.1 Prove Theorem 9.3.2.
9.3.2 Prove Lemma 9.3.5.
9.3.3 Prove parts (e) and (f) of Lemma 9.3.6.
9.3.4 Verify that the functions $\{\lambda_i\}_{i\in I}$ constructed in the proof of part (a) of Theorem 9.3.7 have the required properties. Also prove part (b) of the theorem.
9.3.5 Prove that if D is a subset of a second countable manifold M, and every sequence in D admits a subsequence that converges to a point in D, then D is compact (this is a partial converse of part (b) of Theorem 9.3.2).

9.4 The Tangent and Cotangent Bundles

Recall that if $S = \{g = 0\}$ for a real-valued $\mathcal{C}^\infty$ function g with nonvanishing gradient on an open subset of $\mathbb{R}^3$, then a *tangent vector* to S at a point $p \in S$ is a vector v along which the total differential $(dg)_p$ vanishes. The *tangent plane* to S at p is the plane through p that is parallel to each such tangent vector v. For any such tangent vector v and any real-valued $\mathcal{C}^\infty$ function f on a neighborhood of p in $\mathbb{R}^3$, the value of $(df)_p(v)$ depends only on the values of the restriction of f to a neighborhood of p in S. One may turn this around and view v as a linear functional $f \mapsto (df)_p(v) \in \mathbb{R}$ that is actually defined on the *germs* of $\mathcal{C}^\infty$ functions at p. This point of view provides an efficient approach for defining tangent vectors (complex as well as real) for a smooth manifold.

Definition 9.4.1 Let M be a smooth manifold and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $p \in M$, and let $\sim_p$ be the equivalence relation on the set of $\mathbb{F}$-valued $\mathcal{C}^\infty$ functions that are defined on a neighborhood of p determined by

$$f \sim_p g \quad \iff \quad f = g \text{ on some neighborhood of } p.$$

Each of the associated equivalence classes is called a *germ of an* ($\mathbb{F}$*-valued*) $\mathcal{C}^\infty$ *function at* p. The set of germs of $\mathcal{C}^\infty$ functions at p is called the *stalk of* $\mathcal{C}^\infty$ *at* p and is denoted by $\mathcal{C}^\infty_p$ or $\mathcal{E}_p$. We denote by $\mathrm{germ}_p f$ the germ represented by a local $\mathcal{C}^\infty$ function f on a neighborhood of p; i.e., $\mathrm{germ}_p f$ is the *germ of* f *at* p. We also consider $\mathcal{E}_p$ as an algebra over $\mathbb{F}$ with the natural operations. More precisely, for neighborhoods U and V of p and functions $f \in \mathcal{C}^\infty(U)$ and $g \in \mathcal{C}^\infty(V)$, we define

$$c \cdot \mathrm{germ}_p f \equiv \mathrm{germ}_p(cf) \quad \forall c \in \mathbb{F},$$

$$\mathrm{germ}_p f \cdot \mathrm{germ}_p g \equiv \mathrm{germ}_p(f\restriction_{U\cap V} \cdot g\restriction_{U\cap V}),$$

$$\mathrm{germ}_p f + \mathrm{germ}_p g \equiv \mathrm{germ}_p(f\restriction_{U\cap V} + g\restriction_{U\cap V}),$$

$$\mathrm{germ}_p f - \mathrm{germ}_p g \equiv \mathrm{germ}_p(f\restriction_{U\cap V} - g\restriction_{U\cap V}).$$

(b) For each point $p \in M$, a *tangent vector* (*over* $\mathbb{F}$) *at* p is a linear functional $v : \mathcal{E}_p \to \mathbb{F}$ that satisfies

$$\begin{aligned} v(\mathrm{germ}_p(fg)) &= v(\mathrm{germ}_p f \cdot \mathrm{germ}_p g) \\ &= v(\mathrm{germ}_p f) \cdot g(p) + f(p) \cdot v(\mathrm{germ}_p g) \end{aligned}$$

for all germs $\mathrm{germ}_p f, \mathrm{germ}_p g \in \mathcal{E}_p$ (that is, v is a *linear derivation* on the algebra $\mathcal{E}_p$). For $\mathbb{F} = \mathbb{R}$ ($\mathbb{F} = \mathbb{C}$), v is also called a *real tangent vector* (respectively, a *complex tangent vector*). Given a $\mathcal{C}^\infty$ $\mathbb{F}$-valued function f on a neighborhood of p, we also write $v(f) \equiv v(\mathrm{germ}_p f)$.

(c) For each point $p \in M$, the (real) vector space of real tangent vectors at p is called the *tangent space* (or *real tangent space*) *to* M *at* p and is denoted by $T_p M$. The complex vector space of complex tangent vectors at p is called the *complexified tangent space to* M *at* p and is denoted by $(T_p M)_{\mathbb{C}}$.

(d) For each point $p \in M$ and each $\mathbb{F}$-valued $\mathcal{C}^\infty$ function f on a neighborhood of p, the *differential of* f *at* p is the linear functional $(df)_p$ on the tangent space at p given by $(df)_p(v) \equiv v(f)$ for each tangent vector v at p. If $F = (f_1, \dots, f_m) : U \to \mathbb{F}^m$ is a $\mathcal{C}^\infty$ mapping (equivalently, each of the functions $f_1, \dots, f_m$ is of class $\mathcal{C}^\infty$) of a neighborhood U into $\mathbb{F}^m$, then we define $(dF)_p \equiv ((df_1)_p, \dots, (df_m)_p)$, and for each tangent vector v to M at p, we set $v(F) \equiv (dF)_p(v)$.

(e) If $(U, \Phi = (x_1, \dots, x_n), U')$ is a local $\mathcal{C}^\infty$ chart in M, f is a function defined on a neighborhood of a point $p \in U$, and $j \in \{1, \dots, n\}$, then we define

$$\frac{\partial f}{\partial x_j}(p) \equiv \frac{\partial}{\partial t_j}[f(\Phi^{-1}(t_1, \dots, t_n))]\big|_{(t_1,\dots,t_n)=\Phi(p)},$$

provided this partial derivative exists. The corresponding tangent vector

$$f \mapsto \frac{\partial f}{\partial x_j}(p) \quad \text{is denoted by} \quad \left(\frac{\partial}{\partial x_j}\right)_p.$$

(f) For each $p \in M$, the *cotangent space* and the *complexified cotangent space* to M at p are, respectively, the dual spaces

$$T_p^* M \equiv (T_p M)^* \quad \text{and} \quad (T_p^* M)_{\mathbb{C}} \equiv [(T_p M)_{\mathbb{C}}]^*.$$

(g) If $\Phi : M \to N$ is a $\mathcal{C}^\infty$ mapping of M into a $\mathcal{C}^\infty$ manifold N, then the induced *tangent maps* (or *pushforward* maps) at $p \in M$ are the real linear map $T_p M \to T_{\Phi(p)} N$ and the complex linear map $(T_p M)_{\mathbb{C}} \to (T_{\Phi(p)} N)_{\mathbb{C}}$, which are both denoted by $(\Phi_*)_p$ and are given by

$$(\Phi_*)_p(v)(f) \equiv v(f \circ \Phi)$$

for every $\mathcal{C}^\infty$ function f on a neighborhood of $\Phi(p)$ and for every tangent vector v at p (with f $\mathbb{R}$-valued and $v \in T_p M$ if we are considering the tangent mapping of the real tangent spaces). The associated *pullback mappings* (or *cotangent mappings*) are the real linear map $T^*_{\Phi(p)}N \to T^*_p M$ and the complex linear map $(T^*_{\Phi(p)}N)_{\mathbb{C}} \to (T^*_p M)_{\mathbb{C}}$, which are both denoted by $(\Phi^*)_p$ and are given by the adjoints associated to the tangent mappings. That is,

$$(\Phi^*)_p(\alpha)(v) \equiv \alpha((\Phi_*)_p(v))$$

for every linear functional α on the tangent space to N at $\Phi(p)$ and every tangent vector v to M at p (with α and v real when we are considering the real (co)tangent spaces, and complex when we are considering the complexified (co)tangent spaces).

Remarks 1. If v is a tangent vector to a smooth manifold M at a point $p \in M$ and f is a $\mathcal{C}^\infty$ function that is equal to a constant c on a neighborhood of p, then $v(f) = 0$. For $v(1) = v(1 \cdot 1) = v(1) \cdot 1 + 1 \cdot v(1) = 2v(1)$, and hence $v(c) = v(c \cdot 1) = cv(1) = 0$.

2. *Product rule*. It follows from the definition that if f and g are $\mathcal{C}^\infty$ functions on a neighborhood of a point p in a smooth manifold M, then

$$(d(fg))_p = g(p) \cdot (df)_p + f(p) \cdot (dg)_p.$$

3. *Chain rule*. Let $\Phi\colon M \to N$ and $\Psi\colon N \to P$ be $\mathcal{C}^\infty$ mappings of $\mathcal{C}^\infty$ manifolds M, N, and P, and let $p \in M$. Then

$$([\Psi \circ \Phi]_*)_p = (\Psi_*)_{\Phi(p)} \circ (\Phi_*)_p \quad \text{and} \quad ([\Psi \circ \Phi]^*)_p = (\Phi^*)_p \circ (\Psi^*)_{\Phi(p)}.$$

In particular, if Φ is a diffeomorphism (i.e., Φ is bijective with $\mathcal{C}^\infty$ inverse), then $(\Phi_*)_p$ and $(\Phi^*)_p$ are linear isomorphisms with inverse mappings $(\Phi_*)_p^{-1} = ((\Phi^{-1})_*)_{\Phi(p)}$ and $(\Phi^*)_p^{-1} = ((\Phi^{-1})^*)_{\Phi(p)}$. If $\Psi\colon N \to P = \mathbb{R}$ or $\mathbb{C}$, then $(d(\Psi \circ \Phi))_p = (d\Psi)_{\Phi(p)} \circ (\Phi_*)_p$ (on the real tangent space for $P = \mathbb{R}$ and on the complexified tangent space for $P = \mathbb{C}$). For given a tangent vector v, we have

$$(d(\Psi \circ \Phi))_p(v) = v(\Psi \circ \Phi) = (\Phi_*)_p(v)(\Psi) = (d\Psi)_{\Phi(p)} \circ (\Phi_*)_p(v).$$

A similar argument shows that the above equality also holds if $\Psi\colon N \to P = \mathbb{R}^m$ or $\mathbb{C}^m$. Furthermore, taking $N = P = \mathbb{R}^m$ or $\mathbb{C}^m$ and $\Psi = \mathrm{Id}_P$, we see that the linear maps $(\Phi_*)_p$ and $(d\Phi)_p$ have the same kernel and the same rank.

Proposition 9.4.2 *For any $\mathcal{C}^\infty$ manifold M of dimension n and any point $p \in M$, we have the following*:

(a) *The map $T_pM \to (T_pM)_{\mathbb{C}}$ that associates to each real tangent vector v the complex tangent vector given by $f \mapsto v(\operatorname{Re} f) + iv(\operatorname{Im} f)$ for every $\mathcal{C}^\infty$ function f on a neighborhood of p is an injective linear map. Identifying T_pM with a real vector subspace of $(T_pM)_{\mathbb{C}}$ under the above injection, we get the real vector space direct sum decomposition $(T_pM)_{\mathbb{C}} = T_pM \oplus iT_pM$; and hence we may identify the complexified tangent space $(T_pM)_{\mathbb{C}}$ with the complexification of the real tangent space T_pM (see Definition* 8.1.4) *with the real and* *imaginary projections given by* $\mathrm{Re}\colon u + iv \mapsto u$ *and* $\mathrm{Im}\colon u + iv \mapsto v$ *for each pair* $u, v \in T_pM$. *The map* $T_p^*M \to (T_p^*M)_{\mathbb{C}}$ *that associates to each real linear functional* $\alpha \in T_p^*M$ *the complex linear functional in* $(T_p^*M)_{\mathbb{C}}$ *given by* $v \mapsto \alpha(\mathrm{Re}\, v) + i\alpha(\mathrm{Im}\, v)$ *is an injective real linear map. Identifying* T_p^*M *with a real vector subspace of* $(T_p^*M)_{\mathbb{C}}$ *under the above injection, we get the real vector space direct sum decomposition* $(T_p^*M)_{\mathbb{C}} = T_p^*M \oplus iT_p^*M$; *and we may identify the complexified cotangent space* $(T_p^*M)_{\mathbb{C}}$ *with the complexification of the real cotangent space* T_p^*M *with the real and imaginary projections given by* $\mathrm{Re}\colon \alpha + i\beta \mapsto \alpha$ *and* $\mathrm{Im}\colon \alpha + i\beta \mapsto \beta$ *for each pair* $\alpha, \beta \in T_p^*M$.

(b) *The tangent space* T_pM *is of real dimension* n, *and* $(T_pM)_{\mathbb{C}}$ *is of complex dimension* n. *In fact, if* $(U, \Phi = (x_1, \dots, x_n))$ *is any local* $\mathcal{C}^\infty$ *coordinate neighborhood of* p, *then the partial derivative operators* $\{(\partial/\partial x_j)_p\}_{j=1}^n$ *form a real basis for* T_pM *and a complex basis for* $(T_pM)_{\mathbb{C}}$. *The differentials* $\{(dx_j)_p\}_{j=1}^n$ *form the real dual basis for* T_p^*M *and the complex dual basis for* $(T_p^*M)_{\mathbb{C}}$.

Proof That the mapping $f \mapsto v(\mathrm{Re}\, f) + iv(\mathrm{Im}\, f)$ is a complex tangent vector for each $v \in T_pM$, and that the corresponding mapping $T_pM \to (T_pM)_{\mathbb{C}}$ is an injective real linear map, are left to the reader. Moreover, if $v \in T_pM \setminus \{0\}$, then $v(f) \in \mathbb{R} \setminus \{0\}$ for some real-valued $\mathcal{C}^\infty$ function f on a neighborhood of p. Hence, if $w \in T_pM$, then $iw(f) \in i\mathbb{R}$, so $iw \neq v$. Thus $T_pM \cap iT_pM = \{0\}$. On the other hand, given a nonzero tangent vector $w \in (T_pM)_{\mathbb{C}}$, for each real $\mathcal{C}^\infty$ germ $\mathrm{germ}_p f$, we may set $u(f) \equiv \mathrm{Re}[w(f)]$ and $v(f) \equiv \mathrm{Im}[w(f)]$. The mappings u and v are real tangent vectors at p. For they are obviously real linear, and for each pair of real $\mathcal{C}^\infty$ germs $\mathrm{germ}_p f, \mathrm{germ}_p g \in \mathcal{E}_p$, we have

$$\begin{aligned} u(\mathrm{germ}_p(fg)) &= \mathrm{Re}[w(f) \cdot g(p) + f(p)w(g)] \\ &= u(\mathrm{germ}_p f) \cdot g(p) + f(p) \cdot u(\mathrm{germ}_p g), \end{aligned}$$

and the analogous equalities hold for v. Identifying u and v with their images in $(T_pM)_{\mathbb{C}}$, we also get $w = u + iv$. Thus we have the direct sum decomposition $(T_pM)_{\mathbb{C}} = T_pM \oplus iT_pM$ of $(T_pM)_{\mathbb{C}}$ into real subspaces. Letting $\mathcal{V}$ be the complexification of the real vector space T_pM, we have the real linear map $\mathcal{V} \to (T_pM)_{\mathbb{C}}$ given by $w \mapsto \mathrm{Re}\, w + i\, \mathrm{Im}\, w$. It is now easy to verify that this map is a complex linear isomorphism. The verification that the analogous decomposition of the cotangent spaces holds is left to the reader.

Suppose now that $(U, \Phi = (x_1, \dots, x_n))$ is a local $\mathcal{C}^\infty$ coordinate neighborhood of p. For each pair $i, j = 1, \dots, n$, we have

$$\frac{\partial x_i}{\partial x_j}(p) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j; \end{cases}$$

so the real tangent vectors $\{(\partial/\partial x_j)_p\}_{j=1}^n$ are linearly independent. Conversely, given a real tangent vector $v \in T_pM$, we may set $u \equiv \sum_{j=1}^n v_j \cdot (\partial/\partial x_j)_p \in T_pM$, where $v_j \equiv v(x_j) \in \mathbb{R}$ for each $j = 1, \dots, n$. Given a $\mathcal{C}^\infty$ function f on a neighborhood of p, Lemma 7.2.6 provides constants $\{b_j\}_{j=1}^n$ and $\mathcal{C}^\infty$ functions $\{c_{ij}\}_{i,j=1}^n$ on a neighborhood of p such that

$$f = f(p) + \sum_{j=1}^n b_j(x_j - x_j(p)) + \sum_{i,j=1}^n c_{ij}(x_i - x_i(p))(x_j - x_j(p))$$

on a neighborhood of p. Thus, using the fact that u and v are derivations, one may now easily check that

$$v(f) = \sum_{j=1}^n b_j v_j + 0 = u(f).$$

Thus $v = u$, and hence $\{(\partial/\partial x_j)_p\}_{j=1}^n$ is a basis for T_pM. Clearly, $(dx_j)_p((\partial/\partial x_i)_p) = \delta_{ij}$ for all $i, j = 1, \dots, n$, so $\{(dx_j)_p\}_{j=1}^n$ is the associated real dual basis for T_pM. The corresponding claims regarding the complexifications also follow. □

Remarks 1. It follows from the above proposition that if $(U, (x_1, \dots, x_n))$ is a local $\mathcal{C}^\infty$ coordinate neighborhood in M, v is a tangent vector to M at $p \in U$, α is an element of the cotangent space at p, and f is a $\mathcal{C}^\infty$ function a neighborhood of a point p, then

$$v = \sum_{j=1}^n (dx_j)_p(v) \cdot \left(\frac{\partial}{\partial x_j}\right)_p = \sum_{j=1}^n v(x_j) \cdot \left(\frac{\partial}{\partial x_j}\right)_p,$$

$$\alpha = \sum_{j=1}^n \alpha\left(\left(\frac{\partial}{\partial x_j}\right)_p\right) \cdot (dx_j)_p,$$

and

$$(df)_p = \sum_{j=1}^n df\left(\left(\frac{\partial}{\partial x_j}\right)_p\right)(dx_j)_p = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(p)(dx_j)_p.$$

2. Given a complex tangent vector $w \in (T_pM)_{\mathbb{C}}$, we have $w = u + iv$ with $u, v \in T_pM$. Under the identification with the complexification of T_pM, we have $\operatorname{Re} w \equiv u$ and $\operatorname{Im} w = v$. It should be noted that for a complex $\mathcal{C}^\infty$ function germ $\operatorname{germ}_p f$, $u(f) = u(\operatorname{Re} f) + iu(\operatorname{Im} f)$ need not be equal to $\operatorname{Re}(w(f)) = u(\operatorname{Re} f) - v(\operatorname{Im} f)$.

Definition 9.4.3 Let M be a smooth manifold.

(a) The (*real*) *tangent bundle* and the *complexified tangent bundle* of M are given by

$$TM \equiv \bigcup_{p \in M} T_pM \quad \text{and} \quad (TM)_{\mathbb{C}} \equiv \bigcup_{p \in M} (T_pM)_{\mathbb{C}},$$

respectively. The associated *tangent bundle projections* are the (surjective) mappings

$$\Pi_{TM}\colon TM \to M \quad \text{and} \quad \Pi_{(TM)_{\mathbb{C}}}\colon (TM)_{\mathbb{C}} \to M$$

given by $v \mapsto p$ for each point $p \in M$ and each tangent vector v in T_pM or $(T_pM)_{\mathbb{C}}$.

(b) The (*real*) *cotangent bundle* and the *complexified cotangent bundle* of M are given by

$$T^*M \equiv \bigcup_{p\in M} T_p^*M \quad \text{and} \quad (T^*M)_{\mathbb{C}} \equiv \bigcup_{p\in M} (T_p^*M)_{\mathbb{C}},$$

respectively. The associated *cotangent bundle projections* are the (surjective) mappings

$$\Pi_{T^*M}\colon T^*M \to M \quad \text{and} \quad \Pi_{(T^*M)_{\mathbb{C}}}\colon (T^*M)_{\mathbb{C}} \to M$$

given by $\alpha \mapsto p$ for each point $p \in M$ and each linear functional α in T_p^*M or $(T_p^*M)_{\mathbb{C}}$.

(d) Depending on the context, for each $\mathcal{C}^\infty$ function f on an open set $U \subset M$, the *differential* (or *exterior derivative*) of f is the function $df\colon \Pi_{TM}^{-1}(U) \to \mathbb{R}$ or $df\colon \Pi_{(TM)_{\mathbb{C}}}^{-1}(U) \to \mathbb{C}$ with restriction to each tangent space equal to $(df)_p$; that is, $df(v) = v(f)$ for each tangent vector v at each point $p \in U$. For $F = (f_1, \dots, f_m)$ with $f_1, \dots, f_m \in \mathcal{C}^\infty(U)$, we define $dF \equiv (df_1, \dots, df_m)$.

(e) If $\Phi\colon M \to N$ is a $\mathcal{C}^\infty$ mapping into a $\mathcal{C}^\infty$ manifold N, then the associated *tangent maps* (or *pushforward maps*) are the maps $\Phi_*\colon TM \to TN$ and $\Phi_*\colon (TM)_{\mathbb{C}} \to (TN)_{\mathbb{C}}$ given by $\Phi_*(v) = (\Phi_*)_p(v)$ for every tangent vector v to M at a point $p \in M$. The associated *pullback mappings* (or *cotangent mappings*) are the maps $\Phi^*\colon T^*N \to T^*M$ and $\Phi^*\colon (T^*N)_{\mathbb{C}} \to (T^*M)_{\mathbb{C}}$ given by $\Phi^*\alpha = (\Phi^*)_p\alpha$ for every element α of $T^*_{\Phi(p)}N$ or $(T^*_{\Phi(p)}N)_{\mathbb{C}}$ with $p \in M$.

Remarks 1. The tangent and cotangent bundles of a smooth manifold M admit natural $\mathcal{C}^\infty$ (manifold) structures (see Exercise 9.4.3). They are also examples of $\mathcal{C}^\infty$ *vector bundles* (see, for example [Ns3] or [Wel]).

2. For any open subset Ω of M with inclusion mapping $\iota\colon \Omega \hookrightarrow M$, we identify $T\Omega$ and $T^*\Omega$ with the sets $\Pi_{TM}^{-1}(\Omega) \subset TM$ and $\Pi_{T^*M}^{-1}(\Omega) \subset T^*M$, respectively, under the bijections $\iota_*\colon T\Omega \to \Pi_{TM}^{-1}(\Omega)$ and $\iota^*\colon \Pi_{T^*M}^{-1}(\Omega) \to T^*\Omega$. We make the analogous identifications for the complexified tangent and complexified cotangent bundles.

3. For any open set $\Omega \subset \mathbb{R}^n$, we identify the tangent bundles $T\Omega$ and $(T\Omega)_{\mathbb{C}}$ with the products $\Omega \times \mathbb{R}^n$ and $\Omega \times \mathbb{C}^n$, respectively, under the bijections

$$(\Pi_{T\Omega}, d(\mathrm{Id}_\Omega))\colon T\Omega \to \Omega \times \mathbb{R}^n \quad \text{and} \quad (\Pi_{(T\Omega)_{\mathbb{C}}}, d(\mathrm{Id}_\Omega))\colon (T\Omega)_{\mathbb{C}} \to \Omega \times \mathbb{C}^n.$$

For any $\mathcal{C}^\infty$ map $F\colon \Omega \to \mathbb{R}^m$, we get the corresponding identification of the differential mapping dF on $T\Omega$ with the differential mapping $(p, v) \mapsto (dF)_p(v)$ on

$\Omega \times \mathbb{R}^m$ (see Definition 7.2.2). We make the analogous identification for a $\mathcal{C}^\infty$ mapping $F\colon \Omega \to \mathbb{C}^m$.

4. For any $\mathcal{C}^1$ function f on a neighborhood of a point p in a smooth manifold M, and any tangent vector v to M at p, we define

$$df(v) = v(f) \equiv \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(p),$$

where $(x_1, \dots, x_n)$ are local $\mathcal{C}^\infty$ coordinates in a neighborhood of p and $v_j = dx_j(v)$ for $j = 1, \dots, n$. The verification that $v(f)$ is independent of the choice of local coordinates, and hence that we have an induced *differential* $df\colon v \mapsto df(v)$, is left to the reader. Note also that this definition is consistent with the earlier definitions for f of class $\mathcal{C}^\infty$. Similarly, if $\Phi\colon M \to N$ is a $\mathcal{C}^1$ mapping of smooth manifolds M and N, then we define $\Phi_*\colon TM \to TN$ and $\Phi_*\colon (TM)_{\mathbb{C}} \to (TN)_{\mathbb{C}}$ by $(\Phi_* v)(f) \equiv v(f \circ \Phi)$ for each tangent vector v to M and each local $\mathcal{C}^\infty$ function f in N. It is easy to verify that $\Phi_* v$ is a derivation and that we also have $(\Phi_* v)(f) \equiv v(f \circ \Phi)$ for every $\mathcal{C}^1$ function f. The pullback mappings are given by $\alpha \mapsto \Phi^*\alpha \equiv \alpha \circ \Phi_*$. The mappings Φ_* and Φ^* are well-defined mappings of the corresponding tangent and cotangent bundles, respectively. Moreover, if $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_s)$ are local $\mathcal{C}^\infty$ coordinates in a neighborhood of $p \in M$ and a neighborhood of $\Phi(p)$, respectively, then for every tangent vector $v = \sum_{j=1} v_j \cdot (\partial/\partial x_j)_p$ (i.e., $v_j = dx_j(v)$ for each $j = 1, \dots, n$), we have

$$\Phi_* v = \sum_{k=1}^{s} \sum_{j=1}^{n} v_j \cdot \frac{\partial(y_k \circ \Phi)}{\partial x_j}(p) \cdot \left(\frac{\partial}{\partial y_k}\right)_p,$$

and for every element $\alpha = \sum_{k=1}^{s} a_k (dy_k)_{\Phi(p)}$ of the cotangent space to N at $\Phi(p)$ (i.e., $a_k = \alpha((\partial/\partial y_k)_p)$ for each $k = 1, \dots, s$), we have

$$\Phi^*\alpha = \sum_{j=1}^{n} \sum_{k=1}^{s} a_k \cdot \frac{\partial(y_k \circ \Phi)}{\partial x_j}(p) \cdot (dx_j)_p.$$

For any $\mathcal{C}^1$ mapping $\Psi\colon N \to P$ of N into a $\mathcal{C}^\infty$ manifold P, we have

$$(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_* \quad \text{and} \quad (\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*.$$

For any $\mathcal{C}^1$ function g on N, we have $d(g \circ \Phi) = (dg) \circ \Phi_* = \Phi^* dg$.

5. If $\Phi\colon M \to N$ is a $\mathcal{C}^1$ mapping of smooth manifolds M and N and $\Phi_* \equiv 0$, then Φ is locally constant (see Exercise 9.4.1).

Definition 9.4.4 If $\gamma\colon I \to M$ is a $\mathcal{C}^1$ mapping of an interval I into a smooth manifold M (i.e., γ is $\mathcal{C}^1$ on the interior and γ admits local $\mathcal{C}^1$ extensions at any endpoint contained in I), then the *tangent vector* $\dot{\gamma}(t_0)$ to γ at each $t_0 \in I$ is given by

$$\dot{\gamma}(t_0) \equiv \gamma_* \left(\frac{d}{dt}\right)_{t_0},$$

where t denotes the standard coordinate function on $\mathbb{R}$ (at an endpoint in I, the tangent map is that of a local $\mathcal{C}^1$ extension).

By the remarks preceding this definition, if $\Phi \circ \gamma = (\gamma_1, \dots, \gamma_n)$ in a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, \Phi = (x_1, \dots, x_n))$ of $\gamma(t_0)$, then

$$\dot{\gamma}(t_0) = \sum_{j=1}^{n} \gamma_j{}'(t_0) \left(\frac{\partial}{\partial x_j} \right)_{\gamma(t_0)}.$$

Definition 9.4.5 Let S be a subset of a smooth manifold M of dimension n.

(a) A (*real*) *vector field* (*complex vector field*) on S in M is a map $v\colon S \to TM$ (respectively, $v\colon S \to (TM)_{\mathbb{C}}$) with $\Pi_{TM} \circ v = \mathrm{Id}_S$ (respectively, $\Pi_{(TM)_{\mathbb{C}}} \circ v = \mathrm{Id}_S$). We usually denote the value $v(p)$ by $v_p \in T_pM$ (respectively, $(T_pM)_{\mathbb{C}}$) for each point $p \in S$. For any $\mathcal{C}^1$ function f on an open set $U \subset M$, $df(v) = v(f)$ denotes the function on $S \cap U$ given by $p \mapsto df(v_p) = v_p(f)$ (that is, $df(v) = df \circ v$). If $\Phi\colon U \to N$ is a $\mathcal{C}^1$ mapping of U into a smooth manifold N, then $\Phi_* v$ denotes the mapping $S \cap U \to (TN)_{\mathbb{C}}$ (or TN) given by $p \mapsto \Phi_* v_p$ (that is, $\Phi_* v = \Phi_* \circ v$).

(b) The *coefficient functions* (or simply the *coefficients*) of a vector field v on S with respect to (or in) a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, (x_1, \dots, x_n))$ are the functions on $S \cap U$ given by $v_j \equiv dx_j(v) = v(x_j)$ for $j = 1, \dots, n$ (that is, $v = \sum v_j \frac{\partial}{\partial x_j}$). We say that v is *continuous* if v has continuous coefficients in every local $\mathcal{C}^\infty$ coordinate neighborhood. For S an open set and $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, we say that v is of *class* $\mathcal{C}^k$ if v has $\mathcal{C}^k$ coefficients in every local $\mathcal{C}^\infty$ coordinate neighborhood.

Remarks 1. For the purposes of this book, although it is occasionally convenient to consider vector fields, it is usually more convenient to consider differential forms instead (see Sect. 9.5).

2. A vector field v is of class $\mathcal{C}^k$ if and only if each point admits a local $\mathcal{C}^\infty$ coordinate neighborhood in which the coefficients of v are of class $\mathcal{C}^k$ (see Exercise 9.4.2).

3. For a function f and vector fields u and v, we define vector fields fv and $u + v$ by $p \mapsto f(p)v_p$ and $p \mapsto u_p + v_p$, respectively.

We close this section with some terminology:

Definition 9.4.6 Let $\Phi\colon M \to N$ be a $\mathcal{C}^\infty$ mapping of smooth manifolds M and N. A point $p \in M$ is a *critical point* of Φ if the linear mapping $(\Phi_*)_p\colon T_pM \to T_{\Phi(p)}N$ is *not* surjective. The image of a critical point is called a *critical value* of Φ. Every point in N that is not a critical value is called a *regular value* of Φ (in particular, each point in $N \setminus \Phi(M)$ is a regular value).

Exercises for Sect. 9.4

9.4.1 Prove that if $\Phi\colon M \to N$ is a $\mathcal{C}^1$ mapping of smooth manifolds M and N and $\Phi_* \equiv 0$, then Φ is locally constant.

9.4.2 Let M be a smooth manifold, let $S \subset M$, and let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

(a) For a vector field v on S, prove that the following are equivalent:

(i) The vector field v is of class $\mathcal{C}^k$.

(ii) For every point in S, there exists a local $\mathcal{C}^\infty$ coordinate neighborhood with respect to which the coefficients of v are of class $\mathcal{C}^k$.

(iii) The function $df(v) = v(f)\colon p \mapsto v_p(f)$ is of class $\mathcal{C}^k$ for each local $\mathcal{C}^\infty$ function f.

(b) Prove that any sum of $\mathcal{C}^k$ vector fields, and any product of a $\mathcal{C}^k$ function and a $\mathcal{C}^k$ vector field, is of class $\mathcal{C}^k$.

9.4.3 Let M be a smooth manifold of dimension n.

(a) Prove that there are unique $\mathcal{C}^\infty$ (manifold) structures in TM and in $(TM)_\mathbb{C}$ such that for each local $\mathcal{C}^\infty$ chart $(U, \Phi = (x_1, \dots, x_n), U')$ in M, the triples

$$\left(\Pi_{TM}^{-1}(U), (\Phi \circ \Pi_{TM}, d\Phi\restriction_{\Pi_{TM}^{-1}(U)}), U' \times \mathbb{R}^n\right)$$

and

$$\left(\Pi_{(TM)_\mathbb{C}}^{-1}(U), (\Phi \circ \Pi_{(TM)_\mathbb{C}}, d\Phi\restriction_{\Pi_{(TM)_\mathbb{C}}^{-1}(U)}), U' \times \mathbb{C}^n = U' \times \mathbb{R}^{2n}\right)$$

are local $\mathcal{C}^\infty$ charts in TM and $(TM)_\mathbb{C}$, respectively. That is, the associated local $\mathcal{C}^\infty$ chart in the tangent bundle is given by

$$v \mapsto (x_1(p), \dots, x_n(p), dx_1(v), \dots, dx_n(v))$$

for each point $p \in U$ and each tangent vector v to M at p.

(b) Prove that there are unique $\mathcal{C}^\infty$ (manifold) structures in T^*M and in $(T^*M)_\mathbb{C}$ such that for each local $\mathcal{C}^\infty$ chart $(U, \Phi = (x_1, \dots, x_n), U')$ in M, we get local $\mathcal{C}^\infty$ charts

$$\left(\Pi_{T^*M}^{-1}(U), \Psi, U' \times \mathbb{R}^n\right)$$

and

$$\left(\Pi_{(T^*M)_\mathbb{C}}^{-1}(U), \Psi_\mathbb{C}, U' \times \mathbb{C}^n = U' \times \mathbb{R}^{2n}\right)$$

in T^*M and $(T^*M)_\mathbb{C}$, respectively, where the mappings Ψ and $\Psi_\mathbb{C}$ are given by

$$\alpha \mapsto \left(x_1(p), \dots, x_n(p), \alpha\left(\left(\frac{\partial}{\partial x_1}\right)_p\right), \dots, \alpha\left(\left(\frac{\partial}{\partial x_n}\right)_p\right)\right)$$

for each point $p \in U$ and each element α of T_p^*M and $(T_p^*M)_\mathbb{C}$, respectively.

(c) Prove that for each $\mathcal{C}^\infty$ function f on an open set $\Omega \subset M$, the differentials

$$df : \Pi_{TM}^{-1}(\Omega) \to \mathbb{R} \quad \text{and} \quad df : \Pi_{(TM)_{\mathbb{C}}}^{-1}(\Omega) \to \mathbb{C}$$

are of class $\mathcal{C}^\infty$ with respect to the smooth structures provided by (a).

(d) Prove that if $\Phi : M \to N$ is a $\mathcal{C}^\infty$ mapping of M into a $\mathcal{C}^\infty$ manifold N, then the tangent maps

$$\Phi_* : TM \to TN \quad \text{and} \quad \Phi_* : (TM)_{\mathbb{C}} \to (TN)_{\mathbb{C}}$$

and the pullback maps

$$\Phi^* : T^*N \to T^*M \quad \text{and} \quad \Phi^* : (T^*N)_{\mathbb{C}} \to (T^*M)_{\mathbb{C}}$$

are of class $\mathcal{C}^\infty$ with respect to the smooth structures provided by (a) and (b).

(e) Prove that the inclusion mappings $TM \hookrightarrow (TM)_{\mathbb{C}}$ and $T^*M \hookrightarrow (T^*M)_{\mathbb{C}}$ and the real and imaginary projections $TM_{\mathbb{C}} \to TM$ and $T^*M_{\mathbb{C}} \to T^*M$ are $\mathcal{C}^\infty$ mappings.

9.4.4 Let M be a smooth manifold and let $k \in \mathbb{Z}_{>0} \cup \{\infty\}$. Show that if $f \in \mathcal{C}^k(M)$, then the mapping $df : (TM)_{\mathbb{C}} \to \mathbb{C}$ is of class $\mathcal{C}^{k-1}$, where $(TM)_{\mathbb{C}}$ has the $\mathcal{C}^\infty$ structure provided by Exercise 9.4.3 above. Show also that if $\Phi : M \to N$ is a $\mathcal{C}^k$ mapping of M into a smooth manifold N, then the tangent mapping $\Phi_* : (TM)_{\mathbb{C}} \to (TN)_{\mathbb{C}}$ and the pullback mapping $\Phi^* : (T^*N)_{\mathbb{C}} \to (T^*M)_{\mathbb{C}}$ are of class $\mathcal{C}^{k-1}$.

9.4.5 Let v be a vector field on a smooth manifold M, and let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Prove that v is a $\mathcal{C}^k$ vector field if and only if v is of class $\mathcal{C}^k$ as a mapping of M into $(TM)_{\mathbb{C}}$, where $(TM)_{\mathbb{C}}$ has the $\mathcal{C}^\infty$ structure provided by Exercise 9.4.3 above.

9.5 Differential Forms on Smooth Curves and Surfaces

In this section, we consider differential forms (of degree ≤ 2), that is, objects that are locally exterior products in the cotangent bundle (see Sect. 8.2). For simplicity, we restrict our attention to smooth curves and surfaces (which are the only cases we will need).

Definition 9.5.1 Let M be a smooth manifold of dimension $n = 1$ or 2.

(a) For each $r = 1, 2$, we define

$$\Lambda^r T^*M \equiv \bigcup_{p \in M} \Lambda^r T_p^*M \subset \Lambda^r (T^*M)_{\mathbb{C}} \equiv \bigcup_{p \in M} \Lambda^r (T_p^*M)_{\mathbb{C}}.$$

We also set

$$\Lambda^0 T^*M \equiv M \times \mathbb{R} = \bigcup_{p\in M} \{p\} \times \mathbb{R} = \bigcup_{p\in M} \Lambda^0 T_p^* M$$

$$\subset \Lambda^0 (T^*M)_{\mathbb{C}} \equiv M \times \mathbb{C} = \bigcup_{p\in M} \{p\} \times \mathbb{C} = \bigcup_{p\in M} \Lambda^0 (T_p^* M)_{\mathbb{C}},$$

and for $r \in \mathbb{Z}_{>2}$, we set

$$\Lambda^r T^*M = \Lambda^r (T^*M)_{\mathbb{C}} \equiv M \times \{0\} = \bigcup_{p\in M} \{p\} \times \{0\} = \bigcup_{p\in M} (\Lambda^r T_p^* M)_{\mathbb{C}}.$$

For each $r \in \mathbb{Z}_{\geq 0}$, the corresponding *projections* are the (surjective) mappings

$$\Pi_{\Lambda^r T^*M} \colon \Lambda^r T^*M \to M \quad \text{and} \quad \Pi_{\Lambda^r (T^*M)_{\mathbb{C}}} \colon \Lambda^r (T^*M)_{\mathbb{C}} \to M$$

given by $\alpha \mapsto p$ for each point $p \in M$ and each element α of $\Lambda^r T_p^* M$ or $\Lambda^r (T_p^* M)_{\mathbb{C}}$.

(b) Given a $\mathcal{C}^1$ mapping $\Phi\colon M \to N$ into a $\mathcal{C}^\infty$ manifold N of dimension ≤ 2 and a nonnegative integer r, the associated *pullback mappings* at each point $p \in M$ are the real linear map $\Lambda^r T_{\Phi(p)}^* N \to \Lambda^r T_p^* M$ and the complex linear map $\Lambda^r (T_{\Phi(p)}^* N)_{\mathbb{C}} \to \Lambda^r (T_p^* M)_{\mathbb{C}}$, which are both denoted by $(\Phi^*)_p$ and are defined as follows. For $r = 1$, as in Definition 9.4.1, we set $\Phi_p^* \alpha(v) \equiv \alpha(\Phi_* v)$ for every linear functional α on the tangent space to N at $\Phi(p)$ and every tangent vector v to M at p (with α and v real when we are considering the real (co)tangent spaces, and complex when we are considering the complexified (co)tangent spaces). For $r = 2$, we set $\Phi_p^* \alpha(u, v) \equiv \alpha(\Phi_* u, \Phi_* v)$ for every skew-symmetric bilinear function α on the tangent space to N at $\Phi(p)$ and every pair of tangent vectors u, v to M at p (again, with α, u, and v real when we are considering the real (co)tangent spaces, and complex when we are considering the complexified (co)tangent spaces). For $r = 0$, we set $\Phi_p^*(\Phi(p), \zeta) \equiv (p, \zeta)$ for each scalar ζ, and for $r > 2$, we set $\Phi_p^* \equiv 0$. The pullback mappings $\Phi^* \colon \Lambda^r T^*N \to \Lambda^r T^*M$ and $\Phi^* \colon \Lambda^r (T^*N)_{\mathbb{C}} \to \Lambda^r (T^*M)_{\mathbb{C}}$ are then the mappings with restriction to $\Lambda^r T_{\Phi(p)}^* N$ or $\Lambda^r (T_{\Phi(p)}^* N)_{\mathbb{C}}$ equal to Φ_p^* for $r = 0, 1, 2, \ldots$.

Remarks 1. If $\Phi\colon M \to N$ is a $\mathcal{C}^1$ mapping of $\mathcal{C}^\infty$ manifolds of dimension ≤ 2, $p \in M$, $r, s \in \mathbb{Z}_{\geq 0}$, $\alpha \in \Lambda^r (T_{\Phi(p)}^* N)_{\mathbb{C}}$, and $\beta \in \Lambda^s (T_{\Phi(p)}^* N)_{\mathbb{C}}$, then $\Phi^*(\alpha \wedge \beta) = \Phi^*\alpha \wedge \Phi^*\beta$ (see Exercise 9.5.1).

2. For a given smooth manifold M of dimension $n = 1$ or 2 and a given nonnegative integer r, $\Lambda^r T^*M$ and $\Lambda^r (T^*M)_{\mathbb{C}}$ have natural $\mathcal{C}^\infty$ (manifold) structures (see Exercise 9.5.4). They are also examples of $\mathcal{C}^\infty$ vector bundles (see, for example, [Ns3] or [Wel]).

Definition 9.5.2 Let M be a smooth manifold of dimension $n = 1$ or 2, let $S \subset M$, and let $r \in \mathbb{Z}_{\geq 0}$.

(a) A *real differential form* (*complex differential form*) *of degree* r *in* M *on* S is a map $\alpha\colon S \to \Lambda^r T^*M$ (respectively, $\alpha\colon S \to \Lambda^r(T^*M)_{\mathbb{C}}$) with $\Pi_{\Lambda^r T^*M} \circ \alpha = \mathrm{Id}_S$ (respectively, $\Pi_{\Lambda^r(T^*M)_{\mathbb{C}}} \circ \alpha = \mathrm{Id}_S$). We usually denote the value of α at p by $\alpha_p \in \Lambda^r T_p^* M$ (respectively, $\Lambda^r(T_p^*M)_{\mathbb{C}}$) for each point $p \in S$. For $r = 0$, we identify α with the function $p \mapsto \zeta$ for $\alpha_p = (p, \zeta)$. We also call a differential form of degree r an *r-form*.

(b) Let α be a differential form of degree r on S. The *coefficient functions* (or simply the *coefficients*) of α with respect to (or in) a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, (x_1, \dots, x_n))$ are the functions on $S \cap U$ given by

$$\begin{cases} \alpha & \text{if } r = 0, \\ a_j = \alpha(\partial/\partial x_j) \quad \text{for } j = 1, \dots, n & \\ \quad (\text{i.e., } \alpha = \sum_{j=1}^n a_j\, dx_j) & \text{if } r = 1, \\ a = \alpha(\partial/\partial x_1, \partial/\partial x_2) = \frac{\alpha}{dx_1 \wedge dx_2} & \\ \quad (\text{i.e., } \alpha = a\, dx_1 \wedge dx_2) & \text{if } n = r = 2, \\ 0 & \text{if } r > n. \end{cases}$$

(c) We say that a differential r-form α on S is *continuous* if its coefficients in every local $\mathcal{C}^\infty$ coordinate neighborhood are continuous. For $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, we say that α is of *class* $\mathcal{C}^k$ if the coefficients of α in every local $\mathcal{C}^\infty$ coordinate neighborhood are of class $\mathcal{C}^k$.

(d) For S open, $\mathcal{E}^r(S, \mathbb{F})$ denotes the set of $\mathcal{C}^\infty$ differential forms of degree r with values in $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Depending on the context, $\mathcal{E}^r(S)$ will denote either $\mathcal{E}^r(S, \mathbb{R})$ or $\mathcal{E}^r(S, \mathbb{C})$. The set of $\mathcal{C}^\infty$ $\mathbb{F}$-valued r-forms with compact support (where the support of a differential form is the closure of the set of points in S at which the form is nonzero) in S is denoted by $\mathcal{D}^r(S, \mathbb{F})$ or by $\mathcal{D}^r(S)$.

Remarks 1. Observe that if α and β are differential forms of degree r and s, respectively, on a subset S of a smooth manifold M of dimension 1 or 2, then the exterior product $\alpha \wedge \beta\colon p \mapsto (\alpha \wedge \beta)_p \equiv \alpha_p \wedge \beta_p$ is an $(r+s)$-form, with $\alpha \wedge \beta \equiv 0$ if $r + s > \dim M$ (see Sect. 8.2). For $r = s$, the sum $\alpha + \beta$ is the r-form given by $p \mapsto \alpha_p + \beta_p$.

2. If α is a 1-form and v is a tangent vector at a point in a smooth manifold, then $\overline{\alpha(v)} = \bar{\alpha}(\bar{v})$. If α is a 2-form and u and v are tangent vectors, then $\overline{\alpha(u, v)} = \bar{\alpha}(\bar{u}, \bar{v})$ (see Sects. 8.1 and 8.2).

3. If $\Phi\colon M \to N$ is a $\mathcal{C}^\infty$ mapping of $\mathcal{C}^\infty$ manifolds of dimension ≤ 2 and α is an r-form on a set $S \subset N$, then $\Phi^*\alpha$ is an r-form on $\Phi^{-1}(S)$ in M. For $r = 0$, viewing α as a function on S, we have $\Phi^*\alpha = \alpha \circ \Phi$.

We have the following description of $\mathcal{C}^k$ differential forms and their behavior.

Proposition 9.5.3 *Suppose M is a smooth manifold of dimension $n = 1$ or 2 and $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then we have the following*:

(a) *Let α be a differential form of degree r on a set $S \subset M$. Then α is a continuous (class $\mathcal{C}^k$) differential form if and only if for each point in S, there exists a local $\mathcal{C}^\infty$ coordinate neighborhood with respect to which the corresponding coefficient functions of α are continuous (respectively, of class $\mathcal{C}^k$).*
(b) *If $\Phi\colon N \to M$ is a $\mathcal{C}^\infty$ mapping of a smooth manifold N into M, α is a differential form on a set in M, and α is continuous (of class $\mathcal{C}^k$), then $\Phi^*\alpha$ is continuous (respectively, of class $\mathcal{C}^k$).*
(c) *The sum and exterior product of any two continuous (class $\mathcal{C}^k$) differential forms is continuous (respectively, of class $\mathcal{C}^k$). In particular, for any open set $\Omega \subset M$ and for any pair of nonnegative integers r, s, the exterior product $(\alpha, \beta) \mapsto \alpha \wedge \beta$ yields a mapping $\mathcal{E}^r(\Omega) \times \mathcal{E}^s(\Omega) \to \mathcal{E}^{r+s}(\Omega)$.*

Proof Suppose α is a differential r-form defined at points in a set $S \subset M$ and $(U, x = (x_1, \dots, x_n))$ and $(V, y = (y_1, \dots, y_n))$ are local $\mathcal{C}^\infty$ coordinate neighborhoods in M. For $r = 1$, we have, on $S \cap U \cap V$,

$$\alpha = \sum_{i=1}^n a_i\, dy_i = \sum_{j=1}^n \left[\sum_{i=1}^n a_i \frac{\partial y_i}{\partial x_j} \right] dx_j = \sum_{j=1}^n b_j\, dx_j,$$

where $\{a_i\}$ and $\{b_j\}$ are the coefficients of α in (V, y) and (U, x), respectively. Since $\partial y_i/\partial x_j \in \mathcal{C}^\infty(U \cap V)$ for each pair of indices i and j, it follows that if the coefficients $\{a_i\}$ are continuous (of class $\mathcal{C}^k$), then each of the coefficients $b_j = \sum_{i=1}^n a_i \frac{\partial y_i}{\partial x_j}$ is also continuous (respectively, of class $\mathcal{C}^k$) for each $j = 1, \dots, n$. Similarly, for $n = r = 2$, we have

$$\alpha = a\, dy_1 \wedge dy_2 = a\mathcal{J} \cdot dx_1 \wedge dx_2,$$

where $\mathcal{J}$ is the $\mathcal{C}^\infty$ function given by the (Jacobian) determinant

$$\mathcal{J} \equiv \frac{dy_1 \wedge dy_2}{dx_1 \wedge dx_2} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}.$$

Hence, if the coefficient a is continuous (of class $\mathcal{C}^k$), then the coefficient $a\mathcal{J}$ is continuous (respectively, of class $\mathcal{C}^k$). Part (a) now follows. The proofs of (b) and (c) are left to the reader (see Exercise 9.5.2). □

Remark In particular, the sets $\mathcal{E}^r(\Omega)$ and $\mathcal{D}^r(\Omega)$ associated to an open subset Ω of a smooth manifold are vector spaces.

It is easy to see that if f is a $\mathcal{C}^k$ function on a smooth manifold M of dimension $n = 1$ or 2 with $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then the differential, or exterior derivative, df is a $\mathcal{C}^{k-1}$ differential form of degree 1 on M. In particular, for every open set $\Omega \subset M$, we have

$$d\colon \mathcal{E}^0(\Omega) \to \mathcal{E}^1(\Omega) \quad \text{and} \quad d\colon \mathcal{D}^0(\Omega) \to \mathcal{D}^1(\Omega).$$

The following lemma provides an exterior derivative operator $d\colon \mathcal{E}^1(\Omega) \to \mathcal{E}^2(\Omega)$.

Lemma 9.5.4 *Let M be a 2-dimensional smooth manifold. Then there is a unique operator d on local $\mathcal{C}^1$ differential forms of degree 1 such that:*

(i) *For every $\mathcal{C}^1$ differential form α of degree 1 on an open set $\Omega \subset M$, $d\alpha$ is a $\mathcal{C}^0$ differential form of degree 2 on Ω;*
(ii) *For every pair of $\mathcal{C}^1$ differential forms α and β of degree 1 on an open set $\Omega \subset M$ and every constant $\zeta \in \mathbb{C}$, we have $d(\alpha + \zeta\beta) = d\alpha + \zeta\, d\beta$;*
(iii) *For every local $\mathcal{C}^2$ function f, we have $d^2 f = d(df) = 0$;*
(iv) *For every $\mathcal{C}^1$ function f and every $\mathcal{C}^1$ differential form α of degree 1 on an open set $\Omega \subset M$, we have $d(f\alpha) = df \wedge \alpha + f\, d\alpha$; and*
(v) *For every $\mathcal{C}^1$ differential form α of degree 1 on an open set $\Omega \subset M$ and every open set $U \subset \Omega$, we have $d(\alpha\restriction_U) = (d\alpha)\restriction_U$.*

Moreover, if α is a $\mathcal{C}^1$ differential form of degree 1 on an open set $\Omega \subset M$, $(U, (x, y))$ is a local $\mathcal{C}^\infty$ coordinate neighborhood, and $\alpha = P dx + Q dy$ on $U \cap \Omega$ (i.e., P and Q are the coefficients of α in the coordinate neighborhood), then

$$d\alpha = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy \quad \text{on } U \cap \Omega.$$

In particular, if α is of class $\mathcal{C}^k$ with $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then $d\alpha$ is of class $\mathcal{C}^{k-1}$.

Proof Suppose α is a $\mathcal{C}^k$ 1-form on an open set $\Omega \subset M$ with $k \in \mathbb{Z}_{>0} \cup \{\infty\}$. Let us assume for the moment that an operator d satisfying the conditions (i)–(v) exists. If $(U, (x, y))$ is a local $\mathcal{C}^\infty$ coordinate neighborhood in M, then we have $\alpha = P\, dx + Q\, dy$ on $U \cap \Omega$ for unique functions $P, Q \in \mathcal{C}^k(U \cap \Omega)$ ($P \equiv \alpha(\partial/\partial x)$, $Q \equiv \alpha(\partial/\partial y)$). The conditions (i)–(v) then imply that on $U \cap \Omega$,

$$\begin{aligned} d\alpha &= d(P\, dx) + d(Q\, dy) = (dP) \wedge dx + (dQ) \wedge dy \\ &= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy. \end{aligned}$$

In particular, $d\alpha$ is of class $\mathcal{C}^{k-1}$ and uniqueness follows.

Motivated by the above, given a $\mathcal{C}^1$ 1-form α on an open set $\Omega \subset M$, we now define the $\mathcal{C}^0$ 2-form $d\alpha$ on Ω by setting

$$(d\alpha)\restriction_{U\cap\Omega} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

whenever $(U, (x, y))$ is a local $\mathcal{C}^\infty$ coordinate neighborhood in M, and P and Q are the (unique) $\mathcal{C}^1$ functions with $\alpha = P\, dx + Q\, dy$ on $U \cap \Omega$. The 2-form $d\alpha$ is well defined because, if $(V, (u, v))$ is another local $\mathcal{C}^\infty$ coordinate neighborhood in M and $\alpha = R\, du + S\, dv$ on $V \cap \Omega$, then on $U \cap V \cap \Omega$, we have

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left[\alpha\left(\frac{\partial}{\partial y}\right)\right] = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y} R + \frac{\partial v}{\partial y} S\right]$$

$$= \frac{\partial^2 u}{\partial x \partial y} R + \frac{\partial^2 v}{\partial x \partial y} S + \frac{\partial u}{\partial y}\frac{\partial R}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial S}{\partial x}$$
$$= \frac{\partial^2 u}{\partial x \partial y} R + \frac{\partial^2 v}{\partial x \partial y} S + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x}\frac{\partial R}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\frac{\partial R}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\frac{\partial S}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial x}\frac{\partial S}{\partial v}.$$

Applying the analogous calculation to obtain $\partial P/\partial y$ and subtracting, we get

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \left(\frac{\partial S}{\partial u} - \frac{\partial R}{\partial v}\right) \cdot \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) = \left(\frac{\partial S}{\partial u} - \frac{\partial R}{\partial v}\right) \cdot \det J,$$

where J is the (Jacobian) matrix

$$J \equiv \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

We also have

$$\begin{aligned} dx \wedge dy &= \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right) \wedge \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right) \\ &= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right) \cdot du \wedge dv \\ &= \det(J^{-1}) \cdot du \wedge dv. \end{aligned}$$

It follows that

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot dx \wedge dy = \left(\frac{\partial S}{\partial u} - \frac{\partial R}{\partial v}\right) \cdot du \wedge dv,$$

and hence $d\alpha$ is well defined. As one may verify, the local representation of $d\alpha$ also gives the conditions (ii)–(v) (see Exercise 9.5.5). □

Remark Actually, the conditions (i)–(iv) in the lemma suffice to uniquely determine the operator d (see Exercise 9.5.5).

We now collect together the definition of the *exterior derivative* for a 0-form provided in Sect. 9.4 and the definition for a 1-form suggested by Lemma 9.5.4 above.

Definition 9.5.5 Let α be a differential form of degree $r \in \mathbb{Z}_{\geq 0}$ on an open subset Ω of a smooth manifold M of dimension $n = 1$ or 2.

(a) If α is of class $\mathcal{C}^1$, then the *exterior derivative* of α is the continuous differential form $d\alpha$ of degree $r + 1$ on Ω that is equal to
   (i) The differential of α if $r = 0$ (see Sect. 9.4);
   (ii) The 2-form provided by Lemma 9.5.4 if $r = 1$ and $n = 2$;
   (iii) The trivial (i.e., zero) differential form if $r \geq 2$ or $n = r = 1$.

(b) If α is of class $\mathcal{C}^1$ and we have $d\alpha = 0$, then we say that α is *closed* (or *d-closed*).

(c) If $\alpha = d\beta$ for some $\mathcal{C}^1$ differential form β of degree $r - 1$ on Ω (in particular, $r > 0$), then we say that α is *exact* (or *d-exact*). If β may be chosen to be of class $\mathcal{C}^k$ for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then we also say that α is $\mathcal{C}^k$*-exact*. For $r = 1$, we call the function β a *potential* for α. If for each point in Ω, there exists a $\mathcal{C}^1$ differential form β_0 of degree $r - 1$ on a neighborhood U_0 such that $d\beta_0 = \alpha\restriction_{U_0}$, then we say that α is *locally exact* (or *locally d-exact*) and we call β_0 a *local potential* for α. If for some $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, each of the local forms β_0 may be chosen to be of class $\mathcal{C}^k$, then we also say that α is *locally* $\mathcal{C}^k$*-exact*. It is also convenient to consider the trivial 0-form $\alpha \equiv 0$ to be $\mathcal{C}^\infty$ d-exact and to write $0 = d0$.

Remarks Let M be a smooth manifold of dimension $n = 1$ or 2.

1. For $n = 2$, in terms of local $\mathcal{C}^\infty$ coordinates (x, y) on an open set in M, the exterior derivative of a $\mathcal{C}^1$ differential form u of degree 0 (i.e., a $\mathcal{C}^1$ function) is

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy,$$

and the exterior derivative of a $\mathcal{C}^1$ differential 1-form $\alpha = P\,dx + Q\,dy$ is

$$d\alpha = dP \wedge dx + dQ \wedge dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

2. For $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, a $\mathcal{C}^1$ function u on M is of class $\mathcal{C}^k$ if and only if du is of class $\mathcal{C}^{k-1}$. Consequently, any exact 1-form α of class $\mathcal{C}^{k-1}$ is automatically $\mathcal{C}^k$-exact, and in fact, any potential for α is of class $\mathcal{C}^k$. According to the Poincaré lemma (Lemma 9.5.7 below), any closed $\mathcal{C}^k$ 2-form is locally $\mathcal{C}^k$-exact. For the purposes of this book, characterizations of *global* $\mathcal{C}^k$-exactness of a 2-form are not necessary (although a partial solution is considered in Exercise 9.5.9).

3. Any exact differential form α of class $\mathcal{C}^1$ on M is closed. For $d\alpha = 0$ automatically if $\deg \alpha = 2$. If $\deg \alpha = 1$, then any potential β for α must be of class $\mathcal{C}^2$, and hence $d\alpha = d^2\beta = 0$.

4. If α and β are $\mathcal{C}^1$ differential forms on M, then (see Exercise 9.5.6)

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge d\beta.$$

According to the following fact, the proof of which is left to the reader (see Exercise 9.5.7), exterior differentiation commutes with pullbacks:

Proposition 9.5.6 *If $\Phi\colon M \to N$ is a $\mathcal{C}^2$ mapping of smooth manifolds M and N of dimension* 1 *or* 2 *and β is a $\mathcal{C}^1$ differential form on N, then $d\Phi^*\beta = \Phi^* d\beta$. If* $\deg \beta = 0$, *then this also holds for Φ a $\mathcal{C}^1$ mapping.*

Lemma 9.5.7 (Poincaré lemma) *Let p be a point in a smooth surface M. Then there exists a neighborhood U of p in M such that for each $k \in \mathbb{Z}_{>0} \cup \{\infty\}$ and* *each closed $\mathcal{C}^k$ differential form θ of degree $r > 0$ on U, there exists a $\mathcal{C}^k$ differential form α on U with $d\alpha = \theta$ (in particular, every closed $\mathcal{C}^k$ form is locally $\mathcal{C}^k$-exact).*

Proof The statement is local, so we may assume without loss of generality that $M = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$. Suppose $k \in \mathbb{Z}_{>0} \cup \{\infty\}$ and θ is a $\mathcal{C}^k$ differential form of degree $r > 0$ on M.

If $r = 1$, then there are functions $P, Q \in \mathcal{C}^k(M)$ such that

$$\theta = P\,dx + Q\,dy \quad \text{and} \quad 0 = d\theta = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

The function α on M given by

$$\alpha(x, y) = \int_0^x P(t, 0)\,dt + \int_0^y Q(x, t)\,dt \quad \forall (x, y) \in M$$

then satisfies

$$\begin{aligned}\frac{\partial \alpha}{\partial x}(x, y) &= P(x, 0) + \int_0^y \frac{\partial}{\partial x}[Q(x, t)]\,dt \\ &= P(x, 0) + \int_0^y \frac{\partial}{\partial t}[P(x, t)]\,dt = P(x, y)\end{aligned}$$

(by Proposition 7.2.5) and

$$\frac{\partial \alpha}{\partial y}(x, y) = Q(x, y)$$

for $(x, y) \in M$. Thus $d\alpha = \theta$ on M, and in particular, $\alpha \in \mathcal{C}^k(M)$ (in fact, $\alpha \in \mathcal{C}^{k+1}(M)$).

If $r = 2$, then there is a function $f \in \mathcal{C}^k(M)$ such that $\theta = f\,dx \wedge dy$. The differential 1-form $\alpha = P\,dx + Q\,dy$ on M, where

$$P(x, y) \equiv -\frac{1}{2}\int_0^y f(x, t)\,dt \quad \text{and} \quad Q(x, y) \equiv \frac{1}{2}\int_0^x f(t, y)\,dt \quad \forall (x, y) \in M,$$

is of class $\mathcal{C}^k$ (by Proposition 7.2.5) and satisfies $\partial P/\partial y = -f/2$ and $\partial Q/\partial x = f/2$. Thus $d\alpha = \theta$. □

Remarks 1. In the last part of the above proof, the 1-forms $2P\,dx$ and $2Q\,dy$ also have exterior derivative θ.

2. The Poincaré lemma holds for a closed form of positive degree on a manifold of arbitrary dimension.

The following terminology is used mainly in Chap. 6:

Definition 9.5.8 Let M be a $\mathcal{C}^\infty$ manifold of dimension n, let $p \in M$, and let $k \in \mathbb{Z}_{\geq 0}$. A $\mathcal{C}^k$ function f on a neighborhood of p has *vanishing derivatives of* *order less than or equal to k at p* if

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} f(p) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(p) = 0$$

for every local $\mathcal{C}^\infty$ coordinate neighborhood $(U, x = (x_1, \dots, x_n))$ of p and every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$ with $|\alpha| \leq k$. A $\mathcal{C}^k$ vector field or differential form on a neighborhood of p has *vanishing derivatives of order* $\leq k$ *at* p if its coefficients in every local $\mathcal{C}^\infty$ coordinate neighborhood of p have vanishing derivatives of order $\leq k$ at p. A $\mathcal{C}^\infty$ vector field or differential form with vanishing derivatives of order $\leq m$ at p for *every* positive integer m is said to have *vanishing derivatives of all orders at* p.

In fact, the above condition is independent of the choice of local coordinates.

Proposition 9.5.9 *Let M be an n-dimensional $\mathcal{C}^\infty$ manifold, let $p \in M$, let $k \in \mathbb{Z}_{\geq 0}$, and let f be a $\mathcal{C}^k$ function on a neighborhood of p. Then f has vanishing derivatives of order $\leq k$ at p if (and only if) there exists a local $\mathcal{C}^\infty$ coordinate neighborhood $(V, y = (y_1, \dots, y_n))$ of p such that $(\partial/\partial y)^\alpha f(p) = 0$ for each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$ with $|\alpha| \leq k$. Similarly, a $\mathcal{C}^k$ vector field or differential form on a neighborhood of p has vanishing derivatives of order $\leq k$ at p if (and only if) its coefficients in some local $\mathcal{C}^\infty$ coordinate neighborhood of p have vanishing derivatives of order $\leq k$ at p. If $\theta = f\beta$, where θ is a $\mathcal{C}^k$ vector field or differential form, f is a $\mathcal{C}^k$ function, and β is a nonvanishing $\mathcal{C}^k$ vector field or differential form, then θ has vanishing derivatives of order $\leq k$ at p if and only if f has vanishing derivatives of order $\leq k$ at p.*

Proof The claim concerning functions is clear for $k = 0$. We proceed by induction on k, assuming that $k > 0$ and that the claim holds for nonnegative integers less than k. Assume that we have a local $\mathcal{C}^\infty$ coordinate neighborhood $(V, y = (y_1, \dots, y_n))$ of p such that $(\frac{\partial}{\partial y})^\alpha f(p) = 0$ for each multi-index α with $|\alpha| \leq k$. In particular, by the induction hypothesis, for each $j = 1, \dots, n$, the function $\partial f/\partial y_j$ has vanishing derivatives of order $\leq k - 1$ at p. Given a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, x = (x_1, \dots, x_n))$ of p, a multi-index $\alpha \in (\mathbb{Z}_{\geq 0})^n$ with $|\alpha| = k$, and an index m with $\alpha_m > 0$, setting

$$\beta \equiv (\alpha_1, \dots, \alpha_{m-1}, \alpha_m - 1, \alpha_{m+1}, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n,$$

we get

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} f(p) = \left(\frac{\partial}{\partial x}\right)^{\beta} \frac{\partial f}{\partial x_m}(p) = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x}\right)^{\beta} \left[\frac{\partial y_j}{\partial x_m} \frac{\partial f}{\partial y_j}\right](p).$$

The product rule yields a sum in which each of the terms contains, for some j, a derivative of $\partial f/\partial y_j$ at p of order at most $k - 1$. Therefore, the above must vanish.

The proofs of the claims concerning vector fields and differential forms are left to the reader (see Exercise 9.5.8). $\square$

Exercises for Sect. 9.5

9.5.1 Prove that if $\Phi\colon M \to N$ is a $\mathcal{C}^1$ mapping of $\mathcal{C}^\infty$ manifolds of dimension ≤ 2, $p \in M$, $r, s \in \mathbb{Z}_{\geq 0}$, $\alpha \in \Lambda^r(T^*_{\Phi(p)}N)_{\mathbb{C}}$, and $\beta \in \Lambda^s(T^*_{\Phi(p)}N)_{\mathbb{C}}$, then $\Phi^*(\alpha \wedge \beta) = \Phi^*\alpha \wedge \Phi^*\beta$.

9.5.2 Prove parts (b) and (c) of Proposition 9.5.3.

9.5.3 Let M be a $\mathcal{C}^\infty$ manifold of dimension $n = 1$ or 2, let α be a differential form of degree r on an open set $\Omega \subset M$, and let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Prove that for $r = 1$, α is of class $\mathcal{C}^k$ if and only if the function $\alpha(v)\colon p \mapsto \alpha_p(v_p)$ is of class $\mathcal{C}^k$ for each local $\mathcal{C}^\infty$ vector field v. Prove also that for $r = 2$, α is of class $\mathcal{C}^k$ if and only if the function $\alpha(u, v)\colon p \mapsto \alpha_p(u_p, v_p)$ is of class $\mathcal{C}^k$ for each pair of local $\mathcal{C}^\infty$ vector fields u and v.

9.5.4 Let M be a $\mathcal{C}^\infty$ manifold of dimension $n = 1$ or 2, and let $r \in \mathbb{Z}_{\geq 0}$. For $r = 0$ or $r > 2$, $\Lambda^r T^*M$ and $\Lambda^r(T^*M)_{\mathbb{C}}$ have obvious $\mathcal{C}^\infty$ (manifold) structures. For $r = 1$, they have natural $\mathcal{C}^\infty$ structures provided by Exercise 9.4.3. We now consider the case $r = 2$. Assume that $n = 2$.

(a) Prove that there are unique $\mathcal{C}^\infty$ (manifold) structures in $\Lambda^2 T^*M$ and in $\Lambda^2(T^*M)_{\mathbb{C}}$ such that for each local $\mathcal{C}^\infty$ chart $(U, \Phi = (x_1, x_2), U')$ in M, we get local $\mathcal{C}^\infty$ charts

$$\left(\Pi^{-1}_{\Lambda^2 T^*M}(U), \widehat{\Phi}, U' \times \mathbb{R}\right) \quad \text{and} \quad \left(\Pi^{-1}_{\Lambda^2(T^*M)_{\mathbb{C}}}(U), \widehat{\Phi}_{\mathbb{C}}, U' \times \mathbb{C}\right),$$

in $\Lambda^2 T^*M$ and $\Lambda^2(T^*M)_{\mathbb{C}}$, respectively, where the mappings $\widehat{\Phi}$ and $\widehat{\Phi}_{\mathbb{C}}$ are given by

$$\begin{aligned}\alpha \mapsto &\left(x_1(p), x_2(p), \alpha\big((\partial/\partial x_1)_p, (\partial/\partial x_2)_p\big)\right)\\ &= \left(x_1(p), x_2(p), \frac{\alpha}{(dx_1)_p \wedge (dx_2)_p}\right)\end{aligned}$$

for each point $p \in U$ and each element α of $\Lambda^2 T^*_p M$ and $\Lambda^2(T^*_p M)_{\mathbb{C}}$, respectively.

(b) Prove that if $\Phi\colon M \to N$ is a $\mathcal{C}^k$ mapping of M into a $\mathcal{C}^\infty$ manifold N of dimension ≤ 2 with $k \in \mathbb{Z}_{>0} \cup \{\infty\}$, then the pullback mappings

$$\Phi^*\colon \Lambda^2 T^*N \to \Lambda^2 T^*M \quad \text{and} \quad \Phi^*\colon \Lambda^2(T^*N)_{\mathbb{C}} \to \Lambda^2(T^*M)_{\mathbb{C}}$$

are of class $\mathcal{C}^{k-1}$ with respect to the smooth structures provided by (a).

(c) Prove that the inclusion mapping $\Lambda^2 T^*M \hookrightarrow \Lambda^2(T^*M)_{\mathbb{C}}$ and the real and imaginary projections $\Lambda^2(T^*M)_{\mathbb{C}} \to \Lambda^2 T^*M$ are $\mathcal{C}^\infty$ mappings.

9.5.5 Let M be a 2-dimensional smooth manifold.

(a) Verify that the conditions (ii)–(v) in Lemma 9.5.4 hold for the operator d constructed in the proof.

(b) Prove that in fact, the conditions (i)–(iv) in Lemma 9.5.4 uniquely determine the operator d.

9.5.6 Prove that if α and β are $\mathcal{C}^1$ differential forms on a smooth manifold M of dimension $n = 1$ or 2 and $r = \deg \alpha$, then $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^r \alpha \wedge d\beta$.

9.5.7 Prove Proposition 9.5.6.

9.5.8 Verify the claims concerning vector fields and differential forms appearing in Proposition 9.5.9.

9.5.9 Let M be a second countable smooth surface, let k be a positive integer, and let θ be an exact differential form of degree 2 and class $\mathcal{C}^k$. Prove that there exists a differential form α of class $\mathcal{C}^k$ such that $d\alpha = \theta$.

Hint. By definition, there is a $\mathcal{C}^1$ 1-form β with $d\beta = \theta$. Applying the Poincaré lemma, one gets local $\mathcal{C}^k$ 1-forms $\{\beta_i\}$ with $d\beta_i = \alpha$, and local $\mathcal{C}^2$ functions $\{f_i\}$ with $\beta_i - \beta = df_i$. Forming a suitable $\mathcal{C}^\infty$ partition of unity $\{\lambda_i\}$, one gets a $\mathcal{C}^2$ 1-form $\gamma \equiv \sum(\lambda_i\beta_i + f_i d\lambda_i)$ with $d\gamma = \theta$. Now proceed by induction.

9.5.10 Let M be a smooth manifold, let $p \in M$, let $k \in \mathbb{Z}_{\geq 0}$, and let f be a $\mathcal{C}^{k+1}$ function on M with vanishing derivatives of order $\leq k$ at p. Prove that for every local $\mathcal{C}^\infty$ coordinate neighborhood (U, x) of p, and every compact set $K \subset U$, there exists a constant $C > 0$ such that $|f(q)| \leq C\|x(q) - x(p)\|^{k+1}$ for all points $q \in K$.

9.6 Measurability in a Smooth Manifold

In order to consider Lebesgue integration on manifolds, we must first extend the definition of Lebesgue measurability.

Definition 9.6.1 Let M be a smooth manifold of dimension n, and let $S \subset M$.

(a) S is called (*Lebesgue*) *measurable* if the set $\Phi(S \cap U) \subset \mathbb{R}^n$ is Lebesgue measurable for every local $\mathcal{C}^\infty$ chart (U, Φ, U') in M.
(b) S is of *measure* 0 if $\Phi(S \cap U)$ is a set of Lebesgue measure 0 for every local $\mathcal{C}^\infty$ chart (U, Φ, U') in M.
(c) A statement (P) about points in S is said to hold *almost everywhere* if the set of points in S at which the statement does *not* hold is a set of measure 0.
(d) A mapping $\Psi\colon S \to X$ of S into a topological space X is *measurable* if the inverse image $\Psi^{-1}(U) \subset S$ is measurable for every open set $U \subset X$ (in particular, S is measurable). For $X = [-\infty, \infty]$ or $\mathbb{C}$, Ψ is also called a *measurable function*.

We have the following characterizations of measurable sets and mappings:

Proposition 9.6.2 *In any smooth manifold M, we have the following*:

(a) *A set $S \subset M$ is measurable* (*of measure* 0) *if and only if for each point $p \in S$, there exists a local $\mathcal{C}^\infty$ chart (U, Φ, U') such that $p \in U$ and $\Phi(S \cap U) \subset \mathbb{R}^n$ is measurable* (*respectively, of measure* 0). *The composition $\rho \circ \Psi$ of a continuous mapping ρ and a measurable mapping Ψ is measurable. The image of a measurable subset of M under a diffeomorphism is measurable.*

(b) *The collection of measurable subsets of* M *is a* σ*-algebra that contains the Borel subsets of* M. *Moreover, any subset of a set of measure* 0 *is a set of measure* 0.

(c) *Let* $S \subset M$ *be a measurable set. Then any product or sum of measurable functions* (*with values in* $[-\infty, \infty]$ *or* $\mathbb{C}$) *on* S *is measurable. A mapping* $\Psi = (\psi_1, \dots, \psi_m)$ *of* S *into a product* $X_1 \times \cdots \times X_m$ *of topological spaces* $X_1, \dots, X_m$ *is measurable if and only if* ψ_j *is measurable for each* $j = 1, \dots, m$.

Proof Part (a) follows from the definitions and the change of variables formula (Theorem 7.2.7), and parts (b) and (c) follow from standard arguments. □

In analogy with Definition 9.5.2, we make the following:

Definition 9.6.3 Let S be a measurable subset of a smooth manifold M of dimension $n = 1$ or 2. A differential form on S is *measurable* if its coefficients in every local $\mathcal{C}^\infty$ coordinate neighborhood are measurable.

The proofs of the following facts concerning measurable differential forms are left to the reader (see Exercise 9.6.1):

Proposition 9.6.4 (Cf. Proposition 9.5.3) *Let* M *be a smooth manifold of dimension* $n = 1$ *or* 2, *and let* α *be a differential form on a measurable set* $S \subset M$. *Then we have the following*:

(a) *The differential form* α *is measurable if and only if for every point in* S, *there exists a local* $\mathcal{C}^\infty$ *coordinate neighborhood with respect to which the coefficient functions of* α *are measurable.*

(b) *If* $\Phi\colon N \to M$ *is a diffeomorphism, then* α *is measurable if and only if the pullback differential form* $\Phi^*\alpha$ *on* $\Phi^{-1}(S) \subset N$ *is measurable.*

(c) *The sum and exterior product of any two measurable differential forms on* S *is measurable.*

Remark Similarly, a vector field on a measurable subset of a smooth manifold is *measurable* if its coefficients in any local $\mathcal{C}^\infty$ coordinate neighborhood (or in some local $\mathcal{C}^\infty$ coordinate neighborhood of each point) are measurable.

Exercises for Sect. 9.6

9.6.1 Prove Proposition 9.6.4.

9.6.2 Let α be a differential form of degree r on a measurable subset S of a smooth manifold M of dimension $n = 1$ or 2. Prove that for $r = 1$, α is measurable if and only if the function $\alpha(v)\colon p \mapsto \alpha_p(v_p)$ is measurable for each local $\mathcal{C}^\infty$ vector field v. Also prove that for $r = 2$, α is measurable if and only if the function $\alpha(u, v)\colon p \mapsto \alpha_p(u_p, v_p)$ is measurable for every pair of local $\mathcal{C}^\infty$ vector fields u and v.

9.6.3 Let v be a vector field on a measurable set S in a smooth manifold M. Prove that the following are equivalent:

(i) The vector field v is measurable (i.e., its coefficients in any local $\mathcal{C}^\infty$ coordinate neighborhood are measurable).

(ii) For any local $\mathcal{C}^\infty$ coordinate neighborhood $(U, (x_1, \ldots, x_n))$, the function $dx_j(v)$ is measurable on $S \cap U$ for each $j = 1, \ldots, n$ (i.e., $v = \sum_{j=1}^n v_j \cdot (\partial/\partial x_j)$ on $S \cap U$ with measurable coefficients $v_1, \ldots, v_n$ on $S \cap U$).

(iii) For every point $p \in S$, there exists a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, (x_1, \ldots, x_n))$ in M such that the function $dx_j(v)$ is measurable on $S \cap U$ for each $j = 1, \ldots, n$.

(iv) The function $df(v) = v(f)$ is measurable for each local $\mathcal{C}^\infty$ function f.

9.6.4 Let v be a vector field on a measurable set S in a smooth manifold M. Prove that v is a measurable vector field (i.e., its coefficients in any local $\mathcal{C}^\infty$ coordinate neighborhood are measurable) if and only if v is measurable as a mapping of S into $(TM)_{\mathbb{C}}$, where $(TM)_{\mathbb{C}}$ has the $\mathcal{C}^\infty$ (and hence, topological space) structure provided by Exercise 9.4.3. State and prove the analogous claim for a differential form.

9.6.5 Let $f : M \to N$ be a $\mathcal{C}^\infty$ mapping of second countable $\mathcal{C}^\infty$ manifolds M and N of dimension m and n, respectively. According to Sard's theorem (see, for example, [Mi]), the set of critical values of f is a set of measure 0. Prove this theorem for the case $m = n = 1$.

Hint. One may assume without loss of generality that M is a bounded open interval in $\mathbb{R}$ and $N = \mathbb{R}$. Prove that for every interval $I \subset M$, $f(I)$ is measurable and $\lambda(f(I)) \le \sup_I |f'| \cdot \ell(I)$. Show that one may cover any compact subset of the zero set of f' by a suitable finite collection of disjoint open intervals on which $|f'|$ is small.

9.7 Lebesgue Integration on Curves and Surfaces

A connected 1-dimensional smooth manifold M is *orientable*; that is, one may cover M by coordinate intervals with the same sense of left and right (more precisely, the coordinate transformations have positive derivative). The choice of such an atlas is then one of the two *orientations* in M. One gets the opposite orientation by reversing the direction of each of the intervals in the atlas. For M second countable, the existence of an orientation follows from the fact that every connected second countable 1-dimensional smooth manifold is diffeomorphic to either the circle or the real line (see Theorem 9.10.1). For the general case, one may apply a Zorn's lemma argument. However, in two dimensions (or higher), no orientation need exist.

Example 9.7.1 The *Möbius band* is the $\mathcal{C}^\infty$ surface

$$M = [0, 1] \times (0, 1)/(0, t) \sim (1, 1 - t) \quad \forall t \in (0, 1).$$

The topology is the quotient topology for the quotient map $\Pi\colon [0,1]\times(0,1)\to M$, and a $\mathcal{C}^\infty$ atlas is given by $\{(U_j,\Phi_j,V)\}_{j=1}^2$, where $U_1=\Pi((0,1)\times(0,1))$, $V=(0,1)\times(0,1)$, $\Phi_1=(\Pi\restriction_{(0,1)\times(0,1)})^{-1}$, $U_2=\Pi(([0,1]\setminus\{1/2\})\times(0,1))$, and

$$\Phi_2(\Pi(s,t))=\begin{cases}(s+(1/2),t) & \text{if } (s,t)\in[0,1/2)\times(0,1),\\ (s-(1/2),1-t) & \text{if } (s,t)\in(1/2,1]\times(0,1).\end{cases}$$

We cannot choose an atlas so that on overlaps, the local charts agree on what a right-handed frame should be, because of the twist at the end. The details appear after the precise definitions below.

Throughout this section, M denotes a smooth manifold of dimension $n=1$ or 2 (for a discussion of orientation and integration in higher-dimensional manifolds, the reader may refer to, for example, [Mat], [Ns3], or [Wa]). We recall that the *Jacobian determinant* of a $\mathcal{C}^1$ mapping $F=(f_1,\dots,f_m)\colon U\to\mathbb{R}^m$ of an open subset U of $\mathbb{R}^m$ into $\mathbb{R}^m$ is the continuous function $\mathcal{J}_F : U\to\mathbb{R}$ given by

$$\mathcal{J}_F\equiv\det(dF)=\begin{vmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m}\\ \vdots & \ddots & \vdots\\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m}\end{vmatrix}.$$

In particular, if F is a diffeomorphism, then $\mathcal{J}_F\neq 0$. For $m=1$, we get $\mathcal{J}_F=F'$, and for $m=2$, we get

$$\mathcal{J}_F=\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}-\frac{\partial f_2}{\partial x_1}\frac{\partial f_1}{\partial x_2}.$$

Definition 9.7.2 In the manifold M, we have the following:

(a) Two local $\mathcal{C}^\infty$ charts (U_1,Φ_1,U_1') and (U_2,Φ_2,U_2') in M have *compatible orientations* if each of the associated coordinate transformations has positive Jacobian determinant. That is,

$$\mathcal{J}_{\Phi_1\circ\Phi_2^{-1}}>0\quad\text{on } \Phi_2(U_1\cap U_2)$$

(equivalently, $\mathcal{J}_{\Phi_2\circ\Phi_1^{-1}}=(1/\mathcal{J}_{\Phi_1\circ\Phi_2^{-1}})\circ(\Phi_2\circ\Phi_1^{-1})>0$ on $\Phi_1(U_1\cap U_2)$).

(b) An atlas in M in which the local charts have compatible orientations is said to be *oriented*.

(c) Two oriented atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ in M have *equivalent orientations* if the atlas $\mathcal{A}_1\cup\mathcal{A}_2$ is oriented. The verification that this gives an equivalence relation is left to the reader (see Exercise 9.7.1).

(d) An equivalence class of oriented atlases in M is called an *orientation* in M. If an orientation exists, then M is said to be *orientable*. Otherwise, M is called *nonorientable*. A manifold M together with an orientation is said to be *oriented*. A local chart in a representing atlas for the orientation is then said to be *positively oriented*.

(e) A diffeomorphism $\Lambda: M \to N$ of M onto an oriented smooth manifold N of dimension n is *orientation-preserving* if for each positively oriented local $\mathcal{C}^\infty$ chart (V, Ψ, V') in N, the local $\mathcal{C}^\infty$ chart $(\Lambda^{-1}(V), \Psi \circ \Lambda, V')$ is positively oriented in M (hence, Λ^{-1} is also orientation-preserving).

Lemma 9.7.3 *The Möbius band is nonorientable.*

Proof In the notation of Example 9.7.1, the open subsets U_1 and U_2 are connected and orientable, since we have the diffeomorphism $\Phi_j: U_j \to V \subset \mathbb{R}^2$ for $j = 1, 2$. If M is orientable, then we may choose an oriented $\mathcal{C}^\infty$ atlas $\mathcal{A}$. Since equivalence of orientations is an equivalence relation among oriented atlases, we may choose the orientation on M so that for some map $\rho: \mathbb{R}^2 \to \mathbb{R}^2$ of the form $(x, y) \mapsto (x, \sigma y)$ with $\sigma = \pm 1$, the local $\mathcal{C}^\infty$ charts $(U_1, \rho \circ \Phi_1, \sigma(V))$ and (U_2, Φ_2, V) induce orientations in U_1 and U_2, respectively, that are compatible with the orientation on M. But $U_1 \cap U_2 = \Pi(((0, 1) \setminus \{1/2\}) \times (0, 1))$, and for each point $(s, t) \in \Phi_2(\Pi((0, 1/2) \times (0, 1))) = (1/2, 1) \times (0, 1)$, we have

$$(\rho \circ \Phi_1) \circ \Phi_2^{-1}(s, t) = (s - (1/2), \sigma t),$$

while for each point $(s, t) \in \Phi_2(\Pi((1/2, 1) \times (0, 1))) = (0, 1/2) \times (0, 1)$, we have

$$(\rho \circ \Phi_1) \circ \Phi_2^{-1}(s, t) = (s + (1/2), \sigma(1 - t)).$$

Thus the Jacobian determinant $\mathcal{J}_{\rho \circ \Phi_1 \circ \Phi_2^{-1}}$ is equal to σ on $\Pi((0, 1/2) \times (0, 1))$ and $-\sigma$ on $\Pi((1/2, 1) \times (0, 1))$, and hence the Jacobian determinant cannot be everywhere positive. Thus we have arrived at a contradiction, and therefore M is nonorientable. □

Remarks 1. In the discussion of smooth structures on surfaces in Chap. 6, we will see that a (topological) Möbius band M is nonorientable with respect to *any* $\mathcal{C}^\infty$ structure on M (see Sect. 6.10).

2. If M is an oriented $\mathcal{C}^\infty$ manifold, then any open subset of M has a natural *induced orientation*.

3. If M is a connected orientable $\mathcal{C}^\infty$ manifold, then there are exactly two (opposite) orientations in M (see Exercise 9.7.2).

Proposition 9.7.4 *For each local $\mathcal{C}^\infty$ chart $(U, \Phi = (x_1, \dots, x_n), U')$ in M, let ω_Φ be the nonvanishing $\mathcal{C}^\infty$ differential form of degree n on U given by*

$$\omega_\Phi \equiv \begin{cases} dx_1 & \text{if } n = 1, \\ dx_1 \wedge dx_2 & \text{if } n = 2. \end{cases}$$

(a) *Given local $\mathcal{C}^\infty$ charts $(U, \Phi = (x_1, \dots, x_n), U')$ and $(V, \Psi = (y_1, \dots, y_n), V')$ in M, we have*

$$\frac{\omega_\Psi}{\omega_\Phi} = (\mathcal{J}_{\Psi \circ \Phi^{-1}}) \circ \Phi.$$

In particular, the two local $\mathcal{C}^\infty$ charts have compatible orientations if and only if $\omega_\Psi/\omega_\Phi > 0$ on $U \cap V$.

(b) *If M is second countable and oriented, then there exists a nonvanishing $\mathcal{C}^\infty$ differential form ω of degree n on M such that for every positively oriented local $\mathcal{C}^\infty$ chart (U, Φ, U') in M, we have $\omega/\omega_\Phi > 0$ on U.*

(c) *Given a nonvanishing continuous differential form ω of degree n on M, the collection $\mathcal{A}$ of all local $\mathcal{C}^\infty$ charts (U, Φ, U') in M with $\omega/\omega_\Phi > 0$ on U is an oriented atlas in M (and hence $\mathcal{A}$ determines a unique orientation in M).*

Proof The proof of part (a) and the proof of part (c) are left to the reader (see Exercise 9.7.3). For the proof of part (b), we observe that if M is second countable and oriented, then we may choose a countable locally finite covering of M by positively oriented local $\mathcal{C}^\infty$ charts $\{(U_i, \Phi_i, U_i')\}_{i\in I}$ and a $\mathcal{C}^\infty$ partition of unity $\{\lambda_i\}$ with $\operatorname{supp}\lambda_i \subset U_i$ for each $i \in I$. The $\mathcal{C}^\infty$ differential form

$$\omega \equiv \sum_{i\in I} \lambda_i \cdot \omega_{\Phi_i}$$

is then of degree n (following standard conventions, here we extend $\lambda_i \cdot \omega_{\Phi_i}$ by 0 to a $\mathcal{C}^\infty$ n-form on M for each $i \in I$), and given a positively oriented local $\mathcal{C}^\infty$ chart (U, Φ, U') in M, part (a) implies that

$$\frac{\omega}{\omega_\Phi} = \sum_{i\in I} \lambda_i \cdot \frac{\omega_{\Phi_i}}{\omega_\Phi} > 0$$

on U (in particular, ω is nonvanishing). Thus part (b) is proved. □

Definition 9.7.5 (Cf. Definition 8.2.3) Assume that M is oriented, and let $S \subset M$.

(a) A real differential form ω of degree n defined at points in S is said to be *positive* (*nonnegative*) if for every positively oriented local $\mathcal{C}^\infty$ coordinate neighborhood $(U, \Phi = (x_1, \dots, x_n))$ in M, we have

$$\frac{\omega}{\omega_\Phi} > 0 \quad \text{(respectively, } \geq 0) \quad \text{on } S \cap U,$$

where $\omega_\Phi = dx_1$ if $n = 1$ and $\omega_\Phi = dx_1 \wedge dx_2$ if $n = 2$ (equivalently, each point in S has *some* coordinate neighborhood $(U, \Phi = (x_1, \dots, x_n))$ for which the above holds). We write $\omega > 0$ or $0 < \omega$ (respectively, $\omega \geq 0$ or $0 \leq \omega$). Similarly, ω is said to be *negative* (*nonpositive*) and we write $\omega < 0$ or $0 > \omega$ (respectively, $\omega \leq 0$ or $0 \geq \omega$) if $-\omega > 0$ (respectively, $-\omega \geq 0$). For $p \in M$, an ordered basis $(v_1, \dots, v_n)$ $(= v_1$ or $(v_1, v_2))$ for T_pM is *positively oriented* if each positive $\alpha \in \Lambda^n T_pM$ is positive on the ordered basis (i.e., $\alpha(v_1) > 0$ if $n = 1$, $\alpha(v_1, v_2) > 0$ if $n = 2$).

(b) For two real differential forms α and β of degree n defined at points in S, we write $\alpha \geq \beta$ or $\beta \leq \alpha$ if $\alpha - \beta \geq 0$.

(c) Let α be a real differential form of degree n defined at points in a set $S \subset M$. Given a point $p \in S$, we define the *positive part* and *negative part* of α at p by

$$\alpha_p^+ \equiv \begin{cases} \alpha_p & \text{if } \alpha_p \geq 0, \\ 0 & \text{if } \alpha_p < 0, \end{cases} \quad \text{and} \quad \alpha_p^- \equiv \begin{cases} -\alpha_p & \text{if } \alpha_p \leq 0, \\ 0 & \text{if } \alpha_p > 0, \end{cases}$$

respectively. In other words,

$$(\alpha^+)_p = \left[\frac{\alpha_p}{\omega}\right]^+ \cdot \omega \quad \text{and} \quad (\alpha^-)_p = \left[\frac{\alpha_p}{\omega}\right]^- \cdot \omega$$

for an (arbitrary) positive element $\omega \in \Lambda^n T_p^* M$.

(d) Given a sequence $\{\alpha_\nu\}$ of real differential forms of degree n defined at points in S, and given a point $p \in S$, we define

$$\inf_\nu (\alpha_\nu)_p \equiv \left[\inf_\nu \frac{(\alpha_\nu)_p}{\omega}\right] \cdot \omega, \qquad \liminf_{\nu\to\infty} (\alpha_\nu)_p \equiv \left[\liminf_{\nu\to\infty} \frac{(\alpha_\nu)_p}{\omega}\right] \cdot \omega,$$

$$\sup_\nu (\alpha_\nu)_p \equiv \left[\sup_\nu \frac{(\alpha_\nu)_p}{\omega}\right] \cdot \omega, \qquad \limsup_{\nu\to\infty} (\alpha_\nu)_p \equiv \left[\limsup_{\nu\to\infty} \frac{(\alpha_\nu)_p}{\omega}\right] \cdot \omega,$$

for an arbitrary positive element $\omega \in \Lambda^n T_p^* M$, provided the above defined coefficients exist in $\mathbb{R}$.

Remarks 1. If α is a measurable (continuous) real differential form of degree $n = \dim M$ as in (c) above, then α^+ and α^- are measurable (respectively, continuous). The analogous statement concerning measurability holds for the infimum, supremum, limit inferior, limit superior, and limit of a sequence of real measurable differential forms of degree n as in (d) above.

2. It is easy see that for a sequence $\{\alpha_\nu\}$ as in (d) above and for α as in (c), we have $\lim \alpha_\nu = \alpha$ if and only if $\liminf \alpha_\nu = \limsup \alpha_\nu = \alpha$.

3. A nonvanishing $\mathcal{C}^\infty$ real differential form of degree n on M is also called a *volume form*.

Proposition 9.7.6 *Assume that M is second countable and oriented. Then, for every continuous real differential n-form τ on M, there exists a positive $\mathcal{C}^\infty$ differential n-form ω such that $\omega \geq \tau$ on M.*

Proof Proposition 9.7.4 provides a positive $\mathcal{C}^\infty$ n-form ω' on M, and Proposition 9.3.11 provides a $\mathcal{C}^\infty$ function ρ such that $\rho > |\tau/\omega'|$ on M. The positive $\mathcal{C}^\infty$ n-form $\omega \equiv \rho\omega'$ then satisfies $\omega \geq \tau$ on M. □

Lemma 9.7.7 *Suppose M is oriented, ω is a measurable nonnegative differential form of degree n defined at points in a measurable set $S \subset M$, $(t_1, \ldots, t_n)$ are the standard coordinates in $\mathbb{R}^n$, $\omega_{\mathbb{R}^1} = dt_1$, $\omega_{\mathbb{R}^2} = dt_1 \wedge dt_2$, and $(U, \Phi =$*

$(x_1, \ldots, x_n), U')$ *and* $(V, \Psi = (y_1, \ldots, y_n), V')$ *are two positively oriented local* $\mathcal{C}^\infty$ *charts in* M. *Then*

$$\int_{\Phi(S\cap U\cap V)} \frac{[\Phi^{-1}]^*\omega}{\omega_{\mathbb{R}^n}}\, d\lambda = \int_{\Psi(S\cap U\cap V)} \frac{[\Psi^{-1}]^*\omega}{\omega_{\mathbb{R}^n}}\, d\lambda.$$

Proof This follows from the change of variables formula (Theorem 7.2.7). For if $n = 2$ and $\omega = f\, dx_1 \wedge dx_2 = g\, dy_1 \wedge dy_2$ on $U \cap V$, then

$$f = g \cdot \mathcal{J}_{\Psi\circ\Phi^{-1}} \circ \Phi = g \cdot |\mathcal{J}_{\Psi\circ\Phi^{-1}} \circ \Phi|$$

and

$$[\Phi^{-1}]^*\omega = (f \circ \Phi^{-1})\, dt_1 \wedge dt_2 = (g \circ \Phi^{-1}) \cdot |\mathcal{J}_{\Psi\circ\Phi^{-1}}|\, dt_1 \wedge dt_2.$$

Therefore

$$\begin{aligned}\int_{\Phi(S\cap U\cap V)} \frac{[\Phi^{-1}]^*\omega}{\omega_{\mathbb{R}^2}}\, d\lambda &= \int_{\Phi(S\cap U\cap V)} (f \circ \Phi^{-1})\, d\lambda \\ &= \int_{\Phi\circ\Psi^{-1}(\Psi(S\cap U\cap V))} (g \circ \Psi^{-1} \circ \Psi \circ \Phi^{-1}) \cdot |\mathcal{J}_{\Psi\circ\Phi^{-1}}|\, d\lambda \\ &= \int_{\Psi(S\cap U\cap V)} (g \circ \Psi^{-1})\, d\lambda = \int_{\Psi(S\cap U\cap V)} \frac{[\Psi^{-1}]^*\omega}{\omega_{\mathbb{R}^2}}\, d\lambda.\end{aligned}$$

The proof for $n = 1$ is similar. □

Definition 9.7.8 (Cf. [AhS], [Sp], [Wey]) Suppose M is oriented, ω is a measurable complex differential form of degree n defined at points in a measurable set $S \subset M$, $(t_1, \ldots, t_n)$ are the standard coordinates in $\mathbb{R}^n$, $\omega_{\mathbb{R}^1} = dt_1$, and $\omega_{\mathbb{R}^2} = dt_1 \wedge dt_2$.

(a) If $\omega \geq 0$ on S and $S \subset U$ for some positively oriented local $\mathcal{C}^\infty$ chart (U, Φ, U') in M, then we define

$$\int_S \omega \equiv \int_{\Phi(S)} \frac{[\Phi^{-1}]^*\omega}{\omega_{\mathbb{R}^n}}\, d\lambda \in [0, \infty]$$

(which is independent of the choice of the local chart by Lemma 9.7.7).

(b) If $\omega \geq 0$ on S, then we define the *integral of* ω *over* S by

$$\int_S \omega \equiv \sup \sum_{j=1}^{m} \int_{S_j} \omega \in [0, \infty],$$

where the supremum is taken over all choices of disjoint measurable subsets $S_1, \ldots, S_m$ of S, each of which is contained in a (possibly different) local $\mathcal{C}^\infty$ chart.

(c) We say that ω is *integrable* on S if the nonnegative extended real numbers

$$R_+ \equiv \int_S [\operatorname{Re}\omega]^+, \qquad R_- \equiv \int_S [\operatorname{Re}\omega]^-,$$
$$I_+ \equiv \int_S [\operatorname{Im}\omega]^+, \qquad I_- \equiv \int_S [\operatorname{Im}\omega]^-,$$

are finite. If this is the case, then we define the *integral of ω over S* by

$$\int_S \omega \equiv R_+ - R_- + iI_+ - iI_-.$$

(d) We say that ω is *locally integrable* on S if each point in S admits a neighborhood U in M such that $\omega\restriction_{U\cap S}$ is integrable.

Remarks 1. The definitions in (a) and (b) are consistent; that is, for S contained in a local $\mathcal{C}^\infty$ coordinate neighborhood, the integral of $\omega \geq 0$ defined in (a) is equal to its integral as defined in (b).

2. On a *second countable* oriented manifold, one may simplify the above by using a partition of unity (see, for example, [Mat], [Ns3], or [Wa]).

Proposition 9.7.9 *Assume that M is oriented, let S be a measurable subset of M, let ω be a nonnegative measurable differential form of degree n defined on S, and let α and β be measurable differential forms of degree n defined on S.*

(a) *The function λ_ω on the σ-algebra of measurable subsets of S defined by*

$$\lambda_\omega \colon R \mapsto \int_R \omega$$

is a positive measure.

(b) *If f is a measurable function that satisfies $0 \leq f < \infty$ a.e. in S, then*

$$\int_S f\omega = \int_S f\, d\lambda_\omega.$$

(c) *If $\alpha = 0$ a.e. in $Z \equiv \{x \in S \mid \omega_x = 0\}$ and $\alpha \geq 0$ a.e. in S, then*

$$\int_S \alpha = \int_{S\setminus Z} \frac{\alpha}{\omega} \cdot \omega = \int_{S\setminus Z} \frac{\alpha}{\omega} \cdot d\lambda_\omega.$$

(d) *If $\alpha \geq 0$ and $\beta \geq 0$ a.e. in S, then*

$$\int_S (r\alpha + \beta) = r\int_S \alpha + \int_S \beta \quad \forall r \in [0,\infty).$$

(e) *If α and β are integrable on S and $r \in \mathbb{C}$, then $r\alpha + \beta$ is integrable on S and*

$$\int_S (r\alpha + \beta) = r\int_S \alpha + \int_S \beta.$$

(f) *If $\alpha \geq \beta$ a.e. in S and each of the forms is either nonnegative or integrable on S, then*

$$\int_S \alpha \geq \int_S \beta.$$

(g) *For any open covering $\mathcal{B}$ of M, we have*

$$\int_S \omega = \sup \sum_{j=1}^{m} \int_{S_j} \omega,$$

where the supremum is taken over all choices of disjoint measurable subsets $S_1, \ldots, S_m$ of S each of which is contained in an element of $\mathcal{B}$.

Proof Clearly, we have $\lambda_\omega(\emptyset) = 0$. Suppose R is the union of a sequence $\{S_\nu\}$ of disjoint measurable subsets of S. Given finitely many disjoint measurable subsets $R_1, \ldots, R_m$ of S, each of which is contained in a local $\mathcal{C}^\infty$ coordinate neighborhood in M, we have

$$\sum_{j=1}^{m} \int_{R_j} \omega = \sum_{j=1}^{m} \sum_{\nu=1}^{\infty} \int_{S_\nu \cap R_j} \omega = \sum_{\nu=1}^{\infty} \sum_{j=1}^{m} \int_{S_\nu \cap R_j} \omega \leq \sum_{\nu=1}^{\infty} \int_{S_\nu} \omega = \sum_{\nu=1}^{\infty} \lambda_\omega(S_\nu).$$

Thus $\lambda_\omega(R) \leq \sum_{\nu=1}^{\infty} \lambda_\omega(S_\nu)$. Conversely, given $t \in [0, \infty)$ with $\sum_{\nu=1}^{\infty} \lambda_\omega(S_\nu) > t$, we may choose $N \in \mathbb{Z}_{>0}$ and, for each $\nu = 1, \ldots, N$, disjoint measurable subsets $\{R_j^{(\nu)}\}_{j=1}^{m_\nu}$ of S_ν, each contained in some local $\mathcal{C}^\infty$ coordinate neighborhood, such that

$$t < \sum_{\nu=1}^{N} \sum_{j=1}^{m_\nu} \int_{R_j^{(\nu)}} \omega \leq \lambda_\omega(R).$$

Part (a) now follows.

For (b), observe that by changing f on a set of measure 0, we may assume without loss of generality that $0 \leq f < \infty$ on S. The equality is clear if S is contained in a local $\mathcal{C}^\infty$ coordinate neighborhood (see the remarks following Definition 9.7.8). Applying (a), one gets the equality for a finite union of disjoint measurable subsets, each of which has the above property. Passing to suprema, one gets $\int_S f\omega \leq \int_S f\, d\lambda_\omega$. For the reverse inequality, observe that if $S_1, \ldots, S_m$ are disjoint measurable subsets of S and $r_1, \ldots, r_m$ are nonnegative constants with $f \geq r_j$ on S_j for each j, then

$$\sum_{j=1}^{m} r_j \lambda_\omega(S_j) = \sum_{j=1}^{m} r_j \int_{S_j} \omega \leq \sum_{j=1}^{m} \int_{S_j} f\omega \leq \int_S f\omega$$

(one may verify the above inequalities directly).

The proofs of (c)–(g) are left to the reader (see Exercise 9.7.4). □

Definition 9.7.10 Assume that M is oriented, and let ω be a nonnegative measurable n-form on a measurable set $S \subset M$. Then we denote by λ_ω the positive measure on S provided by Proposition 9.7.9, and we call λ_ω the *measure associated to* ω.

Remark The measure λ_ω is complete if $\omega > 0$ almost everywhere (see Exercise 9.7.9).

Applying the standard convergence theorems for measure spaces, we get the following versions for integrals of differential forms:

Theorem 9.7.11 (Convergence theorems) *Assume that M is oriented. Let ω and $\{\omega_\nu\}_{\nu=1}^\infty$ be measurable n-forms on a measurable set $S \subset M$.*

(a) Monotone convergence theorem. *If $0 \le \omega_1 \le \omega_2 \le \omega_3 \le \cdots$ and $\omega_\nu \to \omega$, then*

$$\int_S \omega = \lim_{\nu\to\infty} \int_S \omega_\nu .$$

(b) Fatou's lemma. *If $\omega_\nu \ge 0$ for each ν and $\omega = \liminf_{\nu\to\infty} \omega_\nu$, then*

$$\int_S \omega \le \liminf_{\nu\to\infty} \int_S \omega_\nu .$$

(c) Dominated convergence theorem. *Suppose $\omega_\nu \to \omega$ and there exists a nonnegative integrable differential form θ of degree n on S such that for each $\nu \in \mathbb{Z}_{>0}$ and each point $x \in S$, we have either $(\omega_\nu)_x = 0$ or $\theta_x > 0$ and $|(\omega_\nu)_x/\theta_x| \le 1$. Then ω is integrable and*

$$\int_S \omega = \lim_{\nu\to\infty} \int_S \omega_\nu .$$

Remark It is not necessary to specify that the form ω be measurable, since the limit or limit inferior of a sequence of measurable differential forms of degree n is measurable.

Proof of Theorem 9.7.11 For the proof of (a), we let $P = \{x \in S \mid \omega_x > 0\}$. By hypothesis, we have $\omega_\nu = 0$ at each point in $S \setminus P$. Therefore, by the standard monotone convergence theorem (i.e., the monotone convergence theorem for integration of functions with respect to measures), we have

$$\int_S \omega_\nu = \int_P \frac{\omega_\nu}{\omega} \cdot \omega = \int_P \frac{\omega_\nu}{\omega} \cdot d\lambda_\omega \to \int_P \frac{\omega}{\omega} \cdot d\lambda_\omega = \int_P \omega = \int_S \omega \quad \text{as } \nu \to \infty.$$

The proofs of (b) and (c) are left to the reader (see Exercise 9.7.5). □

Definition 9.7.12 Let S be a measurable subset of M, and let $p \in [1, \infty]$.

(a) Let α be a measurable differential form of degree r in M on S. For $r=0$, we say that α is in $L^p_{\mathrm{loc}}(S)$ if for every local $\mathcal{C}^\infty$ chart (U, Φ, U'), $\alpha \circ \Phi^{-1}$ is in $L^p_{\mathrm{loc}}(\Phi(S \cap U))$ (that is, each point in $\Phi(S \cap U)$ admits a neighborhood V such that the function $|\alpha \circ \Phi^{-1}|^p$ is integrable on $V \cap \Phi(S \cap U)$). For $r > 0$, we say that α is in $L^p_{\mathrm{loc}}(S)$ if the coefficients of α in every local $\mathcal{C}^\infty$ chart (U, Φ, U') are in $L^p_{\mathrm{loc}}(S \cap U)$. For $p=1$, we also say that α is *locally integrable*, and for $p=2$, we also say that α is *locally square integrable*.

(b) Let $\{\alpha_\nu\}_{\nu=1}^\infty$ and α be differential r-forms in $L^p_{\mathrm{loc}}(S)$. For $r=0$, we say that $\{\alpha_\nu\}$ *converges* in $L^p_{\mathrm{loc}}(S)$ to α if for every local $\mathcal{C}^\infty$ chart (U, Φ, U'), the sequence $\{\alpha_\nu \circ \Phi^{-1}\}$ converges to $\alpha \circ \Phi^{-1}$ in $L^p_{\mathrm{loc}}(\Phi(S \cap U))$ (that is, for each point in $\Phi(S \cap U)$, there exists a neighborhood V in U' such that $\int_{V \cap \Phi(S \cap U)} |\alpha_\nu \circ \Phi^{-1} - \alpha \circ \Phi^{-1}|^p \, d\lambda \to 0$). For $r > 0$, we say that $\{\alpha_\nu\}$ *converges* in $L^p_{\mathrm{loc}}(S)$ to α if in every local $\mathcal{C}^\infty$ chart (U, Φ, U'), the coefficients of the terms of $\{\alpha_\nu\}$ converge in $L^p_{\mathrm{loc}}(S \cap U)$ to the corresponding coefficients of α.

Remarks 1. It is easy to check that the conditions in (a) and (b) hold precisely when they hold in *some* local $\mathcal{C}^\infty$ coordinate neighborhood of each point in S.

2. For M orientable, a measurable n-form α on $S \subset M$ is locally integrable (in the above sense) if and only if each point in S admits a neighborhood U such that α is integrable on $U \cap S$ (as an n-form). The analogous statement for local L^1-convergence also holds.

3. If α and β are differential forms in L^p_{loc} and L^q_{loc}, respectively, with $p, q \in [1, \infty]$ and $p^{-1} + q^{-1} = 1$, then the exterior product $\alpha \wedge \beta$ is locally integrable; that is, $\alpha \wedge \beta$ is in L^1_{loc}.

4. A measurable vector field is said to be in $L^p_{\mathrm{loc}}(S)$ if its coefficients in every local $\mathcal{C}^\infty$ chart (U, Φ, U') are in $L^p_{\mathrm{loc}}(S \cap U)$.

Lemma 9.7.13 *Assume that M is second countable and oriented, and let ω be a nonnegative locally integrable measurable differential form of degree n on M. Then there exists a positive $\mathcal{C}^\infty$ function ρ on M so large that $e^{-\rho}\omega$ is integrable.*

Proof We may assume that M is noncompact (otherwise take $\rho \equiv 1$) and we may choose a locally finite covering $\{U_\nu\}_{\nu=1}^\infty$ by relatively compact open subsets and a $\mathcal{C}^\infty$ partition of unity $\{\lambda_\nu\}$ with $\operatorname{supp} \lambda_\nu \subset U_\nu$ and $0 \le \lambda_\nu \le 1$ for each ν. For each ν, we may choose a constant R_ν such that

$$0 < R_\nu < 1 \quad \text{and} \quad R_\nu \cdot \int_M \lambda_\nu \omega < 2^{-\nu}.$$

The locally finite sum $\sum_{\nu=1}^\infty R_\nu \cdot \lambda_\nu$ then gives a $\mathcal{C}^\infty$ function $\tau \colon M \to (0,1)$, and we may define $\rho \equiv -\log \tau > 0$. We then get

$$\int_M e^{-\rho}\omega = \sum_{\nu=1}^\infty R_\nu \int_M \lambda_\nu \omega < \sum_{\nu=1}^\infty 2^{-\nu} = 1 < \infty.$$

□

Definition 9.7.14 An open subset Ω of a 2-dimensional $\mathcal{C}^\infty$ manifold M is called $\mathcal{C}^\infty$ (or *smooth*) if for each point $p \in M$, there is a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, (x, y))$ such that $\Omega \cap U = \{q \in U \mid x(q) < 0\}$.

Remark The boundary $\partial\Omega$ of a $\mathcal{C}^\infty$ open subset Ω of M is either the empty set or a 1-dimensional $\mathcal{C}^\infty$ submanifold of M.

Lemma 9.7.15 *Let M be an oriented $\mathcal{C}^\infty$ manifold of dimension* 2, *and let Ω be a $\mathcal{C}^\infty$ open subset of M with nonempty boundary $\partial\Omega$. Then $\partial\Omega$ has a* (*natural*) *induced orientation with an oriented $\mathcal{C}^\infty$ atlas $\mathcal{A}$ consisting of all local $\mathcal{C}^\infty$ charts of the form $(U \cap \partial\Omega, \Phi_2\restriction_{U\cap\partial\Omega}, \Phi_2(U \cap \partial\Omega))$, where $(U, \Phi = (\Phi_1, \Phi_2), U')$ is a positively oriented local $\mathcal{C}^\infty$ chart in M with $\Omega \cap U = \{q \in U \mid \Phi_1(q) < 0\}$* (*see Fig.* 9.2). *Furthermore, if $\Lambda\colon M \to N$ is an orientation-preserving diffeomorphism of M onto an oriented $\mathcal{C}^\infty$ manifold N and $\Xi \equiv \Lambda(\Omega)$, then the diffeomorphism $\Lambda\restriction_{\partial\Omega}\colon \partial\Omega \to \partial\Xi$ is orientation-preserving with respect to the induced orientations.*

Proof The collection $\mathcal{A}$ covers $\partial\Omega$, since if $(U, (\Phi_1, \Phi_2), U')$ is a local $\mathcal{C}^\infty$ chart that is *not* positively oriented, U is connected, and $\Omega \cap U = \{\Phi_1 < 0\}$, then the local $\mathcal{C}^\infty$ chart $(U, (\Phi_1, -\Phi_2), U'')$, where $U'' \equiv \{(x, -y) \mid (x, y) \in U'\}$, *is* positively oriented. Given two positively oriented local $\mathcal{C}^\infty$ charts $(U, \Phi = (\Phi_1, \Phi_2) = (x, y), U')$ and $(V, \Psi = (\Psi_1, \Psi_2) = (u, v), V')$ with $\Omega \cap U = \{x < 0\}$ and $\Omega \cap V = \{u < 0\}$, the coordinate transformation $F \equiv \Psi_2 \circ (\Phi_2\restriction_{U\cap\partial\Omega})^{-1}$ of the associated local $\mathcal{C}^\infty$ charts in $\partial\Omega$ satisfies

$$F'(t) = d\Psi_2\left((\Phi_2\restriction_{U\cap\partial\Omega})_*^{-1}\left(\frac{d}{dt}\right)\right) = \frac{\partial v}{\partial y}(\Phi^{-1}(0, t)).$$

On the other hand, we have $u < 0$ on $U \cap V \cap \Omega = \{p \in U \cap V \mid x(p) < 0\}$, $u \equiv 0$ on $U \cap V \cap \partial\Omega = \{p \in U \cap V \mid x(p) = 0\}$, and $u > 0$ on $U \cap V \setminus \overline{\Omega} = \{p \in U \cap V \mid x(p) > 0\}$. So, at each point in $U \cap V \cap \partial\Omega$, we have

$$\frac{\partial u}{\partial x} \geq 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0,$$

and hence

$$0 < \mathcal{J}_{\Psi\circ\Phi^{-1}} \circ \Phi = \begin{vmatrix} \partial u/\partial x & 0 \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y}.$$

It follows that $F'(t) > 0$ and therefore that the atlas is oriented. The proof of the last statement concerning the diffeomorphism $\Lambda\colon M \to N$ is left to the reader (see Exercise 9.7.6). $\square$

Definition 9.7.16 Let Ω be a $\mathcal{C}^\infty$ open subset of an oriented 2-dimensional $\mathcal{C}^\infty$ manifold M with nonempty boundary $\partial\Omega$. The orientation on $\partial\Omega$ given by Lemma 9.7.15 is called the *induced orientation*.

Remark We assume that the boundary of a $\mathcal{C}^\infty$ open set in an oriented 2-dimensional $\mathcal{C}^\infty$ manifold has the induced orientation unless otherwise indicated.

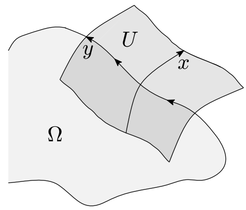

Fig. 9.2 The induced orientation on $\partial\Omega$

Theorem 9.7.17 (Stokes' theorem) *Let M be an oriented $\mathcal{C}^\infty$ surface, let Ω be a nonempty smooth domain in M, and let α be a $\mathcal{C}^1$ differential form of degree 1 on M such that $\overline{\Omega} \cap \operatorname{supp}\alpha$ is compact. Then*

$$\int_{\partial\Omega} \alpha = \int_{\Omega} d\alpha.$$

In particular, if $\operatorname{supp}\alpha \cap \partial\Omega = \emptyset$, *then* $\int_\Omega d\alpha = 0$.

Remark Stokes' theorem should actually be written $\int_{\partial\Omega} \iota^*\alpha = \int_\Omega d\alpha$, where $\iota\colon \partial\Omega \to M$ is the inclusion map; but in an abuse of notation, we will usually leave out the pullback map ι^*.

Proof of Stokes' theorem We may choose a finite collection

$$\{(U_i, \Phi_i = (x_i, y_i), U_i' = (a_i, b_i) \times (0, 1))\}_{i=1}^m$$

of positively oriented local $\mathcal{C}^\infty$ charts such that $\overline{\Omega} \cap \operatorname{supp}\alpha \subset U_1 \cup \cdots \cup U_m \Subset M$ and for each $i = 1, \ldots, m$, $\Omega \cap U_i = \{p \in U_i \mid x_i(p) < 0\} \neq \emptyset$. We may also choose $\mathcal{C}^\infty$ functions $\{\eta_i\}_{i=1}^m$ such that $\sum_{i=1}^m \eta_i \equiv 1$ on $\overline{\Omega} \cap \operatorname{supp}\alpha$ and $\operatorname{supp}\eta_i \subset U_i$ for each $i = 1, \ldots, m$. By reordering if necessary, we may assume that there is an index k such that $a_i < 0 < b_i$ for each $i = 1, \ldots, k$ and $b_i \leq 0$ for each $i = k+1, \ldots, m$. For each $i = 1, \ldots, m$, we have $\beta_i \equiv (\Phi_i^{-1})^*(\eta_i\alpha) = P_i\,dx + Q_i\,dy$ on U_i', where P_i and Q_i are $\mathcal{C}^1$ functions on $\mathbb{R}^2$ with compact support in U_i'. Thus

$$\begin{aligned}
\int_\Omega d\alpha &= \int_\Omega \sum_{i=1}^m d(\eta_i\alpha) = \sum_{i=1}^m \int_\Omega d(\eta_i\alpha) = \sum_{i=1}^m \int_{\Omega\cap U_i} d(\eta_i\alpha) \\
&= \sum_{i=1}^m \int_{\Phi_i(\Omega\cap U_i)} d\beta_i = \sum_{i=1}^k \int_{(a_i,0)\times(0,1)} d\beta_i + \sum_{i=k+1}^m \int_{(a_i,b_i)\times(0,1)} d\beta_i.
\end{aligned}$$

For each $i = 1, \ldots, k$, we have

$$\begin{aligned}
\int_{(a_i,0)\times(0,1)} d\beta_i &= \int_0^1 \int_{a_i}^0 \frac{\partial Q_i}{\partial x}(x, y)\,dx\,dy - \int_{a_i}^0 \int_0^1 \frac{\partial P_i}{\partial y}(x, y)\,dy\,dx \\
&= \int_0^1 [Q_i(0, y) - Q_i(a_i, y)]\,dy - \int_{a_i}^0 [P_i(x, 1) - P_i(x, 0)]\,dx
\end{aligned}$$

$$= \int_0^1 Q_i(0, y)\, dy = \int_{\{0\}\times(0,1)} \beta_i = \int_{\partial\Omega} \eta_i \alpha,$$

and similarly, for $i = k+1, \dots, m$,

$$\int_{(a_i,b_i)\times(0,1)} d\beta_i = \int_0^1 [Q_i(b_i, y) - Q_i(a_i, y)]\, dy$$
$$- \int_{a_i}^{b_i} [P_i(x, 1) - P_i(x, 0)]\, dx = 0.$$

Summing, we get the claim. □

We close this section by briefly recalling the related notion of a *line integral*.

Definition 9.7.18 Let α be a continuous differential form of degree 1 on a $\mathcal{C}^\infty$ manifold M of dimension 1 or 2, and let $\gamma\colon [a, b] \to M$ be a piecewise $\mathcal{C}^1$ path in M (see Definition 9.2.5). Then the (*line*) *integral of* α *along* γ is given by

$$\int_\gamma \alpha \equiv \int_a^b \alpha(\dot\gamma(t))\, dt = \int_{(a,b)} \gamma^*\alpha.$$

Lemma 9.7.19 *Let α be a continuous 1-form on a $\mathcal{C}^\infty$ manifold M of dimension $n = 1$ or 2, and let $\gamma\colon [a, b] \to M$ be a piecewise $\mathcal{C}^1$ path in M.*

(a) *If $(U, (x_1, \dots, x_n))$ is a local $\mathcal{C}^\infty$ coordinate neighborhood in M, $\gamma([a, b]) \subset U$, $c_j = x_j \circ \gamma$ for each $j = 1, \dots, n$, and $\alpha = a_1 dx_1 + \cdots + a_n dx_n$ on U, then*

$$\int_\gamma \alpha = \sum_{j=1}^n \int_a^b a_j(\gamma(t)) c_j'(t)\, dt.$$

(b) *If $N \equiv \gamma((a, b))$ is itself a 1-dimensional $\mathcal{C}^\infty$ manifold for which the restriction $\gamma\restriction_{(a,b)}\colon (a, b) \to N$ is a diffeomorphism and the inclusion map $\iota\colon N \to M$ is of class $\mathcal{C}^\infty$, then $\int_\gamma \alpha = \int_N \iota^*\alpha$, provided N is given the orientation induced by γ.*

The proof is left to the reader (see Exercise 9.7.7). Line integrals are considered in greater depth in Chap. 10.

Example 9.7.20 For polar coordinates (r, θ) in a suitable open subset of $\mathbb{R}^2$, and for rectangular coordinates (x, y), we have $dx \wedge dy = r\, dr \wedge d\theta$. Thus (r, θ) gives positively oriented local $\mathcal{C}^\infty$ coordinates. Consequently, for each $R > 0$ and each $\theta_0 \in \mathbb{R}$, the diffeomorphism $\theta \mapsto Re^{i\theta}$ of $(\theta_0, \theta_0 + 2\pi)$ onto $(\partial\Delta(0; R)) \setminus \{Re^{i\theta_0}\}$ is orientation-preserving (where the boundary $\partial\Delta(0; R)$ of the smooth domain $\Delta(0; R)$ is given the induced orientation). Hence, for any continuous 1-form $\alpha =$

$a\,dr + b\,d\theta$ defined at points of $\partial\Delta(0;R)$, we have

$$\int_{\partial\Delta(0;R)} \alpha = \int_{\theta_0}^{\theta_0+2\pi} b(Re^{i\theta})\,d\theta.$$

Exercises for Sect. 9.7

9.7.1 Prove that equivalence of oriented $\mathcal{C}^\infty$ atlases on a $\mathcal{C}^\infty$ manifold M of dimension $n = 1$ or 2 is an equivalence relation.
9.7.2 Prove that any connected orientable $\mathcal{C}^\infty$ manifold M of dimension $n = 1$ or 2 admits exactly two orientations.
9.7.3 Prove parts (a) and (c) of Proposition 9.7.4.
9.7.4 Prove parts (c)–(g) of Proposition 9.7.9.
9.7.5 Prove parts (b) and (c) of Theorem 9.7.11.
9.7.6 Prove the last part of Lemma 9.7.15. That is, prove that if M and N are oriented 2-dimensional $\mathcal{C}^\infty$ manifolds, Ω is a $\mathcal{C}^\infty$ open subset of M, $\Lambda\colon M \to N$ is an orientation-preserving diffeomorphism, and $\Xi \equiv \Lambda(\Omega)$, then the diffeomorphism $\Lambda\restriction_{\partial\Omega}\colon \partial\Omega \to \partial\Xi$ is orientation-preserving with respect to the induced orientations.
9.7.7 Prove Lemma 9.7.19.
9.7.8 Let M be a *nonorientable* $\mathcal{C}^\infty$ surface. Prove that there exists a surjective $\mathcal{C}^\infty$ mapping $\Upsilon\colon \widehat{M} \to M$ of an orientable $\mathcal{C}^\infty$ surface $\widehat{M}$ onto M such that each point in M admits a connected neighborhood U for which $\Upsilon^{-1}(U)$ has exactly two connected components, each of which is mapped diffeomorphically onto U by Υ ($\Upsilon\colon \widehat{M} \to M$ is called the *orientable double cover of* M).
9.7.9 Let ω be a nonnegative measurable n-form on an orientable $\mathcal{C}^\infty$ manifold M of dimension $n = 1$ or 2. Prove that if $\omega > 0$ a.e., then the measure λ_ω is complete.

9.8 Linear Differential Operators on Manifolds

Throughout this section, M denotes a smooth manifold of dimension n. Linear differential operators on open sets in $\mathbb{R}^n$ were considered in Sect. 7.4. A linear differential operator on M is an operator that may be expressed in local $\mathcal{C}^\infty$ coordinates as a linear differential operator on an open subset of $\mathbb{R}^n$. More precisely:

Definition 9.8.1 Let Ω be an open subset of M. A *linear differential operator A of order $k \in \mathbb{Z}_{\geq 0}$ on Ω in M* is an operator that associates to each open set $V \subset \Omega$ and each $\mathcal{C}^k$ function u on V, a function $Au \in \mathcal{C}^0(V)$, and which has the following property. For each point in Ω, there exist a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, \Phi = (x_1, \dots, x_n), U')$ and a linear differential operator A_Φ of order k on $\Phi(\Omega \cap U)$ such that for each open set $V \subset \Omega$ and each function $u \in \mathcal{C}^k(V)$, we have $[Au]\restriction_{V\cap U} = A_\Phi(u\circ\Phi^{-1})\circ\Phi\restriction_{V\cap U}$. In other words, there exist functions $\{a_\alpha\}_{\alpha\in(\mathbb{Z}_{\geq 0})^n;\, |\alpha|\leq k}$ on $\Omega\cap U$ such that

$$[Au]\restriction_{V\cap U} = \sum_{\alpha\in(\mathbb{Z}_{\geq 0})^n,\, |\alpha|\leq k} a_\alpha\cdot\frac{\partial^{|\alpha|}u}{\partial x^\alpha}$$

on $V\cap U$ for every open set $V\subset\Omega$ and every function $u\in\mathcal{C}^k(V)$ (we have $A_\Phi = \sum(a_\alpha\circ\Phi^{-1})\cdot(\partial/\partial t)^\alpha$, where $(t_1,\dots,t_n)$ are the standard coordinates on $\mathbb{R}^n$). We call this a *local representation* of A with *coefficients* $\{a_\alpha\}$, and we write

$$A = \sum_{\alpha\in(\mathbb{Z}_{\geq 0})^n,\, |\alpha|\leq k} a_\alpha\left(\frac{\partial}{\partial x}\right)^\alpha \quad \text{on } \Omega\cap U.$$

We say that A has $\mathcal{C}^d$ *coefficients* if it has local representations with $\mathcal{C}^d$ coefficients in a neighborhood of each point.

Remarks 1. It is easy to verify that if A is a linear differential operator of order k with $\mathcal{C}^d$ coefficients on an open subset Ω of M, then A has a local representation with $\mathcal{C}^d$ coefficients in *every* local $\mathcal{C}^\infty$ coordinate neighborhood in M.

2. It follows from the definition that if A is a linear differential operator of order k on an open set $\Omega\subset M$, $u\in\mathcal{C}^k(\Omega)$, and V is an open subset of Ω, then $[Au]\restriction_V = A[u\restriction_V]$.

3. Linear differential operators on differential forms are defined similarly (in terms of differential operators on the coefficients in local $\mathcal{C}^\infty$ coordinate neighborhoods).

For *distributional* solutions of the differential equation $Au = v$, one considers distributional solutions of the corresponding equations in local $\mathcal{C}^\infty$ coordinates (cf. Definition 7.4.2).

Definition 9.8.2 Let $k\in\mathbb{Z}_{\geq 0}$, and let A be a linear differential operator of order k with $\mathcal{C}^k$ coefficients on an open set $\Omega\subset M$. For $u, v\in L^1_{\mathrm{loc}}(\Omega)$, we say that Au is equal to v in the *distributional* (or *weak*) *sense*, and we write $A_{\mathrm{distr}}u = v$ (and $A_{\mathrm{distr}}u\in L^1_{\mathrm{loc}}(\Omega)$), if for every local $\mathcal{C}^\infty$ coordinate neighborhood (U,Φ,U') in M with corresponding linear differential operator A_Φ (i.e., $Af = A_\Phi(f\circ\Phi^{-1})\circ\Phi$ for $f\in\mathcal{C}^k$), we have $[A_\Phi]_{\mathrm{distr}}(u\circ\Phi^{-1}) = v\circ\Phi^{-1}$ on $\Phi(\Omega\cap U)$.

This property is local; that is, $A_{\mathrm{distr}}u = v$ on Ω if and only if $A_{\mathrm{distr}}u = v$ on a neighborhood of each point in Ω. Moreover, according to the following, this property is independent of the choice of local $\mathcal{C}^\infty$ coordinates:

Proposition 9.8.3 *Let A be a linear differential operator of order k with $\mathcal{C}^k$ coefficients on an open set $\Omega\subset M$, and let $u, v\in L^1_{\mathrm{loc}}(\Omega)$. Then $A_{\mathrm{distr}}u = v$ if and only if for every point in Ω, there exists a local $\mathcal{C}^\infty$ chart (V,Ψ,V') in M for which the* *associated linear differential operator* A_Ψ *satisfies* $[A_\Psi]_{\text{distr}}(u \circ \Psi^{-1}) = v \circ \Psi^{-1}$ *on* $\Psi(\Omega \cap V)$*. In particular,* $A_{\text{distr}} u = v$ *if and only if for each point in* Ω*, there is a neighborhood* U *in* Ω *such that* $A_{\text{distr}}(u\restriction_U) = v\restriction_U$*.*

Proof Let (U, Φ, U') be a local $\mathcal{C}^\infty$ coordinate neighborhood, and let A_Φ be the associated linear differential operator. Suppose (V, Ψ, V') is a local $\mathcal{C}^\infty$ coordinate neighborhood in M for which the associated linear differential operator A_Ψ satisfies $[A_\Psi]_{\text{distr}}(u \circ \Psi^{-1}) = v \circ \Psi^{-1}$ on $\Psi(\Omega \cap V)$. We then have the coordinate transformation $F \equiv \Phi \circ \Psi^{-1} \colon \Psi(U \cap V) \to \Phi(U \cap V)$, and for each function $\varphi \in \mathcal{D}(\Phi(\Omega \cap U \cap V))$, the change of variables formula (Theorem 7.2.7) gives, for each $f \in \mathcal{C}^\infty(\Phi(\Omega \cap U \cap V))$,

$$\begin{aligned}
\int_{\Phi(\Omega\cap U\cap V)} f \cdot \overline{A_\Phi^*(\varphi)}\, d\lambda &= \int_{\Phi(\Omega\cap U\cap V)} A_\Phi(f) \cdot \overline{\varphi}\, d\lambda \\
&= \int_{\Phi(\Omega\cap U\cap V)} (A(f \circ \Phi) \circ \Phi^{-1}) \cdot \overline{\varphi}\, d\lambda \\
&= \int_{\Psi(\Omega\cap U\cap V)} (A(f \circ \Phi) \circ \Psi^{-1}) \cdot \overline{\varphi \circ F} |\mathcal{J}_F|\, d\lambda \\
&= \int_{\Psi(\Omega\cap U\cap V)} (A_\Psi(f \circ F)) \cdot \overline{\varphi \circ F} |\mathcal{J}_F|\, d\lambda \\
&= \int_{\Psi(\Omega\cap U\cap V)} (f \circ F) \cdot \overline{A_\Psi^*[|\mathcal{J}_F|(\varphi \circ F)]}\, d\lambda \\
&= \int_{\Phi(\Omega\cap U\cap V)} f \cdot \overline{|\mathcal{J}_{F^{-1}}| \cdot (A_\Psi^*[|\mathcal{J}_F|(\varphi \circ F)] \circ F^{-1})}\, d\lambda.
\end{aligned}$$

Therefore, by Lemma 7.3.2, $A_\Phi^*(\varphi) = |\mathcal{J}_{F^{-1}}| \cdot (A_\Psi^*[|\mathcal{J}_F|(\varphi \circ F)] \circ F^{-1})$, and hence

$$\begin{aligned}
\int_{\Phi(\Omega\cap U\cap V)} (u \circ \Phi^{-1}) \cdot \overline{A_\Phi^*(\varphi)}\, d\lambda &= \int_{\Psi(\Omega\cap U\cap V)} (u \circ \Psi^{-1}) \cdot \overline{A_\Psi^*[|\mathcal{J}_F|(\varphi \circ F)]}\, d\lambda \\
&= \int_{\Psi(\Omega\cap U\cap V)} (v \circ \Psi^{-1}) \cdot \overline{\varphi \circ F} |\mathcal{J}_F|\, d\lambda \\
&= \int_{\Phi(\Omega\cap U\cap V)} (v \circ \Phi^{-1}) \cdot \overline{\varphi}\, d\lambda.
\end{aligned}$$

Thus $[A_\Phi]_{\text{distr}}(u \circ \Phi^{-1}) = v \circ \Phi^{-1}$ on $\Phi(\Omega \cap U \cap V)$.

If each point in U admits such a local $\mathcal{C}^\infty$ coordinate neighborhood (V, Ψ, V'), then given a function $\varphi \in \mathcal{D}(\Phi(\Omega \cap U))$, there exist functions $\eta_1, \dots, \eta_m \in \mathcal{D}(\Phi(\Omega \cap U))$ such that $\sum \eta_j \equiv 1$ on $\operatorname{supp} \varphi$ and $[A_\Phi]_{\text{distr}}(u \circ \Phi^{-1}) = v \circ \Phi^{-1}$

on a neighborhood V_j of $\operatorname{supp} \eta_j$ in $\Phi(\Omega \cap U)$ for each $j = 1, \dots, m$. Thus

$$\begin{aligned}\int_{\Phi(\Omega\cap U)} (u \circ \Phi^{-1}) \cdot \overline{A_\Phi^*(\varphi)}\, d\lambda &= \sum_{j=1}^{m} \int_{V_j} (u \circ \Phi^{-1}) \cdot \overline{A_\Phi^*(\eta_j \varphi)}\, d\lambda \\ &= \sum_{j=1}^{m} \int_{V_j} (v \circ \Phi^{-1}) \cdot \overline{\eta_j \varphi}\, d\lambda \\ &= \int_{\Phi(\Omega\cap U)} (v \circ \Phi^{-1}) \cdot \overline{\varphi}\, d\lambda;\end{aligned}$$

and hence $[A_\Phi]_{\text{distr}}(u \circ \Phi^{-1}) = v \circ \Phi^{-1}$ on $\Phi(\Omega \cap U)$. □

9.9 $\mathcal{C}^\infty$ Embeddings

In this section, we consider some criteria that guarantee that a $\mathcal{C}^\infty$ mapping is a (local) diffeomorphism, as well as criteria that guarantee that the zero set of a $\mathcal{C}^\infty$ function, or the image of a $\mathcal{C}^\infty$ mapping, is a smooth submanifold. We will apply these criteria in the study of the topological and holomorphic structure of Riemann surfaces in Sects. 5.10–5.17, and in the study of holomorphic structures on surfaces in Chap. 6. For our purposes, it will suffice to consider curves and surfaces. We first recall the following fact from analysis in $\mathbb{R}^n$:

Theorem 9.9.1 ($\mathcal{C}^\infty$ inverse function theorem) *Suppose $\Omega \subset \mathbb{R}^n$ is an open set, $F\colon \Omega \to \mathbb{R}^n$ is a $\mathcal{C}^\infty$ mapping with Jacobian determinant $\mathcal{J}_F\colon \Omega \to \mathbb{R}$, and $a \in \Omega$ is a point at which $\mathcal{J}_F(a) \neq 0$. Then F maps some neighborhood U of a in Ω diffeomorphically onto some open subset $F(U)$ of $\mathbb{R}^n$.*

Note that conversely, the Jacobian determinant of a diffeomorphism $F\colon U \to V$ of open subsets U and V of $\mathbb{R}^n$ is nonvanishing, because for each point $a \in U$, $\mathcal{J}_F(a)$ is the determinant of the linear operator $(dF)_a\colon \mathbb{R}^n = \mathbb{R}^n \to \mathbb{R}^n$, an isomorphism with inverse linear map $(dF^{-1})_{F(a)}$. The proof of the theorem for $n = 1$ is straightforward. A proof for arbitrary n is outlined in Exercise 9.9.1. The proof can be modified to give a $\mathcal{C}^k$ version, but only the $\mathcal{C}^\infty$ case is applied in this book.

Theorem 9.9.2 *Let M be a smooth manifold of dimension $n = 1$ or 2.*

(a) *If u is a real-valued $\mathcal{C}^\infty$ function on an open set $U \subset M$ and $n = 1$, then the restriction of u to some neighborhood of a point $p \in U$ is a local $\mathcal{C}^\infty$ coordinate if and only if $(du)_p \neq 0$. If u and v are two real-valued $\mathcal{C}^\infty$ functions on an open set $U \subset M$ and $n = 2$, then the restriction of (u, v) to some neighborhood of a point $p \in U$ gives local $\mathcal{C}^\infty$ coordinates if and only if $(du \wedge dv)_p \neq 0$ (i.e., $(du)_p$ and $(dv)_p$ form a basis for $T_p^* M$).*

(b) *Suppose $n = 2$ and $v \in \mathcal{C}^\infty(M, \mathbb{R})$. If $(dv)_p \neq 0$ at some point $p \in M$, then there exists a real-valued $\mathcal{C}^\infty$ function u on a neighborhood U of p such that $(U, (u, v))$ is a local $\mathcal{C}^\infty$ coordinate neighborhood in M. If $dv \neq 0$ at each point in the zero set $Z \equiv \{p \in M \mid v(p) = 0\}$, then Z is a smooth submanifold of dimension 1 in M.*

(c) *Suppose $r \in \{1, 2\}$ and $\Phi\colon N \to M$ is a $\mathcal{C}^\infty$ mapping of an r-dimensional $\mathcal{C}^\infty$ manifold N into M. If $q \in N$ and the tangent map $(\Phi_*)_q\colon T_qN \to T_{\Phi(p)}M$ is injective, then Φ maps a neighborhood V of q in N diffeomorphically onto an r-dimensional submanifold $\Phi(V)$ of some neighborhood U of $\Phi(q)$ in M. If Φ is injective and proper and $(\Phi_*)_q\colon T_qN \to T_{\Phi(q)}M$ is injective for every point $q \in N$, then Φ maps N diffeomorphically onto an r-dimensional smooth submanifold $\Phi(N)$ of M. Conversely, if Φ maps N diffeomorphically onto a smooth submanifold of M, then $\Phi_*\colon TN \to TM$ is injective. If $p \in M$ and $(\Phi_*)_q\colon T_qN \to T_pM$ is surjective for every point $q \in L \equiv \Phi^{-1}(p)$ (i.e., p is a regular value), then L is a $\mathcal{C}^\infty$ submanifold of dimension $r - n$ in N.*

Proof Let $p \in M$. For $n = \dim M = 1$, we may fix a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, \Phi = x)$. For any real-valued $\mathcal{C}^\infty$ function u on a neighborhood of p in U, the Jacobian determinant $\mathcal{J}_u \equiv \partial u/\partial x$ (more precisely, $\mathcal{J}_{u\circ\Phi^{-1}} \circ \Phi = \partial u/\partial x$) satisfies $du = \mathcal{J}_u \cdot dx$. Similarly, for $n = 2$ and for (u, v) a pair of real-valued $\mathcal{C}^\infty$ functions on a neighborhood of p, the Jacobian determinant

$$\mathcal{J}_{(u,v)} \equiv \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

in a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, (x, y))$ of p satisfies $du \wedge dv = \mathcal{J}_{(u,v)} dx \wedge dy$. Part (a) now follows from the inverse function theorem.

If $n = 2$, $p \in M$, and v is a real-valued $\mathcal{C}^\infty$ function on M with $(dv)_p \neq 0$, then fixing a local $\mathcal{C}^\infty$ coordinate neighborhood $(U, (x, y))$ of p, one of the pairs $(dx)_p, (dv)_p$ and $(dy)_p, (dv)_p$ must be a basis for T_p^*M. Setting $u = x$ in the former case and $u = y$ in the latter case, we see that by (a), (u, v) gives local $\mathcal{C}^\infty$ coordinates in a neighborhood of p. In particular, if $dv \neq 0$ at every point in the zero set Z of v, then Z is a $\mathcal{C}^\infty$ submanifold of dimension 1. Thus (b) is proved.

Suppose $r \in \{1, 2\}$ and $\Phi\colon N \to M$ is a $\mathcal{C}^\infty$ mapping of an r-dimensional $\mathcal{C}^\infty$ manifold N into M. If $r = n$, then in local $\mathcal{C}^\infty$ charts, the Jacobian determinant of Φ is precisely the determinant of the tangent map Φ_* (with respect to the bases provided by the partial derivative operators). So the inverse function theorem implies that if $(\Phi_*)_q$ is an isomorphism for some point $q \in N$, then Φ maps a neighborhood of q diffeomorphically onto an open subset of M.

Suppose $r = 1 < 2 = n$ and q is a point at which $(\Phi_*)_q$ is injective. Fixing a local $\mathcal{C}^\infty$ chart $(U, \Psi = (x, y), V \subset \mathbb{R}^2)$ in a neighborhood of $p \equiv \Phi(q)$ with $\Psi(p) = (0, 0) \in V$, we see that by exchanging x and y if necessary, we may assume that the function $\Lambda \equiv x(\Phi)$ satisfies $(d\Lambda)_q \neq 0$. Therefore, by part (a), we may assume that Λ determines a local $\mathcal{C}^\infty$ chart $(W, \Lambda, I \subset \mathbb{R})$ in N such that $q \in W \subset \Phi^{-1}(U)$ and I is an open interval containing $0 = \Lambda(q)$. In fact, by fixing an open interval J with $(0, 0) \in I \times J \subset V$, replacing I with an open subinterval I' containing 0 that is so small that $W' \equiv \Lambda^{-1}(I') \subset \Phi^{-1}(U')$ for $U' = \Psi^{-1}(I' \times J)$, and replacing W with W', U with U', and V with $I' \times J$, we may assume that $V = I \times J$. The $\mathcal{C}^\infty$ function $s \mapsto v(s) \equiv y(s) - y(\Phi(\Lambda^{-1}(x(s))))$ on U satisfies

$$dv = dy - \frac{\partial}{\partial x}\big(y(\Phi(\Lambda^{-1}(x)))\big) \cdot dx - \frac{\partial}{\partial y}\big(y(\Phi(\Lambda^{-1}(x)))\big) \cdot dy$$
$$= dy - \frac{\partial}{\partial x}\big(y(\Phi(\Lambda^{-1}(x)))\big) \cdot dx,$$

and hence $dx \wedge dv = dx \wedge dy \neq 0$. Thus we get a local $\mathcal{C}^\infty$ chart $(U_0, \Psi_0 = (x, v), I_0 \times J_0)$ in M, where $p \in U_0 \subset U$, and I_0 and J_0 are open intervals with $0 \in I_0 \subset I$ and $0 \in J_0 \subset J$. By shrinking I_0 if necessary, we may assume that the neighborhood $W_0 \equiv \Lambda^{-1}(I_0)$ of q in W satisfies $\Phi(W_0) \subset U_0$. Now if $t \in W_0$, then $\Phi(t) \in U_0$ and $v(\Phi(t)) = y(\Phi(t)) - y(\Phi(\Lambda^{-1}(\Lambda(t)))) = 0$. Conversely, if $s \in U_0$ with $v(s) = 0$, then setting $t \equiv \Lambda^{-1}(x(s)) \in W_0$, we get

$$x(\Phi(t)) = \Lambda(t) = x(s) \quad \text{and} \quad v(\Phi(t)) = 0 = v(s),$$

and hence $\Phi(t) = s$. Therefore, $\Phi(W_0)$ is the 1-dimensional submanifold $N_0 \equiv \Psi_0^{-1}(I_0 \times \{0\})$ of U_0. Furthermore, the map $\Phi\restriction_{W_0} = (x\restriction_{N_0})^{-1} \circ \Lambda\restriction_{W_0} \to N_0$ is a composition of diffeomorphisms, and is therefore a diffeomorphism. The verifications of the remaining claims in (c) are left to the reader. □

Definition 9.9.3 A $\mathcal{C}^\infty$ mapping $\Phi \colon N \to M$ of an r-dimensional $\mathcal{C}^\infty$ manifold N into an n-dimensional $\mathcal{C}^\infty$ manifold M with $r, n \in \{1, 2\}$ is

(i) An *immersion* if the tangent map $(\Phi_*)_p \colon T_pN \to T_pM$ is injective for each point $p \in N$;
(ii) An *embedding* (or *imbedding*) if Φ is a proper injective immersion;
(iii) A *submersion* if the tangent map $(\Phi_*)_p \colon T_pN \to T_pM$ is surjective for each point $p \in N$.

Remarks 1. In this book, we require embeddings to be proper, but it is also common practice not to require this as part of the definition.

2. If N is a smooth submanifold of a smooth manifold M, and $\iota \colon N \hookrightarrow M$ is the inclusion map, then we identify TN with $\iota_* TN \subset TM$. If M and N are oriented and α is a differential form of degree $\dim N$ on M, then, in a slight abuse of notation, we often write $\int_N \iota^*\alpha$ simply as $\int_N \alpha$. Note that this does not conflict with the earlier definition of the integral over a measurable subset of M because there, we assumed α to be of degree $\dim M$.

3. Theorem 9.9.2 implies that a $\mathcal{C}^\infty$ mapping $\Phi \colon N \to M$ of $\mathcal{C}^\infty$ manifolds M and N of dimension $n = 1$ or 2 is a local diffeomorphism if and only if the tangent map is an isomorphism at each point.

4. It follows from Theorem 9.9.2 that an open subset Ω of a $\mathcal{C}^\infty$ surface M is smooth if and only if for each point $p \in \partial\Omega$, there exists a real-valued $\mathcal{C}^\infty$ function φ on a neighborhood U of p such that $(d\varphi)_p \neq 0$ and $\Omega \cap U = \{x \in U \mid \varphi(x) < 0\}$.

Exercises for Sect. 9.9

9.9.1 The goal of this exercise is a proof of the $\mathcal{C}^\infty$ inverse function theorem (Theorem 9.9.1). Let Ω be an open subset of $\mathbb{R}^n$, let $F = (f_1, \ldots, f_n) \colon \Omega \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ mapping with Jacobian matrix

$$J_F \equiv \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} : \quad \Omega \to \mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$$

and Jacobian determinant $\mathcal{J}_F = \det(J_F) \colon \Omega \to \mathbb{R}$ (clearly, both are of class $\mathcal{C}^\infty$), and let $a = (a_1, \ldots, a_n) \in \Omega$ be a point at which $\mathcal{J}_F(a) \neq 0$. For each point $b \in \Omega$, let $R_b = (r_b^1, \ldots, r_b^n) \colon \Omega \to \mathbb{R}^n$ be the $\mathcal{C}^\infty$ map given by

$$x \mapsto F(x) - F(b) - (dF)_b(x - b).$$

(a) Show that there is a constant $C_0 > 0$ such that $\|x\| \leq C_0 \|(dF)_a(x)\|$ for every vector $x \in \mathbb{R}^n$.

(b) Prove that for every $\epsilon > 0$, there is a neighborhood U of a in Ω such that $\|R_b(y) - R_b(x)\| \leq \epsilon \|y - x\|$ for all points $b, x, y \in U$.

(c) Prove that there are a constant $C_1 > 0$ and a neighborhood U of a in Ω such that $\|F(y) - F(x)\| \geq C_1 \|y - x\|$ for all points $x, y \in U$. In particular, $F\restriction_U$ is injective.

(d) Prove that if U is a neighborhood of a in Ω on which $\mathcal{J}_F$ is nonvanishing, then for every point $b \in \mathbb{R}^n$, the $\mathcal{C}^\infty$ function $\|F - b\|^2$ cannot attain a positive local minimum value at a point in U.

(e) Prove that if U is a neighborhood of a in Ω such that $\mathcal{J}_F\restriction_U$ is nonvanishing and $F\restriction_U$ is injective, then F maps U homeomorphically onto an open set $F(U)$ in $\mathbb{R}^n$.

Hint. Assuming $V \Subset U$ is open, but $W \equiv F(V)$ is not open, there must exist a point $p \in V$ with $q = F(p) \in \partial W$. Fix an open ball B centered at p with $B \Subset V$. Then, by injectivity, q is not in the compact set $K \equiv F(\partial B)$. Now fix a point $b \in \mathbb{R}^n \setminus W$ with $\|b - q\| < \text{dist}(q, K)/2$ and apply part (d).

(f) Prove that there exists a relatively compact neighborhood U of a in Ω such that $\mathcal{J}_F\restriction_U$ is nonvanishing, F maps U homeomorphically onto an open set $F(U)$ in $\mathbb{R}^n$, and, for $G = (g_1, \ldots, g_n) \equiv (F\restriction_U)^{-1} \colon F(U) \to \mathbb{R}^n$, we have, for some constant $C_2 > 0$, $\|G(y) - G(x)\| \leq C_2 \|y - x\|$ for all $x, y \in F(U)$.

(g) Prove that for U as in part (f), we have

$$\lim_{x \to F(b)} \frac{\|R_b(G(x))\|}{\|x - F(b)\|} = 0 \quad \forall b \in U.$$

(h) Prove that for U as in part (f) and for each point $b \in U$, the first-order partial derivatives of G at $F(b)$ exist and

$$J_G(F(b)) \equiv \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(F(b)) & \cdots & \frac{\partial g_1}{\partial x_n}(F(b)) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1}(F(b)) & \cdots & \frac{\partial g_n}{\partial x_n}(F(b)) \end{pmatrix} = (J_F(b))^{-1}.$$

Hint. We have $x = F(b) + (dF)_b(G(x) - b) + R_b(G(x))$ for each point $x \in F(U)$. Solve for $G(x)$ (in terms of x and $R_b(G(x))$), and apply part (g).

(i) Prove by induction on k that G is of class $\mathcal{C}^k$ for each $k = 0, 1, 2, \ldots$.

9.9.2 Let Ω be an open subset of a second countable smooth manifold M. Prove that Ω is a smooth open set if and only if there exists a $\mathcal{C}^\infty$ function $\varphi\colon M \to \mathbb{R}$ such that $\Omega = \{x \in M \mid \varphi(x) < 0\}$ and $(d\varphi)_p \neq 0$ for each point $p \in \partial\Omega$.

9.10 Classification of Second Countable 1-Dimensional Manifolds

The second countable curves are completely determined by the following:

Theorem 9.10.1 *Every second countable connected* 1*-dimensional manifold is homeomorphic to either the circle* $\mathbb{S}^1$ *or the line* $\mathbb{R}$*. Any second countable connected* 1*-dimensional smooth manifold is diffeomorphic to either* $\mathbb{S}^1$ *or* $\mathbb{R}$.

Remarks 1. It is easy to see that every open interval in $\mathbb{R}$ is diffeomorphic to $\mathbb{R}$. For given $a, b \in \mathbb{R}$ with $a < b$, we have the diffeomorphisms of (a, ∞), $(-\infty, b)$, and (a, b) onto $\mathbb{R}$ given by $t \mapsto \log(t - a)$, $t \mapsto -\log(b - t)$, and $t \mapsto \log((t - a)/(b - t))$, respectively.

2. The theorem is completely believable, but the proof requires some work. One proof is provided in this section. Others may be found in, for example, [Mi] or [GuiP].

3. Elements of the proof also appear in the proof of the existence of smooth structures on second countable topological surfaces in Sect. 6.11.

Throughout this section, M denotes a second countable connected 1-dimensional topological or smooth manifold. The intermediate value theorem implies that a continuous injective real-valued function on an interval I is either strictly increasing or strictly decreasing, and that the function maps the interval homeomorphically onto an interval J (in particular, the image of any endpoint of I is an endpoint of J). Furthermore, if the interval is open and the function is of class $\mathcal{C}^\infty$ with nonvanishing derivative, then the function is a diffeomorphism.

Lemma 9.10.2 *Let* $\alpha\colon I \to M$ *be an injective continuous mapping of an open interval* $I \subset \mathbb{R}$ *into* M *with image* $A = \alpha(I)$*. Then*

(a) *The image A is a connected open subset of M onto which α maps I homeomorphically.*
(b) *For M a smooth manifold and α of class $\mathcal{C}^\infty$ with nonvanishing tangent vector $\dot{\alpha}$, α maps I diffeomorphically onto A.*

Proof Given a point $c \in I$, there is a local chart $(J, \Phi, \mathbb{R})$ in M with $\alpha(c) = \Phi^{-1}(0)$. The connected component H of $\alpha^{-1}(J)$ containing c is then an open subinterval of I and $\Phi \circ \alpha\restriction_H$ maps H homeomorphically onto an open interval. The claims (a) and (b) now follow. $\square$

Lemma 9.10.3 *Let $\alpha\colon [a,b] \to M$ and $\beta\colon [c,d] \to M$ be injective paths, and let $A \equiv \alpha([a,b])$, $A_0 \equiv \alpha((a,b))$, $B \equiv \beta([c,d])$, and $B_0 \equiv \beta((c,d))$. Assume that M is noncompact, $A_0 \cap B_0 \neq \emptyset$, and $A \not\subset B_0$. Then $\beta^{-1}(A) = [c,r]$ or $[r,d]$ for some point $r \in [c,d]$.*

Proof Clearly, $A = \bar{A}_0$ and $B = \overline{B}_0$, and by Lemma 9.10.2, A_0 and B_0 are connected open subsets of M, α maps (a,b) homeomorphically onto A_0, and β maps (c,d) homeomorphically onto B_0. Of course, α maps $[a,b]$ homeomorphically onto A and β maps $[c,d]$ homeomorphically onto B, since these are compact Hausdorff spaces.

Each connected component D of $\beta^{-1}(A)$ is either a singleton or a closed interval, and hence $\alpha^{-1}(\beta(D))$ is either a singleton or a closed interval contained in $[a,b]$. If $D \subset (c,d)$, then $\beta(D)$ is a compact subset of $A \cap B_0$. Since $A \not\subset B_0$, it follows that $\alpha^{-1}(\beta(D))$ is a connected compact subset of $[a,b]$ that is not equal to the entire interval. Hence there is an open interval J such that

$$J \subset (a,b) \cap \alpha^{-1}(B_0) \setminus \alpha^{-1}(\beta(D)) \quad \text{and} \quad \partial J \cap \alpha^{-1}(\beta(D)) \neq \emptyset.$$

Thus $\beta^{-1}(\alpha(J)) \cup D \supsetneqq D$ is a connected subset of $\beta^{-1}(A)$, and we have arrived at a contradiction.

By hypothesis, $\beta^{-1}(A) \subset [c,d]$ has a connected component with nonempty interior, and by the above, such a connected component must be a closed interval with at least one endpoint equal to c or d. Suppose that we have such a connected component of the form $[r,d]$ with $c < r < d$ but $\beta^{-1}(A) \neq [r,d]$. By the above, we then have either $\beta^{-1}(A) = \{u\} \cup [r,d]$ with $u = c$ or $\beta^{-1}(A) = [c,u] \cup [r,d]$ for some $u \in (c,r)$. By replacing α with the injective path $t \mapsto \alpha(-t)$ for $t \in [-b,-a]$ (which has the same image A) if necessary, we may assume that the function $\alpha^{-1} \circ \beta\restriction_{[r,d]}$ is strictly increasing. Clearly, we must have $\beta(u), \beta(r) \in A \setminus A_0$ (otherwise, $\beta^{-1}(A)$ would contain a strictly larger subset of $[c,d]$), so we must have $\beta(r) = \alpha(a)$ and $\beta(u) = \alpha(b)$. In particular, if $u \in (c,r)$, then $A \setminus A_0 = \{\alpha(a), \alpha(b)\} \subset B_0$ and

$$B \setminus B_0 = \{\beta(c), \beta(d)\} \subset A \setminus \{\beta(u), \beta(r)\} = A \setminus \{\alpha(a), \alpha(b)\} = A_0,$$

and hence $A \cup B = A_0 \cup B_0$ is open. Suppose instead that $u = c$. Since $\alpha^{-1} \circ \beta$ is a strictly increasing continuous mapping of $[r,d]$ onto a closed subinterval of $[a,b)$

and we have $\alpha^{-1}(\beta(c)) = b$, we get a continuous injective mapping

$$\gamma : (\alpha^{-1}(\beta(d)), r - c + b) \to A \cup B$$

by setting

$$\gamma(t) = \begin{cases} \alpha(t) & \text{if } t \in (\alpha^{-1}(\beta(d)), b], \\ \beta(t - b + c) & \text{if } t \in [b, r - c + b). \end{cases}$$

Hence, by Lemma 9.10.2, the image of γ is a neighborhood of $\alpha(b) = \beta(c)$ in M, and hence this point is an interior point of $A \cup B$. We also have $\alpha(a) = \beta(r) \in B_0$ and $\beta(d) = \alpha(\alpha^{-1}(\beta(d)) \in A_0$. Thus $A \cup B$ is again open in this case. Therefore, in either case, since M is connected and noncompact, we have arrived at a contradiction. A similar argument shows that any connected component of $\beta^{-1}(A)$ of the form $[c, r]$ with $r > c$ must be equal to $\beta^{-1}(A)$. $\square$

Lemma 9.10.4 *Let $-\infty < a < b < \infty$, and let f and g be real-valued continuous strictly increasing functions on $[a, b]$ with $f(a) < g(b)$. Then there exist numbers c and d with $a < c < d < b$ and a continuous strictly increasing function $h : [a, b] \to \mathbb{R}$ such that $h = f$ on $[a, c]$ and $h = g$ on $[d, b]$. Moreover, if f and g are of class $\mathcal{C}^\infty$ with $f' > 0$ and $g' > 0$ on $[a, b]$, then h may also be chosen to be of class $\mathcal{C}^\infty$ with $h' > 0$.*

Proof We may fix points $c, d \in (a, b)$ with $c < d$ and $f(c) < g(d)$. Hence the function h given by

$$h(t) = \begin{cases} f(t) & \text{if } t \in [a, c], \\ f(c) + \frac{t-c}{d-c} \cdot (g(d) - f(c)) & \text{if } t \in (c, d), \\ g(t) & \text{if } t \in [d, b], \end{cases}$$

is a continuous strictly increasing function with $h = f$ on $[a, c]$ and $h = g$ on $[d, b]$.

If f and g are of class $\mathcal{C}^\infty$ with $f' > 0$ and $g' > 0$, then fixing points u and v with $c < u < v < d$ and $f(u) < g(v)$, we may form a $\mathcal{C}^\infty$ partition of unity $\{\lambda_1, \lambda_2, \lambda_3\}$ on $\mathbb{R}$ (with $0 \leq \lambda_j \leq 1$ for $j = 1, 2, 3$) such that $\operatorname{supp} \lambda_1 \subset (-\infty, u)$, $\operatorname{supp} \lambda_2 \subset (c, d)$, and $\operatorname{supp} \lambda_3 \subset (v, \infty)$ (see Theorem 9.3.7). In particular, we have

$$0 \leq q \equiv f(a) + \int_a^d (\lambda_1(s) f'(s) + \lambda_3(s) g'(s))\, ds \leq f(u) + g(d) - g(v) < g(d).$$

Setting $r \equiv \int_c^d \lambda_2(s)\, ds > 0$, we see that the $\mathcal{C}^\infty$ function h given by

$$h(t) = f(a) + \int_a^t [\lambda_1(s) f'(s) + r^{-1}(g(d) - q)\lambda_2(s) + \lambda_3(s) g'(s)]\, ds \quad \forall t \in [a, b]$$

satisfies $h' > 0$ on (a, b), $h = f$ on $[a, c]$, and $h = g$ on $[d, b]$. $\square$

Lemma 9.10.5 *Suppose Ω_0 and U are open subsets of M; I_0 is an open interval; $\alpha_0\colon I_0 \to \Omega_0$ and $\beta\colon \mathbb{R} \to U$ are homeomorphisms; K_0 is a compact subset of Ω_0; and K is a compact subset of U that meets K_0. Assume that M is noncompact. Then there exists a homeomorphism α of an open interval I onto an open subset Ω of M such that $K_0 \cup K \subset \Omega \subset \Omega_0 \cup U$ and $\alpha^{-1}\restriction_{K_0} = \alpha_0^{-1}\restriction_{K_0}$. Moreover, if M is smooth and α_0 and β are diffeomorphisms, then α may be chosen to be a diffeomorphism.*

Proof If $K \subset \Omega_0$, then we may set $\alpha \equiv \alpha_0\colon I \equiv I_0 \to \Omega \equiv \Omega_0$; so we may assume without loss of generality that $K \not\subset \Omega_0$. We may also assume that K is connected, and we may fix an interval $I_1 = [a, b] \Subset I_0$ with $K_0 \subset \alpha_0((a, b))$ and an interval $J = [c, d] \Subset \mathbb{R}$ with $K \subset \beta((c, d))$.

Let us first suppose that $\Omega_0 \not\subset K$. Then we may choose the intervals I_1 and J so that $\alpha_0(I_1) \not\subset \beta(J)$; and by applying Lemma 9.10.3 (and reparametrizing β if necessary), we may assume that for some $r \in (c, d)$, we have $\beta^{-1}(\alpha_0(I_1)) \cap J = [c, r]$ with either $a < \alpha_0^{-1}(\beta(c)) < \alpha_0^{-1}(\beta(r)) = b$ or $a = \alpha_0^{-1}(\beta(r)) < \alpha_0^{-1}(\beta(c)) < b$. In the former case, we may choose a number $s \in (\alpha_0^{-1}(\beta(c)), b)$ so close to b that $K_0 \subset \alpha_0((a, s))$. By reparametrizing β again, we may assume that $s = \alpha_0^{-1}(\beta(s))$ and $r = \alpha_0^{-1}(\beta(r)) = b$ (in particular, $a < \alpha_0^{-1}(\beta(c)) < s < b < d$ and $c < s < b < d$), and we may apply Lemma 9.10.4 to get constants u and v with $s < u < v < b$ and a continuous strictly increasing function $f\colon [s, b] \to [s, b]$ such that $f(t) = t$ for each point $t \in [s, u]$, $f(t) = \alpha_0^{-1}(\beta(t))$ for each point $t \in [v, b]$, and, for M smooth, f is of class $\mathcal{C}^\infty$ with $f' > 0$. The set

$$\Omega \equiv \alpha_0((a, b]) \cup \beta([b, d)) = \alpha_0((a, b)) \cup \beta((c, d))$$

is then a domain in M, and

$$K_0 \cup K \Subset \alpha_0((a, b)) \cup \beta((c, d)) = \Omega \subset \Omega_0 \cup U.$$

Moreover, we may define a homeomorphism $\alpha\colon I \equiv (a, d) \to \Omega$ (which is a diffeomorphism if M is smooth) by setting

$$\alpha(t) = \begin{cases} \alpha_0(t) & \text{if } t \in (a, s], \\ \alpha_0(f(t)) & \text{if } t \in [s, b], \\ \beta(t) & \text{if } t \in [b, d). \end{cases}$$

In particular, we have $\alpha^{-1} = \alpha_0^{-1}$ on the set $\alpha_0((a, s)) \supset K_0$. In the case in which $a = \alpha_0^{-1}(\beta(r)) < \alpha_0^{-1}(\beta(c)) < b$, we may apply the above arguments to β and the homeomorphism $\hat{\alpha}_0\colon t \mapsto \alpha_0(b - t + a)$ to get a domain Ω and a homeomorphism $\hat{\alpha}\colon (a, d) \to \Omega$. The homeomorphism $\alpha\colon I \equiv (b - d + a, b) \to \Omega$ given by $t \mapsto \hat{\alpha}(b - t + a)$ then has the required properties.

Finally, suppose $\Omega_0 \subset K \subset U$. Then we may choose the homeomorphism (or diffeomorphism) $\beta\colon \mathbb{R} \to U$ and the constants c and d so that $c < a < b < d$,

$\beta([a,b]) = \alpha_0([a,b])$, $\beta(a) = \alpha_0(a)$, and $\beta(b) = \alpha_0(b)$. After choosing numbers s_0 and s_1 with $a < s_0 < s_1 < b$ and $K_0 \subset \alpha_0((s_0, s_1))$, we have

$$\alpha_0^{-1}(\beta(a)) = a < s_0 \quad \text{and} \quad s_1 < b = \alpha_0^{-1}(\beta(b)).$$

Therefore, Lemma 9.10.4 provides constants u_0, v_0, u_1, and v_1 with

$$a < u_0 < v_0 < s_0 < s_1 < u_1 < v_1 < b$$

and continuous strictly increasing functions $f_0 \colon [a, s_0] \to [a, s_0]$ and $f_1 \colon [s_1, b] \to [s_1, b]$ such that $f_0(t) = \alpha_0^{-1}(\beta(t))$ for each point $t \in [a, u_0]$, $f_0(t) = t$ for each point $t \in [v_0, s_0]$, $f_1(t) = t$ for each point $t \in [s_1, u_1]$, and $f_1(t) = \alpha_0^{-1}(\beta(t))$ for each point $t \in [v_1, b]$, and for M smooth, f_0 and f_1 are of class $\mathcal{C}^\infty$ with $f_0' > 0$ and $f_1' > 0$. We then have the domain $\Omega \equiv \beta((c,d))$ that satisfies

$$K_0 \cup K \subset \alpha_0((s_1, s_2)) \cup \beta((c,d)) = \beta((c,d)) = \Omega,$$

and we have the homeomorphism (diffeomorphism for M smooth) $\alpha \colon I \equiv (c,d) \to \Omega$ given by

$$\alpha(t) = \begin{cases} \beta(t) & \text{if } t \in (c,a] \cup [b,d), \\ \alpha_0(f_0(t)) & \text{if } t \in [a, s_0], \\ \alpha_0(t) & \text{if } t \in [s_0, s_1], \\ \alpha_0(f_1(t)) & \text{if } t \in [s_1, b]. \end{cases} \qquad \square$$

Proof of Theorem 9.10.1 We first consider the case in which the manifold M is noncompact. By Lemma 9.3.6, there exist locally finite open coverings $\{U_\nu\}_{\nu=1}^\infty$ and $\{V_\nu\}_{\nu=1}^\infty$ of X such that for each $\nu = 1, 2, 3, \dots$, $V_\nu \Subset U_\nu \Subset M$, U_ν is homeomorphic to $\mathbb{R}$, and for M smooth, U_ν is also diffeomorphic to $\mathbb{R}$. We may also choose the coverings so that for each ν, the set $V_1 \cup \dots \cup V_\nu$ meets $V_{\nu+1}$. For we may let ν_1 be the smallest index ν for which $V_\nu \neq \emptyset$, and given $\nu_1, \dots, \nu_k$, we may let ν_{k+1} be the smallest index $\nu \notin \{\nu_1, \dots, \nu_k\}$ for which V_ν meets $V_{\nu_1} \cup \dots \cup V_{\nu_k}$ (such an index exists because the union of the boundaries of the relatively compact open sets $V_{\nu_1}, \dots, V_{\nu_k}$ cannot be contained in the union of these sets). By induction, we obtain a sequence $\{V_{\nu_k}\}$ that covers M (otherwise, the union would have a boundary point that lies in V_ν for some ν and we would have $\nu < \nu_k$ for $k \gg 0$). Thus we may replace the original coverings $\{V_\nu\}$ and $\{U_\nu\}$ with the coverings $\{V_{\nu_k}\}$ and $\{U_{\nu_k}\}$, respectively.

There exist domains $\{\Omega_\nu\}$ in M and mappings $\{\alpha_\nu\}$ such that for each $\nu = 1, 2, 3, \dots$,

$$\Theta_\nu \equiv V_1 \cup \dots \cup V_\nu \Subset \Omega_\nu \subset \Lambda_\nu \equiv U_1 \cup \dots \cup U_\nu,$$

and α_ν is a homeomorphism of an open interval I_ν onto Ω_ν with $\alpha_\nu^{-1}\restriction_{\Theta_\nu} = \alpha_{\nu+1}^{-1}\restriction_{\Theta_\nu}$. Moreover, for M smooth, each mapping α_ν is a diffeomorphism. For the existence of Ω_1 and $\alpha_1 \colon I_1 \to \Omega_1$ is clear, and one may then proceed to apply Lemma 9.10.5

inductively. Since the sequence of functions $\{\alpha_\mu^{-1}\restriction_{V_\nu}\}_{\mu=\nu}^\infty$ is constant for each ν, we may then define a homeomorphism (a diffeomorphism for M smooth) γ of $M = \bigcup V_\nu$ onto an open interval by setting $\gamma\restriction_{V_\nu} = \alpha_\nu^{-1}\restriction_{V_\nu}$ for each $\nu = 1, 2, 3, \dots$. Fixing a diffeomorphism $\lambda\colon \mathbb{R} \to \gamma(M)$, we get the desired homeomorphism (diffeomorphism for M smooth) $\gamma^{-1} \circ \lambda\colon \mathbb{R} \to M$.

Finally, suppose M is compact. We may fix a homeomorphism $\beta\colon \mathbb{R} \to U$ onto a neighborhood U of a point $p \in M$, with β a diffeomorphism for M smooth. The set $U \setminus \{p\}$ has exactly two connected components U_0 and U_1, and each of the connected components of $M \setminus \{p\}$ must meet, and therefore contain, at least one of these connected components. It follows that $M \setminus \{p\}$ has at most two connected components. However, if $N \approx \mathbb{R}$ is a connected component that contains only one of these connected components, say, U_0, then, fixing a point $q \in U_0$, we see that of the two connected components of $N \setminus \{q\}$, one is relatively compact in N, which is clearly impossible. Thus $M \setminus \{p\}$ is connected, and we may choose a homeomorphism $\alpha\colon (0, 1) \to M \setminus \{p\}$, with α a diffeomorphism for M smooth.

Since U_0 and U_1 are disjoint open connected subsets of $M \setminus \{p\}$ that are not relatively compact in $M \setminus \{p\}$, we may choose U and α so that for some $r \in (0, 1/4)$, we have $\alpha^{-1}(U_0) = (0, 2r)$ and $\alpha^{-1}(U_1) = (1 - 2r, 1)$. We may also choose the homeomorphism (or diffeomorphism) β so that $\beta(-r) = \alpha(1-r)$ and $\beta(r) = \alpha(r)$. In particular, $\beta([-r, r])$ is a connected relatively compact subset of U that meets both U_0 and U_1, and therefore $p \in \beta((-r, r))$, and we may set $t_0 \equiv \beta^{-1}(p)$. We have $\beta((-\infty, t_0)) = U_1$, $\beta((t_0, \infty)) = U_0$, $\alpha^{-1}(\beta(t)) \to 1$ as $t \to t_0^-$, and $\alpha^{-1}(\beta(t)) \to 0$ as $t \to t_0^+$; so $\alpha^{-1} \circ \beta\restriction_{(-\infty,t_0)}$ and $\alpha^{-1} \circ \beta\restriction_{(t_0,\infty)}$ must be strictly increasing functions with $\alpha^{-1} \circ \beta([-r, t_0)) = [1-r, 1)$ and $\alpha^{-1} \circ \beta((t_0, r]) = (0, r]$. Fixing $b_0 \in (-r, t_0) \cap (-r, 0)$ and $b_1 \in (t_0, r) \cap (0, r)$ and applying Lemma 9.10.4, we get continuous strictly increasing functions $f_0\colon [-r, b_0] \to [1 - r, \alpha^{-1}(\beta(b_0))]$ and $f_1\colon [b_1, r] \to [1 + \alpha^{-1}(\beta(b_1)), 1 + r]$ such that $f_0(t) = 1 + t$ for t near $-r$, $f_0(t) = \alpha^{-1}(\beta(t))$ for t near b_0, $f_1(t) = 1 + \alpha^{-1}(\beta(t))$ for t near b_1, and $f_1(t) = 1 + t$ for t near r. Moreover, f_0 and f_1 are of class $\mathcal{C}^\infty$ with $f_0' > 0$ and $f_1' > 0$ for M smooth. The mapping $\gamma\colon (-r, 1 + b_0) \to M$ given by

$$\gamma(t) = \begin{cases} \alpha(t) & \text{if } t \in [r, 1-r], \\ \alpha(f_0(t)) & \text{if } t \in (-r, b_0], \\ \beta(t) & \text{if } t \in [b_0, b_1], \\ \alpha(-1 + f_1(t)) & \text{if } t \in [b_1, r], \\ \alpha(f_0(t-1)) & \text{if } t \in [1-r, 1+b_0), \end{cases}$$

is a surjective continuous mapping that is a local homeomorphism and that satisfies $\gamma(t+1) = \gamma(t)$ for each $t \in (-r, b_0)$. Moreover, γ is a local diffeomorphism if M is smooth. It follows that the mapping $\mathbb{S}^1 \to M$ given by $e^{2\pi i t} \mapsto \gamma(t)$ for $t \in (-r, 1 + b_0)$ is a well-defined homeomorphism, and this mapping is a diffeomorphism if M is smooth. $\square$

9.11 Riemannian Metrics

In this section we consider Riemannian metrics (on smooth surfaces), which in this book are applied directly (only) in the construction of an almost complex structure on an orientable surface (Proposition 6.1.3) in Chap. 6.

Definition 9.11.1 A *Riemannian metric* g on a 2-dimensional smooth manifold M is a choice of a real inner product $g_p(\cdot,\cdot)$ on the real tangent space T_pM at each point $p \in M$ such that for each pair of local $\mathcal{C}^\infty$ vector fields u and v, the function $g(u,v)\colon p \mapsto g_p(u_p, v_p)$ is of class $\mathcal{C}^\infty$. For each point $p \in M$ and each pair of real tangent vectors $u, v \in T_pM$, we also write $g(u,v) = g_p(u,v)$ and $|u|_g^2 = g(u,u)$.

Remarks Riemannian metrics are fundamental objects in the study of the (differential) geometry of manifolds. A Riemannian metric provides a notion of distance, curvature, and volume in a manifold. In this book, we mostly consider related objects, such as *Kähler metrics* on Riemann surfaces and *Hermitian metrics* in holomorphic line bundles. Corresponding notions of volume and curvature do play important roles in the analytic techniques applied throughout this book, but the geometric meaning will not be the main consideration (for much more on Riemannian metrics, see, for example, [KobN1] and [KobN2]).

Observe that if g is a Riemannian metric on a smooth surface M and $(U, (x,y))$ is a local $\mathcal{C}^\infty$ coordinate neighborhood in M, then the functions

$$\begin{aligned} A &\equiv g(\partial/\partial x, \partial/\partial x),\\ B &\equiv g(\partial/\partial x, \partial/\partial y) = g(\partial/\partial y, \partial/\partial x),\\ C &\equiv g(\partial/\partial y, \partial/\partial y), \end{aligned}$$

are of class $\mathcal{C}^\infty$ and

$$g(u,v) = A\,dx(u)\,dx(v) + B\,dx(u)\,dy(v) + B\,dy(u)\,dx(v) + C\,dy(u)\,dy(v)$$

for every pair of tangent vectors u, v at a point in U. In particular, it follows that if u and v are vector fields that are continuous ($\mathcal{C}^k$ with $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, measurable), then the function $g(u,v)$ is continuous (respectively, $\mathcal{C}^k$, measurable).

Proposition 9.11.2 *Every second countable smooth surface M admits a Riemannian metric.*

Proof There exist a locally finite collection of local $\mathcal{C}^\infty$ charts

$$\{(U_\nu, \Phi_\nu = (x_\nu, y_\nu), U'_\nu)\}_{\nu \in N}$$

and a $\mathcal{C}^\infty$ partition of unity $\{\lambda_\nu\}$ with $\operatorname{supp}\lambda_\nu \subset U_\nu$ (and $\lambda_\nu \geq 0$) for each $\nu \in N$. The pairing given by

$$g(u, v) = g_p(u, v) \equiv \sum_{\nu \in N} \lambda_\nu(p)(dx_\nu(u)\, dx_\nu(v) + dy_\nu(u)\, dy_\nu(v)),$$

for all $p \in M$ and $u, v \in T_pM$, is then a Riemannian metric on M. The verification is left to the reader (see Exercise 9.11.1). $\square$

Exercises for Sect. 9.11

9.11.1 Verify that the pairing defined in the proof of Proposition 9.11.2 is a Riemannian metric.

Chapter 10
Background Material on Fundamental Groups, Covering Spaces, and (Co)homology

We recall that a *path* (or *parametrized path* or *curve* or *parametrized curve*) in a topological space X from a point x to a point y is a continuous mapping $\gamma\colon [a,b] \to X$ with $\gamma(a) = x$ and $\gamma(b) = y$ (see Sect. 9.1). We take the domain of a path to be $[0, 1]$, unless otherwise indicated. A *loop* (or *closed curve*) with base point $p \in X$ is a path in X from p to p. In this chapter, we consider the equivalence relation given by *path homotopies*. This leads to the *fundamental group*, which is the group given by the path homotopy equivalence classes of loops at a point, and to *covering spaces*, both of which are important objects in complex analysis and Riemann surface theory. We also consider homology groups, which are essentially Abelian versions of the fundamental group, and cohomology groups, which are groups that are dual to the homology groups.

10.1 The Fundamental Group

Throughout this section, X denotes a nonempty topological space.

Definition 10.1.1 A *path homotopy* in X is a continuous mapping

$$H\colon [a,b] \times [0,1] \to X,$$

where $-\infty < a < b < \infty$ and the paths $s \mapsto H(a,s)$ and $s \mapsto H(b,s)$ are constant. We also call H a *path homotopy* (or simply a *homotopy*) *from* γ_0 *to* γ_1, where for each $s \in [0,1]$, γ_s is the path given by $\gamma_s(t) = H(t,s)$ for each $t \in [a,b]$ (see Fig. 10.1). We also say that γ_0 and γ_1 are *path homotopic* (or simply *homotopic*), and we write $H : \gamma_0 \sim \gamma_1$ or $\gamma_0 \sim \gamma_1$.

Remarks 1. We take the domain of a path homotopy to be $[0,1] \times [0,1]$, unless otherwise indicated.

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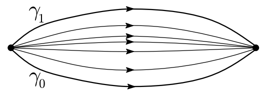

Fig. 10.1 A path homotopy from γ_0 to γ_1

2. *Change of parameter*. If $\gamma\colon [a,b] \to X$ is a path and $\varphi\colon [a,b] \to [a,b]$ is a continuous mapping with $\varphi(a) = a$ and $\varphi(b) = b$, then

$$H : \gamma \sim \gamma \circ \varphi, \quad \text{where } H(t,s) = \gamma(t + s(\varphi(t) - t)) \ \forall (t,s) \in [a,b] \times [0,1].$$

If $\delta\colon [a,b] \to X$ is a path, $H : \delta \sim \gamma$, and $\psi\colon [c,d] \to [a,b]$ is a continuous mapping with $\psi(c) = a$ and $\psi(d) = b$, then

$$G : \delta \circ \psi \sim \gamma \circ \psi, \quad \text{where } G(t,s) = H(\psi(t), s) \ \forall (t,s) \in [c,d] \times [0,1].$$

For this reason, we lose nothing by restricting most of our attention to paths with domain $[0,1]$, since we may always reparametrize so that the domain becomes $[0,1]$.

3. The path homotopy relation $\sim$ is an equivalence relation. To see this, suppose $\gamma_j\colon [a,b] \to X$ is a path in X with $\gamma_j(a) = x$ and $\gamma_j(b) = y$ for $j = 0,1,2$, and $H_j : \gamma_{j-1} \sim \gamma_j$ for $j = 1,2$. Then the mapping

$$(t,s) \mapsto \gamma_0(t) \quad \forall (t,s) \in [a,b] \times [0,1]$$

is a path homotopy from γ_0 to itself;

$$(t,s) \mapsto H_1(t, 1-s) \quad \forall (t,s) \in [a,b] \times [0,1]$$

is a path homotopy from γ_1 to γ_0; and

$$(t,s) \mapsto \begin{cases} H_1(t,2s) & \forall (t,s) \in [a,b] \times [0,1/2], \\ H_2(t,2s-1) & \forall (t,s) \in [a,b] \times [1/2,1], \end{cases}$$

is a path homotopy from γ_0 to γ_2.

Definition 10.1.2 Let $\gamma_0\colon [0,1] \to X$ and $\gamma_1\colon [0,1] \to X$ be paths in X with $\gamma_0(1) = \gamma_1(0)$.

(a) The *product path* of γ_0 and γ_1 is the path $\gamma_0 * \gamma_1\colon [0,1] \to X$ from $\gamma_0(0)$ to $\gamma_1(1)$ given by

$$\gamma_0 * \gamma_1(t) \equiv \begin{cases} \gamma_0(2t) & \text{if } t \in [0,1/2], \\ \gamma_1(2t-1) & \text{if } t \in [1/2,1]. \end{cases}$$

(b) The *reverse path* of γ_0 is the path $\gamma_0^-\colon [0,1] \to X$ from $\gamma_0(1)$ to $\gamma_0(0)$ given by

$$\gamma_0^-(t) \equiv \gamma_0(1-t) \quad \forall t \in [0,1].$$

Remarks 1. The product path is obtained by tracing the first path at twice the speed, and then tracing the second path at twice the speed. The paths are traced at twice the speed so that the resulting domain is again [0, 1].

2. The reverse path is obtained by retracing the path in the reverse direction. Clearly, for any path γ (with domain [0, 1]), we have $(\gamma^-)^- = \gamma$.

3. Occasionally, it is convenient to consider product and reverse paths with domains other than [0, 1]. If $\gamma_j \colon [a_j, b_j] \to X$ is a path in X for $j = 0, 1$ and $\gamma_0(b_0) = \gamma_1(a_1)$, then by $\gamma_0 * \gamma_1 \colon [a, b] \to X$ we will mean a path obtained by tracing γ_0 and then γ_1, after some (nonunique, noncanonical) linear reparametrizations. That is, after choosing an interval $[a, b]$ and a point $c \in (a, b)$, we have

$$\gamma_0 * \gamma_1(t) \equiv \begin{cases} \gamma_0(a_0 + \frac{t-a}{c-a}(b_0 - a_0)) & \text{if } t \in [a, c], \\ \gamma_1(a_1 + \frac{t-c}{b-c}(b_1 - a_1))) & \text{if } t \in [c, b]. \end{cases}$$

Similarly, by $\gamma_0^- \colon [a, b] \to X$ we will mean the path given by

$$\gamma_0^-(t) \equiv \gamma_0\left(b_0 + \frac{t-a}{b-a}(a_0 - b_0)\right) \quad \forall t \in [a, b].$$

4. For paths γ_0 and γ_1 in X and a continuous mapping $\Phi \colon X \to Y$ into a topological space Y, we have $\Phi(\gamma_0 * \gamma_1) = \Phi(\gamma_0) * \Phi(\gamma_1)$ and $\Phi(\gamma_0^-) = (\Phi(\gamma_0))^-$.

5. The product operation on paths is not associative. Given n paths $\gamma_1, \dots, \gamma_n$, we will set

$$\gamma_1 * \cdots * \gamma_n \equiv (\cdots((\gamma_1 * \gamma_2) * \gamma_3) \cdots) * \gamma_n.$$

On the other hand, the product is associative up to path homotopy. In fact, we have the following lemma, the proof of which is left to the reader (see Exercise 10.1.1).

Lemma 10.1.3 *Let $\gamma_j \colon [0, 1] \to X$ be a path with $\gamma_j(0) = x_j$ and $\gamma_j(1) = y_j$ for $j = 0, 1, 2$. For each point $p \in X$, let $e_p \colon [0, 1] \to X$ denote the constant loop $t \mapsto p$.*

(a) *We have $\gamma_0 * \gamma_0^- \sim e_{x_0}$, $\gamma_0^- * \gamma_0 \sim e_{y_0}$, $e_{x_0} * \gamma_0 \sim \gamma_0$, and $\gamma_0 * e_{y_0} \sim \gamma_0$. Furthermore, if $\gamma_0 \sim \gamma_1$, then $\gamma_0^- \sim \gamma_1^-$.*
(b) *If $\gamma_0(1) = \gamma_1(0)$ and $\gamma_1(1) = \gamma_2(0)$, then $(\gamma_0 * \gamma_1) * \gamma_2 \sim \gamma_0 * (\gamma_1 * \gamma_2)$.*
(c) *If $\gamma_0(1) = \gamma_1(1) = \gamma_2(0)$ and $\gamma_0 \sim \gamma_1$, then $\gamma_0 * \gamma_2 \sim \gamma_1 * \gamma_2$.*
(d) *If $\gamma_0(1) = \gamma_1(0) = \gamma_2(0)$ and $\gamma_1 \sim \gamma_2$, then $\gamma_0 * \gamma_1 \sim \gamma_0 * \gamma_2$.*
(e) *If $\Phi \colon X \to Y$ is a continuous mapping of topological spaces and $H : \gamma_0 \sim \gamma_1$, then $\Phi(H) \colon \Phi(\gamma_0) \sim \Phi(\gamma_1)$.*

Lemma 10.1.3 allows us to associate a group to each point in X as follows:

Definition 10.1.4 Let $x_0 \in X$.

(a) We denote the set of loops in X based at x_0 with domain [0, 1] by $\mathcal{L}(X, x_0)$. For each loop $\gamma \in \mathcal{L}(X, x_0)$, we denote the path homotopy equivalence class of γ by $[\gamma]$ or $[\gamma]_{x_0}$ or $[\gamma]_{\pi_1(X, x_0)}$.

(b) The *fundamental group of X with base point x_0* is the group $\pi_1(X, x_0)$ with elements given by the set of equivalence classes of loops in $\mathcal{L}(X, x_0)$, with the *product* and *inverse* given by $[\gamma_0] \cdot [\gamma_1] \equiv [\gamma_0 * \gamma_1]$ and $[\gamma_0]^{-1} \equiv [\gamma_0^-]$, respectively, for all $\gamma_0, \gamma_1 \in \mathcal{L}(X, x_0)$, and the *identity element* given by $[e_{x_0}]$, where $e_{x_0} \in \mathcal{L}(X, x_0)$ is the constant loop based at x_0.
(c) If $\Phi\colon X \to Y$ is a continuous mapping of topological spaces and $y_0 = \Phi(x_0)$, then the map $\Phi_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $[\gamma]_{x_0} \mapsto [\Phi(\gamma)]_{y_0}$ is the induced *pushforward* homomorphism of fundamental groups.

Remarks If $\Phi\colon X \to Y$ and $\Psi\colon Y \to Z$ are continuous mappings of topological spaces, $x_0 \in X$, $y_0 = \Phi(x_0)$, and $z_0 = \Psi(y_0)$, then the induced homomorphisms $\Phi_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $\Psi_*\colon \pi_1(Y, y_0) \to \pi_1(Z, z_0)$ satisfy

$$\Psi_* \circ \Phi_* = (\Psi \circ \Phi)_*\colon \pi_1(X, x_0) \to \pi_1(Z, z_0).$$

In particular, if Φ is a homeomorphism, then $\Phi_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism with inverse mapping $(\Phi^{-1})_*$.

Lemma 10.1.5 *Let $\eta\colon [0, 1] \to X$ be a path from x_0 to x_1 in X. Then the map $\pi_1(X, x_0) \to \pi_1(X, x_1)$ given by $[\gamma]_{x_0} \mapsto [\eta^- * \gamma * \eta]_{x_1}$ is a well-defined surjective isomorphism with inverse map given by $[\gamma]_{x_1} \mapsto [\eta * \gamma * \eta^-]_{x_0}$. Moreover, if $\Phi\colon X \to Y$ is a continuous mapping into a topological space Y, $y_j = \Phi(x_j)$ for $j = 0, 1$, and $\lambda = \Phi(\eta)$, then the above isomorphism and the isomorphism $[\gamma]_{y_0} \mapsto [\lambda^- * \gamma * \lambda]_{y_1}$ together give a commutative diagram*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\Phi_*"] \arrow[d, "\cong"'] & \Phi_*\pi_1(X, x_0) \subset \arrow[d, "\cong"] & \pi_1(Y, y_0) \arrow[d, "\cong"] \\
\pi_1(X, x_1) \arrow[r, "\Phi_*"] & \Phi_*\pi_1(X, x_1) \subset & \pi_1(Y, y_1)
\end{tikzcd}$$

Proof This follows from Lemma 10.1.3. □

Remark Because all fundamental groups of a path connected topological space X are isomorphic (although not canonically isomorphic), the associated groups are (or more precisely, the associated isomorphism class of groups is) often denoted by $\pi_1(X)$.

Definition 10.1.6 We say that X is *simply connected* if X is path connected and $\pi_1(X)$ is trivial (that is, $\pi_1(X, x_0) = [e_{x_0}]_{x_0}$ for some, hence for every, point $x_0 \in X$). We say that X is *locally simply connected* if for every point $p \in X$ and every neighborhood U of p, there exists a simply connected neighborhood V of p in U.

For example, $\mathbb{R}^n$ is simply connected, and for $n > 1$, the unit sphere $\mathbb{S}^n$ is simply connected (see Exercise 10.1.3). We postpone consideration of examples of nontrivial fundamental groups until Sect. 10.4, since they are most easily considered within the context of covering spaces and deck transformations.

The proof of the following consequence of Lemma 10.1.3 is left to the reader (see Exercise 10.1.2):

Lemma 10.1.7 *If X is simply connected, then any two paths in X with the same initial and terminal points are path homotopic.*

We will also need the following easy observation:

Lemma 10.1.8 *Let α be a path in X, let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of $[0, 1]$, and for each $i = 1, \dots, n$, let $\alpha_i : [0, 1] \to X$ be a reparametrization of the path $\alpha\restriction_{[t_{i-1}, t_i]}$. Then the path $\beta \equiv \alpha_1 * \cdots * \alpha_n$ is a reparametrization of α. In particular, $\alpha \sim \beta$ (see the remarks following Definition 10.1.1).*

Proof We prove by induction that $\beta_i \equiv \alpha_1 * \cdots * \alpha_i$ is a reparametrization of $\alpha\restriction_{[0,t_i]}$ for each $i = 1, \dots, n$, the case $i = 1$ being trivial. Assuming that β_{i-1} is a reparametrization of $\alpha\restriction_{[0,t_{i-1}]}$, we have continuous mappings $\varphi\colon [0, 1] \to [0, t_{i-1}]$ and $\psi\colon [0, 1] \to [t_{i-1}, t_i]$ such that $\varphi(0) = 0$, $\varphi(1) = t_{i-1}$, $\beta_{i-1} = \alpha \circ \varphi$, $\psi(0) = t_{i-1}$, $\psi(1) = t_i$, and $\alpha_i = \alpha \circ \psi$. Thus

$$\beta_i = \beta_{i-1} * \alpha_i = (\alpha \circ \varphi) * (\alpha \circ \psi) = \alpha \circ \rho,$$

where $\rho\colon [0, 1] \to [0, t_i]$ is the continuous mapping given by

$$\rho(t) \equiv \begin{cases} \varphi(2t) & \text{if } t \in [0, 1/2], \\ \psi(2t - 1) & \text{if } t \in [1/2, 1]. \end{cases}$$

The claim now follows by induction, and the lemma is then the case $i = n$. □

Lemma 10.1.9 *Let X be a connected locally simply connected locally compact Hausdorff space.*

(a) *For every compact set $K \subset X$, there is a path connected relatively compact neighborhood Ω of K in X such that $\operatorname{im}(\pi_1(\Omega) \to \pi_1(X))$ is finitely generated.*
(b) *If X is compact, then $\pi_1(X)$ is finitely generated.*
(c) *If X is second countable, then $\pi_1(X)$ is countable.*

Proof For the proof of (a), suppose K is a compact subset of X and x_0 is a point in K. By applying Lemma 9.3.6 and replacing K with the closure of a relatively compact connected neighborhood in X, we may assume that K is connected. The collection of relatively compact simply connected open subsets of X forms a basis for the topology. Therefore, by Lemma 9.3.6, there exist finite coverings of K by nonempty connected open sets $\{B_i\}_{i\in I}$ and $\{B'_i\}_{i\in I}$ in X such that for each $i \in I$, $B_i \cap K \neq \emptyset$, B'_i is simply connected, and $B_i \Subset B'_i$, and for every pair of indices $i, j \in I$ with $\overline{B}_i \cap \overline{B}_j \neq \emptyset$, we have $B_i \Subset B'_j$. In particular, K is contained in the domain $\Omega \equiv \bigcup_{i\in I} B_i \Subset X$. For each pair of indices $i, j \in I$ with $B_i \cap B_j \neq \emptyset$, we may choose a point $p_{ij} = p_{ji} \in B_i \cap B_j$ and a path $\alpha_{ij} = \alpha_{ji}$ in Ω from x_0 to p_{ij}. Let us also choose these paths so that α_{ii} lies in B_i whenever $i \in I$ with $x_0 \in B_i$. Finally, for each triple of indices $i, j, k \in I$ with $B_i \cap B_j \neq \emptyset$ and $B_j \cap B_k \neq \emptyset$, we may choose a path β_{ijk} in B_j from p_{ij} to p_{jk}.

Given a loop λ in Ω with base point x_0, we may form a partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$$

such that for each $\nu = 1, \dots, m$, we have $\lambda([t_{\nu-1}, t_\nu]) \subset B_{i_\nu}$ for some index $i_\nu \in I$. In particular, for each $\nu = 1, \dots, m$, we may choose a reparametrization $\lambda_\nu \colon [0, 1] \to B_{i_\nu}$ of $\lambda\restriction_{[t_{\nu-1}, t_\nu]}$. Set $i_0 = i_1$ and $i_{m+1} = i_m$. For each $\nu = 1, \dots, m+1$ we may choose a path γ_ν in B_{i_ν} from $p_{i_{\nu-1}i_\nu}$ to $\lambda(t_{\nu-1})$. For each $\nu = 1, \dots, m$, since $B_{i_\nu} \cup B_{i_{\nu+1}}$ is contained in the simply connected set B'_{i_ν}, the path $\gamma_\nu * \lambda_\nu * \gamma_{\nu+1}^-$ is path homotopic in X to the path $\beta_{i_{\nu-1}i_\nu i_{\nu+1}}$ (which lies in Ω). Moreover, γ_1^- and $\alpha_{i_0 i_1} = \alpha_{i_1 i_1}$ are paths in B_{i_1} with the same endpoints and are therefore path homotopic, while γ_{m+1} and $\alpha_{i_m i_{m+1}}^- = \alpha_{i_m i_m}^-$ are paths in B_{i_m} with the same endpoints and are therefore also path homotopic. Hence we have the following path homotopy equivalences in X:

$$\begin{aligned}\lambda &\sim \gamma_1^- * \gamma_1 * \lambda_1 * \gamma_2^- * \gamma_2 * \lambda_2 * \gamma_3^- * \cdots * \gamma_{m-1} \\ &\quad * \lambda_{m-1} * \gamma_m^- * \gamma_m * \lambda_m * \gamma_{m+1}^- * \gamma_{m+1} \\ &\sim \gamma_1^- * \beta_{i_0 i_1 i_2} * \beta_{i_1 i_2 i_3} * \cdots * \beta_{i_{m-1} i_m i_{m+1}} * \gamma_{m+1} \\ &\sim \alpha_{i_0 i_1} * \beta_{i_0 i_1 i_2} * \alpha_{i_1 i_2}^- * \alpha_{i_1 i_2} * \beta_{i_1 i_2 i_3} \\ &\quad * \alpha_{i_2 i_3}^- * \cdots * \alpha_{i_{m-1} i_m} * \beta_{i_{m-1} i_m i_{m+1}} * \alpha_{i_m i_{m+1}}^-.\end{aligned}$$

It follows that the image $\mathrm{im}(\pi_1(\Omega, x_0) \to \pi_1(X, x_0))$ is (finitely) generated by the finite set of elements

$$\{[\alpha_{ij} * \beta_{ijk} * \alpha_{jk}^-]_{x_0} \mid i, j, k \in I,\ B_i \cap B_j \neq \emptyset,\ \text{and } B_j \cap B_k \neq \emptyset\}.$$

Thus part (a) is proved.

Part (b) follows immediately from (a). Finally, for the proof of part (c), we observe that part (a) and σ-compactness (Lemma 9.3.6) provide a sequence of domains $\{\Omega_\nu\}$ and a point $x_0 \in \Omega_1$ such that $X = \bigcup_\nu \Omega_\nu$ and such that for each ν, $\Omega_\nu \Subset \Omega_{\nu+1}$ and the image Γ_ν of $\pi_1(\Omega_\nu, x_0)$ in $\pi_1(X, x_0)$ is finitely generated and therefore countable. Furthermore, $\pi_1(X, x_0)$ is equal to the countable set $\bigcup_{\nu=1}^\infty \Gamma_\nu$. For given an element $[\gamma] \in \pi_1(X, x_0)$, we have $\gamma([0, 1]) \subset \Omega_\nu$ for some $\nu \gg 0$, and hence $[\gamma] \in \Gamma_\nu$. □

Exercises for Sect. 10.1

10.1.1 Prove Lemma 10.1.3.

10.1.2 Prove Lemma 10.1.7.

10.1.3 Prove that for $n > 1$, the unit sphere $\mathbb{S}^n$ is simply connected.

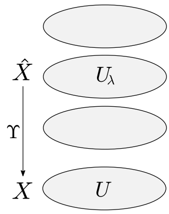

Fig. 10.2 A covering space and an evenly covered set

10.2 Elementary Properties of Covering Spaces

Definition 10.2.1 Let $\Upsilon : \widehat{X} \to X$ be a surjective continuous mapping of topological spaces.

(a) An open set $U \subset X$ is *evenly covered by* Υ if $\widehat{U} \equiv \Upsilon^{-1}(U) = \bigcup_{\lambda \in \Lambda} U_\lambda$, where $\{U_\lambda\}_{\lambda \in \Lambda}$ is a collection of disjoint open subsets of $\widehat{X}$ and Υ maps U_λ homeomorphically onto U for each index $\lambda \in \Lambda$ (see Fig. 10.2).
(b) Υ is called a *covering map* and $\widehat{X}$ is called a *covering space* of X if every point in X has a neighborhood that is evenly covered by Υ.
(c) If $\widehat{X}$ and X are $\mathcal{C}^\infty$ manifolds and $\Upsilon : \widehat{X} \to X$ is a covering map that is a local diffeomorphism (i.e., for $\widehat{U} \equiv \Upsilon^{-1}(U) = \bigcup_{\lambda \in \Lambda} U_\lambda$ as in (a), Υ maps each of the sets U_λ diffeomorphically onto U), then $\Upsilon : \widehat{X} \to X$ is called a $\mathcal{C}^\infty$ *covering space* (or a $\mathcal{C}^\infty$ *covering manifold*) of X.

Remarks 1. The map $\mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^3$ is a covering map (in fact, a homeomorphism) that is of class $\mathcal{C}^\infty$, but this is not a $\mathcal{C}^\infty$ covering space because the map is not a local diffeomorphism.

2. If $\widehat{X}$ is a locally connected topological space, $\Upsilon : \widehat{X} \to X$ is a surjective continuous mapping of $\widehat{X}$ onto a topological space X, and U is an evenly covered domain in X, then Υ maps each connected component of $\Upsilon^{-1}(U)$ homeomorphically onto U.

Example 10.2.2 The map $\Upsilon : \mathbb{R} \to \mathbb{S}^1$ given by $t \mapsto e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t))$ is a $\mathcal{C}^\infty$ covering map. For as considered in Examples 7.2.8, 9.2.8, and 9.7.20, for each point $t_0 \in \mathbb{R}$, we get an open arc $U \equiv \mathbb{S}^1 \setminus \{e^{2\pi i t_0}\}$ and we have

$$\Upsilon^{-1}(U) = \bigcup_{n \in \mathbb{Z}} U_n,$$

where $\{U_n\}$ is the collection of disjoint open intervals given by

$$U_n \equiv (t_0 + (n-1)2\pi, t_0 + n2\pi) \quad \forall n \in \mathbb{Z},$$

and Υ maps U_n diffeomorphically onto U for each n.

Definition 10.2.3 Let $\Phi : X \to Z$ and $\Psi : Y \to Z$ be continuous mappings of topological spaces. A *lifting* of Φ (relative to Ψ) is a continuous map $\widehat{\Phi} : X \to Y$ such that $\Psi \circ \widehat{\Phi} = \Phi$. In other words, the following diagram commutes:

$$\begin{tikzcd} & Y \arrow[d, "\Psi"] \\ X \arrow[ur, "\widehat{\Phi}"] \arrow[r, "\Phi"'] & Z \end{tikzcd}$$

Lemma 10.2.4 *Let X, Y, and $\widehat{Y}$ be connected topological spaces, let $\Phi\colon X \to Y$ be a continuous map, let $\Upsilon\colon \widehat{Y} \to Y$ be a covering space, and let $\widehat{\Phi}\colon X \to \widehat{Y}$ be a lifting of Φ.*

(a) *If $\widehat{\Phi}'$ is a lifting of Φ that agrees with $\widehat{\Phi}$ at some point in X, then $\widehat{\Phi}' = \widehat{\Phi}$.*
(b) *If Φ is a local homeomorphism, then $\widehat{\Phi}$ is a local homeomorphism.*
(c) *If Φ is a covering map and Y is locally connected, then $\widehat{\Phi}$ is a covering map.*
(d) *Suppose X and Y are C^∞ manifolds, Φ is of class C^∞, and $\widehat{Y}$ is a C^∞ covering space. Then the lifting $\widehat{\Phi}$ is of class C^∞. Moreover, if Φ is a local diffeomorphism, then $\widehat{\Phi}$ is a local diffeomorphism. If $\Phi\colon X \to Y$ is a C^∞ covering space, then $\widehat{\Phi}\colon X \to \widehat{Y}$ is a C^∞ covering space.*

Proof For the proof of (a), let $\widehat{\Phi}'$ be a lifting of Φ and assume that the set

$$A \equiv \left\{x \in X \mid \widehat{\Phi}'(x) = \widehat{\Phi}(x)\right\}$$

is nonempty. Since X is connected, it suffices to show that A is both closed and open. Given a point $x_0 \in X$, we may choose a neighborhood U of $\Phi(x_0)$ such that

$$\widehat{U} \equiv \Upsilon^{-1}(U) = \bigcup_{\lambda\in\Lambda} U_\lambda,$$

where $\{U_\lambda\}_{\lambda\in\Lambda}$ is a collection of disjoint open subsets of $\widehat{Y}$, and Υ maps U_λ homeomorphically onto U for each index $\lambda \in \Lambda$. Hence there exist unique indices λ and λ' with $\widehat{\Phi}(x_0) \in U_\lambda$ and $\widehat{\Phi}'(x_0) \in U_{\lambda'}$, that is,

$$\widehat{\Phi}(x_0) = (\Upsilon\restriction_{U_\lambda})^{-1}(\Phi(x_0)) \quad \text{and} \quad \widehat{\Phi}'(x_0) = (\Upsilon\restriction_{U_{\lambda'}})^{-1}(\Phi(x_0)).$$

The set $V \equiv \widehat{\Phi}^{-1}(U_\lambda) \cap \left(\widehat{\Phi}'\right)^{-1}(U_{\lambda'})$ is then a neighborhood of x_0. If $x_0 \notin A$, then $\lambda \neq \lambda'$, so $\widehat{\Phi}(V) \cap \widehat{\Phi}'(V) = \emptyset$, and hence $V \subset X \setminus A$. If $x_0 \in A$, then $\lambda = \lambda'$, and on V, we have $\widehat{\Phi} = (\Upsilon\restriction_{U_\lambda})^{-1} \circ \Phi = \widehat{\Phi}'$, and hence $V \subset A$. Thus A is both closed and open, and therefore $A = X$. The proofs of (b)–(d) are left to the reader (see Exercise 10.2.1). □

Theorem 10.2.5 (Lifting theorem) *Let $\Upsilon\colon \widehat{X} \to X$ be a connected covering space of a connected topological space X, let $x_0 \in X$, and let $\widehat{x_0} \in \Upsilon^{-1}(x_0)$.*

(a) Path lifting property. *If $\gamma\colon [a, b] \to X$ is a path in X with $\gamma(a) = x_0$, then there is a unique lifting $\hat{\gamma}\colon [a, b] \to \widehat{X}$ of γ to a path in $\widehat{X}$ with $\hat{\gamma}(a) = \hat{x}_0$.*

(b) Path homotopy lifting property. *If* $H\colon [a,b]\times[0,1]\to X$ *is a path homotopy in* X *with* $H(a,\cdot)\equiv x_0$, *then there is a unique lifting*

$$\widehat{H}\colon [a,b]\times[0,1]\to\widehat{X}$$

of H *to a path homotopy in* $\widehat{X}$ *with* $\widehat{H}(a,\cdot)\equiv\hat{x}_0$ *(in particular,* $\widehat{H}(b,\cdot)$ *is constant).*

(c) *The induced homomorphism* $\Upsilon_*\colon \pi_1(\widehat{X},\hat{x}_0)\to\pi_1(X,x_0)$ *is injective.*

(d) General lifting property. *If* $\Phi\colon Y\to X$ *is a continuous mapping of a connected locally path connected topological space* Y *to* X, $y_0\in\Phi^{-1}(x_0)$, *and* $\Phi_*\pi_1(Y,y_0)\subset\Upsilon_*\pi_1(\widehat{X},\hat{x}_0)$, *then there is a unique lifting* $\widehat{\Phi}\colon Y\to\widehat{X}$ *of* Φ *with* $\widehat{\Phi}(y_0)=\hat{x}_0$. *In fact, for every point* $y\in Y$ *and every path* $\gamma\colon[a,b]\to Y$ *from* y_0 *to* y, *we have* $\widehat{\Phi}(y)=\hat{\lambda}(b)$, *where* $\hat{\lambda}\colon[a,b]\to\widehat{X}$ *is the unique lifting of the path* $\lambda\equiv\Phi(\gamma)$ *with* $\hat{\lambda}(a)=\hat{x}_0$. *Conversely, if such a lifting* $\widehat{\Phi}$ *exists, then* $\Phi_*\pi_1(Y,y_0)\subset\Upsilon_*\pi_1(\widehat{X},\hat{x}_0)$.

Proof We first prove uniqueness in part (a). Suppose $\gamma\colon[a,b]\to X$ is a path and $\Upsilon(\hat{x}_0)=x_0=\gamma(a)$ as in (a). For any evenly covered neighborhood U of x_0, it is clear that if γ lies in U, then there is a unique lifting of γ with initial point $\widehat{x}_0$. In general, if α and β are two liftings with initial point $\hat{x}_0$, then letting

$$c\equiv\sup\{t\in[a,b]\mid\alpha=\beta\text{ on }[a,t]\}\in(a,b]$$

and forming an evenly covered neighborhood U of $\gamma(c)$, we may choose $\epsilon>0$ so that $a<c-\epsilon$ and $\gamma([c-\epsilon,c+\epsilon]\cap[a,b])\subset U$. But then, $\alpha(c-\epsilon)=\beta(c-\epsilon)$, and hence $\alpha=\beta$ on $[c-\epsilon,c+\epsilon]\cap[a,b]$ and therefore on $[a,c+\epsilon]\cap[a,b]$. It follows that $c=b$ and $\alpha=\beta$. Thus we have uniqueness. A similar argument shows that for

$$d\equiv\sup\{t\in[a,b]\mid\gamma\restriction_{[a,t]}\text{ admits a lifting with initial point }\hat{x}_0\},$$

we have $d=b$ and d lies in the above set. Thus we have existence as well.

For the proof of (b), observe that by part (a), for each $s\in[0,1]$, we have a unique lifting $\widehat{H}_s\colon[a,b]\to\widehat{X}$ of the path $H_s\colon t\mapsto H(t,s)$ to a path in $\widehat{X}$ with $\widehat{H}_s(a)=\hat{x}_0$. Thus we may define a map $\widehat{H}\colon[a,b]\times[0,1]\to\widehat{X}$ by $(t,s)\mapsto\widehat{H}_s(t)$; and we get $\widehat{H}(a,\cdot)\equiv\hat{x}_0$ and $\Upsilon\circ\widehat{H}=H$. If the set $A\subset[a,b]\times[0,1]$ of points of discontinuity for $\widehat{H}$ is nonempty, then we may set $c\equiv\inf\{t\in[a,b]\mid(\{t\}\times[0,1])\cap A\neq\emptyset\}$, and we may choose $r\in[0,1]$ with $(c,r)\in\overline{A}$. If $\epsilon>0$ is sufficiently small and

$$D\equiv([c-\epsilon,c+\epsilon]\cap[a,b])\times([r-\epsilon,r+\epsilon]\cap[0,1]),$$

then $H(D)$ is contained in some evenly covered open set U. If $c=a$, then Υ maps a neighborhood U_0 of $\hat{x}_0$ homeomorphically onto U, and we get a (continuous) lifting $(\Upsilon\restriction_{U_0})^{-1}\circ H\restriction_D\colon D\to U_0$ that is equal to $\hat{x}_0$ along $\{a\}\times([r-\epsilon,r+\epsilon]\cap[0,1])$, and therefore must be equal to $\widehat{H}$ on D. However, the point $(c,r)=(a,r)$ lies in the set $((c-\epsilon,c+\epsilon)\cap[a,b])\times((r-\epsilon,r+\epsilon)\cap[0,1])$, which is open in $[a,b]\times[0,1]$ and is contained in D. Since $(c,r)\in\overline{A}$, this is not possible. Assuming that $c>a$, we may then choose the number ϵ to be in the interval $(0, c-a)$. But then $\widehat{H}$ is continuous on the connected subset $[c-\epsilon, c) \times ([r-\epsilon, r+\epsilon] \cap [0,1])$ of D, and hence the image of this set must be contained in some open set U_1 that Υ maps homeomorphically onto U. Again, applying the local inverse, we get a continuous lifting of $H\restriction_D$ that agrees with $\widehat{H}$ along the set $\{c-\epsilon\} \times ([r-\epsilon, r+\epsilon] \cap [0,1])$ and therefore by construction, must agree with $\widehat{H}$ on all of D. Thus we have again arrived at a contradiction, and we see that $\widehat{H}$ must be continuous. Moreover, the connected set $\widehat{H}(\{b\} \times [0,1])$ lies in the fiber $\Upsilon^{-1}(H(b,0))$, and is therefore a singleton. Thus $\widehat{H}$ is a path homotopy. Uniqueness follows from (a).

Part (c) now follows easily, and part (d) is left to the reader (see Exercise 10.2.2). □

Corollary 10.2.6 *Let $\Upsilon\colon \widehat{X} \to X$ and $\check{\Upsilon}\colon \check{X} \to X$ be connected covering spaces of a connected locally path connected topological space X, let $x_0 \in X$, let $\hat{x}_0 \in \Upsilon^{-1}(x_0)$, and let $\check{x}_0 \in \check{\Upsilon}^{-1}(x_0)$. If $\Upsilon_*\pi_1(\widehat{X}, \hat{x}_0) \subset \check{\Upsilon}_*\pi_1(\check{X}, \check{x}_0)$, then there exists a unique commutative diagram of covering maps*

$$\begin{tikzcd} & \check{X} \arrow[d, "\check{\Upsilon}"] \\ \widehat{X} \arrow[ur, "\widehat{\Upsilon}"] \arrow[r, "\Upsilon"'] & X \end{tikzcd}$$

with $\widehat{\Upsilon}(\hat{x}_0) = \check{x}_0$. Moreover, if $\Upsilon_*\pi_1(\widehat{X}, \hat{x}_0) = \check{\Upsilon}_*\pi_1(\check{X}, \check{x}_0)$, then $\widehat{\Upsilon}$ is a homeomorphism. If X, $\widehat{X}$, and $\check{X}$ are $\mathcal{C}^\infty$ manifolds and Υ and $\check{\Upsilon}$ are $\mathcal{C}^\infty$ covering maps, then $\widehat{\Upsilon}$ is a $\mathcal{C}^\infty$ covering map (and hence $\widehat{\Upsilon}$ is a diffeomorphism if $\Upsilon_*\pi_1(\widehat{X}, \hat{x}_0) = \check{\Upsilon}_*\pi_1(\check{X}, \check{x}_0)$).

Proof The lifting theorem (Theorem 10.2.5) and Lemma 10.2.4 give the existence and uniqueness of the commutative diagram. If the image groups are equal, then we also have a corresponding covering map $\check{X} \to \widehat{X}$. The composition with $\widehat{\Upsilon}$ is then a lifting of Υ to a covering map $\widehat{X} \to \widehat{X}$ that fixes x_0, and therefore by uniqueness, the composition must be the identity. Thus $\widehat{\Upsilon}$ is a homeomorphism in this case. The claims in the $\mathcal{C}^\infty$ case follow from the above and Lemma 10.2.4. □

Corollary 10.2.7 *Let $\Upsilon\colon \widehat{X} \to X$ be a connected covering space of a connected locally path connected topological space X, let U be a connected open subset of X, and let V be a connected component of $\Upsilon^{-1}(U)$. Then the restriction $\Upsilon\restriction_V\colon V \to U$ is a covering space.*

Proof Fix a point $\hat{x}_0 \in V$, and let $x_0 \equiv \Upsilon(\hat{x}_0)$. Then, given a point $x \in U$, there is a path γ in U from x_0 to x. Hence the unique lifting $\hat{\gamma}$ to a path in $\widehat{X}$ with $\hat{\gamma}(0) = \hat{x}_0$ must lie in V, and we have $\Upsilon(\hat{\gamma}(1)) = x$. Thus $\Upsilon(V) = U$, and it follows that $\Upsilon\restriction_V\colon V \to U$ is a covering space. □

Definition 10.2.8 Two covering spaces $\Upsilon\colon \widehat{X} \to X$ and $\check{\Upsilon}\colon \check{X} \to X$ of a topological space X are *equivalent* if there exists a homeomorphism $\widehat{\Upsilon}\colon \widehat{X} \to \check{X}$ that is a lifting of Υ; i.e., $\widehat{\Upsilon}$ is a fiber-preserving homeomorphism.

Remarks 1. By Lemma 10.2.4, if the above are $\mathcal{C}^\infty$ covering manifolds, then the homeomorphism $\widehat{\Upsilon}$ is actually a diffeomorphism, and we say that the coverings are $\mathcal{C}^\infty$ *equivalent*.

2. Equivalence of coverings is an equivalence relation (see Exercise 10.2.3).

3. Corollary 10.2.6 immediately gives the following:

Proposition 10.2.9 *Let $\Upsilon\colon \widehat{X} \to X$ and $\check{\Upsilon}\colon \check{X} \to X$ be connected coverings of a connected locally path connected topological space X, and let $x_0 \in X$ and $\hat{x}_0 \in \Upsilon^{-1}(x_0)$. Then the two coverings are equivalent if and only if there exists a point $\check{x}_0 \in \check{\Upsilon}^{-1}(x_0)$ such that $\Upsilon_*\pi_1(\widehat{X}, \widehat{x_0}) = \check{\Upsilon}_*\pi_1(\check{X}, \check{x}_0)$.*

Any covering space of a $\mathcal{C}^\infty$ manifold has a natural induced $\mathcal{C}^\infty$ structure.

Proposition 10.2.10 *Let $\Upsilon\colon \widehat{M} \to M$ be a covering space.*

(a) *If M is a Hausdorff space, then $\widehat{M}$ is a Hausdorff space.*
(b) *If M is a topological manifold, then $\widehat{M}$ is a topological manifold of the same dimension.*
(c) *If M is a $\mathcal{C}^\infty$ manifold, then there is a unique $\mathcal{C}^\infty$ structure on $\widehat{M}$ with respect to which Υ is a local diffeomorphism (i.e., with respect to which $\Upsilon\colon \widehat{M} \to M$ is a $\mathcal{C}^\infty$ covering manifold).*

Remark Given a covering space $\Upsilon\colon \widehat{M} \to M$ of a $\mathcal{C}^\infty$ manifold M, we assume that $\widehat{M}$ is a $\mathcal{C}^\infty$ manifold with the associated induced $\mathcal{C}^\infty$ structure unless otherwise indicated.

Proof of Proposition 10.2.10 The proofs of parts (a) and (b) are left to the reader (see Exercise 10.2.4). For the proof of (c), observe that if M is a $\mathcal{C}^\infty$ manifold of dimension n, then $\widehat{M}$ is Hausdorff (by (a)), and we may form a $\mathcal{C}^\infty$ atlas $\mathcal{A} = \{(U_i, \Phi_i, U_i')\}_{i\in I}$ on M such that U_i is connected and evenly covered for each $i \in I$. For each $i \in I$, $\widehat{U}_i \equiv \Upsilon^{-1}(U_i)$ is the union of a collection of disjoint connected open sets $\{U_i^{(\lambda)}\}_{\lambda\in\Lambda_i}$ in $\widehat{M}$, each of which is mapped homeomorphically onto U_i. Setting $\Phi_i^{(\lambda)} \equiv \Phi_i \circ \Upsilon\restriction_{U_i^{(\lambda)}}$, we get an n-dimensional atlas $\widehat{\mathcal{A}} \equiv \{(U_i^{(\lambda)}, \Phi_i^{(\lambda)}, U_i')\}_{\lambda\in\Lambda_i, i\in I}$ on $\widehat{M}$. Suppose $i \in I$, $j \in I$, $\lambda \in \Lambda_i$, $\mu \in \Lambda_j$, and $U_i^{(\lambda)} \cap U_j^{(\mu)} \neq \emptyset$. Then the coordinate transformation

$$\Phi_i^{(\lambda)} \circ [\Phi_j^{(\mu)}]^{-1} = \Phi_i \circ \Phi_j^{-1}\restriction_{\Phi_j^{(\mu)}(U_i^{(\lambda)}\cap U_j^{(\mu)})}\colon \Phi_j^{(\mu)}(U_i^{(\lambda)} \cap U_j^{(\mu)})$$
$$\longrightarrow \Phi_i^{(\lambda)}(U_i^{(\lambda)} \cap U_j^{(\mu)}) \subset \Phi_i(U_i \cap U_j)$$

is of class $\mathcal{C}^\infty$, and hence the atlas determines a $\mathcal{C}^\infty$ structure on $\widehat{M}$ with respect to which Υ is a local diffeomorphism.

Moreover, given an arbitrary $\mathcal{C}^\infty$ structure on $\widehat{M}$ with respect to which Υ is a local diffeomorphism, and given indices $i \in I$ and $\lambda \in \Lambda_i$, the mapping $\Phi_i^{(\lambda)} = \Phi_i \circ \Upsilon\restriction_{U_i^{(\lambda)}}$ is a composition of a $\mathcal{C}^\infty$ diffeomorphism on $U_i \subset M$ and a $\mathcal{C}^\infty$ diffeomorphism on $U_i^{(\lambda)}$, and is therefore a $\mathcal{C}^\infty$ diffeomorphism. Thus the elements of the atlas $\widehat{\mathcal{A}}$ are $\mathcal{C}^\infty$ compatible with the local $\mathcal{C}^\infty$ charts on $\widehat{M}$, and uniqueness follows. □

Recall that a continuous mapping of Hausdorff spaces is *proper* if the inverse image of every compact subset of the target is compact (see Definition 9.3.9). A sequence in a manifold converges to a point p if for every neighborhood U of p, all but finitely many terms of the sequence lie in U. Recall also that a compact subset of a manifold has the Bolzano–Weierstrass property (see Theorem 9.3.2). These properties allow one to prove the following useful fact concerning *finite* covering spaces (i.e., covering spaces with finite fibers) of manifolds:

Lemma 10.2.11 *For any local homeomorphism $\Phi\colon M \to N$ of connected topological manifolds M and N, we have the following*:

(a) *The mapping Φ is proper if and only if Φ is a finite covering map.*
(b) *If K is a nonempty connected compact subset of $\Phi(M)$ with connected compact inverse image $\widehat{K} \equiv \Phi^{-1}(K)$, Ω is a sufficiently small connected neighborhood of K in N, and $\widehat{\Omega}$ is the connected component of $\Phi^{-1}(\Omega)$ containing $\widehat{K}$, then the restriction $\Upsilon \equiv \Phi\restriction_{\widehat{\Omega}}\colon \widehat{\Omega} \to \Omega$ is a finite covering map.*

Proof For the proof of (a), let us first assume that Φ is a proper mapping. The map Φ is then surjective. For if $y \in \overline{\Phi(M)}$, then we may choose a sequence of points $\{x_\nu\}$ in M such that $y_\nu = \Phi(x_\nu) \to y$. Since Φ is proper, $\{x_\nu\}$ must lie in a compact subset of M, and hence some subsequence must converge to some point $x \in M$. Continuity then gives $\Phi(x) = y$. Thus $\Phi(M)$ is a nonempty subset of N that is closed (by the above) and open (since Φ is a local homeomorphism); and therefore, since N is connected, we have $\Phi(M) = N$.

Now given a point $y_0 \in N$, we may choose a finite covering of the compact fiber $F \equiv \Phi^{-1}(y_0)$ by relatively compact connected open subsets $V_1, \dots, V_n$ of M, each of which is mapped homeomorphically onto a connected neighborhood of y_0 by Φ. Each of these sets contains exactly one element of F, and since M is Hausdorff, we may choose the sets to be disjoint. We may also choose a connected neighborhood U of y_0 in $\Phi(V_1) \cap \cdots \cap \Phi(V_n)$ such that $\Phi^{-1}(U) \subset V \equiv V_1 \cup \cdots \cup V_n$. For if this were not the case, then as in the above proof of surjectivity, there would exist a sequence of points $\{x_\nu\}$ in $M \setminus V$ that converges to a point $x \in M \setminus V$ with $\Phi(x) = y_0$, which is impossible, since $F \subset V$. Thus the desired neighborhood U must exist. Hence $\Phi^{-1}(U) = U_1 \cup \cdots \cup U_n$, where for each $i = 1, \dots, n$, $U_i \equiv \Phi^{-1}(U) \cap V_i = (\Phi\restriction_{V_i})^{-1}(U)$ is a connected neighborhood that is mapped homeomorphically onto U by Φ. Since the sets $U_1, \dots, U_n$ are disjoint, it follows that U is evenly covered, and hence that $\Phi\colon M \to N$ is a finite covering map.

For the proof of the converse, suppose $\Phi\colon M \to N$ is a finite covering map. Then, for each point $p \in N$, the inverse image $\Phi^{-1}(U)$ of some (in fact, any) evenly covered neighborhood U of p has only finitely many connected components. Fixing a relatively compact neighborhood V of p in U, we see that $\Phi^{-1}(\overline{V})$ is compact. Since any compact set $D \subset N$ admits a finite covering by such sets V, $\Phi^{-1}(D)$ must be compact.

For the proof of (b), it suffices to show that for any sufficiently small connected neighborhood Ω of K and for $\widehat{\Omega}$ the connected component of $\Phi^{-1}(\Omega)$ containing the connected compact set $\widehat{K} \equiv \Phi^{-1}(K)$, the restriction $\Upsilon \equiv \Phi\restriction_{\widehat{\Omega}}\colon \widehat{\Omega} \to \Omega$ is proper; for we may then apply part (a). Clearly, for this, we may assume that $K \neq N$. Since $\widehat{K}$ is connected and compact, we may choose a relatively compact connected neighborhood V of $\widehat{K}$ in M. Since $\Phi(M)$ is open, we may require that $\Omega \Subset \Phi(M)$. Moreover, if Ω is sufficiently small, then Ω will not meet the compact set $\Phi(\partial V)$, and hence the connected component $\widehat{\Omega}$ of $\Phi^{-1}(\Omega)$ containing $\widehat{K}$ will be a connected open subset of M that meets V, but not ∂V. Thus we will have $\widehat{\Omega} \subset V$. Moreover, if C is a compact subset of Ω, then $\Phi^{-1}(C)$ is a closed subset of M that does not meet $\partial\widehat{\Omega}$ (since any point in $\Phi^{-1}(C) \cap \mathrm{cl}(\widehat{\Omega})$ has a connected neighborhood in $\Phi^{-1}(\Omega)$ that meets, and hence is contained in, $\widehat{\Omega}$). Thus $\Phi^{-1}(C) \cap \widehat{\Omega}$ is compact, the restriction $\Upsilon \equiv \Phi\restriction_{\widehat{\Omega}}\colon \widehat{\Omega} \to \Omega$ is a proper mapping, and the claim follows. □

Exercises for Sect. 10.2

10.2.1 Prove parts (b)–(d) of Lemma 10.2.4.
10.2.2 Prove part (d) of Theorem 10.2.5.
10.2.3 Prove that equivalence of coverings is an equivalence relation.
10.2.4 Prove parts (a) and (b) of Proposition 10.2.10.

10.3 The Universal Covering

Definition 10.3.1 A simply connected covering space $\Upsilon\colon \widetilde{X} \to X$ of a connected topological space X is called a *universal covering space* of X.

Corollary 10.2.6 immediately gives the following:

Lemma 10.3.2 (Uniqueness of universal coverings) *A connected locally path connected topological space X has at most one universal covering space up to equivalence of coverings. In fact, if $x_0 \in X$ and for $j = 1, 2$, $\Upsilon_j\colon X_j \to X$ is a universal covering space of X and $x_j \in \Upsilon_j^{-1}(x_0)$, then there is unique homeomorphism $\Phi\colon X_1 \to X_2$ such that $\Upsilon_2 \circ \Phi = \Upsilon_1$ and $\Phi(x_1) = x_2$.*

The main goal of this section is the following:

Theorem 10.3.3 (Existence of universal coverings) *Every connected locally simply connected topological space X has a universal covering space $\Upsilon\colon \widetilde{X} \to X$.*

Proof The idea of the construction is to separate the terminal points of paths from a fixed initial point whenever the paths are not homotopic. Fix a point $x_0 \in X$, and let $\mathcal{P}$ be the set of paths $\alpha\colon [0,1] \to X$ with initial point $\alpha(0) = x_0$. Then path homotopy equivalence determines an equivalence relation $\sim$ on $\mathcal{P}$, and we may set $\widetilde{X} \equiv \mathcal{P}/\sim$. Denoting the path homotopy equivalence class of each path α by $[\alpha]$, we may define a surjective map $\Upsilon\colon \widetilde{X} \to X$ by $[\alpha] \mapsto \alpha(1)$.

We must define a topology on $\widetilde{X}$ with respect to which this is a universal covering. Let $\mathcal{A}$ be the collection of all pairs $([\alpha], U)$, where $[\alpha] \in \widetilde{X}$ and U is a simply connected neighborhood of $\Upsilon([\alpha]) = \alpha(1)$ in X. For each element $\xi = ([\alpha], U) \in \mathcal{A}$, we get a well-defined map $\Phi_\xi\colon U \to \widetilde{X}$ with $\Phi_\xi(\alpha(1)) = [\alpha]$ by setting, for each point $x \in U$,

$$\Phi_\xi(x) \equiv [\alpha * \beta],$$

where $\beta\colon [0,1] \to U$ is an arbitrary path in U from $\alpha(1)$ to x. For if $\alpha \sim \alpha'$ and β' is a path in U from $\alpha(1) = \alpha'(1)$ to $x \in U$, then since U is simply connected, β and β', and therefore $\alpha * \beta$ and $\alpha' * \beta'$, are path homotopic. Clearly, $\Upsilon \circ \Phi_\xi = \mathrm{Id}_U$, and hence Φ_ξ maps U bijectively onto $\Phi_\xi(U)$ with inverse mapping $\Upsilon\restriction_{\Phi_\xi(U)}$. Observe also that if $\xi = ([\alpha], U)$, $\zeta = ([\alpha], V) \in \mathcal{A}$ with $U \subset V$, then $\Phi_\xi = \Phi_\zeta\restriction_U$.

One may verify that the collection $\mathcal{T}$ of subsets $V \subset \widetilde{X}$ such that $\Phi_\xi^{-1}(V)$ is an open set in X for every element $\xi \in \mathcal{A}$ is a topology on $\widetilde{X}$ (see Exercise 10.3.1). Moreover, $\Phi_\xi(U) \in \mathcal{T}$ for each $\xi = ([\alpha], U) \in \mathcal{A}$. For given an element $\zeta = ([\beta], V) \in \mathcal{A}$ and a point

$$x_1 \in \Phi_\zeta^{-1}(\Phi_\xi(U)) = \Upsilon(\Phi_\zeta(V) \cap \Phi_\xi(U)) \subset U \cap V,$$

we may choose a simply connected neighborhood W of x contained in $U \cap V$, a path $\gamma\colon [0,1] \to U$ from $\alpha(1)$ to x_1, and a path $\delta\colon [0,1] \to V$ from $\beta(1)$ to x_1. We then get

$$[\alpha * \gamma] = \Phi_\xi(x_1) = [\Upsilon\restriction_{\Phi_\zeta(V) \cap \Phi_\xi(U)}]^{-1}(x_1) = \Phi_\zeta(x_1) = [\beta * \delta].$$

Hence if $x \in W$ and $\eta\colon [0,1] \to W$ is a path from x_1 to x, then

$$\Phi_\zeta(x) = [\beta * \delta * \eta] = [\alpha * \gamma * \eta] = \Phi_\xi(x) \in \Phi_\xi(U).$$

Thus $W \subset \Phi_\zeta^{-1}(\Phi_\xi(U))$, and hence $\Phi_\zeta^{-1}(\Phi_\xi(U))$ is open.

Clearly, with respect to the topology $\mathcal{T}$, Φ_ξ is a continuous bijection of U onto the open set $\Phi_\xi(U)$ for each $\xi = ([\alpha], U) \in \mathcal{A}$. Moreover, if $V \subset \widetilde{X}$ is open, then

$$\Upsilon(V) = \bigcup_{\xi = ([\alpha], U) \in \mathcal{A}} \Upsilon(V \cap \Phi_\xi(U)) = \bigcup_{\xi \in \mathcal{A}} \Phi_\xi^{-1}(V)$$

is open; hence Υ is an open mapping. Thus $\Upsilon\colon \widehat{X} \to X$ is a surjective local homeomorphism (with local continuous inverses given by the mappings $\{\Phi_\xi\}_{\xi \in \mathcal{A}}$). In fact, given a point $x_1 \in X$ and an element $([\alpha], U) \in \mathcal{A}$ with $x_1 = \alpha(1)$, $\Upsilon^{-1}(U)$ is equal to the union of disjoint connected open sets

$$\{\Phi_{([\beta*\alpha],U)}(U)\}_{[\beta]_{x_0} \in \pi_1(X,x_0)},$$

each of which is mapped homeomorphically onto U by Υ. For if $[\gamma] \in \Upsilon^{-1}(U)$ and $\delta\colon [0,1] \to U$ is a path from x_1 to $\gamma(1)$, then $\beta \equiv \gamma * \delta^- * \alpha^-$ is a loop based at x_0 and $[\gamma] = [\beta * \alpha * \delta] = \Phi_{([\beta*\alpha],U)}(\gamma(1))$. If, on the other hand, β_1 and β_2 are loops based at x_0 and there exist points $y_1, y_2 \in U$ with $\Phi_{([\beta_1*\alpha],U)}(y_1) = \Phi_{([\beta_2*\alpha],U)}(y_2)$, then, applying Υ, we see that $y_1 = y_2$. Fixing a path $\delta\colon [0,1] \to U$ from x_1 to $y_1 = y_2$, we get

$$[\beta_1 * \alpha * \delta] = \Phi_{([\beta_1*\alpha],U)}(y_1) = \Phi_{([\beta_2*\alpha],U)}(y_2) = [\beta_2 * \alpha * \delta],$$

and hence

$$[\beta_1]_{x_0} = [\beta_1 * \alpha * \delta * \delta^- * \alpha^-] = [\beta_2 * \alpha * \delta * \delta^- * \alpha^-] = [\beta_2]_{x_0}.$$

Thus the sets are disjoint, and $\Upsilon\colon \widetilde{X} \to X$ is a covering space.

It remains to show that $\widetilde{X}$ is simply connected. Let e_{x_0} be the constant loop at x_0. Given a path $\alpha \in \mathcal{P}$, we may choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ so that for each $i = 1, \dots, n$, $\alpha([t_{i-1}, t_i])$ is contained in some simply connected open set U_i. For each $i = 1, \dots, n$, we may also choose a reparametrization $\alpha_i\colon [0,1] \to U_i$ of $\alpha\restriction_{[t_{i-1},t_i]}$. In particular, by Lemma 10.1.8, $\alpha \sim \beta \equiv \alpha_1 * \cdots * \alpha_n$. Let $\beta_i = \alpha_1 * \cdots * \alpha_i$ for $i = 1, \dots, n$, and let $x_i = \alpha(t_i)$ for $i = 0, \dots, n$. For $i = 2, \dots, n$, we have

$$\Phi_{([\beta_{i-1}],U_{i-1})}(x_{i-1}) = [\beta_{i-1}] = [\beta_{i-1} * \alpha_i * \alpha_i^-] = \Phi_{([\beta_i],U_i)}(x_{i-1}).$$

Thus we get a lifting of β to a path

$$\tilde{\beta} \equiv \Phi_{([\beta_1],U_1)}(\alpha_1) * \Phi_{([\beta_2],U_2)}(\alpha_2) * \cdots * \Phi_{([\beta_n],U_n)}(\alpha_n)$$

in $\widetilde{X}$ from

$$\tilde{\beta}(0) = \Phi_{([\beta_1],U_1)}(x_0) = [\alpha_1 * \alpha_1^-] = [e_{x_0}]$$

to

$$\tilde{\beta}(1) = \Phi_{([\beta_n],U_n)}(x_n) = [\beta_n] = [\beta] = [\alpha].$$

In particular, $\widetilde{X}$ is path connected. Moreover, if γ is a loop in $\widetilde{X}$ based at $[e_{x_0}]$ and we set $\alpha = \Upsilon(\gamma)$ and form β as above, then by the path homotopy lifting property (Theorem 10.2.5), we have

$$[\alpha]_{x_0} = [\alpha] = [\beta] = \tilde{\beta}(1) = \gamma(1) = [e_{x_0}] = [e_{x_0}]_{x_0}.$$

Thus α is path homotopic to the constant loop e_{x_0}, and therefore, again by the path homotopy lifting property, the lifting γ is path homotopic in $\widetilde{X}$ to the constant loop at $[e_{x_0}]$. $\square$

Exercises for Sect. 10.3

10.3.1 Verify that the collection of sets $\mathcal{T}$ constructed in the proof of Theorem 10.3.3 is a topology.

10.4 Deck Transformations

Definition 10.4.1 A *deck transformation* (or *covering transformation*) of a covering space $\Upsilon\colon \widehat{X} \to X$ is a homeomorphism $\Phi\colon \widehat{X} \to \widehat{X}$ such that $\Upsilon \circ \Phi = \Upsilon$ (i.e., Φ preserves the fibers). The group determined by the set of deck transformations for the covering with multiplication given by composition is denoted by $\mathrm{Deck}(\Upsilon)$.

Theorem 10.4.2 *Let $\Upsilon\colon \widetilde{X} \to X$ be a universal covering of a connected locally path connected topological space X, let $x_0 \in X$, let $F \equiv \Upsilon^{-1}(x_0)$, and let $\tilde{x}_0 \in F$. We then have the following*:

(a) *The map* $\mathrm{Deck}(\Upsilon) \to F$ *given by* $\Phi \mapsto \Phi(\tilde{x}_0)$ *is a bijection.*
(b) *The map* $\chi\colon \mathrm{Deck}(\Upsilon) \to \pi_1(X, x_0)$ *given by* $\Phi \mapsto [\Upsilon(\tilde{\alpha})]_{x_0}$, *where $\tilde{\alpha}$ is an arbitrary path in $\widetilde{X}$ from $\tilde{x}_0$ to $\Phi(\tilde{x}_0)$, is a well-defined group isomorphism.*
(c) *Let* $\sigma \equiv \chi^{-1}\colon \pi_1(X, x_0) \to \mathrm{Deck}(\Upsilon)$ *be the inverse isomorphism. If $x \in \widetilde{X}$, $\tilde{\gamma}$ is a path from $\tilde{x}_0$ to x, $[\alpha]_{x_0} \in \pi_1(X, x_0)$, $\tilde{\alpha}$ is the unique lifting of α to a path in $\tilde{X}$ with $\tilde{\alpha}(0) = \tilde{x}_0$, and $\tilde{\gamma}_1$ is the unique lifting of $\gamma \equiv \Upsilon(\tilde{\gamma})$ to a path with $\tilde{\gamma}_1(0) = \tilde{\alpha}(1)$, then $\sigma([\alpha]_{x_0})(x) = \tilde{\gamma}_1(1)$ (that is, $\sigma([\alpha]_{x_0})(x)$ is equal to the terminal point of the lifting of $\alpha * \gamma$ with initial point $\tilde{x}_0$).*
(d) *The composition* $\pi_1(X, x_0) \to \mathrm{Deck}(\Upsilon) \to F$ *is the bijection given by*

$$[\alpha]_{x_0} \mapsto \sigma([\alpha]_{x_0})(\tilde{x}_0).$$

(e) *Each element* $\Phi \in \mathrm{Deck}(\Upsilon) \setminus \{\mathrm{Id}\}$ *has no fixed points. In fact, if U is an evenly covered domain in X and U_0 is a component of $\widetilde{U} \equiv \Upsilon^{-1}(U)$, then the set $U_1 \equiv \Phi(U_0)$ is a component of $\widetilde{U}$ that is distinct (hence disjoint) from U_0.*

Proof By uniqueness of universal coverings up to equivalence (Lemma 10.3.2), for each point $y \in F$, there is a unique homeomorphism $\Phi\colon \widetilde{X} \to \widetilde{X}$ such that $\Upsilon \circ \Phi = \Upsilon$ and $\Phi(\tilde{x}_0) = y$; that is, the map $\mathrm{Deck}(\Upsilon) \to F$ is bijective. Thus (a) is proved.

Suppose that for $j = 1, 2$, $\Phi_j \in \mathrm{Deck}(\Upsilon)$, $y_j = \Phi_j(\tilde{x}_0)$, and $\tilde{\alpha}_j\colon [0, 1] \to \widetilde{X}$ is a path with initial point $\tilde{\alpha}_j(0) = \tilde{x}_0$ and terminal point $\tilde{\alpha}_j(1) = y_j$. In particular, $\alpha_j \equiv \Upsilon(\tilde{\alpha}_j)$ is a loop based at x_0 for $j = 1, 2$. If $y_1 = y_2$ (i.e., $\Phi_1 = \Phi_2$), then since $\widetilde{X}$ is simply connected, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, and therefore α_1 and α_2, are path homotopic. Hence χ is well defined. In general, $\Phi_1(\tilde{\alpha}_2)$ is a path in $\widetilde{X}$ from y_1 to $\Phi_1 \circ \Phi_2(\tilde{x}_0)$, and hence

$$\chi(\Phi_1 \circ \Phi_2) = [\Upsilon(\tilde{\alpha}_1 * \Phi_1(\tilde{\alpha}_2))]_{x_0} = [\alpha_1 * \alpha_2]_{x_0} = [\alpha_1]_{x_0} \cdot [\alpha_2]_{x_0} = \chi(\Phi_1) \cdot \chi(\Phi_2).$$

Thus χ is a homomorphism. If α_1 is path homotopic to the trivial loop, then the lifting $\tilde{\alpha}_1$ is also a loop (by the path homotopy lifting property in Theorem 10.2.5) and hence $\Phi_1(\tilde{x}_0) = y_1 = \tilde{x}_0$. Thus $\Phi_1 = \mathrm{Id}$ and it follows that χ is injective. Given a loop α based at x_0, there are a unique lifting to a path $\tilde{\alpha}$ in $\widetilde{X}$ with $\tilde{\alpha}(0) = \tilde{x}_0$ and a unique deck transformation Φ with $\Phi(\tilde{x}_0) = \tilde{\alpha}(1)$. Clearly, $\chi(\Phi) = [\alpha]_{x_0}$, so χ is surjective and (b) is proved.

The proofs of (c)–(e) are left to the reader (see Exercise 10.4.1). □

Definition 10.4.3 Let Γ be a subgroup of the group of self-homeomorphisms of a locally compact Hausdorff space X.

(a) We denote by $\sim_\Gamma$ the equivalence relation on X given by

$$x \sim_\Gamma y \quad \Longleftrightarrow \quad \gamma(x) = y \text{ for some } \gamma \in \Gamma;$$

by $\Gamma\backslash X$ the quotient space $X/{\sim_\Gamma}$, which is given the quotient topology; and by $\Upsilon_\Gamma : X \to \Gamma\backslash X$ the corresponding quotient map.

(b) The group Γ acts *properly discontinuously* on X if for every compact set $K \subset X$, $\gamma(K) \cap K = \emptyset$ for all but finitely many elements γ of Γ.

(c) The group Γ acts *freely* if for every element $\gamma \in \Gamma$ with $\gamma \neq \mathrm{Id}$ and every point $x \in X$, we have $\gamma(x) \neq x$.

Lemma 10.4.4 *Let Γ be a subgroup of the group of self-homeomorphisms of a locally compact Hausdorff space X. Then Γ acts properly discontinuously on X if and only if for each pair of points $x, y \in X$, there exist neighborhoods U of x and V of y such that $\gamma(U) \cap V = \emptyset$ for all but finitely many elements γ of Γ.*

Proof If Γ acts properly discontinuously and $x, y \in X$, then fixing relatively compact neighborhoods U of x and V of y, we see that

$$\gamma(U) \cap V \subset \gamma(\overline{U} \cup \overline{V}) \cap (\overline{U} \cup \overline{V}) = \emptyset$$

for all but finitely many elements γ of Γ.

For the proof of the converse, suppose that $K \subset X$ is compact and that for each pair of points $x, y \in K$, there exist neighborhoods U_{xy} of x and V_{xy} of y such that

$$G_{xy} \equiv \{\gamma \in \Gamma \mid \gamma(U_{xy}) \cap V_{xy} \neq \emptyset\}$$

is finite. For each point $x \in K$, we may choose a finite set $Y_x \subset K$ such that $K \subset \bigcup_{y \in Y_x} V_{xy}$, and we may set W_x equal to the neighborhood $\bigcap_{y \in Y_x} U_{xy}$ of x. Thus there exist points $x_1, \dots, x_n \in K$ such that $K \subset \bigcup_{j=1}^n W_{x_j}$. If $\gamma \in \Gamma$ and $\gamma(x) = y$ for some pair of points $x, y \in K$, then $x \in W_{x_j} \subset U_{x_j z}$, $y \in V_{x_j z}$, and $\gamma \in G_{x_j z}$ for some index $j \in \{1, \dots, n\}$ and some point $z \in Y_{x_j}$. It follows that

$$\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \emptyset\} \subset \bigcup_{j=1}^n \bigcup_{z \in Y_{x_j}} G_{x_j z}$$

is a finite set. □

Theorem 10.4.5 *Let $\Upsilon\colon \widetilde{X} \to X$ be the universal covering space of a connected locally simply connected locally compact Hausdorff space X. Then $\Gamma \equiv \mathrm{Deck}(\Upsilon)$ acts properly discontinuously and freely on $\widetilde{X}$, the quotient map $\Upsilon_\Gamma\colon \widetilde{X} \to \Gamma\backslash\widetilde{X}$ is a covering map, and we have a commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\Upsilon"'] \arrow[dr, "\Upsilon_\Gamma"] & \\ X & \Gamma\backslash\widetilde{X} \arrow[l, "\cong"'] \end{tikzcd}$$

in which the mapping $\Gamma\backslash\widetilde{X} \to X$ is a homeomorphism. Moreover, if X is a $\mathcal{C}^\infty$ manifold, then Γ is a group of self-diffeomorphisms of $\widetilde{X}$ (with respect to the induced $\mathcal{C}^\infty$ structure) and $\Gamma\backslash\widetilde{X}$ has a unique $\mathcal{C}^\infty$ structure with respect to which Υ_Γ is a $\mathcal{C}^\infty$ covering map and the map $\Gamma\backslash\widetilde{X} \to X$ is a diffeomorphism.

In other words, $\Upsilon_\Gamma\colon \widetilde{X} \to \Gamma\backslash\widetilde{X}$ is a covering that we may identify with the original covering $\Upsilon\colon \widetilde{X} \to X$. It follows that if $\widehat{\Upsilon}\colon \widetilde{X} \to \widehat{X}$ is the universal covering space of a connected locally simply connected locally compact Hausdorff space (or a connected $\mathcal{C}^\infty$ manifold) $\widehat{X}$ and $\mathrm{Deck}(\widehat{\Upsilon}) = \mathrm{Deck}(\Upsilon)$, then $\widehat{X} \cong \Gamma\backslash\widetilde{X} \cong X$ and we may identify the coverings $\widehat{\Upsilon}\colon \widetilde{X} \to \widehat{X}$ and $\Upsilon\colon \widetilde{X} \to X$.

Proof of Theorem 10.4.5 Part (e) of Theorem 10.4.2 and Lemma 10.4.4 together imply that Γ acts properly discontinuously and freely. Part (a) of Theorem 10.4.2 implies that Υ descends to a continuous bijection $\Gamma\backslash\widetilde{X} \to X$. The inverse mapping $X \to \Gamma\backslash\widetilde{X}$ is also continuous because Υ is a surjective open mapping. Lemma 10.2.4 implies that Υ_Γ is a covering map. Finally, if X is a $\mathcal{C}^\infty$ manifold and $\Upsilon\colon \widetilde{X} \to X$ is the $\mathcal{C}^\infty$ universal covering, then Lemma 10.2.4 implies that every deck transformation is a diffeomorphism. Moreover, we get a unique $\mathcal{C}^\infty$ structure on $\Gamma\backslash\widetilde{X}$ with respect to which the map $\Gamma\backslash\widetilde{X} \to X$ is a diffeomorphism (for example, by Proposition 10.2.10) and Lemma 10.2.4 then implies that Υ_Γ is a $\mathcal{C}^\infty$ covering map. □

We also have the following partial converse:

Theorem 10.4.6 *Let $\widetilde{X}$ be a simply connected locally simply connected locally compact Hausdorff space, let Γ be a subgroup of the group of self-homeomorphisms of $\widetilde{X}$ that acts properly discontinuously and freely on $\widetilde{X}$, and let $\Upsilon = \Upsilon_\Gamma\colon \widetilde{X} \to X = \Gamma\backslash\widetilde{X}$ be the corresponding quotient space mapping. We then have the following:*

(a) *The quotient space X is connected, locally simply connected, and locally compact Hausdorff; $\Upsilon\colon \widetilde{X} \to X$ is the universal covering space; and $\Gamma = \mathrm{Deck}(\Upsilon)$.*
(b) *If $\widehat{\Gamma}$ is a subgroup of Γ, then we have a commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\Upsilon"'] \arrow[dr, "\Upsilon_{\widehat{\Gamma}}"] & \\ X & \widehat{X} \equiv \widehat{\Gamma}\backslash\widetilde{X} \arrow[l, "\widehat{\Upsilon}"'] \end{tikzcd}$$

of covering maps. Moreover, if $\tilde{x}_0 \in \widetilde{X}$, $x_0 = \Upsilon(\tilde{x}_0)$, *and* $\hat{x}_0 = \Upsilon_{\widehat{\Gamma}}(\tilde{x}_0)$, *and*

$$\chi\colon \Gamma = \mathrm{Deck}(\Upsilon) \to \pi_1(X, x_0) \quad \textit{and} \quad \widehat{\chi}\colon \widehat{\Gamma} = \mathrm{Deck}(\Upsilon_{\widehat{\Gamma}}) \to \pi_1(\widehat{X}, \hat{x}_0)$$

are the corresponding group isomorphisms, then

$$\chi \circ \widehat{\chi}^{-1} = \widehat{\Upsilon}_*\colon \pi_1(\widehat{X}, \hat{x}_0) \to \pi_1(X, x_0).$$

In other words, the action of $[\hat{\gamma}] \in \pi_1(\widehat{X}, \hat{x}_0)$ *on* $\widetilde{X}$ *is the same as that of* $\widehat{\Upsilon}_*[\hat{\gamma}] = [\widehat{\Upsilon}(\hat{\gamma})] \in \pi_1(X, x_0)$.

(c) *If* $\widetilde{X}$ *is a topological manifold, then* X *is a topological manifold of the same dimension.*

(d) *If* $\widetilde{X}$ *is a* $\mathcal{C}^\infty$ *manifold and each element of* Γ *is a diffeomorphism, then* X *has a unique* $\mathcal{C}^\infty$ *structure with respect to which* $\Upsilon\colon \widetilde{X} \to X$ *is a* $\mathcal{C}^\infty$ (*universal*) *covering space. Moreover, for* $\widehat{\Gamma} \subset \Gamma$ *as in part* (b), $\Upsilon_{\widehat{\Gamma}}$ *and* $\widehat{\Upsilon}$ *are* $\mathcal{C}^\infty$ *covering maps* (*with respect to a unique* $\mathcal{C}^\infty$ *structure on* $\widehat{X}$).

Proof Υ is an open mapping because given an open set $W \subset \widetilde{X}$, $\Upsilon^{-1}(\Upsilon(W)) = \Gamma \cdot W$ is the union of the open sets $\{\gamma(W)\}_{\gamma \in \Gamma}$. Since $\widetilde{X}$ is locally compact Hausdorff and Γ acts properly discontinously, given a point $x_0 \in X$, we may fix a point $y_0 \in \Upsilon^{-1}(x_0)$, a relatively compact neighborhood V of y_0 in $\widetilde{X}$, and a finite set $\Gamma_1 \subset \Gamma$ such that $\gamma(V) \cap V = \emptyset$ for each element $\gamma \in \Gamma$ with $\gamma \notin \Gamma_1$. Since $\widetilde{X}$ is Hausdorff, for each $\gamma \in \Gamma_1$ with $\gamma \neq \mathrm{Id}$, we may choose disjoint neighborhoods V_γ and W_γ of y_0 and $\gamma(y_0)$, respectively. For $\gamma = \mathrm{Id}$, we set $V_\gamma = W_\gamma = V$. Finally, we may choose a connected neighborhood U_0 of y_0 with

$$U_0 \subset V \cap \bigcap_{\gamma \in \Gamma_1} V_\gamma \cap \gamma^{-1}(W_\gamma).$$

Setting $U \equiv \Upsilon(U_0)$, we get

$$\Upsilon^{-1}(U) = \Gamma \cdot U_0 = \bigcup_{\gamma \in \Gamma} \gamma(U_0);$$

and hence, in order to show that U is evenly covered by Υ, it suffices to show that the sets $\{\gamma(U_0)\}_{\gamma \in \Gamma}$ are disjoint and that Υ maps each of these sets homeomorphically onto U. For this, we first observe that if $\alpha \in \Gamma$ with $U_0 \cap \alpha(U_0) \neq \emptyset$, then $\alpha = \mathrm{Id}$. For we have $V \cap \alpha(V) \neq \emptyset$, and hence $\alpha \in \Gamma_1$ and

$$U_0 \cap \alpha(U_0) \subset V_\alpha \cap \alpha^{-1}(W_\alpha) \cap \alpha(V_\alpha) \cap W_\alpha \subset V_\alpha \cap W_\alpha.$$

Thus $\alpha = \mathrm{Id}$, and it follows that the open sets $\{\gamma(U_0)\}_{\gamma \in \Gamma}$ are disjoint and Υ maps each of these sets bijectively and therefore homeomorphically (since Υ is continuous and open) onto U.

To see that X is Hausdorff, suppose x_1 and x_2 are distinct points in X. Then, for each $j = 1, 2$, we may choose an evenly covered connected neighborhood U_j of x_j

and a point $y_j \in \Upsilon^{-1}(x_j)$ such that the component V_j of $\Upsilon^{-1}(U_j)$ containing y_j is relatively compact in $\widetilde{X}$. Thus the set $K \equiv \overline{V}_1 \cup \overline{V}_2$ is compact, and since Γ acts properly discontinuously, we get a finite set $\Gamma_1 \subset \Gamma$ such that $\gamma(K) \cap K = \emptyset$ for each element $\gamma \in \Gamma \setminus \Gamma_1$. Since $\widetilde{X}$ is Hausdorff and $y_2 \notin \Gamma \cdot y_1$, for each element $\gamma \in \Gamma_1$ we may choose neighborhoods P_γ and Q_γ of $\gamma(y_1)$ and y_2, respectively, that are disjoint. Thus we get neighborhoods

$$R_1 \equiv V_1 \cap \bigcap_{\gamma \in \Gamma_1} \gamma^{-1}(P_\gamma) \quad \text{and} \quad R_2 \equiv V_2 \cap \bigcap_{\gamma \in \Gamma_1} Q_\gamma$$

of y_1 and y_2, respectively, and neighborhoods $W_1 \equiv \Upsilon(R_1) \subset U_1$ and $W_2 \equiv \Upsilon(R_2) \subset U_2$ of x_1 and x_2, respectively. The neighborhoods W_1 and W_2 are disjoint because

$$\begin{aligned}\Upsilon^{-1}(W_1) \cap \Upsilon^{-1}(W_2) \cap V_2 &= (\Gamma \cdot R_1) \cap R_2 = (\Gamma_1 \cdot R_1) \cap R_2 \\ &\subset \left(\bigcup_{\gamma \in \Gamma_1} P_\gamma\right) \cap \left(\bigcap_{\gamma \in \Gamma_1} Q_\gamma\right) = \emptyset.\end{aligned}$$

Thus X is Hausdorff. Moreover, since Υ is a surjective local homeomorphism, X is also connected, locally simply connected, and locally compact.

For the proof that $\Gamma = \mathrm{Deck}(\Upsilon)$ (and for the proof of (b)), let us fix a point $\tilde{x}_0 \in \widetilde{X}$ and let $x_0 = \Upsilon(\tilde{x}_0)$. Given an element $\Phi \in \Gamma$, we have $\Upsilon \circ \Phi = \Upsilon$ and therefore $\Phi \in \mathrm{Deck}(\Upsilon)$. Conversely, if $\Phi \in \mathrm{Deck}(\Upsilon)$, then $\Upsilon(\Phi(\tilde{x}_0)) = \Upsilon(\tilde{x}_0)$, and hence there is an element $\Psi \in \Gamma \subset \mathrm{Deck}(\Upsilon)$ with $\Psi(\tilde{x}_0) = \Phi(\tilde{x}_0)$. Therefore, by Theorem 10.4.2, $\Phi = \Psi \in \Gamma$. Thus $\Gamma = \mathrm{Deck}(\Upsilon)$.

For the proof of (b), suppose $\widehat{\Gamma}$ is a subgroup of Γ, $\Upsilon_{\widehat{\Gamma}} : \widetilde{X} \to \widehat{X} = \widehat{\Gamma} \backslash \widetilde{X}$ is the corresponding quotient covering space, and $\hat{x}_0 = \Upsilon_{\widehat{\Gamma}}(\tilde{x}_0)$. The surjective mapping $\widehat{\Upsilon} : \widehat{X} \to X$ given by $\Upsilon_{\widehat{\Gamma}}(x) \mapsto \Upsilon(x)$ is well defined and therefore continuous with respect to the quotient topology. If U is a nonempty domain in X that is evenly covered by Υ, and U_0 is a connected component of $\widehat{\Upsilon}^{-1}(U)$, then each connected component V of $\Upsilon_{\widehat{\Gamma}}^{-1}(U_0) \subset \Upsilon^{-1}(U)$ is equal to a connected component of $\Upsilon^{-1}(U)$. For if W is the component of $\Upsilon^{-1}(U)$ containing V, then $\Upsilon_{\widehat{\Gamma}}(W)$ is a connected subset of $\widehat{\Upsilon}^{-1}(U)$ ($\widehat{\Upsilon}(\Upsilon_{\widehat{\Gamma}}(W)) = \Upsilon(W) = U$) that meets, and is therefore contained in, U_0. So $W = V$. Since Υ maps V homeomorphically onto U and $\Upsilon_{\widehat{\Gamma}}$ maps V locally homeomorphically onto U_0 (by Corollary 10.2.7), $\widehat{\Upsilon}$ must map U_0 homeomorphically onto U. Thus $\widehat{\Upsilon}$ is a covering map.

To complete the proof of (b), we observe that given an element $[\hat{\gamma}] \in \pi_1(\widehat{X}, \hat{x}_0)$, we may set $\gamma = \widehat{\Upsilon}(\hat{\gamma})$ and we may form the common lifting $\tilde{\gamma}$ of γ and $\hat{\gamma}$ to a path in $\widetilde{X}$ with $\tilde{\gamma}(0) = \tilde{x}_0$. We then have

$$[\gamma] \cdot \tilde{x}_0 = \chi^{-1}([\gamma])(\tilde{x}_0) = \tilde{\gamma}(1) = \widehat{\chi}^{-1}([\hat{\gamma}])(\tilde{x}_0) = [\hat{\gamma}] \cdot \tilde{x}_0,$$

so $[\gamma] = \widehat{\Upsilon}_*[\hat{\gamma}] = \chi(\widehat{\chi}^{-1}([\hat{\gamma}]))$.

The proofs of (c) and (d) are left to the reader (see Exercise 10.4.2). $\square$

Corollary 10.4.7 *Let X be a connected locally simply connected locally compact Hausdorff space, let $x_0 \in X$, and let Γ be a subgroup of $\pi_1(X, x_0)$. Then, up to equivalence of covering spaces, there is a unique connected covering space $\widehat{\Upsilon}\colon \widehat{X} \to X$ such that for some point $\hat{x}_0 \in \widehat{\Upsilon}^{-1}(x_0)$, $\widehat{\Upsilon}_*\colon \pi_1(\widehat{X}, \hat{x}_0) \to \pi_1(X, x_0)$ maps $\pi_1(\widehat{X}, \hat{x}_0)$ isomorphically onto Γ. In fact, if $\Upsilon\colon \widetilde{X} \to X$ is the universal covering space, then we have the commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\Upsilon_\Gamma"] \arrow[d, "\Upsilon"'] & \widehat{X} = \Gamma\backslash\widetilde{X} \arrow[dl, "\widehat{\Upsilon}"] \\ X & \end{tikzcd}$$

(*here, we identify* $\pi_1(X, x_0) \supset \Gamma$ *with* $\mathrm{Deck}(\Upsilon)$). *Moreover, if X is a $\mathcal{C}^\infty$ manifold, and $\widetilde{X}$ has the induced $\mathcal{C}^\infty$ structure, then there is a unique $\mathcal{C}^\infty$ structure on $\widehat{X}$ with respect to which Υ_Γ and $\widehat{\Upsilon}$ are $\mathcal{C}^\infty$ covering maps.*

Proof This follows immediately from Theorem 10.3.3 (existence of the universal cover), Theorem 10.4.6 (the construction of a covering from a properly discontinuous free group action), and Proposition 10.2.9 (equivalence of coverings with the same image fundamental group). □

Example 10.4.8 For $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, the $\mathcal{C}^\infty$ universal covering space is given by $\Upsilon\colon \mathbb{R} \to \mathbb{S}^1$, where $\Upsilon(t) = (\cos(2\pi t), \sin(2\pi t)) = e^{2\pi i t}$ for each $t \in \mathbb{R}$ (see Example 10.2.2). If $\Phi \in \mathrm{Deck}(\Upsilon)$, then $e^{2\pi i \Phi(t)} = e^{2\pi i t}$ for all $t \in \mathbb{R}$. Thus $t \mapsto (\Phi(t) - t)/(2\pi)$ is a $\mathcal{C}^\infty$ function with values in $\mathbb{Z}$, and is therefore a constant function. Conversely, any translation of the form $t \mapsto t + n$ for $n \in \mathbb{Z}$ is clearly a deck transformation. Thus $\pi_1(\mathbb{S}^1) \cong \mathrm{Deck}(\Upsilon) \cong \mathbb{Z}$. In fact, for $x_0 = (1, 0)$, $\pi_1(\mathbb{S}^1, x_0)$ is the infinite cyclic (free) group generated by $[\gamma]_{x_0}$, where $\gamma(t) = e^{2\pi i t}$ for each $t \in [0, 1]$ (since γ has the lifting $\tilde{\gamma}\colon t \mapsto t$ with $\tilde{\gamma}(0) = 0$ and $\tilde{\gamma}(1) = 0 + 1 = 1$).

Any subgroup Γ of $\mathbb{Z} \cong \pi_1(\mathbb{S}^1, x_0)$ must be of the form $\Gamma = m\mathbb{Z}$ for some nonnegative integer m; that is, Γ is the infinite cyclic subgroup generated by $[\gamma]_{x_0}^m$. We also have a universal covering map $\mathbb{R} \to \mathbb{S}^1$ given by $t \mapsto e^{2\pi i t/m}$ (the proof is similar to the above), and this covering map has deck transformation group $m\mathbb{Z} = \Gamma \subset \mathbb{Z}$. Therefore, $\mathbb{S}^1 \cong \Gamma\backslash\mathbb{R}$ and the covering space of $\mathbb{S}^1$ associated to the subgroup Γ as in Corollary 10.4.7 is the finite mapping $\mathbb{S}^1 \to \mathbb{S}^1$ given by $e^{2\pi i t/m} \mapsto e^{2\pi i t}$, that is, by $z \mapsto z^m$ (see Example 1.6.2).

Exercises for Sect. 10.4

10.4.1 Prove parts (c)–(e) of Theorem 10.4.2.

10.4.2 Prove parts (c) and (d) of Theorem 10.4.6.

10.5 Line Integrals on $\mathcal{C}^\infty$ Surfaces

Throughout this section, M denotes a 2-dimensional $\mathcal{C}^\infty$ manifold. We first recall that for a continuous differential form θ of degree 1 on M and a piecewise $\mathcal{C}^1$ path $\gamma\colon [a,b] \to M$, the (*line*) *integral of* θ *along* γ is given by $\int_\gamma \theta = \int_a^b \theta(\dot{\gamma}(t))\,dt$ (see Definition 9.7.18). For any point $a \in \mathbb{R}$ and any map $\gamma\colon \{a\} \to M$, we define $\int_\gamma \theta \equiv 0$. The chain rule and the fundamental theorem of calculus give the following (see Exercise 10.5.1):

Lemma 10.5.1 *For any continuous* 1*-form* θ *on* M *and any piecewise* $\mathcal{C}^1$ *path* $\gamma\colon [a,b] \to M$:

(a) *We have* $\int_{\gamma^-} \theta = -\int_\gamma \theta$.
(b) *If* $a < c < b$, *then* $\int_\gamma \theta = \int_{\gamma\restriction_{[a,c]}} \theta + \int_{\gamma\restriction_{[c,d]}} \theta$.
(c) *If* $\theta = df$ *for a* $\mathcal{C}^1$ *function* f *on a neighborhood of* $\gamma([a,b])$, *then*

$$\int_\gamma \theta = f(\gamma(b)) - f(\gamma(a)).$$

Part (c) of the above lemma suggests that one may define integration of a locally exact 1-form (for example, a closed $\mathcal{C}^1$ form) along a *continuous* path as a sum of changes of local potentials. The following lemma will imply that the integral is well defined.

Lemma 10.5.2 *Let* θ *be a locally exact* 1*-form on* M (*in particular,* θ *is continuous*), *let* $\gamma\colon [a,b] \to M$ *be a* (*continuous*) *path in* M, *let* $\mathcal{S}\colon\ a = s_0 < \cdots < s_m = b$ *and* $\mathcal{T}\colon\ a = t_0 < \cdots < t_n = b$ *be partitions of* $[a,b]$, *and for each* $i = 1,\ldots,m$ *and each* $j = 1,\ldots,n$, *let* α_i *and* β_j *be a* $\mathcal{C}^1$ *function on a neighborhood* U_i *of* $\gamma([s_{i-1}, s_i])$ *in* M *and on a neighborhood* V_j *of* $\gamma([t_{j-1}, t_j])$ *in* M, *respectively, such that* $\theta = d\alpha_i$ *on* U_i *and* $\theta = d\beta_j$ *on* V_j. *Then*

$$\sum_{i=1}^m \big[\alpha_i(\gamma(s_i)) - \alpha_i(\gamma(s_{i-1}))\big] = \sum_{j=1}^n \big[\beta_j(\gamma(t_j)) - \beta_j(\gamma(t_{j-1}))\big].$$

Furthermore, if γ *is a piecewise* $\mathcal{C}^1$ *path, then the above number is equal to the integral of* θ *along* γ.

Proof By considering a common refinement of the two partitions, we see that we may assume without loss of generality that $\mathcal{T}$ is a refinement of $\mathcal{S}$; that is, $\{s_0,\ldots,s_m\} \subset \{t_0,\ldots,t_n\}$. Thus, for each $j = 1,\ldots,n$, we have, for some unique index i, $[t_{j-1}, t_j] \subset [s_{i-1}, s_i]$ and $d\beta_j = d\alpha_i = \theta$ on some connected neighborhood W of $\gamma([t_{j-1}, t_j])$ in $U_i \cap V_j$. In particular, α_i and β_j differ by a constant on W, and hence $\alpha_i(\gamma(t_j)) - \alpha_i(\gamma(t_{j-1})) = \beta_j(\gamma(t_j)) - \beta_j(\gamma(t_{j-1}))$. Therefore,

$$\sum_{i=1}^{m}\big[\alpha_i(\gamma(s_i)) - \alpha_i(\gamma(s_{i-1}))\big]$$
$$= \sum_{i=1}^{m} \sum_{\substack{1\le j\le n \\ [t_{j-1},t_j]\subset[s_{i-1},s_i]}} \big[\alpha_i(\gamma(t_j)) - \alpha_i(\gamma(t_{j-1}))\big]$$
$$= \sum_{j=1}^{n}\big[\beta_j(\gamma(t_j)) - \beta_j(\gamma(t_{j-1}))\big].$$

Finally, Lemma 10.5.1 implies that the above gives the integral when γ is a piecewise $\mathcal{C}^1$ path. □

Definition 10.5.3 Let θ be a locally exact 1-form on M. For any (continuous) path $\gamma\colon [a,b]\to M$, the *integral of θ along γ* is given by

$$\int_\gamma \theta \equiv \sum_{i=1}^{n}\big[\alpha_i(\gamma(t_i)) - \alpha_i(\gamma(t_{i-1}))\big],$$

where $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ is a partition of $[a,b]$ and for each $i = 1,\dots,n$, α_i is a $\mathcal{C}^1$ function with $d\alpha_i = \theta$ on a neighborhood of $\gamma([t_{i-1},t_i])$ in M.

Lemma 10.5.4 *For any locally exact* 1*-form θ on M and any path $\gamma\colon[a,b]\to M$:*

(a) *We have $\int_{\gamma^-}\theta = -\int_\gamma\theta$.*
(b) *If $a < c < b$, then $\int_\gamma\theta = \int_{\gamma\restriction_{[a,c]}}\theta + \int_{\gamma\restriction_{[c,d]}}\theta$.*
(c) *Suppose $a\le c\le d\le b$; $I = \{t\in\mathbb{R}\mid c\le t\le d\}$; $s_0,s_1,s_2,\dots,s_m$ are arbitrary points in $[a,b]$ with $s_0 = c$ and $s_m = d$; and α_i is $\mathcal{C}^1$ function with $d\alpha_i = \theta$ on a neighborhood of $\{\gamma(t)\mid s_{i-1}\le t\le s_i$ or $s_{i-1}\ge t\ge s_i\}$ in M for $i = 1,\dots,m$. Then $\int_{\gamma\restriction_I}\theta = \sum_{i=1}^m[\alpha_i(\gamma(s_i)) - \alpha_i(\gamma(s_{i-1}))]$.*
(d) Invariance under continuous reparametrization. *If $\varphi\colon[r,s]\to[a,b]$ is a continuous map with $c\equiv\varphi(r)\le d\equiv\varphi(s)$ and $I = \{t\in\mathbb{R}\mid c\le t\le d\}$, then $\int_{\gamma\circ\varphi}\theta = \int_{\gamma\restriction_I}\theta$.*
(e) *If $\Phi\colon N\to M$ is a $\mathcal{C}^1$ mapping of a* 2*-dimensional $\mathcal{C}^\infty$ manifold N into M, then the pullback $\Phi^*\theta$ is a locally exact* 1*-form, and for every path τ in N, we have $\int_\tau\Phi^*\theta = \int_{\Phi(\tau)}\theta$.*

Proof The proofs of parts (a), (b), and (e) are left to the reader (see Exercise 10.5.2). We prove part (c) for any choice of points $s_0,s_1,s_2,\dots,s_m$ and functions $\alpha_1,\dots,\alpha_m$ by induction on m, the case $m = 1$ being trivial. Assume now that $m > 1$ and that the claim holds for positive integers less than m. If there is an index $k\in\{1,\dots,m-1\}$ with $s_k < s_0 = c$, then applying the induction hypothesis to the finite sequences $s_k, s_{k+1},\dots,s_m = d$ and $s_k, s_{k-1}, s_{k-2},\dots,s_1,s_0 = c$, we get

$$\int_{\gamma\restriction_I}\theta = \int_{\gamma\restriction_{[s_k,d]}}\theta - \int_{\gamma\restriction_{[s_k,c]}}\theta$$

$$= \sum_{i=k+1}^{m} \big[\alpha_i(\gamma(s_i)) - \alpha_i(\gamma(s_{i-1}))\big]$$
$$- \sum_{i=1}^{k} \big[\alpha_{k-i+1}(\gamma(s_{k-i})) - \alpha_{k-i+1}(\gamma(s_{k-i+1}))\big]$$
$$= \sum_{i=1}^{m} \big[\alpha_i(\gamma(s_i)) - \alpha_i(\gamma(s_{i-1}))\big].$$

If there is an index $k \in \{1, \dots, m-1\}$ with $c = s_0 \le s_k \le s_m = d$, then for $J \equiv \{t \in \mathbb{R} \mid c \le t \le s_k\}$ and $K \equiv \{t \in \mathbb{R} \mid s_k \le t \le d\}$, the induction hypothesis gives

$$\int_{\gamma\restriction_I} \theta = \int_{\gamma\restriction_J} \theta + \int_{\gamma\restriction_K} \theta$$
$$= \sum_{i=1}^{k} \big[\alpha_i(\gamma(s_i)) - \alpha_i(\gamma(s_{i-1}))\big] + \sum_{i=k+1}^{m} \big[\alpha_i(\gamma(s_i)) - \alpha_i(\gamma(s_{i-1}))\big]$$
$$= \sum_{i=1}^{m} \big[\alpha_i(\gamma(s_i)) - \alpha_i(\gamma(s_{i-1}))\big].$$

Finally, if $s_m < s_k$ for $k = 1, \dots, m-1$, then the induction hypothesis gives

$$\int_{\gamma\restriction_I} \theta = \int_{\gamma\restriction_{[c,s_{m-1}]}} \theta - \int_{\gamma\restriction_{[d,s_{m-1}]}} \theta$$
$$= \sum_{i=1}^{m-1} \big[\alpha_i(\gamma(s_i)) - \alpha_i(\gamma(s_{i-1}))\big] - \big[\alpha_m(\gamma(s_{m-1})) - \alpha_m(\gamma(s_m))\big]$$
$$= \sum_{i=1}^{m} \big[\alpha_i(\gamma(s_i)) - \alpha_i(\gamma(s_{i-1}))\big].$$

Thus (c) is proved.

For the setup in (d), we may choose a partition $r = t_0 < t_1 < \cdots < t_m = s$ such that for each $i = 1, \dots, m$, there is a $\mathcal{C}^1$ function α_i with $d\alpha_i = \theta$ on a neighborhood of $\gamma(\varphi([t_{i-1}, t_i])) \supset \gamma(\{s \in [a,b] \mid \varphi(t_{i-1}) \le s \le \varphi(t_i) \text{ or } \varphi(t_{i-1}) \ge s \ge \varphi(t_i)\})$. Setting $s_i = \varphi(t_i)$ for $i = 0, \dots, m$ and applying part (c), we get

$$\int_{\gamma\circ\varphi} \theta = \sum_{i=1}^{m} \big[\alpha_i(\gamma(\varphi(t_i))) - \alpha_i(\gamma(\varphi(t_{i-1})))\big] = \int_{\gamma\restriction_I} \theta. \qquad \square$$

Proposition 10.5.5 *For any locally exact* 1*-form* θ *on* M, *the following are equivalent*:

(i) *The form* θ *is exact.*
(ii) $\int_\gamma \theta = 0$ *for every loop* γ *in* M.
(iii) *The line integrals are independent of path*; *that is*, $\int_{\gamma_0} \theta = \int_{\gamma_1} \theta$ *for every pair of paths* γ_0 *and* γ_1 *with the same endpoints.*

Proof We may assume without loss of generality that M is connected. That (i) implies (ii) follows immediately from the definition of the line integral. That (ii) implies (iii) follows immediately from Lemma 10.5.4.

Assuming now that the condition (iii) holds, and fixing a point $p \in M$, for each point $x \in M$ we may define $f(x) \equiv \int_\gamma \theta$, where γ is an arbitrary path from p to x in M. Given $q \in M$, we may fix a $\mathcal{C}^1$ function α with $d\alpha = \theta$ on some connected neighborhood U of q, and a path η from p to q. Given a point $x \in U$, we may choose a path γ from q to x in U, and we get

$$f(x) = \int_\gamma \theta + \int_\eta \theta = \alpha(x) - \alpha(q) + \int_\gamma \theta.$$

Hence f and α differ by a constant on U, and therefore $f \in \mathcal{C}^1(M)$ and $df = \theta$. $\square$

One may greatly reduce the collection of loops along which one must check that the integral vanishes in order to get exactness.

Theorem 10.5.6 *Let θ be a locally exact 1-form on M, and let γ_0 and γ_1 be two path homotopic paths in M. Then $\int_{\gamma_0} \theta = \int_{\gamma_1} \theta$.*

Proof Let $H: [a, b] \times [0, 1] \to M$ be a path homotopy from γ_0 to γ_1, and for each $s \in [0, 1]$, let $\gamma_s \equiv H(\cdot, s): [a, b] \to M$. We may choose partitions

$$a = t_0 < t_1 < t_2 < \cdots < t_m = b \quad \text{and} \quad 0 = s_0 < s_1 < s_2 < \cdots < s_n = 1$$

such that for each pair $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, there is a $\mathcal{C}^1$ function α_{ij} with $d\alpha_{ij} = \theta$ on a connected neighborhood U_{ij} of $H([t_{i-1}, t_i] \times [s_{j-1}, s_j])$ in M (see Fig. 10.3). In particular, if $i \geq 2$ and $j \geq 1$, then α_{ij} and $\alpha_{i-1,j}$ differ by a constant on some neighborhood of $H(\{t_{i-1}\} \times [s_{j-1}, s_j])$. For $j = 1, \dots, n$,

$$\begin{aligned}
\int_{\gamma_{s_{j-1}}} \theta &= \sum_{i=1}^{m} \big[\alpha_{ij}(\gamma_{s_{j-1}}(t_i)) - \alpha_{ij}(\gamma_{s_{j-1}}(t_{i-1}))\big] \\
&= \sum_{i=1}^{m} \Big[\big(\alpha_{ij}(\gamma_{s_{j-1}}(t_i)) - \alpha_{ij}(\gamma_{s_j}(t_i))\big) + \big(\alpha_{ij}(\gamma_{s_j}(t_i)) - \alpha_{ij}(\gamma_{s_j}(t_{i-1}))\big) \\
&\qquad + \big(\alpha_{ij}(\gamma_{s_j}(t_{i-1})) - \alpha_{ij}(\gamma_{s_{j-1}}(t_{i-1}))\big)\Big] \\
&= \sum_{i=1}^{m-1} \big[\alpha_{ij}(\gamma_{s_{j-1}}(t_i)) - \alpha_{ij}(\gamma_{s_j}(t_i))\big] + \int_{\gamma_{s_j}} \theta \\
&\qquad + \sum_{i=2}^{m} \big[\alpha_{i-1,j}(\gamma_{s_j}(t_{i-1})) - \alpha_{i-1,j}(\gamma_{s_{j-1}}(t_{i-1}))\big] \\
&= \int_{\gamma_{s_j}} \theta
\end{aligned}$$

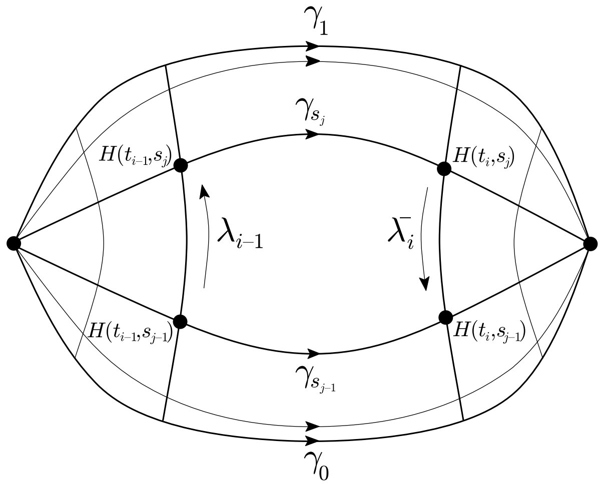

Fig. 10.3 Invariance of line integrals under path homotopy

(here we have used the fact that $\gamma_s(a)$ and $\gamma_s(b)$ are each constant in s). The claim now follows. $\square$

Remark One may view the equality

$$\begin{aligned}\alpha_{ij}(\gamma_{s_{j-1}}(t_i)) - \alpha_{ij}(\gamma_{s_{j-1}}(t_{i-1})) = &\big(\alpha_{ij}(\gamma_{s_{j-1}}(t_i)) - \alpha_{ij}(\gamma_{s_j}(t_i))\big)\\ &+ \big(\alpha_{ij}(\gamma_{s_j}(t_i)) - \alpha_{ij}(\gamma_{s_j}(t_{i-1}))\big)\\ &+ \big(\alpha_{ij}(\gamma_{s_j}(t_{i-1})) - \alpha_{ij}(\gamma_{s_{j-1}}(t_{i-1}))\big)\end{aligned}$$

as the replacement of the integral of θ along the segment $\gamma_{s_{j-1}}\restriction_{[t_{i-1},t_i]}$ with the integral along the path $\lambda_{i-1} * \gamma_{s_j}\restriction_{[t_{i-1},t_i]} * \lambda_i^-$, where λ_i is the path $s \mapsto H(t_i, s)$ from $\gamma_{s_{j-1}}(t_i)$ to $\gamma_{s_j}(t_i)$ for $i = 0, \dots, m$ and $j = 1, \dots, m$ (see Fig. 10.3). The integrals are equal because θ is exact on the neighborhood U_{ij}. The integrals along the λ_i's cancel in the telescoping sum.

Corollary 10.5.7 *If M is simply connected, then any locally exact 1-form on M is exact.*

Corollary 10.5.8 *If M is connected, $x_0 \in M$, and θ is a locally exact 1-form on M, then the map $\pi_1(X, x_0) \to (\mathbb{C}, +)$ given by $[\gamma] \mapsto \int_\gamma \theta$ is a well-defined group homomorphism.*

Exercises for Sect. 10.5

10.5.1 Prove Lemma 10.5.1.
10.5.2 Prove parts (a), (b), and (e) of Lemma 10.5.4.

10.5.3 Let $\theta = P\,dx + Q\,dy$ for continuous functions P, Q on an open set $\Omega \subset \mathbb{R}^2$. Show that if

$$\left(\frac{\partial Q}{\partial x}\right)_{\text{distr}} = \left(\frac{\partial P}{\partial y}\right)_{\text{distr}} \in L^1_{\text{loc}}(\Omega),$$

then θ is locally exact.

10.6 Homology and Cohomology of Second Countable $\mathcal{C}^\infty$ Surfaces

Differential forms and their appropriate generalizations allow one to associate certain Abelian groups and vector spaces to a $\mathcal{C}^\infty$ surface. In a sense, these groups measure the number of loops (and forms) in the surface when we identify loops (and forms) that are identical up to integration. We will need these groups only for second countable surfaces, so we restrict our attention to that case for simplicity. The approach taken here is similar to the approach in the book of Weyl [Wey]. Throughout this section, $\mathbb{F}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$.

Definition 10.6.1 For a second countable $\mathcal{C}^\infty$ surface M and an integer $r \in \{0, 1, 2\}$, the *rth de Rham cohomology group with coefficients in* $\mathbb{F}$ is the quotient group (and quotient vector space)

$$\begin{aligned} H^r_{\text{deR}}(M, \mathbb{F}) \equiv \ker\bigl(&\mathcal{E}^r(M, \mathbb{F}) \xrightarrow{d} \mathcal{E}^{r+1}(M, \mathbb{F})\bigr) \\ &/\text{im}\bigl(\mathcal{E}^{r-1}(M, \mathbb{F}) \xrightarrow{d} \mathcal{E}^r(M, \mathbb{F})\bigr) \end{aligned}$$

(here, we set $d \equiv 0$ on $\mathcal{E}^{-1}(M) \equiv 0$). We also denote $H^r_{\text{deR}}(M, \mathbb{F})$ simply by $H^r_{\text{deR}}(M)$. For each closed differential form $\alpha \in \mathcal{E}^r(M, \mathbb{F})$, we call the corresponding equivalence class in $H^r_{\text{deR}}(M, \mathbb{F})$ the *de Rham cohomology class* of α, and we denote this class by $[\alpha]_{H^r_{\text{deR}}(M,\mathbb{F})}$, by $[\alpha]_{\text{deR}}$, or simply by $[\alpha]$. We also say that two differential forms are *cohomologous* if they represent the same de Rham cohomology class (i.e., if their difference is $\mathcal{C}^\infty$-exact). We also let

$$\begin{aligned} Z^1_{\text{deR}}(M, \mathbb{F}) &\equiv \ker\bigl(\mathcal{E}^1(M, \mathbb{F}) \xrightarrow{d} \mathcal{E}^2(M, \mathbb{F})\bigr), \\ B^1_{\text{deR}}(M, \mathbb{F}) &\equiv \text{im}\bigl(\mathcal{E}^0(M, \mathbb{F}) \xrightarrow{d} \mathcal{E}^1(M, \mathbb{F})\bigr). \end{aligned}$$

In other words, $H^r_{\text{deR}}(M, \mathbb{F})$ is given by the closed $\mathcal{C}^\infty$ $\mathbb{F}$-valued differential forms of degree r on M, where we identify two closed $\mathcal{C}^\infty$ differential forms α and β if and only if $\alpha - \beta$ is $\mathcal{C}^\infty$-exact. Clearly, $H^r_{\text{deR}}(M, \mathbb{F})$ is an Abelian group with respect to the induced sum operation $[\alpha]_{\text{deR}} + [\beta]_{\text{deR}} = [\alpha + \beta]_{\text{deR}}$. In fact, $H^r_{\text{deR}}(M, \mathbb{F})$ is a vector space over $\mathbb{F}$ with respect to the induced sum and the induced scalar multiplication $\zeta \cdot [\alpha]_{\text{deR}} = [\zeta\alpha]_{\text{deR}}$. For our purposes, we will mainly consider the case $r = 1$.

Although we often use the same notation $[\alpha]_{\mathrm{deR}}$ to denote the de Rham cohomology class of a closed form α in $H^r_{\mathrm{deR}}(M,\mathbb{R})$ or $H^r_{\mathrm{deR}}(M,\mathbb{C})$, there is no real danger of confusion, because according to the following proposition, we may view $H^r_{\mathrm{deR}}(M,\mathbb{R})$ as a real vector subspace (and as a subgroup) of $H^r_{\mathrm{deR}}(M,\mathbb{C})$.

Proposition 10.6.2 *For a second countable $\mathcal{C}^\infty$ surface M and an integer $r \in \{0, 1, 2\}$, the map $H^r_{\mathrm{deR}}(M,\mathbb{R}) \to H^r_{\mathrm{deR}}(M,\mathbb{C})$ given by $[\alpha]_{H^r_{\mathrm{deR}}(M,\mathbb{R})} \mapsto [\alpha]_{H^r_{\mathrm{deR}}(M,\mathbb{C})}$ is a well-defined injective real linear map. In particular, identifying $H^r_{\mathrm{deR}}(M,\mathbb{R})$ with its image in $H^r_{\mathrm{deR}}(M,\mathbb{C})$, we get the real direct sum decomposition $H^r_{\mathrm{deR}}(M,\mathbb{C}) = H^r_{\mathrm{deR}}(M,\mathbb{R}) \oplus i\,H^r_{\mathrm{deR}}(M,\mathbb{R})$ given by*

$$[\alpha]_{\mathrm{deR}} = [\operatorname{Re}\alpha]_{\mathrm{deR}} + i[\operatorname{Im}\alpha]_{\mathrm{deR}} \quad \forall [\alpha]_{\mathrm{deR}} \in H^r_{\mathrm{deR}}(M,\mathbb{C})$$

(*thus we may identify $H^r_{\mathrm{deR}}(M,\mathbb{C})$ with the complexification $(H^r_{\mathrm{deR}}(M,\mathbb{R}))_{\mathbb{C}}$*).

Proof Let α and β be closed $\mathcal{C}^\infty$ real differential forms of degree r. If $\alpha - \beta = d\gamma$ for some $\mathcal{C}^\infty$ real differential form γ of degree $r-1$, then clearly, $[\alpha]_{\mathrm{deR}} = [\beta]_{\mathrm{deR}}$ in $H^r_{\mathrm{deR}}(M,\mathbb{C})$. Thus the map is well defined. It is also easy to check that the map is linear. Finally, if $\alpha = d\gamma$ for some $\mathcal{C}^\infty$ complex differential form γ, then we have $\alpha = d\operatorname{Re}\gamma$ (and $d\operatorname{Im}\gamma = 0$). It follows that the map is injective.

For the direct sum decomposition, observe that if $[\alpha]_{\mathrm{deR}} \in H^r_{\mathrm{deR}}(M,\mathbb{C})$, then $d\operatorname{Re}\alpha = d\operatorname{Im}\alpha = 0$ and $[\alpha]_{\mathrm{deR}} = [\operatorname{Re}\alpha]_{\mathrm{deR}} + i[\operatorname{Im}\alpha]_{\mathrm{deR}}$. If ρ and τ are closed $\mathcal{C}^\infty$ real r-forms with $[\rho]_{\mathrm{deR}} = i[\tau]_{\mathrm{deR}} \in H^r_{\mathrm{deR}}(M,\mathbb{R}) \cap i\,H^r_{\mathrm{deR}}(M,\mathbb{R})$, then we have $\rho - i\tau = d\beta$ for some $\mathcal{C}^\infty$ complex $(r-1)$-form β ($\rho - i\tau = 0$ for $r = 0$). But then $\rho = d(\operatorname{Re}\beta)$ and $\tau = d(-\operatorname{Im}\beta)$, and hence $[\rho]_{\mathrm{deR}} = i[\tau]_{\mathrm{deR}} = 0$. $\square$

Remark In Sect. 10.7, we will see that for any second countable $\mathcal{C}^\infty$ surface M, $H^1_{\mathrm{deR}}(M,\mathbb{F})$ is canonically isomorphic to a vector space $H^1(M,\mathbb{F})$ (the first *cohomology group*) that depends only on the topology of M. In fact, we will see that

$$H^1_{\mathrm{deR}}(M,\mathbb{F}) \cong H^1(M,\mathbb{F}) \cong \operatorname{Hom}(\pi_1(M),\mathbb{F})$$

(where $\operatorname{Hom}(\pi_1(M),\mathbb{F})$ is the group of homomorphisms of $\pi_1(M)$ into $\mathbb{F}$, which has a natural vector space structure). For this reason, we identify $H^1_{\mathrm{deR}}(M,\mathbb{F})$ with $H^1(M,\mathbb{F})$. Although for the sake of convenience, we will define $H^1(M,\mathbb{F})$ as the space of classes of *Čech 1-forms* (which are natural topological analogues of closed 1-forms), the resulting space is actually canonically isomorphic to the *singular cohomology group* with coefficients in $\mathbb{F}$ (see the remarks at the end of Sect. 10.7 and Exercise 10.7.13). We will also consider, in Sect. 10.7, the cohomology group $H^1(M,\mathbb{A})$ with coefficients in an arbitrary subring $\mathbb{A}$ of $\mathbb{C}$ with $1 \in \mathbb{A}$, which is an $\mathbb{A}$-submodule of $H^1(M,\mathbb{C})$. It turns out that under the above isomorphisms, $H^1(M,\mathbb{A})$ may be identified with the submodule $\operatorname{Hom}(\pi_1(M),\mathbb{A})$ of $\operatorname{Hom}(\pi_1(M),\mathbb{C})$ and with the submodule of $H^1_{\mathrm{deR}}(M,\mathbb{C})$ consisting of those classes $[\theta]_{\mathrm{deR}} \in H^1_{\mathrm{deR}}(M,\mathbb{C})$ for which $\int_\gamma \theta \in \mathbb{A}$ for every loop γ in M. In this section, we focus mainly on cohomology (and homology) over $\mathbb{F}$, and set aside consideration of coefficients in other rings until Sect. 10.7.

We now turn to homology. For this, we first consider the vector space consisting of all formal finite linear combinations of paths. It will later be convenient (in Sect. 10.7) to have considered formal finite linear combinations over more general rings.

Definition 10.6.3 Let M be a second countable topological surface, and let $\mathbb{A}$ be a subring of $\mathbb{C}$ that contains 1 (i.e., that contains $\mathbb{Z}$).

(a) A 1-*chain in* M *with coefficients in* $\mathbb{A}$ is a formal finite linear combination

$$\xi = \sum_{i=1}^{m} a_i \cdot \gamma_i,$$

where $a_i \in \mathbb{A}$ and $\gamma_i \colon [0,1] \to M$ is a (continuous) path for $i = 1, \dots, m$. Following standard conventions, we identify the above 1-chain ξ with another 1-chain $\zeta = \sum_{j=1}^{n} b_j \beta_j$ in M if for each $i = 1, \dots, m$,

$$\sum_{\substack{1 \le l \le m \\ \gamma_l = \gamma_i}} a_l = \begin{cases} \sum_{\substack{1 \le j \le n \\ \beta_j = \gamma_i}} b_j & \text{if } \gamma_i \in \{\beta_1, \dots, \beta_n\}, \\ 0 & \text{if } \gamma_i \notin \{\beta_1, \dots, \beta_n\}, \end{cases}$$

and for each $j = 1, \dots, n$, the analogous statement, with the roles of ξ and ζ switched, holds.

A 0-*chain in* M *with coefficients in* $\mathbb{A}$ is a formal finite linear combination

$$\xi = \sum_{i=1}^{m} a_i \cdot p_i,$$

where $a_i \in \mathbb{A}$ and $p_i \in X$ for $i = 1, \dots, m$ (and we have the identifications analogous to those described above for 1-chains). For $q = 0, 1$, the q-chains in M, together with the obvious addition and scalar multiplication by elements of $\mathbb{A}$, form an $\mathbb{A}$-module (a vector space when $\mathbb{A}$ is a field), which we denote by $C_q(M, \mathbb{A})$.

(b) The *boundary operator* $\partial \colon C_1(M, \mathbb{A}) \to C_0(M, \mathbb{A})$ is the homomorphism (a linear map when $\mathbb{A}$ is a field) given by

$$\xi = \sum_{i=1}^{m} a_i \cdot \gamma_i \mapsto \sum_{i=1}^{m} a_i \cdot (\gamma_i(1) - \gamma_i(0))$$

(which, as one may easily check, is well defined).

(c) A 1-*cycle* in M is a 1-chain ξ for which $\partial\xi = 0$. The set of 1-cycles in $C_1(M, \mathbb{A})$ is a submodule (or subspace), which we denote by

$$Z_1(M, \mathbb{A}) \equiv \ker(\partial \colon C_1(M, \mathbb{A}) \to C_0(M, \mathbb{A})).$$

Remarks 1. Let $\mathcal{P}$ be the set of paths in M, and let G be the $\mathbb{A}$-module consisting of the $\mathbb{A}$-valued functions on $\mathcal{P}$ that vanish everywhere except at finitely many points. Then we may identify G with $C_1(M, \mathbb{A})$ under the correspondence

$$f \leftrightarrow \sum_{\gamma \in P \setminus f^{-1}(0)} f(\gamma) \cdot \gamma$$

(which we set equal to 0 if $f \equiv 0$). Similarly, we may identify the $\mathbb{A}$-module D consisting of the $\mathbb{A}$-valued functions on M that vanish everywhere except at finitely many points with $C_0(M, \mathbb{A})$ under the correspondence

$$f \leftrightarrow \sum_{p \in M \setminus f^{-1}(0)} f(p) \cdot p$$

(which we set equal to 0 if $f \equiv 0$). This point of view together with linearity makes it clear that the boundary operator ∂ is well defined by (b).

2. For any pair of subrings $\mathbb{A}_1$ and $\mathbb{A}_2$ of $\mathbb{C}$ with $1 \in \mathbb{A}_1 \subset \mathbb{A}_2$, $C_r(M, \mathbb{A}_1)$ is a submodule of $C_r(M, \mathbb{A}_1)$ for $r = 0, 1$ and $Z_1(M, \mathbb{A}_1)$ is a submodule of $Z_1(M, \mathbb{A}_2)$.

Definition 10.6.4 Let M be a second countable $\mathcal{C}^\infty$ surface.

(a) For every 1-chain $\xi = \sum_{i=1}^m a_i \cdot \gamma_i \in C_1(M, \mathbb{C})$ and every closed $\mathcal{C}^\infty$ 1-form (in fact, for every locally exact 1-form) θ on M, the *integral* (or *line integral*) of θ along ξ is given by

$$\int_\xi \theta \equiv \sum_{i=1}^m a_i \int_{\gamma_i} \theta$$

(which, as one may easily check, is well defined).

(b) We denote by $B_1(M, \mathbb{F})$ the subspace of $Z_1(M, \mathbb{F})$ consisting of all 1-cycles ξ for which $\int_\xi \theta = 0$ for every closed $\mathcal{C}^\infty$ 1-form θ on M (note that the definition yields the same space regardless of whether we require the forms θ to be real or complex).

(c) The *first homology group of M with coefficients in* $\mathbb{F}$ is the quotient vector space

$$H_1(M, \mathbb{F}) \equiv Z_1(M, \mathbb{F})/B_1(M, \mathbb{F}).$$

We call the equivalence class in $H_1(M, \mathbb{F})$ represented by $\xi \in Z_1(M, \mathbb{F})$ the *homology class* of ξ, and we denote this class by $[\xi]_{H_1(M,\mathbb{F})}$, by $[\xi]_{H_1}$, or simply by $[\xi]$. Two 1-cycles $\xi, \zeta \in Z_1(M, \mathbb{F})$ are said to be *homologous* if they represent the same homology class, that is, if $\int_\xi \theta = \int_\zeta \theta$ for every closed $\mathcal{C}^\infty$ 1-form θ on M.

Remarks 1. If $\xi \in C_1(M, \mathbb{F})$ and $\int_\xi \theta = 0$ for every closed $\mathcal{C}^\infty$ 1-form θ on M, then ξ is a 1-cycle and hence $\xi \in B_1(M, \mathbb{F})$ (see Exercise 10.6.1). In other words, the assumption in part (b) that the 1-chain ξ is a 1-cycle is superfluous.

2. It is more standard to define $B_1(M, \mathbb{F})$ as the image of a certain *boundary operator* ∂ on a space of 2-*chains* (see the remarks at the end of Sect. 10.7), but it turns out that the above definition, which is in a more convenient form for our purposes, is equivalent to the standard definition (see the exercises for Chaps. 5 and 6). Consequently, $H_1(M, \mathbb{F})$ is equal to the first *singular homology group* with coefficients in $\mathbb{F}$.

3. We will consider homology and cohomology groups of topological surfaces in Sect. 10.7. In particular, we will see that the first homology group and the first de Rham cohomology group actually depend only on the topology of the surface.

4. Given a subring $\mathbb{A}$ of $\mathbb{C}$ with $1 \in \mathbb{A}$, the natural analogue of the above definition gives a submodule of $Z_1(M, \mathbb{A})$ and a corresponding quotient; but without further hypotheses, the quotient need not coincide with the singular homology group. However, it turns out that the two do coincide for an *orientable* second countable $\mathcal{C}^\infty$ surface. With this in mind, for any orientable second countable $\mathcal{C}^\infty$ surface M and any subring $\mathbb{A}$ of $\mathbb{C}$ with $1 \in \mathbb{A}$, we define the $\mathbb{A}$-modules

$$B_1(M, \mathbb{A}) \equiv \left\{ \xi \in Z_1(M, \mathbb{A}) \,\middle|\, \int_\xi \theta = 0 \;\forall \theta \in Z^1_{\mathrm{deR}}(M, \mathbb{C}) \right\}$$

and

$$H_1(M, \mathbb{A}) \equiv Z_1(M, \mathbb{A})/B_1(M, \mathbb{A}).$$

On the other hand, in this section, we focus mostly on homology (and cohomology) over $\mathbb{F}$, and set aside consideration of coefficients in other rings until Sect. 10.7.

For a second countable $\mathcal{C}^\infty$ surface M, we have the real direct sum decomposition

$$Z_1(M, \mathbb{C}) = Z_1(M, \mathbb{R}) \oplus i Z_1(M, \mathbb{R}) = (Z_1(M, \mathbb{R}))_{\mathbb{C}}$$

given by $\xi = \operatorname{Re}\xi + i \operatorname{Im}\xi$ for each $\xi \in Z_1(M, \mathbb{C})$, where for any q-chain $\eta = \sum_{j=1}^m a_j \lambda_j$, $\operatorname{Re}\eta \equiv \sum_j (\operatorname{Re} a_j)\lambda_j$ and $\operatorname{Im}\eta \equiv \sum_j (\operatorname{Im} a_j)\lambda_j$. According to the following proposition, the proof of which is left to the reader (see Exercise 10.6.2), this also holds at the level of homology:

Proposition 10.6.5 *For a second countable $\mathcal{C}^\infty$ surface M, the real linear map $H_1(M, \mathbb{R}) \to H_1(M, \mathbb{C})$ given by $[\xi]_{H_1(M,\mathbb{R})} \mapsto [\xi]_{H_1(M,\mathbb{C})}$ is well defined and injective. In particular, we may identify $H_1(M, \mathbb{R})$ with its image in $H_1(M, \mathbb{C})$, and we have the real direct sum decomposition $H_1(M, \mathbb{C}) = H_1(M, \mathbb{R}) \oplus i H_1(M, \mathbb{R})$ given by*

$$[\xi]_{H_1(M,\mathbb{C})} = [\operatorname{Re}\xi]_{H_1(M,\mathbb{R})} + i[\operatorname{Im}\xi]_{H_1(M,\mathbb{R})} \quad \forall [\xi]_{H_1} \in H_1(M, \mathbb{C})$$

(*thus we may identify $H_1(M, \mathbb{C})$ with the complexification* $(H_1(M, \mathbb{R}))_{\mathbb{C}}$).

Lemma 10.6.6 and Proposition 10.6.7 below often allow one to work with loops in place of general paths when considering homology:

Lemma 10.6.6 *Let M be a second countable $\mathcal{C}^\infty$ surface, let $\mathbb{A}$ be a subring of $\mathbb{C}$ with $1 \in \mathbb{A}$, and let $x_0 \in M$. Then, for every 1-cycle $\xi \in Z_1(M, \mathbb{A})$, there exists a 1-cycle $\eta \in Z_1(M, \mathbb{A})$ such that η is a linear combination over $\mathbb{A}$ of loops based at x_0 and $\int_\xi \theta = \int_\eta \theta$ for every $\theta \in Z^1_{\mathrm{deR}}(M, \mathbb{C})$ (that is, ξ and η are homologous as 1-cycles over $\mathbb{C}$). Moreover, for ξ an integral 1-cycle (that is, for $\mathbb{A} = \mathbb{Z}$), one may choose η itself to be a loop based at x_0.*

Proof For $\xi = \sum_{i=1}^m a_i \gamma_i \in Z_1(M, \mathbb{A})$, we prove by induction on m that ξ is homologous to a linear combination of loops. For $m = 1$, we have $0 = \partial\xi = a_1 \cdot (\gamma_1(1) - \gamma_1(0))$, and hence $a_1 = 0$ or γ_1 is a loop. Assume now that the claim holds for any linear combination of strictly fewer than m paths. If $a_j = 0$ or γ_j is a loop for some j, then $\xi - a_j\gamma_j = \sum_{i \neq j} a_i\gamma_i$ is a 1-cycle, which, by the induction hypothesis, is homologous over $\mathbb{C}$ to a linear combination of loops. Thus we may assume that $a_j \neq 0$ and γ_j is *not* a loop for each $j = 1, \dots, m$. Hence we may fix distinct indices $i_1, \dots, i_k \in \{1, \dots, m\}$ and a path $\lambda_\nu = \gamma_{i_\nu}$ or $\gamma_{i_\nu}^-$ for each $\nu = 1, \dots, k$ such that for all $\mu, \nu = 1, \dots, k$, we have $\lambda_\mu(0) \neq \lambda_\nu(0)$ and $\lambda_\mu(1) \neq \lambda_\nu(1)$ if $\mu \neq \nu$, and $\lambda_\mu(1) = \lambda_\nu(0)$ if and only if $\nu = \mu + 1$. Furthermore, we may choose k to be the largest integer for which it is possible to make such a choice. In particular, the point $\lambda_k(1)$, which is equal to $\gamma_{i_k}(0)$ or $\gamma_{i_k}(1)$, is not an endpoint for any of the paths $\gamma_{i_1}, \dots, \gamma_{i_{k-1}}$; and hence, since $\partial\xi = 0$, $\lambda_k(1)$ must be an endpoint for the path $\gamma_{i_{k+1}}$ for some index $i_{k+1} \in \{1, \dots, m\} \setminus \{i_1, \dots, i_k\}$. Setting λ_{k+1} equal to $\gamma_{i_{k+1}}$ if $\lambda_k(1) = \gamma_{i_{k+1}}(0)$, $\gamma_{i_{k+1}}^-$ if $\lambda_k(1) = \gamma_{i_{k+1}}(1)$, we see that by the maximality of k, $\lambda_{k+1}(1) = \lambda_\mu(0)$ for some unique $\mu \in \{1, \dots, k\}$. Letting λ be the loop $\lambda_\mu * \lambda_{\mu+1} * \cdots * \lambda_{k+1}$, and for each $\nu = \mu, \dots, k+1$, letting σ_ν be equal to 1 if $\lambda_\nu = \gamma_{i_\nu}$, -1 if $\lambda_\nu = \gamma_{i_\nu}^-$, we see that ξ is homologous over $\mathbb{C}$ to the 1-cycle

$$\eta \equiv \sum_{i \in \{1,\dots,m\} \setminus \{i_\mu, \dots, i_{k+1}\}} a_i\gamma_i + \sum_{\nu=\mu+1}^{k+1} (a_{i_\nu} - \sigma_\mu a_{i_\mu} \sigma_\nu)\gamma_{i_\nu} + \sigma_\mu a_{i_\mu} \lambda.$$

Applying the induction hypothesis to the 1-cycle $\eta - \sigma_\mu a_\mu \lambda$, we get the claim. Moreover, given a loop τ in M, we may fix a path ρ from x_0 to $\tau(0)$, and τ will be homologous over $\mathbb{C}$ to the loop $\rho * \tau * \rho^-$ with base point at x_0. It follows that ξ is actually homologous over $\mathbb{C}$ to a linear combination $\eta \equiv \sum_{j=1}^n b_j\rho_j$ of loops $\{\rho_j\}_{j=1}^n$ *based at* x_0. Moreover, if $\mathbb{A} = \mathbb{Z}$, then may assume that $b_j = 1$ for each $j = 1, \dots, n$ (replacing b_j with $-b_j$ and ρ_j with ρ_j^- whenever $b_j < 0$, we get $\eta = \sum_{j=1}^n \sum_{k=1}^{b_j} \rho_j$). Hence ξ is then homologous over $\mathbb{C}$ to the loop $\rho_1 * \cdots * \rho_n$. □

Proposition 10.6.7 *Let M be a second countable $\mathcal{C}^\infty$ surface, and let $x_0 \in M$. Then:*

(a) *The mapping $[\alpha]_{x_0} = [\alpha]_{\pi_1(M,x_0)} \mapsto [\alpha]_{H_1(M,\mathbb{F})}$ gives a well-defined group homomorphism $\pi_1(M, x_0) \to H_1(M, \mathbb{F})$.*
(b) *The image of the fundamental group satisfies*

$$\mathrm{Span}_{\mathbb{F}}\big[\mathrm{im}(\pi_1(M, x_0) \to H_1(M, \mathbb{F}))\big] = H_1(M, \mathbb{F}).$$

(c) *The image of any set of generators of the group* $\pi_1(M, x_0)$ *spans* $H_1(M, \mathbb{F})$ *as a vector space, and consequently,* $H_1(M, \mathbb{F})$ *has a countable basis. In particular, if* $\pi_1(M)$ *is finitely generated (for example, if* M *is compact), then* $H_1(M, \mathbb{F})$ *is finite-dimensional.*
(d) $\int_\xi \theta = 0$ *for every exact* $\mathcal{C}^\infty$ 1*-form* θ *on* M *and every* 1*-cycle* $\xi \in Z_1(M, \mathbb{C})$.

Proof Theorem 10.5.6 implies that the mapping in (a) is well defined. Moreover, given two elements $[\alpha]_{x_0}, [\beta]_{x_0} \in \pi_1(M, x_0)$ and a closed $\mathcal{C}^\infty$ 1-form θ on M, Lemma 10.5.4 implies that

$$\int_{\alpha * \beta} \theta = \int_\alpha \theta + \int_\beta \theta = \int_{1\cdot\alpha + 1\cdot\beta} \theta.$$

Thus $[\alpha * \beta]_{H_1(M,\mathbb{F})} = [\alpha]_{H_1(M,\mathbb{F})} + [\beta]_{H_1(M,\mathbb{F})}$, and it follows that the mapping is a group homomorphism. Part (b) follows from Lemma 10.6.6, part (c) follows from part (b) and Lemma 10.1.9, and part (d) follows from part (b) and Proposition 10.5.5. □

Definition 10.6.8 Let M be a second countable $\mathcal{C}^\infty$ surface. We call the bilinear pairing $H^1_{\mathrm{deR}}(M, \mathbb{F}) \times H_1(M, \mathbb{F}) \to \mathbb{F}$ (or $H_1(M, \mathbb{F}) \times H^1_{\mathrm{deR}}(M, \mathbb{F}) \to \mathbb{F}$) given by

$$([\theta]_{\mathrm{deR}}, [\xi]_{H_1})_{\mathrm{deR}} = ([\xi]_{H_1}, [\theta]_{\mathrm{deR}})_{\mathrm{deR}} \equiv \int_\xi \theta$$

for every $[\theta]_{\mathrm{deR}} \in H^1_{\mathrm{deR}}(M, \mathbb{F})$ and $[\xi]_{H_1} \in H_1(M, \mathbb{F})$ the *de Rham pairing* on M (Definition 10.6.4 and Proposition 10.6.7 imply that this pairing is well defined).

Remark In Sect. 10.7, we will see that we may identify $H^1_{\mathrm{deR}}(M, \mathbb{F})$ with the dual space $(H_1(M, \mathbb{F}))^*$ under the linear isomorphism $[\theta]_{\mathrm{deR}} \mapsto ([\theta]_{\mathrm{deR}}, \cdot)_{\mathrm{deR}}$ (see Theorem 10.7.16).

For a $\mathcal{C}^\infty$ mapping $\Phi \colon M \to N$ of second countable $\mathcal{C}^\infty$ surfaces M and N, recall that we have the *pullback* (linear) mapping $\Phi^* \colon \mathcal{E}^r(N, \mathbb{F}) \to \mathcal{E}^r(M, \mathbb{F})$ for each r (see Definition 9.5.1), and because $d \circ \Phi^* = \Phi^* \circ d$ (by Proposition 9.5.6), the pullback of a closed (exact, $\mathcal{C}^\infty$-exact) differential form is also closed (respectively, exact, $\mathcal{C}^\infty$-exact). For $r = 0, 1$, we denote by $\Phi_* \colon C_r(M, \mathbb{F}) \to C_r(N, \mathbb{F})$ the induced *pushforward* linear map given by

$$\xi = \sum_{i=1}^m a_i \gamma_i \mapsto \sum_{i=1}^m a_i \Phi(\gamma_i),$$

where $a_1, \dots, a_m \in \mathbb{F}$ and $\gamma_1, \dots, \gamma_m$ are paths in M if $q = 1$, points in M if $q = 0$. As is easy to check, we have $\partial \circ \Phi_* = \Phi_* \circ \partial$, and hence $\Phi_*(Z_1(M, \mathbb{F})) \subset Z_1(N, \mathbb{F})$. It is also easy to check that if θ is a closed $\mathcal{C}^\infty$ 1-form (or even a locally exact continuous 1-form) on N, then $\int_\xi \Phi^*\theta = \int_{\Phi_*(\xi)} \theta$ for every 1-chain $\xi \in C_1(M, \mathbb{F})$. In particular, $\Phi_* B_1(M, \mathbb{F}) \subset B_1(N, \mathbb{F})$. If Φ is a diffeomorphism, then Φ_* and Φ^* are isomorphisms with inverse mappings $(\Phi_*)^{-1} = (\Phi^{-1})_*$ and $(\Phi^*)^{-1} = (\Phi^{-1})^*$.

The above mappings induce linear mappings of the corresponding de Rham cohomology and homology spaces, which have the expected functoriality properties.

Proposition 10.6.9 *For any $\mathcal{C}^\infty$ mapping $\Phi\colon M \to N$ of second countable $\mathcal{C}^\infty$ surfaces M and N, we have the following*:

(a) *The mapping $H_1(M, \mathbb{F}) \to H_1(N, \mathbb{F})$ given by $[\xi]_{H_1(M,\mathbb{F})} \mapsto [\Phi_* \xi]_{H_1(N,\mathbb{F})}$ is a well-defined linear map, and the diagram*

$$\begin{tikzcd} H_1(M,\mathbb{R}) \arrow[r] \arrow[d] & H_1(M,\mathbb{C}) \arrow[d] \\ H_1(N,\mathbb{R}) \arrow[r] & H_1(N,\mathbb{C}) \end{tikzcd}$$

commutes. If Φ is a diffeomorphism, then the map $H_1(M, \mathbb{F}) \to H_1(N, \mathbb{F})$ is an isomorphism with inverse mapping $[\zeta]_{H_1(N,\mathbb{F})} \mapsto [\Phi_*^{-1} \zeta]_{H_1(M,\mathbb{F})}$.

(b) *If $y_0 = \Phi(x_0)$ for some point $x_0 \in M$, then the diagram of induced mappings*

$$\begin{tikzcd} \pi_1(M,x_0) \arrow[r] \arrow[d] & H_1(M,\mathbb{F}) \arrow[d] \\ \pi_1(N,y_0) \arrow[r] & H_1(N,\mathbb{F}) \end{tikzcd}$$

commutes. Moreover, the image in $H_1(N, \mathbb{F})$ of (any generating set for) $\pi_1(M, x_0)$ spans the image (vector space) of $H_1(M, \mathbb{F})$.

(c) *The mapping $H^1_{\mathrm{deR}}(N, \mathbb{F}) \to H^1_{\mathrm{deR}}(M, \mathbb{F})$ given by $[\theta]_{\mathrm{deR}} \mapsto [\Phi^*\theta]_{\mathrm{deR}}$ is well defined and linear, and the diagram*

$$\begin{tikzcd} H^1_{\mathrm{deR}}(N,\mathbb{R}) \arrow[r] \arrow[d] & H^1_{\mathrm{deR}}(N,\mathbb{C}) \arrow[d] \\ H^1_{\mathrm{deR}}(M,\mathbb{R}) \arrow[r] & H^1_{\mathrm{deR}}(M,\mathbb{C}) \end{tikzcd}$$

commutes. Moreover, the map $H^1_{\mathrm{deR}}(N, \mathbb{F}) \to H^1_{\mathrm{deR}}(M, \mathbb{F})$ is an isomorphism if Φ is a diffeomorphism.

(d) *For each $[\theta]_{H^1_{\mathrm{deR}}(N,\mathbb{F})} \in H^1_{\mathrm{deR}}(N, \mathbb{F})$ and each $[\xi]_{H_1(M,\mathbb{F})} \in H_1(M, \mathbb{F})$, we have*

$$\big([\Phi^*\theta]_{H^1_{\mathrm{deR}}(M,\mathbb{F})}, [\xi]_{H_1(M,\mathbb{F})}\big)_{\mathrm{deR}} = \big([\theta]_{H^1_{\mathrm{deR}}(N,\mathbb{F})}, [\Phi_*\xi]_{H_1(N,\mathbb{F})}\big)_{\mathrm{deR}}.$$

The proof is left to the reader (see Exercise 10.6.4). Note that the linear map in (c) may be defined in exactly the same way for de Rham cohomology in degrees 0 and 2.

Definition 10.6.10 For $\Phi\colon M \to N$ as in Proposition 10.6.9, we call the corresponding mappings of homology and de Rham cohomology the induced *pushfor-*

ward and *pullback* mappings, respectively, and we denote these mappings by

$$\Phi_*: H_1(M, \mathbb{F}) \to H_1(N, \mathbb{F}) \quad \text{and} \quad \Phi^*: H^1_{\mathrm{deR}}(N, \mathbb{F}) \to H^1_{\mathrm{deR}}(M, \mathbb{F}).$$

Remark If $\Phi: M \to N$ and $\Psi: N \to P$ are $\mathcal{C}^\infty$ mappings of second countable $\mathcal{C}^\infty$ surfaces, then $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$ and $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$ both at the level of q-chains and 1-forms, and at the level of homology and de Rham cohomology (see Exercise 10.6.5).

Exercises for Sect. 10.6

10.6.1 Prove that if ξ is a 1-chain on a $\mathcal{C}^\infty$ surface M and $\int_\xi \theta = 0$ for every closed $\mathcal{C}^\infty$ 1-form θ on M, then ξ is a 1-cycle.

10.6.2 Prove Proposition 10.6.5.

10.6.3 Let θ be an exact 1-form of class $\mathcal{C}^1$ on a second countable $\mathcal{C}^\infty$ surface M. Prove that $\int_\xi \theta = 0$ for every 1-cycle $\xi \in Z_1(M, \mathbb{C})$.

10.6.4 Prove Proposition 10.6.9.

10.6.5 Prove that if $\Phi: M \to N$ and $\Psi: N \to P$ are $\mathcal{C}^\infty$ mappings of second countable $\mathcal{C}^\infty$ surfaces, then $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$ and $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$ both at the level of q-chains and 1-forms and at the level of homology and de Rham cohomology.

10.6.6 Prove that the map $([\alpha]_{\mathrm{deR}}, [\beta]_{\mathrm{deR}}) \mapsto [\alpha \wedge \beta]_{\mathrm{deR}}$ determines a well-defined bilinear pairing $H^1_{\mathrm{deR}}(M, \mathbb{C}) \times H^1_{\mathrm{deR}}(M, \mathbb{C}) \to H^2_{\mathrm{deR}}(M, \mathbb{C})$ for any second countable $\mathcal{C}^\infty$ manifold M.

10.6.7 Prove that for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $H_1(\mathbb{R}^2 \setminus \{0\}, \mathbb{F}) \cong \mathbb{F}$ and $H^1_{\mathrm{deR}}(\mathbb{R}^2 \setminus \{0\}, \mathbb{F}) \cong \mathbb{F}$.

10.7 Homology and Cohomology of Second Countable $\mathcal{C}^0$ Surfaces

In Sect. 10.6, homology and cohomology groups were defined using a $\mathcal{C}^\infty$ structure on a surface. As we will see in this section, these groups are actually, up to isomorphism, topological; that is, one may define analogues on a second countable topological surface that are isomorphic to the corresponding $\mathcal{C}^\infty$ versions when M is given a $\mathcal{C}^\infty$ structure. This is particularly appealing because, as is shown in Chap. 6, every second countable topological surface admits a $\mathcal{C}^\infty$ structure. The fact that these groups do not depend on the choice of the $\mathcal{C}^\infty$ structure is not crucial for most of our purposes, but it does allow one to see that certain complex analytic objects associated to a Riemann surface are actually topological in nature. For this reason, a proof is outlined in this section.

We also consider homology and cohomology with coefficients in an arbitrary subring $\mathbb{A}$ of $\mathbb{C}$ containing $\mathbb{Z}$. When considering homology with coefficients in $\mathbb{A}$, in order to guarantee that the natural generalization of Definition 10.6.4 (with $\mathbb{A}$ in place of $\mathbb{F}$) yields a group that coincides with the first singular homology group (see the remarks at the end of this section and the exercises for Chaps. 5 and 6), we will restrict our attention to second countable topological surfaces that admit *orientable* smooth structures (in fact, as is shown in Chap. 6, this is actually a topological condition).

There are many different standard ways of defining homology and cohomology on topological spaces, and with enough restrictions on the spaces, the resulting groups are isomorphic. Here, we consider a variant of *Čech cohomology* (see, for example, [Wa]) as considered in, for example, [Fa] (for an equivalent approach using potentials on covering spaces, see [For]). The idea is to produce a generalization of the notion of a locally exact 1-form, and to then proceed as before. The discussion in Sect. 10.5 of line integrals along continuous paths suggests that one may do this by considering the local potentials themselves, without mentioning the associated differential 1-form one gets by applying d. Throughout this section, $\mathbb{F}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$, and $\mathbb{A}$ denotes a subring of $\mathbb{C}$ that contains 1 (i.e., that contains $\mathbb{Z}$).

Definition 10.7.1 Let M be a second countable topological surface.

(a) We denote by $P^1(M, \mathbb{A})$ (P stands for "potentials") the collection of families of pairs $\{(f_i, U_i)\}_{i\in I}$ for which $\{U_i\}_{i\in I}$ is an open covering of M; for each $i \in I$, $f_i : U_i \to \mathbb{C}$ is a continuous function on U_i, with f_i real-valued if $\mathbb{A} \subset \mathbb{R}$; and the function $f_i - f_j$ is locally constant on the overlap $U_i \cap U_j$ with values in $\mathbb{A}$ for each pair of indices $i, j \in I$. Given two elements $F = \{(f_i, U_i)\}_{i\in I}$, $G = \{(g_j, V_j)\}_{j\in J} \in P^1(M, \mathbb{A})$ and a ring element $\zeta \in \mathbb{A}$, we denote by ζF and $F \dotplus G$ the elements of $P^1(M, \mathbb{A})$ given by

$$\zeta F \equiv \{(\zeta f_i, U_i)\}_{i\in I}$$

and

$$F \dotplus G \equiv \left\{(f_i\restriction_{U_i\cap V_j} + g_j\restriction_{U_i\cap V_j}, U_i \cap V_j)\right\}_{(i,j)\in I\times J}.$$

(b) Let $\sim$ be the equivalence relation in $P^1(M, \mathbb{C})$ given by

$$F = \{(f_i, U_i)\}_{i\in I} \sim G = \{(g_j, V_j)\}_{j\in J}$$

if and only if $f_i - g_j$ is locally constant on $U_i \cap V_j$ for all indices $i \in I$ and $j \in J$; that is, the union of the two families is an element of $P^1(M, \mathbb{C})$ (note that for $F, G \in P^1(M, \mathbb{A})$, we do not require that each of the complex-valued functions $f_i - g_j$ take values in $\mathbb{A}$, just that they be locally constant).

(c) We denote the corresponding quotient mapping and quotient space by

$$\Pi_{Z^1(M,\mathbb{C})} : P^1(M, \mathbb{C}) \to Z^1(M, \mathbb{C}) \equiv P^1(M, \mathbb{C})/\sim,$$

we let

$$Z^1(M, \mathbb{A}) \equiv P^1(M, \mathbb{A})/\sim = \Pi_{Z^1(M,\mathbb{C})}(P^1(M, \mathbb{A})) \subset Z^1(M, \mathbb{C})$$

and

$$\Pi_{Z^1(M,\mathbb{A})} \equiv \Pi_{Z^1(M,\mathbb{C})}\restriction_{P^1(M,\mathbb{A})} : P^1(M, \mathbb{A}) \to Z^1(M, \mathbb{A}),$$

and we call any element of $Z^1(M, \mathbb{A})$ a *Čech* 1-*form* (or a *continuous closed* 1-*form*) on M over $\mathbb{A}$ (or an *integral Čech* 1-*form* for $\mathbb{A} = \mathbb{Z}$ or a *real Čech* 1-*form* for $\mathbb{A} = \mathbb{R}$ or a *complex Čech* 1-*form* for $\mathbb{A} = \mathbb{C}$). We also equip $Z^1(M, \mathbb{A})$ with the $\mathbb{A}$-module structure with zero element $\Pi_{Z^1(M,\mathbb{A})}(\{(0, M)\})$ and multiplication (by elements of $\mathbb{A}$) and addition given by

$$\zeta \cdot \Pi_{Z^1(M,\mathbb{A})}(F) \equiv \Pi_{Z^1(M,\mathbb{A})}(\zeta F)$$

and

$$\Pi_{Z^1(M,\mathbb{A})}(F) + \Pi_{Z^1(M,\mathbb{A})}(G) \equiv \Pi_{Z^1(M,\mathbb{A})}(F \dot{+} G)$$

for all $F, G \in P^1(M, \mathbb{A})$ and $\zeta \in \mathbb{A}$ (the verification that the above give a well-defined module structure is left to the reader in Exercise 10.7.1). For $F = \{(f_i, U_i)\}_{i \in I} \in P^1(M, \mathbb{C})$ and $\theta = \Pi_{Z^1(M,\mathbb{C})}(F)$, the *support* of θ is the closed subset $\operatorname{supp}\theta$ of M with complement equal to the union of all open sets $U \subset M$ for which $f_i\restriction_{U \cap U_i}$ is locally constant for each $i \in I$ (it is easy to see that this set is independent of the choice of the representative F). Equivalently, $M \setminus \operatorname{supp}\theta$ is the largest open set U_0 for which there is a representative $F = \{(f_i, U_i)\}_{i \in I} \in P^1(M, \mathbb{C})$ (we may take $F \in P^1(M, \mathbb{R})$ if $\theta \in Z^1(M, \mathbb{R})$) with $0 \in I$ and $f_0 \equiv 0$.

(d) For any Čech 1-form θ on M and any path $\gamma : [a, b] \to M$, the *integral of θ along γ* (or the *line integral of θ along γ*) is given by

$$\int_\gamma \theta \equiv \sum_{\nu=1}^{m} \left[f_{i_\nu}(\gamma(t_\nu)) - f_{i_\nu}(\gamma(t_{\nu-1})) \right]$$

for any representative $F = \{(f_i, U_i)\}_{i \in I} \in P^1(M, \mathbb{C})$ of θ and any partition $\mathcal{P} : a = t_0 < \cdots < t_m = b$ with $\gamma([t_{\nu-1}, t_\nu]) \subset U_{i_\nu}$ for some $i_\nu \in I$ for each $\nu = 1, \ldots, m$ (the verification that this number is independent of the choice of the representative F, the partition $\mathcal{P}$, and the indices $i_1, \ldots, i_m$ is identical to the proof of Lemma 10.5.2, and the details are left to the reader in Exercise 10.7.2). For any map $\gamma : \{a\} \to M$, we set $\int_\gamma \theta \equiv 0$.

(e) A Čech 1-form θ on M over $\mathbb{A}$ is *exact* if $\theta = \Pi_{Z^1(M,\mathbb{C})}(\{(f, M)\})$ for some continuous function $f : M \to \mathbb{C}$ (which we may take to be real-valued if $\mathbb{A} \subset \mathbb{R}$ simply by replacing f with $\operatorname{Re} f$); that is, θ is represented by the singleton family $\{(f, M)\}$ (observe that $\{(f, M)\} \in P^1(M, \mathbb{A})$ and $\theta = \Pi_{Z^1(M,\mathbb{A})}(\{(f, M)\})$, provided we take f to be real-valued if $\mathbb{A} \subset \mathbb{R}$). The module of exact Čech 1-forms over $\mathbb{A}$ is denoted by $B^1(M, \mathbb{A})$.

Remarks 1. One may view the representing functions $\{(f_i, U_i)\}_{i \in I} \in P^1(M, \mathbb{A})$ for a Čech 1-form θ as generalized local potentials. The requirement that f_i be real-valued if $\mathbb{A} \subset \mathbb{R}$ is convenient for our purposes, but not absolutely necessary. Of course, it would be too restrictive to require that f_i be $\mathbb{A}$-valued (for example, any integer or rational-valued continuous function is locally constant).

2. For a Čech 1-form θ over $\mathbb{A}$, one may not be able to choose a representative $\{(f_i, U_i)\}_{i\in I} \in P^1(M, \mathbb{A})$ with $f_i \equiv 0$ and $U_i = M \setminus \operatorname{supp}\theta$ for some $i \in I$. For example, choosing a real-valued continuous function f on $\mathbb{R}^2$ with $f \equiv 0$ on $\Delta(0; 1)$ and $f \equiv 1/2$ on $\mathbb{R}^2 \setminus \Delta(0; 2)$, we get the Čech 1-form θ represented by $\{(f, \mathbb{R}^2)\} \in P^1(\mathbb{R}^2, \mathbb{Z})$. If there is a representative $\{(f_i, U_i)\}_{i\in I} \in P^1(M, \mathbb{Z})$ as above with $f_i \equiv 0$ and $U_i = M \setminus \operatorname{supp}\theta$ for some $i \in I$, then according to Lemma 10.7.2 below, there is a constant $\zeta \in \mathbb{C}$ such that $f - (f_i - \zeta)$ is integer-valued. But then, evaluating on $\mathbb{R}^2 \setminus \Delta(0; 2)$ and $\Delta(0; 1)$, we see that both $(1/2) - \zeta$ and ζ must be integers, which is impossible.

3. By definition, two elements $\{(f_i, U_i)\}_{i\in I}, \{(g_j, V_j)\}_{j\in J} \in P^1(M, \mathbb{A})$ represent the same Čech 1-form θ over $\mathbb{A}$ if $f_i - g_j$ is locally constant as a $\mathbb{C}$-valued function for all $i \in I$ and $j \in J$. Since f_i and g_j roughly correspond to local potentials for θ (in a generalized sense), it is natural that the addition of constants not affect θ, even constants not in $\mathbb{A}$. Thus, although it may seem more natural to require that $f_i - g_j$ take values in $\mathbb{A}$, such a requirement, which would introduce some complications (for example, the exact 1-forms represented by $f \equiv 1/2$ and $g \equiv 0$ would differ as integral Čech 1-forms, but not as real Čech 1-forms), is not necessary. Moreover, we have the following fact:

Lemma 10.7.2 *Given a second countable topological surface M and two representatives $F = \{(f_i, U_i)\}_{i\in I}, G = \{(g_j, V_j)\}_{j\in J} \in P^1(M, \mathbb{A})$ for a Čech 1-form $\theta = \Pi_{Z^1(M,\mathbb{A})}(F) = \Pi_{Z^1(M,\mathbb{A})}(G)$ over $\mathbb{A}$, there exists a constant $\zeta \in \mathbb{C}$ (with $\zeta \in \mathbb{R}$ if $\mathbb{A} \subset \mathbb{R}$) such that the representative $H \equiv \{(g_j - \zeta, V_j)\}_{j\in J}$ of θ is in $P^1(M, \mathbb{A})$ and the function $f_i - (g_j - \zeta)$ is locally constant with values in $\mathbb{A}$ on $U_i \cap V_j$ for all $i \in I$ and $j \in J$.*

Proof Fix a point $x_0 \in M$ and indices $i_0 \in I$ and $j_0 \in J$ with $x_0 \in U_{i_0} \cap V_{j_0}$, and set $\zeta \equiv g_{j_0}(x_0) - f_{i_0}(x_0)$. The set Q of points $x \in M$ for which $f_i(x) - (g_j(x) - \zeta) \in \mathbb{A}$ for some (and therefore for every) choice of $i \in I$ and $j \in J$ with $x \in U_i \cap V_j$ is then nonempty. Moreover, given a point $x \in M$ and indices $i \in I$ and $j \in J$ with $x \in U_i \cap V_j$, the function $f_i - (g_j - \zeta)$ will be constant on some neighborhood W of x in $U_i \cap V_j$. It follows that Q is both open and closed in M, and hence $Q = M$. □

Analogues of the arguments in Sect. 10.5 give the following:

Theorem 10.7.3 *For any Čech 1-form θ on a second countable topological surface M, we have the following*:

(a) *For any path $\gamma : [a, b] \to M$:*
   (i) *We have $\int_{\gamma^-} \theta = -\int_\gamma \theta$.*
   (ii) *If $a < c < b$, then $\int_\gamma \theta = \int_{\gamma\restriction_{[a,c]}} \theta + \int_{\gamma\restriction_{[c,d]}} \theta$.*
   (iii) *If $a \le c \le d \le b$, $J = \{t \in \mathbb{R} \mid c \le t \le d\}$, $s_0, s_1, s_2, \ldots, s_m$ are points in $[a, b]$ with $s_0 = c$ and $s_m = d$, and $F = \{(f_i, U_i)\}_{i\in I} \in P^1(M, \mathbb{A})$ is a*

representative of θ *such that for each* $\nu = 1, \dots, m$,

$$\{\gamma(t) \mid s_{\nu-1} \leq t \leq s_\nu \text{ or } s_{\nu-1} \geq t \geq s_\nu\} \subset U_{i_\nu}$$

for some $i_\nu \in I$, *then* $\int_{\gamma\restriction_J} \theta = \sum_{\nu=1}^m [f_{i_\nu}(\gamma(s_\nu)) - f_{i_\nu}(\gamma(s_{\nu-1}))]$.

(iv) Invariance under continuous reparametrization. *For any continuous mapping* $\varphi\colon [r,s] \to [a,b]$ *with* $c \equiv \varphi(r) \leq d \equiv \varphi(s)$ *and for* $J \equiv \{t \in \mathbb{R} \mid c \leq t \leq d\}$, *we have* $\int_{\gamma\circ\varphi} \theta = \int_{\gamma\restriction_J} \theta$.

(b) *The following are equivalent*:

(i) *The Čech* 1*-form* θ *is exact.*

(ii) $\int_\gamma \theta = 0$ *for every loop* γ *in* M.

(iii) *The line integrals are independent of path*; *that is*, $\int_{\gamma_0} \theta = \int_{\gamma_1} \theta$ *for every pair of paths* γ_0 *and* γ_1 *with the same endpoints.*

(c) *If* γ_0 *and* γ_1 *are two path homotopic paths in* M, *then* $\int_{\gamma_0} \theta = \int_{\gamma_1} \theta$.

(d) *If* M *is simply connected, then* θ *is exact.*

(e) *Assuming that* $\theta \in Z^1(M, \mathbb{R})$ *if* $\mathbb{A} \subset \mathbb{R}$, *given a point* $x_0 \in M$, *we have* $\theta \in Z^1(M, \mathbb{A})$ *if and only if* $\int_\gamma \theta \in \mathbb{A}$ *for every loop* γ *based at* x_0. *Moreover, if* $\theta \in Z^1(M, \mathbb{A})$, *then the mapping* $[\gamma] \mapsto \int_\gamma \theta$ *determines a well-defined group homomorphism* $\pi_1(X, x_0) \to (\mathbb{A}, +)$.

(f) *If* $\theta \in Z^1(M, \mathbb{A})$ *and* $\mathcal{B}$ *is a basis for the topology on* M, *then* θ *has a representative* $F = \{(f_i, U_i)\}_{i\in I} \in P^1(M, \mathbb{A})$ *such that* $\{U_i\}_{i\in I}$ *is a countable locally finite family of elements of* $\mathcal{B}$ (*which covers* M).

Proof We prove part (e) and leave the proofs of the remaining parts to the reader (see Exercise 10.7.3). For this, let us fix a point $x_0 \in M$ and an element $F = \{(f_i, U_i)\}_{i\in I} \in P^1(M, \mathbb{C})$ representing θ, with $F \in P^1(M, \mathbb{R})$ if $\mathbb{A} \subset \mathbb{R}$. If we may choose F to be in $P^1(M, \mathbb{A})$, then given a loop γ based at x_0 and a partition $\mathcal{P}\colon a = t_0 < \cdots < t_m = b$ with $\gamma([t_{\nu-1}, t_\nu]) \subset U_{i_\nu}$ for some $i_\nu \in I$ for each $\nu = 1, \dots, m$, we have

$$\begin{aligned}\int_\gamma \theta &= \sum_{\nu=1}^m \big[f_{i_\nu}(\gamma(t_\nu)) - f_{i_\nu}(\gamma(t_{\nu-1}))\big] \\ &= [f_{i_m}(x_0) - f_{i_1}(x_0)] + \sum_{\nu=1}^{m-1} \big[-f_{i_{\nu+1}}(\gamma(t_\nu)) + f_{i_\nu}(\gamma(t_\nu))\big] \in \mathbb{A}.\end{aligned}$$

For the proof of the converse, for each $i \in I$, let us assume that U_i is nonempty and connected (as we may, for example, by part (f)), let us fix a point $p_i \in U_i$ and a path γ_i from x_0 to p_i, and let us set $g_i \equiv f_i - f_i(p_i) + \int_{\gamma_i} \theta$. Then $F \sim G \equiv \{(g_i, U_i)\}_{i\in I}$. Moreover, given indices $i, j \in I$ and a point $x \in U_i \cap U_j$, we may choose a path α in U_i from p_i to p and β in U_j from p_j to p. We then have

$$g_i(x) - g_j(x) = \left[f_i(x) - f_i(p_i) + \int_{\gamma_i} \theta\right] - \left[f_j(x) - f_j(p_j) + \int_{\gamma_j} \theta\right] = \int_\eta \theta,$$

where $\eta \equiv \gamma_i * \alpha * \beta^- * \gamma_j^-$. Hence, if the integral of θ along each loop based at x_0 lies in $\mathbb{A}$, then $G \in P^1(M, \mathbb{A})$ and therefore $\theta \in Z^1(M, \mathbb{A})$. That the mapping $[\gamma] \mapsto \int_\gamma \theta$ determines a well-defined group homomorphism $\pi_1(X, x_0) \to (\mathbb{A}, +)$ then follows from parts (a) and (c). $\square$

On a second countable $\mathcal{C}^\infty$ surface, every locally exact 1-form is a Čech 1-form. In fact, we have the following:

Lemma 10.7.4 *Let M be a second countable $\mathcal{C}^\infty$ surface. To any locally exact 1-form θ over $\mathbb{F}$ (i.e., θ is a real 1-form if $\mathbb{F} = \mathbb{R}$) on M, we may associate a unique Čech 1-form $\hat{\theta} = \Pi_{Z^1(M,\mathbb{F})}(F) \in Z^1(M, \mathbb{F})$, where $F \equiv \{(f_i, U_i)\}_{i \in I} \in P^1(M, \mathbb{F})$ for an arbitrary choice of an open covering $\{U_i\}_{i \in I}$ of M and functions $\{f_i\}_{i \in I}$ with $f_i \in \mathcal{C}^1(U_i, \mathbb{F})$ and $df_i = \theta \restriction_{U_i}$ for each $i \in I$ (i.e., θ is independent of the choice of the collection F of local potentials for θ). Moreover, we have the following:*

(i) *The map $\theta \mapsto \hat{\theta}$ determines a linear isomorphism of $Z^1_{\text{deR}}(M, \mathbb{F})$ onto the vector subspace of $Z^1(M, \mathbb{F})$ consisting of all Čech 1-forms represented by a family of $\mathbb{F}$-valued local $\mathcal{C}^\infty$ functions.*
(ii) *For every path γ in M, we have $\int_\gamma \theta = \int_\gamma \hat{\theta}$ for every choice of θ.*
(iii) *The given 1-form θ is exact (as a 1-form) if and only if $\hat{\theta}$ is exact (as a Čech 1-form), and if this is the case, then for some $\mathbb{F}$-valued function $f \in \mathcal{C}^1(M)$, we have $\theta = df$ and $\hat{\theta} = \Pi_{Z^1(M,\mathbb{F})}(\{(f, M)\})$.*
(iv) *The given 1-form θ has compact support if and only if $\hat{\theta}$ has compact support.*

The proof is left to the reader (see Exercise 10.7.4). On a second countable $\mathcal{C}^\infty$ surface, we will identify any locally exact 1-form θ with its corresponding Čech 1-form, and we will give the Čech 1-form the same name θ (instead of $\hat{\theta}$).

Definition 10.7.5 For a second countable topological surface M, the *first cohomology group with coefficients in* $\mathbb{A}$ is the quotient $\mathbb{A}$-module

$$H^1(M, \mathbb{A}) \equiv Z^1(M, \mathbb{A})/B^1(M, \mathbb{A}) = Z^1(M, \mathbb{A})/(B^1(M, \mathbb{C}) \cap Z^1(M, \mathbb{A})).$$

For each Čech 1-form $\theta \in Z^1(M, \mathbb{A})$, we call the corresponding equivalence class in $H^1(M, \mathbb{A})$ the *cohomology class* of θ, and we denote this class by $[\theta]_{H^1(M,\mathbb{A})}$, by $[\theta]_{H^1}$, or simply by $[\theta]$. We also say that two Čech 1-forms are *cohomologous* if they represent the same cohomology class (i.e., if their difference is exact).

Given a Čech 1-form $\theta = \Pi_{Z^1(M,\mathbb{C})}(F)$ with $F = \{(f_i, U_i)\}_{i \in I} \in P^1(M, \mathbb{C})$ on a second countable topological surface M, we get well-defined Čech 1-forms $\operatorname{Re}\theta$ and $\operatorname{Im}\theta$ in $Z^1(M, \mathbb{R})$ represented by $\operatorname{Re} F \equiv \{(\operatorname{Re} f_i, U_i)\}_{i \in I} \in P^1(M, \mathbb{R})$ and $\operatorname{Im} F \equiv \{(\operatorname{Im} f_i, U_i)\}_{i \in I} \in P^1(M, \mathbb{R})$, respectively, and we have $\theta = \operatorname{Re}\theta + i \operatorname{Im}\theta$. It follows that we have a real direct sum decomposition $Z^1(M, \mathbb{C}) = Z^1(M, \mathbb{R}) \oplus i Z^1(M, \mathbb{R})$, and that we may identify $Z^1(M, \mathbb{C})$ with the complexification $(Z^1(M, \mathbb{R}))_{\mathbb{C}}$ (see Exercise 10.7.5). This also holds at the level of cohomology:

Proposition 10.7.6 *For a second countable topological surface M, the mapping $H^1(M,\mathbb{A}) \to H^1(M,\mathbb{C})$ given by $[\theta]_{H^1(M,\mathbb{A})} \mapsto [\theta]_{H^1(M,\mathbb{C})}$ is a well-defined injective $\mathbb{A}$-module homomorphism, and identifying $H^1(M,\mathbb{A})$ with its image in $H^1(M,\mathbb{C})$, we have*

$$H^1(M,\mathbb{A}) = \left\{ [\theta]_{H^1(M,\mathbb{C})} \in H^1(M,\mathbb{C}) \,\middle|\, \int_\gamma \theta \in \mathbb{A} \textit{ for every loop } \gamma \textit{ in } M \right\}.$$

Moreover, we have the real direct sum decomposition

$$H^1(M,\mathbb{C}) = H^1(M,\mathbb{R}) \oplus i H^1(M,\mathbb{R})$$

given by $[\theta]_{H^1(M,\mathbb{C})} = [\operatorname{Re}\theta]_{H^1(M,\mathbb{R})} + i[\operatorname{Im}\theta]_{H^1(M,\mathbb{R})}$ for each $[\theta]_{H^1(M,\mathbb{C})} \in H^1(M,\mathbb{C})$ (thus we may identify $H^1(M,\mathbb{C})$ with the complexification $(H^1(M,\mathbb{R}))_{\mathbb{C}}$).

Proof The claims regarding the map $H^1(M,\mathbb{A}) \hookrightarrow H^1(M,\mathbb{C})$ follow from the definitions and Theorem 10.7.3 (observe that if $\mathbb{A} \subset \mathbb{R}$, $[\theta] \in H^1(M,\mathbb{A})$, and $\int_\gamma \theta \in \mathbb{A}$ for every loop γ in M, then $\int_\gamma \operatorname{Im}\theta = 0$ for every loop γ, and hence $[\theta]$ has a real representative), so it remains to show that each cohomology class ξ in $H^1(M,\mathbb{R}) \cap i H^1(M,\mathbb{R})$ is trivial. We have $\xi = [\theta]_{H^1(M,\mathbb{R})} = i[\eta]_{H^1(M,\mathbb{R})}$, where $\theta = \Pi_{Z^1(M,\mathbb{R})}(F)$ and $\eta = \Pi_{Z^1(M,\mathbb{R})}(G)$ for some choice of $F = \{(f_j, U_j)\}_{j\in J}, G = \{(g_n, V_n)\}_{n\in N} \in P^1(M,\mathbb{R})$ (in particular, the functions $\{f_j\}$ and $\{g_n\}$ are real-valued). Consequently, there exists a continuous function u on M such that for each $j \in J$ and $n \in N$, the function $f_j - i g_n - u$ is locally constant on $U_j \cap V_n$. It follows that the functions $f_j - \operatorname{Re} u$ and $g_n - \operatorname{Im} u$ are locally constant on U_j and V_n, respectively, and hence that $\xi = 0$. □

For a second countable surface, although the first cohomology with coefficients in $\mathbb{F}$ is clearly a topological object, for any $\mathcal{C}^\infty$ structure on the surface, it is canonically isomorphic to the first de Rham cohomology.

Theorem 10.7.7 (de Rham) *Let M be a second countable $\mathcal{C}^\infty$ surface. Then, for every Čech 1-form α over $\mathbb{F}$ on M, there exists a closed $\mathcal{C}^\infty$ 1-form β over $\mathbb{F}$ on M such that $\alpha - \beta$ is exact. Consequently, we have a well-defined (surjective) linear isomorphism $H^1_{\text{deR}}(M,\mathbb{F}) \xrightarrow{\cong} H^1(M,\mathbb{F})$ given by $[\beta]_{\text{deR}} \mapsto [\beta]_{H^1}$, and under this isomorphism, assuming that $\mathbb{A} \subset \mathbb{F}$, we get an $\mathbb{A}$-module isomorphism*

$$\left\{ [\theta]_{\text{deR}} \in H^1_{\text{deR}}(M,\mathbb{F}) \,\middle|\, \int_\gamma \theta \in \mathbb{A} \textit{ for every loop } \gamma \textit{ in } M \right\} \xrightarrow{\cong} H^1(M,\mathbb{A}).$$

Moreover, if α is a Čech 1-form with compact support, then $\alpha - \beta$ is exact for some closed $\mathcal{C}^\infty$ 1-form β over $\mathbb{F}$ with compact support.

Proof Suppose α is a Čech 1-form over $\mathbb{F}$ on M. Applying part (f) of Theorem 10.7.3, we get a representing family $F = \{(f_i, U_i)\}_{i\in I}$ for α such that $U_i \Subset M$

for each $i \in I$ and $\{U_i\}_{i\in I}$ is a countable locally finite open covering of M. We may also choose F so that $f_i \equiv 0$ whenever $i \in I$ and $U_i \cap (\operatorname{supp} \alpha) = \emptyset$. Applying Theorem 9.3.7, we get a $\mathcal{C}^\infty$ partition of unity $\{\lambda_i\}_{i\in I}$ on M such that $\operatorname{supp} \lambda_i \subset U_i$ for each $i \in I$. Thus we get a continuous function $f \equiv \sum_{i\in I} \lambda_i f_i : M \to \mathbb{F}$. For each $i \in I$, we have, on U_i,

$$f_i - f = \sum_{j\in I} \lambda_j (f_i - f_j) \in \mathcal{C}^\infty(U_i),$$

since $f_i - f_j$ is locally constant on $U_i \cap U_j$ for each $j \in J$. Therefore, by Lemma 10.7.4, the Čech 1-form $\beta \equiv \alpha - \Pi_{Z^1(M,\mathbb{F})}(\{(f, X)\})$, which is cohomologous to α, is actually a closed $\mathcal{C}^\infty$ 1-form on M (with compact support if α has compact support). Lemma 10.7.4 and Theorem 10.7.3 now give the remaining claims. □

For a group G and an Abelian group A, the set of group homomorphisms from G to A, together with the natural zero element and addition, is an Abelian group denoted by $\operatorname{Hom}(G, A)$ (similarly, for modules $\mathcal{M}$ and $\mathcal{N}$ over a ring R, the set of module homomorphisms from $\mathcal{M}$ to $\mathcal{N}$ is an R-module, which is denoted by $\operatorname{Hom}(\mathcal{M}, \mathcal{N})$). In fact, $\operatorname{Hom}(G, \mathbb{A})$ is an $\mathbb{A}$-module. The point of view provided by the Čech 1-forms leads to the following characterization of the first cohomology group:

Theorem 10.7.8 (de Rham) *Let M be a second countable topological surface, and let $x_0 \in M$. Then the mapping $H^1(M, \mathbb{A}) \to \operatorname{Hom}(\pi_1(M, x_0), \mathbb{A})$ given by $[\theta]_{H^1} \mapsto \rho$, where $\rho: [\gamma]_{x_0} \mapsto \int_\gamma \theta$, is a well-defined module isomorphism. Consequently, for M a second countable smooth surface, the analogous mapping $[\theta]_{\mathrm{deR}} \mapsto \rho$ yields a vector space isomorphism $H^1_{\mathrm{deR}}(M, \mathbb{F}) \to \operatorname{Hom}(\pi_1(M, x_0), \mathbb{F})$.*

Proof Theorem 10.7.3 implies that the mapping is a well-defined injective module homomorphism, so it remains to verify surjectivity. For this, let us fix a locally finite covering $\{U_i\}_{i\in I}$ of M by simply connected open sets and a continuous partition of unity $\{\eta_i\}_{i\in I}$ with $\operatorname{supp} \eta_i \subset U_i$ for each $i \in I$, and for each index $i \in I$, let us fix a point $p_i \in U_i$ and a path γ_i from x_0 to p_i. Given a group homomorphism $\rho: \pi_1(M, x_0) \to \mathbb{A}$, let us set, for each pair of indices $i, j \in I$ and each point $p \in U_i \cap U_j$,

$$f_{ij}(p) \equiv \rho([\gamma_j * \beta * \alpha^- * \gamma_i^-]_{x_0}) \in \mathbb{A},$$

where α is an arbitrary path in U_i from p_i to p, and β is an arbitrary path in U_j from p_j to p. The functions $\{f_{ij}\}$ are well defined because U_i is simply connected for each $i \in I$, and since ρ is a group homomorphism, they satisfy the *cocycle relation*

$$f_{ij} + f_{jk} = f_{ik} \text{ on } U_i \cap U_j \cap U_k \quad \forall i, j, k \in I.$$

Moreover, as is easy to verify, the functions are locally constant. For each $j \in I$, we get a continuous function $f_j \equiv \sum_{i\in I} \eta_i f_{ij}$ on U_j (here, we have extended the function $\eta_i f_{ij}$ by 0 from a function on $U_i \cap U_j$ to a continuous function on U_j for each $i \in I$), and we have

$$f_j - f_i = \sum_{k\in I} \eta_k(f_{kj} - f_{ki}) = \sum_{k\in I} \eta_k f_{ij} = f_{ij} \colon U_i \cap U_j \to \mathbb{A} \quad \forall i, j \in I.$$

Moreover, f_i is real-valued for each $i \in I$ if $\mathbb{A} \subset \mathbb{R}$. Thus the family $F \equiv \{(f_i, U_i)\}_{i\in I}$ is an element of $P^1(M, \mathbb{A})$, and hence F determines a Čech 1-form $\theta \equiv \Pi_{Z^1(M,\mathbb{A})}(F)$.

Given a loop λ based at x_0, we may fix a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that for each $\nu = 1, \ldots, m$, $\lambda([t_{\nu-1}, t_\nu]) \subset U_{i_\nu}$ for some $i_\nu \in I$. In particular, for each $\nu = 1, \ldots, m$, we may fix paths β_ν and α_ν in U_{i_ν} from p_{i_ν} to $\lambda(t_{\nu-1})$ and $\lambda(t_\nu)$, respectively. Thus

$$\begin{aligned}\lambda &\sim \beta_1^- * \alpha_1 * \beta_2^- * \alpha_2 * \cdots * \beta_m^- * \alpha_m \\ &\sim \beta_1^- * \gamma_{i_1}^- * (\gamma_{i_1} * \alpha_1 * \beta_2^- * \gamma_{i_2}^-) * \cdots \\ &\quad * (\gamma_{i_{m-1}} * \alpha_{m-1} * \beta_m^- * \gamma_{i_m}^-) * \gamma_{i_m} * \alpha_m,\end{aligned}$$

and hence

$$\begin{aligned}\rho([\lambda]) &= \rho([\beta_1^- * \gamma_{i_1}^-]) + \rho([\gamma_{i_1} * \alpha_1 * \beta_2^- * \gamma_{i_2}^-]) \\ &\quad + \cdots + \rho([\gamma_{i_{m-1}} * \alpha_{m-1} * \beta_m^- * \gamma_{i_m}^-]) + \rho([\gamma_{i_m} * \alpha_m]) \\ &= \rho([\gamma_{i_m} * \alpha_m * \beta_1^- * \gamma_{i_1}^-]) + \rho([\gamma_{i_1} * \alpha_1 * \beta_2^- * \gamma_{i_2}^-]) \\ &\quad + \cdots + \rho([\gamma_{i_{m-1}} * \alpha_{m-1} * \beta_m^- * \gamma_{i_m}^-]) \\ &= (f_{i_m}(\lambda(t_m)) - f_{i_1}(\lambda(t_0))) + (f_{i_1}(\lambda(t_1)) - f_{i_2}(\lambda(t_1))) \\ &\quad + \cdots + (f_{i_{m-1}}(\lambda(t_{m-1})) - f_{i_m}(\lambda(t_{m-1}))) \\ &= \sum_{\nu=1}^{m} (f_{i_\nu}(\lambda(t_\nu)) - f_{i_\nu}(\lambda(t_{\nu-1}))) = \int_\lambda \theta.\end{aligned}$$

Thus ρ is the image of $[\theta]_{H^1}$, and we have surjectivity.

The claim concerning a second countable $\mathcal{C}^\infty$ surface follows from the above together with Theorem 10.7.7. □

The definitions of the homology groups with coefficients in $\mathbb{F}$ for a second countable topological surface are analogous to the definitions in the $\mathcal{C}^\infty$ case provided in Sect. 10.6. However, in this section, we will also consider homology over the ring $\mathbb{A}$, and when doing so, we will add the condition that the surface admit an orientable smooth surface structure.

Definition 10.7.9 (Cf. Definition 10.6.4) Let M be a second countable topological surface.

(a) For every 1-chain $\xi = \sum_{i=1}^{m} a_i \cdot \gamma_i \in C_1(M, \mathbb{C})$ (see Definition 10.6.3) and every Čech 1-form θ on M, the *integral* (or *line integral*) of θ along ξ is given by

$$\int_{\xi} \theta \equiv \sum_{i=1}^{m} a_i \int_{\gamma_i} \theta$$

(which, as one may easily check, is well defined).

(b) We denote by $B_1(M, \mathbb{F})$ the subspace of $Z_1(M, \mathbb{F})$ consisting of all 1-cycles ξ for which $\int_{\xi} \theta = 0$ for every Čech 1-form θ on M (note that the definition yields the same space regardless of whether we require the Čech 1-forms θ to be real or complex). The *first homology group of M with coefficients in* $\mathbb{F}$ is the quotient vector space

$$H_1(M, \mathbb{F}) \equiv Z_1(M, \mathbb{F})/B_1(M, \mathbb{F}).$$

We call the equivalence class in $H_1(M, \mathbb{F})$ represented by $\xi \in Z_1(M, \mathbb{F})$ the *homology class* of ξ, and we denote this class by $[\xi]_{H_1(M,\mathbb{F})}$, by $[\xi]_{H_1}$, or simply by $[\xi]$. Two 1-cycles $\xi, \zeta \in Z_1(M, \mathbb{F})$ are said to be *homologous* if they represent the same homology class; that is, if $\int_{\xi} \theta = \int_{\zeta} \theta$ for every Čech 1-form θ on M.

(c) Assuming that M admits an orientable 2-dimensional $\mathcal{C}^{\infty}$ structure, we denote by $B_1(M, \mathbb{A})$ the $\mathbb{A}$-submodule of $Z_1(M, \mathbb{A})$ consisting of all 1-cycles ξ for which $\int_{\xi} \theta = 0$ for every Čech 1-form θ on M (again, the definition yields the same module regardless of whether we require the Čech 1-forms θ to be real or complex). The *first homology group of M with coefficients in* $\mathbb{A}$ is the quotient $\mathbb{A}$-module

$$H_1(M, \mathbb{A}) \equiv Z_1(M, \mathbb{A})/B_1(M, \mathbb{A}).$$

The equivalence class in $H_1(M, \mathbb{A})$ represented by $\xi \in Z_1(M, \mathbb{A})$ is the *homology class* of ξ, denoted by $[\xi]_{H_1(M,\mathbb{A})}$, by $[\xi]_{H_1}$, or simply by $[\xi]$. Two 1-cycles $\xi, \zeta \in Z_1(M, \mathbb{A})$ are *homologous* if they represent the same homology class.

Remark Although we defined $H_1(M, \mathbb{A})$ only for M admitting an orientable $\mathcal{C}^{\infty}$ surface structure, this module is clearly a topological object (in particular, for most purposes, we may as well consider M to be an orientable $\mathcal{C}^{\infty}$ surface in this case). The extra condition is included in order to make $H_1(M, \mathbb{A})$ coincide with the first singular homology group (see the remarks at the end of this section and the exercises for Chaps. 5 and 6). In Chap. 6, it is proved that every second countable topological surface admits a smooth structure, and that orientability is actually a topological condition.

For a second countable topological surface M and for subrings $\mathbb{A}_1$ and $\mathbb{A}_2$ of $\mathbb{C}$ with $\mathbb{Z} \subset \mathbb{A}_1 \subset \mathbb{A}_2$, we have $Z_1(M, \mathbb{A}_1) \subset Z_1(M, \mathbb{A}_2)$. We also have the real direct sum decomposition

$$Z_1(M, \mathbb{C}) = Z_1(M, \mathbb{R}) \oplus i Z_1(M, \mathbb{R}) = (Z_1(M, \mathbb{R}))_{\mathbb{C}}$$

given by $\xi = \operatorname{Re}\xi + i \operatorname{Im}\xi$ for each $\xi \in Z_1(M, \mathbb{C})$, where for any q-chain $\eta = \sum_{j=1}^m a_j \lambda_j$, $\operatorname{Re}\eta \equiv \sum_j (\operatorname{Re} a_j)\lambda_j$ and $\operatorname{Im}\eta \equiv \sum_j (\operatorname{Im} a_j)\lambda_j$. According to the following fact, the proof of which is left to the reader (see Exercise 10.7.6), these inclusions also hold at the level of homology:

Proposition 10.7.10 (Cf. Proposition 10.6.5) *For a second countable topological surface M, the real linear map $H_1(M, \mathbb{R}) \to H_1(M, \mathbb{C})$ given by $[\xi]_{H_1(M,\mathbb{R})} \mapsto [\xi]_{H_1(M,\mathbb{C})}$ is well defined and injective; and identifying $H_1(M, \mathbb{R})$ with its image in $H_1(M, \mathbb{C})$, we have the real direct sum decomposition $H_1(M, \mathbb{C}) = H_1(M, \mathbb{R}) \oplus i H_1(M, \mathbb{R})$ given by*

$$[\xi]_{H_1(M,\mathbb{C})} = [\operatorname{Re}\xi]_{H_1(M,\mathbb{R})} + i[\operatorname{Im}\xi]_{H_1(M,\mathbb{R})} \quad \forall [\xi]_{H_1} \in H_1(M, \mathbb{C})$$

(*thus we may identify $H_1(M, \mathbb{C})$ with the complexification $(H_1(M, \mathbb{R}))_{\mathbb{C}}$*). *Furthermore, for M an orientable second countable smooth surface, the $\mathbb{A}$-module homomorphism $H_1(M, \mathbb{A}) \to H_1(M, \mathbb{C})$ given by $[\xi]_{H_1(M,\mathbb{A})} \mapsto [\xi]_{H_1(M,\mathbb{C})}$ is well defined and injective. In particular, $H_1(M, \mathbb{A})$ is torsion-free, and we may identify $H_1(M, \mathbb{A})$ with its image in $H_1(M, \mathbb{C})$.*

Recall that an element g of a group is called a *torsion* element if g is of finite order (i.e., $g^m = e$ for some positive integer m). A group in which only the identity is a torsion element is said to be *torsion-free*.

Also as in the $\mathcal{C}^\infty$ case, one may often reduce questions regarding homology to questions regarding integration along loops by applying Lemma 10.7.11 and Proposition 10.7.12 below, the proofs of which are left to the reader (see Exercises 10.7.7 and 10.7.8):

Lemma 10.7.11 (Cf. Lemma 10.6.6) *Let M be a second countable topological surface, and let $x_0 \in M$. Then, for every 1-cycle $\xi \in Z_1(M, \mathbb{A})$, there exists a 1-cycle $\eta \in Z_1(M, \mathbb{A})$ such that η is a linear combination over $\mathbb{A}$ of loops based at x_0 and $\int_\xi \theta = \int_\eta \theta$ for every $\theta \in Z^1(M, \mathbb{C})$ (that is, ξ and η are homologous as 1-cycles over $\mathbb{C}$). In particular, $\int_\xi \theta \in \mathbb{A}$ for every $\theta \in Z^1(M, \mathbb{A})$. Moreover, for ξ an integral 1-cycle (that is, for $\mathbb{A} = \mathbb{Z}$), one may choose η itself to be a loop based at x_0.*

Proposition 10.7.12 (Cf. Proposition 10.6.7) *Let M be a second countable topological surface, and let $x_0 \in M$. Then:*

(a) *The mapping $[\alpha]_{x_0} = [\alpha]_{\pi_1(M,x_0)} \mapsto [\alpha]_{H_1(M,\mathbb{F})}$ gives a well-defined group homomorphism $\pi_1(M, x_0) \to H_1(M, \mathbb{F})$. For M an orientable second countable $\mathcal{C}^\infty$ surface, the mapping $[\alpha]_{x_0} \mapsto [\alpha]_{H_1(M,\mathbb{Z})}$ gives a well-defined surjective group homomorphism $\pi_1(M, x_0) \to H_1(M, \mathbb{Z})$, and in particular, the image of any set of generators of the group $\pi_1(M, x_0)$ generates $H_1(M, \mathbb{Z})$ as a group (equivalently, as a $\mathbb{Z}$-module) and $H_1(M, \mathbb{Z})$ is countable.*

(b) *The image of the fundamental group satisfies*

$$\operatorname{Span}_{\mathbb{F}}\left[\operatorname{im}(\pi_1(M, x_0) \to H_1(M, \mathbb{F}))\right] = H_1(M, \mathbb{F}),$$

and consequently, the image of any set of generators of the group $\pi_1(M, x_0)$ *spans* $H_1(M, \mathbb{F})$ *as a vector space and* $H_1(M, \mathbb{F})$ *has a countable basis. In particular, if* $\pi_1(M)$ *is finitely generated (for example, if* M *is compact), then* $H_1(M, \mathbb{F})$ *is finite-dimensional. For* M *an orientable second countable* $\mathcal{C}^\infty$ *surface, we have*

$$\operatorname{Span}_{\mathbb{A}}\big[\operatorname{im}(\pi_1(M, x_0) \to H_1(M, \mathbb{A}))\big] = H_1(M, \mathbb{A}),$$

and consequently, the image of any set of generators of the group $\pi_1(M, x_0)$ *generates* $H_1(M, \mathbb{A})$ *as an* $\mathbb{A}$*-module and* $H_1(M, \mathbb{A})$ *is countably generated. In particular, if* $\pi_1(M)$ *is finitely generated, then* $H_1(M, \mathbb{A})$ *is finitely generated.*

(c) $\int_\xi \theta = 0$ *for every exact Čech* 1*-form* θ *on* M *and every* 1*-cycle* $\xi \in Z_1(M, \mathbb{C})$.

Definition 10.7.13 Let M be a second countable topological surface. We call the bilinear pairing $H^1(M, \mathbb{F}) \times H_1(M, \mathbb{F}) \to \mathbb{F}$ (or $H_1(M, \mathbb{F}) \times H^1(M, \mathbb{F}) \to \mathbb{F}$) given by

$$([\theta]_{H^1}, [\xi]_{H_1})_{\mathrm{deR}} = ([\xi]_{H_1}, [\theta]_{H^1})_{\mathrm{deR}} \equiv \int_\xi \theta$$

for every $[\theta]_{H^1} \in H^1(M, \mathbb{F})$ and $[\xi]_{H_1} \in H_1(M, \mathbb{F})$ the *de Rham pairing over* $\mathbb{F}$ on M (Definition 10.7.9 and Proposition 10.7.12 imply that this pairing is well defined). If M also admits an orientable 2-dimensional $\mathcal{C}^\infty$ structure, then we call the bilinear pairing of $\mathbb{A}$-modules $H_1(M, \mathbb{A}) \times H^1(M, \mathbb{A}) \to \mathbb{A}$ given by the restriction of the de Rham pairing over $\mathbb{C}$ the *de Rham pairing over* $\mathbb{A}$.

Remark Theorem 10.7.7 and Proposition 10.7.12 imply that although, in the $\mathcal{C}^\infty$ case, $H_1(M, \mathbb{F})$ was defined (in Definition 10.6.4) using the $\mathcal{C}^\infty$ structure, the topological definition (Definition 10.7.9) actually yields the same vector space (the spaces are literally the same, not just canonically isomorphic). Moreover, under the identification given by the isomorphism $H^1_{\mathrm{deR}}(M, \mathbb{F}) \cong H^1(M, \mathbb{F})$, the versions of the de Rham pairing $(\cdot, \cdot)_{\mathrm{deR}}$ in Definitions 10.6.8 and 10.7.13 coincide.

A continuous mapping $\Phi\colon M \to N$ of second countable topological surfaces M and N induces a *pullback* map $\Phi^*\colon P^1(N, \mathbb{A}) \to P^1(M, \mathbb{A})$ given by

$$\{(f_i, U_i)\}_{i \in I} \mapsto \{(f_i \circ \Phi, \Phi^{-1}(U_i))\}_{i \in I}.$$

This map determines a well-defined *pullback* map $\Phi^*\colon Z^1(N, \mathbb{A}) \to Z^1(M, \mathbb{A})$ (which we give the same name) given by $\Pi_{Z^1(N,\mathbb{A})}(F) \mapsto \Pi_{Z^1(M,\mathbb{A})}(\Phi^* F)$, and this map is an $\mathbb{A}$-module homomorphism. Moreover, $\Phi^* B^1(N, \mathbb{A}) \subset B^1(M, \mathbb{A})$. For $r = 0, 1$, we denote by $\Phi_*\colon C_r(M, \mathbb{A}) \to C_r(N, \mathbb{A})$ the induced *pushforward* module homomorphism given by

$$\xi = \sum_{i=1}^m a_i \gamma_i \mapsto \sum_{i=1}^m a_i \Phi(\gamma_i),$$

where $a_1, \ldots, a_m \in \mathbb{A}$ and $\gamma_1, \ldots, \gamma_m$ are paths in M if $q = 1$, points in M if $q = 0$. As is easy to check, we have $\partial \circ \Phi_* = \Phi_* \circ \partial$, and hence $\Phi_*(Z_1(M, \mathbb{A})) \subset Z_1(N, \mathbb{A})$. It is also easy to check that if θ is a Čech 1-form on N, then $\int_\xi \Phi^*\theta = \int_{\Phi_*(\xi)} \theta$ for every 1-chain $\xi \in C_1(M, \mathbb{A})$. In particular, $\Phi_* B_1(M, \mathbb{A}) \subset B_1(N, \mathbb{A})$, assuming that M and N admit orientable $\mathcal{C}^\infty$ surface structures unless $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$. If Φ is a homeomorphism, then Φ_* and Φ^* are isomorphisms and $(\Phi_*)^{-1} = (\Phi^{-1})_*$ and $(\Phi^*)^{-1} = (\Phi^{-1})^*$.

The above mappings induce module homomorphisms of the corresponding cohomology and homology spaces that have the expected functoriality properties.

Proposition 10.7.14 *For any continuous mapping $\Phi\colon M \to N$ of second countable topological surfaces M and N, we have the following*:

(a) *The mapping $H_1(M, \mathbb{A}) \to H_1(N, \mathbb{A})$ given by $[\xi]_{H_1(M,\mathbb{A})} \mapsto [\Phi_*\xi]_{H_1(N,\mathbb{A})}$ is a well-defined module homomorphism, and the diagram*

$$\begin{array}{ccc} H_1(M, \mathbb{A}) & \longrightarrow & H_1(M, \mathbb{C}) \\ \downarrow & & \downarrow \\ H_1(N, \mathbb{A}) & \longrightarrow & H_1(N, \mathbb{C}) \end{array}$$

commutes, provided M and N are orientable $\mathcal{C}^\infty$ surfaces if $\mathbb{A}$ is neither $\mathbb{R}$ nor $\mathbb{C}$. If Φ is a homeomorphism, then the map $H_1(M, \mathbb{A}) \to H_1(N, \mathbb{A})$ is an isomorphism with inverse mapping $[\zeta]_{H_1(N,\mathbb{A})} \mapsto [\Phi_*^{-1}\zeta]_{H_1(M,\mathbb{A})}$.

(b) *If $y_0 = \Phi(x_0)$ for some point $x_0 \in M$, then the diagram of induced mappings*

$$\begin{array}{ccc} \pi_1(M, x_0) & \longrightarrow & H_1(M, \mathbb{A}) \\ \downarrow & & \downarrow \\ \pi_1(N, y_0) & \longrightarrow & H_1(N, \mathbb{A}) \end{array}$$

commutes, provided M and N are orientable $\mathcal{C}^\infty$ surfaces if $\mathbb{A}$ is neither $\mathbb{R}$ nor $\mathbb{C}$. Moreover, the image in $H_1(N, \mathbb{A})$ of (any generating set for) $\pi_1(M, x_0)$ generates the image of $H_1(M, \mathbb{A})$ as an $\mathbb{A}$-module.

(c) *The mapping $H^1(N, \mathbb{A}) \to H^1(M, \mathbb{A})$ given by $[\theta]_{H^1} \mapsto [\Phi^*\theta]_{H^1}$ is a well-defined module homomorphism, and the diagram*

$$\begin{array}{ccc} H^1(N, \mathbb{A}) & \longrightarrow & H^1(N, \mathbb{C}) \\ \downarrow & & \downarrow \\ H^1(M, \mathbb{A}) & \longrightarrow & H^1(M, \mathbb{C}) \end{array}$$

commutes. Moreover, the map $H^1(N, \mathbb{A}) \to H^1(M, \mathbb{A})$ is an isomorphism if Φ is a homeomorphism.

(d) *For each* $[\theta]_{H^1(N,\mathbb{F})} \in H^1(N,\mathbb{F})$ *and each* $[\xi]_{H_1(M,\mathbb{F})} \in H_1(M,\mathbb{F})$, *we have*

$$\big([\Phi^*\theta]_{H^1(M,\mathbb{F})}, [\xi]_{H_1(M,\mathbb{F})}\big)_{\mathrm{deR}} = \big([\theta]_{H^1(N,\mathbb{F})}, [\Phi_*\xi]_{H_1(N,\mathbb{F})}\big)_{\mathrm{deR}}.$$

The proof is left to the reader (see Exercise 10.7.9).

Definition 10.7.15 For $\Phi\colon M \to N$ as in Proposition 10.7.14, we call the corresponding mappings of homology and cohomology the induced *pushforward* and *pullback* mappings, respectively, and we denote these module homomorphisms by

$$\Phi_*\colon H_1(M,\mathbb{A}) \to H_1(N,\mathbb{A}) \quad \text{and} \quad \Phi^*\colon H^1(N,\mathbb{A}) \to H^1(M,\mathbb{A}).$$

Remark If $\Phi\colon M \to N$ and $\Psi\colon N \to P$ are continuous mappings of second countable topological surfaces, then $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$ and $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$ both at the level of q-chains and Čech 1-forms and at the level of homology and cohomology (see Exercise 10.7.10).

We may identify the cohomology space with the dual space of the homology space in a natural way:

Theorem 10.7.16 (de Rham) *For any second countable topological surface* M:

(a) *The mapping* $[\theta]_{H^1(M,\mathbb{F})} \mapsto ([\theta]_{H^1(M,\mathbb{F})}, \cdot)_{\mathrm{deR}}$ *gives a linear isomorphism*

$$H^1(M,\mathbb{F}) \xrightarrow{\cong} (H_1(M,\mathbb{F}))^* = \mathrm{Hom}(H_1(M,\mathbb{F}),\mathbb{F}).$$

(b) *The mapping* $[\xi]_{H_1(M,\mathbb{F})} \mapsto ([\xi]_{H_1(M,\mathbb{F})}, \cdot)_{\mathrm{deR}}$ *gives an injective linear map*

$$H_1(M,\mathbb{F}) \hookrightarrow (H^1(M,\mathbb{F}))^* = \mathrm{Hom}(H^1(M,\mathbb{F}),\mathbb{F}),$$

and this map is surjective if and only if $H_1(M,\mathbb{F})$ *is finite-dimensional. If* $H_1(M,\mathbb{F})$ *is infinite-dimensional, then* $H_1(M,\mathbb{F})$ *has a countable basis, but* $H^1(M,\mathbb{F})$ *does not.*

Proof It is easy to see from the definitions that the mappings in (a) and (b) are linear maps. The injectivity of the mapping in (a) follows from Theorem 10.7.3, and the injectivity of the mapping in (b) follows from the definition of $H_1(M,\mathbb{F})$.

For the proof of surjectivity of the mapping in (a), observe that any linear functional $\tau \in (H_1(M,\mathbb{F}))^*$ determines a group homomorphism $\hat{\tau}\colon \pi_1(M) \to \mathbb{F}$ given by $[\gamma] \mapsto \tau([\gamma]_{H_1})$. Thus Theorem 10.7.8 provides a Čech 1-form $\theta \in Z^1(M,\mathbb{F})$ such that

$$\tau([\gamma]_{H_1}) = \hat{\tau}([\gamma]) = \int_\gamma \theta \quad \forall [\gamma] \in \pi_1(M),$$

and Proposition 10.7.12 implies that

$$\tau([\xi]_{H_1}) = ([\theta]_{H^1}, [\xi]_{H_1})_{\mathrm{deR}} \quad \forall [\xi]_{H_1} \in H_1(M,\mathbb{F}).$$

Thus part (a) is proved.

For the proof of (b), observe that if $H_1(M,\mathbb{F})$ is finite-dimensional, then by (a), $H^1(M,\mathbb{F})$ and $H_1(M,\mathbb{F})$ must have the same dimension, and therefore the injective linear map in (b) must be surjective. Conversely, if $H_1(M,\mathbb{F})$ is infinite-dimensional, then by Proposition 10.7.12, $H_1(M,\mathbb{F})$ has a countably infinite basis. But the dual space of any infinite-dimensional vector space cannot have a countable basis (see Exercise 8.1.3), so $H^1(M,\mathbb{F})$ does not have a countable basis. In particular, $(H^1(M,\mathbb{F}))^*$ also does not have a countable basis, so the linear map in (b) cannot be surjective. □

Theorem 10.7.16 and Proposition 10.7.12 give the following:

Corollary 10.7.17 *If M is a compact topological surface, then*

$$\dim H^1(M,\mathbb{F}) = \dim H_1(M,\mathbb{F}) < \infty.$$

Remark According to Theorem 10.7.7, for M a second countable $\mathcal{C}^\infty$ surface, we may identify $H^1(M,\mathbb{F})$ with $H^1_{\mathrm{deR}}(M,\mathbb{F})$; so Theorem 10.7.16 and Corollary 10.7.17 hold in this case with $H^1_{\mathrm{deR}}(M,\mathbb{F})$ in place of $H^1(M,\mathbb{F})$.

According to the fundamental theorem of finitely generated Abelian groups, every finitely generated torsion-free Abelian group $(G,+)$ is isomorphic to $\mathbb{Z}^n$ for some unique integer n called the *rank* of G; that is, as a $\mathbb{Z}$-module, G has a basis with n elements. Consequently, a modification of the proof of Theorem 10.7.16 gives the following (weaker) version for coefficients in the subring $\mathbb{A}$, the proof of which is left to the reader (see Exercise 10.7.12):

Theorem 10.7.18 (de Rham) *For any orientable second countable $\mathcal{C}^\infty$ surface M:*

(a) *The mapping $[\theta]_{H^1(M,\mathbb{A})} \mapsto ([\theta]_{H^1(M,\mathbb{A})}, \cdot)_{\mathrm{deR}}$ gives a module isomorphism*

$$H^1(M,\mathbb{A}) \xrightarrow{\cong} \mathrm{Hom}(H_1(M,\mathbb{A}),\mathbb{A}).$$

(b) *If $\pi_1(M)$ is finitely generated (which is the case if, for example, M is compact), then we have the following:*
   (i) *The mapping $[\xi]_{H_1(M,\mathbb{A})} \mapsto ([\xi]_{H_1(M,\mathbb{A})}, \cdot)_{\mathrm{deR}}$ gives a module isomorphism*

$$H_1(M,\mathbb{A}) \xrightarrow{\cong} \mathrm{Hom}(H^1(M,\mathbb{A}),\mathbb{A}).$$

   (ii) *If $\{[\zeta_i]_{H_1}\}_{i=1}^m$ are linearly independent (over $\mathbb{Z}$) elements of $H_1(M,\mathbb{Z})$, then $\{[\zeta_i]_{H_1}\}_{i=1}^m$ are linearly independent (over $\mathbb{A}$) in $H_1(M,\mathbb{A})$.*
   (iii) *Every basis $\{[\xi_j]_{H_1}\}_{j=1}^n$ for $H_1(M,\mathbb{Z})$ is also a basis for $H_1(M,\mathbb{A})$, and the (dual) basis $\{[\theta_j]_{H_1}\}_{j=1}^n$ for $H^1(M,\mathbb{Z}) \cong \mathrm{Hom}(H_1(M,\mathbb{Z}),\mathbb{Z})$ determined by*

$$([\theta_i]_{H^1},[\xi_j]_{H_1})_{\mathrm{deR}} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i \neq j, \end{cases} \qquad \forall i,j=1,\ldots,n$$

   is also a (dual) basis for $H^1(M,\mathbb{A})$.

Remark In the exercises for Chaps. 5 and 6, the reader is asked to prove that the above homomorphism $H_1(M, \mathbb{A}) \to \mathrm{Hom}(H^1(M, \mathbb{A}), \mathbb{A})$ is injective for *any* orientable second countable $\mathcal{C}^\infty$ surface M (even if $\pi_1(M)$ is not finitely generated).

Theorem 10.7.18 and Proposition 10.7.12 give the following:

Corollary 10.7.19 *If M is a compact orientable $\mathcal{C}^\infty$ surface, then*

$$\operatorname{rank} H_1(M, \mathbb{Z}) = \operatorname{rank} H^1(M, \mathbb{Z}) = \dim H_1(M, \mathbb{F}) = \dim H^1(M, \mathbb{F}) < \infty.$$

Remarks We close this section with some instructive, but nonessential, remarks concerning other (equivalent) versions of homology and cohomology on a second countable topological surface M.

1. *Čech cohomology*. A reader familiar with Čech (sheaf) cohomology (see, for example, [For]) will recognize the space $Z^1(M, \mathbb{C})$ of Čech 1-forms on M as the space of global sections $\check{H}^0(M, \mathcal{C}^0/\mathbb{C})$, where $\mathcal{C}^0$ is the sheaf of germs of continuous functions and $\mathbb{C}$ is the sheaf of germs of locally constant functions. The space $B^1(M, \mathbb{C})$ of exact Čech 1-forms is the image of $\check{H}^0(M, \mathcal{C}^0)$. The exact sequence of sheaves $0 \to \mathbb{C} \to \mathcal{C}^0 \to \mathcal{C}^0/\mathbb{C} \to 0$ yields the exact sequence of Čech cohomology spaces

$$\begin{aligned} 0 &\to \check{H}^0(M, \mathbb{C}) \to \check{H}^0(M, \mathcal{C}^0) \to \check{H}^0(M, \mathcal{C}^0/\mathbb{C}) \\ &\to \check{H}^1(M, \mathbb{C}) \to \check{H}^1(M, \mathcal{C}^0) \to \check{H}^1(M, \mathcal{C}^0/\mathbb{C}) \to \cdots. \end{aligned}$$

Since $\check{H}^1(M, \mathcal{C}^0) = 0$, we get the isomorphism

$$\begin{aligned} Z^1(M, \mathbb{C})/B^1(M, \mathbb{C}) &= \check{H}^0(M, \mathcal{C}^0/\mathbb{C})/\mathrm{im}(\check{H}^0(M, \mathcal{C}^0) \to \check{H}^0(M, \mathcal{C}^0/\mathbb{C})) \\ &\xrightarrow{\cong} \check{H}^1(M, \mathbb{C}). \end{aligned}$$

2. *Singular homology and cohomology*. The *standard* 2-*simplex* is the closed triangular region $\Delta^2 \equiv \{(x, y) \in (\mathbb{R}_{\geq 0})^2 \mid x + y \leq 1\} \subset \mathbb{R}^2$; a *singular* 2-*simplex in M* is a continuous mapping $\sigma\colon \Delta^2 \to M$; and a (*singular*) 2-*chain in M with coefficients in* $\mathbb{A}$ is any formal finite linear combination

$$\xi = \sum_{i=1}^{m} a_i \cdot \sigma_i,$$

where $a_i \in \mathbb{A}$ and $\sigma_i\colon [0, 1] \to M$ is a singular 2-simplex for $i = 1, \dots, m$. We identify the above 2-chain ξ with another 2-chain $\zeta = \sum_{j=1}^{n} b_j \hat{\sigma}_j$ in M if for each $i = 1, \dots, m$,

$$\sum_{\substack{1 \leq l \leq m \\ \sigma_l = \sigma_i}} a_l = \begin{cases} \sum_{\substack{1 \leq j \leq n \\ \hat{\sigma}_j = \sigma_i}} b_j & \text{if } \sigma_i \in \{\hat{\sigma}_1, \dots, \hat{\sigma}_n\}, \\ 0 & \text{if } \sigma_i \notin \{\hat{\sigma}_1, \dots, \hat{\sigma}_n\}, \end{cases}$$

and for each $j = 1, \ldots, n$, the analogous statement, with the roles of ξ and ζ switched, holds. The module formed by the 2-chains is denoted by $C_2(M, \mathbb{A})$.

For $\nu = 0, 1, 2$, the *νth edge* of a singular 2-simplex σ in M is the path $\sigma^{(\nu)}$ in M given by

$$t \mapsto \begin{cases} \sigma(1-t, t) & \text{if } \nu = 0, \\ \sigma(0, t) & \text{if } \nu = 1, \\ \sigma(t, 0) & \text{if } \nu = 2, \end{cases} \qquad \forall t \in [0, 1].$$

The *boundary operator* $\partial \colon C_2(M, \mathbb{A}) \to C_1(M, \mathbb{A})$ is the homomorphism given by

$$\xi = \sum_{i=1}^{m} a_i \cdot \sigma_i \mapsto \sum_{i=1}^{m} a_i \left(\sigma_i^{(0)} - \sigma_i^{(1)} + \sigma_i^{(2)}\right),$$

the image is denoted by

$$B_1^{\Delta}(M, \mathbb{A}) \equiv \operatorname{im}(\partial \colon C_2(M, \mathbb{A}) \to C_1(M, \mathbb{A})),$$

and each element of $B_1^{\Delta}(M, \mathbb{A})$ is called a (*singular*) *1-boundary*. We have $\partial^2 = 0$ (as is easy to verify), so $B_1^{\Delta}(M, \mathbb{A})$ is a submodule of $Z_1(M, \mathbb{A})$, and we may define the *first singular homology group of M with coefficients in* $\mathbb{A}$ to be the quotient module

$$H_1^{\Delta}(M, \mathbb{A}) \equiv Z_1(M, \mathbb{A}) \big/ B_1^{\Delta}(M, \mathbb{A}).$$

For $q = 0, 1, 2$, we set

$$C_{\Delta}^{q}(M, \mathbb{A}) \equiv \operatorname{Hom}(C_q(M, \mathbb{A}), \mathbb{A}),$$

and we call any element of the above module a *q-cochain in M with coefficients in* $\mathbb{A}$. For $q = 0, 1$, the homomorphism $\delta \colon C_{\Delta}^{q}(M, \mathbb{A}) \to C_{\Delta}^{q+1}(M, \mathbb{A})$ given by

$$(\delta\varphi)(\xi) \equiv \varphi(\partial\xi) \quad \forall \varphi \in C_{\Delta}^{q}(M, \mathbb{A}), \xi \in C_{q+1}(M, \mathbb{A})$$

is called the *coboundary operator*. We set

$$Z_{\Delta}^{1}(M, \mathbb{A}) \equiv \ker(\delta \colon C_{\Delta}^{1}(M, \mathbb{A}) \to C_{\Delta}^{2}(M, \mathbb{A})),$$

$$B_{\Delta}^{1}(M, \mathbb{A}) \equiv \operatorname{im}(\delta \colon C_{\Delta}^{0}(M, \mathbb{A}) \to C_{\Delta}^{1}(M, \mathbb{A})),$$

we call each element of $Z_{\Delta}^{1}(M, \mathbb{A})$ a (*singular*) 1-*cocycle*, and we call each element of $B_{\Delta}^{1}(M, \mathbb{A})$ a (*singular*) 1-*coboundary*. Clearly, $\delta^2 = 0$, so $B_{\Delta}^{1}(M, \mathbb{A})$ is a submodule of $Z_{\Delta}^{1}(M, \mathbb{A})$, and we may define the *first singular cohomology group of M with coefficients in* $\mathbb{A}$ to be the quotient module

$$H_{\Delta}^{1}(M, \mathbb{A}) \equiv Z_{\Delta}^{1}(M, \mathbb{A}) \big/ B_{\Delta}^{1}(M, \mathbb{A}).$$

Further properties of $H_1^{\Delta}(M, \mathbb{A})$ and $H_{\Delta}^{1}(M, \mathbb{A})$, and how they relate to $H_1(M, \mathbb{A})$ and $H^1(M, \mathbb{A})$, are considered in Exercises 10.7.13–10.7.16 and in the exercises for Chaps. 5 and 6.

Exercises for Sect. 10.7

10.7.1 Verify that the operations assigned to $Z^1(M, \mathbb{A})$ (as in Definition 10.7.1) determine a well-defined module structure.

10.7.2 Verify that line integrals of closed Čech 1-forms, as defined in Definition 10.7.1, are well defined.

10.7.3 Prove parts (a)–(d) and part (f) of Theorem 10.7.3.

10.7.4 Prove Lemma 10.7.4.

10.7.5 Verify that for any second countable topological surface M, we have a real direct sum decomposition $Z^1(M, \mathbb{C}) = Z^1(M, \mathbb{R}) \oplus i Z^1(M, \mathbb{R})$.

10.7.6 Prove Proposition 10.7.10.

10.7.7 Prove Lemma 10.7.11.

10.7.8 Prove Proposition 10.7.12.

10.7.9 Prove Proposition 10.7.14.

10.7.10 Prove that if $\Phi\colon M \to N$ and $\Psi\colon N \to P$ are continuous mappings of second countable topological surfaces, then $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$ and $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$ both at the level of q-chains and Čech 1-forms and at the level of homology and cohomology.

10.7.11 In the situation of Theorem 10.7.16, assuming that the homology vector space $H_1(M, \mathbb{F})$ is infinite-dimensional, give a direct proof that the associated injection $[\xi]_{H_1} \mapsto ([\xi]_{H_1}, \cdot)_{\mathrm{deR}}$ is not surjective using the fact that, for any infinite-dimensional vector space $\mathcal{V}$ over $\mathbb{F}$, we have $\mathcal{V} \subsetneqq (\mathcal{V}^*)^*$ (see Sect. 8.1).

10.7.12 Prove Theorem 10.7.18.

Some properties of $H_1^{\Delta}(M, \mathbb{A})$ and $H^1_{\Delta}(M, \mathbb{A})$, and how they relate to $H_1(M, \mathbb{A})$ and $H^1(M, \mathbb{A})$, are considered in Exercises 10.7.13–10.7.16 below.

10.7.13 Let M be a second countable topological surface, and let $\mathbb{A}$ be a subring of $\mathbb{C}$ containing $\mathbb{Z}$.

(a) Suppose α and β are paths in M.

(i) Prove that if $\alpha(1) = \beta(0)$, then $\alpha + \beta - \alpha * \beta = \partial\sigma$ for some singular 2-simplex σ in M (in particular, $\alpha + \beta - \alpha * \beta \in B_1^{\Delta}(M, \mathbb{Z})$).

(ii) Prove that if $\alpha(1) = \beta(0)$, then $-\alpha + \beta^- + \alpha * \beta = \partial\sigma$ for some singular 2-simplex σ in M (in particular, $-\alpha + \beta^- + \alpha * \beta \in B_1^{\Delta}(M, \mathbb{Z})$).

(iii) Prove that the constant loop $e_{p_0}\colon t \mapsto p_0$ at any point $p_0 \in M$ satisfies $e_{p_0} = \partial\sigma$ for the constant singular 2-simplex $\sigma\colon (x, y) \mapsto p_0$.

(iv) Prove that if α and β are path homotopic and $p_1 \equiv \alpha(1) = \beta(1)$, then there is a singular 2-simplex σ in M such that $\sigma^{(0)} = e_{p_1}$ (the constant loop at p_1), $\sigma^{(1)} = \beta$, and $\sigma^{(2)} = \alpha$; and hence $\alpha - \beta = \partial(\sigma - \hat{\sigma})$, where $\hat{\sigma}$ is the constant singular 2-simplex given by $\hat{\sigma}\colon (x, y) \mapsto p_1$ (in particular, $\alpha - \beta \in B_1^{\Delta}(M, \mathbb{Z})$).

(v) Prove that $\alpha + \alpha^- \in B_1^{\Delta}(M, \mathbb{Z})$.

(vi) Prove that if σ is a singular 2-simplex with $\alpha = \sigma^{(2)}$ and $\beta = \sigma^{(0)}$, then $\sigma^{(1)}$ is path homotopic to $\alpha * \beta$.

(b) Prove that for each point $p_0 \in M$, the mapping $[\gamma]_{p_0} \mapsto [\gamma]_{H_1^\Delta(M,\mathbb{A})}$ gives a well-defined homomorphism $\pi_1(M, p_0) \to H_1^\Delta(M, \mathbb{A})$. Prove also that the image of this homomorphism generates $H_1^\Delta(M, \mathbb{A})$ as an $\mathbb{A}$-module and that the map is surjective if $\mathbb{A} = \mathbb{Z}$.

(c) Prove that $H_\Delta^1(M, \mathbb{A})$ is (naturally) isomorphic to $H^1(M, \mathbb{A})$.

Hint. Fix a point $p_0 \in M$, and applying part (a) above, show that the mapping $H_\Delta^1(M, \mathbb{A}) \to \mathrm{Hom}(\pi_1(M, p_0), \mathbb{A})$ determined by $[\varphi]_{H_\Delta^1} \cdot [\gamma] = \varphi(\gamma)$ for all $[\varphi]_{H_\Delta^1} \in H_\Delta^1(M, \mathbb{A})$ and $[\gamma] \in \pi_1(M, p_0)$ is a well-defined injective homomorphism. For surjectivity, fix a path λ_x from p_0 to x for each point $x \in M$, with $\lambda_{p_0} = e_{p_0}$. Show that given a homomorphism $F \in \mathrm{Hom}(\pi_1(M, p_0), \mathbb{A})$, there is a unique singular 1-cocycle φ determined by $\varphi(\gamma) \equiv F([\lambda_{\gamma(0)} * \gamma * \lambda_{\gamma(1)}^-])$ for each path γ.

(d) Prove that $B_1^\Delta(M, \mathbb{A}) \subset B_1(M, \mathbb{A})$, provided M admits an orientable $\mathcal{C}^\infty$ surface structure if $\mathbb{A} \neq \mathbb{R}$ and $\mathbb{A} \neq \mathbb{C}$. Conclude that there is a (natural) surjective homomorphism $H_1^\Delta(M, \mathbb{A}) \to H_1(M, \mathbb{A})$ (the reader is asked to prove bijectivity in the exercises for Chaps. 5 and 6).

10.7.14 Let M be a second countable topological surface, let σ be a singular 2-simplex in M, and let $p_0 \in M$. Prove that:

(a) If λ_0 is a path in M from p_0 to $\sigma(0, 0)$, then there exists a singular 2-simplex σ_0 in M such that $\sigma_0^{(0)} = \sigma^{(0)}$, $\sigma_0^{(1)} = \lambda_0 * \sigma^{(1)}$, and $\sigma_0^{(2)} = \lambda_0 * \sigma^{(2)}$.

(b) If λ_1 is a path in M from p_0 to $\sigma(1, 0)$, then there exists a singular 2-simplex σ_1 in M such that $\sigma_1^{(0)} = \lambda_1 * \sigma^{(0)}$, $\sigma_1^{(1)} = \sigma^{(1)}$, and $\sigma_1^{(2)} = \sigma^{(2)} * \lambda_1^-$.

(c) If λ_2 is a path in M from p_0 to $\sigma(0, 1)$, then there exists a singular 2-simplex σ_2 in M such that $\sigma_2^{(0)} = \sigma^{(0)} * \lambda_2^-$, $\sigma_2^{(1)} = \sigma^{(1)} * \lambda_2^-$, and $\sigma_2^{(2)} = \sigma^{(2)}$.

10.7.15 Suppose α and β are path homotopic paths in a second countable topological surface M, σ is a singular 2-simplex in M, $\nu \in \{0, 1, 2\}$, and $\alpha = \sigma^{(\nu)}$. Prove that there exists a singular 2-simplex $\hat{\sigma}$ in M such that $\hat{\sigma}^{(\nu)} = \beta$ and $\hat{\sigma}^{(\mu)} = \sigma^{(\mu)}$ for $\mu \in \{0, 1, 2\} \setminus \{\nu\}$. Conclude that in particular,

$$\partial\sigma = (-1)^\nu\alpha - (-1)^\nu\beta + \partial\hat{\sigma}.$$

10.7.16 The *real projective plane* is the quotient space $\mathbb{RP}^2 \equiv \mathbb{S}^2/[x \sim -x \ \forall x \in \mathbb{S}^2]$. Let $\Upsilon : \mathbb{S}^2 \to \mathbb{RP}^2$ be the corresponding quotient map.

(a) Prove that $\mathbb{RP}^2$ admits a $\mathcal{C}^\infty$ surface structure for which Υ is a $\mathcal{C}^\infty$ covering map.

(b) Prove that $\mathbb{RP}^2$ is nonorientable (thus $\Upsilon : \mathbb{S}^2 \to \mathbb{RP}^2$ is an orientable double cover as considered in Exercise 9.7.8).

(c) Letting $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, prove the following equalities (which demonstrate that the version of integral homology $H_1(M, \mathbb{Z})$ considered in this section for orientable $\mathcal{C}^\infty$ surfaces would be inadequate for nonorientable surfaces):

$$
\begin{aligned}
&H_1^{\Delta}(\mathbb{RP}^2, \mathbb{Z}) \cong \pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}, \\
&H_1(\mathbb{RP}^2, \mathbb{F}) = H_1^{\Delta}(\mathbb{RP}^2, \mathbb{F}) = \{0\}, \\
&H^1(\mathbb{RP}^2, \mathbb{Z}) = H^1_{\Delta}(\mathbb{RP}^2, \mathbb{Z}) = H^1(\mathbb{RP}^2, \mathbb{F}) = H^1_{\Delta}(\mathbb{RP}^2, \mathbb{F}) = \{0\}.
\end{aligned}
$$

Hint. For the claim concerning the homology groups, let γ be a generating loop for $\pi_1(\mathbb{RP}^2)$. If $\gamma = \partial\xi$ for some integral 2-cochain ξ, then by applying Exercises 10.7.14 and 10.7.15, reduce to the case in which each of the boundary edges of the singular 2-simplices in ξ is either γ or the constant loop. Applying Exercise 10.7.13, obtain a contradiction (one may also prove this fact by observing that the first singular homology group is equal to the Abelianization of the fundamental group, as proved in, for example, [Hat]).

Chapter 11
Background Material on Sobolev Spaces and Regularity

This chapter is required for the proof of integrability of almost complex structures in Chap. 6. Throughout this chapter, we use the operator notation:

$$D_j = \frac{\partial}{\partial x_j} \quad \text{for } j = 1, \dots, n$$

and

$$D^\alpha = \left(\frac{\partial}{\partial x}\right)^\alpha = (D_1)^{\alpha_1} \cdots (D_n)^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

for each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$ (see Sect. 7.4). We also use the following notation. Let us denote the coordinates in $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ by

$$(x, y) = ((x_1, \dots, x_n), (y_1, \dots, y_n)) = (x_1, \dots, x_n, y_1, \dots, y_n).$$

Given a linear differential operator $L = \sum b_\beta D^\beta$ on an open set $\Omega \subset \mathbb{R}^n$, we denote by L_x and L_y the linear differential operators on $\Omega \times \Omega$ given by

$$L_x[h(x, y)] = \sum b_\beta(x) \left(\frac{\partial}{\partial x}\right)^\beta [h(x, y)]$$

and

$$L_y[h(x, y)] = \sum b_\beta(y) \left(\frac{\partial}{\partial y}\right)^\beta [h(x, y)].$$

For example, for $\Omega = \mathbb{R}^2$, we have

$$D_x^{(1,2)}[x_1^2 x_2^3 y_1^4 y_2^5] = (2x_1)(6x_2) y_1^4 y_2^5$$

and

$$D_y^{(1,2)}[x_1^2 x_2^3 y_1^4 y_2^5] = x_1^2 x_2^3 (4y_1^3)(20y_2^3).$$

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For $L = x_1 D^{(1,2)}$, we have

$$\begin{aligned} L(x_1^2 x_2^3) &= x_1(2x_1)(6x_2), \\ L_x[x_1^2 x_2^3 y_1^4 y_2^5] &= x_1(2x_1)(6x_2)y_1^4 y_2^5, \\ L_y[x_1^2 x_2^3 y_1^4 y_2^5] &= y_1 x_1^2 x_2^3 (4y_1^3)(20y_2^3). \end{aligned}$$

The main goal of this chapter is the following regularity theorem:

Theorem 11.0.1 (General first-order regularity) *Let A be a linear differential operator of order 1 with $\mathcal{C}^\infty$ coefficients on an open set $\Omega \subset \mathbb{R}^n$. Assume that there exists a constant $C > 0$ such that*

$$\|u\|^2_{L^2(\Omega)} + \sum_{j=1}^{n} \|D_j u\|^2_{L^2(\Omega)} \le C \cdot \left(\|u\|^2_{L^2(\Omega)} + \|Au\|^2_{L^2(\Omega)}\right) \quad \forall u \in D(\Omega). \quad (1)$$

Then, for every function $u \in L^2_{\text{loc}}(\Omega)$ with $A_{\text{distr}} u \in \mathcal{C}^\infty(\Omega)$, we have $u \in \mathcal{C}^\infty(\Omega)$.

Remarks 1. The analogous version holds in higher dimensions (see, for example, [Hö]) and for higher-order operators (see, for example, [GiT] or [Ns3]).

2. The term $\|u\|^2_{L^2(\Omega)}$ on the left-hand side is redundant. However, it will be convenient to have the theorem stated in this form, since the left-hand side represents the square of the first-order Sobolev norm of u (see Definition 11.1.1), and Sobolev spaces are applied in the proof.

A natural approach to the regularity theorem (and to the study of differential equations in general) is to complete the space of $\mathcal{C}^\infty$ functions with respect to the norm obtained from L^p norms of derivatives. One obtains the so-called *Sobolev spaces*. One proves the regularity theorem by applying the *Sobolev lemma*, which gives an embedding of Sobolev spaces for a large number of derivatives into the Banach space $\mathcal{C}^k$. For our purposes, $p = 2$ suffices, but Sobolev spaces for arbitrary $p \ge 1$ are, in general, very useful. The reader may refer to, for example, [Ad] and [GiT] for more about this important class of spaces.

11.1 Sobolev Spaces

Definition 11.1.1 Let Ω be an open subset of $\mathbb{R}^n$, and let $k \in \mathbb{Z}_{\ge 0}$.

(a) The associated *Sobolev space of order k* is given by

$$W^k(\Omega) \equiv \{u \in L^2(\Omega) \mid D^\alpha_{\text{distr}} u \in L^2(\Omega)\ \forall \alpha \in \mathbb{Z}^n_{\ge 0} \text{ with } |\alpha| \le k\}.$$

(b) The *Sobolev inner product* and *Sobolev norm* are given by

$$\langle u, v\rangle_{W^k(\Omega)} \equiv \sum_{|\alpha| \le k} \langle D^\alpha_{\text{distr}} u, D^\alpha_{\text{distr}} v\rangle_{L^2(\Omega)}$$

and $\|u\|_{W^k(\Omega)} \equiv \langle u, u\rangle^{1/2}_{W^k(\Omega)}$, respectively, for all $u, v \in W^k(\Omega)$.

(c) The *local Sobolev space of order k* is given by

$$W^k_{\text{loc}}(\Omega) \equiv \{u \in L^2_{\text{loc}}(\Omega) \mid D^\alpha_{\text{distr}} u \in L^2_{\text{loc}}(\Omega)\ \forall \alpha \text{ with } |\alpha| \le k\}.$$

For all $u, v \in W^k_{\text{loc}}(\Omega)$ and any measurable subset E of Ω, we set

$$\langle u, v\rangle_{W^k(E)} \equiv \sum_{|\alpha|\le k} \langle D^\alpha_{\text{distr}} u, D^\alpha_{\text{distr}} v\rangle_{L^2(E)}$$

and $\|u\|_{W^k(E)} \equiv \langle u, u\rangle^{1/2}_{W^k(E)}$, provided these numbers exist in $\mathbb{C}$ (or $[0, \infty]$).

Remark Clearly, $\langle u, v\rangle_{W^k(E)}$ and $\|u\|_{W^k(E)}$ are independent of the choice of the neighborhood Ω of E.

Completeness of $L^2(\Omega)$ gives the following fact, the proof of which is left to the reader (see Exercise 11.1.1).

Proposition 11.1.2 $(W^k(\Omega), \langle\cdot,\cdot\rangle_{W^k(\Omega)})$ *is a Hilbert space for every open set* $\Omega \subset \mathbb{R}^n$ *and every nonnegative integer* k.

The following lemma allows one to manipulate distributional values of differential operators in ways similar to the ways in which one treats standard values. In particular, it allows one to cut off in order to get compact support.

Lemma 11.1.3 *Let A and B be linear differential operators of order* $k \ge 1$ *and* l, *respectively, with* $\mathcal{C}^\infty$ *coefficients on an open set* $\Omega \subset \mathbb{R}^n$, *and let* $m \equiv k + l - 1$.

(a) *If* $u \in W^l_{\text{loc}}(\Omega)$, *then*

$$B_{\text{distr}} u = \left[\sum_{|\beta|\le l} b_\beta \cdot D^\beta_{\text{distr}} u\right] \in L^2_{\text{loc}}(\Omega),$$

where $B = \sum_{|\beta|\le l} b_\beta D^\beta$. *In particular, for every compact set* $K \subset \Omega$, *there is a constant* $C > 0$ *such that for every function* $u \in W^l_{\text{loc}}(\Omega)$, *we have*

$$\|B_{\text{distr}} u\|_{L^2(K)} \le C\|u\|_{W^l(K)}.$$

(b) *The commutator* $[A, B] \equiv AB - BA$ *is a linear differential operator of order* m.

(c) *If* $u \in W^m_{\text{loc}}(\Omega)$ *and* $A_{\text{distr}} u \in W^l_{\text{loc}}(\Omega)$, *then we have* $B_{\text{distr}} u \in W^{k-1}_{\text{loc}}(\Omega)$, $(AB)_{\text{distr}} u \in L^2_{\text{loc}}(\Omega)$, $(BA)_{\text{distr}} u \in L^2_{\text{loc}}(\Omega)$, $[A, B]_{\text{distr}} u \in L^2_{\text{loc}}(\Omega)$, *and*

$$\begin{aligned}(AB)_{\text{distr}}(u) = A_{\text{distr}} B_{\text{distr}} u &= (BA)_{\text{distr}} u + [A, B]_{\text{distr}} u \\ &= B_{\text{distr}} A_{\text{distr}} u + [A, B]_{\text{distr}} u \in L^2_{\text{loc}}(\Omega).\end{aligned}$$

In particular, for any nonnegative integer s, the product of a function in $\mathcal{C}^\infty(\Omega)$ *and a function in* $W^s_{\text{loc}}(\Omega)$ *is in* $W^s_{\text{loc}}(\Omega)$.

(d) *We have* ${}^tB = (-1)^l B + L$ *for some linear differential operator* L *of order* $\max(l-1, 0)$ *with* $\mathcal{C}^\infty$ *coefficients on* Ω (${}^tB = B$ *if* $l = 0$).

Remark As noted above, the *commutator* of two linear differential operators A and B with $\mathcal{C}^\infty$ coefficients is given by $[A, B] = AB - BA$. If A and B are of order 0, then $[A, B] = 0$.

Proof of Lemma 11.1.3 Part (a) follows immediately from Lemma 7.4.3 (note that $L^2_{\mathrm{loc}} \subset L^1_{\mathrm{loc}}$). The product rule gives part (b) (see Exercise 11.1.2). For the proof of part (c), let $u \in W^m_{\mathrm{loc}}(\Omega)$. Parts (a) and (b) imply that $[A, B]_{\mathrm{distr}} u \in L^2_{\mathrm{loc}}(\Omega)$. Furthermore, for each multi-index α with $|\alpha| \leq k-1$, we have, by Lemma 7.4.3, $D^\alpha_{\mathrm{distr}}[B_{\mathrm{distr}} u] = (D^\alpha B)_{\mathrm{distr}} u \in L^2_{\mathrm{loc}}(\Omega)$ ($D^\alpha B$ is of order $k-1+l = m$), and hence $B_{\mathrm{distr}} u \in W^{k-1}_{\mathrm{loc}}(\Omega)$. If $A_{\mathrm{distr}} u \in W^l_{\mathrm{loc}}(\Omega)$, then (a) and Lemma 7.4.3 together give

$$(BA)_{\mathrm{distr}} u = B_{\mathrm{distr}} A_{\mathrm{distr}} u \in L^2_{\mathrm{loc}}(\Omega),$$

and hence

$$A_{\mathrm{distr}} B_{\mathrm{distr}} u = (AB)_{\mathrm{distr}} u = (BA)_{\mathrm{distr}} u + [A, B]_{\mathrm{distr}} u \in L^2_{\mathrm{loc}}(\Omega).$$

The proof of (d) is left to the reader (see Exercise 11.1.2). □

Exercises for Sect. 11.1

11.1.1 Prove Proposition 11.1.2.
11.1.2 Prove parts (b) and (d) of Lemma 11.1.3.

11.2 Uniform Convergence of Derivatives

In the context of differential equations, it is also natural to consider uniform convergence of derivatives in place of L^p convergence. For this, it is useful to have the following well-known fact:

Proposition 11.2.1 *Let* k *be a nonnegative integer, let* $\{u_\nu\}$ *be a sequence of* $\mathcal{C}^k$ *functions on an open set* $\Omega \subset \mathbb{R}^n$, *and let* $\{v_\alpha\}_{\alpha \in (\mathbb{Z}_{\geq 0})^n, |\alpha| \leq k}$ *be a collection of functions such that* $D^\alpha u_\nu \to v_\alpha$ *uniformly on compact subsets of* Ω *for each multi-index* α *with* $|\alpha| \leq k$. *Then, for* $v \equiv v_{(0,\ldots,0)}$, *we have* $v \in \mathcal{C}^k(\Omega)$ *and* $D^\alpha v = v_\alpha$ *for each* α *with* $|\alpha| \leq k$.

The proof is left to the reader (see Exercise 11.2.1).

Corollary 11.2.2 *If* $k \in \mathbb{Z}_{\geq 0}$, Ω *is an open subset of* $\mathbb{R}^n$, *and* $\{u_\nu\}$ *is a sequence of functions in* $\mathcal{C}^k(\Omega)$ *such that for every compact set* $K \subset \Omega$, $\{D^\alpha u_\nu \restriction_K\}$ *is a Cauchy sequence in* $(\mathcal{C}^0(K), \|\cdot\| \equiv \sup_K |\cdot|)$ *for each multi-index* α *with* $|\alpha| \leq k$, *then there exists a function* $u \in \mathcal{C}^k(\Omega)$ *such that* $D^\alpha u_\nu \to D^\alpha u$ *uniformly on compact subsets of* Ω *for each* α *with* $|\alpha| \leq k$.

The proof is again left to the reader (see Exercise 11.2.2).

Exercises for Sect. 11.2

11.2.1 Prove Proposition 11.2.1.
11.2.2 Prove Corollary 11.2.2.

11.3 The Strong Friedrichs Lemma

The goal of this section is the following strong version of the Friedrichs regularization lemma (Lemma 7.3.1):

Lemma 11.3.1 (Friedrichs) *Let κ be a nonnegative $\mathcal{C}^\infty$ function with compact support in the unit ball $B(0;1)$ in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \kappa\, d\lambda = 1$, let Ω be an open subset of $\mathbb{R}^n$, let k be a nonnegative integer, and let A be a linear differential operator of order k with $\mathcal{C}^\infty$ coefficients on Ω. For each function $u \in L^1_{\text{loc}}(\Omega)$ and for each $\delta > 0$, let $\Omega_\delta \equiv \{x \in \Omega \mid \text{dist}(x, \mathbb{R}^n \setminus \Omega) > \delta\}$, let $\kappa^\delta(x) \equiv \delta^{-n}\kappa(x/\delta)$ for each point $x \in \mathbb{R}^n$, and let*

$$u_\delta(x) \equiv \int_\Omega u(y)\kappa^\delta(x-y)\, d\lambda(y) = \int_{B(0;\delta)} u(x-y)\kappa^\delta(y)\, d\lambda(y)$$
$$= \int_{B(0;1)} u(x-\delta y)\kappa(y)\, d\lambda(y)$$

for every $x \in \Omega_\delta$. Finally, let K be a compact subset of Ω, let δ_0 be a constant with $0 < \delta_0 < \text{dist}(K, \mathbb{R}^n \setminus \Omega)$ (hence $K \subset \Omega_{\delta_0}$), and let $K^\delta \equiv \{x \in \mathbb{R}^n \mid \text{dist}(x, K) \le \delta\}$ for each $\delta > 0$. Then we have the following:

(a) *For every function $u \in L^1_{\text{loc}}(\Omega)$ and every $\delta > 0$, we have $u_\delta \in \mathcal{C}^\infty(\Omega_\delta)$.*
(b) *If $u \in \mathcal{C}^k(\Omega)$, then $A(u_\delta) \to Au$ uniformly on K as $\delta \to 0^+$.*
(c) *We have $\|u_\delta\|_{W^k(K)} \le \|u\|_{W^k(K^{\delta_0})}$ for every $u \in W^k_{\text{loc}}(\Omega)$ and $\delta \in (0, \delta_0)$.*
(d) *If $u \in W^k_{\text{loc}}(\Omega)$, then $\|u_\delta - u\|_{W^k(K)} \to 0$ as $\delta \to 0^+$ (i.e., $u_\delta \to u$ in $W^k_{\text{loc}}(\Omega)$).*
(e) *There is a constant $C > 0$ such that for every $\delta \in (0, \delta_0)$ and every function $u \in W^{\max(k-1,0)}(\Omega)$ with $A_{\text{distr}}u \in L^2_{\text{loc}}(\Omega)$, we have*

$$\|A(u_\delta) - (A_{\text{distr}}u)_\delta\|_{L^2(K)} \le C\|u\|_{W^{\max(k-1,0)}(\Omega)}.$$

(f) *If $u \in W^{\max(k-1,0)}_{\text{loc}}(\Omega)$ and $A_{\text{distr}}u \in L^2_{\text{loc}}(\Omega)$, then $\|A(u_\delta) - A_{\text{distr}}u\|_{L^2(K)} \to 0$ as $\delta \to 0^+$.*

Proof The proof given here is similar to the proof of Lemma 5.2.2 in [Hö]. Part (a) follows from Lemma 7.3.1 (or differentiation past the integral). For the proof of (b), we observe that if $u \in \mathcal{C}^k(\Omega)$, α is a multi-index with $|\alpha| \le k$, and a is a $\mathcal{C}^\infty$ function on Ω, then by Lemma 7.3.1 and Lemma 7.4.4, we have

$$\max_K |a \cdot D^\alpha(u_\delta) - a \cdot D^\alpha u| = \max_K |a \cdot [D^\alpha u]_\delta - a \cdot [D^\alpha u]|$$
$$\leq \max_K |a| \cdot \max_K |[D^\alpha u]_\delta - [D^\alpha u]| \to 0 \quad \text{as } \delta \to 0^+.$$

The claim (b) now follows.

For the proof of (c), suppose $u \in L^2_{\mathrm{loc}}(\Omega)$ and $\delta \in (0, \delta_0)$. Then, for each point $x \in K$, the Schwarz inequality gives

$$\begin{aligned}
|u_\delta(x)|^2 &= \left| \int_{B(0;1)} u(x - \delta y)\kappa(y)\, d\lambda(y) \right|^2 \\
&\leq \left[\int_{B(0;1)} |u(x - \delta y)|^2 \kappa(y)\, d\lambda(y) \right] \cdot \left[\int_{B(0;1)} \kappa(y)\, d\lambda(y) \right] \\
&= \int_{B(0;1)} |u(x - \delta y)|^2 \kappa(y)\, d\lambda(y).
\end{aligned}$$

Applying Fubini's theorem (Theorem 7.1.10), we get

$$\begin{aligned}
\|u_\delta\|^2_{L^2(K)} &\leq \int_{B(0;1)} \left[\int_K |u(x - \delta y)|^2\, d\lambda(x) \right] \kappa(y)\, d\lambda(y) \\
&= \int_{B(0;1)} \left[\int_{K+(-\delta y)} |u(x)|^2\, d\lambda(x) \right] \kappa(y)\, d\lambda(y) \\
&\leq \int_{B(0;1)} \left[\int_{K^{\delta_0}} |u(x)|^2\, d\lambda(x) \right] \kappa(y)\, d\lambda(y) = \|u\|^2_{L^2(K^{\delta_0})}.
\end{aligned}$$

If $u \in W^k_{\mathrm{loc}}(\Omega)$, then applying Lemma 7.4.4, we get

$$\begin{aligned}
\|u_\delta\|^2_{W^k(K)} &= \sum_{|\alpha| \leq k} \|D^\alpha(u_\delta)\|^2_{L^2(K)} = \sum_{|\alpha| \leq k} \|[(D^\alpha)_{\mathrm{distr}} u]_\delta\|^2_{L^2(K)} \\
&\leq \sum_{|\alpha| \leq k} \|(D^\alpha)_{\mathrm{distr}} u\|^2_{L^2(K^{\delta_0})} = \|u\|^2_{W^k(K^{\delta_0})}.
\end{aligned}$$

Thus (c) is proved.

For the proof of (d), we first observe that if $v \in \mathcal{C}^0(\Omega)$, then

$$\|v_\delta - v\|_{L^2(K)} \leq \lambda(K)^{1/2} \max_K |v_\delta - v| \to 0 \quad \text{as } \delta \to 0^+$$

by part (b) (or Lemma 7.3.1). Given $u \in L^2_{\mathrm{loc}}(\Omega)$ and $\epsilon > 0$, we may choose a continuous function v on Ω such that $\|v - u\|_{L^2(K^{\delta_0})} < \epsilon/3$ (see Exercise 7.1.4). Hence, by part (c), we have

$$\|v_\delta - u_\delta\|_{L^2(K)} = \|(v - u)_\delta\|_{L^2(K)} < \epsilon/3$$

for each $\delta \in (0, \delta_0)$. On the other hand, for $\delta > 0$ sufficiently small, we also have $\|v_\delta - v\|_{L^2(K)} < \epsilon/3$. Therefore

$$\|u_\delta - u\|_{L^2(K)} \le \|u_\delta - v_\delta\|_{L^2(K)} + \|v_\delta - v\|_{L^2(K)} + \|v - u\|_{L^2(K)} < \epsilon.$$

Thus $\|u_\delta - u\|_{L^2(K)} \to 0$ as $\delta \to 0^+$. If $u \in W^k_{\text{loc}}(\Omega)$, then applying Lemma 7.4.4 (as above), we get

$$\begin{aligned}\|u_\delta - u\|^2_{W^k(K)} &= \sum_{|\alpha|\le k} \|D^\alpha(u_\delta) - (D^\alpha)_{\text{distr}} u\|^2_{L^2(K)} \\ &= \sum_{|\alpha|\le k} \|[(D^\alpha)_{\text{distr}} u]_\delta - [(D^\alpha)_{\text{distr}} u]\|^2_{L^2(K)} \to 0 \quad \text{as } \delta \to 0^+.\end{aligned}$$

Thus (d) is proved.

We now prove (e) by induction on k. If $k = 0$, then the differential operator A is given by $v \mapsto a \cdot v$ for some $\mathcal{C}^\infty$ function a on Ω. Therefore, if $u \in L^2(\Omega)$, then for each point $x \in K$ and each $\delta \in (0, \delta_0)$, the Schwarz inequality gives

$$\begin{aligned}&|A(u_\delta)(x) - (A_{\text{distr}} u)_\delta(x)|^2 \\ &\quad = \left| \int_{B(0;1)} (a(x) - a(x - \delta y)) u(x - \delta y) \kappa(y)\, d\lambda(y) \right|^2 \\ &\quad \le C^2 \cdot \left[\int_{B(0;1)} |u(x - \delta y)|^2 \kappa(y)\, d\lambda(y) \right] \cdot \int_{B(0;1)} \kappa(y)\, d\lambda(y) \\ &\quad = C^2 \cdot \left[\int_{B(0;1)} |u(x - \delta y)|^2 \kappa(y)\, d\lambda(y) \right],\end{aligned}$$

where $C \equiv 2 \max_{K^{\delta_0}} |a|$. Applying Fubini's theorem (Theorem 7.1.10), we get

$$\begin{aligned}&\|A(u_\delta) - (A_{\text{distr}} u)_\delta\|^2_{L^2(K)} \\ &\quad \le C^2 \cdot \int_{B(0;1)} \left[\int_K |u(x - \delta y)|^2\, d\lambda(x) \right] \kappa(y)\, d\lambda(y) \\ &\quad = C^2 \cdot \int_{B(0;1)} \left[\int_{K + (-\delta y)} |u(x)|^2\, d\lambda(x) \right] \kappa(y)\, d\lambda(y) \\ &\quad \le C^2 \cdot \int_{B(0;1)} \left[\int_\Omega |u(x)|^2\, d\lambda(x) \right] \kappa(y)\, d\lambda(y) = C^2 \|u\|^2_{L^2(\Omega)}.\end{aligned}$$

Assume now that $k > 0$ and that the inequality holds for operators of order $< k$. We have $A = A' + A''$, where A' and A'' are linear differential operators, each term in A' is of the form aD^α with $|\alpha| = k$, and A'' is of order $k - 1$. If $u \in W^{k-1}(\Omega)$ and $A_{\text{distr}} u \in L^2_{\text{loc}}(\Omega)$, then in particular, $A''_{\text{distr}} u$ exists and is in $L^2_{\text{loc}}(\Omega)$ by Lemma 11.1.3. Hence, by the induction hypothesis, there is a constant

$C'' > 0$ such that for each $\delta \in (0, \delta_0)$, we have

$$\|A''(u_\delta) - (A''_{\text{distr}}u)_\delta\|_{L^2(K)} \le C''\|u\|_{W^{\max(k-2,0)}(\Omega)} \le C''\|u\|_{W^{\max(k-1,0)}(\Omega)}$$

for every such function u. Furthermore, $A'_{\text{distr}}u = A_{\text{distr}}u - A''_{\text{distr}}u \in L^2_{\text{loc}}(\Omega)$ and

$$\begin{aligned}\|A(u_\delta) - (A_{\text{distr}}u)_\delta\|_{L^2(K)} \le \|A'(u_\delta) - (A'_{\text{distr}}u)_\delta\|_{L^2(K)} \\ + \|A''(u_\delta) - (A''_{\text{distr}}u)_\delta\|_{L^2(K)}.\end{aligned}$$

Thus we may assume without loss of generality that $A = A'$; i.e., $A = \sum_{|\alpha|=k} a_\alpha D^\alpha$ with $a_\alpha \in \mathcal{C}^\infty(\Omega)$ for each α.

For each $\delta \in (0, \delta_0)$ and each point $x \in K$, differentiation past the integral (Proposition 7.2.5) gives, for $u \in W^{k-1}(\Omega)$ with $A_{\text{distr}}u \in L^2_{\text{loc}}(\Omega)$,

$$\begin{aligned}&A(u_\delta)(x) - (A_{\text{distr}}u)_\delta(x) \\ &\quad= \int_{B(x;\delta)} u(y)A_x[\kappa^\delta(x-y)]\,d\lambda(y) - \int_{B(x;\delta)} [A_{\text{distr}}u(y)]\kappa^\delta(x-y)\,d\lambda(y) \\ &\quad= \int_{B(x;\delta)} u(y)A_x[\kappa^\delta(x-y)]\,d\lambda(y) - \int_{B(x;\delta)} u(y)\cdot({}^tA)_y[\kappa^\delta(x-y)]\,d\lambda(y).\end{aligned}$$

According to Lemma 11.1.3, we have ${}^tA = (-1)^kA + L$ for some linear differential operator L of order $k-1$ with $\mathcal{C}^\infty$ coefficients on Ω. We also have

$$\begin{aligned}A_x[\kappa^\delta(x-y)] &= \sum a_\alpha(x)D^\alpha_x[\kappa^\delta(x-y)] \\ &= \sum (-1)^k a_\alpha(x)D^\alpha_y[\kappa^\delta(x-y)] \\ &= (-1)^kA_y[\kappa^\delta(x-y)] + M[\kappa^\delta(x-y)],\end{aligned}$$

where M is the linear differential operator on $\Omega \times \Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ given by

$$M[h(x,y)] = \sum_{\alpha\in(\mathbb{Z}_{\ge 0})^n,\ |\alpha|=k} (-1)^k(a_\alpha(x) - a_\alpha(y))D^\alpha_y[h(x,y)].$$

Thus $A(u_\delta)(x) - (A_{\text{distr}}u)_\delta(x) = F(x,\delta) - G(x,\delta)$, where

$$F(x,\delta) \equiv \int_{B(x;\delta)} u(y)M[\kappa^\delta(x-y)]\,d\lambda(y)$$

and

$$G(x,\delta) \equiv \int_{B(x;\delta)} u(y)L_y[\kappa^\delta(x-y)]\,d\lambda(y).$$

Since ${}^tL$ is of order $k-1$ and $u \in W^{k-1}(\Omega)$, Lemma 11.1.3 implies that ${}^tL_{\text{distr}}u \in L^2_{\text{loc}}(\Omega)$ and $\|{}^tL_{\text{distr}}u\|_{L^2(K^{\delta_0})} \le C_0\|u\|_{W^{k-1}(\Omega)}$ for some constant $C_0 > 0$ that is independent of the choice of u. In particular, we have $G(x, \delta) = ({}^t L_{\mathrm{distr}} u)_\delta(x)$ for each $x \in K$, and, applying part (c), we get

$$\|G(\cdot, \delta)\|_{L^2(K)} \leq \|{}^t L_{\mathrm{distr}} u\|_{L^2(K^{\delta_0})} \leq C_0 \|u\|_{W^{k-1}(\Omega)}.$$

It remains to bound the L^2 norm of $F(\cdot, \delta)$. For each multi-index α with $|\alpha| = k$, we have $D^\alpha = D^{\alpha'} D_{j_\alpha}$ for some multi-index α' with $|\alpha'| = k - 1$ and some index $j_\alpha \in \{1, \dots, n\}$. Thus, if $h(x, y)$ is a $\mathcal{C}^\infty$ function on an open subset of $\Omega \times \Omega$, then

$$\begin{aligned}
M[h(x, y)] &= \sum_{|\alpha|=k} (-1)^k (a_\alpha(x) - a_\alpha(y)) \cdot D_y^{\alpha'}\big((D_{j_\alpha})_y (h(x, y))\big) \\
&= \sum_{|\alpha|=k} (-1)^k D_y^{\alpha'}\big((a_\alpha(x) - a_\alpha(y)) \cdot (D_{j_\alpha})_y (h(x, y))\big) \\
&\quad + \sum_{|\alpha|=k} (-1)^k \big[(a_\alpha(x) - a_\alpha(y)), D_y^{\alpha'}\big]\big((D_{j_\alpha})_y (h(x, y))\big) \\
&= \sum_{|\alpha|=k} (-1)^k D_y^{\alpha'}\big((a_\alpha(x) - a_\alpha(y)) \cdot (D_{j_\alpha})_y (h(x, y))\big) \\
&\quad - \sum_{|\alpha|=k} (-1)^k \big[a_\alpha(y), D_y^{\alpha'}\big]\big((D_{j_\alpha})_y (h(x, y))\big) \\
&= \sum_{|\alpha|=k} (-1)^k D_y^{\alpha'}\big((a_\alpha(x) - a_\alpha(y)) \cdot (D_{j_\alpha})_y (h(x, y))\big) + N_y(h(x, y)),
\end{aligned}$$

where $[\cdot, \cdot]$ denotes the commutator (see Lemma 11.1.3) and N is a linear differential operator of order $k - 1$ with $\mathcal{C}^\infty$ coefficients on Ω ($N = 0$ if $k = 1$). Note that the third equality holds because the function $a_\alpha(x)$ is constant with respect to the variables $y = (y_1, \dots, y_n)$ for each α. Therefore, since $N = {}^t({}^t N)$ and $u \in W^{k-1}(\Omega)$, Lemma 11.1.3 implies that for each point $x \in K$ and each $\delta \in (0, \delta_0)$, we have

$$F(x, \delta) = -\sum_{|\alpha|=k} H_\alpha(x, \delta) + ({}^t N_{\mathrm{distr}} u)_\delta(x),$$

where for each α,

$$H_\alpha(x, \delta) \equiv \int_{B(x;\delta)} (D_{\mathrm{distr}}^{\alpha'} u(y)) \cdot (a_\alpha(x) - a_\alpha(y)) \cdot (D_{j_\alpha})_y [\kappa^\delta(x - y)] \, d\lambda(y).$$

A construction similar to that of the bound on the L^2 norm of $G(\cdot, \delta)$ yields a constant $C_1 > 0$, which is independent of the choice of u and δ, such that

$$\|({}^t N_{\mathrm{distr}} u)_\delta\|_{L^2(K)} \leq C_1 \|u\|_{W^{k-1}(\Omega)}.$$

To bound the L^2 norm of each $H_\alpha(\cdot, \delta)$, we first observe that Lemma 7.2.4 implies that there is a constant $C_2 > 0$ independent of the choice of u and δ such that for each multi-index α with $|\alpha| = k$, for each point $x \in K$, and each point $y \in \overline{B(x;\delta)} \subset K^{\delta_0}$, we have $|a_\alpha(x) - a_\alpha(y)| \leq C_2\delta$ and

$$(D_{j_\alpha})_y[\kappa^\delta(x-y)] = (D_{j_\alpha})_y\left[\frac{1}{\delta^n}\kappa\left(\frac{x-y}{\delta}\right)\right] = -\frac{1}{\delta^{n+1}}(D_{j_\alpha}\kappa)\left(\frac{x-y}{\delta}\right).$$

Setting $v \equiv \mathrm{vol}(B(0;1))$ and $C_3 \equiv C_2 \max_{x\in\mathbb{R}^n,\,|\alpha|=k} |D_{j_\alpha}\kappa(x)|$, and applying the Schwarz inequality, we get, for each point $x \in K$ and each multi-index α with $|\alpha| = k$,

$$\begin{aligned} |H_\alpha(x,\delta)|^2 &\leq C_3^2\left[\int_{B(x;\delta)} |D_{\mathrm{distr}}^{\alpha'}u(y)|\delta^{-n}\,d\lambda(y)\right]^2 \\ &= C_3^2\left[\int_{B(0;1)} |D_{\mathrm{distr}}^{\alpha'}u(x-\delta y)|\,d\lambda(y)\right]^2 \\ &\leq C_3^2 \cdot v \cdot \int_{B(0;1)} |D_{\mathrm{distr}}^{\alpha'}u(x-\delta y)|^2\,d\lambda(y). \end{aligned}$$

Applying Fubini's theorem (Theorem 7.1.10), we get

$$\begin{aligned} \|H_\alpha(\cdot,\delta)\|_{L^2(K)}^2 &\leq C_3^2 \cdot v \cdot \int_{B(0;1)}\left[\int_K |D_{\mathrm{distr}}^{\alpha'}u(x-\delta y)|^2\,d\lambda(x)\right]d\lambda(y) \\ &= C_3^2 \cdot v \cdot \int_{B(0;1)}\left[\int_{K+(-\delta y)} |D_{\mathrm{distr}}^{\alpha'}u(x)|^2\,d\lambda(x)\right]d\lambda(y) \\ &\leq C_3^2 \cdot v^2 \cdot \|u\|_{W^{k-1}(\Omega)}^2. \end{aligned}$$

The claim (e) now follows.

For the proof of (f), we fix an open set Ω' with $K \subset \Omega' \Subset \Omega$ and a constant δ_0' with $0 < \delta_0' < \mathrm{dist}(K, \mathbb{R}^n \setminus \Omega')$. According to part (e), there is a constant $C > 0$ such that for every $\delta \in (0, \delta_0')$ and every function $u \in W_{\mathrm{loc}}^{\max(k-1,0)}(\Omega)$ with $A_{\mathrm{distr}}u \in L_{\mathrm{loc}}^2(\Omega)$, we have

$$\|A(u_\delta) - (A_{\mathrm{distr}}u)_\delta\|_{L^2(K)} \leq C\|u\|_{W^{\max(k-1,0)}(\Omega')}.$$

Given such a function u and a number $\epsilon > 0$, part (d) implies that we may choose a function $v \in \mathcal{C}^\infty(\Omega)$ such that $\|u - v\|_{W^{\max(k-1,0)}(\Omega')} < \epsilon/(4C)$. For each $\delta \in (0, \delta_0')$, we get

$$\begin{aligned} &\|A(u_\delta) - A_{\mathrm{distr}}u\|_{L^2(K)} \\ &\quad\leq \|A(u_\delta) - (A_{\mathrm{distr}}u)_\delta\|_{L^2(K)} + \|(A_{\mathrm{distr}}u)_\delta - A_{\mathrm{distr}}u\|_{L^2(K)} \\ &\quad\leq \|A((u-v)_\delta) - (A_{\mathrm{distr}}(u-v))_\delta\|_{L^2(K)} + \|A(v_\delta) - Av\|_{L^2(K)} \\ &\qquad + \|(Av)_\delta - Av\|_{L^2(K)} + \|(A_{\mathrm{distr}}u)_\delta - A_{\mathrm{distr}}u\|_{L^2(K)} \end{aligned}$$

$$< \frac{\epsilon}{4} + \|A(v_\delta) - Av\|_{L^2(K)} + \|(Av)_\delta - Av\|_{L^2(K)}$$
$$+ \|(A_{\mathrm{distr}}u)_\delta - A_{\mathrm{distr}}u\|_{L^2(K)}.$$

By part (b), for $\delta > 0$ sufficiently small, we have

$$\|A(v_\delta) - Av\|_{L^2(K)} < \epsilon/4,$$

and by part (d), for $\delta > 0$ sufficiently small, we also have

$$\|(Av)_\delta - Av\|_{L^2(K)} < \epsilon/4 \quad \text{and} \quad \|(A_{\mathrm{distr}}u)_\delta - A_{\mathrm{distr}}u\|_{L^2(K)} < \epsilon/4.$$

Thus $\|A(u_\delta) - A_{\mathrm{distr}}u\|_{L^2(K)} < \epsilon$, and the claim (f) is proved. □

11.4 The Sobolev Lemma

The goal of this section is the following fact, which allows one to obtain differentiability of functions possessing a high number of L^2 distributional derivatives:

Lemma 11.4.1 (Sobolev lemma) *For every open set $\Omega \subset \mathbb{R}^n$ and every $k \in \mathbb{Z}_{\geq 0}$, we have $W^{k+n}_{\mathrm{loc}}(\Omega) \subset \mathcal{C}^k(\Omega)$; that is, each function $u \in W^{k+n}_{\mathrm{loc}}(\Omega)$ is equal almost everywhere to a function of class $\mathcal{C}^k$.*

Proof Multiplying the given function by a $\mathcal{C}^\infty$ cutoff function and applying part (c) of Lemma 11.1.3, we see that we may assume without loss of generality that

$$\operatorname{supp} u \subset \overset{\circ}{K} \subset K \subset \Omega$$

for some compact set K. Lemma 11.3.1 then provides a sequence of functions $\{u_\nu\}$ in $\mathcal{D}(\Omega)$ such that $\operatorname{supp} u_\nu \subset K$ for each ν and

$$\|u_\nu - u\|_{W^{k+n}(\Omega)} = \|u_\nu - u\|_{W^{k+n}(K)} \to 0 \quad \text{as } \nu \to \infty.$$

Applying the inequality

$$\sup |\varphi| \leq \int_{\mathbb{R}^n} \left| \frac{\partial^n \varphi}{\partial x_1 \cdots \partial x_n} \right| \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$$

(see Exercise 11.4.1), we see that for every multi-index α with $|\alpha| \leq k$ and for all positive integers μ and ν, we have, for $\beta = \alpha + (1, \ldots, 1)$,

$$\sup_\Omega |D^\alpha u_\mu - D^\alpha u_\nu| \leq \|D^\beta u_\mu - D^\beta u_\nu\|_{L^1(K)} \leq (\lambda(K))^{1/2} \cdot \|D^\beta u_\mu - D^\beta u_\nu\|_{L^2(K)}$$
$$\leq (\lambda(K))^{1/2} \cdot \|u_\mu - u_\nu\|_{W^{k+n}(\Omega)} \to 0 \quad \text{as } \mu, \nu \to \infty.$$

Proposition 11.2.1 now implies that $u \in \mathcal{C}^k(\Omega)$. □

Exercises for Sect. 11.4

11.4.1 Prove the inequality

$$\sup|\varphi| \le \int_{\mathbb{R}^n} \left| \frac{\partial^n \varphi}{\partial x_1 \cdots \partial x_n} \right| \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

11.5 Proof of the Regularity Theorem

We now come to the proof of Theorem 11.0.1. Throughout this section, A denotes a linear differential operator of order 1 with $\mathcal{C}^\infty$ coefficients on an open set $\Omega \subset \mathbb{R}^n$. The first step is the following observation:

Lemma 11.5.1 *If the inequality* (1) *in Theorem* 11.0.1 *holds for some constant* $C > 0$ *and for every function* $u \in \mathcal{D}(\Omega)$, *then for every positive integer* k *and every compact set* $K \subset \Omega$, *there is a constant* $C(K, k) > 0$ *such that*

$$\|u\|^2_{W^k(K)} \le C(K,k) \cdot \left(\|u\|^2_{L^2(\Omega)} + \|Au\|^2_{W^{k-1}(\Omega)}\right) \quad \forall u \in D(\Omega). \tag{2}$$

Proof We proceed by induction on $k = 1, 2, 3, \ldots$. The case $k = 1$ follows from the (stronger) inequality (1). Thus we may assume that $k > 1$ and that the inequality (2) holds for every compact subset of Ω (for some choice of a constant) for strictly lower orders. We may fix a compact set K' with $K \subset K' \subset \Omega$ and a function $\rho \in \mathcal{D}(\Omega)$ with $\rho \equiv 1$ on K and $\operatorname{supp} \rho \subset K'$. Let L denote the linear differential operator of order 0 given by multiplication by ρ. Given a multi-index α with $|\alpha| = k$, we have $D^\alpha = D^{\alpha'} D_{j_\alpha}$ for some $j_\alpha \in \{1, \ldots, n\}$ and some multi-index α' with $|\alpha'| = k - 1$. For each function $u \in \mathcal{D}(\Omega)$, we get

$$A(D_{j_\alpha}(\rho u)) = D_{j_\alpha} L(Au) + [A, D_{j_\alpha} L]u,$$

where $D_{j_\alpha} L$ and $[A, D_{j_\alpha} L]$ are of order 1. Applying part (a) of Lemma 11.1.3, we get a constant $C_1(K, k) > 0$ independent of the choice of u and α such that

$$\begin{aligned}
\|D^\alpha u\|^2_{L^2(K)} &= \|D^{\alpha'} D_{j_\alpha}(\rho u)\|^2_{L^2(K)} \\
&\le C(K, k-1) \cdot \left(\|D_{j_\alpha}(\rho u)\|^2_{L^2(\Omega)} + \|A D_{j_\alpha}(\rho u)\|^2_{W^{k-2}(\Omega)}\right) \\
&= C(K, k-1) \cdot \left(\|D_{j_\alpha}(\rho u)\|^2_{L^2(K')} + \|A D_{j_\alpha}(\rho u)\|^2_{W^{k-2}(K')}\right) \\
&\le C_1(K, k) \cdot \left(\|u\|^2_{W^{k-1}(K')} + \|Au\|^2_{W^{k-1}(K')}\right) \\
&\le C_1(K, k) \cdot \Big[C(K', k-1)\left(\|u\|^2_{L^2(\Omega)} + \|Au\|^2_{W^{k-2}(\Omega)}\right) \\
&\qquad + \|Au\|^2_{W^{k-1}(K')}\Big] \\
&\le C_1(K, k) \cdot (C(K', k-1) + 1) \cdot \left[\|u\|^2_{L^2(\Omega)} + \|Au\|^2_{W^{k-1}(\Omega)}\right].
\end{aligned}$$

The claim now follows. $\square$

Theorem 11.0.1 now follows immediately from the Sobolev lemma (Lemma 11.4.1) together with the following fact:

Lemma 11.5.2 *If the inequality* (1) *in Theorem* 11.0.1 *holds for some constant* $C > 0$ *and for every function* $u \in \mathcal{D}(\Omega)$, *then for every nonnegative integer* k, *we have*

$$\left\{u \in L^2_{\mathrm{loc}}(\Omega) \mid A_{\mathrm{distr}}u \in W^{\max(k-1,0)}_{\mathrm{loc}}(\Omega)\right\} \subset W^k_{\mathrm{loc}}(\Omega).$$

Proof Given a nonnegative integer k and a function $u \in L^2_{\mathrm{loc}}(\Omega)$ with $A_{\mathrm{distr}}u \in W^{\max(k-1,0)}_{\mathrm{loc}}(\Omega)$, we must show that $u \in W^k_{\mathrm{loc}}(\Omega)$. We proceed by induction on k. The case $k = 0$ is trivial, so let us assume that $k > 0$ and the claim holds for the Sobolev spaces of order $< k$. In particular, $u \in W^{k-1}_{\mathrm{loc}}(\Omega)$. In order to show that $u \in W^k_{\mathrm{loc}}(\Omega)$, we may assume without loss of generality that u has compact support in Ω. For given a compact subset of Ω, we may choose a (cutoff) function $\rho \in \mathcal{D}(\Omega)$ such that $\rho \equiv 1$ on the subset. Lemma 11.1.3 then gives

$$A_{\mathrm{distr}}(\rho u) = \rho \cdot A_{\mathrm{distr}}u + [A, \rho]u \in W^{k-1}(\Omega),$$

since $v \mapsto \rho v$ and $[A, \rho]$ are operators of order 0 and $A_{\mathrm{distr}}u$, $u \in W^{k-1}_{\mathrm{loc}}(\Omega)$. Thus we may assume that $\operatorname{supp} u \subset \overset{\circ}{K} \subset K \subset \Omega$ for some compact set K.

Applying the (strong) Friedrichs lemma (Lemma 11.3.1), we get a sequence of functions in $\{u_\nu\}$ in $\mathcal{D}(\Omega)$ such that $\operatorname{supp} u_\nu \subset K$ for each ν and, as $\nu \to \infty$,

$$\|u_\nu - u\|_{W^{k-1}(\Omega)} = \|u_\nu - u\|_{W^{k-1}(K)} \to 0 \quad \text{and} \quad \|Au_\nu - A_{\mathrm{distr}}u\|_{W^{k-1}(\Omega)} \to 0.$$

Note that for the convergence of the first sequence, we have applied part (d) of the lemma, while for the second sequence, we have applied part (f) to the operator $D^\alpha A$ for each multi-index α with $|\alpha| \le k - 1$, and we have also used the fact that $(D^\alpha A)_{\mathrm{distr}}u = D^\alpha_{\mathrm{distr}}A_{\mathrm{distr}}u$ (Lemma 7.4.3). By Lemma 11.5.1, for some constant $C' > 0$ and for all positive integers μ and ν,

$$\|u_\mu - u_\nu\|^2_{W^k(\Omega)} = \|u_\mu - u_\nu\|^2_{W^k(K)} \le C' \cdot \left(\|u_\mu - u_\nu\|^2_{L^2(\Omega)} + \|Au_\mu - Au_\nu\|^2_{W^{k-1}(\Omega)}\right).$$

Therefore, since the right-hand side converges to 0 as $\mu, \nu \to \infty$ and $W^k(\Omega)$ is a Hilbert space, we have $u \in W^k(\Omega)$, and the lemma follows. □

Exercises for Sect. 11.5

11.5.1 Let A be a linear differential operator of order 1 with $\mathcal{C}^\infty$ coefficients on an open set $\Omega \subset \mathbb{R}^n$. Assume that the inequality (1) in Theorem 11.0.1 holds for some constant $C > 0$ and for every function $u \in \mathcal{D}(\Omega)$. Assume also that the coefficients of A and their partial derivatives of all orders are bounded on Ω. Prove that for every positive integer k, there is a constant $C_k > 0$ such that

$$\|u\|^2_{W^k(\Omega)} \le C_k \cdot \left(\|u\|^2_{L^2(\Omega)} + \|Au\|^2_{W^{k-1}(\Omega)}\right) \quad \forall u \in D(\Omega).$$

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Notation Index

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