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As isothermic as possible surface parameterization

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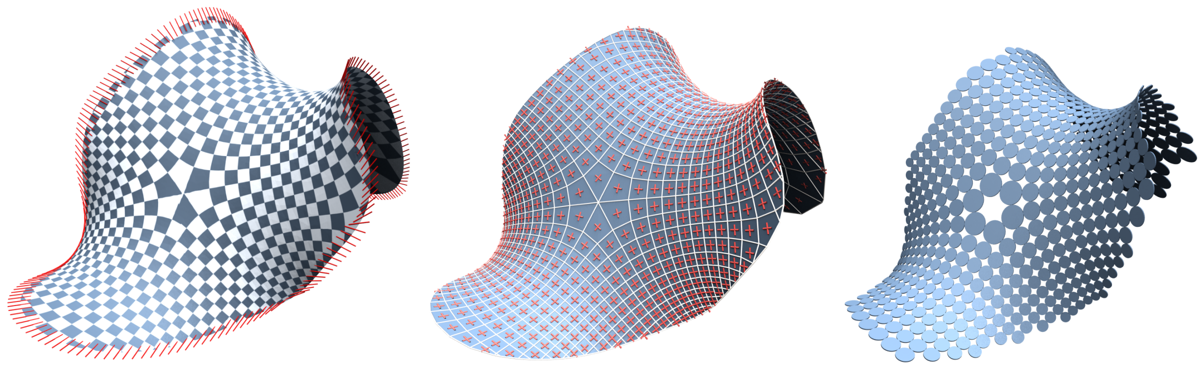

Figure 1: *For a triangulated surface we calculate texture coordinates by solving a boundary value problem for principle curvature directions on boundary edges (checker board texture and red directions). The edges of the corresponding quad mesh align with the curvature directions (red crosses). The mesh is then optimized towards planar quads with touching incircles.*

Abstract

Isothermic surfaces are surfaces that allow for conformal curvature line parameterizations. They include surfaces of constant mean curvature, minimal surfaces, surfaces of revolution, and conic sections.
Starting from a triangulation of an arbitrary surface with disk topology, we construct a parameterization that is as-isothermic-as-possible. We use special boundary conditions for the discrete conformal parameterization and state a new variational principle for discrete isothermic surfaces. When applied to a triangulation of an isothermic surface our approach recovers the exact structure of the conformal curvature line parameterization. The resulting quad mesh consists of planar faces with touching incircles. No integration of vector fields is needed.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Geometric algorithms, languages, and systems

1. Introduction

Our work is mainly motivated by applications to architectural models. One key problem is to convert triangle meshes created by form finding methods or physical simulation to quadrilateral meshes, which are preferred for glass-steel structures. Roofs that act shell-like turn out to have almost constant mean curvature. These are the kinds of surfaces that initiated our study of conformal curvature line parameterizations. Both conformality and alignment with curvature lines turn out to be favorable properties for parameterizations needed in this context. A recent trend in free-form architecture is also looking for visually pleasing mesh structures and cost optimized geometries. Repetitive patterns like circle patterns or packings on surfaces have been investigated and used on real buildings. For references to research in this direction we refer to the next section.

Different kinds of parameterizations of triangulated surfaces have been studied by many researchers in the field. As in the smooth theory, it is often easier to investigate properties of a surface patch in the parameter domain than di-

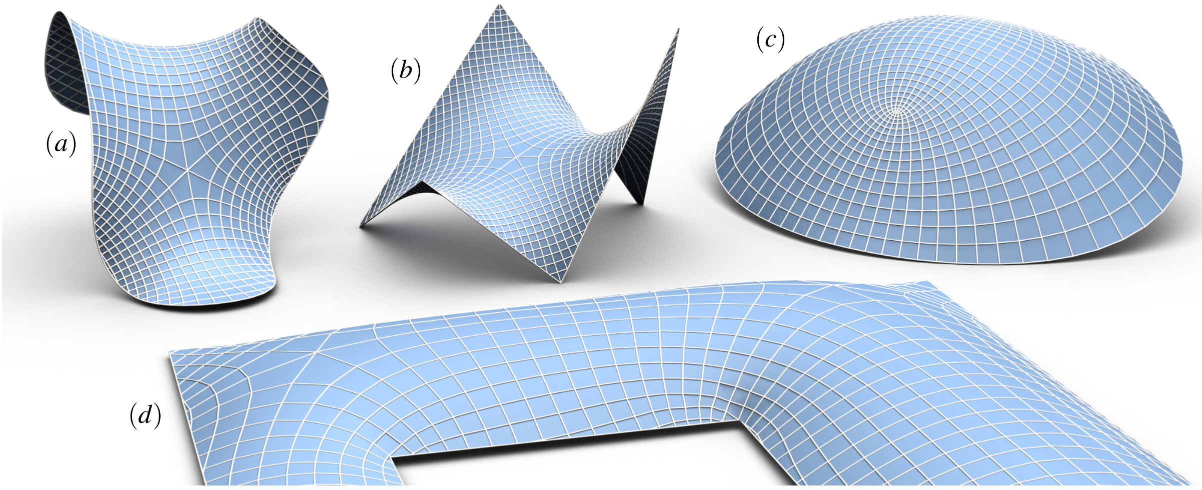

Figure 2: *The surfaces of this paper gathered for a group shot. All have been parameterized and remeshed. (a) The **Teaser** surface is the minimizer of a spring energy with a smooth fixed boundary curve. (b) A **Minimal** surface with polygonal boundary curve. (c) **Dome**: Part of a NURBS surface exhibiting positive curvature and two curvature field singularities. (d) Architectural **Roof** structure with planar boundary curve and regions of positive and negative curvature*

rectly on the embedding. The research concerning parameterizations is driven by a wide range of applications in computer graphics, geometry processing, or architectural geometry. Special parameterizations adapted from differential geometry serve various purposes. Conformal mappings are angle preserving and a good candidate for example for texture mapping. Whereas curvature line parameterizations capture the geometry of the surface. Parameterizations along conjugate directions allow for planar faces.

We propose a surface parameterization technique that creates parameterizations that are on the one hand conformal and on the other hand capture curvature information. In case of isothermic surfaces we obtain a discrete conformal curvature line parameterizations in the sense of discrete differential geometry. For non-isothermic surfaces the parameter lines coincide with the curvature directions on the boundary. This leads to a definition of a quality measure for the "isothermicity" of a parametrization. To construct almost isothermic parameterizations, we define special boundary conditions for a discrete conformal mapping to the plane. Our scheme uses curvature information on the boundary only and creates curvature lines in the interior of the surface automatically without explicit integration. This approach is build on top of the conformal mapping technique of Springborn, Schröder, and Pinkall [SSP08]. Hence we inherit the speed and superior projective mapping properties of their parameterizations.

We use the obtained parameterization for remeshing and optimize the quadrilaterals to have touching incircles by minimizing a novel energy. These S-isothermic meshes have been studied in discrete differential geometry and possess some remarkable properties, e.g., minimal S-isothermic surfaces may be deformed isometrically retaining the same Gauß map. We show how to construct this 1-parameter family of isometric surfaces even if the surface is only approximately S-isothermic minimal.

The rest of the paper is organized as follows: Section 2 gives an overview over existing parameterization schemes and their relation to our approach. We also give reference to the related mathematical literature in discrete differential geometry. In Section 3 we define a quality measure for as-isothermic-as-possible parameterizations and describe an algorithm to obtain isothermic parameterizations for isothermic surfaces. Further we discuss how to deal with singularities and non-isothermic surfaces. A variational principle to generate S-isothermic surfaces is presented in Section 4. Moreover we explain how to deform minimal surfaces isometrically. At the end of the section we show the effect of our optimization on several examples from different classes of surfaces. In Section 5 we sum up the results and propose extensions and enhancements subject to further research.

2. Related Work

There has been considerable work on conformal parameterization as well as curvature line parameterizations both of which are of course related to our as-isothermic-as-possible parameterizations. For general background on mesh parameterization we refer to the surveys by Floater and Hormann [FH05] and Sheffer, Praun, and Rose [SPR06]. Our approach is based on the discrete conformal equivalence of triangle meshes introduced in Springborn, Schröder, and Pinkall [SSP08] (see Bobenko, Pinkall, and Springborn [BPS10] for the mathematical background). The convex functional optimized in Springborn *et al.* constructs a conformally equivalent flat mesh for specified boundary conditions and singularities. A different approach based on circle patterns is used by Kharevych, Springborn, and Schröder [KSS06] to produce conformal mappings from a surface to the plane. Global conformal parameterizations for surfaces of arbitrary topology were constructed by Gu and Yau [GY03]. Parameterization aligning with lines of principle curvature were constructed by Alliez *et al.* [ACSD*03]. Global parameterizations following arbitrary frame fields (including in particular principle curvature fields) are constructed in Kälberer, Nieser, and Polthier [KNP07]. They use discrete Hodge decomposition and harmonic vector fields to obtain a globally consistent parameterization. Their QuadCover algorithm can deal with surfaces of arbitrary genus and treats singularities using a suitable branched cover.

The use of variational principles to enforce desired properties (e.g. planarity of quadrilateral faces) for surface meshes has been successfully used in geometry processing and in particular in architectural geometry. Liu *et al.* [LPW*06] propose an algorithm to optimize a quadrilateral mesh to become planar and even conical. Pottmann *et al.* [PSB*08] use functionals to approximate freeform surfaces with single curved panels. The energy minimized in Section 4 is a combination of a new functional with an energy recently described by Schiftner *et al.* [SHWP09]. They construct circle packing triangle meshes that approximate a given surface by minimizing a combination of energies. Discrete S-isothermic minimal surfaces are defined in terms of their Gauß map in Bobenko, Hoffmann, and Springborn [BHS06]. This Gauß map is a Koebe polyhedron with edges tangent to a sphere. These Koebe polyhedra also occur in the study of edge offset meshes by Pottmann *et al.* [PLW*07], which again use a variational approach to obtain support structures. Another parametrization technique creating quad dominant meshes guided by conjugate parameter direction is given by Zadravec, Schiftner, and Wallner [ZSW10]. Their algorithm includes a very nice level set approach to circumvent the integration of a vector field.

The notion of discrete S-isothermic meshes was introduced in the mathematical context by Bobenko and Pinkall [BP99] as a special class of quad meshes. The mathematical theory of these meshes has since then been an active field of research in discrete differential geometry. A good overview of the recent development and literature can be found in the book [BS08]. The treatment of minimal surfaces and their associated family of isometric surfaces can be found in Bobenko, Hoffmann, and Springborn [BHS06]. An algorithm to construct triangulated minimal surfaces with given boundary conditions is defined in Pinkall and Polthier [PP93].

3. As-isothermic-as-possible parameterization

We start out with a triangle mesh $M = (V,E,F)$ with disk topology and vertices V, edges E, and faces F. Because of the simple topology the mesh may be oriented consistently by choosing normals for the faces or ordering the vertices of the triangles. We call a map $\Phi : V \to \mathbb{R}^2$, $v \mapsto \Phi(v_i) = (x_i, y_i)$ preserving the orientation a *discrete parameterization* of the surface M. This map naturally extends to edges and faces of the mesh.

We recall the definition of discrete conformal equivalence of triangle meshes given by Springborn *et al.* [SSP08]. It is stated in terms of edge lengths of the surface edges and corresponding parameter edges in the xy-plane.

Definition 3.1 A discrete parameterization Φ is *conformal* if there exist a function $u : V \to \mathbb{R}$, $v_i \mapsto u_i := u(v_i)$ such that the following condition for the edge lengths l and $\tilde{l}$ of the edges $e = (v_i, v_j)$ and $\Phi(e)$ holds

$$\tilde{l} = e^{(u_i+u_j)/2} l. \tag{1}$$

Springborn *et al.* [SSP08] show how to find a function u for a given triangle mesh by minimizing a convex functional.

To construct parameterizations aligning with given directions, in particular principle curvature directions, we consider a function $\alpha : E \to [-\frac{\pi}{2}, \frac{\pi}{2}[$ on the edges of the surface. We call a parameterization Φ an α-*parameterization* if for any $e = (v_i, v_j) \in E$

$$\tan\alpha(e) = \frac{x_i - x_j}{y_i - y_j} \tag{2}$$

with $\Phi(v_i) = (x_i, y_i)$ and $\Phi(v_j) = (x_j, y_j)$, i.e., the angle between the edges $\Phi(e)$ in the plane and the y-axis is exactly $\alpha(e)$. Moreover, if the function α is defined by principle curvature directions we call the corresponding α-parameterization a *discrete curvature line parameterization*. Summing up the above definitions we obtain the following

Definition 3.2 A parameterization Φ is a *discrete isothermic parameterization* if it is a discrete conformal curvature line parameterization.

As not all surfaces admit a conformal curvature line parameterization we propose the following quality measure for a discrete conformal parametrization Φ and an angle function α. For an edge e_i let α_i be the prescribed angle from the curvature vector field. Let $\tilde{\alpha}_i$ be the angle between the edge $\Phi(e)$ and the y-axis in the parameter domain. Then

$$Q^{\alpha}(e) \; := \; |\alpha_i - \tilde{\alpha}_i|$$

expresses how good the parametrization matches the prescribed angle constraints at a single edge. By summing over all edges of the input mesh we obtain a measure to determine the isothermicity of a parametrization:

$$Q^{\alpha}(\Phi) = \frac{1}{\#E} \sum_{e_i \in E} |\alpha_i - \tilde{\alpha}_i| \,. \tag{3}$$

Surely, the functional depends on the quality of the curvature directions encoded by α. But once the angle function α is fixed, we obtain a functional on the space of conformal parameterizations. We call a minimizer of this functional an *as-isothermic-as-possible parameterization*. The parameters are the data needed to uniquely determine a discrete conformal parametrization, e.g., the boundary angles and the singularities with associated curvatures.

Idea. If we specify appropriate conditions on the boundary arising from one of the principle curvature directions and place singularities with suitable curvatures, we can construct a conformal parameterization that aligns with these principle curvature directions. For isothermic surfaces this will produce a discrete isothermic parameterization for the entire surface in the sense of the stated functional, i.e., $Q^{\alpha}(\Phi)$ will be small.

3.1. Curvature Boundary Conditions

Let Y be a map that assigns a direction Y_i to every edge e_i of the triangulation. This information can be transformed into an angle map $\alpha : E \to [-\frac{\pi}{2}, \frac{\pi}{2}[$ of signed angles between the edges $e_i \in E$ and the directions Y_i as used in the previous section. Let e_i and e_j be two consecutive boundary edges with common vertex v and let γ_v be the sum of interior angles on the surface. To align the associated directions $Y_i = Y(e_i)$ and $Y_j = Y(e_j)$ with the y-axis in the plane we calculate θ_{ij} between the edges $\Phi(e_i)$ and $\Phi(e_j)$ in the parameter plane such that Y_i and Y_j are parallel and $\left|\gamma_v - \theta_{ij}\right| \leq \frac{\pi}{2}$. See Figure 3 for an illustration of the alignment in the parameter plane. These θ_{ij} serve as an input for the discrete conformal parameterization. For a field of curvature directions on boundary edges we assume that a smooth curvature field does not turn more that $\pi/2$ per boundary vertex. Therefore $\left|\gamma_v - \theta_{ij}\right| \leq \frac{\pi}{2}$ induces a consistent orientation around the boundary.

3.2. Singularities

After calculating the boundary angles, we need to introduce some cone singularities in the interior of the surface to satisfy the discrete Gauß-Bonnet Theorem. There exists an analog of the smooth Gauß-Bonnet for discrete surfaces that relates the Gaussian and the boundary curvature to the Euler characteristic. During parameterization we construct a metric that is flat everywhere except for cone singularities where positive or negative curvatures are introduced. For the purpose of curvature line parameterizations we can only have cone points with discrete curvatures of π, 0, or $-k\pi$ at singularities of the curvature direction field. The Gaussian curvature κ_i at interior vertices v_i is the angle defect, i.e., $\kappa_i = 2\pi - \theta_i$, where θ_i is the sum of the angles at the vertex v_i. For a boundary vertex w_j the corresponding geodesic curvature is defined by $\kappa_j^g = \pi - \theta_j$. So if we split the vertex set $V = V_b \cup V_i$ into boundary vertices V_b and interior vertices

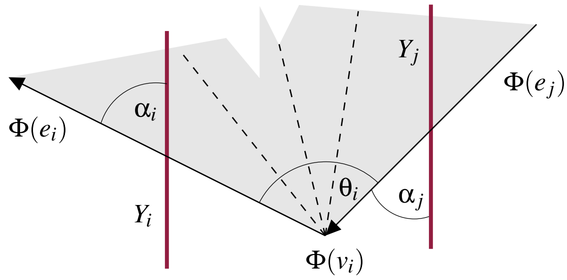

Figure 3: *The angle $\theta_i = \angle(-\Phi(e_j), \Phi(e_i))$ at the vertex $\Phi(v_i)$ has to be chosen such that curvature directions Y_i and Y_j align with the y-axes and that θ_i does not differ by more than $\frac{\pi}{2}$ from the interior angle sum on the surface.*

V_i the discrete Gauß-Bonnet Theorem becomes:

$$\sum_{v_i \in V_i} \kappa_i + \sum_{w_j \in V_b} \kappa_j^g = 2\pi\chi, \tag{4}$$

where χ is the Euler characteristic of the surface ($\chi = 1$ for disks). Since all Y directions in parameter space are parallel, the boundary curvature adds up to a multiple of π. If this sum happens to differ from 2π, there must be singularities in the curvature field and we have to compensate the deficit at interior vertices to satisfy Eq. (4). In Figure 4 the boundary curvature sum of the domain is 4π. So we had to pick two singularities each of curvature $-\pi$ to satisfy the Gauß-Bonnet equation.

3.3. Implementation

The algorithm to compute a new parameterization of a surface is the following:

1. Generate curvature directions
2. Compute boundary angles
3. Compute conformal parameterization
4. Create new mesh using projective texture coordinates

To estimate principle curvature fields we use the method of Cohen-Steiner and Morvan [CSM03], where the curvature tensor is averaged over a disk of a given radius centered at edge midpoints. This defines the angle function α. From the directions and angles we deduce the angles θ for the boundary vertices as described in Section 3.1. These angles are the boundary curvatures we plug into the algorithm of Springborn *et al.* to obtain a conformal parameterization. If necessary we pick singularities for the curvature field and prescribe corresponding cone angles. A consistent singularity choice can easily be checked using Eq. (4). We layout the new edges in the texture coordinate plane such that an arbitrary boundary edge $\Phi(e)$ intersects the y-axis in the desired angle $\alpha(e)$. By construction the intersection angles coincide with the prescribed α's for all boundary edges. Since the domain of parameterization is not free of curvature we have

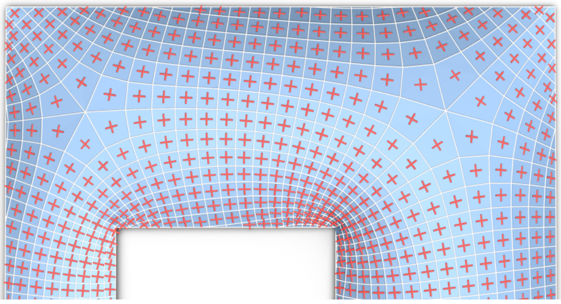

Figure 4: *Parameterization of the* ***Roof*** *model. The discrete curvature lines approximate curvature directions with high quality. See Section 3.4 for a discussion.*

to cut along paths from cone points to the boundary of the domain. The layout overlaps if singularities with negative curvature are used. To create seamless parameter lines we use the rectification approach described in [SSP08]. Finally we create a new mesh based on a regular xy-grid in $\mathbb{R}^2$. The remeshing process is carried out as a subdivision step followed by some cleanup and regluing: We use the projective interpolation in the texture domain as proposed in [SSP08] to increase the quality of the result. Previously cut paths from singularities to the boundary are sewed up to obtain the final remesh.

3.4. Examples and Quality

With the functional $Q^{\alpha}(\Phi)$ introduced in Eq. (3) we are now able to measure the quality of our parameterizations. There are two kinds of examples to consider: The first class of meshes stems from smooth surfaces that admit conformal curvature line parameterizations, i.e., triangulations approximating isothermic surfaces. The second class consists of arbitrary non-isothermic surfaces. For an almost isothermic surfaces we expect our parameterization to reconstruct the isothermic coordinates up to numerical precision and hence the functional to be reasonably small. For non-isothermic surfaces we achieve the correct directions on the boundary but lack accuracy in the interior. In Table 1 we summarize the numerical results obtained from the surfaces of Figure 2.

Isothermic surfaces. The class of smooth isothermic surfaces contains surfaces of constant mean curvature, surfaces of revolution, and conic sections. We use the **Minimal** example as an instance of an isothermic surface with mean curvature zero (Figure 2b). It was created using the software JavaView [PHPR07] and is a minimizer of the Dirichlet energy introduced in [PP93]. As expected this surface exhibits the highest curvature line quality of all tested meshes. The error however cannot vanish completely since the surface's curvature field contains singularities. In the vicinity of these points the numerical curvature directions contain significant amounts of noise.

	$\#E$	$\#\partial E$	$Q^{\alpha}(\Phi)$	$Q_{\max}$	Q_{σ}
Minimal	6260	450	0.036	0.603	0.033
Teaser	17550	1000	0.057	1.20	0.066
Roof	3766	470	0.051	0.610	0.059
Dome	1900	350	0.133	1.52	0.157

Table 1: *The curvature line approximation quality of the examples. $\#\partial E$ are the number of boundary edges. Q_{σ} is the standard deviation of $Q^{\alpha}(e)$.*

Non-isothermic surfaces. Non-isothermic surfaces are surfaces that do not admit a parameterization with conformal curvature lines. We investigate the properties of surfaces (a), (c), and (d) displayed in Figure 2.

The **Teaser** surface was created as a minimizer of a spring functional fixing the boundary and modelling interior edges as springs of rest length zero. It is not far away from a minimal surface with the same boundary. The curvature line pattern however differs substantially as it contains singularities whereas the minimal surface with this boundary curve does not. The quality of the curvature line pattern is also very high. The mean angle error of the numerical directions is 3.2 degrees. Note that the deviation Q_{σ} from the mean value as it is also very low. For this surface the coordinates generated by our algorithm are a globally good approximation to conformal curvature lines.

A quality plot of the **Roof** surface (Figures 2d and 4) is shown in Figure 5. Surprisingly the quality of the curvature lines is a high as in the **Teaser** or the **Minimal** case. This suggests that a slight variation of the surface yields an isothermic surface. See also Figure 4 for a visual impression of the quality of the curvature lines.

The **Dome** model (Figure 2c) is created from a NURBS parameterization. The quality plot (Figure 5) reveals areas of high angle deviation especially around the singularities. Here we have a mean error of 7.6 degrees and a deviation of 9.0 degrees. Other areas, in particular those near the boundary, are of high curvature line quality. The distance to the nearest isothermic surface is expected to be larger than in the previous examples. More evidence for this is given in Section 4.

Discussion. Our parameterization scheme works well for surfaces that are not too far away from surfaces that possess isothermic coordinates. In the case of surfaces stemming from minimal or constant mean curvature surfaces we get almost perfect approximation quality of curvature lines. These are surfaces that are particularly interesting when designing beam layouts for roof structures. For other surfaces the parameterization is conformal and the parameter line pattern captures the combinatorics of the curvature line pattern while approximating the curvature line geometry.

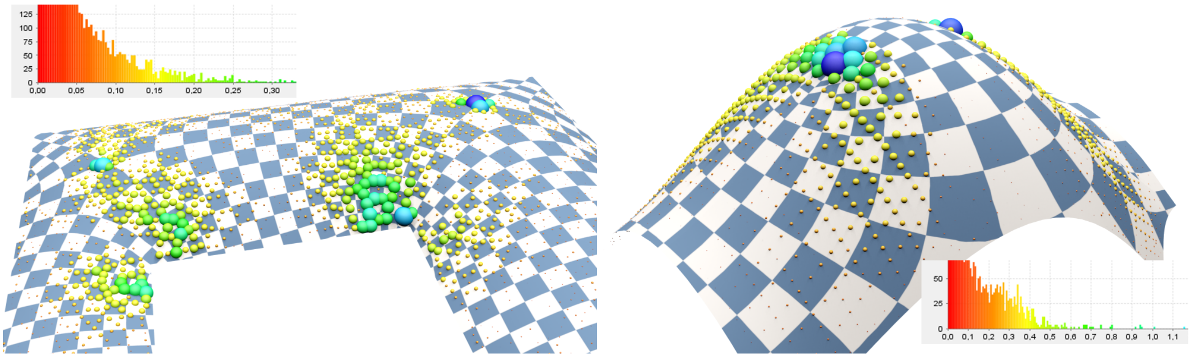

Figure 5: *The quality of the parameterization is measured in radians per edge. Small beads represent good curvature direction quality, big beads represent high deviation. The color of the histogram corresponds to the color of the beads. Note that the mean error of the* ***Roof****'s surface is half the error of the* ***Dome****. See also Table 1 for detailed quality measures of the other surfaces.*

4. Discrete S-isothermic Surfaces

Starting with an isothermically or almost isometrically parameterized mesh using the described algorithm we aim to create discrete S-isothermic surfaces that stay in the vicinity of the input surface. S-isothermic surfaces were introduced by Bobenko and Pinkall [BP99]: A quadrilateral mesh is *S-isothermic* if (i) all the quadrilaterals are planar, (ii) all faces have incircles, and (iii) the incircles of adjacent quadrilaterals touch. Figure 8 displays S-isothermic surfaces derived from our parametrization shown in Figure 2 as described in the previous section.

4.1. Variational Principle

In this section we introduce an energy whose minimizers are S-isothermic surfaces. The S-isothermic energy E_S consists of three parts:

$$E_S := \lambda_1 E_{\text{planar}} + \lambda_2 E_{\text{incircle}} + \lambda_3 E_{\text{touch}} \tag{5}$$

The planarity energy E_{planar} penalizes non-planar quadrilateral faces. For each quad it can be defined either by the distance of the diagonals (an idea attributed to Schröder in Pottmann *et al.* [PSB*08]) or the volume of the tetrahedron spanned by the four vertices. For E_{incircle} we use the energy defined by Schiftner *et al.* [SHWP09] based on the fact that the sum of opposite edge lengths must be equal for a planar quad to possess an incircle.

The energy E_{touch} is a new energy that enforces touching incircles if the faces are planar and possess incircles. It is defined per edge, see Figure 6 for the exact labeling of the angles at one edge. We define

$$E_{\text{touch}}(e) = \left(\cot\frac{\alpha_i}{2}\cot\frac{\beta_j}{2} - \cot\frac{\alpha_j}{2}\cot\frac{\beta_i}{2}\right)^2. \tag{6}$$

for an interior edge of the triangulation. On boundary edges the energy is zero. The energy can be easily formulated in terms of the vertex coordinates and the derivatives can be calculated explicitly.

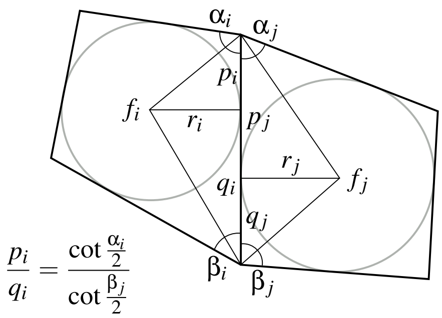

Figure 6: *Labels for the touching-circles-functional at an edge. The circles touch if the ratio of p and q is equal on both sides of the edge.*

We need a good initial guess to find meaningful minimizers of E_S. S-isothermic minimal surfaces converge to isothermic parameterizations of smooth minimal surfaces [BHS06]. For general S-isothermic surfaces this is an open conjecture. The parameterizations obtained in Section 3 are good candidates to start from with the optimization of the functional.

We use the non-linear optimization package PETSc/TAO [BBB*11, BMM*07] and its java binding [Som10] to find minimizers of E_S. Figure 7 shows convergence plots of E_S for the four models that were discussed in the previous section.

As seen in the quality analysis of Section 3.4 the **Teaser**, the **Minimal**, and the **Roof** models are close-to-isothermic surfaces. For these models the corresponding S-isothermic surface is also close to the input surface. Figure 7 shows the energy during the optimization. Here the three close-to-isothermic meshes start with a lower energy than the **Dome**

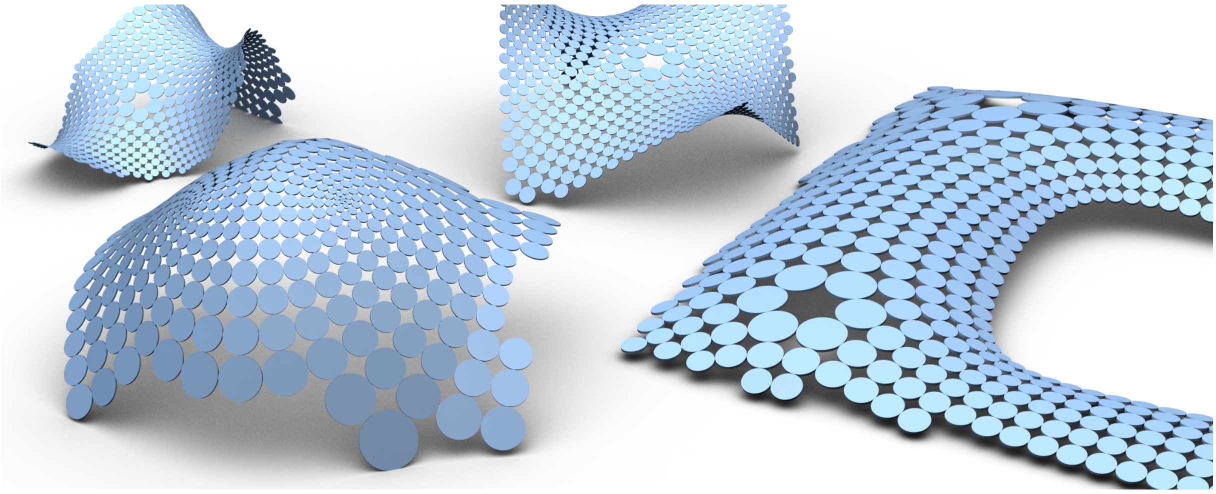

Figure 8: *S-isothermic meshes created from the models presented in Figure 2. The inner quadrilaterals are optimized towards touching incircles. A series of touching circles in a row can be interpreted as discrete curvature line.*

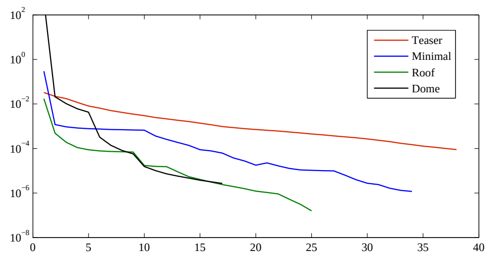

Figure 7: *Convergence behavior of E_S during optimization. We use the meshes displayed in Figure 2 as initial guesses for the minimization. The convergence of the Teaser geometry is slower due to the high complexity*

model. After the **Dome** has passed some iterations it exhibits convergence properties similar to the other models. As the surface converges against a discrete S-isothermic surface, we observe a considerable change in shape during the first iterations especially around the singularities.

Concluding we emphasize that the shapes obtained by optimization of the functional are S-isothermic by definition. We have demonstrated for the **Dome** shape how this method can approximate non-isothermic surfaces by nearby S-isothermic meshes.

4.2. Isometric Deformations of Minimal Surfaces

Every smooth minimal surface possesses a 1-parameter (associated) family of non-trivial isometric deformations. All surfaces in this family have the same Gauß map. For discrete S-isothermic minimal surfaces this construction is discretized in [BHS06]: Edge lengths and the conformality of the parameterization are preserved. We need to introduce the concept of dual surface to construct the family of isometric surfaces.

The Dual Surface. In differential geometry for isothermic surfaces there is a notion of a dual surface. This dual or Christoffel transform is also an isothermic surface. Both surfaces are parameterized with isothermic coordinates. This setup can be discretized using the definition of discrete S-isothermic surfaces. The dual surface can be constructed using the incircle structure of S-isothermic meshes. We introduce consistent signs on edges on a discrete S-isothermic surface such that opposite sides of the quads have the same sign and consecutive edges in a quad have different signs. The dual mesh of a mesh M will have parallel edges calculated as follows: Let $e_{ij} = v_j - v_i$ be an edge vector of M. Then the dual edge vector e^*_{ij} satisfies the following equation:

$$e^*_{ij} = \pm \frac{1}{r_i \cdot r_j} e_{ij}$$

where r_i and r_j are the distances from the vertex v_i and v_j to the touching point of the incircles of incident faces at the edge e_{ij}. For a given discrete S-isothermic mesh we can easily calculate the dual mesh by enumerating the vertices along a spanning tree of edges. In Figure 9 the dual surface of the **Minimal** model is shown in the upper left hand corner. Via the dual surface we can now construct isometric deformations of minimal surfaces.

Deformation. The dual of a smooth minimal surface coincides with its Gauß map which is a part of a sphere. This sphere may be multiply covered. For discrete S-isothermic minimal surfaces this Gauß map is a part of a Koebe poly-

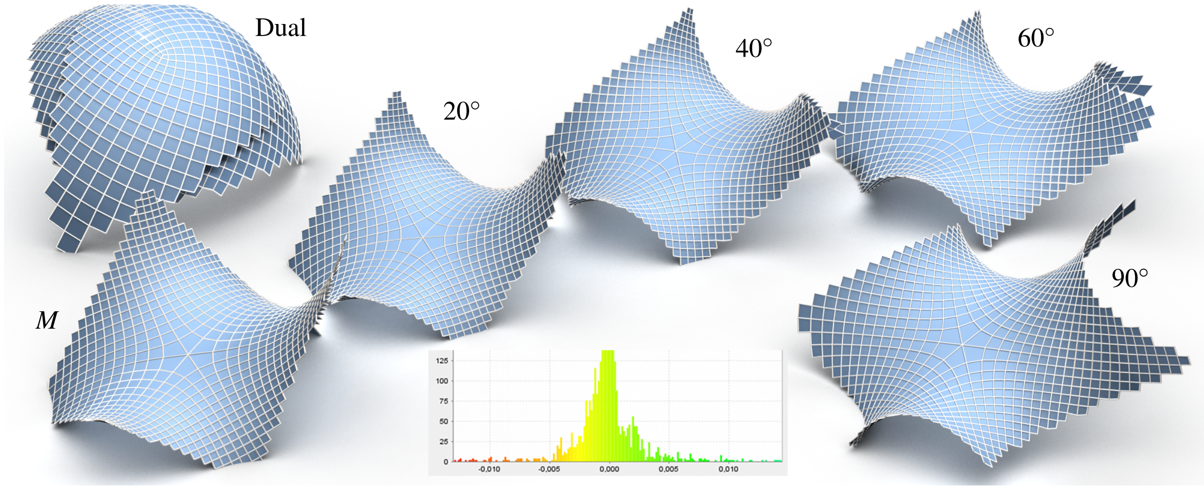

Figure 9: *Non-trivial isometric deformations of a minimal surface. The edges of the dual surface are rotated to create a* 1*-parameter family of isometrically deformed surfaces with period* 2π*. The histogram shows the edge length error of the* 90° *model when compared with the initial surface. The mean edge length deviation of this model is* 1.28%*.*

hedron, i.e., a polyhedral surfaces with edges tangent to a sphere. On the Koebe polyhedron every edge is rotated by a fixed angle in the tangent space of the sphere at the points of tangency of the edge. The resulting edge vectors again form closed quads and can be dualized. This dual surface has the same edge lengths as the initial minimal surface.

For minimal quad meshes with touching incircles that were created using our parameterization and the optimization step, the dual surface will be close to a Koebe polyhedron. We use a least-squares-sphere to define a consistent tangent space at the touching points of incircles with the edges. The resulting deformation of a given minimal surface is then close to isometric. To distribute the isometry error on the edges we average over different roots of the layout spanning tree. Surfaces that do not possess exact isometric deformations are deformed approximative.

We apply this procedure to the **Minimal** model (Figure 2a). Figure 9 shows the surface together with its dual and isometrically deformed versions with different turning angles.

5. Conclusions and Future Research

The main contributions of this article are, on the one hand, a new algorithm to compute special parameterization of surfaces, and on the other hand, a new energy for meshes with touching incircles.

Previous approaches to the problem of curvature line parameterization have been proposed by many authors: Kälberer, Nieser, and Polthier [KNP07] are able to produce high quality parameterizations for general frame fields on triangulated surfaces of arbitrary genus. Zadravec, Schiftner, and Wallner [ZSW10] are also motivated by applications to architecture and optimize the structure of the mesh using conjugate directions and a level set method for remeshing.

We see the main advantage of the algorithm presented in Section 3 in its simplicity and its applicability to shell-like roof structures which arise in architectural models. Since these models often have almost constant mean curvature and thus allow for an almost isothermic parameterization our algorithm performs particularly well on these examples.

We use fixed boundary conditions during parameterization, thus the quality of the meshes in terms of the quality functional can be improved. This leads to a control problem where the boundary angles become variables in an optimization step. We are in the process of investigating this control problem. Our current approach yields good approximations to the real minimizers of the functional and in case of isothermic surfaces they coincide. One could also propose different quality functionals that take into account that the curvature directions are unstable in the vicinity of nearly umbilic regions.

The new energy described in Section 4 is closely related to the article of Schiftner *et al.* [SHWP09] dealing with circle packings meshes. They explicitly do not treat quadrilateral meshes since they are aware of the shape restrictions and focus on triangle meshes instead. This leads to beautiful circle packing meshes with additional statical benefits. The shape restriction lies in the core of isothermic surfaces but did not influence our results dramatically for the surfaces in our focus. How to approximate arbitrary surfaces by isothermic surfaces is unknown and will be subject to future research. The results of Section 4 suggest that this might be possible using related methods.

Our new functional generates quad circle packing meshes in the sense of Schiftner *et al.*. For surfaces arising in architectural context (in particular for shell-like roofs) we are able to construct aesthetically pleasing quad meshes supporting a circle packing. This additional structure may be used to deform the surface if it is close to a minimal surface. We know from the smooth theory that such deformations are also possible for general constant mean curvature surfaces but this has not been discretized, yet. For non-isothermic surfaces we were also able to construct nearby planar quad circle packing surfaces.

The study of different mesh types has recently been a very active field of research in discrete differential geometry. It would be interesting to see further connections between the considered meshes in mathematics and applications in computer graphics and geometry processing. Also deformations of special mesh classes would be interesting for modelling or for constructing mobile structures in architecture.

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