Discrete differential geometry of surfaces. Variational principles, algorithms, and implementation

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Discrete Uniformization

The discrete uniformization theory persented here is based on the notion of discrete conformal eqivalence of triangle meshes. The Euclidean definition was first considered by [Luo04], the variational principle and applications in computer graphics is due to [SSP08, BPS10]. The notion of conformal equivalence of non-Euclidean metrics and corresponding variational principles were first defined in [BPS10]. [Guo10] investigate the gradient flow of this principle. Most of the material presented here can be found in [BSS].

2.1 Discrete Riemann surfaces

Definition 1. *A* discrete surface *is a collection of triangles equipped with a metric of constant Gaussian curvature and geodesic edges. Triangles are glued along edges to form a surface.*

Generically a discrete surface can have boundary components. We consider this case in Section 2.7. By glueing triangles equipped with a metric of constant curvature we obtain a surface that has constant curvature except for points where the metric has cone-like singularities (Figure 2.1). A discrete surface is called Euclidean for $K = 0$, hyperbolic for $K = -1$, and spherical if $K = 1$.

Definition 2. *The map* $l : E \to \mathbb{R}$ *of triangle edge lengths of a discrete surface is called a* discrete Euclidean, hyperbolic, or spherical metric *respectively.*

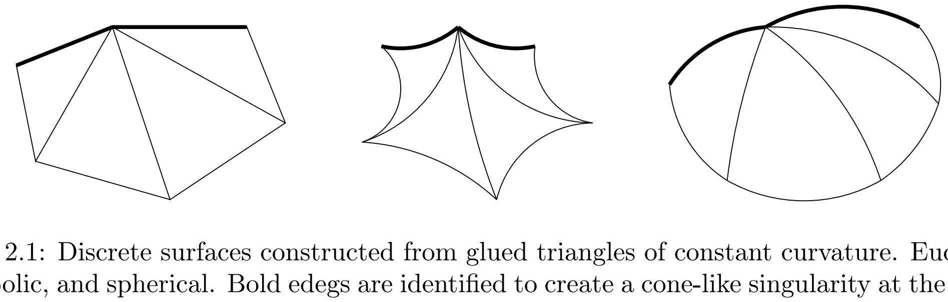

Figure 2.1: Discrete surfaces constructed from glued triangles of constant curvature. Euclidean, hyperbolic, and spherical. Bold edegs are identified to create a cone-like singularity at the vertex.

2.2 Variational principles

2.2.1 Discrete conformal equivalence

Definition 3. *Two Euclidean triangulations T and $\tilde{T}$ are* discretely conformally equivalent *if there is a map $u : V \to \mathbb{R}$ such that for any edge ij it is*

$$l_{ij} = e^{u_i + u_j} \tilde{l}_{ij}$$

where l_{ij} is the length of the edge ij.

Definition 4. *A* discrete flat Euclidean metric *is a map $l : E \to \mathbb{R}_+$ such that triangle inequalities are satisfied and angle sums around each inner vertex are equal to 2π.*

2.2.2 Variational principles for discrete metrics in $\mathbb{E}^2$, $\mathbb{H}^2$, and $\mathbb{S}^2$

Construction of discrete flat metrics. A discrete Euclidean flat metric is the minimizer of a convex functional.

$$\begin{aligned}
\lambda_{ij} &:= 2 \log l_{ij} && (2.1)\\
\tilde{\lambda}_{ij} &:= \lambda_{ij} + u_i + u_j && (2.2)\\
f_{Euc}(u_i, u_j, u_k) &:= \alpha_i \tilde{\lambda}_{jk} + \alpha_j \tilde{\lambda}_{ki} + \alpha_k \tilde{\lambda}_{ij} + 2\left(Л(\alpha_i) + Л(\alpha_j) + Л(\alpha_k)\right) && (2.3)
\end{aligned}$$

Definition 5.

$$E_{Euc}(u) := \sum_{ijk \in F} \left(f_{Euc}(u_i, u_j, u_k) - \frac{\pi}{2} \left(\tilde{\lambda}_{jk} + \tilde{\lambda}_{ki} + \tilde{\lambda}_{ij} \right) \right) + \sum_{i \in V} \Theta_i u_i \qquad (2.4)$$

This definition and the derivatives can be found in [BPS10]

For the hyperbolic case λ and $\tilde{\lambda}$ are defined as before. Further define

$$\begin{aligned}
\beta_i &:= \frac{1}{2}(\pi + \alpha_i - \alpha_j - \alpha_k) && (2.5)\\
\beta_j &:= \frac{1}{2}(\pi - \alpha_i + \alpha_j - \alpha_k) && (2.6)\\
\beta_k &:= \frac{1}{2}(\pi - \alpha_i - \alpha_j + \alpha_k) && (2.7)\\
f_{Hyp}(u_i, u_j, u_k) &:= \beta_i \tilde{\lambda}_{jk} + \beta_j \tilde{\lambda}_{ki} + \beta_k \tilde{\lambda}_{ij} && (2.8)\\
&\quad + Л(\alpha_i) + Л(\alpha_j) + Л(\alpha_k) + Л(\beta_i) + Л(\beta_j) + Л(\beta_k) && (2.9)\\
&\quad + Л\left(\frac{1}{2}(\pi - \alpha_i - \alpha_j - \alpha_k)\right) && (2.10)
\end{aligned}$$

Definition 6.

$$E_{Hyp}(u) := \sum_{ijk \in F} \left(f_{Hyp}(u_i, u_j, u_k) - \frac{\pi}{2} \left(\tilde{\lambda}_{jk} + \tilde{\lambda}_{ki} + \tilde{\lambda}_{ij} \right) \right) + \sum_{i \in V} \Theta_i u_i \qquad (2.11)$$

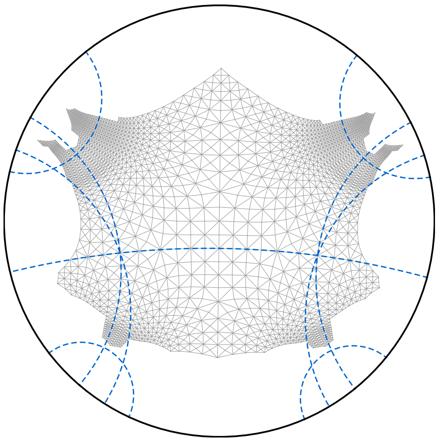

Figure 2.2: Hyperbolic flat metric on a genus 2 surface and the axes of the associated hyperbolic motions.

2.2.3 Realization

2.3 Uniformization of surfaces of higher genus

Triangulated surfaces of genus $g \geq 2$ without boundary can be equipped with a discretely conformally equivalent flat hyperbolic metric [BPS10]. By flat hyperbolic metric we mean that the edge length are hyperbolic and for any vertex the angle sum is 2π. To realize this metric in the hyperbolic plane e.g. in the Poicaré disk model one has to introduce cuts along a basis of the homotopy. This creates a simply connected domain in $\mathbb{H}^2$. Matching cut paths are realated by a hyperbolic motion i.e. the Möbius transformations that leave the unit disk invariant (Figure 2.2).

2.3.1 The cut-graph and fuchsian groups

Want so say here: the number of transformations generated by the mapping of corresponding edges equals the number of path segments in the homotopy-cut-graph. They generate a fuchsian group with #vertices relations

Proposition 1.

2.3.2 Minimal presentation

2.4 Canonical fundamental domains of fuchsian groups

2.4.1 Separated handles

2.4.2 Opposite sides identified

2.5 Uniformization of elliptic and hyperelliptic surfaces

2.5.1 Elliptic Functions

2.5.2 The moduli space

2.5.3 Numerical convergence analysis

2.5.4 The modulus of the Wente torus

2.5.5 Construction of hyperelliptic surfaces

Any hyperelliptic Riemann surface can be expressed as an algebraic curve of the form

$$\mu^2 = \prod_{i=1}^{n} (\lambda - \lambda_i)^2 \qquad n \geq 3, \quad \lambda_i \neq \lambda_j \forall i \neq j.$$

Here λ_i are the branch points of the doubly covered Riemann sphere.

2.5.6 Weierstrass points on hyperelliptic surfaces

A hyperelliptic surface comes together with a holomorphic involution h called the hyperelliptic involution. The branch points are fixed points under this transformation. For a hyperelliptic algebraic curve it is $h(\mu, \lambda) = (-\mu, \lambda)$

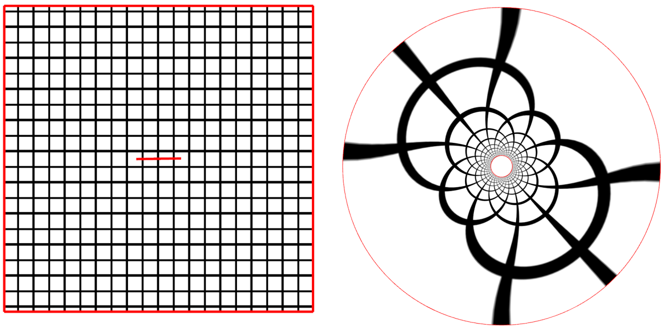

Figure 2.3: Square with symmetric slit to the circle

2.5.7 Canonical domains

2.5.8 Lawsons surface

2.6 Conformal Mapping to $\hat{\mathbb{C}}$ (Planned)

2.6.1 Selection of Branch Data

2.6.2 Examples

2.7 Simply and multiply connected domains

2.7.1 Variation of edge length

2.7.2 Examples

2.7.3 Comparison with Examples of the Schwarz-Christoffel community

Chapter 3

Discrete Surface Parameterization

3.1 Discrete quasiisothermic parametrizations

The notion of quasiconformal parameterizations

3.1.1 Discrete quasiisothermic parameterizations

3.1.2 Formulation as boundary value problem

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3.1.4 Variational principle for S-isothermic surfaces

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3.2 Gridshells and Applications in Architecture

3.2.1 Tschebyscheff Meshes

3.2.2 Variational Principle

3.2.3 Examples

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