## 0.1 Periodic boundary conditions

{sec:boundary}

If we want to construct periodic conformal maps we are allowed to specify angle sums $\theta_v$ at boundary vertices. The condition for the sums of boundary angles differs from the plane case in the following way: The curvature at a boundary vertex $v$ is given by $\kappa_v = \pi - \theta_v$, where $\theta_v$ is the angle sum of the adjacent triangles in the target mesh. For the two boundary loops $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ we have:

$$\sum_{i=1}^{n} \kappa_{v_i} + \sum_{j=1}^{m} \kappa_{w_j} = 0. \tag{1}$$ {eq:theta}

This condition makes sure that the two boundary curves "bend" the same amount and can hence be wrapped around a cone. We will now show how boundary conditions can be used to construct periodic patterns on the studied models. We start with a discrete conformal map of the doubly-curved model from Figure **??** to a standard cylinder.

*Straight cylinder.* The simplest way to generate a map to the cylinder is to set the target angles for all boundary vertices to $\pi$. Hence the curvatures at the boundary vertices are zero and the two boundary loops are mapped to "straight" curves. In this case both angle sums of equation 1 vanish and the target mesh can be wrapped around a cylinder, see Figure **??**, left. The new edge lengths computed with the variational principle correspond to the lengths on a cylinder. This cylinder can be unrolled in the plane preserving angles and lengths. So the two boundary polygons are mapped to straight lines in the plane. These two straight lines have to be parallel and of equal lengths. If the lengths of the boundary curves in the original model differ a lot, then a map to a cylinder induces a lot of conformal stretch. This stretch can be reduced by specifying special boundary conditions for a parameterization on a cone of revolution.

*Cone of revolution.* As long as equation (1) is satisfied we obtain a map to a general cone of revolution. In our case, we require that the periodic parameterization is adapted to the target pattern. This means that the two sums of equation (1) need to be (the same) multiples of $\frac{\pi}{3}$ (triangle or hex) or multiples of $\frac{\pi}{2}$ (quad). We present two methods to achieve this requirement: a uniform distribution and a concentration of curvature.

If the boundary of the mesh should to be aligned with the pattern, then boundary angles need to be quantized, i.e., multiples of $\frac{\pi}{3}$ or $\frac{\pi}{2}$ need to be chosen as target angles. In Figure **??**, middle, three vertices of the top and bottom boundary curve were manually assigned to $\frac{4}{3}\pi$ and $\frac{2}{3}\pi$, respectively. All other boundary angles are set to $\pi$, i.e., straight. Such a map can be used as a starting point to obtain a tesselation with equal hexagons as described in Section **??**.

It is known that a confomal map that does not change the length of the boundary exhibits the least stretch in the interior of the surface. To obtain such a parameterization we first construct a periodic conformal mapping onto an arbitrary cone such that the lengths of the boundary edges are not changed. The resulting angle sums

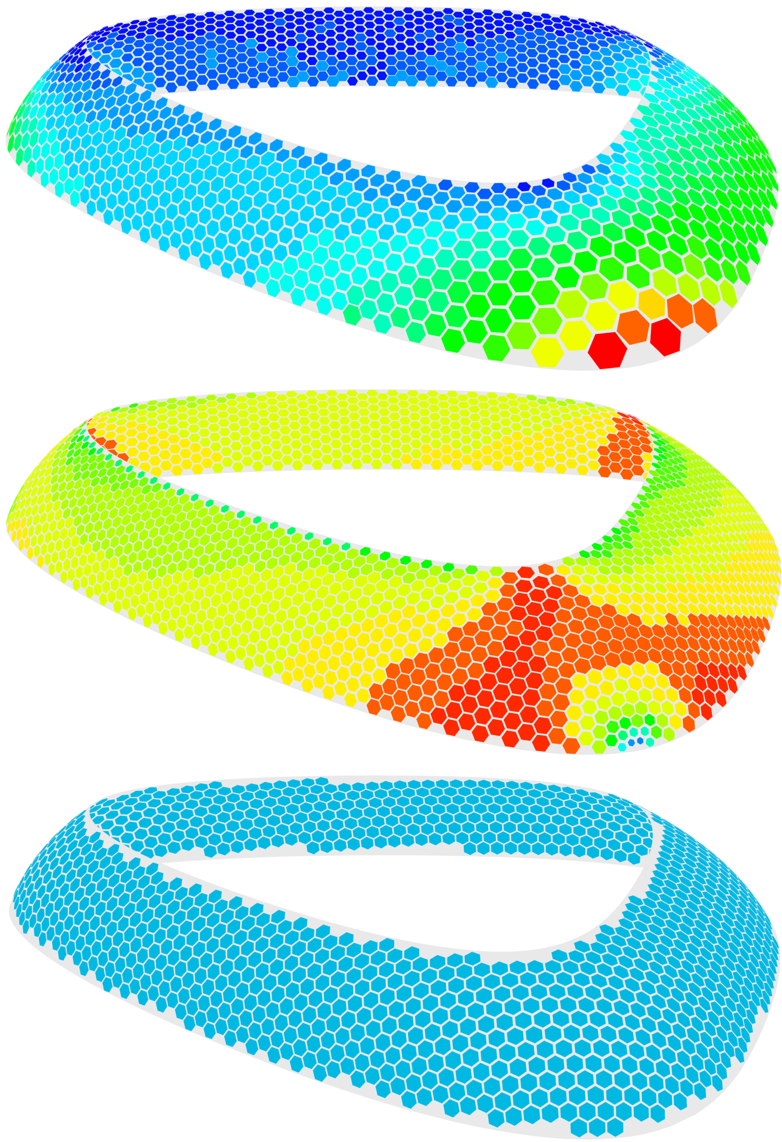

Figure 1: Quantized periodic hexagonal panelizations. Boundary conditions affect the amount of stretch in the interior of the surface. Top: Hexagonal pattern aligns with the boundary, a strong condition that produces large deviation of edge lengths. Middle: Map to a pattern-adapted polygon on a cone of revolution. The pattern contains exceptional points at the boundary. The stretch is minimized while at the same time the pattern alignes with the boundary. Bottom: Conformal map with the least stretch in the interior, pattern can be optimized to consist of congruent hexagons alone. In all images, panels with the same color are congruent. The corresponding domains of parameterization are shown in Figure **??**.

{fig:hex_example}

at boundary vertices of the target mesh determine the cone angle of the map. The cone angle of a pattern adapted periodic parameterization is the closest multiple of the desired quantization. We distribute the difference to the closest quantized angle uniformly to the individual boundary vertices and recompute the map with these angle conditions. The obtained map is periodic and exhibits the lowest stretch of all periodic conformal maps (see Figure **??**, right).

*Design and structural opportunities.* It is also possible to use special boundary conditions to support structural purposes or design requirements. If one aims for a panelization with boundary aligned patterns, then the target boundary angles must be quantized.

To include entrances in a facade it is possible to incorporate special boundary conditions. An example with special boundary vertices with domain angles $\frac{4}{3}\pi$ and $\frac{\pi}{3}$ is shown in Figure 2, left. In the remeshed surface, the lower boundary curve bends inside at the vertices with angle $\frac{4}{3}\pi$ around the vertex with angle $\frac{\pi}{3}$. This incorporates a natural entrance into the facade.

Another effect of such angle conditions is a densification of the pattern at the vertices with small angles. Such a concentration of elements can be used to enforce structural properties of a geometry. An example of a diagrid generated with such boundary conditions is shown in Figure 2, right.

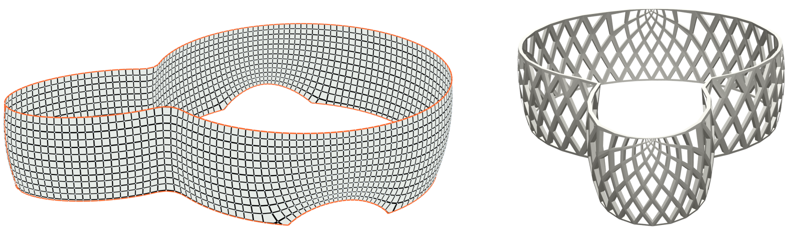

Figure 2: A periodic conformal map onto a cylinder with special vertices creates the opportunity to incorporate entrances (left) or concentration of support structure (right).

{fig:entrance}