Discrete differential geometry of surfaces. Variational principles, algorithms, and implementation

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1 Introduction

2 Discrete conformal maps

Definition 1. *Two triangulations T and $\tilde{T}$ are* discrete conformal equivalent *if there is a map $u : V \to \mathbb{R}$ such that for any edge ij it is*

$$l_{ij} = e^{u_i + u_j} \tilde{l}_{ij}$$

where l_{ij} is the length of the edge ij.

Definition 2. *A* discrete flat metric *is a map $l : E \to \mathbb{R}_+$ such that triangle inequalities are satisfied and angle sums around each inner vertex are equal to 2π.*

2.1 Euclidean

Construction of discrete flat metrics. A discrete Euclidean flat metric is the minimizer of a convex functional.

$$\begin{aligned} \lambda_{ij} &:= 2 \log l_{ij} && (1) \\ \tilde{\lambda}_{ij} &:= \lambda_{ij} + u_i + u_j && (2) \\ f_{Euc}(u_i, u_j, u_k) &:= \alpha_i \tilde{\lambda}_{jk} + \alpha_j \tilde{\lambda}_{ki} + \alpha_k \tilde{\lambda}_{ij} + 2\left(\text{Л}(\alpha_i) + \text{Л}(\alpha_j) + \text{Л}(\alpha_k)\right) && (3) \end{aligned}$$

Definition 3.

$$E_{Euc}(u) := \sum_{ijk \in F} f_{Euc}(u_i, u_j, u_k) - \frac{\pi}{2}\left(\tilde{\lambda}_{jk} + \tilde{\lambda}_{ki} + \tilde{\lambda}_{ij}\right) + \sum_{i \in V} \theta_i u_i \tag{4}$$

The gradient of the Euclidean functional is then givien by

$$\frac{\partial E_{Euc}}{\partial u_i} = \sum_{ij \in E}$$

2.2 Hyperbolic

For the hyperbolic case λ and $\tilde{\lambda}$ are defined as before. Further define

$$\begin{aligned} \beta_i &:= \frac{1}{2}\left(\pi + \alpha_i - \alpha_j - \alpha_k\right) && (5) \\ \beta_j &:= \frac{1}{2}\left(\pi - \alpha_i + \alpha_j - \alpha_k\right) && (6) \\ \beta_k &:= \frac{1}{2}\left(\pi - \alpha_i - \alpha_j + \alpha_k\right) && (7) \\ f_{Hyp}(u_i, u_j, u_k) &:= \beta_i \tilde{\lambda}_{jk} + \beta_j \tilde{\lambda}_{ki} + \beta_k \tilde{\lambda}_{ij} && (8) \\ &\quad + \text{Л}(\alpha_i) + \text{Л}(\alpha_j) + \text{Л}(\alpha_k) + \text{Л}(\beta_i) + \text{Л}(\beta_j) + \text{Л}(\beta_k) && (9) \\ &\quad + \text{Л}\left(\frac{1}{2}(\pi - \alpha_i - \alpha_j - \alpha_k)\right) && (10) \end{aligned}$$

Definition 4.

$$E_{Hyp}(u) := \sum_{ijk \in F} f_{Hyp}(u_i, u_j, u_k) - \frac{\pi}{2}\left(\tilde{\lambda}_{jk} + \tilde{\lambda}_{ki} + \tilde{\lambda}_{ij}\right) + \sum_{i \in V} \theta_i u_i \tag{11}$$

3 Uniformization of elliptic curves

3.1 Elliptic Functions

3.2 Numerical convergence analysis

4 Uniformization of hyperelliptic curves

4.1 Construction

Any hyperelliptic Riemann surface can be expressed as an algebraic curve of the form

$$\mu^2 = \prod_{i=1}^{n} (\lambda - \lambda_i)^2 \qquad n \geq 3, \quad \lambda_i \neq \lambda_j \forall i \neq j.$$

Here λ_i are the branch points of the doubly covered Riemann sphere.

4.2 Canonical domains

5 Discrete isothermic parametrizations

5.1 A discrete ellipsoid and its dual surface