UNIFORMIZATION, MODULI, AND KLEINIAN GROUPS†

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On May 18, 1967 I had the honour of giving the first G. H. Hardy Lecture before the London Mathematical Society, and I was to submit an expository article based on the lecture for publication by the Society. I apologize that it took me so long to do this; as reparation, the present survey covers more ground than the lecture did and takes into account the developments of the last three years.

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1. OLD AND NEW UNIFORMIZATION THEOREMS

1. *Uniformization*

To uniformize means to represent parametrically by single-valued holomorphic or meromorphic functions.

More precisely, let Σ be a set in $\mathbb{C}^N$, the complex N-space, and let $f_1, \ldots, f_N$ be N meromorphic functions defined in some domain D of $\mathbb{C}$ (or of $\mathbb{C}^m$, for some $m > 1$). We say that the functions $f_1, \ldots, f_N$ *uniformize* Σ if the following conditions are satisfied: (i) Let $D_0 \subset D$ be the set on which $f = (f_1, \ldots, f_N)$ is defined as an N-tuple of complex numbers; the set Σ is the *closure* of $f(D_0)$.

(ii) There is a *discrete* group G of holomorphic selfmappings of D such that $f(t_1) = f(t_2)$ if and only if $t_2 = g(t_1)$ for some $g \in G$. (There is a similar definition for subsets of a projective space $\mathbb{P}_N$.)

For instance, the circle $\zeta_1{}^2 + \zeta_2{}^2 = 1$ can be uniformized by the transcendental functions $\zeta_1 = \cos\theta$, $\zeta_2 = \sin\theta$ and also, which is more interesting, by the rational functions

$$\zeta_1 = \frac{1-t^2}{1+t^2}, \quad \zeta_2 = \frac{2t}{1+t^2}.$$

In the first case, $D_0 = \mathbb{C}$ and G is generated by the translation $z \mapsto z + 2\pi$. In the second case, $D_0 = \mathbb{C} - \{i, -i\}$ and $G = 1$, the trivial group.

A cubic curve

$$\zeta_2{}^2 = 4(\zeta_1 - e_1)(\zeta_1 - e_2)(\zeta_1 - e_3),$$

where e_1, e_2, e_3 are distinct numbers, with $e_1 + e_2 + e_3 = 0$, cannot be uniformized by rational functions. But it admits a parametric representation $\zeta_1 = \wp(z)$, $\zeta_2 = \wp'(z)$ where $\wp$ is the Weierstrass $\wp$-function with periods w_1 and w_2

$$\wp(z) = z^{-2} + \Sigma\{(z - mw_1 - nw_2)^{-2} - (mw_1 - nw_2)^{-2}\},$$

Here w_1 and w_2 are appropriately chosen complex numbers, $\tau = w_1/w_2$ has a positive imaginary part, and the summation is extended over all $(m, n) \in Z^2$, $(m, n) \neq (0, 0)$. In this case, G is the elliptic group generated by the two translations $z \mapsto z + w_1$, $z \mapsto z + w_2$.

The uniformization problem for *arbitrary* plane algebraic curves reads: represent all pairs of complex numbers (ζ_1, ζ_2) satisfying the equation

$$P(\zeta_1, \zeta_2) = 0,$$

as single-valued functions of a complex variable. Here P is an irreducible polynomial with complex coefficients.

With an algebraic curve, there is associated a non-negative integer p, its genus, and the solution of the uniformization problem depends on this number. If $p = 0$, the curve can be uniformized by rational functions; if $p = 0$, the curve can be uniformized by rational functions; if $p = 1$, the curve can be uniformized by elliptic functions. If $p > 1$, uniformization necessarily requires functions defined only on proper subsets of $\mathbb{C}$. Indeed, *all* algebraic curves can be uniformized by functions defined in the upper half-plane $U = \{z \mid z = x+iy \in \mathbb{C}, y > 0\}$.

The solution set of every polynomial equation in two unknowns can be uniformized by meromorphic functions defined in U, or is a finite union of sets with this property.

(The second case occurs when $P(\zeta_1, \zeta_2)$ is not irreducible but a product of distinct factors.)

This theorem (or rather a more precise form of it, to be stated later) was conceived, by Klein and by Poincaré, independently, about 90 years ago. Poincaré also generalized the theorem to arbitrary *analytic* curves, that is, to the case where P is not a polynomial, but a convergent power series in two variables, together with all its analytic continuations. It took some of the best mathematicians of the world 25 years to give rigorous proofs of this and some related theorems; the first complete proofs were found by Poincaré and by Koebe in 1907. Many basic concepts and methods of contemporary mathematics were developed explicitly for establishing uniformization. The list includes covering spaces, methods for solving non-linear elliptic partial differential equations, existence and distortion theorems for conformal mappings, topological dimension, invariance of domain.

Accounts of the classical uniformization theory will be found in the second volume of Fricke–Klein [57] (monumental and hard to read), in the second volume of Appel–Goursat [18] (written by Fatou), in the treatises by Ford [55], and by Nevanlinna [95], and in most standard texts on Riemann surfaces (Weyl [111], Pfluger [99], Springer [106], Ahlfors–Sario [16]). It is of interest to consult also the original works by Klein (see [67] which contain also the amusing correspondence with Poincaré), Poincaré [100, 101] and Koebe [69–72].

Recently, the theory of quasiconformal mappings was found to provide an easy access to uniformization theory [25]. Furthermore, quasiconformal mappings can be used to prove a stronger statement.

It is possible to uniformize simultaneously, by functions of several variables, all algebraic curves of a fixed genus p with a fixed number n of distinguished points.

(The precise meaning of "uniformizing simultaneously" and the proof of the statement just made will be explained in 3, §4. The classical model for simultaneous uniformization is the Weierstrass $\wp$-function, considered as a function of z and τ.)

An important application of simultaneous uniformization of curves to higher dimensional agebraic varieties has been discovered by P. A. Griffiths **[61]**. His approach (which we sketch in 3, §6) implies the following.

The set of common zeros of any number of polynomial equations in $N \geqslant 2$ unknown can be uniformized by N meromorphic functions defined in some bounded domain in some $\mathbb{C}^r$, or is a finite union of sets with this property.

It would be interesting to know whether or when the adjective "polynomial" can be replaced by "analytic".

A basic tool in establishing any uniformization theorem is the concept of a Riemann surface.

2. *Riemann Surfaces*

A Riemann surface is a connected one-dimensional complex manifold, that is, an orientable surface, in the sense of topology, on which certain complex valued continuous functions, defined on open subsets, are designated as *holomorphic*. Among these, there must be functions which map sufficiently small neighbourhoods of any point on the surface homeomorphically onto domains in the complex plane. Such functions are called *local parameters*, and a function defined near a point on the surface is holomorphic at this point if and only if it can be written as a power series in a local parameter. The system of holomorphic functions (or, in today's language, the sheaf of germs of holomorphic functions) is called the *complex structure* of the Riemann surface. One way of defining such a structure is to construct, on a given smooth orientable surface, a Riemannian metric and to define local parameters as conformal (orientation- and angle-preserving) homeomorphisms of domains on the surface into $\mathbb{C}$. A conformal mapping between two Riemann surface is a homeomorphism which preserves the complex structures.

One defines, in an obvious way, global meromorphic functions on a Riemann surface. If (and only if) the surface is *compact*, then any two meromorphic functions are connected by an algebraic equation, and may be treated as the coordinates of a plane algebraic curve.

Conversely, given a plane algebraic curve, there is a definite and well-known procedure of associating with it a compact Riemann surface such that the coordinates of the curve become meromorphic functions on the surface, and the field of rational functions on the curve becomes identical with the field of meromorphic functions on the surface.

The Riemann surface of an algebraic curve is topologically a sphere with p handles; the number p is called the *genus* of the surface and of the curve. Two algebraic curves are birationally equivalent, that is, have the same function field, if and only if their Riemann surfaces are conformally equivalent. A necessary condition is that the two curves have the same genus; for $p > 0$, however, this condition is not sufficient.

Now let $\zeta_2 = F_0(\zeta_1) = \Sigma A_j(\zeta_1 - a)^j$ be a power series with positive radius of convergence. We consider all power series (and all so-called Puiseux series) F_α from which F_0 can be obtained by analytic continuation, and all pairs (ζ_1, ζ_2) connected by $\zeta_2 = F_\alpha(\zeta_1)$. There is a definite and well-known procedure of associating with F_0 a Riemann surface on which ζ_1 and ζ_2 become single-valued meromorphic functions. If F_0 is not an algebraic function, the Riemann surface is *not compact*.

A *puncture* on a Riemann surface S is defined by a subdomain $\Delta \subset S$ such that there is a conformal bijection h of Δ onto the domain $0 < |z| < 1$ in $\mathbb{C}$, with the property: if $\lim z_j = 0$, then the sequence $h^{-1}(z_j)$ diverges on S. Two such subdomains define the same puncture, if there is a third contained in both. There is an obvious canonical way of filling in a puncture by adding a point to S. A Riemann surface is said to be of *finite type* (p, n) if it can be made into a compact surface of genus p by filling in n punctures.

A Riemann surface S is said to have (ideal) boundary curves if there is a Riemann surface Σ and an anti-conformal (angle-preserving but orientation-reversing) involution J of Σ such that the complement of the fixed point set b of J has two components, one of which is (conformally equivalent to) S. If we require that Σ have no boundary curves, then Σ, J and b are determined uniquely, up to conformal equivalence. In this case Σ is called the *double* of S, and every component of $b = b(S)$ is called an *ideal boundary curve* of S. Every ideal boundary curve is homeomorphic to a circle or to an interval, and S is embedded in $S \cup b(S)$ as an open dense subset.

Example. The double of U is $\mathbb{C}$, the ideal boundary curve of U is $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

It will be convenient to define a *Riemann surface with ramification points* as a Riemann surface S together with a discrete set of distinguished points $P_1, P_2, \ldots$ on S, to each of which there is assigned a *ramification number* ν_j which may be either an integer >1 or the symbol ∞. It is plain what is meant by a conformal mapping of surfaces with ramification points. The significance of this concept becomes apparent when one represents Riemann surfaces by discontinuous groups of Möbius transformations.

3. *Kleinian and Fuchsian groups*

A *Möbius transformation* g is a conformal automorphism of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; it is of the form

$$z \mapsto g(z) = \frac{az+b}{cz+d}, \quad ad-bc = 1,$$

with $a, b, c, d, \in \mathbb{C}$. A Möbius transformation $g \neq id$ is either parabolic (conjugate in the group of all Möbius transformations to $z \mapsto z+1$) or elliptic (conjugate to $z \mapsto \lambda z$, $|\lambda| = 1$) or loxodromic (conjugate to $z \mapsto \lambda z$, $|\lambda| \neq 1$); a loxodromic transformation is called hyperbolic if λ is real and positive.

A discrete group G of Möbius transformations is called *Kleinian* if its *limit set* $\Lambda = \Lambda(G)$ is not the whole of $\hat{\mathbb{C}}$; the limit set is defined as the set of accumulation points of orbits; it is also the closure of the set of fixed points of non-elliptic elements of G, distinct from the identity. The *region of discontinuity* of a Kleinian group $\Omega = \Omega(G) = \hat{\mathbb{C}} - \Lambda(G)$ is open and dense in $\hat{\mathbb{C}}$, and G acts properly discontinuously in Ω. This means that, for every compact set $\kappa \subset \Omega$, $\kappa \neq \emptyset$, one has $g(\kappa) \cap \kappa = \emptyset$ for all but finitely many $g \in G$.

A Kleinian group G is called *elementary* if Λ is finite. It is easy to enumerate all elementary groups.

A Kleinian group G is called *Fuchsian* if all its loxodromic elements are hyperbolic and G leaves a disc or a half-plane fixed; one can achieve, by conjugation, that this be the upper half-plane U, so that all $g \in G$ have real coefficients a, b, c, d. In talking about Fuchsian groups, we shall *always* assume that this has been done, unless the contrary is explicitly stated.

A *fundamental region* for a Kleinian group G is a set $\omega \subset \Omega$ such that its boundary (in Ω) has measure 0, no two interior points of ω are G-equivalent, and every point of Ω is G-equivalent to a point of the closure of ω.

The quotient $\Omega(G)/G$ has a canonical complex structure determined by the condition: the projection $\pi\colon \Omega \to \Omega/G$ is holomorphic. Every component of Ω/G is therefore a Riemann surface. We make it into a Riemann surface with ramification points by filling in all the punctures and assigning to the added points the ramification number ∞, and by assigning to a point $P \in \Omega/G$ the ramification number ν if $P = \pi(z_0)$ and the stabilizer of z_0 in G is a group of order $\nu > 1$. In this way, G represents a disjoint union of Riemann surfaces with ramification points. If $\Delta \subset \Omega$ is a G-invariant open set, $\Delta/G = \pi(\Delta) \subset \Omega/G$ is also a union of Riemann surfaces with ramification points.

Concerning the classical theory of discontinuous groups and automorphic functions of one variable, see Fricke–Klein **[56, 57]**, Appell–Goursat **[18]**, Ford **[55]**; a more recent book is Lehner **[80]**.

Examples of Kleinian groups.

(i) The *elliptic group* generated by $z \mapsto z + w_1$ $z \mapsto z + w_2$ with $\operatorname{Im}(w_2/w_1) > 0$. The group is elementary, since $\Lambda = \{\infty\}$, $\Omega = \mathbb{C}$. A parallelogram with vertices $0, w_1, w_2, w_1 + w_2$ is a fundamental region. The group represents a compact Riemann surface of genus 1.

(ii) Let $C_1, \ldots, C_{2p}$ be $2p \geqslant 2$ smooth Jordan curves such that the domain exterior to each C_j contains all C_k, $k \neq j$, and let $g_1, \ldots, g_p$ be Möbius transformations such that g_j maps the domain exterior to C_j onto the domain interior to C_{p+j}. Then the g_j generate a Kleinian group G, called a *Schottky group*. The domain exterior to all C_j is a fundamental region; Ω/G is a compact Riemann surface of genus p without ramification points.

(iii) Every discrete group G of real Möbius transformations with $ad - bc > 0$ is Fuchsian. In this case, either $\Lambda = \hat{\mathbb{R}}$ (and G is called of the *first kind*) or Λ is a nowhere dense subset of $\hat{\mathbb{R}}$ (and G is called of the *second kind*).

Here and hereafter, we denote by L the lower half-plane. If G is of the first kind, it represents two Riemann surfaces U/G and L/G; they are mirror images of each other and have no ideal boundary curves. If G is of the second kind, $\Omega(G)/G$ is a single surface, the double of U/G, and U/G has boundary curves. Indeed,

$$b(U/G) = (\hat{\mathbb{R}} - \Lambda(G))/G.$$

(iv) Here and hereafter, we think of the upper half-plane U as a model of *non-Euclidean geometry* based on the *Poincaré metric*

$$ds = y^{-1}\,|dz|.$$

An important special case of (iii) is a finitely generated Fuchsian group of the first kind; such a group will be called *of finite type*. A group of finite type has a fundamental region consisting of a non-Euclidean polygon in U and its mirror image in L. The polygon has finitely many sides and a finite Poincaré area which can be written as

$$A = 2\pi\left\{2p-2+\sum_{j=1}^{n}\left(1-\frac{1}{\nu_j}\right)\right\}.$$

The set of numbers $\{p, n;\ \nu_1, \ldots, \nu_n\}$, called the *signature* of G, has the following significance. There are precisely n non-conjugate (in G) maximal cyclic subgroups $G_1, \ldots, G_n$ of G generated by elliptic or parabolic elements; they can be numbered so that G_j has order ν_j if $2 \leqslant \nu_j < \infty$ and is parabolic if $\nu_j = \infty$; the number p is a non-negative integer. The group acts freely on $\Omega = U \cup L$ if and only if no ν is finite; the closure of the fundamental polygon is compact in U if and only if no ν is infinite. Finally, U/G is a Riemann surface of genus p, with ramification numbers $\nu_1, \ldots, \nu_n$.

There are *no* restrictions on the signature beyond the requirement that $A > 0$. If this condition is satisfied, for a given set $(p, n;\ \nu_1, \ldots, \nu_n)$, there is a geometric method, going back to Poincaré, for constructing a Fuchsian group with this signature. Curiously enough, correct proofs of this classical method have been given only very recently, *cf.* Siegel **[105]**, Coldewey and Zieschang **[44]**, Maskit **[89]**.

(v) Let C be a directed Jordan curve, and G a discrete group of Möbius transformations leaving C fixed. Then G is Kleinian and is called a *quasi-Fuchsian* group. We have that $\Lambda \subset C$; if $\Lambda = C$, G is called of the first kind. The fixed curve of a quasi-Fuchsian group of the first kind contains no rectifiable arcs, except when the group is Fuchsian.

The classical construction of a quasi-Fuchsian group is as follows. Let

$$C_1, C_2, \ldots, C_{N-1}, C_N = C_0,$$

be circles such that C_i is externally tangent to C_{i-1} and to C_{i+1}, and is disjoint from all other C_k. Let G be the group of products of even numbers of reflections about the C_i. Then G is quasi-Fuchsian. The fixed curve lies in the union of the discs bounded by the C_i, and passes through all tangency points. A different approach to quasi-Fuchsian groups is discussed in §6 below.

(vi) Let $G_1, G_2, \ldots, G_r$ be Fuchsian groups of finite type with signatures $(p_j, 0)$, $j = 1, \ldots, r$, acting not on U but on discs $\Delta_1, \ldots, \Delta_r$. If the discs lie sufficiently far apart, the group G generated by $G_1, \ldots, G_r$ is Kleinian and Ω/G consists of $r+1$ compact Riemann surfaces (without ramification points) of genera $p_1, p_2, \ldots, p_r$ and $p = p_1 + \ldots + p_r$. This follows from Klein's combination theorem, to be stated in 4, §1.

(vii) A non-elementary Kleinian group G is called (*totally*) *degenerated* if Ω is connected and simply connected. Such groups exist, see 4, §6, but no example has been constructed.

4. *The limit circle theorem*

In view of the connection between algebraic curves and Riemann surfaces noted in §2, the Klein–Poincaré uniformization theorem for algebraic curves is an immediate consequence of the following 'limit circle theorem'.

I. *Let S be a compact Riemann surface of genus $p \geqslant 0$, with $n \geqslant 0$ distinguished points and with ramification numbers $\nu_1, \ldots, \nu_n$. Assume that the number A associated with the signature $(p, n;\ \nu_1, \ldots, \nu_n)$ is positive.*

Then there exists a Fuchsian group G such that U/G is (*conformally equivalent to*) *S with the given ramification points.*

The group G is determined uniquely, except for a conjugation in the group of Möbius transformations. It is of finite type and has signature $(p, n;\ \nu_1, \ldots, \nu_n)$.

Poincaré's assertion about uniformization of analytic non-algebraic curves is contained in the following proposition.

II. *Every non-compact Riemann surface with ramification points* (*which cannot be made compact by filling in punctures*) *can be represented as U/G where G is a Fuchsian group; the group is determined up to a conjugation.*

Some (but not all) proofs of I yield also II. One can also obtain II from I by a limiting process. Theorems I and II yield the following

COROLLARY. *Every Riemann surface S which is not* (*conformally equivalent to*) *$\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}-\{0\}$ or a closed Riemann surface of genus 1 can be represented as U/G where G is a Fuchsian group without elliptic elements.*

The uniqueness in Theorem I (and II) is proved rather easily, using analytic continuation and the monodromy theorem. The difficulties occur in the existence proof.

There are three classical methods for proving the existence statement in I. One, suggested by H. A. Schwarz, proceeds as follows. Given S and the ramification points $P_1, P_2, \ldots$, let σ denote the set of all P_j with $\nu_j = \infty$. Construct, topologically, the universal covering surface $\tilde{S}$ of $S-\sigma$, if there are no P_j with $\nu_j < \infty$, or a covering surface $\tilde{S}$ which is branched over each P_j with $\nu_j < \infty$ of order ν_j. Next, show that $\tilde{S}$ is not compact and simply connected. Now transfer the conformal structure from $S-\sigma$ to $\tilde{S}$ via the covering. Then the covering $\tilde{S} \to S-\sigma$ becomes a holomorphic mapping and is equivalent to factoring $\tilde{S}$ by a group G of conformal automorphisms. Finally, use the Riemann mapping theorem to verify that $\tilde{S}$ is (conformally equivalent to) either $\mathbb{C}$ or U. The hypotheses of the theorem imply that the first case does not occur, and in the second case the group G is Fuchsian. This method works also for proving II.

The second method, also suggested by Schwarz, is based on the observation that the desired covering $U \to S-\sigma$ would be known if we knew, on $S-\sigma$, the Riemannian line element $ds = \lambda(\zeta)|d\zeta|$, ζ a local parameter, obtained by transferring to S the Poincaré metric $y^{-1}|dz|$ on U. For a compact S, $\lambda(\zeta)$ is uniquely determined by having Gauss curvature -1, which means that $u = \log \lambda$ satisfies

$$u_{\xi\xi}+u_{\eta\eta} = e^{2u},$$

and by having certain prescribed singularities at the points P_j. The proof of I is thus reduced to solving a non-linear partial differential equation of elliptic type.

Originally, Klein and Poincaré wanted to prove I by a continuity method, using the fact that the number $6p-6+2n$ is twice the number of complex "moduli" of a curve of genus p with n distinguished points (this was well known since Riemann) and also the number of real parameters determining a Fuchsian group of signature $(p, n; \nu_1, \ldots, \nu_n)$. Whether or not a really complete continuity proof was ever given, the success of the other methods consigned the continuity arguments to limbo. But a completely different version of the continuity argument appears within the framework of quasiconformal mappings. It gives, with one stroke, a simple proof of I and of all other classical uniformization theorems. It also yields new results, seemingly inaccessible by the old methods [25].

5. *Quasiconformal mappings*

A Riemannian metric on a Riemann surface S (in particular, on a plane domain) is defined by a line element

$$ds = A(z)\,|dz+\mu(z)\,d\bar{z}| \quad (z = \text{local parameter})$$

where $A > 0$ and μ are functions, with

$$|\mu| \leqslant k < 1,$$

and $A(z)|dz|$ and $\mu(z)\,d\bar{z}/dz$ are invariant under changes of local parameters. The quantity $\mu(z)\,d\bar{z}/dz$ is called a *Beltrami differential* (and if $S \subset \mathbb{C}$, z the standard coordinate, $\mu(z)$ is called a *Beltrami coefficient*). One denotes by $\|\mu\|_\infty$ the smallest k for which $|\mu| \leqslant k$, a.e. We assume from now on that the function $\mu(z)$ is *measurable*; the scale factor $A(z)$ will play no role in what follows.

A homeomorphism $f: S \to S'$ is called *μ-conformal* if, in terms of local parameters $z = x+iy$ on S and $\zeta = \xi+i\eta$ on S', it satisfies the *Beltrami equation*

$$\zeta_{\bar z} = \mu(z)\,\zeta_z.$$

Here the partial derivatives $\zeta_z = (\zeta_x - i\zeta_y)/2$ and $\zeta_{\bar z} = (\zeta_x + i\zeta_y)/2$, taken in the sense of distribution theory, are required to be measurable, locally square integrable functions. That a Beltrami equation can be solved without requiring any smoothness of μ has been first recognized by Morrey [**92**]. This fact is very important for the application we have in mind.

A mapping which is μ-conformal for some μ, $\|\mu\|_\infty < 1$, is called *quasiconformal*, with *dilatation*

$$K(f) = \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}.$$

The dilatation obeys the rules: $K(f) \geqslant 1$, with equality only for conformal f, $K(f^{-1}) = K(f)$, $K(f_1 f_2) \leqslant K(f_1)\,K(f_2)$.

It is obvious that if S is a Riemann surface with ramification points, $S' = f(S)$ can be considered, in a natural way, as a Riemann surface with ramification points. Quasiconformal equivalence is, in general, weaker than conformal equivalence (any two compact Riemann surfaces of the same genus are quasiconformally equivalent), but stronger than topological equivalence (U and $\mathbb{C}$ are not quasiconformally equivalent).

The standard reference on quasiconformal mapping in two dimensions is Lehto–Virtanen [**81**], containing an excellent bibliography; for a short presentation see Ahlfors [**11**].

The following global result is of basic importance.

III. *Let $\mu(z)$, $z \in \mathbb{C}$, be a Beltrami coefficient. There exist a unique μ-conformal automorphism $z \to w^\mu(z)$ of $\hat{\mathbb{C}}$ which leaves 0, 1, ∞ fixed; every other μ-conformal automorphism of $\hat{\mathbb{C}}$ is of the form $\alpha \circ w^\mu$, α a Möbius transformation.*

If μ depends holomorphically (as an element of the Banach space $L_\infty(\mathbb{C})$) on complex parameters, so does $w^\mu(z)$, for every z. In particular

$$\left.\frac{\partial w^{\varepsilon\mu}(z)}{\partial \varepsilon}\right|_{\varepsilon=0} = -\frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\mu(\zeta)\,d\xi\,d\eta}{\zeta(\zeta-1)(\zeta-z)}. \tag{*}$$

A self-contained proof can be found in a paper by Ahlfors and Bers [**15**]. Relation (*) is derived in [**29**] and in [**11**].

6. *Quasiconformal deformation of Kleinian groups*

If G is a Kleinian group, a Beltrami coefficient μ on $\mathbb{C}$ is called a Beltrami coefficient on G provided that

$$\mu(g(z))\,\overline{g'(z)}/g'(z) = \mu(z), \quad g \in G, \text{ and } \mu \mid \Lambda = 0.$$

Assume that this is so. Then for every $g \in G$, the function $w^\mu(g(z))$ is again a μ-conformal automorphism of $\hat{\mathbb{C}}$, as is verified by a direct calculation. Hence there is a Möbius transformation g_1 with $w^\mu \circ g = g_1 \circ w^\mu$. We conclude that

$$G^\mu = w^\mu\, G(w^\mu)^{-1},$$

is again a Kleinian group.

The mapping $G \to G^\mu$ given by $g \mapsto w^\mu \circ g \circ (w^\mu)^{-1}$ is called a *quasiconformal isomorphism* defined by μ, or a *μ-conformal deformation.* Observe that

$$\Omega(G^\mu) = w^\mu(\Omega(G)),$$

and that the mapping $w^\mu : \Omega(G) \to \Omega(G^\mu)$ induces quasiconformal mappings of the components of $\Omega(G)/G$ onto the corresponding components of $\Omega(G^\mu)/G^\mu$.

If G is a Fuchsian group and μ is a Beltrami coefficient on G, then G^μ is a quasi-Fuchsian group with fixed curve $w^\mu(\hat{\mathbb{R}})$. If μ also satisfies the symmetry condition

$$\mu(\bar{z}) = \overline{\mu(z)},$$

then G^μ is again Fuchsian. If G has signature $(p, n;\ \nu_1, \ldots, \nu_n)$, so does G^μ. All these statements follow at once from III.

We now state a general principle which implies all classical uniformization theorems.

IV. *Let G be a Kleinian group and let $\Omega(G)/G = S_1 \cup S_2 \cup \ldots$ where the S_j are Riemann surfaces (with ramification points). Let $f_j : S_j \to S_j'$ be quasiconformal mappings such that $K(f_j) \leqslant K < \infty$. Then there is a quasiconformal deformation $G \to G'$ such that $\Omega(G')/G' = S_1' \cup S_2' \cup \ldots$.*

Sketch of proof. Pull back the Beltrami differentials on $S_1, S_2, \ldots$ defined by the mappings $f_1, f_2, \ldots$ to $\Omega(G)$; this yields a Beltrami coefficient $\mu(z)$, $z \in \Omega(G)$; note that $\|\mu\|_\infty < 1$, since all $K(f_j) \leqslant K$. We obtain a Beltrami coefficient μ on $\mathbb{C}$ by setting $\mu \mid \Lambda = 0$. One checks that μ is a Beltrami coefficient on G, and that G^μ has the required properties. Here are some applications.

Every compact Riemann surface of genus p is diffeomorphic and hence quasi-conformally equivalent to any other such surface. Therefore, every such surface can be represented as $\Omega(G)/G$ where G is a Schottky group on p generators. This is the classical *retrosection theorem.*

Every Riemann surface of genus 1 is diffeomorphic to the unit square with opposite sides identified. Hence every such surface is the quotient of $\mathbb{C}$ by an elliptic group.

Let $p_1, p_2, \ldots, p_{r+1}$ be integers with $p_j \geqslant 2$ and $p_{r+1} = p_1 + \ldots + p_r$. Let $S_1, \ldots, S_{r+1}$ be given compact Riemann surfaces with these genera. There is a Kleinian group G with $\Omega/G = S_1 \cup \ldots \cup S_{r+1}$. This follows from IV and Example (vi) in §3.

In order to prove Theorem I using the principle IV, consider a given compact Riemann surface S of genus p with distinguished points $P_1, \ldots, P_n$ and ramification numbers $\nu_1, \ldots, \nu_n$ (assuming, of course, that $A > 0$). Construct *some* fixed Fuchsian group G_0 of signature $(p, n; \nu_1, \ldots, \nu_n)$, and set $S_0 = U/G_0$, $\bar{S}_0 = L/G_0$; $\bar{S}_0$ is the "mirror image" of S_0. It is easy to find a diffeomorphic, hence quasiconformal, mapping f_0 of the Riemann surface (with ramification points) S_0 onto the given Riemann surface S; there is an obvious "mirror image" of this mapping, $\mathrm{f}_0 : \bar{S}_0 \to \bar{S}$. Pulling back the Beltrami differentials of the mapping f_0 and f_0 to U and L, respectively, one obtains a Beltrami coefficient on G satisfying the symmetry condition $\mu(\bar{z}) = \overline{\mu(z)}$. Hence G^μ is a Fuchsian group, and by IV we have that $U/G^\mu = S$.

Now let G_0, S_0, $\bar{S}_0$, S and f_0 be as before, and let S_1 be another compact Riemann surface with ramification points (and the same value of the number p, n, $\nu_1, \ldots, \nu_n$). Let f_1 be a diffeomorphism of $\bar{S}_0$ onto S_1. Proceeding by the prescription of IV, we obtain a theorem on simultaneous uniformization of two distinct Riemann surfaces [23, 25].

V. *Given two compact Riemann surfaces S and S_1 with ramification points and with the same values of the numbers $(p, n; \nu_1, \ldots, \nu_n)$, there is a quasi-Fuchsian group G of the first kind with $\Omega(G)/G$ conformally equivalent to $S \cup S_1$.*

The theorem can be strengthened so as to include a *uniqueness statement*, and it can be extended to non-compact Riemann surfaces. The far-reaching consequences of this construction of quasi-Fuchsian groups will be discussed below.

2. TEICHMÜLLER SPACES

1. *Teichmüller space, an analogue of the upper half-plane*

Every algebraic curve of genus 1 can be uniformized by rational functions of the Weierstrass $\wp$-function $\wp(z)$ and its derivative $\wp'(z)$. For these functions, there are explicit formulas which exhibit their analytic dependence not only on the uniformizing parameter z but also on the periods w_1 and w_2 and thus on the complex modulus $\tau = w_1/w_2 \in U$. It is well known that two algebraic curves of genus 1 are birationally equivalent if and only if they are uniformized by elliptic functions with moduli τ and τ' such that

$$\tau' = \frac{a\tau+b}{c\tau+d}, \quad ad-bc = 1; \quad a, d, b, c \in Z.$$

Thus every point of the upper half-plane U defines a (conformal equivalence class of a) Riemann surface of genus 1, and two points of U determine conformally equivalent Riemann surfaces if and only if they are equivalent under the elliptic modular group $\mathrm{SL}(2, \mathbb{Z})/\{\pm I\}$.

For algebraic curves of genus $p > 1$, there are *two* analogues of the upper half-plane: the Siegel generalized upper half-plane H_p and the Teichmüller space T_p. H_p is the set of all symmetric $p \times p$ matrices with positive definite imaginary part; it is (isomorphic to) a bounded symmetric domain in $\mathbb{C}^{p(p+1)/2}$. A Riemann surface S of genus p, together with a canonical homology basis, determines a point in H_p;

a change of the homology basis induces an automorphism of H_p by an element of the Siegel modular group $\mathrm{Sp}\,(2p, \mathbb{Z})/\{\pm I\}$. However, only part of a $3p-3$ dimensional analytic subset of H_p corresponds to Riemann surfaces.

The space T_p appears implicitly in the early continuity arguments by Klein and Poincaré; it has been constructed as a real manifold of $6p-6$ dimension by Fricke **[56]** (who proved it to be a cell) and by Fenchel–Nielsen **[54]**. Teichmüller **[107, 108]** introduced a metric in T_p. Later it turned out that T_p has a natural complex structure and can be embedded in $\mathbb{C}^{3p-3}$ as a bounded domain (Ahlfors **[5, 8, 9]**, Bers **[22, 27, 28]**). A compact Riemann surface of genus p, together with a standard system of generators for its fundamental group, determines a point in T_p; every point arises in this way. There is a (discrete) group of biholomorphic automorphisms of T_p, the modular group, which identifies points corresponding to conformally equivalent Riemann surfaces. This group is a homomorphic, and for $p > 2$ an isomorphic, image of the so-called mapping class group of a closed orientable surface of genus p.

The definition of Teichmüller space can be extended to Riemann surfaces with ramification points, to non-compact Riemann surfaces and to arbitrary Fuchsian groups **[28]**. This leads to infinite dimensional Teichmüller spaces and to non-discrete modular groups, but also introduces a certain harmony in the theory. In particular, all Teichmüller spaces considered (and their modular groups) turn out to be subspaces of a "universal" Teichmüller space (and subgroups of a "universal" modular group). The universal Teichmüller space is a bounded domain in a Banach space and is homogeneous, which the finite dimensional Teichmüller spaces are not.

Remark. For comprehensive, though partly out of date, presentations, see **[29]** and Ahlfors **[11]**. There are (for the case of compact surfaces) other approaches to Teichmüller space theory (Rauch **[103]**, Grothendieck **[62]**, Earle and Eells **[51, 52]**) which we shall not discuss. We also neglect the theory of extremal quasiconformal mappings with which Teichmüller initiated the whole modern development. Concerning extremal mappings, see Ahlfors **[4]**, Bers **[21, 35]**, Hamilton **[63]**, Krushkal **[79]**.

2. *Teichmüller spaces of Riemann surfaces*

Let S be a Riemann surface, compact or not, which is not (conformally equivalent to) $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}-\{0\}$ or a compact surface of genus 1, so that S can be identified with U/G; G is a Fuchsian group. It is not difficult to verify that, if S has ideal boundary curves, then every quasiconformal bijection $S \to S'$ extends, by continuity, to a homeomorphism $S \cup b(S) \to S' \cup b(S')$.

Two quasiconformal bijections, $f\colon S \to S'$ and $g\colon S \to S''$, are called *equivalent* if there is a conformal mapping $h\colon S' \to S''$ such that $g^{-1} \circ h \circ f$ is homotopic to the identity, by a homotopy which leaves every point on $b(S)$ fixed (the latter condition is vacuous if S has no boundary curves). Let $\{f\}$ denote the equivalence class of f. The set of all $\{f\}$ is the *Teichmüller spaces* $T(S)$ of S.

Every quasiconformal bijection $g\colon S_0 \to S$ induces an *allowable bijection* $T(S) \to T(S_0)$ which takes a $\{f\} \in T(S)$ into $\{f \circ g\} \in T(S_0)$.

We define the *modular group* Mod (S) as the factor group of the group of all quasiconformal selfmappings g of S over the normal subgroup of those homotopic to the identity modulo $b(S)$. The element of Mod (S) defined by a selfmapping g will be denoted by $[g]$. This group acts on $T(S)$ as follows: $[g] \in \text{Mod}(S)$ induces the allowable selfmapping $\{f\} \mapsto \{f \circ g^{-1}\}$ of $T(S)$. The action is not necessarily effective; it may happen that $id \neq [g] \in \text{Mod}(S)$ but $\{f \circ g^{-1}\} = \{f\}$ for all $\{f\} \in T(S)$. For instance, if $S = \mathbb{C} - \{0, 1\}$, then $T(S)$ is a point but Mod (S) has order 6. (This definition of Mod (S), a slight departure from the traditional one, turns out to be quite useful.)

The *Teichmüller distance* between two points $\{f\}$ and $\{f'\}$ of $T(S)$ is defined as the infimum of $\log K(g' \circ g^{-1})$ where $g \in \{f\}$, $g' \in \{f'\}$. It is not difficult to verify that $T(S)$, with this distance, is a *complete connected metric space*, and that allowable isomorphisms are *isometries*.

If we repeat the preceding definitions, omitting all references to ideal boundary curves, we obtain, instead of the Teichmüller space $T(S)$, the so-called *reduced Teichmüller space* $T^{\#}(S)$, and instead if the modular group Mod (S) the *reduced modular group* $\text{Mod}^{\#}(S)$. There is an obvious canonical surjection $T(S) \to T^{\#}(S)$ which defines a Teichmüller metric on $T^{\#}(S)$ and induces an epimorphism $\text{Mod}(S) \to \text{Mod}^{\#}(S)$.

One verifies that two elements of $T(S)$, $\{f_1\}$ and $\{f_2\}$, are equivalent under the group Mod (S) if and only if the Riemann surfaces $f_1(S)$ and $f_2(S)$ are conformally equivalent. Hence

$$X(S) = T(S)/\text{Mod}(S),$$

is the space of conformal equivalence classes (*space of moduli*) of Riemann surfaces quasiconformally equivalent to S. One can also show that $X(S) = T^{\#}(S)/\text{Mod}^{\#}(S)$.

3. *Teichmüller spaces of Fuchsian groups*

Let Q denote the group of quasiconformal self-mappings of the upper half-plane U, Q_n the subgroup of those $w \in Q$ which keep the points 0. 1, ∞ fixed, and Q_0 the (normal) subgroup of Q consisting of elements which keep every $x \in \mathbb{R}$ fixed. Two elements, w_1 and w_2, of Q will be called *equivalent* if $w_1 \circ w_2{}^{-1} \in Q_0$. The equivalence class of $w \in Q$ will be denoted by $[w]$.

Let G be a Fuchsian group acting on U, $Q(G)$ the set of quasiconformal deformations of G into Fuchsian groups, i.e., the set of $w \in G$ for which wGw^{-1} is a Fuchsian group, and let $N(G)$ be the normalizer of G in Q, i.e., the set of $\omega \in Q$ with $\omega G \omega^{-1} = G$. Also set $Q_n(G) = Q(G) \cap Q_n$.

The *Teichmüller space* $T(G)$ of G is, by definition, the image of $Q_n(G)$ under the canonical mapping $Q_n \to Q_n/Q_0$, i.e., the set of equivalence classes $[w]$ of $w \in Q_n(G)$. The *Teichmüller distance* between two elements $[w_1]$ and $[w_2]$ of $T(G)$ is defined as the infimum of $\log K(w)$, $w \in [w_1 \circ w_2{}^{-1}]$.

Every $u \in Q(G)$ induces an *allowable bijection* $T(uGu^{-1}) \to T(G)$ which takes a $[w] \in T(uGu^{-1})$ into $[\alpha \circ w \circ u^{-1}] \in T(G)$, where α is a real Möbius transformation chosen so that $\alpha \circ w \circ u^{-1} \in Q_n$. This mapping depends only on $[w]$; it is an *isometry* in the two relevant Teichmüller metrics.

The *modular group* Mod (G) of G is defined as Mod $(G) =$ mod $(G)/G$ where mod $(G) = N(G)/((G) \cap Q_0)$. (By abuse of language G may be considered a subgroup of mod (G).) In other words, elements of Mod (G) are defined by elements of $N(G)$, and two elements, ω and ω', of $N(G)$ define the same element $\langle\omega\rangle$ of Mod (G) if and only if there is a $g \in G$ with $\omega' \circ \omega^{-1} \circ g \in Q_0$. This group acts on $T(G)$ as a group of allowable selfmappings: $\langle\omega\rangle \in$ Mod (G) induces the mapping $[w] \to [\alpha \circ w \circ \omega^{-1}]$, α an appropriate Möbius transformation.

One verifies that two elements, $[w_1]$ and $[w_2]$, of $T(G)$ are equivalent under Mod (G) if and only if the groups $w_1 G w_1{}^{-1}$ and $w_2 G w_2{}^{-1}$ are conjugate in the group of all Möbius transformations. Hence

$$X(G) = T(G)/\text{Mod}\,(G),$$

is the *space of conjugacy classes of quasiconformal images of G.*

If we repeat the preceding definitions, replacing the concept of equivalence ($w_1 \circ w_2{}^{-1}$ leaves all points of $\mathbb{R}$ fixed) by that of *G-equivalence* ($w_1 \circ w_2{}^{-1}$ leaves all points of $\Lambda(G)$ fixed), we arrive at the *reduced Teichmüller space* $T^{\#}(G)$ and the *reduced modular group* Mod$^{\#}(G)$. As before, there are canonical sirjections Mod $(G) \to$ Mod$^{\#}(G)$, and as before, we have that $X(G) = T^{\#}(G)/\text{Mod}^{\#}(G)$.

4. *Isomorphism theorems*

The following isomorphism theorems show that Teichmüller spaces of Riemann surfaces and of Fuchsian groups are not essentially distinct.

VI. *Let G be a Fuchsian group without elliptic elements. There is a canonical isometry between* $T(G)$ *and* $T(U/G)$ *which induces an isomorphism between* Mod (G) *and* Mod (U/G).

The word " isometry " refers, of course, to the two Teichmüller metrics.

The proof is an exercise; one must only remember that every quasiconformal bijection $f\colon U/G \to f(U/G)$ can be lifted to a quasiconformal automorphism of U, since the universal covering surface of $f(U/G)$ is again U. One sees easily that there is a natural correspondence between allowable mappings of $T(G)$ and those of $T(U/G)$.

VII. *Let G be a Fuchsian group with elliptic elements, and let* U_G *denote the upper half-plane with the fixed points of all elliptic elements removed. Then there is a canonical isometry between* $T(G)$ *and* $T(U_G/G)$ *which induces an isomorphism between* Mod (G) *and a subgroup of* Mod (U_G/G).

There are two distinct proofs of this result (due to Bers—L. Greenberg **[39]** and to A. Marden **[91]**); both are surprisingly difficult.

The subgroup mentioned in the theorem is, in general, a proper subgroup. For instance, if G is of finite type and of signature $(p, n;\ \nu_1, \ldots, \nu_n)$, then U_G/G is a compact surface S with n punctures $P_1, P_2, \ldots, P_n$. Every orientation-preserving topological selfmapping f of S which leaves the set $\{P_1, \ldots, P_n\}$ fixed induces an element of Mod (U_G/G), but only those f which send every P_i into a P_j with $\nu_j = \nu_i$ correspond to elements of Mod (G).

The isomorphism theorems have a

COROLLARY. *If G_1 and G_2 are two Fuchsian groups, every conformal mapping $U_{G_1}/G_1 \to U_{G_2}/G_2$ induces, canonically, an isometry $T(G_1) \to T(G_2)$.*

There are corresponding isomorphism theorems for reduced Teichmüller spaces.

5. *Complex structure*

An element $w \in Q_n$ is uniquely determined by its Beltrami coefficient $\mu = w_{\bar{z}}/w_z$, an element of the Banach space $L_\infty(U)$. We write $w = w_\mu$; in other words, in the notation of 1, §5, $w_\mu = w^{\tilde{\mu}}$ where $\tilde{\mu} \mid U = \mu$, and μ is extended to the lower half-plane L by the symmetry condition $\tilde{\mu}(\bar{z}) = \overline{\mu(z)}$, $z \in U$. We give $T(G)$ a complex structure by requiring that whenever μ depends holomorphically on complex parameters, as an element of the complex Banach space $L_\infty(U)$, the same be true of $[w_\mu] \in T(G)$. This definition makes $T(G)$ only into a so-called ringed space; we do not yet know know whether the complex structure is " good ". It turns out that it is. In particular, if G is of finite type, $T(G)$ is a complex manifold. (Ahlfors **[5]**, *cf.* Rauch **[103]**, Weil **[110]**, Bers **[22]**). This can be demonstrated by an explicit construction of local complex coordinates. But there is also a global way **[27]** of obtaining the same result, and this way works for all G.

It turns out that allowable mappings preserve the complex structure, and so do the isometries described in Theorem VII and in its Corollary. We now use Theorem VI to give a complex structure to $T(S)$. It turns out that, in the case of compact surfaces, this complex structure is *natural*; it conforms to all other methods of introducing a complex structure in a space of Riemann surfaces (using periods of Abelian integrals, branch points of " concrete " Riemann surfaces, etc.).

Actually, much more is true. Let us denote by $B_2(U, G)$ the Banach space of holomorphic functions $\phi(z)$, $z \in L$, which satisfy the functional equation

$$\phi(g(z))\, g'(z)^2 = \phi(z), \quad g \in G,$$

(characterizing automorphic forms of weight -4 or, which is the same, *quadratic differentials*) and the growth condition

$$\|\phi\|_B = \|\phi\|_{B_2(L)} = \sup |y^2 \phi(z)| < \infty,$$

(characterizing so-called bounded forms). This Banach space is finite dimensional if and only if G is of finite type.

VIII. *There exists a canonical biholomorphic embedding of $T(G)$ into a domain $B_2(L, G)$. This domain contains the ball $\|\phi\|_B < \frac{1}{2}$ and is contained in the ball $\|\phi\|_B < \frac{3}{2}$.*

This is the central result of the theory. To prove it, associate with each $w \in Q_n(G)$ a quasiconformal automorphism $w^\#$ of $\mathbb{C}$ determined by the conditions

$$w^\# \circ w^{-1} \,|U \text{ and } w^\# |\, L \text{ are conformal.}$$

In other words, define the Beltrami coefficient μ on G by the requirements: $\mu \,|\, U = w_{\bar{z}}/w_z$ (so that $w = w_\mu$) and $\mu \,|\, L = 0$; then $w^\# = w^\mu$.

We observe that $w^\# \,|\, L$ depends only on the equivalence class $[w]$ of w. Indeed, let $w_1 \in [w]$. Consider the automorphism h of $\hat{\mathbb{C}}$ defined by

$$h = \begin{cases} w^\# \circ w^{-1} \circ w_1 \circ (w_1{}^\#)^{-1} \text{ in } w_1{}^\#(U). \\ w^\# \circ (w_1{}^\#)^{-1} \text{ in } w_1{}^\#(L) \end{cases}$$

Since h is conformal off $w^\#(\mathbb{R})$ and quasiconformal everywhere, it is conformal; since it keeps 0, 1, ∞ fixed, it is the identity. Thus $w^\# \,|L = w_1{}^\# |\, L$.

Now let $\phi^\mu(z)$, $z \in L$, be the *Schwarzian derivative* of the schlicht holomorphic function $w^\mu(z)$, $z \in L$; thus

$$\phi^\mu(z) = \{w^\mu, z\} = u'(z) - \tfrac{1}{2}u(z)^2 \text{ where } u(z) = \frac{d}{dz} \log \frac{dw^\mu(z)}{dz} .$$

By a theorem of Nehari [94], $|y^2 \phi^\mu(z)| \leqslant \frac{3}{2}$ for all $z \in L$. Using the fact that, for every $g \in G$, $w^\mu \circ g \circ (w^\mu)^{-1}$ is a Möbius transformation, one computes that ϕ^μ is a quadratic differential. Thus $\phi^\mu \in B_2(L, G)$.

It is classical that knowing ϕ^μ, we can find $w^\mu \,|\, L$, modulo a Möbius transformation. More precisely, let $\eta_1(z)$ and $\eta_2(z)$, $z \in L$, be two linearly independent solutions of the ordinary differential equation

$$2\eta''(z) + \phi^\mu(z)\eta(z) = 0.$$

and set

$$w^\mu(z) = \eta_1(z)/\eta_2(z), \quad z \in L.$$

Then W^μ has the Schwarzian derivative ϕ^μ. Therefore $w^\mu \,|\, L = \alpha \circ W^\mu$, α a Möbius transformation.

But w^μ maps the points 0. 1, ∞ onto themselves. Hence we know α, thus also the Jordan domain $w^\mu(L)$ and hence also the complementary Jordan domain $w^\mu(U)$. Now let h_μ: $w^\mu(U) \to U$ be the conformal mapping which keeps 0, 1, ∞ fixed. Then $h_\mu \circ w^\mu \,|\, U$ is a μ-conformal mapping $U \to U$ and keeps 0, 1, ∞ fixed. Hence $w_\mu = h_\mu \circ w^\mu \,|\, U$, so that the knowledge of ϕ^μ determines $w_\mu \,|\, \mathbb{R} = h_\mu \circ w^\mu \,|\, \mathbb{R}$ and therefore $[w_\mu]$. In other words

$$[w_\mu] \mapsto \phi^\mu,$$

is an injection $T(G) \to B_2(L, G)$. Using Theorem III in 1, §5, one verifies that this injection is holomorphic.

We must show next that the image of $T(G)$, which is certainly connected, is open. This is easy if G is of finite type. In this case, G has a signature, say $(p, n;\ \nu_1, \ldots, \nu_n)$ and one knows from other considerations that $T(G)$ is homeomorphic to $\mathbb{R}^{6p-6+2n}$. One also knows, from the Riemann–Roch theorem, that $\dim_{\mathbb{C}} B_2(L, G) = 3p-3+n$.

The desired conclusion follows from Brouwer's theorem on invariance of domain. In the general case, the proofs (Ahlfors [9], Bers, [28, 29, 31] Earle [47]) are much longer and involve " hard " analysis.

From now on we identify $T(G)$ with its canonical image in $B_2(L, G)$.

The fact that $T(G)$ contains the ball $\|\phi\|_B < \frac{1}{2}$ follows from a refinement of a theorem of Nehari due to Ahlfors–Weill [17].

Assume that $\phi \in B_2(L, G)$ *and* $\|\phi\|_B < \frac{1}{2}$. *For* $z \in U$, *set* $\mu(z) = -2y^2\phi(\bar{z})$. *Then* $w_\mu \in Q_n(G)$ *and* $\phi^\mu = \phi$.

It is natural to ask whether one can strengthen this result and obtain a holomorphic mapping $\phi \mapsto \mu$ of the *whole* Teichmüller space $T(G)$ into $L_\infty(U)$ such that μ is a Beltrami coefficient for G in U, and $\phi^\mu = \phi$. Earle [50] proved that this is *impossible*, for dim $T(G) > 1$.

The reduced Teichmüller space $T^{\#}(G)$ has a canonical real-analytic structure and (as observed by Earle [46]) a canonical embedding as a domain in a real Banach space.

6. *Universal Teichmüller space*

The preceding considerations apply, in particular, to the case $G = 1$, the trivial group. In view of Theorem V, $T(1)$ can be identified with $T(U)$, and Mod (1) with Mod (U). Note also that $T^{\#}(1)$ reduces to a point.

The definitions of Teichmüller space imply at once that for two Fuchsian groups G_1 and $G_2 \subset G_1$, there is an inclusion $T(G_1) \subset T(G_2)$. Hence every $T(G)$ is a subset of the universal Teichmüller space $T(1)$.

IX. *Every* $T(G)$ *is closed in* $T(1)$ *and the inclusion* $T(G) \subset T(1)$ *is a homeomorphism. Also* $T(G)$ *is the component of* 0 *in* $T(1) \cap B_2(L, G)$.

It is not known whether the Teichmüller metrics of $T(1)$ and on $T(G) \subset T(1)$ coincide. It is also not known whether Theorem IX can be strengthened to the statement

$$T(G) = T(1) \cap B_2(L, G).$$

This is so if $T(G)$ is finite dimensional, and also (as proved by Kra [73]) if G is finitely generated and of the second kind.

The universal Teichmüller space $T(1) = Q_n/Q_0$ is a metric space and a group, but it is *not a topological group.* It can also be described directly as the group (under composition) of strictly increasing real-valued functions $x \mapsto \xi(x)$, satisfying

$$\xi(-\infty) = -\infty, \quad \xi(0) = 0, \quad \xi(1) = 1, \quad \xi(+\infty) = +\infty,$$

which are *boundary values of quasiconformal automorphisms of* U. The latter condition is equivalent, by a theorem of Beurling and Ahlfors [40], to the existence of a constant M, depending on ξ, such that

$$0 < \frac{1}{M} < \frac{\xi(x+y)-\xi(x)}{\xi(x)-\xi(x-y)} < M$$

for all x and y. Using the Beurling–Ahlfors proof of this criterion, Earle [45] showed that $T(1)$ *is contractible.*

One can also describe $T(1) \subset B_2(L, 1)$ as the set of all Schwarzian derivatives of schlicht functions $W(z)$, $z \in L$, such that $W(L)$ is a Jordan domain, and its boundary curve is *quasi-circle* (a quasiconformal image of a circle). Ahlfors [9] gave a *geometric criterion* for quasi-circles: a Jordan curve which passes through ∞ (this can be achieved by a Möbius transformation) is a quasi-circle if and only if there is a constant M such that for any three points a, b, c, on C, with b between a and c,

$$|b-a| \leqslant M\,|c-a|.$$

X. *Every Teichmüller space is holomorphically convex.*

This has been proved (by Bers–Ehrenpreis [38]) using the Ahlfors criterion and Theorem IX.

The universal modular group Mod (1) contains a subgroup of (right) *translations* in $T(1)$: $[w] \mapsto [w \circ w_0]$ where $[w_0]$ is a fixed element of $T(1)$. Also, Mod (1) contains a subgroup (isomorphic to the real Möbius group) of so-called *rotations*; an element of this subgroup is of the form $[w] \mapsto [\beta \circ w \circ \alpha]$ where α is a fixed real Möbius transformation and β a real Möbius transformation chosen so that $\beta \circ w \circ \alpha \in Q_n$. The two subgroups have only the identity in common, and generate Mod (1).

Every rotation is a restriction to $T(1)$ of a linear isometry of $B_2(L, 1)$; this isometry takes $\phi(z) \in B_2(L, G)$ into $\phi(\alpha(z))\,\alpha'(z)^2$. The description of translations in terms of $B_2(L, 1)$ has been given by F. Gardiner [58]; it is rather complicated, and it is not known whether the translations act on the boundary of $T(1)$ in $B_2(L, 1)$.

It can be shown that Mod (G) *can be identified with the subgroup of* Mod (1) *which maps* $T(G)$ *onto itself.*

It is easy to see that the set of Schwarzian derivatives of schlicht functions is closed in $B_2(L, 1)$.

CONJECTURE. *Every Schwarzian derivative of a schlicht function lies in the closure of* $T(1)$.

A proof would have important consequences.

7. *Finite dimensional Teichmüller spaces and spaces of moduli*

The (complex) dimension of $T(G)$ equals the dimension of $B_2(L, G)$ and is finite if and only if G is of finite type. (On the other hand, dim $T^{\#}(G) < \infty$ whenever G is finitely generated.) If dim $T(G) < \infty$, G has signature $(p, n;\ \nu_1, \ldots, \nu_n)$; we call (p, n) the type of G and note that dim $T(G) = 3p-3+n$.

Two Fuchsian groups of the same signature are always quasiconformally equivalent, so that there are allowable isomorphisms between their Teichmüller spaces. These isomorphisms are determined up to actions of the respective modular groups and induce isomorphisms of these groups. Also, $T(G_1)$ and $T(G_2)$ are isomorphic if they have the same type (in view of Theorem VII and its corollary).

Hence, it is legitimate to talk about the Teichmüller space $T(p, n)$, by which we mean a $T(G)$ with G of type (p, n), about the modular group $M(p, n)$, by which we mean Mod (G) for G of signature $(p, n;\ \infty, \dots, \infty)$, and to consider Mod (G_1) for G_1 of signature $(p, n;\ \nu_1, \dots, \nu_n)$ as a subgroup Mod $(p, n;\ \nu_1, \dots, \nu_n) \subset$ Mod (p, n). By Theorem VI, $T(p, n)$ and Mod (p, n) are also the Teichmüller space and modular group, respectively, of a Riemann surface of type (p, n) (compact surface of genus p punctured at n points points). One sometimes writes T_p instead of $T(p, 0)$.

XI. *$T(p, n)$ is a cell, and* Mod (p, n) *is discrete.*

This is a classical result of Fricke **[56, 57]**; the first statement is harder to prove than the second. Modernized proofs of that statement are due to Fenchel–Nielsen **[54]**, to L. Keen **[65, 66]**, and to Coldewey–Zieschang **[44]**. A proof also follows from the Teichmüller theory of extremal quasiconformal mappings.

The Teichmüller metric in $T(n, p)$ has been the subject of a deep investigation by Kravets **[78]** (see also M. Linch's thesis **[82]**). He showed that the Teichmüller space is a *straight line space* in the sense of Busemann, and studied its curvature properties.

Recently Royden **[104]** showed that the Teichmüller metric is completely determined by the complex structure of $T(n, p)$, it is the so-called Kobayashi **[68]** metric. Another important result of Royden **[104]** is

XII. *If* dim $T(p, n) > 2$, *then the action of* Mod (p, n) *is the full group of holomorphic automorphisms of $T(p, n)$.*

This shows that $T(p, n)$ is not a homogeneous domain, for dim $T(p, n) \geqslant 3$.

The Teichmüller metric is, of course, not a Hermitian metric. There exists also a Hermitian metric, invariant under Mod (p, n), in $T(p, n)$, the so-called *Weil–Petersson metric.* It is preserved by allowable isomorphisms and by the isomorphisms described in Theorems VI, VII, so that it can be completely characterized by the following property:

Let $T(p, n) = T(G)$, let $d = \dim T(G)$, and let $\phi_1, \dots, \phi_d$ be a basis in $B_2(L, G)$ such that

$$\iint_{G/L} \phi_i(z)\, \overline{\phi_j(z)}\, y^2\, dx\, dy = \delta_{ij}.$$

The Weil–Petersson distance from 0 to a point $c_1 \phi_1 + \dots + c_d \phi_d \in T(G)$ is, for small $t^2 = |c_1|^2 + \dots + |c_d|^2$, equal to $t + o(t)$.

It is shown that this metric is *Kâhlerian* (Weil **[110]**, Ahlfors **[6]**), that holomorphic sections have negative curvature (Ahlfors **[7]**), and that there is a constant $c_{p,n}$ such that the Weil–Petersson distance between two points in $T(p, n)$ does not exceed $c_{p,n}$ times the Teichmüller distance (Linch **[82]**.

CONJECTURE. *The Weil–Petersson metric is identical with the Bergman metric and is complete.*

CONJECTURE. *The Weil–Petersson volume of a fundamental domain for* Mod (p, n) *in* $T(p, n)$ *is finite.*

Recall now that
$$X(p, n) = T(p, n)/\text{Mod}\,(p, n),$$
is the space of moduli (conformal equivalence classes) of Riemann surfaces of type (p, n), and
$$X(p, n;\ v_1, \ldots, v_n) = T(p, n)/\text{Mod}\,(p, n;\ v_1, \ldots, v_n),$$
is the space of moduli (conjugacy classes) of Fuchsian groups of signature $(p, n;\ v_1, \ldots, v_n)$. One sees easily that $X(p, n;\ v_1, \ldots, v_n)$ is a finitely-sheeted ramified holomorphic covering space of $X(p, n)$.

A discrete group of holomorphic automorphisms of a bounded domain acts properly discontinuously. Using a general theorem of H. Cartan [**42**], one obtains from Theorem XI the

COROLLARY. *Each* $X(p, n)$ *and each* $X(p, n;\ v_1, \ldots, v_n)$ *is an irreducible normal complex space.*

By completely different methods, Baily [**19**] proved that $X(p, 0)$ is *quasi-projective.* This is probably true for all moduli spaces. In this connection, *cf.* the recent thesis by J. Gilman [**59**].

The moduli spaces have *non-uniformizable singularities.* Indeed, in most cases, every point of $X(p, n)$ corresponding to a Riemann surface with conformal automorphisms is such a singularity (Rauch [**102**], *cf.* also Harvey [**64**]).

On the other hand, $X(p, 0)$ is *simply connected.* The proof of this fact (Maclachlan [**83**]) uses Theorem XI and the work of Birman [**83**]; it involves exhibiting a set of elements of finite order of Mod $(p, 0)$ which generate this group. It is not known whether the moduli spaces $X(p, n)$, $n > 0$, are simply connected. We note, in this connection, that if $n > 2p+2$ and the numbers $v_1, \ldots, v_n$ are all distinct, then $X(p, n;\ v_1, \ldots, v_n)$ is a manifold and not simply connected.

3. FIBER SPACES OVER TEICHMÜLLER SPACES

1. *The Teichmüller space as a space of differential equations*

Let G be, as before, a Fuchsian group acting on the upper half-plane U. We repeat, and make more precise, the description of the Teichmüller space $T(G)$ as a subset of the Banach space $B_2(L, G)$. Let $\phi \in B_2(L, G)$; we associate with ϕ the meromorphic function
$$W_\phi = \eta_1/\eta_2,$$
where η_1 and η_2 are solutions of the differential equation
$$2\eta'' + \phi\eta = 0,$$

normalized by the initial conditions

$$\eta_1(-i) = \eta_2'(-i) = 1, \quad \eta_1'(-i) = \eta_2(-i) = 0.$$

This function is always locally schlicht in L, and, in view of the G-invariance of $\phi(z)\,dz^2$, there always exists a group G^ϕ of Möbius transformations (the *monodromy group* of ϕ) and an isomorphism χ_ϕ: $G \to G^\phi$ such that

$$W_\phi \circ g = \chi_\phi(g) \circ W_\phi \text{ for } g \in G.$$

The theory of ordinary differential equations in the complex domain implies at once that, for every fixed $g \in G$, the Möbius transformation $\chi_\phi(g)$ depends *holomorphically* on $\phi \in B_2(L, G)$.

Now $\phi \in T(G)$ if and only if (i) $W_\phi(L)$ is a *Jordan domain*, so that the group G^ϕ is a quasi-Fuchsian group with fixed curve C_ϕ = boundary curve of $W_\phi(L)$, and (ii) the homeomorphism W_ϕ has a *quasiconformal extension* to $\hat{\mathbb{C}}$ which conjugates G into G^ϕ.

If G is of finite type, then it can be shown that condition (i) implies (ii). In this case, $T(G) \subset B_2(L, G)$ can be defined without any reference to quasiconformal mappings. At present, however, we know of no way of establishing the main theorems without using these mappings.

For $\phi \in T(G)$, the Jordan curve C_ϕ depends *holomorphically* on ϕ, in the sense that it admits a parametric representation $z = W_\phi(x)$, $-\infty \leqslant x \leqslant +\infty$, where x is the curve parameter, and for each fixed x, including $x = \infty$, the complex number $W_\phi(x)$ is a holomorphic function of $\phi \in T(G)$. We will denote by $D(\phi)$ the Jordan domain *complementary* to $W_\phi(L)$.

For each $\phi \in T(G)$, there is a Möbius transformation α_ϕ such that $\alpha_\phi \circ W_\phi$ keeps 0, 1, ∞ fixed; $\alpha_\phi \circ W_\phi$ is, in the notations of 2, §5, the function $w^\# \mid L$, and ϕ "is" the point $[w]$ of $T(G)$. (This means that ϕ is the Schwarzian derivative of $w^\# \mid L$.) Of course, $w \in Q_n(G)$ is not determined by the equivalence class $[w]$, but the Fuchsian group $G_\phi = wGw^{-1}$ and the isomorphisms $g \mapsto w \circ g \circ w^{-1}$ of G onto wGw^{-1} are. Let h_ϕ be the conformal mapping $\alpha_\phi(D(\phi)) \to U$ which keeps 0, 1, ∞ fixed; then $G_\phi = \alpha_\phi \circ W_\phi G W_\phi^{-1} \circ \alpha_\phi^{-1}$ and the isomorphism is conjugation by $h_\phi \circ \alpha_\phi \circ W_\phi$.

Unfortunately, there is no known method to decide whether a given $\phi \in B_2(L, G)$ belongs to $T(G)$. This is so even if $d = \dim B_2(L, G) < \infty$ (in this case, the components of ϕ with respect to a basis $\phi_1, \phi_2, \ldots, \phi_d$ are classically known as *accessory parameters*). Even the case $d = 1$ is untractable.

Remark. The accessory parameters appear in classical uniformization theory. Suppose that we want to find the holomorphic universal covering π: $U \to S - \sigma$ where S is a compact Riemann surface of genus p, thought of as a plane algebraic curve and thus as a "concrete" branched covering surface of $\hat{\mathbb{C}}$, and σ is a set of n points. The inverse mapping $\omega = \pi^{-1}$ will be a multiple valued *linearly polymorphic* function. This means that ω undergoes a Möbius transformation when it is continued analytically along any closed path on S, avoiding σ. These Möbius transformations form a group G. Now every linearly polymorphic function can be written as $\omega = \eta_1/\eta_2$ where η_1, η_2 are linearly independent solutions of a linear second order differential equation " on S ". In fact, the theory of ordinary differential equations permits one to write down this equation explicitly, except for $d = 3p-3+n$ constants, called accessory parameters. The problem is to determine these constants so that all elements of G become *real* Möbius transformations. The creators of uniformization notices at once that the number of accessory parameters equals the number of conformal moduli of $S-\sigma$. Now we know why this is so: the accessory parameters, on the mirror image of the curve, *are* the moduli.

2. *Fiber spaces*

The *fiber space* $F(G)$ over the Teichmüller space $T(G)$ is defined as the set of pairs (ϕ, z) where $\phi \in T(G)$ and $z \in D(\phi)$; it is a domain in $B_2(L, G) \oplus \mathbb{C}$. Note that the fiber space $F(G)$ is the *restriction* to $T(G)$ of the universal fiber space $F(1)$ over $T(1)$.

One can, if desired, consider a different (biholomorphically equivalent) fiber space, replacing $D(\phi)$ by $\alpha_\phi(D(\phi))$. But the present definition has several advantages. For instance, applying Koebe's one-quarter-theorem to the schlicht function

$$\frac{i}{2}\, W_\phi\left(\frac{i\zeta+i}{\zeta-1}\right)^{-1} = \zeta + a_2\,\zeta^2 + a_3\,\zeta^3 + \ldots, \quad |\zeta| < 1,$$

we see that $W_\phi(L)$ contains all points z with $|z| > 2$, so that $D(\phi)$ is contained in the disc $|z| < 2$. Thus $F(G)$ is a *bounded domain* in $B_2(L, G) \oplus \mathbb{C}$.

Let $\partial T(G)$ denote the *boundary* of $T(G)$ in $B_2(L, G)$. The group G^ϕ, which is defined for all $\phi \in B_2(L, G)$, need not, in general, be discrete. But *if* $\phi \in \partial T(G)$, *then* G^ϕ *is a Kleinian group*. We shall exploit this fact in a later section.

The group G *operates on* $F(G)$, effectively, holomorphically and discretely, by the rule

$$g(\phi, z) = \left(\phi,\, W_\phi \circ g \circ W_\phi^{-1}(z)\right), \quad g \in G.$$

The quotient $F(G)/G$ is again a *fiber space* over $T(G)$. The fiber over a point ϕ is the quotient $D(\phi)/G^\phi$ which is isomorphic to the quotient U/G_ϕ. If G has no elliptic elements, $\phi \in T(G)$ is an equivalence class of quasiconformal mappings

$$f_\phi\colon U/G \to f_\phi(U/G);$$

in this case, the fiber of $F(G)/G$ over ϕ is conformally equivalent to $f_\phi(U/G)$.

The fiber space $F(G)$ has several applications which we intend to discuss. First we make a digression on automorphic forms.

3. *Automorphic forms for Fuchsian and quasi-Fuchsian groups*

If G is a Fuchsian group acting on U, a *holomorphic automorphic form* of weight $(-2q)$ for G in U is a function $\psi(z)$, $z \in U$, satisfying the functional equation

$$\psi(g(z))\, g'(z)^q = \psi(z), \quad g \in G.$$

We consider only integers q with
$$q \geqslant 2,$$
unless the contrary is specifically stated. Holomorphic automorphic forms with
$$\|\psi\|_{A_q(U,G)} = \iint\limits_{U/G} y^{q-2} |\psi(z)| \, dx \, dy < \infty,$$
form the Banach space $A_q(U, G)$ of *integrable* automorphic forms. (The integral makes sense since $y^{q-2} |\psi| \, dx \, dy$ is G-invariant.) Those with
$$\|\psi\|_{B_q(U)} = \sup y^q |\psi(z)| < \infty,$$
form the Banach space $B_q(U, G)$ of *bounded* forms. The *Petersson* [97] *scalar* product
$$\langle \psi_1, \psi_2 \rangle = \iint\limits_{U/G} y^{2q-2} \psi_1(z) \overline{\psi_2(z)} \, dx \, dy,$$
exists whenever $\psi_1 \in A_q(U, G)$ and $\psi_2 \in B_q(U, G)$ (note that $y^{2q-2} \psi_1 \bar{\psi}_2 \, dx \, dy$ is invariant). The definitions apply also if $G = 1$, the trivial group. If $\Psi \in A_q(U, 1)$, the Poincaré *theta series*
$$(\theta \Psi)(z) = \sum_{g \in G} \Psi(g(z)) \, g'(z)^q$$
converges uniformly and absolutely on compact subsets of U.

We state two basic results on automorphic forms ([30] *cf.* also Earle [48,49], Kra [76], Appendix).

XIII. *θ is a continuous linear surjection $A_q(U, 1) \to A_q(U, G)$.*

XIV. *The Petersson scalar product establishes an antilinear topological isomorphism between $B_q(U, G)$ and the dual space to $A_q(U, G)$.*

The situation becomes much simpler if G is of finite type. Then $A_q(U, G) = B_q(U, G)$, is finite dimensional, and is called the space of *cusp forms.* If G has signature $(p, n; \ \nu_1, \ldots, \nu_n)$, then
$$d_q = \dim A_q(U, G) = (2q-1)(p-1) + \sum_{j=1}^{n} \left[q - \frac{q}{\nu_j} \right],$$
where $[x]$ is the integral part of x and it is agreed that $[q - q/\infty] = q - 1$. This formula is a well-known corollary of the Riemann-Roch theorem.

The definitions, and the theorems, extend at once to a more general situation, namely to the case where G is a quasi-Fuchsian group and the holomorphic solutions of the functional equation for automorphic forms are considered in the two Jordan domains Δ_1 and Δ_2 complementary to the fixed curve C of the group. We again have, for $j = 1$ and for $j = 2$, spaces of integrable and of bounded holomorphic forms, $A_q(\Delta_j, G)$ and $B_q(\Delta_j, G)$, paired by the Petersson product $\langle , \rangle$.

But now the integration in the definitions of the norms and of the scalar product extend over Δ_j/G, and the factors y^{q-2}, y^q and y^{2q-2} are to be replaced by λ^{2-q}, λ^{-q} and λ^{2-2q}, respectively, where $\lambda(z)|dz|$, $z \in \Delta_j$, is the Poincaré metric in Δ_j (the unique complete conformal metric of constant Gaussian curvature (-1)).

Let Δ_1 be the component of $\hat{\mathbb{C}} - C$ lying to the left of C. We define an operator $L_C^{(q)}$ mapping functions in Δ_2 into functions in Δ_1 by the formula

$$(L_C^{(q)}\psi)(z) = \frac{1-2q}{\pi} \iint_{\Delta_2} \frac{\lambda(\zeta)^{2-2q}\overline{\psi(\zeta)}\,d\xi\,d\eta}{(\zeta - z)^{2q}}, \quad z \in \Delta_1.$$

The operator $L_{-C}^{(q)}$ is defined analogously. (By $-C$ we mean, of course, the curve C with the orientation reversed.)

XV. *Assume that C is a quasi-circle and G a quasi-Fuchsian group with fixed curve C. Then $L_C^{(q)}$ is a continuous antilinear bijection of $A_q(\Delta_2, G)$ and of $B_q(\Delta_2, G)$ onto $A_q(\Delta_1, G)$ and onto $B_q(\Delta_2, G)$, respectively. Also $L_{-C}^{(q)}$ is adjoint to $L_C^{(q)}$ with respect to the Petersson scalar product.*

The proof **[31]** is rather delicate and uses a surprising property of quasi-circles discovered by Ahlfors **[9]**: if C passes through ∞, then there is an involution of $\hat{\mathbb{C}}$ which leaves every point of C fixed and satisfies a uniform Lipschitz condition. The proof simplifies considerably if G is of finite type. Theorem XV can be used to prove the openness of $T(G)$ in $B_2(L, G)$.

Remark. For G of finite type, Theorem XIII has been known to Poincaré. The question arises, how to choose d_q functions Ψ_j in $A_q(U, 1)$ so as to obtain a basis of cusp forms of the form $\{\theta\Psi_j\}$. It is a little more convenient to consider the same question for $A_q(D, G)$ where D is the unit disc $|z| < 1$. In this case, d_q out of the functions θz^j, $j = 0, \ldots, (2q-2)(p-1)+n$ form a basis; which ones depends on the so-called Weierstrass gap sequence at 0. In particular, if $z = 0$ is not a *Weierstrass point*, i.e. if no $\psi \in A_q(D, G)$ vanishes at $z = 0$ together with its first $d_q - 1$ derivatives without vanishing identically, then the functions θz^j, $j = 0, 1, \ldots, d_q - 1$, form a basis.

If $z = 0$ is a Weierstrass point, one can read off, from the Weierstrass gap sequence, which exponents j one needs **[34]**. The connection between Weierstrass gaps and bases of theta series has been first noted by Petersson **[98]**.

We recall that for every q there are Weierstrass points, and only finitely many of those. On the other hand, every open set contains Weierstrass points for some q (Mumford, Olsen **[96]**).

4. *Simultaneous uniformization*

We are now in a position to state the main theorems on simultaneous uniformization **[28, 29]**.

XVI. *Let G be a Fuchsian group, acting on U. To every ψ in $A_q(U, G)$ (or in $B_q(U, G)$), one can associate, canonically, a holomorphic function $\Psi(\phi, z)$, $(\phi, z) \in F(G)$, such that: $\Psi(0, z) = \psi(z)$, and, for every fixed $\phi \in T(G)$, the mapping $\psi \mapsto \Psi(\phi, \cdot)$ is a linear topological isomorphism of $A_q(U, G)$ onto $A_q(D(\phi), G^\phi)$ (or of $B_q(U, G)$ onto $B_q(D(\phi), G^\phi)$).*

The function Ψ is constructed explicitly. Given ψ, set $\psi_1(z) = \overline{\psi(\bar{z})}$, $z \in L$. Then find, for a given ϕ, the function $\psi_2(z), z \in W_\phi(L)$, by setting $\psi_2(W_\phi(t)) = \psi_1(t)\, W_\phi'(t)^{-q}$. Then ψ_2 belongs to $A_q(W_\phi(L), G^\phi)$ or to $B_q(W_\phi(L), G^\phi)$. Finally, define $\Psi = L_{C_\phi}{}^{(q)}\psi_2$. The resulting formula reads:

$$\Psi(\phi, z) = -\frac{2q-1}{\pi} \iint_{\eta<0} \frac{\eta^{2q-2}\,\psi(\zeta)\, W_\phi'(\zeta)^q\, d\xi\, d\eta}{(W_\phi(\zeta)-z)^{2q}}.$$

(If one wants to obtain an automorphic form $\hat{\psi}$ "corresponding" to $\Psi(\phi, z)$ and belonging to the space $A_q(U, G_\phi)$, one must set $\hat{\psi}(h_\phi(t)) = \Psi(\phi, t)\, h_\phi'(t)^{-q}$ where $h_\phi\colon D(\phi) \to U$ is a conformal mapping such that $h_\phi \circ W_\phi$ keeps 0, 1, ∞ fixed. Note that the dependence of $\hat{\psi}(z)$ on ϕ is *not* holomorphic.)

Assume next that G is of finite type and purely hyperbolic, so that U/G is a compact surface of some genus $p > 1$. Choose a basis $\psi_1, \ldots, \psi_{5p-5}$ for $A_3(U, G)$, and construct the functions $\Psi_1, \ldots, \Psi_{5p-5}$ as in Theorem XVI. Finally, recall that by taking cubic holomorphic differentials on a compact Riemann surface as homogeneous coordinates, one obtains a holomorphic embedding of the surface into the complex projective space $\mathbb{P}_{5_g-4}$. This leads to the following result.

XVII. *For G a hyperbolic Fuchsian group of type $(p, 0)$, there are $5p-5$ holomorphic functions $\Psi_1(\phi, z), \ldots, \Psi_{5p-5}(\phi, z)$, $(\phi, z) \in F(G)$, with the following property: every algebraic curve of genus p in $\mathbb{P}_N$ $(N = 2, 3, \ldots)$ admits a parametric representation of the form*

$$\zeta_j = P_j(\Psi_1(\phi, z), \ldots, \Psi_{5p-5}(\phi, z)), \quad j = 0, 1, \ldots, N,$$

where $\phi \in T(G)$ is fixed, z varies over $D(\phi)$ and the P_j are homogenous polynomials in $5p-5$ variables, of the same degree.

Noting that ϕ in the theorem is a point in $\mathbb{C}^{3p-3}$, we recognize that the functions Ψ_j accomplish for curves of genus p what the Weierstrass $\wp$-function does for $p = 1$. Unfortunately, Theorem XVII contains a purely existential element—for reasons explained at the end of §1.

For G of finite type, Theorem XVI can be extended to the case $q = 1$ **[26]**. For the sake of brevity, we explain this only for a G as in Theorem XVII. Let $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_p$ be a set of *standard generators* of G. These elements generate G with the single defining relation

$$\prod_{j=1}^{p} \alpha_j \circ \beta_j \circ \alpha_j^{-1} \circ \beta_j^{-1} = id.$$

For $\phi \in T(G)$, set $\alpha_j^\phi = \chi_\phi(\alpha_j)$, $\beta_j^\phi = \chi_\phi(\beta_j)$. For every $\phi \in T(G)$, there are p holomorphic functions of $z \in D(\phi)$, called the normalized Abelian integrals of the first kind, $I_1(\phi, z), \ldots, I_p(\phi, z)$, such that

$$I_i(\phi, \alpha_j^\phi(z)) - I_i(\phi, z) = \delta_{ij} \text{ for } z \in D(\phi).$$

The derivatives $\partial_1(\phi, z)/\partial z, \ldots, \partial I_p(\phi, z)/\partial z$ are uniquely determined, and form, for every ϕ, a basis for the automorphic forms of weight (-2) for the group G^ϕ in $D(\phi)$. It can be shown, using Theorem XVI, that each $\partial I_i(\phi, z)/\partial z$ *is a holomorphic function of* $(\phi, z) \in F(G)$.

Also, the p^2 functions

$$Z_{ij}(\phi) = I_i(\phi, \beta_j^\phi(z)) - I_i(\phi, z), \quad z \in D(\phi),$$

are well defined (i.e., the differences above do not depend on z). The mapping $\phi \to (Z_{ij}(\phi))$ is *a holomorphic mapping* $T_p \to H_p$. This is the so-called period mapping which sends a closed Riemann surface of genus p endowed with a standard system of generators for the fundamental group into the period matrix belonging to the corresponding canonical homology basis.

5. *Quotients and isomorphisms*

The group mod (G), *cf.* 2, §3, can be made to operate on the fiber space $F(G)$. An element ω^* of mod (G) is induced by a quasiconformal self-mapping ω of U with $\omega G \omega^{-1} = G$; this ω also induces an element $\langle\omega\rangle$ of the modular group

$$\text{Mod}\,(G) = \text{mod}\,(G)/G.$$

For $\phi \in T(G)$, set $\psi = \langle\omega\rangle(\phi)$. One can show that among all conformal mappings of $D(\phi)$ onto $D(\psi)$ there is one, f, which satisfies

$$f(W_\phi(x)) = W_\psi(\omega(x)), \quad x \in R.$$

(Proof of all statements in this section will appear in a forthcoming paper [37].) For (ϕ, z) of $F(G)$ we define:

$$\omega^*(\phi, z) = (\psi, f(z)).$$

XVII. mod (G) *acts as a group of holomorphic automorphisms of* $F(G)$; *this group contains the action of G as a normal subgroup.*

Suppose now that G is finitely generated and of the first kind, of signature $(p, n;\ \nu_1, \ldots, \nu_n)$. Then mod (G) is discrete, so that

$$\tilde{X}(p, n;\ \nu_1, \ldots, \nu_n) = F(G)/\text{mod}\,(G),$$

is a normal complex space. There is an obvious mapping

$$\tilde{X}(p, n;\ \nu_1, \ldots, \nu_n) \to X(p, n;\ \nu_1, \ldots, \nu_n) = T(G)/\text{mod}\,(G),$$

and one obtains the following result.

XIX. *For every signature* $(p, n;\ \nu_1, \ldots, \nu_n)$, *there is a normal complex fiber space* $\tilde{X}(p, n;\ \nu_1, \ldots, \nu_n)$ *over the moduli space* $X(p, n;\ \nu_1, \ldots, \nu_n)$ *such that the fiber over a point* $x \in X(p, n;\ \nu_1, \ldots, \nu_n)$ *representing the conjugacy class of a Fuchsian group* Γ *is isomorphic to* $U/\hat{\Gamma}$, $\hat{\Gamma}$ *being the normalizer of* Γ *in the group of all real Möbius transformations.*

If we specialize to the case when all $\nu_j = \infty$, we obtain the

COROLLARY. *For all values of p and n with $3p-3+n \geqslant 0$, there is a normal complex fiber space $\tilde{X}(p, n)$ over the moduli space $X(p, n)$ such that the fiber over a point $x \in X(p, n)$ representing the conformal equivalence class of a Riemann surface Σ is isomorphic to* $\Sigma/\mathrm{Aut}\,(\Sigma)$, $\mathrm{Aut}\,(\Sigma)$ *being the group of all conformal automorphisms of* Σ.

For $n = 0$, this has been stated, without proof, by Teichmüller [**109**], and proved, in an entirely different way, by Baily [**20**].

XX. *If G is a finitely generated Fuchsian group of the first kind, without elliptic elements, and V another group, with U/V conformally equivalent to $(U/G)-\{P\}$, for some point P, then there is a canonical isomorphism*

$$T(V) \simeq F(G),$$

which induces an isomorphism between a subgroup of Mod (V) *and* mod (G).

If G is of type (p, n), the subgroup is of index n and we obtain the

COROLLARY. *$\tilde{X}(p, 0)$ is isomorphic to $X(p, 1)$. For $n > 0$, $\tilde{X}(p, n)$ is an $(n+1)$ sheeted ramified covering space of $X(p, n+1)$.*

For G as above, we write $T(p, n)$ instead of $T(G)$ and $F(p, n)$ instead of $F(G)$. The isomorphism Theorem XX leads to a new embeddeding of $T(p, n)$ into $\mathbb{C}^{3p-3+n}$, for low genera p. Indeed, $T(0, 3)$ is a points, $T(0, 4) = F(0, 3)$ is conformally equivalent to the unit disc, $T(0, 5) = F(0, 4)$, $T(0, 6) = F(0, 5)$, etc. On the other hand, $T(0, 4)$ and $T(0, 6)$ can be identified with $T(1, 1)$ and $T(2, 0)$, respectively, and we have $T(1, 2) = F(1, 1)$, $T(2, 1) = F(2, 0)$, etc.

Let us call a domain in $\mathbb{C}^N$ a *Bergman domain* if it is the set of N-tuples $(z_1, z_2, \ldots, z_N)$ such that $|z_1| < 1$ and $z_{j+1} \in D_{j+1}(z_1, \ldots, z_j)$, where D_{j+1} is a bounded Jordan domain whose boundary curve admits a parametric representation

$$z_{j+1} = W(z_1, \ldots, z_j;\; e^{i\theta}), \quad 0 \leqslant \theta \leqslant 2\pi,$$

W being, for each fixed θ, a holomorphic function of $z_1, \ldots, z_j$ for

$$|z_1| < 1, \quad z_2 \in D_2(z_1), \ldots, z_j \in D_j(z_1, \ldots, z_{j-1}).$$

The preceding argument establishes

XXI. *$T(p, n)$ can be represented as a bounded Bergman domain in $\mathbb{C}^{3p-3+n}$, for* $p = 0, 1, 2$.

6. *Uniformization of algebraic varieties*

The following striking application of quasi-Fuchsian groups and the fiber spaces over Teichmüller spaces to algebraic varieties of arbitrary dimensions is due to Griffiths [**61**].

XXII. *Let A_0 be an n-dimensional irreducible non-singular projective algebraic variety (over $\mathbb{C}$), $A_1 \subset A_0$ a non-empty Zariski open subset. Then there is a non-empty Zariski open subset $A \subset A_1$ such that the universal covering space $\tilde{A}$ of A is (biholomorphically equivalent to) a bounded Bergman domain in $\mathbb{C}^n$.*

Recall that a projective algebraic variety (over $\mathbb{C}$) is the set, in some complex projective space $\mathbb{P}_N$, of common zeros of a set of homogeneous polynomial equations. Such a variety is called irreducible if it is not the union of two subvarieties, none of which is a subvariety of the other, non-singular if it is a complex manifold. A Zariski open subset of a variety is the complement of a subvariety. Note that the sets A_1 and A of the theorem are open and dense in A_0, in the ordinary topology.

Theorem XXII implies the uniformization statement made in 1, §1. Indeed, given a set of polynomial equations in any number of variables, we can make it into a set of homogeneous equations by adding a new variable. Let V be the projective variety defined by these equations. Now apply XXII to each irreducible component of V. In view of Hironaka's theorem, we may assume these components to be non-singular. However, the assumption of non-singularity may be eliminated from XXII.

We shall outline the main steps in the proof of Theorem XXII; it proceeds by induction on the (complex) dimension n of A_0. If $n = 1$, A_0 is a compact Riemann surface; A_1 and A are obtained by removing finitely many points. The assertion of XXII is the classical uniformization theorem (*cf.* the Corollary in 1, §4),.

Assume that $n > 1$ and $A_0 \subset \mathbb{P}_N$ for some N. It is not difficult (at least not for algebraic geometers) to find homogeneous polynomials $P_0, \ldots, P_{n-1}$, of the same degree, in $N+1$ variables, and non-empty Zariski open subsets, $A \subset A_1$ and $B \subset \mathbb{P}_{n-1}$, such that the mapping f which sends a point $\zeta = (\zeta_0, \ldots, \zeta_N)$ in $\mathbb{P}_N$ into

$$(P_0(\zeta), \ldots, P_{n-1}(\zeta)),$$

is a well-defined mapping of A onto B, of maximal rank at all points of A, and with the property: for every $b \in B$, $S_b = f^{-1}(b) \cap A$ is a Riemann surface of (fixed) finite type (p, m), with $3p-3+m \geqslant 0$. (If $A_0 = \mathbb{P}_n$, the P_α may be taken as linear forms.)

In view of the induction hypothesis, we may assume, replacing if need be B by a smaller Zariski open set, that the universal covering space $\tilde{B}$ of B is a bounded Bergman domain in $\mathbb{C}^{n-1}$.

Let G be a Fuchsian group of finite type and without elliptic elements, with U/G of type (p, m). Let $b_0 \in B$ and let $\phi_0 \in T(G)$ be such that $D(\phi_0)/G^{\phi_0}$ is conformally equivalent to S_{b_0}. In a sufficiently small neighbourhood β of b_0 in B, there is a uniquely determined continuous mapping $F_0\colon \beta \to T(G)$ such that $F_0(b_0) = \phi_0$ and $D(F_0(b))/G^{F_0(b)}$ is conformally equivalent to S_b. One can verify that this mapping is holomorphic, and one then concludes that it can be continued analytically along every path in B.

Let $\pi\colon \tilde{B} \to B$ be the covering map and $\tilde{b}_0 \in \tilde{B}$ a point with $\pi(\tilde{b}_0) = b_0$. Since $\tilde{B}$ is simply connected, there is a holomorphic mapping $F\colon \tilde{B} \to T(G)$ with $F(b_0) = \phi_0$ and such that, for every $Z \in \tilde{B}$, $D(F(Z))/G^{F(Z)}$ is conformally equivalent to $S_{\pi(Z)}$.

Now let M be the set of pairs (Z, z) with $Z \in \tilde{B}$ and $z \in D(F(Z))$; this M is a bounded Bergman domain in $\mathbb{C}^n$. The group G operates on M as a discrete fixed-point-free group of holomorphic automorphisms, by the rule:

$$g(Z, z) = \left(Z,\, W_{F(Z)} \circ g \circ W^{-1}{}_{F(Z)}(z)\right), \quad g \in G.$$

The quotient M/G is a manifold and the canonical holomorphic mapping $M \to M/G$ is a universal covering. A point of M/G may be thought of as a pair (Z, t) with $Z \in \tilde{B}$, $t \in S_{\pi(Z)} \subset A$, such that $f(t) = \pi(Z)$. Follow $M \to M/G$ by the holomorphic mapping which sends (Z, t) into t. The combined mapping $M \to A$ is seen to be a universal covering. Hence $M = \tilde{A}$.

4. GENERAL THEORY OF KLEINIAN GROUPS

1. *Finding Kleinian groups*

For various reasons, the general theory of Kleinian groups has lagged behind the theory of Fuchsian groups. Recently there has been a revival of interest, primarily because of the stimulus provided by quasiconformal mappings. Still there are many more open problems than definitive answers.

There are various methods for obtaining Kleinian groups, other than Fuchsian. One does not know, however, of a method for constructing all such groups, and one is as yet far removed from a complete classification. This is true even for the simplest and most important case of finitely generated groups.

One way of constructing Kleinian groups ought to come from *solid non-Euclidean geometry*. Consider $\mathbb{R}^3$ as the set of pairs $(z = x+iy, t)$, $z \in \mathbb{C}$, $t \in \mathbb{R}$, and the upper half space $\mathbb{R}^3_+ = [t > 0]$, endowed with the Poincaré line element

$$ds^2 = (dx^2 + dy^2 + dt^2)/t^2,$$

as a model of the non-Euclidean space. Every Möbius transformation

$$z \mapsto (az+b)/(cz+d),$$

has an extension to the upper half space as a non-Euclidean motion, and every non-Euclidean motion has a "trace" on $\hat{\mathbb{C}}$ which is a Möbius transformation. Thus every Kleinian group G may be thought of as a discrete group of non-Euclidean motions. It gives rise to a space $\mathbb{R}^3_+/G$, which is a 3-manifold, even if G contains elliptic elements. This way of looking at Kleinian groups goes back to Poincaré and is acquiring, in conjunction with 3-dimensional topology, an ever-increasing importance in recent investigations (Maskit, A. Marden, and others). Still it is fair to say that non-Euclidean geometry has not yet led to any new types of Kleinian groups.

An important theorem of Klein's, the so-called *combination theorem*, gives a sufficient condition in order that the group $\langle G_1, G_2 \rangle$ generated by two Kleinian groups G_1 and G_2, with $G_1 \cap G_2 = 1$, be Kleinian (and their free product). The condition reads: there should be fundamental regions ω_1 and ω_2 for the two groups such that each contains the closure of the complement of the other (*cf.* Ford [55]). Klein's theorem can be applied repeatedly, to more than two groups. For instance, a Schottky group on p generators is the free product of p cyclic loxodromic groups.

Maskit [84, 85, 87] refined Klein's theorem and supplemented it by two more combination theorems which permit one to construct new Kleinian groups from old ones. One combination theorem concerns a group $\langle G_1, G_2\rangle$ generated by two Kleinian groups G_1 and G_2 with $G_1 \cap G_2 \neq 1$; under proper geometric conditions, $\langle G_1, G_2\rangle$ turns out to be Kleinian, and a *free product with amalgamated subgroup*. The other theorem gives geometric conditions in order that $\langle G_1, \gamma\rangle$, where γ conjugates two hitherto *non-conjugate subgroups* of G_1, be Kleinian. The combination theorems are a powerful tool for constructing interesting examples.

Once a Kleinian group has been constructed, one obtains a family of other Kleinian groups by *quasiconformal deformations*. In fact, according to Theorem IV, every example of a Kleinian group gives rise to a uniformization theorem for one or several Riemann surfaces.

Finally, it is possible to obtain new Kleinian groups as *limits* of sequences of " known " Kleinian groups. More precisely, let there be given a Kleinian group G and a sequence of isomorphisms χ_j of G such that $\chi_j(G)$ is a Kleinian group and for every parabolic $g \in G$, $\chi_j(g)$ is parabolic if and only if g is, and for every $g \in G$, $\chi(g) = \lim \chi_j(g)$ exists, as a Möbius transformation. V. Chuckrow [43] proved that under those circumstances, χ: $G \to \chi(G)$ is an isomorphism (and not merely a homomorphism), and that, for every loxodromic $g \in G$, $\chi(g)$ is either loxodromic or parabolic. The limit group $\chi(G)$ need not be Kleinian. If it is, it may be strikingly different from all $\chi_j(G)$. Examples will be given in §6.

2. *The limit set*

A challenging and difficult question of the general theory of Kleinian groups concerns the nature of the limit set $\Lambda(G)$. Many years ago P. T. Myrberg [93] observed that every non-elementary Kleinian group G contains non-cyclic Schottky subgroups. He concluded from this that $\Lambda(G)$ cannot be " too small "; its *logarithmic capacity must be positive.* (Myrberg's observation is also crucial for the proof of Chuckrow's theorem mentioned above.)

It is natural to ask whether the (2-dimensional) measure of Λ can be positive. For infinitely generated groups, this is possible. Abikoff [1] constructed a quasi-Fuchsian group of the first kind whose fixed curve has *positive measure.* This group is not a quasiconformal deformation of a Fuchsian group.

Ahlfors proved that $\operatorname{mes} \Lambda(G) = 0$ if G, viewed as a group of motions in non-Euclidean space, has a fundamental region in $\mathbb{R}^3_+$ bounded by a non-Euclidean polyhedron with *finitely many sides* [12]. Such a group is necessarily finitely generated, but there are finitely generated groups which do not satisfy Ahlfors' condition (L. Greenberg [60]).

On the other hand, Maskit [87] showed that if one starts with Kleinian groups, finitely generated or not, whose limit sets have measure 0, the same holds for groups obtained by using the Klein-Maskit *combination theorem.* The proof uses an idea of Koebe.

An interesting part of the limit set Λ is the *residual limit set* $\Lambda_0 \subset \Lambda$, consisting of points not on the boundary of any component of Ω. This set has been discovered and studied by Abikoff [2]. If $\Lambda_0 = \varnothing$, then either Ω has a G-invariant component, or Ω/G is connected and G has a quasi-Fuchsian subgroup of index 2.

We now proceed to analytic methods in the theory of Kleinian groups.

3. *Fuchsian equivalents and automorphic forms for Kleinian groups*

Let G be a non-elementary Kleinian group. The components of $\Omega(G)$ are also called *components of* G. Two components, Δ_1 and Δ_2, are called conjugate if there is a $g \in G$ with $g(\Delta_1) = \Delta_2$.

Let $\Delta_1, \Delta_2, \Delta_3, \ldots$ be a *complete* list of non-conjugate components of G, and let G_j be the largest subgroups of G which leaves Δ_j fixed. Since G is not elementary, there exist holomorphic universal coverings $h_j\colon U \to \Delta_j$. For each j, let H_j be the covering group of h_j, and let Γ_j be the group of all real Möbius transformations γ with the property: there is a $g = \chi_j(\gamma) \in G_j$ with $h_j \circ \gamma = g \circ h_j$. Then H_j and Γ_j are Fuchsian group; there are exact sequences

$$1 \to H_j \hookrightarrow \Gamma_j \xrightarrow{\chi_j} G_j \to 1,$$

and

$$\Omega/G = \Delta_1/G_1 \cup \Delta_2/G_2 \cup \ldots = U/\Gamma_1 \cup U/\Gamma_2 \cup \ldots$$

with $\Delta_j/G_j = U/\Gamma_j$ as Riemann surfaces with ramification points. The set

$$(\Gamma_1, \Gamma_2, \ldots)$$

is called a *Fuchsian equivalent* of G. It is uniquely determined by G, except that each Γ_j is determined only up to a conjugation in the group of all real Möbius transformations.

It is known that conjugacy classes of maximal parabolic subgroups of a Fuchsian group Γ are in a one-to-one correspondence with the punctures of U/Γ. In the case of Kleinian groups, the situation is slightly more complicated.

Let Σ be a maximal parabolic subgroup of Γ_j, γ a generator of Σ. Then $g = \chi_j(\gamma)$ is also parabolic and generates a maximal parabolic subgroup of G_j. We may assume that γ is the translation $z \mapsto z+1$; this can be achieved by conjugation. Let a and $b > 1$ be real numbers and let $M \subset U$ be the half-strip $a \leqslant x < a+1$, $y > b$. Then no two distinct points of M are Γ_j-equivalent, and the image of M in $U/\Gamma_j = \Delta_j/G_j$ is conformally equivalent to a punctured disc, and therefore defines a *puncture* on Ω/G. Every element of the form g^n is said to *belong* to this puncture.

A given parabolic element of G may belong to *no* puncture, to *one* puncture, or to *two* punctures.

In what follows, an important part will be played by the function $\lambda(z)$, $z \in \Omega$, defined by the condition: if $z \in \Delta$, Δ a component of G, then $\lambda(z)|dz|$ is the Poincaré line element in Δ. We note that $\lambda(g(z))\,|g'(z)| = \lambda(z)$ for $g \in G$, and that

$$\lambda(h_j(z))\,|h_j'(z)| = y^{-1},$$

for all j. For every open G-invariant set $\omega \subset \Omega$, one can define the *Poincaré area* of ω/G,

$$\iint_{\omega/G} \lambda(z)^2\,dx\,dy.$$

The total Poincaré area of the quotient,

$$\text{Area}\,(\Omega/G) = \iint_{\Omega/G} \lambda(z)^2\,dx\,dy,$$

is the sum of Poincaré areas of the components of Ω/G. It is finite if and only if Ω/G is a finite union of Riemann surfaces of finite type.

An *automorphic form* ϕ (of weight $(-2q)$) for G is, of course, a function defined on Ω, holomorphic, except perhaps for isolated singularities, and satisfying the functional equation $\phi(g(z))\,g'(z)^q = \phi(z)$, $g \in G$. We also require that the isolated singularities of ϕ, if any, should not " accumulate to a puncture of Ω/G ". An automorphic form ϕ is called *integrable* if $|\lambda^{2-q}\phi|$ is integrable over a fundamental region; such a ϕ has no singularities except perhaps simple poles. The everywhere holomorphic integrable forms are the elements of the Banach space $A_q(\Omega, G)$. The complex conjugate of its dual space is canonically isomorphic to the space $B_q(A, G)$ of *bounded* forms, i.e. those for which $|\lambda^{-q}\phi|$ is bounded, the isomorphism being accomplished by the Petersson *scalar product*

$$\langle\phi, \psi\rangle = \iint_{\omega/G} \lambda(z)^{2-2q}\,\phi(z)\,\overline{\psi(z)}\,dx\,dy.$$

This follows, without difficulty, from Theorem XIII in 3, §3.

4. *Cohomology*

Let $q \geqslant 2$ be an integer and G, as before, a non-elementary Kleinian group. This group, like the group of all Möbius transformations, operates, from the right, on functions $f(z)$, $z \in \Omega$, by the rule

$$(fg)(z) = f(g(z))\,g'(z)^{1-q}, \quad g \in G.$$

In particular, if Π_{2q-2} is the vector space of polynomials in one variable of degree $\leqslant 2q-2$, then $\Pi_{2q-2}\,G \subset \Pi_{2q-2}$ and one can define the first Eichler **[53]** *cohomology group* $H^1(G, \Pi_{2q-2})$ which we will denote simply by $H(G, q)$. Actually, $H(G, q)$ is a complex vector space; its elements are cocycles modulo coboundaries. Here a *cocycle* is a mapping $\sigma\colon g \mapsto \sigma_g$ of G into G_{2q-2} satisfying the condition

$$\sigma_{g\circ h} = \sigma_g h + \sigma_h; \quad g, h \in G_{2q-2}.$$

A *coboundary* (of an element $P \in \Pi_{2q-2}$) is a cocycle of the form

$$(\delta P)_g = Pg - P, \quad g \in G.$$

A cocycle σ (and its cohomology class) is called *parabolic* if, for every parabolic $g_0 \in G$ belonging to a puncture on Ω/G, there is a $P_0 \in \Pi_{2q-2}$ with $\sigma_{g_0} = P_0 g_0 - P_0$. A cocycle (and its cohomology class) is called *strongly parabolic* if the same is true for every parabolic $g_0 \in G$.

We are concerned with a de Rham-Hodge type question: how to represent cohomology by functions defined on $\Omega(G)$? Note first that the definition of coboundary makes sense for any function $f(z)$, $z \in \Omega$. The function f will be called an *automorphic integral* if $(\delta f)_g \in \Pi_{2q-2}$ for all $g \in G$. Then δf is a cocycle; its cohomology class is called the *cohomology class* (or period) of f. An automorphic integral is called parabolic (strongly parabolic) if its cohomology class is. There are two distinguished families of automorphic integrals: potentials of bounded automorphic forms and Eichler integrals.

A *generalized Beltrami coefficient* μ for G is a measurable function such that $\mu \mid \Lambda = 0$,

$$\mu(g(z))\,\overline{g'(z)}\,g'(z)^{1-q} = \mu(z) \text{ for all } g \in G,$$

and $|\lambda^{q-2}\mu|$ is bounded. A generalized Beltrami coefficient is called *canonical* if

$$\mu(z) = \lambda(z)^{2-2q}\,\overline{\phi(z)}, \quad z \in \Omega$$

where $\phi \in B_q(\Omega, G)$. Every generalized Beltrami coefficient μ defines a continuous linear functional l_μ on $A_q(\Omega, G)$ by the formula

$$l_\mu(\psi) = \iint_{\Omega/G} \psi(z)\,\mu(z)\,dx\,dy,$$

and every such functional can be so represented. Two (generalized) Beltrami coefficients, μ_1 and μ_2, will be called *equivalent* if $l_{\mu_1} = l_{\mu_2}$. It follows from §3 that every equivalence class contains a unique canonical Beltrami coefficient.

A *potential* for a Beltrami coefficient μ is a continuous function $F(z)$, $z \in \mathbb{C}$, with $F(z) = O(|z|^{2q-2})$, $z \to \infty$, and

$$\frac{\partial F}{\partial \bar{z}} = \mu,$$

in the sense of distribution theory. Such a potential can be constructed explicitly:

$$F(z) = -\frac{Q(z)}{\pi} \iint_{\Omega} \frac{\mu(\zeta)\,d\xi\,d\eta}{Q(\zeta)(\zeta - z)},$$

where $Q(t)$ is a polynomial with precisely $2q-1$ or $2q-2$ distinct zeros lying in Λ **[32]**. The potential is a strongly parabolic automorphic integral and its cohomology class depends only on the equivalence class of μ. Thus there is a *canonical anti-linear mapping*

$$B_q(\Omega, G) \to H(G, q),$$

which sends a bounded automorphic form ϕ into the cohomology class of $\lambda^{2-2q}\bar{\phi}$. We call this class the cohomology class of ϕ.

LEMMA. *The canonical mapping $B_2(\Omega, G) \to H(G, 2)$ is an injection.*

This was proved, in another formulation, by Ahlfors [10]. The proof involves a very delicate " mollifier ".

Automorphic forms on G, and their potentials, depend on the action of G on the components of Ω. There is another type of automorphic integrals which reflect the way G permutes the components.

An *Eichler integral* is a function $E(z)$, $z \in \Omega$, holomorphic except perhaps for isolated singularities, which is an automorphic integral. We require that the isolated singularities, if any, should not " accumulate at a puncture". Two Eichler integrals are called *equivalent* if their difference is a polynomial of degree $\leqslant 2q-2$; in this case, their cohomology classes coincide.

If $E(z)$ is an Eichler integral, then the $(2q-1)$-st derivative $E^{(2q-1)}(z)$ is an automorphic form; it depends only on the equivalence class of E. It may happen, however, that $E^{(2q-1)} = 0$, so that $E \mid \Delta \in \Pi_{2q-2}$ for every component Δ of Ω, but E is not equivalent to 0.

Let $E(z)$ be an Eichler integral, and assume that a component S of Ω/G has a puncture. Let $g \in G$ be a generator of a corresponding maximal parabolic subgroup; we can achieve by conjugation that g is the translation $z \mapsto z+1$ and that the inverse image in Ω of a properly chosen neighbourhood of the puncture is the half-plane $y = \operatorname{Im}(z) > 1$. For $y > 1$, we have a Fourier expansion

$$E^{(2q-1)}(z) = \sum_{j=-\infty}^{+\infty} a_j e^{2\pi ijz} .$$

One says that E is *regular* at the puncture of $a_j = 0$ for $j \leqslant 0$, and that E is *quasi-regular* if $a_j = 0$ for $j < 0$. One says that E is *parabolic* at the puncture, if $a_0 = 0$. The Fourier series $\sum_{n<0} a_n e^{2\pi inz}$ is called the *principal part* of E at the puncture considered. Ahlfors [**14**] noted that E is parabolic if and only if it is parabolic at every puncture.

XXIII. *There exist parabolic Eichler integrals which have prescribed principal parts at finitely many non-equivalent points $a_1, \ldots, a_r$ of Ω (and are holomorphic at all points in Ω not G-equivalent to $a_1, \ldots, a_r$), and also have prescribed principal parts at finitely many punctures (and are regular at all other punctures).*

The proof [**36**] is by a direct construction using a linear functional on integrable automorphic forms defined by the given principal parts, and generalized Beltrami coefficients with small supports.

5. *Finitely generated groups*

A satisfactory Hodge-type theorem for Eichler cohomology is available, at present, only for finitely generated groups. For these, we have the Ahlfors **[10]** finiteness theorem:

XXIV. *If G is a non-elementary finitely generated Kleinian group, $\Omega(G)/G$ is a finite union of Riemann surfaces of finite type.*

We assume now that

G is non-elementary and has N generators.

This implies, at once, that

$$\dim H(G,q) \leqslant (2q-1)(N-1);$$

equality holds if G is a free group on N generators. Now the lemma in §4 shows that $\dim B_2(\Omega, G) \leqslant 3N-3$; this yields the essential part of Ahlfors' finiteness theorem: every component of Ω/G is a Riemann surface of finite type. Knowing that much, it is easy to obtain the theorem **[32]**:

XXV. *The canonical mapping $B_q(\Omega, G) \rightarrow H(G,q)$ is an injection for $q \geqslant 2$.*

Hence

$$\dim B_q(\Omega, G) \leqslant (2q-1)(N-1), \tag{*}$$

for all $q \geqslant 2$. Ahlfors' finiteness theorem follows. (We note that the converse of XXIV is not true.)

It is not difficult to conclude from the finiteness theorem that $A_q(\Omega, G) = B_q(\Omega, G)$. This space is called the space of *cusp forms* on G.

XXVI. *Every element of $H(G,q)$ is the cohomology class of an Eichler integral.*

This result is due to Ahlfors **[14]**. Actually, Ahlfors proved a much more precise theorem; he showed that one obtains the whole group $H(G,q)$ if one restricts oneself to Eichler integrals which are quasi-regular at the punctures and have no singularities in Ω except for poles whose locations and order can be prescribed within certain limits.

XXVII. *Every cohomology class is, uniquely, the sum of a cohomology class of a cusp form and a cohomology class of a holomorphic Eichler integral which is quasi-regular at all punctures.*

This definitive theorem is due to Kra **[75–77]**. The proof involves the construction, via an appropriate partition of unity, of a C_∞ automorphic integral θ, cohomologous to a given cocycle, such that $\partial\theta/\partial\bar{z}$ is a generalized Beltrami coefficient. One then forms a potential F of $\partial\theta/\partial\bar{z}$ and verifies that $\theta - F$ is a quasi-regular Eichler integral.

Theorem XXVI, and all its refinements, follow from XXVII by reversing the proof of Theorem XXIII.

Let G represent K distinct Riemann surfaces, let $\Delta_1, \ldots, \Delta_K$ be a complete list of non-conjugate components of Ω, and let $(\Gamma_1, \ldots, \Gamma_K)$ be the Fuchsian equivalent of G. Each Fuchsian group Γ_j is of finite type. Let their signatures be

$$(p_j, n_j;\ \nu_1{}^j, \ldots, \nu_{n_j}{}^j), \quad j = 1, \ldots, K.$$

This set is called the *signature* of G. One obtains from the Gauss-Bonnet formula, or from elementary non-Euclidean geometry, that

$$\text{Area}\,(\Omega/G) = 2\pi \sum_{j=1}^{K} \left\{2p_j - 2 + \sum_{i=1}^{n_j} (1 - 1/\nu_i{}^j)\right\},$$

where $1/\infty = 0$.

On the other hand, one verifies that the space $A_q(\Omega, G) = B_q(\Omega, G)$ is isomorphic to the direct sum of K spaces $A_q(U;\ \Gamma_j)$. Computing the dimension of these spaces by the Riemann-Roch theorem (*cf.* 3, §3), and inserting into (*) we obtain that

$$\sum_{j=1}^{K} \left\{(2q-1)(p_j-1) + \sum_{i=1}^{n_j} [q - q/\nu_i{}^j]\right\} \leqslant (2q-1)(N-1). \qquad (**)$$

Dividing this inequality by q and letting $q \to \infty$, one gets the *first area inequality* [32],

$$\text{Area}\,(\Omega/G) \leqslant 4\pi(N-1),$$

which contains and refines the finiteness theorem. (It is sharp; equality holds for Schottky groups.)

Since for any Fuchsian group Γ of finite type Area $(U/\Gamma) \geqslant \pi/21$, we see that

$$K \leqslant 84(N-1).$$

Using (**) directly, Ahlfors [13] improved this to

$$K \leqslant 18(N-1).$$

Maskit conjectured that

$$K \leqslant 2(N-1);$$

he also showed that this would be a sharp inequality. One can prove this inequality from the area inequality if G has no elliptic elements, since in this case Area $(U/\Gamma_j) \geqslant 2\pi$. On the other hand, A. Marden [92] showed, using 3-dimensional topology, that if G has only loxodromic elements, then

$$K \leqslant N/2.$$

This result is, at present, inaccessible to cohomological arguments.

We consider next homomorphisms of G into the group of all Möbius transformations. We call such a homomorphism *allowable* if it takes parabolic elements into parabolic ones. Two such homomorphisms will be called *equivalent* if they differ by a conjugation by a fixed Möbius transformation. Since G is finitely generated, the set Def (G) of equivalence classes of allowable homomorphisms of G has the structure of an affine algebraic variety in some $\mathbb{C}^r$. The *quasiconformal deformation space*

$\mathrm{Def}_{qc}(G) \subset \mathrm{Def}(G)$ consists of (equivalence classes of) quasiconformal deformation, that is, conjugations by quasiconformal automorphisms of $\hat{\mathbb{C}}$.

We say that G is *stable* (or, *conditionally stable*) if every allowable homomorphism of G (or, every quasiconformal deformation of G) sufficiently near the identity is a conjugation by a quasiconformal automorphism of C with dilatation arbitrary close to 1. (This is a slight variation of a previous definition.) We shall see later that there are unstable groups.

CONJECTURE. *A group obtained from stable groups by the use of the Klein–Maskit combination theorems is stable.*

CONJECTURE. *Every (finitely generated) Kleinian group is conditionally stable.*

The method of proving the Lemma in §4, in conjunction with Theorem III in §1, leads to the following result [33].

XXVIII. *If, at the point id,* $\mathrm{Def}(G)$ *is locally irreducible and of dimension* $\dim A_2(\Omega, G)$ *(or is contained in a variety with these properties), then G is stable.*

Recently, Gardiner and Kra proved a different stability criterion:

XXIX. *If* $\dim A_2(\Omega, G) = \dim H_{\text{parabolic}}(G, 2)$, *G is stable.*

Example. All quasi-Fuchsian groups are stable.

XXX. *Let* $(\Gamma_1, \ldots, \Gamma_K)$ *be a Fuchsian equivalent of G. Assume that G and all its quasiconformal images are stable or, at least, conditionally stable. Then the quasiconformal deformation space* $\mathrm{Def}_{qc}(G)$ *is a complex manifold and (isomorphic to) the quotient of* $\hat{T} = T(\Gamma_1) \times \ldots \times T(\Gamma_K)$ *by a freely acting subgroup* $\hat{M}$ *of*

$$\mathrm{Mod}(\Gamma_1) \times \ldots \times \mathrm{Mod}(\Gamma_K).$$

[34].

(Maskit [88] proved, without any stability hypotheses, that there always is a holomorphic bijection $\hat{T}/\hat{M} \rightarrow \mathrm{Def}_{qc}(G)$.)

XXXI. *Under the hypothesis of Theorem XXX, assume that all components of* Ω *are simply connected. Then* $\mathrm{Def}_{qc}(G)$ *is (isomorphic to)* $T(\Gamma_1) \times \ldots \times T(\Gamma_K)$.

The proof [34] depends on the theory of extremal quasiconformal mappings.

6. *Finitely generated function groups*

We assume now that G (a finitely generated non-elementary Kleinian group) is a function group, that is, that there is a component Δ_0 of Ω invariant under G. (The classical theory was concerned only with such groups.) If $\Delta_0 \neq \Omega$, then all other components of Ω are *simply connected* (Accola [3]).

Let Ω_0 be any G-invariant union of components of Ω; we denote by $A_q(\Omega_0, G)$ the subspace of $A_q(\Omega, G)$ consisting of those ψ for which $\psi \mid \Omega - \Omega_0 = 0$. If

$\psi \in A_q(\Omega_0, G)$, let F be a potential of ψ. Then $F \mid \Omega - \Omega_0$ is holomorphic. Define $\hat{\psi}$ by the conditions:

$$\hat{\psi} \mid \Omega_0 = 0, \quad \hat{\psi} \mid \Omega - \Omega_0 = d^{2q-1}(F \mid \Omega - \Omega_0)/dz^{2q-1}.$$

It can be shown that $\psi \to \hat{\psi}$ is a canonical antilinear mapping

$$A_q(\Omega_0, G) \to A_q(\Omega - \Omega_0, G).$$

It is easy to write down an explicit formula for this mapping: except for a constant factor, it reads just like the integral defining L in 3, §3, the integration being extended over Ω_0. All this does not require G to be a function group.

XXXII. *If Δ_0 is an invariant component of G, the canonical mapping*

$$A_q(\Omega - \Delta_0, \Delta_0) \to A_q(\Delta_0, G),$$

is injective. [**32**].

Hence

$$\dim A_q(\Omega, G) \leqslant 2 \dim A_q(\Delta_0, G),$$

for all $q \geqslant 2$. Dividing by q and letting $q \to \infty$, we obtain, as in §5, the *second area inequality* [**32**].

$$\text{Area } (\Omega/G) \leqslant 2 \text{ Area } (\Delta_0/G).$$

This shows, in particular, that G has at most 2 invariant components.

XXXIII. *If G has two invariant components, G is quasi-Fuchsian.*

This deep result is due to Maskit [**86**]. (The condition that G be finitely generated is essential.)

A finitely generated function group with a simply connected invariant component will be called a *b-group,* provided it is not quasi-Fuchsian.

A yet unpublished result of Maskit's reads:

XXXIV. *If G is such that the equality sign holds in the second area inequality then G is either quasi-Fuchsian or a b-group, and G can be constructed from elementary and Fuchsian groups using the combination theorems.*

We call a b-group *non-degenerate* if equality holds in the second area inequality, *partially degenerate* if

$$1 < \text{Area}(\Omega/G)/\text{Area } (\Delta_0/G) < 2,$$

and (*totally*) *degenerate* if $\Omega_0 = \Delta_0$.

The theory of b-groups is closely connected with Teichmüller spaces. Let Γ be a Fuchsian group of finite type, $T(\Gamma)$ its Teichmüller space and $\partial T(\Gamma)$ the boundary of $T(\Gamma)$ in $B_2(L, \Gamma)$.

XXXV. *If $\phi \in \partial T(\Gamma)$, then $\Gamma^\phi = W_\phi \Gamma (W_\phi)^{-1}$ is a b-group* [**33**].

Concerning the definition of W_ϕ, *cf.* 3, §1.

A $\phi \in \partial T(\Gamma)$ is called a *cusp* if there is a hyperbolic element $\gamma \in \Gamma$ such that $W_\phi \circ \gamma \circ W_\phi{}^{-1}$ is parabolic.

XXXVI. *If* $0 < \dim T(\Gamma) < \infty$, *then* $\partial T(\Gamma)$ *contains cusps, but the set of cusps has positive (real) codimension in* $\partial T(\Gamma)$ [**33**].

In proving this result, one shows that a cusp can be obtained by " contracting a cycle on U/Γ into a puncture".

CONJECTURE. *There is a fundamental region* ω *for* Mod (Γ) *in* $T(\Gamma)$ *such that the intersection* Σ *of the closure of* ω *with* $\partial T(\Gamma)$ *consists only of cusps.*

An unpublished result of Mayer and Mumford on the compactification of the space of moduli suggests that one may even require that Γ^{ϕ} be non-degenerate for $\phi \in \Sigma$.

Theorem XXXI of §3 suggests the following

CONJECTURE. *Let* $\phi \in \partial T(\Gamma)$. *Then there is a complex manifold* M, *isomorphic to a product of Teichmüller spaces, with* $\phi \in M \subset \partial T(\Gamma)$ *and* Γ^{ψ} *a quasiconformal deformation of* Γ^{ϕ} *for all* $\psi \in M$.

The next conjecture (as well as Theorem XXXIII above), would follow at once from the more general conjecture about Schwarzian derivatives of schlicht functions stated in 2, §6.

CONJECTURE. *Every b-group is a* Γ^{ϕ} *for some Fuchsian group* Γ *and some* $\phi \in \partial T(\Gamma)$.

There is strong secondary evidence for this conjecture. Maskit [**86**] determined all possible signatures for *b*-groups; it turns out that these are precisely the signatures one would expect if the conjecture were true. Maskit also constructed, without reference to Teichmüller spaces, all possible non-degenerate *b*-groups, and was able to construct partially degenerate *b*-groups *assuming* the existence of degenerate ones. But at this point, one must fall back on Teichmüller space theory.

XXXVII. *If* $\phi \in \partial T(\Gamma)$ *is not a cusp,* Γ^{ϕ} *is unstable and degenerate* [**33**].

Theorems XXXVI and XXXVII show that there are many degenerate groups. One can show that, for any compact Riemann surface of genus >1, there are uncountably many non-conjugate purely loxodromic degenerate groups G such that Ω/G is conformally equivalent to the given surface.

PROBLEM. *Is there an algorithm for constructing degenerate groups?*

References

1. W. Abikoff, " Some remarks on Kleinian groups", *Advances in the theory of Riemann surfaces, Ann. of Math. Studies*, 66 (1971), 1-5.
2. ———, " Residual limit sets of Kleinian groups ", to appear.
3. R. D. M. Accola, " Invariant domains for Kleinian groups ", *Amer. J. Math.*, 88 (1966) 329–336.
4. L. V. Ahlfors, " On quasiconformal mappings ", *J. Analyse Math.*, 3 (1953–54), 1–58.
5. ———, " The complex analytic structure of the space of closed Riemann surfaces ", *Analytic Functions* (Princeton University Press, 1960), 349–376.
6. ———, Some remarks on Teichmüller's space of Riemann surfaces ", *Ann. of Math.*, 74 (1961), 171–191.
7. ———, " Curvature properties of Teichmüller's space ", *Journal d'Analyse*, 9 (1961), 161–176.
8. ———, " Teichmüller spaces ", *Proc. Internat. Congress Math.*: (1962) (Institute Mittag-Leffler, Djursholm, Sweden, 1963), 3–9.
9. ———, " Quasiconformal reflections ", *Acta. Math.*, 109 (1963), 291–301.
10. ———, " Finitely generated Kleinian groups ", *Amer. J. Math.*, 86 (1964), 413–429; 87 (1965), 759.
11. ———, *Lectures on quasiconformal mappings* (Van Nostrand, Princeton, 1966).
12. ———, " Fundamental polyhedrons and limit point sets of Kleinian groups ", *Proc. Nat. Acad. Sci.*, 55 (1966), 251–254.
13. ———, " Eichler integrals and Bers' area theorem ", *Mich. Math. J.*, 15 (1968), 257–263.
14. ———, " The structure of finitely generated Kleinian groups ", *Acta Math.*, 122 (1969), 1–17.
15. ——— and L. Bers, " Riemann's mapping theorem for variable metrics ", *Ann. of Math.*, 72 (1960), 385–404.
16. ——— and L. Sario, *Riemann Surfaces* (Princeton University Press, Princeton, N.J., 1960).
17. ——— and G. Weill, " A uniqueness theorem for Beltrami equations ", *Proc. Amer. Math. Soc.*, 13 (1962), 975–978.
18. P. Appell and E. Goursat, *Théorie des fonctions algébriques et de leurs integrales*, Vol. 2 (Gauthiers–Villars, 1930).
19. W. L. Baily, Jr., " On moduli of Jacobian varieties ", *Ann. of Math.*, 71 (1960), 303–314.
20. ——— " On the theory of θ-functions, the moduli of abelian varieties, and the moduli of curves ", *Ann. of Math.*, 75 (1962), 342–381.
21. L. Bers, " Quasiconformal mappings and Teichmüller's theorem ", *Analytic Functions* (Princeton University Press, Princeton, 1960), 89–119.
22. ———, " Spaces of Riemann surfaces ", *Proc. Internat. Congress Math.*, (Edinburgh, 1958), 349–61.
23. ———, " Simultaneous uniformization ", *Bull. Amer. Math. Soc.*, 66 (1960), 94–97.
24. " Completeness theorems for Poincaré series in one variable ", *Proc. Internat. Symp. on Linear Spaces* (Jerusalem, 1960), 88–100.
25. ———, " Uniformization by Beltrami equation ", *Comm. Pure Appl. Math.*, 14 (1961), 215–228.
26. ———, " Holomorphic differentials as functions of moduli ", *Bull. Amer. Math. Soc.*, 67 (1961), 206–210.
27. ———, " Correction to Spaces of Riemann surfaces as bounded domains ", *Bull. Amer. Math. Soc.*, 67 (1961), 465–466.
28. ———, " Automorphic forms and general Teichmüller spaces," *Proc. Conf. Complex Analysis* (Minneapolis, 1964) (Springer, 1965), 109–113.
29. ———, *On moduli of Riemann surfaces*, Lectures at Forschunginsittut für Mathematik, Eidgenössische Technische Hochschule (Zurich, Summer 1964), mimeographed.
30. ———, " Automorphic forms and Poincaré series for infinitely generated Fuchsian groups ", *Amer. J. Math.*, 87 (1965), 169–214.

31. ———, " A non-standard integral equation with applications to quasiconformal mappings ", *Acta Math.*, 116 (1966), 113–134.
32. ———, " Inequalities for finitely generated Kleinian groups ", *J. Analyse Math.*, 18 (1967), 23–41.
33. ———, " On boundaries of Teichmüller spaces and on Kleinian groups I ", *Ann. of Math.*, 91 (1970), 570–600.
34. ———, " Spaces of Kleinian groups ", *Several Complex Variables, I* (Maryland, 1970) Lecture Notes on Mathematics 155 (Springer-Verlag, 1970), 9–34.
35. " Extremal quasiconformal mappings ", *Advances in the Theory of Riemann Surfaces, Ann. of Math. Studies*, 66, (Princeton, 1971), 27–52.
36. ———, " Eichler integrals with singularities ", *Acta. Math.*, 127 (1971), 11–22.
37. ———, " Fiber spaces over Teichmüller spaces ", to appear.
38. ——— and L. Ehrenpreis, " Holomorphic convexity of Teichmüller spaces ", *Bull. Amer. Math. Soc.*, 70 (1964), 761–764.
39. ——— and L. Greenberg, " Isomorphisms between Teichmüller spaces ", *Advances in the theory of Riemann Surfaces, Ann. of Math. Studies*, 66 (1971), 53–79.
40. A. Beurling and L. V. Ahlfors, " The boundary correspondence for quasiconformal mappings ", *Acta. Math.*", 96 (1956), 125–142.
41. J. Birman, " Automorphisms of the fundamental group of a closed orientable 2-manifold ", *Proc. Amer. Math. Soc.*, 21 (1969), 351–354.
42. H. Cartan, " Quotient d'un espace analytique par un groupe d'automorphismes," *Algebraic Geometry and Algebraic Topology, A Symposium in Honor of S. Lefschetz* (Princeton University Press, 1957), 90–102.
43. V. Chuckrow, " On Schottky groups with applications to Kleinian groups ", *Ann. of Math.*, 88 (1968), 47–61.
44. H. D. Coldewey and H. Zieschang, " Der Raum der markierten Riemannschen Flächen ", appendix to H. Zieschang, E. Vogt, H. D. Coldewey: " *Flächen und ebene Diskontinuierliche Gruppen* ", Lecture Notes in Mathematics, 122 (Springer Verlag, 1970).
45. C. J. Earle, " The contractability of certain Teichmüller spaces ", *Bull. Amer. Math. Soc.* 73 (1967), 434–437.
46. ———, " Reduced Teichmüller spaces ", *Trans. Amer. Math. Soc.*, 126 (1967), 54–63.
47. ———, " The Teichmüller spaces for arbitrary Fuchsian groups ", *Bull. Amer. Math. Soc.*, 70 (1964), 699–701.
48. ———, " A reproducing formula for integrable automorphic forms ", *Amer. J. Math.*, 88 (1966), 867–870.
49. ———, " Some remarks on Poincaré series ", *Compos. Math.*, 21 (1969), 167–176.
50. ———, " On holomorphic cross-sections in Teichmüller spaces ", *Duke Math. J.*, 33 (1969), 409–416.
51. ——— and J., Eels, Jr., " On the differential geometry of Teichmüller spaces ", *J. Analyse Math.*, 19 (1967), 35–52.
52. ———, ———, " A fibre bundle description of Teichmüller theory ", *Journal Diff. Geom.*, 3 (1969), 19–43.
53. M. Eichler, " Eine Verallgemeinerung der Abelschen Integrale ", *Math. Z.*, 67 (1957), 267–298.
54. W. Fenchel and J. Nielsen, *Discontinuous Groups of Non-euclidean Motions*, unpublished manuscript.
55. L. R. Ford, *Automorphic Functions*, 2nd ed. (Chelsea, New York, 1951).
56. R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Funktionen*, 1 (B. G. Teubner, 1926).
57. ———, ———, *Vorlesungen über die Theorie der automorphen Funktionen*, 2 (B. G. Teubner, 1926).

58. F. Gardiner, " An analysis of the group operation in universal Teichmüller space ", *Trans. Math. Soc.*, 132 (1968), 471–486.

59. J. P. Gilman, *Relative modular groups in Teichmüller space*, Ph.D. Thesis (Columbia University, 1971).

60. L. Greenberg, " Fundamental polyhedra for Kleinian groups ", *Ann. of Math.*, 84 (1966), 433–41.

61. P. A. Griffiths, " Complex analytic properties of certain Zariski open sets on algebraic varieties ", *Ann. of Math.*, 94 (1971), 21-51.

62. A. Grothendieck, " Techniques de construction en géométrie analytique ", *Séminaire H. Cartan* (Paris, 1960–61), Exp. 7–8

63. R. S. Hamilton, " Extremal quasiconformal mappings with prescribed boundary values ", *Trans. Amer. Math. Soc.*, 138 (1969), 399–406.

64. W. J. Harvey, " Cyclic groups of automorphisms of a compact Riemann surface ", *Quart. J. Math.*, 17 (1966), 86–97.

65. L. Keen, " Intrinsic moduli on Riemann surfaces ", *Ann. of Math.*, 84 (1966), 404–420.

66. ———, " On Fricke moduli ", *Advances in the theory of Riemann Surfaces, Ann. of Math. Studies*, 66 (1971), 205–224.

67. F. Klein, *Gesammelte Mathematische Abhandlungen*, 3 (Springer, 1923).

68. S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings* (Marcel Dekker).

69. P. Koebe, " Über die Uniformisierung beliebiger analytischer Kurven ", *Göttinger Nachr* (1907), 191–210, 633–669.

70. ———, " Über die Uniformisierung algebraischer Kurven ", *Math. Annal.*, 67 (1909), 145–224; 69 (1910), 1–81; 72 (1912), 437–516; 75 (1914), 42–129.

71. ———, " Über die Uniformisierung beliebiger analytischer Kurven ", *J. reine angew. Math.*, 138 (1910), 192–253, 139 (1911), 251–292.

72. ———, " Methoden der konformen Abbildung und Uniformisierung ", *Intern. Cong.-Math.*, Bologna 3 (1928), 195–203.

73. I. Kra, " On Teichmüller spaces for finitely generated Fuchsian groups ", *Amer. J. Math.*, 91 (1969), 67–74.

74. ———, " On cohomology of Kleinian groups ", *Ann. of Math.*, 89 (1969), 533–556.

75. ———, " On cohomology of Kleinian groups: II ", *Ann. of Math.*, 90 (1969), 576–590.

76. ———, " Eichler cohomology and the structure of finitely generated Kleinian groups ", *Advances in the Theory of Riemann Surfaces, Ann. of Math. Studies*, 66 (1971), 225–623.

77. ———, " Cohomology of Kleinian groups: III ", *Acta. Math.*, to appear.

78. S. Kravetz, " On the geometry of Teichmüller spaces and the structure of their modular groups ", *Ann. Acad. Sci. Fenn.*, 278 (1959), 1–35.

79. S. L. Krushkal, " On Teichmüller's theorem on extremal quasi-conformal mappings ", *Sibirsk Mat. Žurnal*, 8 (1967), 313–332. (Text in Russian).

80. J. Lehner, " *Discontinuous groups and automorphic functions* ", *Amer. Math. Soc.*, (1964.)

81. O. Lehto and K. I. Virtanen, *Quasikonforme Abbildungen*, (Springer-Verlag, Berlin, 1965).

82. M. R. Linch, *On Metrics in Teichmüller Spaces*, Ph.D. Thesis (Columbia University, 1971).

83. C. Machlachlan, " Moduli space is simply connected ", *Proc. Amer. Math. Soc.*, to appear.

84. B. Maskit, " On Klein's combination theorem I ", *Trans. Amer. Math. Soc.*, 120 (1965), 499–509

85. ———, " On Klein's combination theorem II ", *Trans. Amer. Math. Soc.*, 131 (1968), 32–39.

86. ———, " On boundaries of Teichmüller spaces and on Kleinian groups II ", *Ann. of Math.*, 91 (1970), 608–638.

87. ———, " On Klein's combination theorem III ", *Advances in the theory of Riemann surfaces, Ann. of Math. Studies*, 66, (1971), 297–316.

88. ———, " Selfmaps on Kleinian groups ", *Amer. J. Math.*, 93 (1971), 840-856.

89. ———, " On Poincaré's theorem for fundamental polygons ", *Advances in Math.*, 7, (1971) 219-230.

90. A. Marden, " On homotopic mappings of Riemann surfaces ", *Ann. of Math.*, 90 (1969), 1–8.
91. ———, " An inequality for Kleinian groups"; *Advances in the theory of Riemann surfaces Ann. of Math. Studies*, 66 (1971), 295–296.
92. C. B. Morrey, " On the solutions of quasi-linear elliptic partial differential equations ", *Trans. Amer. Math. Soc.*, 43 (1938), 43 (1938), 126–166.
93. P. J. Myreberg, " Die Kapazität der singulären Menge der linearen Gruppen ", *Ann. Acad. Sci. Fenn.*, 10 (1941), 1–19.
94. Z. Nehari, " Schwarzian derivatives and schlicht functions ", *Bull. Amer. Math. Soc.*, 55 (1949), 545–551.
95. R. Nevanlinna, *Uniformisierung* (Springer, Berlin, 1953).
96. B. A. Olsen, " On higher order Weierstrass points ", *Ann. of Math.*, 95 (1972), 357–364.
97. H. Petersson, " Über eine Metrisierung der automorphen Formen und die Theorie der Poincaré-schen Reihen ", *Math. Ann.*, 117 (1940), 453–537.
98. ———, " Über Weierstrass Punkte und die expliziten Darstellungen der automorphen Formen von reeller Dimension ", *Math. Z.*, 52 (1949), 32–59.
99. A. Pfluger, *Theorie der Riemannschen Flächen* (Springer-Verlag, Berlin, 1957).
100. H. Poincaré, *Oevres*, Vol. 2 (Gauthiers-Villar, 1916).
101. ———, " Sur l'uniformisation des fonctions analytiques ", *Acta. Math.* 31 (1907) 1-63.
102. H. E. Rauch, " Weierstrass points, branch points, and the moduli of Riemann surfaces ", *Comm. Pure Appl. Math.*, 12, (1959), 543–560.
103. ———, " A transcendential view of the space of algebraic Riemann surfaces ", *Bull. Amer. Math. Soc.*, 71 (1965), 1–39.
104. H. L. Royden, " Automorphisms and isometries of Teichmüller spaces ", *Advances in the Theory of Riemann Surfaces* (Princeton University Press, Princeton, N.J., 1971), 369–383.
105. C. L. Siegel, *Topics in Complex Function Theory*, Vol. 2 (Wiley-Interscience, 1971).
106. G. Springer, *Introduction to Riemann Surfaces* (Addison-Wesley, 1957).
107. O. Teichmüller, " Extremale quasikonforme Abbildungen und quadratische Differentiale ", *Abh. Preuss. Akad Wiss. Math.-Nat. Kl.*, 22, (1939).
108. ———, " Bestimmung der extremalen quasikonformen Abbildung bei geschlossenen orientierten Riemannschen Flächen ", *Preussische Akad. Ber.*, 4, (1943).
109. ———, " Veräänderliche Riemannsche Flächen ", *Deutsche Math.*, 9 (1944), 344–359.
110. A. Weil, " Sur les modules des surfaces de Riemann ", *Séminaire Bourbaki* (1958).
111. H. Weyl, *Die Idee der Riemannschen Fläche* (Teubner, Leipzig, 1955).

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