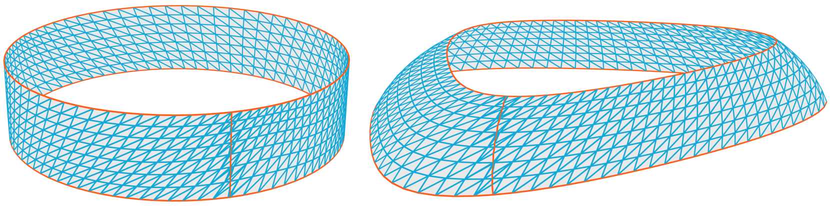


Figure 1: A discrete periodic map from a cylinder to a triangulated surface. On the cylinder all edges of the triangulation are geodesic arcs. If the cylinder is cut at the vertical orange path, then it can be unrolled to the plane creating a rectangular domain.

{fig:discrete_map}

1 Periodic conformal parametrization

{sec:conformal-parameterization}

In this section we describe our algorithm for the creation of periodic conformal maps for cylindrical meshes/surfaces, i.e., surfaces with the same topology as a cylinder. First we will review the discrete conformal maps of [Springborn et al. 2008]. Then we show how it can be generalized to yield periodic maps to cylinders or cones.

A smooth *conformal map* between two surfaces is a map that preserves angles. Intuitively, one can think of a conformal map as a map that preserves the shape but not the scale of small figures. For conformal surface parameterization, one looks for conformal maps from the plane to a surface and vice-versa. These can be used to map different patterns onto surfaces in a way that only isotropic stretch/uniform scaling is applied to the pattern elements. The method described in [Springborn et al. 2008] is a triangle mesh based discretization of conformal maps. For a given triangle mesh and prescribed angle sums at the vertices, i.e., for each vertex v of the surface – interior or boundary – we may prescribe an angle θ_v that corresponds to

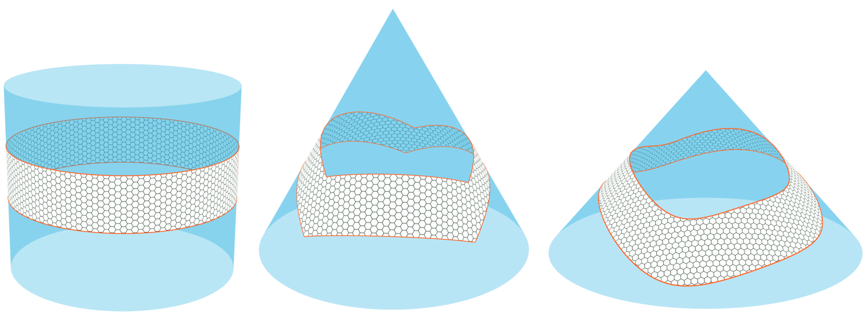


Figure 2: Periodic domains of parameterization of the surfaces shown in Figure **??**. Left: Map to a cylinder with geodesic boundary curves. Middle: Map to a cone of revolution with hex-pattern-adapted angle. The domain is a polygon with quantized angles. Right: Isometric boundary on a cone with hex-pattern-adapted angle. Panalizations created with the help of these maps are shown in Figure 3.

{fig:cone_maps_teaser}

the angle sum of adjacent triangles in the target mesh. Starting from an input mesh and target angles θ_v the method calculates new edge lengths for the triangles of the target mesh such that the angle sums at the target vertices are as prescribed. This goal is achieved by minimizing a convex functional. The prescribed angles have to satisfy a Gauss-Bonnet type condition, i.e., the angles at interior vertices have to match the angles at the boundary vertices. We will state the condition for the special cases treated later in the article, see Equation 1.

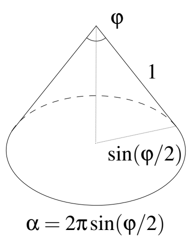
For the parameterization problem, we want to construct a map from a surface to the plane. To get a planar target mesh, the target angles have to be set to 2π for all interior vertices, i.e., the angles of the triangles adjacent to every interior vertex sum up to 2π. Thus the computed target triangles can be laid out in the plane. At the boundaries there is still a certain degree of freedom, which allows to map the surface to different shapes, e.g., a rectangle or a more general polygon with prescribed angles. An alternative choice of boundary conditions yields a target mesh whose boundary edges have the same lengths as the original mesh. Then the control over the boundary angles is no longer possible.

This method for the parameterization of triangle meshes can be generalized to triangle meshes with cylinder topology, see Figure 1. Instead of constructing a discrete conformal map from the surface to the plane, we construct a map to a cylinder or cone, whose image is isometric to a polygonal region in the plane, see Figure 2. This works with an approach very similar to the previous one. We start with the definition of a periodic parameterization.

Definition 1 *Let $M = (V,E,F)$ be a cylindrical triangle mesh with vertices V, edges E, and triangles F. Let $D \subset C$ be a region on a cone/cylinder of revolution. A continuous bijection $\Phi : D \to M$ is called a* discrete periodic parameterization*. D is called the* domain of parameterization*.*

In the latter we always assume that the preimages of edges of M are geodesic arcs on the cone/cylinder C. For panelization of periodic surfaces we need to make sure, that different patterns match around the cone or cylinder. This yields certain restrictions on the cone that serves as domain of parameterization.

Definition 2 *Let C be a cone with aperture φ and $\Phi : C \supset D \to M$ be a discrete periodic parameterization of a triangle mesh M with domain D. The map Φ is called* triangle adapted *if the cone angle α is a multiple of $\frac{\pi}{3}$ and* quad-pattern adapted *if it is a multiple of $\frac{\pi}{2}$.*



This definition ensures that either a quad-, triangle-, or hex-pattern fits seamlessly onto the surface after parameterization.

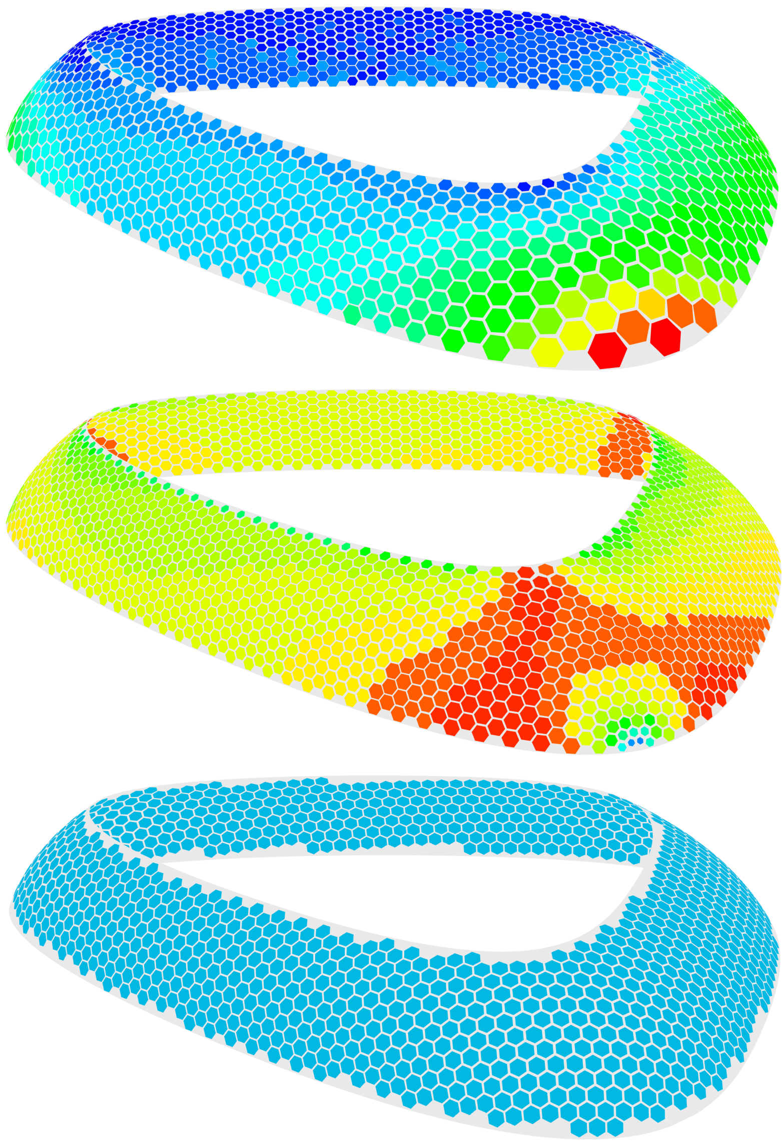


Figure 3: Quantized periodic hexagonal panelizations. Boundary conditions affect the amount of stretch in the interior of the surface. Top: Hexagonal pattern aligns with the boundary, a strong condition that produces large deviation of edge lengths. Middle: Map to a pattern-adapted polygon on a cone of revolution. The pattern contains exceptional points at the boundary. The stretch is minimized while at the same time the pattern alignes with the boundary. Bottom: Conformal map with the least stretch in the interior, pattern can be optimized to consist of congruent hexagons alone. In all images, panels with the same color are congruent. The corresponding domains of parameterization are shown in Figure 2.

{fig:hex_example}

1.1 Periodic boundary conditions

{sec:boundary}

If we want to construct periodic conformal maps we are allowed to specify angle sums θ_v at boundary vertices. The condition for the sums of boundary angles differs from the plane case in the following way: The curvature at a boundary vertex v is given by $\kappa_v = \pi - \theta_v$, where θ_v is the angle sum of the adjacent triangles in the target mesh. For the two boundary loops $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ we have:

{eq:theta}

$$\sum_{i=1}^{n} \kappa_{v_i} + \sum_{j=1}^{m} \kappa_{w_j} = 0. \tag{1}$$

This condition makes sure that the two boundary curves "bend" the same amount and can hence be wrapped around a cone. We will now show how boundary conditions can be used to construct periodic patterns on the studied models. We start with a discrete conformal map of the doubly-curved model from Figure **??** to a standard cylinder.

Straight cylinder. The simplest way to generate a map to the cylinder is to set the target angles for all boundary vertices to π. Hence the curvatures at the boundary vertices are zero and the two boundary loops are mapped to "straight" curves. In this case both angle sums of Equation 1 vanish and the target mesh can be wrapped around a cylinder, see Figure 2, left. The new edge lengths computed with the variational principle correspond to the lengths on a cylinder. This cylinder can be unrolled in the plane preserving angles and lengths. So the two boundary polygons are mapped to straight lines in the plane. These two straight lines have to be parallel and of equal lengths. If the lengths of the boundary curves in the original model differ a lot, then a map to a cylinder induces a lot of conformal stretch. This stretch can be reduced by specifying special boundary conditions for a parameterization on a cone of revolution.

Cone of revolution. As long as Equation (1) is satisfied we obtain a map to a general cone of revolution. In our case, we require that the periodic parameterization is adapted to the target pattern. This means that the two sums of Equation (1) need to be (the same) multiples of $\frac{\pi}{3}$ (triangle or hex) or multiples of $\frac{\pi}{2}$ (quad). We present two methods to achieve this requirement: a uniform distribution and a concentration of curvature.

If the boundary of the mesh should to be aligned with the pattern, then boundary angles need to be quantized, i.e., multiples of $\frac{\pi}{3}$ or $\frac{\pi}{2}$ need to be chosen as target angles. In Figure 2, middle, three vertices of the top and bottom boundary curve were manually assigned to $\frac{4}{3}\pi$ and $\frac{2}{3}\pi$, respectively. All other boundary angles are set to π, i.e., straight. Such a map can be used as a starting point to obtain a tesselation with equal hexagons as described in Section **??**.

It is known that a confomal map that does not change the length of the boundary exhibits the least stretch in the interior of the surface. To obtain such a parameterization we first construct a periodic conformal mapping onto an arbitrary cone such that the lengths of the boundary edges are not changed. The resulting angle sums at boundary vertices of the target mesh determine the cone angle of the map. The cone angle of a pattern adapted periodic parameterization is the closest multiple of the desired quantization. We distribute the difference to the closest quantized angle uniformly to the individual boundary vertices and recompute the map with these angle conditions. The obtained map is periodic and exhibits the lowest stretch of all periodic conformal maps (see Figure 2, right).

Design and structural opportunities. It is also possible to use special boundary conditions to support structural purposes or design requirements. If one aims for a panelization with boundary aligned patterns, then the target boundary angles must be quantized.

To include entrances in a facade it is possible to incorporate special boundary conditions. An example with special boundary vertices with domain angles $\frac{4}{3}\pi$ and $\frac{\pi}{3}$ is shown in Figure 4, left. In the remeshed surface, the lower boundary curve bends inside at the vertices with angle $\frac{4}{3}\pi$ around the vertex with angle $\frac{\pi}{3}$. This incorporates a natural entrance into the facade.

Another effect of such angle conditions is a densification of the pattern at the vertices with small angles. Such a concentration of elements can be used to enforce structural properties of a geometry. An example of a diagrid generated with such boundary conditions is shown in Figure 4, right.

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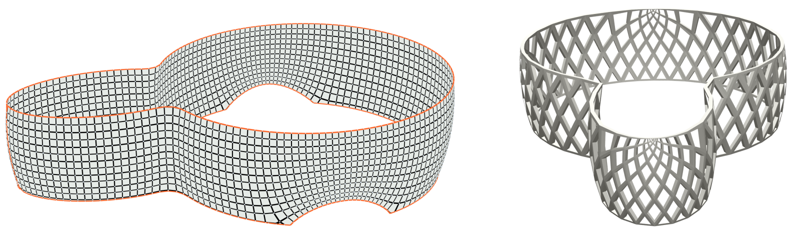

Figure 4: A periodic conformal map onto a cylinder with special vertices creates the opportunity to incorporate entrances (left) or concentration of support structure (right).

{fig:entrance}