

Chapter 9

Fourier Series Methods

Project 9.2

Computer Algebra Calculation of Fourier Coefficients

A computer algebra system can greatly ease the burden of calculation of the Fourier coefficients of a given function $f(t)$. In the case of a function defined "piecewise," we must take care to "split" the integral according to the different intervals of definition of the function. In the paragraphs that follow we illustrate the use of *Maple*, *Mathematica*, and *MATLAB* in deriving the Fourier series

$$f(t) = \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin nt}{n} \quad (1)$$

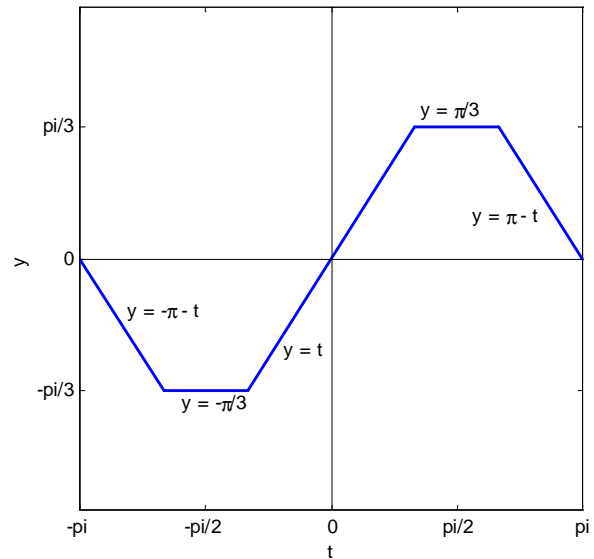
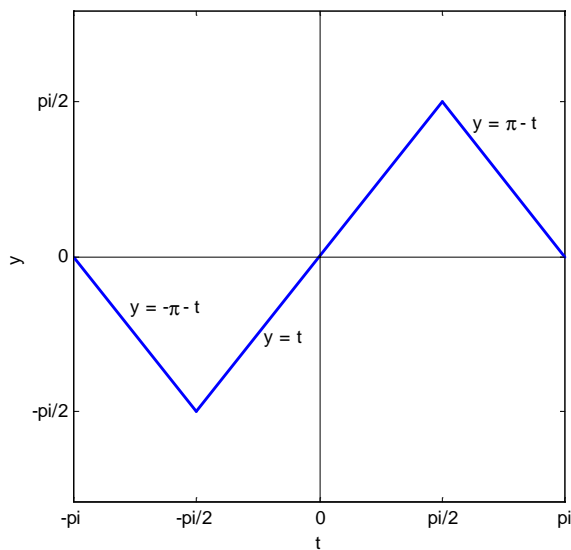
of the period 2π square wave function defined on $(-\pi, \pi)$ by

$$f(t) = \begin{cases} -1 & \text{if } -\pi < t < 0, \\ +1 & \text{if } 0 < t < \pi. \end{cases} \quad (2)$$

In this case the function is defined by different formulas on two different intervals, so each Fourier coefficient integral from $-\pi$ to π must be calculated as the sum of two integrals:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^0 (-1) \cos nt \, dt + \frac{1}{\pi} \int_0^{\pi} (+1) \cos nt \, dt, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^0 (-1) \sin nt \, dt + \frac{1}{\pi} \int_0^{\pi} (+1) \sin nt \, dt. \end{aligned} \quad (3)$$

To practice the symbolic derivation of Fourier series in this manner, you can begin by verifying the Fourier series calculated manually in Examples 1 and 2 of Section 9.2 in the text. Then Problems 1 through 21 there are fair game. Finally, the period 2π triangular wave and trapezoidal wave functions illustrated in the figures at the top of the next page have especially interesting Fourier series that we invite you to discover for yourself.



Using Maple

We can define the cosine coefficients in (3) as functions of n by

```
a := n -> (1/Pi)*(int(-cos(n*t), t=-Pi..0) +
               int(+cos(n*t), t=0..Pi)):
```

```
a(n);
```

0

Of course, our odd function has no cosine terms in its Fourier series. The sine coefficients are defined by

```
b := n -> (1/Pi)*(int(-sin(n*t), t=-Pi..0) +
               int(+sin(n*t), t= 0..Pi)):
```

```
b(n);
```

$$-2 \frac{-1 + \cos(\pi n)}{\pi n}$$

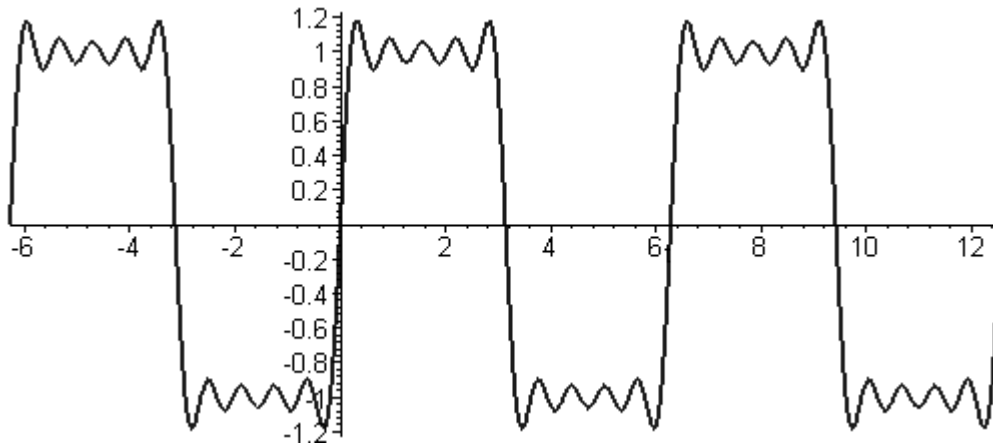
Then a typical partial sum of the Fourier (sine) series is given by

```
fourierSum := sum('b(n)*sin(n*t)', 'n'=1..9);
```

$$fourierSum := 4 \frac{\sin(t)}{\pi} + \frac{4}{3} \frac{\sin(3t)}{\pi} + \frac{4}{5} \frac{\sin(5t)}{\pi} + \frac{4}{7} \frac{\sin(7t)}{\pi} + \frac{4}{9} \frac{\sin(9t)}{\pi}$$

and we can proceed to plot its graph.

```
plot(fourierSum, t=-2*Pi..4*Pi);
```



Using *Mathematica*

We can define the cosine coefficients in (3) as functions of n by

```
a[n_] = (1/Pi)*(Integrate[-Cos[n*t], {t,-Pi, 0}] +
           Integrate[+Cos[n*t], {t, 0, Pi}])
0
```

Of course, our odd function has no cosine terms in its Fourier series. The sine coefficients are defined by

```
b[n_] = (1/Pi)*(Integrate[-Sin[n*t], {t,-Pi, 0}] +
           Integrate[+Sin[n*t], {t, 0, +Pi}]);
b[n] // Simplify

$$-\frac{2(\cos(n\pi)-1)}{n\pi}$$

```

Then a typical partial sum of the Fourier (sine) series is given by

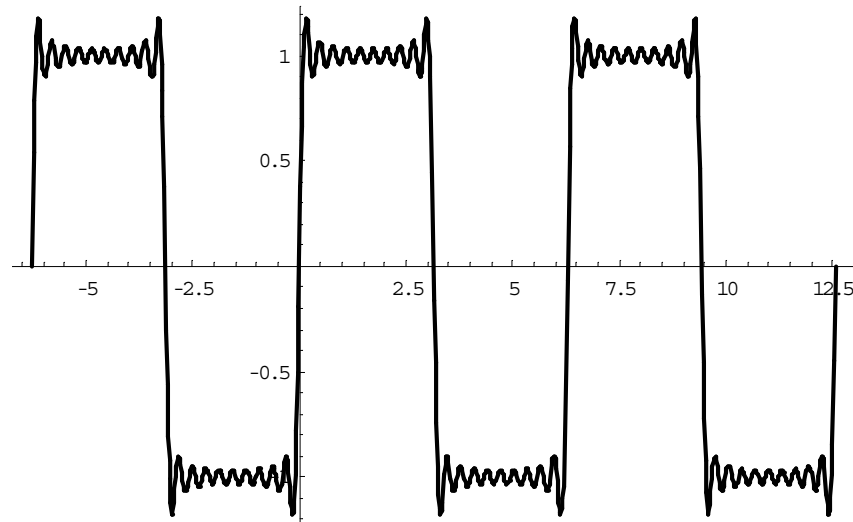
```
fourierSum = Sum[b[n] Sin[n t], {n,1,19}]
```

$$\frac{4 \sin(t)}{\pi} + \frac{4 \sin(3t)}{3\pi} + \frac{4 \sin(5t)}{5\pi} + \frac{4 \sin(7t)}{7\pi} + \frac{4 \sin(9t)}{9\pi} +$$

$$\frac{4 \sin(11t)}{11\pi} + \frac{4 \sin(13t)}{13\pi} + \frac{4 \sin(15t)}{15\pi} + \frac{4 \sin(17t)}{17\pi} + \frac{4 \sin(19t)}{19\pi}$$

and we can proceed to plot its graph.

```
Plot[fourierSum, {t, -2 Pi, 4 Pi}];
```



Using MATLAB

We can define the cosine coefficients in (3) as functions of n by

```
syms n t pi
an = (1/pi)*(int(-cos(n*t), -pi, 0)+int(cos(n*t), 0, pi))
an =
    0
```

Of course, our odd function has no cosine terms in its Fourier series. The sine coefficients are defined by

```
bn = (1/pi)*(int(-sin(n*t), -pi, 0)+int(sin(n*t), 0, pi));
pretty(bn)
```

$$-2 \frac{-1 + \cos(\pi n)}{\pi n}$$

MATLAB does not yet know that n is an integer, but *we* do:

```
bn = subs(bn, '(-1)^(n)', 'cos(pi*n)');
pretty(bn)
```

$$-2 \frac{-1 + (-1)^n}{\pi n}$$

So now it's obvious that

$$b_n = \begin{cases} 4/n\pi & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

We proceed to set up a typical Fourier sum,

```
FourierSum = (4/pi)*sin(t);
for k = 3:2:25
    FourierSum = FourierSum+subs((4/pi)*sin(n*t)/n,k,n);
end
FourierSum
FourierSum =
4/pi*sin(t)+4/3/pi*sin(3*t)+4/5/pi*sin(5*t)+
4/7/pi*sin(7*t)+ 4/9/pi*sin(9*t)+4/11/pi*sin(11*t)+
4/13/pi*sin(13*t)+4/15/pi*sin(15*t)+4/17/pi*sin(17*t)+
4/19/pi*sin(19*t)+4/21/pi*sin(21*t)+4/23/pi*sin(23*t)+
4/25/pi*sin(25*t)
```

and to plot its graph.

```
ezplot(FourierSum, 3.1416*[-2 4])
```

