Assignment 1

CSCI 4113/6101: Design and Analysis of Algorithms II

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Question 1 (Integer linear programming; 10 marks) The TRAVELLING SALESMAN problem (TSP) is a classical NP-hard problem. The input is a complete graph G with vertex set V and a weight function $w : V \times V \to \mathbb{R}$ assigning a weight $w(u, v)$ to every edge (u, v) in G. The goal is to find a cycle in G that visits every vertex exactly once and such that the total weight of the edges in the cycle is minimized. Express this problem as an integer linear program and argue that an optimal solution to the ILP you provide corresponds to an optimal solution to the TSP.

Question 2 (Simplex algorithm; 10 marks) Consider the following LP:

$$\begin{aligned} \text{Minimize } & 8x + 2y - z \\ \text{s.t. } & 4x + y - z \le 6 \\ & 3x - y + z \ge 4 \\ & x - 2y - 2z \le -6 \\ & 4x - y + 2z \le 11 \\ & x, z \ge 0 \\ & y \le 0 \end{aligned}$$

Solve it using the Simplex Algorithm. Specifically,

(a) Convert the LP into one in standard form.

(b) Convert the LP in standard form into a tableau.

(c) Show the steps the Simplex Algorithm takes to find an initial BFS.

(d) Given the BFS determined in part (c), show the steps the Simplex Algorithm takes to find an optimal solution.

(e) State the final optimal solution and its objective function value in the form $x = \ldots, y = \ldots, z = \ldots, 8x + 2y - z = \ldots$.

(f) State the dual of the above LP and the optimal dual solution corresponding to the final tableau in part (d).

(g) Use complementary slackness to verify that, indeed, the primal and dual solutions reported in parts (e) and (f) are optimal.

For parts (c) and (d), you should list the sequence of tableaux the algorithm produces. For every pair of consecutive tableaux, you should list the transformation that was applied to produce the second tableau from the first one.

Question 1.

Assign variables

① Let $n = |V|$.

② Let w_{ij} be the weight from vertex i to vertex j.

③ Let x be the determing variable for a vertex is inclued in the circle or not. If $x_{ij} = 1$, then the edge from vetex i to j is inclued in the circle otherwise, it's not.

Construct the ILP.

Objective function: minimize $\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} x_{ij} w_{ij}$

For the objective function, the goal is to minimize the total weight of edges in the cycle. Therefore, the objective function is a minimization of the product of the edges and their weights in the cycle.

Constraints (s.t.)

① $\sum_{\substack{j=1 \\ i \neq j}}^{n} x_{ij} = 1 \qquad \forall i \in \{1, 2, \ldots, n-1, n\}$

Since the goal is to make sure that the sellsman only travels a city once. This constraints makes sure that the satesman only leaves a city once.

② $\sum_{\substack{i=1 \\ i \neq j}}^{n} x_{ij} = 1 \qquad \forall j \in \{1, 2, \ldots, n-1, n\}$

Again, the salesman is only allowed to travel a city once. Therefore, after we have made sure that the salesman only leaves a city once using the first constraints. We also need this second constraint makes sure that the salesman only arrives a city once.

③ Introducing a new variable $m = |S|$

$$\sum_{\substack{i \in S \\ i \neq j}}^{n} \sum_{j \in S}^{n} x_{ij} \geq 2 \qquad \forall S \subsetneq V,\ m \geq 2$$

or $x_{ij} \leq m-1$

Since each vertex is only allowed to be visited once; We have to make sure there is no sub tour in the route. Therefore, for edges in a cut, the sum of the deteming variable x has to be at least 2, which makes sure that any tour enters the cut and eventually leaves. Also, the no sub tour constraint can be considered as no sub cycle constraint. Therefore, the number of edges is at most the number of vertices in the sub set minus one.

Question 2.

(a) original LP

$$\text{Minimize} \quad 8x + 2y - z$$

$$\begin{aligned} \text{s.t.} \quad & 4x + y - z \le 6 \\ & 3x - y + z \ge 4 \\ & x - 2y - 2z \le -6 \\ & 4x - y + 2z \le 11 \\ & x, z \ge 0 \\ & y \le 0 \end{aligned}$$

canonical form

$$\text{Maxmize} \quad -8x + 2y' + z$$

$$\begin{aligned} \text{s.t.} \quad & 4x - y' - z \le 6 \\ & -3x - y' - z \le -4 \\ & x + 2y' - 2z \le -6 \\ & 4x + y' + 2z \le 11 \\ & x, y', z \ge 0 \end{aligned}$$

Replace y with $-y'$

standard form

$$\text{Maximize} \quad -8x + 2y' + z$$

$$\text{s.t.} \quad \begin{aligned} y_1 &= 6 - 4x + y' + z \\ y_2 &= -4 + 3x + y' + z \\ y_3 &= -6 - x - 2y' + 2z \\ y_4 &= 11 - 4x - y' - 2z \\ &x, y', z \geq 0 \\ &y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

(b)

First, move all the non-basic variables to the same side of the basic variables.

$$\text{Maximize} \quad -8x + 2y' + z$$

$$\text{s.t.} \quad \begin{aligned} y_1 \quad\quad\quad\quad &+4x - y' - z = 6 \\ y_2 \quad\quad\quad &-3x - y' - z = -4 \\ y_3 \quad &+x + 2y' - 2z = -6 \\ y_4 &+4x + y' + 2z = 11 \\ &x, y', z \geq 0 \\ &y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

Unify the basic variables and non-basic variables.

Basic variable: $\{z_1, z_2, z_3, z_4\}$

Non-Basic variable: $\{z_5, z_6, z_7\}$

$$\text{Maximize } -8z_5 + 2z_6 + z_7$$

$$\begin{aligned} \text{s.t. } z_1 \quad\quad & +4z_5 - z_6 - z_7 = 6 \\ z_2 \quad & -3z_5 - z_6 - z_7 = -4 \\ z_3 & +z_5 + 2z_6 - 2z_7 = -6 \\ z_4 & +4z_5 + z_6 + 2z_7 = 11 \end{aligned}$$

$$z_i \geq 0 \quad \forall\, 1 \leq i \leq 7$$

Tableau

	Basic variables				Non-basic variables		
	z_1	z_2	z_3	z_4	z_5	z_6	z_7
6	1				4	-1	-1
-4		1			-3	-1	-1
-6			1		1	2	-2
11				1	4	1	2
0	0	0	0	0	-8	2	1

(c)

① constructing the auxiliary LP

a non-basic variable S is add to the original LP to form a auxiliary LP

Auxiliary LP

	Basic variables				Non-basic variables			
	z_1	z_2	z_3	z_4	z_5	z_6	z_7	S
6	1				4	-1	-1	-1
-4		1			-3	-1	-1	-1
-6			1		1	2	-2	-1
11				1	4	1	2	-1
0								-1

These two coefficients are negative, which cause the basic solution is infeasible.

② Use a speical pivot to obtain Auxiliary LP with a BFS.

Move S out of non-basis to basis, and move -6 out of basis to non-basis.

Pivoted

	Basic variables				Non-basic variables				
	z_1	z_2	s	z_4	z_5	z_6	z_7	z_3	
6	1		-1		4	-1	-1		+1
-4		1	-1		-3	-1	-1		+1
-6			-1		1	2	-2	1	×-1
11			-1	1	4	1	2		+1
0			-1						+1

Auxiliary LP with BFS

	Basic variables				Non-basic variables			
	z_1	z_2	s	z_4	z_5	z_6	z_7	z_3
12	1				3	-3	1	-1
2		1			-4	-3	1	-1
6			1		-1	-2	2	-1
17				1	3	-1	4	-1
6					-1	-2	2	-1

$\frac{12}{4} = \min\left\{\frac{12}{1}, \frac{2}{1}, \frac{6}{2}, \frac{17}{4}\right\} \Rightarrow z_2$ leaves

coefficient is positive

③ Find the optimal solution of the current LP

Swap z_2 and z_7

	Basic variables				Non-basic variables				
	z_1	z_7	S	z_4	z_5	z_6	z_2	z_3	
12	1	1			3	-3		-1	-1
2		1			-4	-3	1	-1	×1
6		2	1		-1	-2		-1	-2
17		4		1	3	-1		-1	-4
6		2			-1	-2		-1	-2

Restore it to standard form

	Basic variables				Non-basic variables			
	z_1	z_7	S	z_4	z_5	z_6	z_2	z_3
10	1				7		-1	
2		1			-4	-3	1	-1
2			1		7	4	-2	1
9				1	19	11	-4	3
2					7	4	-2	1

$\frac{1}{2} = \min\left\{\frac{2}{4}, \frac{9}{11}\right\} \Rightarrow S$ leaves

↑ choose z_6 to basis

Swap S and z_6

	Basic variables				Non-basic variables				
	z_1	z_7	z_6	z_4	z_5	S	z_2	z_3	
10	1				7		-1	0	=
2		1	-3		-4		1	-1	+3
2			4		7	-1	-2	1	$\times\frac{1}{4}$
9			11	1	19		-4	3	-11
2			4		7		-2	1	-4

Restore it to normal form

		Basic variables				Non-basic variables			
		z_1	z_7	z_6	z_4	z_5	S	z_2	z_3
	10	1				7		-1	0
≥ 0	$\frac{7}{2}$		1			$\frac{5}{4}$	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$
	$\frac{1}{2}$			1		$\frac{7}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$
	$\frac{7}{2}$				1	$-\frac{1}{4}$	$\frac{11}{4}$	$\frac{3}{2}$	$\frac{1}{4}$
$=0$	0						-2		

↑ optimal solution of this LP

↑ drop this

Since S is non-basic, we just drop S and restore the original objective function!

	Basic variables				Non-basic variables		
	z_1	z_7	z_6	z_4	z_5	z_2	z_3
10	1				7	-1	0
$\frac{7}{2}$		1			$\frac{5}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$
$\frac{1}{2}$			1		$\frac{7}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$
$\frac{7}{2}$				1	$-\frac{1}{4}$	$\frac{3}{2}$	$\frac{1}{4}$
0		1	2		-8		

As a result, we get the initial BFS of LP

(d) ① Restore the standard form of the original LP.

	Basic variables				Non-basic variables		
	z_1	z_7	z_6	z_4	z_5	z_2	z_3
10	1				7	-1	0
$\frac{7}{2}$		1			$\frac{5}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$
$\frac{1}{2}$			1		$\frac{7}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$
$\frac{7}{2}$				1	$-\frac{1}{4}$	$\frac{3}{2}$	$\frac{1}{4}$
$-\frac{9}{2}$					$-\frac{51}{4}$	$\frac{3}{2}$	$-\frac{1}{4}$

$-$Row2
-2Row3

$\frac{7}{3} = \min\left\{\frac{7/2}{3/2}\right\}$, Z_4 enter non-basis

Z_2 is positive, enter basis

③ use pivoting to find the optimal solution of original LP.

swap z_4 and z_2

	Basic variables				Non-basic variables			
	z_1	z_7	z_6	z_2	z_5	z_4	z_3	
10	1			-1	7		0	$+1$
$\frac{7}{2}$		1		$-\frac{1}{2}$	$\frac{5}{4}$		$-\frac{1}{4}$	$+\frac{1}{2}$
$\frac{1}{2}$			1	$-\frac{1}{2}$	$\frac{7}{4}$		$\frac{1}{4}$	$+\frac{1}{2}$
$\frac{7}{2}$				$\frac{3}{2}$	$-\frac{1}{4}$	1	$\frac{1}{4}$	$\times\frac{2}{3}$
$-\frac{9}{2}$				$\frac{3}{2}$	$-\frac{51}{4}$		$-\frac{1}{4}$	$-\frac{3}{2}$

Restore it to standard form.

	Basic variables				Non-basic variables		
	z_1	z_7	z_6	z_2	z_5	z_4	z_3
$\frac{37}{3}$	1				$\frac{41}{6}$	$\frac{2}{3}$	$\frac{1}{6}$
$\frac{14}{3}$		1			$\frac{7}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$
$\frac{5}{3}$			1		$\frac{5}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{7}{3}$				1	$-\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$
-8					$-\frac{25}{2}$	-1	$-\frac{1}{2}$

Therefore, this tableau represents the optimal solution of the original LP.

All coefficient ≤ 0
$\Rightarrow$ BFS is optimal

(e) Final optimal solution and its objective function values

$z_1 = \frac{37}{3}$, $z_2 = \frac{7}{3}$, $z_3 = 0$, $z_4 = 0$, $z_5 = 0$,

$z_6 = \frac{5}{3}$, $z_7 = \frac{14}{3}$.

$-8z_5 + 2z_6 + z_7 = \frac{5}{3} \times 2 + \frac{14}{3} = \frac{24}{3} = 8$ objective function value

we can also obtain this from the tableau

(f) Primal LP:

Maximize $-8x + 2y' + z$

s.t.

$$\begin{aligned} 4x - y' - z &\le 6 \quad & w_1 \\ -3x - y' - z &\le -4 \quad & w_2 \\ x + 2y' - 2z &\le -6 \quad & w_3 \\ 4x + y' + 2z &\le 11 \quad & w_4 \\ x, y', z &\ge 0 \end{aligned}$$

Dual LP:

Minimize $6w_1 - 4w_2 - 6w_3 + 11w_4$

s.t.

$$\begin{aligned} 4w_1 - 3w_2 + w_3 + 4w_4 &\ge -8 \\ -w_1 - w_2 + 2w_3 + w_4 &\ge 2 \\ -w_1 - w_2 - 2w_3 + 2w_4 &\ge 1 \\ w_1, w_2, w_3, w_4 &\ge 0 \end{aligned}$$

The final tableau of primal LP.

	Basic variables				Non-basic variables		
	z_1	z_7	z_6	z_2	z_5	z_4	z_3
$\frac{31}{3}$	1				$\frac{41}{6}$	$\frac{2}{3}$	$\frac{1}{6}$
$\frac{14}{3}$		1			$\frac{7}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$
$\frac{5}{3}$			1		$\frac{5}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{7}{3}$				1	$-\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$
-8					$-\frac{25}{2}$	-1	$-\frac{1}{2}$

Since $\hat{y}_i = -c_i'' \geq 0 \quad \forall\, 1 \leq i \leq m$,
$\hat{y}_1 = 0$, $\hat{y}_2 = 0$, $\hat{y}_3 = \frac{1}{2}$, $\hat{y}_4 = 1$.

Therefore, $\sum_{i=1}^{m} b_i \hat{y}_i = 6 \cdot 0 - 4 \cdot 0 - 6 \cdot \frac{1}{2} + 11 \cdot 1 = 8$

According to part (e), the feasibe primal solution $\sum_{j=1}^{m} c_j \hat{x}_j = 8$. Hence, $\sum_{j=1}^{m} c_j \hat{x}_j = 8 = \sum_{i=1}^{m} b_i \hat{y}_i$. Using the strong duality of LP, we prove that 8 is the optimal primal and dual solution of LP.

Final: optimal dual solution = 8

objective function variables: $w_1 = 0$, $w_2 = 0$, $w_3 = \frac{1}{2}$, $w_4 = 1$.

(g) primal Complementary slackness check:

Value of the primal variable is 0 ; The corrsponding dual constriaint is tignt ;

$x = 0$ ✓

$y' = \frac{5}{3}$ ✗ ⇒ constriaint? | $-1\cdot 0 - 1\cdot 0 + 2\cdot\frac{1}{2} + 1\cdot 1 = 2$ ✓

$z = \frac{14}{3}$ ✗ ⇒ constriaint? | $-1\cdot 0 - 1\cdot 0 - 2\cdot\frac{1}{2} + 2\cdot 1 = 1$ ✓

overall, satisfied.

Value of the dual variable is 0 ; The corrsponding primal constriaint is tignt ;

$W_1 = 0$ ✓

$W_2 = 0$ ✓

$W_3 = \frac{1}{2}$ ✗ ⇒ constriaint? | $1\cdot 0 + 2\cdot\frac{5}{3} - 2\cdot\frac{14}{3} = \frac{18}{3} = -6$ ✓

$W_4 = 1$ ✗ ⇒ constriaint? | $4\cdot 0 + 1\cdot\frac{5}{3} + 2\cdot\frac{14}{3} = \frac{33}{3} = 11$ ✓

overall, satisfied.

Therefore, both primal complementary slackness and dual complementary slackness are satisfied, which means the primal and dual solutions are ineed optimal.