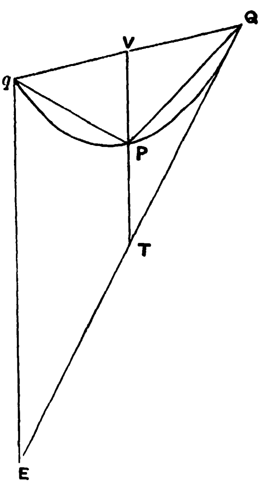
Proposition 17.

It is now manifest that *the area of any segment of a parabola is four-thirds of the triangle which has the same base as the segment and equal height.*

Let Qq be the base of the segment, P its vertex. Then PQq is the inscribed triangle with the same base as the segment and equal height.



Since P is the vertex* of the segment, the diameter through P bisects Qq. Let V be the point of bisection.

Let VP, and qE drawn parallel to it, meet the tangent at Q in T, E respectively.

Then, by parallels,

$$qE = 2VT,$$

and $\qquad PV = PT, \qquad$ [Prop. 2]

so that $\qquad VT = 2PV.$

Hence $\triangle EqQ = 4\triangle PQq.$

But, by Prop. 16, the area of the segment is equal to $\frac{1}{3}\triangle EqQ$.

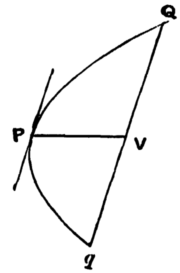
Therefore $\qquad$ (area of segment) $= \frac{4}{3}\triangle PQq.$

DEF. "In segments bounded by a straight line and any curve I call the straight line the **base**, and the **height** the greatest perpendicular drawn from the curve to the base of the segment, and the **vertex** the point from which the greatest perpendicular is drawn."

* It is curious that Archimedes uses the terms *base* and *vertex* of a segment here, but gives the definition of them later (at the end of the proposition). Moreover he assumes the converse of the property proved in Prop. 18.

Proposition 18.

If Qq be the base of a segment of a parabola, and V the middle point of Qq, and if the diameter through V meet the curve in P, then P is the vertex of the segment.

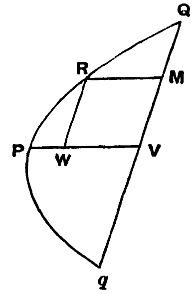


For Qq is parallel to the tangent at P [Prop. 1]. Therefore, of all the perpendiculars which can be drawn from points on the segment to the base Qq, that from P is the greatest. Hence, by the definition, P is the vertex of the segment.

Proposition 19.

If Qq be a chord of a parabola bisected in V by the diameter PV, and if RM be a diameter bisecting QV in M, and RW be the ordinate from R to PV, then

$$PV = \tfrac{4}{3}RM.$$



For, by the property of the parabola,

$$PV : PW = QV^2 : RW^2$$
$$= 4RW^2 : RW^2,$$

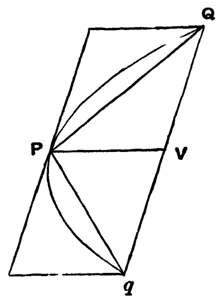
so that $$PV = 4PW,$$

whence $$PV = \tfrac{4}{3}RM.$$

Proposition 20.

If Qq be the base, and P the vertex, of a parabolic segment, then the triangle PQq is greater than half the segment PQq.

For the chord Qq is parallel to the tangent at P, and the triangle PQq is half the parallelogram formed by Qq, the tangent at P, and the diameters through Q, q.



Therefore the triangle PQq is greater than half the segment.

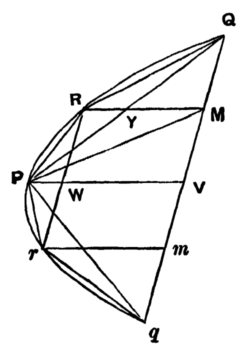
Cor. It follows that *it is possible to inscribe in the segment a polygon such that the segments left over are together less than any assigned area.*

Proposition 21.

If Qq be the base, and P the vertex, of any parabolic segment, and if R be the vertex of the segment cut off by PQ, then

$$\triangle PQq = 8 \triangle PRQ.$$

The diameter through R will bisect the chord PQ, and therefore also QV, where PV is the diameter bisecting Qq. Let the diameter through R bisect PQ in Y and QV in M. Join PM.



By Prop. 19,

$$PV = \tfrac{4}{3} RM.$$

Also $$PV = 2YM.$$

Therefore $$YM = 2RY,$$

and $$\triangle PQM = 2 \triangle PRQ.$$

Hence $$\triangle PQV = 4 \triangle PRQ,$$

and $$\triangle PQq = 8 \triangle PRQ.$$

Also, if RW, the ordinate from R to PV, be produced to meet the curve again in r,

$$RW = rW,$$

and the same proof shows that

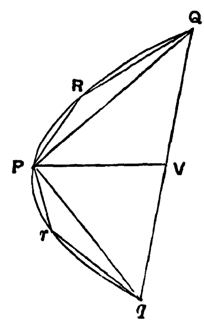
$$\triangle PQq = 8 \triangle Prq.$$

Proposition 22.

If there be a series of areas A, B, C, D, ... each of which is four times the next in order, and if the largest, A, be equal to the triangle PQq inscribed in a parabolic segment PQq and having the same base with it and equal height, then

$$(A + B + C + D + \ldots) < (\textit{area of segment } PQq).$$

For, since $\triangle PQq = 8 \triangle PRQ = 8 \triangle Pqr$, where R, r are the vertices of the segments cut off by PQ, Pq, as in the last proposition,



$$\triangle PQq = 4 (\triangle PQR + \triangle Pqr).$$

Therefore, since $\triangle PQq = A$,

$$\triangle PQR + \triangle Pqr = B.$$

In like manner we prove that the triangles similarly inscribed in the remaining segments are together equal to the area C, and so on.

Therefore $A + B + C + D + \ldots$ is equal to the area of a certain inscribed polygon, and is therefore less than the area of the segment.

Proposition 23.

Given a series of areas A, B, C, D, ... Z, of which A is the greatest, and each is equal to four times the next in order, then

$$A + B + C + \ldots + Z + \tfrac{1}{3}Z = \tfrac{4}{3}A.$$

Take areas $b, c, d, \ldots$ such that

$$b = \tfrac{1}{3}B,$$

$$c = \tfrac{1}{3}C,$$

$$d = \tfrac{1}{3}D, \text{ and so on.}$$

Then, since $$b = \tfrac{1}{3}B,$$

and $$B = \tfrac{1}{4}A,$$

$$B + b = \tfrac{1}{3}A.$$

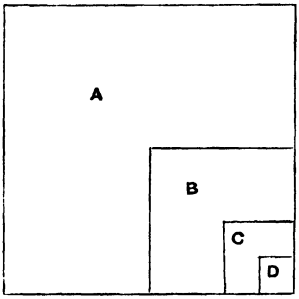
Similarly $$C + c = \tfrac{1}{3}B.$$

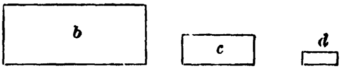
....

Therefore

$$B + C + D + \ldots + Z + b + c + d + \ldots + z = \tfrac{1}{3}(A + B + C + \ldots + Y).$$

But $$b + c + d + \ldots + y = \tfrac{1}{3}(B + C + D + \ldots + Y).$$





Therefore, by subtraction,

$$B + C + D + \ldots + Z + z = \tfrac{1}{3}A$$

or $$A + B + C + \ldots + Z + \tfrac{1}{3}Z = \tfrac{4}{3}A.$$

[The algebraical equivalent of this result is of course

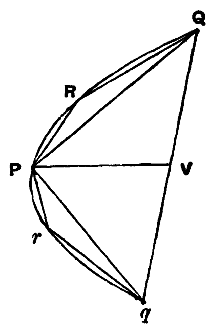
$$1+\tfrac{1}{4}+(\tfrac{1}{4})^2+\dots+(\tfrac{1}{4})^{n-1}=\tfrac{4}{3}-\tfrac{1}{3}(\tfrac{1}{4})^{n-1}$$
$$=\frac{1-(\tfrac{1}{4})^n}{1-\tfrac{1}{4}}.]$$

Proposition 24.

Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.

Suppose $$K=\tfrac{4}{3}\triangle PQq,$$

where P is the vertex of the segment; and we have then to prove that the area of the segment is equal to K.



For, if the segment be not equal to K, it must either be greater or less.

I. Suppose the area of the segment greater than K.

If then we inscribe in the segments cut off by PQ, Pq triangles which have the same base and equal height, i.e. triangles with the same vertices R, r as those of the segments, and if in the remaining segments we inscribe triangles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment PQq exceeds K.

Therefore the polygon so formed must be greater than the area K; which is impossible, since [Prop. 23]

$$A+B+C+\dots+Z<\tfrac{4}{3}A,$$

where $$A=\triangle PQq.$$

Thus the area of the segment cannot be greater than K.

II. Suppose, if possible, that the area of the segment is less than K.

If then $\triangle PQq = A$, $B = \frac{1}{4}A$, $C = \frac{1}{4}B$, and so on, until we arrive at an area X such that X is less than the difference between K and the segment, we have

$$A + B + C + \ldots + X + \tfrac{1}{3}X = \tfrac{4}{3}A \qquad \text{[Prop. 23]}$$
$$= K.$$

Now, since K exceeds $A + B + C + \ldots + X$ by an area less than X, and the area of the segment by an area greater than X, it follows that

$$A + B + C + \ldots + X > \text{(the segment)};$$

which is impossible, by Prop. 22 above.

Hence the segment is not less than K.

Thus, since the segment is neither greater nor less than K,

$$(\text{area of segment } PQq) = K = \tfrac{4}{3}\triangle PQq.$$