

SN: 587623

PHYC90012 General Relativity

Assignment 2

By

Braden MOORE

Master of Science
The University of Melbourne

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2 Klein's geometry

A two-dimensional surface is covered by coordinates (u, v) in the domain $u^2 + v^2 = 1$. The independent components of the metric are given by

$$g_{11} = \frac{a^2(1 - v^2)}{(1 - u^2 - v^2)^2}, \quad (1)$$

$$g_{12} = \frac{a^2 uv}{(1 - u^2 - v^2)^2}, \quad (2)$$

$$g_{22} = \frac{a^2(1 - u^2)}{(1 - u^2 - v^2)^2}, \quad (3)$$

the independent components of the inverse metric are given by

$$g^{11} = a^{-2}(1 - u^2)(1 - u^2 - v^2), \quad (4)$$

$$g^{12} = -a^{-2}uv(1 - u^2 - v^2), \quad (5)$$

$$g^{22} = a^{-2}(1 - v^2)(1 - u^2 - v^2), \quad (6)$$

and the independent, non-zero Christoffel symbols are given by

$$\Gamma_{11}^1 = \frac{2u}{1 - u^2 - v^2}, \quad (7)$$

$$\Gamma_{12}^1 = \frac{v}{1 - u^2 - v^2}, \quad (8)$$

$$\Gamma_{12}^2 = \frac{u}{1 - u^2 - v^2}, \quad (9)$$

$$\Gamma_{22}^2 = \frac{2v}{1 - u^2 - v^2}. \quad (10)$$

Remember that $g_{\alpha\beta}$, $g^{\alpha\beta}$, and $\Gamma_{\alpha\beta}^\lambda$ are all symmetric in α and β .

(a) Starting from (1)-(6), derive the expression (7) for Γ_{11}^1 .

We begin with the expression for the Christoffel symbols in terms of the metric

$$\Gamma_{\alpha\beta}^\lambda = \frac{1}{2}g^{\lambda\mu}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}) \quad (11)$$

We now calculate the values of $g_{\alpha\beta,\mu}$ from (1)-(3)

$$g_{11,1} = \frac{\partial g_{11}}{\partial x^1} = \frac{\partial \left(\frac{a^2(1-v^2)}{(1-u^2-v^2)^2} \right)}{\partial u} \quad (12)$$

$$= \frac{4a^2u(1-v^2)}{(1-u^2-v^2)^3} \quad (13)$$

$$g_{12,1} = \frac{\partial g_{12}}{\partial x^1} = \frac{\partial \left(\frac{a^2uv}{(1-u^2-v^2)^2} \right)}{\partial u} \quad (14)$$

$$= \frac{a^2v(3u^2-v^2+1)}{(1-u^2-v^2)^3} \quad (15)$$

$$= g_{21,1} \quad \text{by symmetry} \quad (16)$$

$$g_{22,1} = \frac{\partial g_{22}}{\partial x^1} = \frac{\partial \left(\frac{a^2(1-u^2)}{(1-u^2-v^2)^2} \right)}{\partial u} \quad (17)$$

$$= \frac{2a^2u(1-u^2+v^2)}{(1-u^2-v^2)^3} \quad (18)$$

By inspecting the components of the metric above, we see that $g_{11,1} \mapsto g_{22,2}$ with $u \leftrightarrow v$, similarly $g_{12,1} \mapsto g_{12,2}$ with $u \leftrightarrow v$, and $g_{11,2} \mapsto g_{22,1}$ with $u \leftrightarrow v$. Hence

$$g_{11,2} = \frac{2a^2v(1-v^2+u^2)}{(1-u^2-v^2)^3} \quad (19)$$

$$g_{12,2} = \frac{a^2u(3v^2-u^2+1)}{(1-u^2-v^2)^3} \quad (20)$$

$$= g_{21,2} \quad \text{by symmetry} \quad (21)$$

$$g_{22,2} = \frac{4a^2v(1-u^2)}{(1-u^2-v^2)^3} \quad (22)$$

So now we can evaluate the Christoffel symbols.

$$\Gamma_{11}^1 = \frac{1}{2} g^{1\mu} (g_{\mu 1,1} + g_{\mu 1,1} - g_{11,\mu}) \quad (23)$$

$$= \frac{1}{2} [g^{11} g_{11,1} + g^{12} (2g_{21,1} - g_{11,2})] \quad (24)$$

$$= \frac{1}{2} \left[a^{-2}(1-u^2)(1-u^2-v^2) \frac{4a^2u(1-v^2)}{(1-u^2-v^2)^3} \right. \\ \left. + -a^{-2}uv(1-u^2-v^2) \left(\frac{2a^2v(3u^2-v^2+1) - 2a^2v(1-v^2+u^2)}{(1-u^2-v^2)^3} \right) \right] \quad (25)$$

$$= \frac{1}{2} \left[\frac{(1-u^2) \cdot 4u(1-v^2) - uv \cdot 2v(3u^2-u^2)}{(1-u^2-v^2)^2} \right] \quad (26)$$

$$= \frac{1}{2} \left[\frac{4u(1-u^2)(1-v^2) - 4u^2v^2}{(1-u^2-v^2)^2} \right] \quad (27)$$

$$= 2u \left[\frac{1-u^2-v^2 + \cancel{u^2v^2} - \cancel{u^2v^2}}{(1-u^2-v^2)^2} \right] \quad (28)$$

$$= \frac{2u}{1-u^2-v^2} \quad (29)$$

as required. As an exercise, I have further derived the remaining Christoffel symbols

$$\Gamma_{12}^1 = \frac{1}{2} g^{1\mu} (g_{\mu 1,2} + g_{\mu 2,1} - g_{12,\mu}) \quad (30)$$

$$= \frac{1}{2} [g^{11} g_{11,2} + g^{12} g_{22,1}] \quad (31)$$

$$= \frac{1}{2} \left[a^{-2} (1-u^2)(1-u^2-v^2) \frac{2a^2 v(1-v^2+u^2)}{(1-u^2-v^2)^3} + -a^{-2} uv(1-u^2-v^2) \frac{2a^2 u(1-u^2+v^2)}{(1-u^2-v^2)^3} \right] \quad (32)$$

$$= \frac{1}{2} \left[\frac{(1-u^2) \cdot 2v(1-v^2-u^2) - uv \cdot 2u(1-u^2+v^2)}{(1-u^2-v^2)^2} \right] \quad (33)$$

$$= v \left[\frac{(1-u^2)(1-v^2+u^2) - u^2(1-u^2+v^2)}{(1-u^2-v^2)^2} \right] \quad (34)$$

$$= v \left[\frac{1-u^2-v^2}{(1-u^2-v^2)^2} \right] \quad (35)$$

$$= \frac{v}{1-u^2-v^2} = \Gamma_{21}^1, \quad \text{by symmetry} \quad (36)$$

as given.

$$\Gamma_{12}^2 = \frac{1}{2} g^{2\mu} (g_{\mu 1,2} + g_{\mu 2,1} - g_{12,\mu}) \quad (37)$$

$$= \frac{1}{2} [g^{21} g_{11,2} + g^{22} g_{22,1}] \quad (38)$$

Now, we deduced earlier that $g_{11,2} = g_{22,1}|_{u \leftrightarrow v}$, we see by inspection of (6) that $g^{22} = g^{11}|_{u \leftrightarrow v}$, and by symmetry of the metric we have $g^{21} = g^{12}$. We note also that g^{12} is symmetric under interchange of u and v . Combining these results we find

$$\Gamma_{12}^2 = \frac{1}{2} [g^{12} g_{22,1} + g^{11} g_{11,2}]|_{u \leftrightarrow v} \quad (39)$$

$$= \Gamma_{12}^1|_{u \leftrightarrow v} \quad (40)$$

$$= \frac{u}{1-u^2-v^2} \quad (41)$$

as given.

$$\Gamma_{22}^2 = \frac{1}{2} g^{2\mu} (g_{\mu 2,2} + g_{\mu 2,2} - g_{22,\mu}) \quad (42)$$

$$= \frac{1}{2} [g^{21} (2g_{12,2} - g_{22,1}) + g^{22} g_{22,2}] \quad (43)$$

Similar to our approach for Γ_{12}^2 , we note by observation that $g_{12,2} = g_{21,1}|_{u \leftrightarrow v}$, $g_{22,1} = g_{11,2}|_{u \leftrightarrow v}$, $g_{22} = g_{11}|_{u \leftrightarrow v}$, and $g_{22,2} = g_{11,1}$, also noting that $g^{21} = g^{12} = g^{12}|_{u \leftrightarrow v}$. Thus we find

$$\Gamma_{22}^2 = \frac{1}{2} [g^{12} (2g_{21,2} - g_{11,2}) + g^{11} g_{11,1}]|_{u \leftrightarrow v} \quad (44)$$

$$= \Gamma_{11}^1|_{u \leftrightarrow v} \quad (45)$$

$$= \frac{2v}{1-u^2-v^2} = \Gamma_{21}^2, \quad \text{by symmetry} \quad (46)$$

as given.

(b) Prove that the Riemann tensor with all indices lowered, $R_{\alpha\beta\gamma\delta}$, contains four nonzero elements, any three of which can be written in terms of the fourth.

We recall the Bianchi identities for the Riemann tensor

$$R_{\alpha\beta\mu\nu} = \frac{1}{2} (g_{\alpha\nu, \beta\mu} + g_{\beta\mu, \alpha\nu} - g_{\alpha\mu, \beta\nu} - g_{\beta\nu, \alpha\mu}) \quad (47)$$

Since $R_{\alpha\beta\mu\nu}$ is anti-symmetric under exchange $\alpha \leftrightarrow \beta$, and $\mu \leftrightarrow \nu$ also¹, we know

$$R_{\alpha\alpha\mu\nu} = 0 \quad (48)$$

$$R_{\alpha\beta\mu\mu} = 0 \quad (49)$$

for all α, β, μ, ν . Using this we greatly reduce the number of components to investigate. We find

$$R_{11\mu\nu} = 0 \quad R_{\alpha\beta 11} = 0 \quad (50)$$

$$R_{22\mu\nu} = 0 \quad R_{\alpha\beta 22} = 0 \quad (51)$$

We also know the Riemann tensor is symmetric under exchange of pairs $\alpha\beta \leftrightarrow \mu\nu$, i.e.

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \quad (52)$$

Hence we find

$$R_{1212} = -R_{1221} = R_{2121} = -R_{2112} \quad (53)$$

We can calculate the component R_{1212} (and hence determine the remaining components) as

$$R_{1212} = \frac{1}{2} (g_{12,21} + g_{21,12} - g_{11,22} - g_{22,11}) \quad (54)$$

$$= g_{12,12} \quad (55)$$

$$= \frac{\partial}{\partial v} (g_{12,1}) \quad (56)$$

$$= \frac{\partial}{\partial v} \left(\frac{a^2 v (3u^2 - v^2 + 1)}{(1 - u^2 - v^2)^3} \right) \quad (57)$$

$$= \frac{-a^2 [3v^2 - 2v^2(9u^2 + 1) + (u^2 - 1)(3u^2 + 1)]}{(1 - u^2 - v^2)^4} \quad (58)$$

which is, in general, non-zero (note that 58) is symmetric under interchange $u \leftrightarrow v$, as we would expect).

¹This can be easily proved from (47) by considering the symmetries of $g_{\alpha\beta, \mu\nu}$; however for brevity this proof is omitted.

(c) Prove that, in Klein's geometry, the Ricci tensor satisfies

$$R_{\alpha\beta} = -\frac{g_{\alpha\beta}}{a^2}, \quad (59)$$

and the Ricci scalar satisfies

$$R = -\frac{2}{a^2}. \quad (60)$$

We can find the Ricci tensor by contracting the first and third indices of the Riemann tensor

$$R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta} \quad (61)$$

$$= \Gamma^\mu{}_{\alpha\beta,\mu} - \Gamma^\mu{}_{\alpha\mu,\beta} + \Gamma^\mu{}_{\nu\mu}\Gamma^\nu{}_{\alpha\beta} - \Gamma^\mu{}_{\nu\beta}\Gamma^\nu{}_{\alpha\mu} \quad (62)$$

$$= \Gamma^1{}_{\alpha\beta,1} + \Gamma^2{}_{\alpha\beta,2} - \Gamma^1{}_{\alpha 1,\beta} - \Gamma^2{}_{\alpha 2,\beta} + \Gamma^1{}_{\nu 1}\Gamma^\nu{}_{\alpha\beta} + \Gamma^2{}_{\nu 2}\Gamma^\nu{}_{\alpha\beta} - \Gamma^1{}_{\nu\beta}\Gamma^\nu{}_{\alpha 1} - \Gamma^2{}_{\nu\beta}\Gamma^\nu{}_{\alpha 2} \quad (63)$$

$$= \Gamma^1{}_{\alpha\beta,1} + \Gamma^2{}_{\alpha\beta,2} - \Gamma^1{}_{\alpha 1,\beta} - \Gamma^2{}_{\alpha 2,\beta} + \Gamma^1{}_{11}\Gamma^1{}_{\alpha\beta} + \Gamma^1{}_{21}\Gamma^2{}_{\alpha\beta} + \Gamma^2{}_{12}\Gamma^1{}_{\alpha\beta} + \Gamma^2{}_{22}\Gamma^2{}_{\alpha\beta} \\ - \Gamma^1{}_{1\beta}\Gamma^1{}_{\alpha 1} - \Gamma^1{}_{2\beta}\Gamma^2{}_{\alpha 1} - \Gamma^2{}_{1\beta}\Gamma^1{}_{\alpha 2} - \Gamma^2{}_{2\beta}\Gamma^2{}_{\alpha 2} \quad (64)$$

from Schutz (6.63).

We can easily calculate the values of $\Gamma^\alpha_{\beta\mu,\nu}$:

$$\Gamma^1_{11,1} = \frac{2(1-u^2+v^2)}{(1-u^2-v^2)^2} \quad \Gamma^1_{11,2} = \frac{4uv}{(1-u^2-v^2)^2} \quad (65)$$

$$\Gamma^1_{12,1} = \frac{2uv}{(1-u^2-v^2)^2} \quad \Gamma^1_{12,2} = \frac{1-u^2+v^2}{(1-u^2-v^2)^2} \quad (66)$$

$$\Gamma^2_{12,1} = \frac{1+u^2-v^2}{(1-u^2-v^2)^2} \quad \Gamma^2_{12,2} = \frac{2uv}{(1-u^2-v^2)^2} \quad (67)$$

$$\Gamma^2_{22,1} = \frac{4uv}{(1-u^2-v^2)^2} \quad \Gamma^2_{22,2} = \frac{2(1-u^2+v^2)}{(1-u^2-v^2)^2} \quad (68)$$

with all others zero.

We shall calculate each component of $R_{\alpha\beta}$ below using (64)

$$R_{11} = \cancel{\Gamma^1_{11,1}} + \Gamma^2_{11,2} - \cancel{\Gamma^1_{11,1}} - \Gamma^2_{12,1} + \cancel{\Gamma^1_{11}\Gamma^1_{11}} + \cancel{\Gamma^1_{21}\Gamma^2_{11}} + \Gamma^2_{12}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^2_{11} \\ - \cancel{\Gamma^1_{11}\Gamma^1_{11}} - \cancel{\Gamma^1_{21}\Gamma^2_{11}} - \Gamma^2_{11}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^2_{12} \quad (69)$$

by cancellations,

$$= \cancel{\Gamma^2_{11,2}} - \Gamma^2_{12,1} + \Gamma^2_{12}\Gamma^1_{11} + \cancel{\Gamma^2_{22}\Gamma^2_{11}} - \cancel{\Gamma^2_{11}\Gamma^1_{12}} - \Gamma^2_{12}\Gamma^2_{12} \quad (70)$$

since $\Gamma^2_{11} = 0$,

$$= \Gamma^2_{12} \Gamma^1_{11} - \Gamma^2_{12} \Gamma^2_{12} - \Gamma^2_{12,1} \quad (71)$$

$$= \frac{1}{(1 - u^2 - v^2)} [u \cdot 2u - u \cdot u - (1 + u^2 - v^2)] \quad (72)$$

$$= \frac{1}{(1 - u^2 - v^2)^2} [2u^2 - u^2 - 1 - u^2 + v^2] \quad (73)$$

$$= -\frac{(1 - v^2)}{(1 - u^2 - v^2)^2} \quad (74)$$

$$= -\frac{1}{a^2} \frac{a^2(1 - v^2)}{(1 - u^2 - v^2)^2} \quad (75)$$

$$= -\frac{g_{11}}{a^2} \quad (76)$$

as required.

$$R_{12} = \cancel{\Gamma^1_{12,1} + \Gamma^2_{12,2}} - \cancel{\Gamma^1_{11,2} - \Gamma^2_{12,2}} + \cancel{\Gamma^1_{11} \Gamma^1_{12} + \Gamma^1_{21} \Gamma^2_{12}} + \cancel{\Gamma^2_{12} \Gamma^1_{12} + \Gamma^2_{22} \Gamma^2_{12}} - \cancel{\Gamma^1_{12} \Gamma^1_{11} - \Gamma^1_{22} \Gamma^2_{11} - \Gamma^2_{12} \Gamma^1_{12} - \Gamma^2_{22} \Gamma^2_{12}} \quad (77)$$

$$= \Gamma^1_{12,1} - \Gamma^1_{11,2} + \Gamma^1_{21} \Gamma^2_{12} \quad (78)$$

$$= \frac{1}{(1 - u^2 - v^2)} [2uv - 4uv + v \cdot u] \quad (79)$$

$$= -\frac{uv}{(1 - u^2 - v^2)^2} \quad (80)$$

$$= -\frac{1}{a^2} \frac{a^2 uv}{(1 - u^2 - v^2)^2} \quad (81)$$

$$= -\frac{g_{12}}{a^2} \quad (82)$$

as required (note by symmetry, $R_{12} = R_{21}$, so $R_{21} = -\frac{g_{21}}{a^2}$).

$$R_{22} = \cancel{\Gamma^1_{22,1} + \Gamma^2_{22,2}} - \cancel{\Gamma^1_{21,2} - \Gamma^2_{22,2}} + \cancel{\Gamma^1_{11} \Gamma^1_{22} + \Gamma^1_{21} \Gamma^2_{22}} + \cancel{\Gamma^2_{12} \Gamma^1_{22} + \Gamma^2_{22} \Gamma^2_{22}} - \cancel{\Gamma^1_{12} \Gamma^1_{21} - \Gamma^1_{22} \Gamma^2_{21} - \Gamma^2_{12} \Gamma^1_{22} - \Gamma^2_{22} \Gamma^2_{22}} \quad (83)$$

$$= \Gamma^1_{21} \Gamma^2_{22} - \Gamma^1_{12} \Gamma^1_{21} - \Gamma^1_{21,2} \quad (84)$$

$$= \frac{1}{(1 - u^2 - v^2)^2} [v \cdot 2v - v \cdot v - (1 - u^2 + v^2)] \quad (85)$$

$$= \frac{1}{(1 - u^2 - v^2)^2} [2v^2 - v^2 - 1 + u^2 - v^2] \quad (86)$$

$$= -\frac{(1 - u^2)}{(1 - u^2 - v^2)^2} \quad (87)$$

$$= -\frac{1}{a^2} \frac{a^2(1 - u^2)}{(1 - u^2 - v^2)^2} \quad (88)$$

$$= -\frac{g_{22}}{a^2} \quad (89)$$

as required.

$$(90)$$

Hence we have proved that, in Klen's geometry,

$$R_{\alpha\beta} = -\frac{g_{\alpha\beta}}{a^2} \quad (91)$$

The Ricci scalar is formed by contracting the Ricci tensor:

$$R = g^{\alpha\beta} R_{\alpha\beta} \quad (92)$$

$$= -g^{\alpha\beta} \frac{g_{\alpha\beta}}{a^2} \quad (93)$$

$$= -\frac{2}{a^2} \quad (94)$$

since $g^{\alpha\beta} g_{\alpha\beta} = g^{\alpha\beta} g_{\beta\alpha} = \delta^\alpha_\alpha = 2$ in Klein's geometry.

(d) Answer each of the following questions in one or two sentences.

i. In what fundamental way does Klein's geometry differ from a two-sphere?

A 2-sphere has positive curvature (the geodesics converge; also note a triangle on the surface would have the sum of internal angles $> 180^\circ$ - see Figure 4). By contrast, Klein's geometry has negative curvature!

- ii. The hyperbola $x^2 - y^2 = 1$ is rotated around the y -axis to form a three-dimensional hyperboloid of revolution. Does it possess positive or negative curvature? Justify your answer physically with a diagram; do not attempt to calculate anything.

Along surfaces with positive curvature, geodesics will converge as they travel along the surface. By contrast, geodesics along surfaces of negative curvature will diverge.

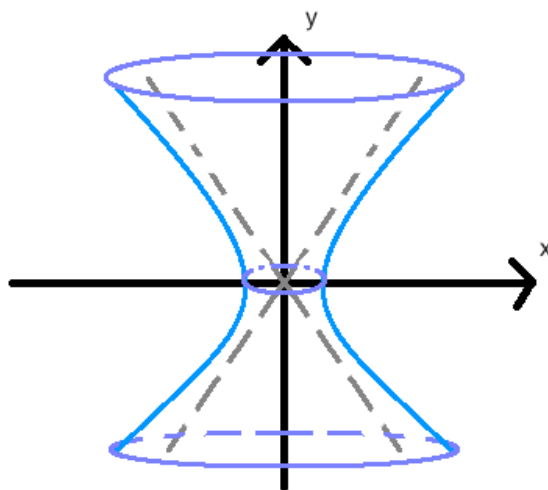


Figure 1: $x^2 - y^2 = 1$ rotated about the y -axis

We see that geodesics travelling along the surface would diverge; hence this hyperboloid possesses negative curvature. A triangle on the surface would look like Figure 2, with the sum of its angles $< 180^\circ$.



Figure 2: Triangle with $\Sigma < 180^\circ$

iii. The hyperbola $x^2 - y^2 = 1$ is now rotated around the x -axis. What is the sign of the curvature this time? Why?

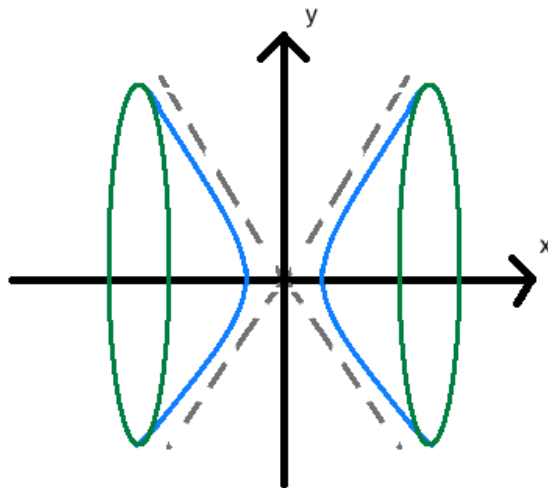


Figure 3: $x^2 - y^2 = 1$ rotated about the x -axis

We see that geodesics travelling along the surface would converge; hence this hyperboloid possesses positive curvature. A triangle on the surface would look like Figure 4 below, with the sum of its angles $> 180^\circ$.

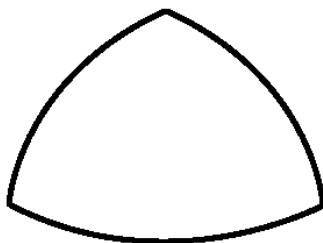


Figure 4: Triangle with $\Sigma < 180^\circ$

iv. Setting aside their dimensionality, in what fundamental way do the hyperboloids of revolution in parts (d)(ii) and (d)(iii) differ from Klein's geometry? Justify your answer in words; don't try to calculate anything.

The hyperboloids differ from Klein's geometry in that geodesics can "traverse" around the space and return to their origin (for example, by travelling on a geodesic around the hyperboloid). Further, for these hyperboloids we have non-constant curvature.

In the hyperboloid in Figure 3, we note that there are obvious discontinuities. There are no discontinuities in Klein's geometry, which is an infinite plane.

v. Identify a spacetime manifold, that resembles Klein's geometry. Don't worry too much about the precise mathematical meaning of "resembles", a qualitative justification is fine.

We can consider anti-de Sitter space to resemble Klein's geometry. We would find that both have negative scalar curvature. The anti-de Sitter space is a maximally symmetric Lorentzian manifold (hence have a different signature metric), but "locally" the manifold would resemble Klein's geometry.

(e) Consider the triangle $\triangle ABC$, whose sides are “straight lines” (geodesics) joining the points $A(0,0)$, $B(b,0)$, and $C(0,b)$, with $b < 1$. It is easy to show (you don’t need to!) that the sides AB and AC are just the curves $v = 0$ and $u = 0$ respectively.

i. What is the equation of the geodesic joining B and C ?

From Schutz (6.51), letting λ be the parameter of the geodesic with $x^1 = u$ and $x^2 = v$ we have the geodesic equation

$$\frac{d}{d\lambda} \left(\frac{dx^\alpha}{d\lambda} \right) + \Gamma^\alpha_{\mu\beta} \frac{dx^\mu}{d\lambda} \frac{dx^\beta}{d\lambda} = 0 \quad (95)$$

For $\alpha = 1$ we have

$$\frac{d}{d\lambda} \left(\frac{du}{d\lambda} \right) = -\Gamma^1_{\mu\beta} \frac{dx^\mu}{d\lambda} \frac{dx^\beta}{d\lambda} \quad (96)$$

$$= -\Gamma^1_{11} \frac{dx^1}{d\lambda} \frac{dx^1}{d\lambda} - \Gamma^1_{12} \frac{dx^1}{d\lambda} \frac{dx^2}{d\lambda} - \Gamma^1_{21} \frac{dx^2}{d\lambda} \frac{dx^1}{d\lambda} - \cancel{\Gamma^1_{22} \frac{dx^2}{d\lambda} \frac{dx^2}{d\lambda}} \quad (97)$$

$$= -\Gamma^1_{11} \left(\frac{du}{d\lambda} \right)^2 - 2\Gamma^1_{12} \frac{du}{d\lambda} \frac{dv}{d\lambda} \quad (98)$$

which can be written as

$$\frac{d}{d\lambda} \left(\frac{du}{d\lambda} \right) + \Gamma^1_{11} \left(\frac{du}{d\lambda} \right)^2 + 2\Gamma^1_{12} \frac{du}{d\lambda} \frac{dv}{d\lambda} = 0 \quad (99)$$

$$\Rightarrow \frac{d}{d\lambda} \left(\frac{du}{d\lambda} \right) + 2u(1-u^2-v^2)^{-1} \left(\frac{du}{d\lambda} \right)^2 + 2v(1-u^2-v^2)^{-1} \frac{du}{d\lambda} \frac{dv}{d\lambda} = 0 \quad (100)$$

$$\Rightarrow \frac{d}{d\lambda} \left(\frac{du}{d\lambda} \right) (1-u^2-v^2)^{-1} + 2u(1-u^2-v^2)^{-2} \left(\frac{du}{d\lambda} \right)^2 + 2v(1-u^2-v^2)^{-2} \frac{du}{d\lambda} \frac{dv}{d\lambda} = 0 \quad (101)$$

$$\Rightarrow \frac{d}{d\lambda} \left[(1-u^2-v^2)^{-1} \frac{du}{d\lambda} \right] = 0 \quad (102)$$

We shall prove that going from (101) to (102) is correct.

$$\frac{d}{d\lambda} \left[(1-u^2-v^2)^{-1} \frac{du}{d\lambda} \right] = \frac{d}{d\lambda} \left[(1-u^2-v^2)^{-1} \right] \frac{du}{d\lambda} + (1-u^2-v^2)^{-1} \frac{d^2u}{d\lambda^2} \quad (103)$$

$$= \left[-\frac{d}{d\lambda} (-u^2) + -\frac{d}{d\lambda} (-v^2) \right] (1-u^2-v^2)^{-2} \frac{du}{d\lambda} + (1-u^2-v^2)^{-1} \frac{d^2u}{d\lambda^2} \quad (104)$$

$$= \left[\frac{du}{d\lambda} \frac{d(u^2)}{du} + \frac{dv}{d\lambda} \frac{d(v^2)}{dv} \right] (1-u^2-v^2)^{-2} \frac{du}{d\lambda} + (1-u^2-v^2)^{-1} \frac{d^2u}{d\lambda^2} \quad (105)$$

$$= \left[2u \frac{du}{d\lambda} + 2v \frac{dv}{d\lambda} \right] (1-u^2-v^2)^{-2} \frac{du}{d\lambda} + (1-u^2-v^2)^{-1} \frac{d^2u}{d\lambda^2} \quad (106)$$

$$= \text{LHS of (101)} \quad (107)$$

Hence this is indeed correct.

For $\alpha = 2$ we have

$$\frac{d}{d\lambda} \left(\frac{dv}{d\lambda} \right) = -\Gamma^2_{\mu\beta} \frac{dx^\mu}{d\lambda} \frac{dx^\beta}{d\lambda} \quad (108)$$

$$= -\Gamma^2_{11} \frac{dx^1}{d\lambda} \frac{dx^1}{d\lambda} - \Gamma^2_{12} \frac{dx^1}{d\lambda} \frac{dx^2}{d\lambda} - \Gamma^2_{21} \frac{dx^2}{d\lambda} \frac{dx^1}{d\lambda} - \Gamma^2_{22} \frac{dx^2}{d\lambda} \frac{dx^2}{d\lambda} \quad (109)$$

$$= -\Gamma^2_{22} \left(\frac{dv}{d\lambda} \right)^2 - 2\Gamma^2_{12} \frac{du}{d\lambda} \frac{dv}{d\lambda} \quad (110)$$

$$= -2v(1 - u^2 - v^2)^{-1} \left(\frac{dv}{d\lambda} \right)^2 - 2u(1 - u^2 - v^2)^{-1} \frac{du}{d\lambda} \frac{dv}{d\lambda} \quad (111)$$

By comparison with (100)-(102), we see that this implies

$$\frac{d}{d\lambda} \left[(1 - u^2 - v^2)^{-1} \frac{dv}{d\lambda} \right] = 0 \quad (112)$$

We now find

$$\frac{d}{d\lambda} \left[(1 - u^2 - v^2)^{-1} \frac{du}{d\lambda} \right] = 0 \Rightarrow (1 - u^2 - v^2)^{-1} \frac{du}{d\lambda} = c_1 \quad (113)$$

$$\frac{d}{d\lambda} \left[(1 - u^2 - v^2)^{-1} \frac{dv}{d\lambda} \right] = 0 \Rightarrow (1 - u^2 - v^2)^{-1} \frac{dv}{d\lambda} = c_2 \quad (114)$$

Dividing (113) by (114) we find

$$\frac{du}{dv} = \frac{c_1}{c_2} \equiv c_3 \quad (115)$$

$$\Rightarrow v = c_3 u + c_4 \quad (116)$$

Since the point $(0, b)$ lies on the geodesic we have,

$$b = c_4 \quad (117)$$

$$\Rightarrow c_4 = b \quad (118)$$

Similarly using the point $(b, 0)$,

$$0 = bc_3 + b \quad (119)$$

$$\Rightarrow c_3 = -1 \quad (120)$$

Thus we conclude that equation of the geodesic joining B and C is

$$v = -u + b \quad (121)$$

ii. Prove that the sum of the interior angles of ΔABC is

$$\Sigma = \angle ABC + \angle BCA + \angle CAB = \frac{\pi}{2} + 2 \cos^{-1} \left(\frac{1}{\sqrt{2-b^2}} \right). \quad (122)$$

The sum of the angles is less than 180 degrees!

The cosine of the angle between two vectors \vec{M} and \vec{N} is given by

$$\cos \theta = \frac{\vec{M} \cdot \vec{N}}{|\vec{M}| |\vec{N}|} \quad (123)$$

In component notation we have

$$\theta = \cos^{-1} \left(\frac{g_{\alpha\beta} M^\alpha N^\beta}{\sqrt{g_{ij} M^i M^j g_{kl} N^k N^l}} \right) \quad (124)$$

The angle between two curves at a point is equal to the angle between curves pointing in the same direction;

The equations bounding the triangle parametrised by λ are given as

$$\vec{a} = (\lambda, 0) \quad (125)$$

$$\vec{b} = (0, \lambda) \quad (126)$$

$$\vec{c} = (\lambda, b - \lambda) \quad (127)$$

where $\lambda \leq b$.

$$\Rightarrow \vec{a} = (1, 0) \quad (128)$$

$$\vec{b} = (0, 1) \quad (129)$$

$$\vec{c} = (1, -1) \quad (130)$$

We shall use these direction vectors WHY???.

Calculating $\angle ABC$

We use $\vec{M} = \vec{a}$ and $\vec{N} = \vec{c}$. At the point B(b, 0),

$$g_{11} = \frac{a^2}{(1-b^2)^2} \quad (131)$$

$$g_{12} = 0 \quad (132)$$

$$g_{22} = \frac{a^2}{1-b^2} \quad (133)$$

Hence we find

$$\angle ABC = \cos^{-1} \left(\frac{g_{11}}{\sqrt{g_{11}g_{11} + g_{11}g_{22}}} \right) \quad (134)$$

$$= \cos^{-1} \left(\frac{\frac{a^2}{(1-b^2)^2}}{\sqrt{\frac{a^4}{(1-b^2)^4} + \frac{a^4}{(1-b^2)^3}}} \right) \quad (135)$$

$$= \cos^{-1} \left(\frac{a^2}{(1-b^2)^2} \sqrt{\frac{1+(1-b^2)}{(1-b^2)^4}} \right) \quad (136)$$

$$= \cos^{-1} \left(\frac{1}{\sqrt{2-b^2}} \right) \quad (137)$$

Calculating $\angle BCA$

We use $\vec{M} = \vec{b}$ and $\vec{N} = \vec{c}$. At the point C(0, b),

$$g_{11} = \frac{a^2}{1-b^2} \quad (138)$$

$$g_{12} = 0 \quad (139)$$

$$g_{22} = \frac{a^2}{(1-b^2)^2} \quad (140)$$

Hence we find

$$\angle BCA = \cos^{-1} \left(\frac{g_{22}}{\sqrt{g_{22}g_{11} + g_{22}g_{22}}} \right) \quad (141)$$

$$= \cos^{-1} \left(\frac{\frac{a^2}{(1-b^2)^2}}{\sqrt{\frac{a^4}{(1-b^2)^4} + \frac{a^4}{(1-b^2)^3}}} \right) \quad (142)$$

$$= \cos^{-1} \left(\frac{a^2}{(1-b^2)^2} \sqrt{\frac{1+(1-b^2)}{(1-b^2)^4}} \right) \quad (143)$$

$$= \cos^{-1} \left(\frac{1}{\sqrt{2-b^2}} \right) \quad (144)$$

Calculating $\angle ABC$

We use $\vec{M} = \vec{a}$ and $\vec{N} = \vec{c}$. At the point B(b, 0),

$$g_{11} = \frac{a^2}{(1-b^2)^2} \quad (145)$$

$$g_{12} = 0 \quad (146)$$

$$g_{22} = \frac{a^2}{1-b^2} \quad (147)$$

Hence we find

$$\angle ABC = \cos^{-1} \left(\frac{g_{11}}{\sqrt{g_{11}g_{11} + g_{11}g_{22}}} \right) \quad (148)$$

$$= \cos^{-1} \left(\frac{\frac{a^2}{(1-b^2)^2}}{\sqrt{\frac{a^4}{(1-b^2)^4} + \frac{a^4}{(1-b^2)^3}}} \right) \quad (149)$$

$$= \cos^{-1} \left(\frac{a^2}{(1-b^2)^2} \sqrt{\frac{1+(1-b^2)}{(1-b^2)^4}} \right) \quad (150)$$

$$= \cos^{-1} \left(\frac{1}{\sqrt{2-b^2}} \right) \quad (151)$$

Calculating $\angle CAB$

We use $\vec{M} = \vec{a}$ and $\vec{N} = \vec{b}$. At the point A(0,0),

$$g_{11} = a^2 \quad (152)$$

$$g_{12} = 0 \quad (153)$$

$$g_{22} = a^2 \quad (154)$$

Hence we find

$$\angle BCA = \cos^{-1} \left(\frac{\cancel{g_{12}}}{\sqrt{g_{11}g_{22}}} \right) \quad (155)$$

$$= \cos^{-1}(0) \quad (156)$$

$$= \frac{\pi}{2} \quad (157)$$

Summing these angles we indeed find

$$\Sigma = \angle ABC + \angle BCA + \angle CAB = \frac{\pi}{2} + 2 \cos^{-1} \left(\frac{1}{\sqrt{2-b^2}} \right) \quad (158)$$

as required.

iii. Triangles in Klein's geometry can have $\Sigma = 0$! Without proof, sketch what such a triangle might look like. Your sketch by necessity will be an incomplete representation; there is no way to draw a Klein triangle faithfully on a flat page.

A triangle in space with negative curvature will have $\Sigma < 180^\circ$, as shown in Figure 2. One can imagine a triangle such as Figure 5 below.

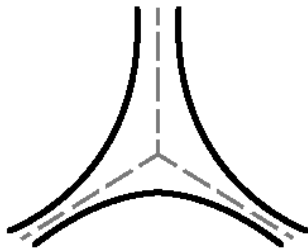


Figure 5: Triangle with $\Sigma = 0^\circ$

If the dotted lines extended to infinity, we would have a triangle with $\Sigma = 0^\circ$. In flat space, the lines would look somewhat parallel.

(f) i. Write down a closed form expression for the area A of $\triangle ABC$ as an integral over a subset of the (u, v) domain.

Surface area depends not on the parameterisation of the space, but only on the surface itself. We have an expression for area

$$A = \int_{\text{surface}} \sqrt{\det g} \, dA \quad (159)$$

where g is the metric. We have

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad (160)$$

$$\Rightarrow \det g = g_{11}g_{22} - g_{12}g_{21} \quad (161)$$

$$= \frac{a^4(1-v^2)(1-u^2) - a^4u^2v^2}{(1-u^2-v^2)^4} \quad (162)$$

$$= \frac{a^4}{(1-u^2-v^2)^4} \left[1 - u^2 - v^2 + \cancel{u^2v^2} - \cancel{u^2v^2} \right] \quad (163)$$

$$= \frac{a^4}{(1-u^2-v^2)^3} \quad (164)$$

We can now determine an expression for the area of $\triangle ABC$ as

$$A = \iint \frac{a^4}{(1-u^2-v^2)^3} \, du \, dv \quad (165)$$

$$= \int_{v=0}^{v=b} \int_{u=0}^{u=b-v} \frac{a^2}{(1-u^2-v^2)^{3/2}} \, du \, dv \quad (166)$$

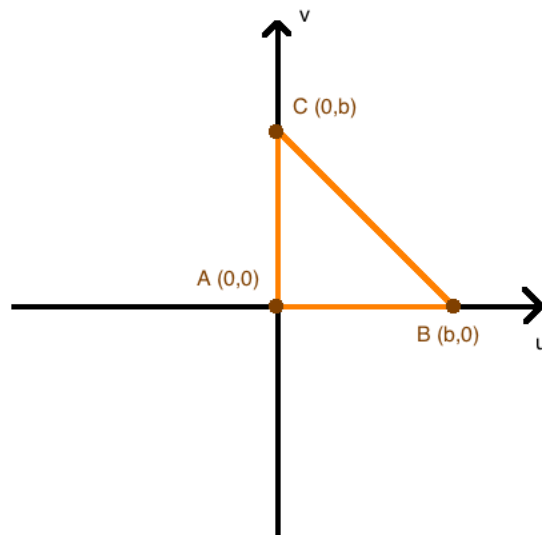


Figure 6: The triangle $\triangle ABC$ on the $u - v$ axis

ii. By changing variables to $y = v + u$ and $z = v - u$, recast your integral in the form

$$A = 2a^2 \int_0^b \frac{dy y}{(2 - y^2)\sqrt{1 - y^2}}. \quad (167)$$

Hence show that one has

$$A = a^2(\pi - \Sigma). \quad (168)$$

We begin by calculating the Jacobian,

$$J(y, z) = \begin{pmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (169)$$

$$\Rightarrow \det J = \frac{1}{2} \quad (170)$$

Next we consider the bounds of the integral; that is, the surface we are integrating over.

$$(u, v) = (0, 0) \Rightarrow (y, z) = (0, 0) \quad (171)$$

$$(u, v) = (b, 0) \Rightarrow (y, z) = (b, -b) \quad (172)$$

$$(u, v) = (0, b) \Rightarrow (y, z) = (b, b) \quad (173)$$

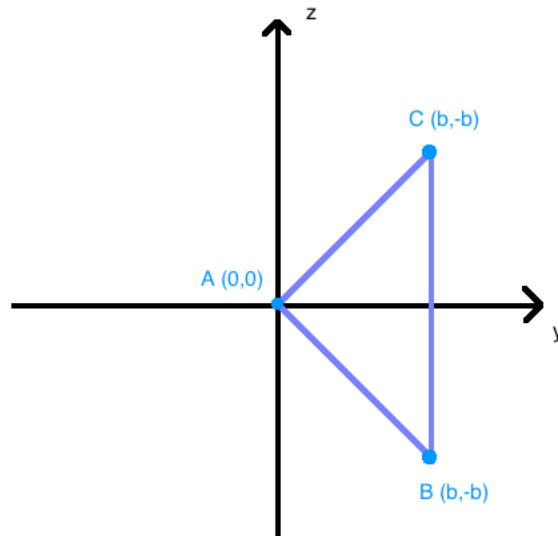


Figure 7: The triangle $\triangle ABC$ on the $y - z$ axis

In $y - z$ space, our surface looks like Figure 7 above. Just as we integrated over first u in terms of v then integrated over v in (166), we shall integrate from $z = -y$ to $z = y$, then over $y = 0$ to $y = b$.

Also note that we can express u and v in terms of y and z as

$$u = \frac{1}{2}(y - z) \quad (174)$$

$$v = \frac{1}{2}(y + z) \quad (175)$$

Hence we can rewrite A as

$$A = 2a^2 \int_{y=0}^{y=b} \int_{z=-y}^{z=y} \left(1 - \left[\frac{1}{2}(y-z) \right]^2 - \left[\frac{1}{2}(y+z) \right]^2 \right)^{-3/2} \frac{1}{4} dy dz \quad (176)$$

$$= \frac{a^2}{2} \int_{y=0}^{y=b} \int_{z=-y}^{z=y} \left(1 - \frac{1}{2}y^2 - \frac{1}{2}z^2 \right)^{-3/2} dy dz \quad (177)$$

We now let $m = 1 - \frac{1}{2}y^2$, and let $z = \sqrt{2m} \sin r \Rightarrow dz = \sqrt{2m} \cos r dr$

$$A = \frac{a^2}{2} \int_{y=0}^{y=b} \int_{r=-\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)}^{r=\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)} \left(\underbrace{m - m \sin^2 r}_{=m(\cos^2 r)} \right)^{-3/2} dy \sqrt{2m} \cos r dr \quad (178)$$

$$= \frac{a^2}{2} \iint \sqrt{2m}^{-1} \cos^{-2} r dy dr \quad (179)$$

$$= \frac{a^2}{\sqrt{2}} \int_{y=0}^{y=b} m^{-1} \left[\tan(r) \right]_{-\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)}^{\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)} dy \quad (180)$$

$$= \frac{a^2}{\sqrt{2}} \int m^{-1} \left[\frac{2y}{\sqrt{2-2y^2}} \right] dy \quad (181)$$

In moving from (180) to (181) we use the identity

$$\tan [\sin^{-1}(ax)] = \frac{ax}{\sqrt{1-a^2x^2}} \quad (182)$$

$$\Rightarrow \tan \left[\sin^{-1} \left(\frac{y}{\sqrt{2m}} \right) \right] = \frac{\frac{y}{\sqrt{2m}}}{\sqrt{1-\frac{y^2}{2m}}} \quad (183)$$

$$= \frac{y}{\sqrt{2m-y^2}} \quad (184)$$

Since $m = 1 - \frac{1}{2}y^2$ we find

$$\tan \left[\sin^{-1} \left(\frac{y}{\sqrt{2m}} \right) \right] = \frac{y}{\sqrt{2-2y^2}} \quad (185)$$

Continuing from (181),

$$A = \frac{a^2}{\sqrt{2}} \int_0^b m^{-1} \left[\frac{2y}{\sqrt{2-2y^2}} \right] dy \quad (186)$$

$$= a^2 \int_0^b \frac{1}{1-\frac{1}{2}y^2} \frac{y}{\sqrt{1-y^2}} dy \quad (187)$$

$$= 2a^2 \int_0^b \frac{dy y}{(2-y^2)\sqrt{1-y^2}} \quad (188)$$

as required.

We will now show that $A = a^2(\pi - \Sigma)$.

$$A = 2a^2 \int_0^b \frac{dy y}{(2-y^2)\sqrt{1-y^2}} \quad (189)$$

Let $p = \sqrt{1 - y^2}$

$$\Rightarrow \frac{dp}{dy} = \frac{-2y \times \frac{1}{2}}{\sqrt{1 - y^2}} = -\frac{y}{\sqrt{1 - y^2}} = -\frac{y}{p} \quad (190)$$

$$\Rightarrow A = 2a^2 \int_{p=1}^{p=\sqrt{1-b^2}} \frac{1}{1+p^2} \frac{1}{p} \times \frac{-p}{y} dp \quad (191)$$

$$= -2a^2 \int_1^{\sqrt{1-b^2}} \frac{1}{1+p^2} dp \quad (192)$$

$$= 2a^2 \int_{\sqrt{1-b^2}}^1 \frac{1}{1+p^2} dp \quad (193)$$

$$= 2a^2 [\tan^{-1}(p)]_{\sqrt{1-b^2}}^1 \quad (194)$$

$$= 2a^2 \left[\tan^{-1}(1) - \tan^{-1}(\sqrt{1-b^2}) \right] \quad (195)$$

$$= 2a^2 \left[\frac{\pi}{4} - \tan^{-1}(\sqrt{1-b^2}) \right] \quad (196)$$

We now consider the remaining \tan^{-1} term.

$$\tan^{-1}(x) = \sin^{-1} \left(\frac{x}{\sqrt{x^2 + 1}} \right) \quad (197)$$

$$\Rightarrow \tan^{-1}(\sqrt{1-b^2}) = \sin^{-1} \left(\frac{\sqrt{1-b^2}}{\sqrt{2-b^2}} \right) \quad (198)$$

$$= \sin^{-1} \left(\sqrt{\frac{2-b^2-1}{2-b^2}} \right) \quad (199)$$

$$= \sin^{-1} \left(\sqrt{1 - \frac{1}{2-b^2}} \right) \quad (200)$$

$$\cos^{-1}(x) = \sin^{-1}(\sqrt{1-x^2}), \quad \text{if } 0 \leq x \leq 1 \quad (201)$$

$$\Rightarrow \tan^{-1}(\sqrt{1-b^2}) = \sin^{-1} \left(\sqrt{1 - \frac{1}{2-b^2}} \right) = \cos^{-1} \left(\frac{1}{\sqrt{2-b^2}} \right) \quad (202)$$

Substituting this into (196) we conclude

$$A = 2a^2 \left[\frac{\pi}{4} - \cos^{-1} \left(\frac{1}{\sqrt{2-b^2}} \right) \right] \quad (203)$$

$$= a^2 \left[\frac{\pi}{2} - 2\cos^{-1} \left(\frac{1}{\sqrt{2-b^2}} \right) \right] \quad (204)$$

$$= a^2 \left(\frac{\pi}{2} - \Sigma \right) \quad (205)$$

as required.

iii. Explain briefly, in one or two sentences, why (168) guarantees the *nonexistence* of similar triangles in Klein's geometry.

The equation

$$A = a^2 \left(\frac{\pi}{2} - \Sigma \right)$$

tells us that in Klein's geometry, the area of a triangle is linked directly to the sum of its internal angles. Similar triangles possess the same internal angles and hence the same sum of internal angles; however since equation (168) gives only a single value of area for any Σ , we see that the area is fixed, and hence similar triangles cannot exist in Klein's geometry.

(g) A vector \vec{W} with equal components W^1 and W^2 at the point $A(0,0)$ is parallel transported along the geodesic AB . Show that its components, when it reaches the point $B(b,0)$, are in the ratio

$$\frac{W^1}{W^2} = (1 - b^2)^{1/2} \quad (206)$$

Parallel transport of a vector \vec{V} along a curve s is given by

$$U^\beta V^\alpha_{;\beta} = 0 \quad (207)$$

As V does not change over the geodesic AB , we know $x^\beta = u = x^1$.

We have two geodesic equations for the transport of \vec{W} along AB :

$$\frac{\partial W^1}{\partial u} + \Gamma^1_{11} W^1 + \Gamma^1_{21} W^2 = 0 \quad (208)$$

$$\frac{\partial W^2}{\partial u} + \Gamma^2_{21} W^2 = 0 \quad (209)$$

Solving the second equation gives us

$$\frac{\partial W^2}{\partial u} + \frac{u}{1 - u^2 - v^2} W^2 = 0 \quad (210)$$

$$\Rightarrow \frac{dW^2}{W^2} = -\frac{u \, du}{1 - u^2 - v^2} \quad (211)$$

$$\log W^2 = \frac{1}{2} \log(1 - u^2 - v^2) + c_1 \quad (212)$$

$$W^2 = c_1 \sqrt{1 - u^2 - v^2} \quad (213)$$

Now, substituting (213) into (208) we find

$$\frac{\partial W^1}{\partial u} + \frac{2u}{1 - u^2 - v^2} W^1 = \frac{c_1 v}{\sqrt{1 - u^2 - v^2}} = 0 \quad (214)$$

$$(1 - u^2 - v^2)^{-1} \frac{dW^1}{du} + \frac{2u}{(1 - u^2 - v^2)^2} W^1 + \frac{c_1 v}{(1 - u^2 - v^2)^{3/2}} = 0 \quad (215)$$

$$\frac{\partial}{\partial u} \left(\frac{W^1}{1 - u^2 - v^2} \right) = -\frac{c_1 v}{(1 - u^2 - v^2)^{3/2}} \quad (216)$$

$$\frac{W^1}{1 - u^2 - v^2} = -c_1 \int \frac{v \, dv}{(1 - u^2 - v^2)^{3/2}} \quad (217)$$

We shall solve this integral through some clever substitution. We let

$$k = 1 - v^2, \quad (218)$$

$$m = \sqrt{k} \sin n \quad (219)$$

Then, we have

$$\Rightarrow \int \frac{v \, dv}{(1 - u^2 - v^2)^{3/2}} = \frac{1}{(k - u^2)^{3/2}} \quad (220)$$

$$= k^{-3/2} \int \sec^2 n \, dn \quad (221)$$

$$= k^{-3/2} \tan n + c_2 \quad (222)$$

$$= k^{-3/2} \frac{n}{\sqrt{k^2 - n^2}} + c_2 \quad (223)$$

$$= (1 - v^2)^{-3/2} \frac{u}{\sqrt{1 - u^2 - v^2}} + c_2 \quad (224)$$

So now we have

$$(1 - u^2 - v^2)^{-1} W^1 = -c_1 (1 - v^2)^{-3/2} \frac{vu}{\sqrt{1 - u^2 - v^2}} + c_3 \quad (225)$$

$$\Rightarrow W^1 = -c_1 (1 - v^2)^{-3/2} vu \sqrt{1 - u^2 - v^2} + c_3 (1 - u^2 - v^2) \quad (226)$$

We can now combine (226) and (213) to find the ratio $\frac{W^1}{W^2}$.

At $(u, v) = (0, 0)$, we have $\vec{W} = (w, w)$ (where w is some constant; the components are equal at this point) so

$$W^1(0, 0) = w = c_3 \quad (227)$$

$$W^2(0, 0) = w = c_1 \quad (228)$$

Considering the point $(u, v) = (b, 0)$ we find

$$W^1(b, 0) = w(1 - b^2) \quad (229)$$

$$W^2(b, 0) = w\sqrt{1 - b^2} \quad (230)$$

$$\Rightarrow \frac{W^1}{W^2}(b, 0) = \frac{w(1 - b^2)}{w\sqrt{1 - b^2}} \quad (231)$$

$$= (1 - b^2)^{1/2} \quad (232)$$

as required.