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PHYC90012 General Relativity

Assignment 3

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3 Playing tennis in a four-dimensional "brane world"

Projectile motion in a constant gravitational field is a classic introductory problem in Newtonian mechanics. One way to make a uniform gravitational field is to assemble an infinite, flat sheet of matter. If the mass per unit area in the $x-y$ plane is σ, then the Newtonian gravitational potential is given by

$$\Phi_{\text{Newton}} = 2\pi G\sigma|z|, \tag{1}$$

where z is the Cartesian coordinate normal to the sheet. [Convince yourself that (1) is true by analogy with electrostatics or otherwise.] Below we ask whether it is possible to recreate this system in the strong-gravity regime in general relativity.

(a) By appealing to symmetry, argue that the most general metric of a plane-parallel, four-dimensional spacetime takes the form

$$ds^2 = -e^{2\Phi(z)}dt^2 + e^{2\Psi(z)}(dx^2+dy^2) + dz^2, \tag{2}$$

where $\Phi(t,z)$ (not necessarily the same as Φ_{Newton}), $\Psi(t,z)$, and $\Lambda(t,z)$ are functions determined by Einstein's field equations. determined by Einstein's field equations.

The plane is symmetric under x and $y \Rightarrow g_{xx} = g_{yy}$. By symmetry, the spacetime is unchanged under $x \to -x$, $y \to -y$, $z \to -z$, $t \to -t$ transformations $\Rightarrow$ cross-terms are zero, i.e. $g_{xy} = -g_{xy} = 0$.

Recall that the spacetime interval ds^2 can be written as,

$$ds^2 = g_{00}dt^2 + g_{11}dx^2 + g_{22}dy^+g_{33}dz^2 \tag{3}$$

so that we have

$$ds^2 = -e^{2\Phi(z,t)}dt^2 + e^{2\Psi(z,t)}(dx^2+dy^2) + e^{2\Lambda(z,t)}dz^2 \tag{4}$$

where $g_{00} < 0$ and $g_{11}, g_{22} > 0$ everywhere.

We have introduced functions Φ, Ψ, Λ which depend on z, t by the geometry given.

(b) i. If the spacetime is static (caution: such a solution may not necessarily exist), construct explicitly a coordinate transformation, that puts (2) into the form

$$ds^2 = -e^{2\Phi(z)}dt^2 + e^{2\Psi(z)}(dx^2+dy^2) + dz^2, \tag{5}$$

where t, x, y, and z are suitably relabelled from (2).

For a static spacetime (no time dependence), we can rewrite the functions independent of time, i.e.

$$\begin{aligned} e^{\Phi(z)}dt' &= e^{\Phi(z,t)}dt \\ e^{\Psi(z)}dx' &= e^{\Psi(z,t)}dx \\ e^{\Psi(z)}dy' &= e^{\Psi(z,t)}dy \\ dx' &= e^{\Lambda(z,t)}dz \end{aligned}$$

Assuming the functions Φ, Ψ can be re-expressed in a time independent way, we will only need the transformation $dz' = e^{\Lambda(z,t)}dz$

This leaves us with

$$ds^2 = -e^{2\Phi(z)}dt^2 + e^{2\Psi(z)}(dx^2 + dy^2) + dz^2 \tag{6}$$

as required.

(b) ii. Explain why it is impossible to get rid of the $e^{2\Lambda(r)}$ factor multiplying dr^2 in the "standard" Schwarzschild metric in the same way, such that the metric contains only one undetermined function $e^{2\Phi(r)}$.

Using Schwarzchild metric $dr' = \left(1 - \frac{2M}{r}\right)^{-1/2} dr$ we find

$$r' = \sqrt{r^2 - 2Mr} + M\log\left(\sqrt{r^2 - 2Mr} + r - m\right) \tag{7}$$

At $r = 0$ the second term becomes $M\log(-m)$, which is undefined since $m > 0$.

Wrong?

(c) Nine of the Christoffel symbols associated with (5) are nonzero. Calculate them. Here and henceforth, please feel free to use a symbolic algebra package like *Mathematica* to make your life easier.

These Christoffel symbols have been calculated used Mathematica (see attached code).

$$\Gamma^1_{14} = \Gamma^1_{41} = \Phi'(z) \tag{8}$$
$$\Gamma^2_{42} = \Gamma^2_{24} = \Psi'(z) \tag{9}$$
$$\Gamma^3_{43} = \Gamma^3_{34} = \Psi'(z) \tag{10}$$
$$\Gamma^4_{11} = e^{2\Phi}\Phi' \tag{11}$$
$$\Gamma^4_{22} = -e^{2\Psi}\Psi' \tag{12}$$
$$\Gamma^4_{33} = -e^{2\Psi}\Psi' \tag{13}$$

(d) Show that the nonzero components of the Ricci tensor $R_{\mu\nu}$ are

$$R_{tt} = e^{2\Phi}[(\Phi')^2 + 2\Phi'\Psi' + \Phi''], \tag{14}$$
$$R_{xx} = -e^{2\Phi}[2(\Psi')^2 + \Psi'\Phi' + \Psi''] \tag{15}$$
$$R_{yy} = R_{xx}, \tag{16}$$
$$R_{zz} = -(\Phi')^2 - 2(\Psi')^2 - \Phi'' - 2\Psi''. \tag{17}$$

Primes denote derivatives with respect to z.

We recall the Ricci tensor $R_{\alpha\beta}$ is a contraction on the first and third indices of the Riemann tensor, written as

$$R_{\alpha\beta} = R^{i}{}_{\alpha i\beta} \tag{18}$$

We also recall that the Riemann tensor in the form $R^{\alpha}_{\beta\mu\nu}$ can be written as

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma} \tag{19}$$

The non-zero components of the Ricci tensor are calculated in Mathematica as shown in the code, and we find

$$R_{tt} = e^{2\Phi}[(\Phi')^2 + 2\Phi'\Psi' + \Phi''], \tag{20}$$
$$R_{xx} = -e^{2\Phi}[2(\Psi')^2 + \Psi'\Phi' + \Psi''] \tag{21}$$
$$R_{yy} = R_{xx}, \tag{22}$$
$$R_{zz} = -(\Phi')^2 - 2(\Psi')^2 - \Phi'' - 2\Psi'', \tag{23}$$

as required.

(e) Show that the nonzero contravariant components of Einstein's field equations with cosmological constant Λ and stress-energy tensor $T_{\mu\nu}$ are

$$8\pi T^{tt} = -e^{-2\Phi}[3(\Psi')^2 + 2\Psi'' + \Lambda], \tag{24}$$
$$8\pi T^{xx} = e^{-2\Psi}[(\Phi')^2 + \Phi'\Psi' + (\Psi')^2 + \Phi'' + \Psi'' + \Lambda], \tag{25}$$
$$8\pi T^{yy} = 8\pi T^{xx}, \tag{26}$$
$$8\pi T^{zz} = \Psi'(2\Phi' + \Psi') + \Lambda. \tag{27}$$

You can do this with pen and paper but will find *Mathematica* more soothing.

We know

$$8\pi T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda \tag{28}$$
$$\Rightarrow 8\pi T^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}8\pi T_{\mu\nu} \tag{29}$$

The values of $8\pi T^{\alpha\alpha}$ are calculated as shown in the code, and we find

$$8\pi T^{tt} = -e^{-2\Phi}[3(\Psi')^2 + 2\Psi'' + \Lambda], \tag{30}$$
$$8\pi T^{xx} = e^{-2\Psi}[(\Phi')^2 + \Phi'\Psi' + (\Psi')^2 + \Phi'' + \Psi'' + \Lambda], \tag{31}$$
$$8\pi T^{yy} = 8\pi T^{xx}, \tag{32}$$
$$8\pi T^{zz} = \Psi'(2\Phi' + \Psi') + \Lambda, \tag{33}$$

as required.

(f) Einstein's field equations place strong constraints on the physical form of the stress-energy consistent with the metric (5).

i. Explain why it is impossible to generate (5) with a sheet of cold matter (dust).

Starting with

$$8\pi T^{tt} = -e^{-2\Phi}\left(3(\Psi')^2 + 2\Psi'' + \Lambda\right) \tag{34}$$

we let $T^{tt} = \delta(z)$ as we are confied to the sheet $z = 0$. As the LHS has a delta-function, the right hand side must also be proportional to a delta-function.

As a first derivative cannot be a delta-function (proof not shown), as as $\Lambda = 0$, we see

$$\Phi'' \sim \delta(z) \tag{35}$$

We also have

$$8\pi T^{xx} = 0 = \Phi'^2 + \Phi'\Psi' + \Psi'^2 + \Phi'' + \Psi'' + \Lambda \tag{36}$$

$$8\pi T^{zz} = 0 = \Psi'(2\Phi' + \Psi') + \Lambda \tag{37}$$

Subtracting (37) from (36) we find

$$\Phi'' = -\Psi'' \tag{38}$$

which implies that Φ'' must be a delta-function.

Similarly, subtracting (36) from (37) we find

$$0 = (\Phi')^2 - \Phi'\Psi' \tag{39}$$

$$\therefore \Psi' = 0 \text{ or } \Phi' = \Psi' \tag{40}$$

However, both of these possibilities contradict with our previous findings! Hence we cannot generate (5) with a sheet of cold matter.

ii. By thinking about a suitable initial value problem qualitatively, speculate why this seemingly natural system - a static dust sheet - is impossible to assemble "from the ground up" in general relativity, even though there is no obstacle in a Newtonian context. What might "go wrong"? Do not try calculating anything, unless you are in the mood to win a Nobel Prize.

If we are to build a sheet from the ground up, i.e. particle by particle, we would find that as we add a particle, a gravitational or electric force would act on the other particles currently in the "sheet" and distory the structure. Hence, a flat sheet structure would be impossible to form.

(g) Consider the special case $\Psi = 0$, i.e. warped time, Euclidean space, no preferential warping of space in the z direction relative to the x and y directions.

i. In the bulk ($z = 0$), where there is zero stress-energy, show that $\Lambda = 0$ must hold to obtain a self-consistent solution of the form (5).

Zero stree energy leads to

$$8\pi T^{tt} = 0 = \Psi'(2\Phi' + \Psi) + \Lambda = \Lambda \tag{41}$$

since $\Psi = 0 \Rightarrow \Psi' = 0$.

$$\therefore \Lambda = 0 \tag{42}$$

ii. Hence show that the stress-energy must vanish on the sheet $z = 0$ too.

At $z = 0$, there will be no mass as $\Lambda = 0, \Psi = 0$

$\Rightarrow$ no invariant mass E_m

$\Rightarrow$ no energy density

$\therefore T^{tt} = 0$

$\therefore$ momentum flow must also be zero

Wrong?

iii. Prove that (5) reduces to the Rindler metric for a uniformly accelerated frame. What is the implied acceleration?

In the special case $\Phi = 0$, a similar bulk-then-sheet approach yields the Minkowski metric.

$$ds^2 = -e^{2\Phi(z)}dt^2 + dx^2 + dy^2 + dz^2 \tag{43}$$

If z is small we can Taylor expand

$$\therefore ds^2 = -\left(1 + 2\Phi'(z)\right)dt^2 + dx^2 + dy^2 + dz^2 \tag{44}$$

This is of the form of an accelerating frame with acceleration $2\Phi'(z)$.

(h) Consider the special case $\Psi = \Phi$. This corresponds to preferentially warping space in z relative to x and y, i.e. the "symmetric" coordinates (t, x, y) are treated as a Minkowski subspace with warping in the z coordinate. The system is the four dimensional analogue of the five-dimensional *Randall-Sundrum 1-brane* spacetime, which revolutionized the study of brane worlds

i. In the bulk, show that one obtains

$$\Phi = g|z|, \tag{45}$$

if $\Phi(z)$ vanishes at $z = 0$ without loss of generality. In (45), g is an integration constant with units of acceleration.

$$\Phi = \Psi \Rightarrow 0 = 3(\Phi') + \Lambda \tag{46}$$

$$\therefore \frac{d\Phi}{dz} = \pm\sqrt{-\frac{\Lambda}{3}} \tag{47}$$

$$\therefore \Phi = \pm\sqrt{-\frac{\Lambda}{3}}|z|, \quad \text{as z is symmetric} \tag{48}$$

$$= g|z| \tag{49}$$

ii. Prove that g satisfies the fine-tuning condition

$$\Lambda = -3g^2. \tag{50}$$

Comment on the physical significance of the sign of Λ.

$$g = \pm\sqrt{-\frac{\Lambda}{3}} \therefore \Lambda = -3g^2 \tag{51}$$

If Λ is positive $\Rightarrow$ imaginary g which is not physical. $\Rightarrow$ Λ is negative.

Wrong?

iii. Assume that the stress-energy is confined to the sheet, i.e. $T^{\mu\nu} \propto \delta(z)$. Prove that the stress-energy tensor takes the physically unusual form

$$T^{\mu\nu} = (2\pi)^{-1} g\delta(z)\text{diag }(-1, 1, 1, 0). \tag{52}$$

Equation (52) implies fine tuning between $T^{\mu\nu}$ and Λ via g.

$$\Phi = g|z| \therefore \Phi'' = 2g\delta(z) \tag{53}$$

as

$$\frac{d\Phi}{dz} = \text{sgn}(z) \tag{54}$$

$$\frac{d}{dz}\text{sgn}(z) = z\delta(z) \tag{55}$$

Also,

$$
\begin{aligned}
3(\Phi')^2 &= -\Lambda && (56)\\
\therefore 8\pi T^{tt} &= -e^{2g|z|}(4\Phi'') && (57)\\
\therefore Ttt &= -e^{2g|z|}(2\pi)^{-1}\delta(z) && (58)\\
8\pi T^{xx} &= e^{2g|z|}(4\Phi'') && (59)\\
T^{xx} &= (2\pi)^{-1}e^{2g|z|}\delta(z) && (60)\\
T^{yy} &= T^{xx} && (61)\\
8\pi T^{zz} &= 0 && (62)\\
\Rightarrow T^{zz} &= 0 && (63)
\end{aligned}
$$

which leaves us with

$$T^{\mu\nu} = (2\pi)^{-1}g\delta(z)\text{diag }(-1,1,1,0), \tag{64}$$

as required.

iv. Interpret physically the two cases $g < 0$ and $g > 0$.

$g < 0$ positive energy density, normal stress (pressure) is negative.

$g > 0$ negative energy density, normal stress (pressure) is positive.

v. Interpret physically the signs of T^{tt} and $T^{xx} = T^{yy}$ for $g > 0$.

For $g > 0$, T^{tt} is negative, $T^{xx} = T^{yy}$ is positive.

vi. Explain physically why $T^{zz} = 0$ makes sense in terms of a plausible microscopic ("particulate") model of the sheet.

T^{zz} is pressure in the $z-$direction. But if there was pressure, this leads to motion in z of the sheet, and the sheet would buckle. But since our sheet is flat, there must not be any pressure in $z \Rightarrow T^{zz} = 0$.

vii. Prove that it is impossible to build the sheet out of a perfect fluid or dust.

For Schutz Ch. 4, for a perfect fluid the stress-energy tensor is given by

$$T^{\mu\nu} = \text{diag}(\rho, p, p, p) \tag{65}$$

But our $T^{\mu\nu}$ is confined to the sheet

$$T^{\mu\nu} \propto \text{diag}(-1, 1, 1, 0) \tag{66}$$

In the perfect fluid (**??**), we see the last two elements of the tensor are identical. However, on the sheet (**??**) we see the last two elements are different ($1 \neq 0$) $\Rightarrow$ we cannot build a sheet out of a perfect fluid!

(i) A tennis player standing on the sheet hits a tennis ball into the $z > 0$ half-space with initial 4-velocity $u(0)$.

i. Starting from the geodesic equations in the form

$$\frac{du_\nu}{d\tau} = \frac{1}{2} g_{\alpha\beta,\nu} u^\alpha u^\beta, \tag{67}$$

identify three constants of the motion. In (65), τ denotes the proper time as measured in the ball's momentarily comoving reference frame.

ii. From the z component of (65), show that

$$\frac{du^z}{d\tau} = -g[1 + (u^z)^2]. \tag{68}$$

What is unusual physically about this acceleration?

iii. Similarly or otherwise, prove the equivalent result

$$\left(\frac{dz}{d\tau}\right)^2 = -1 + e^{-2gz}\{1 + [u^z(0)]^2\}. \tag{69}$$

iv. Solve either (66) or (67) to obtain

$$z = g^{-1} \ln[\cos g\tau + u^z(0) \sin g\tau]. \tag{70}$$

Please note that there are many valid ways to prove (66)-(68).

(j) Show that the ball reaches the apex of its trajectory at proper time

$$\tau_{\text{top}} = g^{-1} \tan^{-1} u^z(0), \tag{71}$$

and that the associated coordinate time is

$$t_{\text{top}} = \frac{u^z(0)\{1 + [u^x(0)]^2 + [u^z(0)]^2\}^{1/2}}{g\{1 + [u^z(0)]^2\}}, \tag{72}$$

in the coordinates in which (5) is written. Interestingly, τ_{top} is finite, no matter how hard the ball is struck initially. Does the same hold true for t_{top}?

(k) Show that the horizontal distance traversed by the ball from its launch point to where it lands on the sheet, as measured in the coordinates associated with (5), is given by

$$x_{\text{land}} = \frac{2u^x(0)u^z(0)}{g\{1 + [u^z(0)]^2\}}. \tag{73}$$

How does (71) differ qualitatively from Newtonian projectile motion? Should we expect x_{land} to equal the x-component of the initial 3-velocity multiplied by $2t_{\text{top}}$, since the metric does not depend on x?

(l) Finally, suppose that instead of a tennis ball we launch a laser beam into the $z > 0$ half-space with initial 4-momentum $p(0)$ per photon.

i. Let λ be an affine parameter tracing out the light ray. Starting from the geodesic equations or otherwise, solve for the photon's trajectory. You should find

$$z = g^{-1} \ln[1 + p^z(0) g\lambda]. \tag{74}$$

In other words, the photon never falls back to the sheet, no matter how much stress-energy the sheet contains.

ii. Consider an intergalactic variant of the Pound-Rebka experiment, in which we fly horizontally (in the x direction, say) in a rocket at constant speed V [as measured in the coordinates associated with (5)] while maintaining a constant altitude z_{rocket} above the sheet. Show that the laser frequency measured by the stationary emitter ν_{em}, and the frequency measured by the experimentalist in the rocket, ν_{rec}, are related according to

$$\frac{\nu_{\text{rec}}}{\nu_{\text{em}}} = \frac{e^{-gz_{\text{rocket}}}}{(1 - V^2)^{1/2}} \left[1 - \frac{Vp^x(0)}{\{[p^x(0)]^2 + [p^z(0)]^2\}^{1/2}} \right]. \tag{75}$$

Most of the physics in this question applies in five dimensions too. It is interesting to think about it in the context of Randall-Sundrum brane worlds.