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PHYC90012 General Relativity

Assignment 2

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2 Klein's geometry

A two-dimensional surface is covered by coordinates (u, v) in the domain $u^2 + v^2 = 1$. The independent components of the metric are given by

$$g_{11} = \frac{a^2(1-v^2)}{(1-u^2-v^2)^2}, \tag{1}$$
$$g_{12} = \frac{a^2uv}{(1-u^2-v^2)^2}, \tag{2}$$
$$g_{22} = \frac{a^2(1-u^2)}{(1-u^2-v^2)^2}, \tag{3}$$

the independent components of the inverse metric are given by

$$g^{11} = a^{-2}(1-u^2)(1-u^2-v^2), \tag{4}$$
$$g^{12} = -a^{-2}uv(1-u^2-v^2), \tag{5}$$
$$g^{22} = a^{-2}(1-v^2)(1-u^2-v^2), \tag{6}$$

and the independent, non-zero Christoffel symbols are given by

$$\Gamma^1_{11} = \frac{2u}{1-u^2-v^2}, \tag{7}$$
$$\Gamma^1_{12} = \frac{v}{1-u^2-v^2}, \tag{8}$$
$$\Gamma^2_{12} = \frac{u}{1-u^2-v^2}, \tag{9}$$
$$\Gamma^2_{22} = \frac{2v}{1-u^2-v^2}. \tag{10}$$

Remember that $g_{\alpha\beta}$, $g^{\alpha\beta}$, and $\Gamma^\lambda_{\alpha\beta}$ are all symmetric in α and β.

(a) Starting from (1)-(6), derive the expression (7) for Γ^1_{11}.

We begin with the expression for the Christoffel symbols in terms of the metric

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2}g^{\lambda\mu}\left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right) \tag{11}$$

We now calculate the values of $g_{\alpha\beta,\mu}$ from (1)-(3)

$$g_{11,1} = \frac{\partial g_{11}}{\partial x^1} = \frac{\partial\left(\frac{a^2(1-v^2)}{(1-u^2-v^2)^2}\right)}{\partial u} \tag{12}$$
$$= \frac{4a^2u(1-v^2)}{(1-u^2-v^2)^3} \tag{13}$$

$$g_{12,1} = \frac{\partial g_{12}}{\partial x^1} = \frac{\partial\left(\frac{a^2uv}{(1-u^2-v^2)^2}\right)}{\partial u} \tag{14}$$
$$= \frac{a^2v(3u^2-v^2+1)}{(1-u^2-v^2)^3} \tag{15}$$
$$= g_{21,1} \quad \text{by symmetry} \tag{16}$$

$$g_{22,1} = \frac{\partial g_{22}}{\partial x^1} = \frac{\partial\left(\frac{a^2(1-u^2)}{(1-u^2-v^2)^2}\right)}{\partial u} \tag{17}$$
$$= \frac{2a^2u(1-u^2+v^2)}{(1-u^2-v^2)^3} \tag{18}$$

By inspecting the components of the metric above, we see that $g_{11,1} \mapsto g_{22,2}$ with $u \leftrightarrow v$, similarly $g_{12,1} \mapsto g_{12,2}$ with $u \leftrightarrow v$, and $g_{11,2} \mapsto g_{22,1}$ with $u \leftrightarrow v$. Hence

$$g_{11,2} = \frac{2a^2v(1-v^2+u^2)}{(1-u^2-v^2)^3} \tag{19}$$
$$g_{12,2} = \frac{a^2u(3v^2-u^2+1)}{(1-u^2-v^2)^3} \tag{20}$$
$$= g_{21,2} \quad \text{by symmetry} \tag{21}$$
$$g_{22,2} = \frac{4a^2v(1-u^2)}{(1-u^2-v^2)^3} \tag{22}$$

So now we can evaluate the Christoffel symbols.

$$\Gamma^1_{11} = \frac{1}{2}g^{1\mu}\left(g_{\mu 1,1} + g_{\mu 1,1} - g_{11,\mu}\right) \tag{23}$$
$$= \frac{1}{2}\left[g^{11}g_{11,1} + g^{12}\left(2g_{21,1} - g_{11,2}\right)\right] \tag{24}$$
$$= \frac{1}{2}\left[a^{-2}(1-u^2)(1-u^2-v^2)\frac{4a^2u(1-v^2)}{(1-u^2-v^2)^3}\right.$$
$$\left. + -a^{-2}uv(1-u^2-v^2)\left(\frac{2a^2v(3u^2-v^2+1) - 2a^2v(1-v^2+u^2)}{(1-u^2-v^2)^3}\right)\right] \tag{25}$$
$$= \frac{1}{2}\left[\frac{(1-u^2)\cdot 4u(1-v^2) - uv\cdot 2v(3u^2-u^2)}{(1-u^2-v^2)^2}\right] \tag{26}$$
$$= \frac{1}{2}\left[\frac{4u(1-u^2)(1-v^2) - 4u^2v^2}{(1-u^2-v^2)^2}\right] \tag{27}$$
$$= 2u\left[\frac{1-u^2-v^2+\cancel{u^2v^2-u^2v^2}}{(1-u^2-v^2)^2}\right] \tag{28}$$
$$= \frac{2u}{1-u^2-v^2} \tag{29}$$

as required. As an exercise, I have further derived the remaining Christoffel symbols

$$\begin{aligned}
\Gamma^1_{12} &= \frac{1}{2} g^{1\mu} \left(g_{\mu 1,2} + g_{\mu 2,1} - g_{12,\mu} \right) && (30)\\
&= \frac{1}{2} \left[g^{11} g_{11,2} + g^{12} g_{22,1} \right] && (31)\\
&= \frac{1}{2} \left[a^{-2}(1-u^2)(1-u^2-v^2) \frac{2a^2 v(1-v^2+u^2)}{(1-u^2-v^2)^3} \right. \\
&\quad \left. + -a^{-2} uv (1-u^2-v^2) \frac{2a^2 u(1-u^2+v^2)}{(1-u^2-v^2)^3} \right] && (32)\\
&= \frac{1}{2} \left[\frac{(1-u^2)\cdot 2v(1-v^2-u^2) - uv \cdot 2u(1-u^2+v^2)}{(1-u^2-v^2)^2} \right] && (33)\\
&= v \left[\frac{(1-u^2)(1-v^2+u^2) - u^2(1-u^2+v^2)}{(1-u^2-v^2)^2} \right] && (34)\\
&= v \left[\frac{\cancel{1-u^2-v^2}}{(1-u^2-v^2)^{\cancel{2}}} \right] && (35)\\
&= \frac{v}{1-u^2-v^2} = \Gamma^1_{21}, \quad \text{by symmetry} && (36)
\end{aligned}$$

as given.

$$\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2} g^{2\mu} \left(g_{\mu 1,2} + g_{\mu 2,1} - g_{12,\mu} \right) && (37)\\
&= \frac{1}{2} \left[g^{21} g_{11,2} + g^{22} g_{22,1} \right] && (38)
\end{aligned}$$

Now, we deduced earlier that $g_{11,2} = g_{22,1}|_{u\leftrightarrow v}$, we see by inspection of (6) that $g^{22} = g^{11}|_{u\leftrightarrow v}$, and by symmetry of the metric we have $g^{21} = g^{12}$. We note also that g^{12} is symmetric under interchange of u and v. Combining these results we find

$$\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2} \left[g^{12} g_{22,1} + g^{11} g_{11,2} \right] |_{u\leftrightarrow v} && (39)\\
&= \Gamma^1_{12}|_{u\leftrightarrow v} && (40)\\
&= \frac{u}{1-u^2-v^2} && (41)
\end{aligned}$$

as given.

$$\begin{aligned}
\Gamma^2_{22} &= \frac{1}{2} g^{2\mu} \left(g_{\mu 2,2} + g_{\mu 2,2} - g_{22,\mu} \right) && (42)\\
&= \frac{1}{2} \left[g^{21} \left(2g_{12,2} - g_{22,1} \right) + g^{22} g_{22,2} \right] && (43)
\end{aligned}$$

Similar to our approach for Γ^2_{12}, we note by observation that $g_{12,2} = g_{21,1}|_{u\leftrightarrow v}$, $g_{22,1} = g_{11,2}|_{u\leftrightarrow v}$, $g_{22} = g_{11}|_{u\leftrightarrow v}$, and $g_{22,2} = g_{11,1}$, also noting that $g^{21} = g^{12} = g^{12}|_{u\leftrightarrow v}$. Thus we find

$$\begin{aligned}
\Gamma^2_{22} &= \frac{1}{2} \left[g^{12} (2g_{21,2} - g_{11,2}) + g^{11} g_{11,1} \right] |_{u\leftrightarrow v} && (44)\\
&= \Gamma^1_{11}|_{u\leftrightarrow v} && (45)\\
&= \frac{2v}{1-u^2-v^2} = \Gamma^2_{21}, \quad \text{by symmetry} && (46)
\end{aligned}$$

as given.

(b) Prove that the Riemann tensor with all indices lowered, $R_{\alpha\beta\gamma\delta}$, contains four nonzero elements, any three of which can be written in terms of the fourth.

We recall the Bianchi identities for the Riemann tensor

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu} + g_{\beta\mu,\alpha\nu} - g_{\alpha\mu,\beta\nu} - g_{\beta\nu,\alpha\mu}\right) \tag{47}$$

Since $R_{\alpha\beta\mu\nu}$ is anti-symmetric under exchange $\alpha \leftrightarrow \beta$, and $\mu \leftrightarrow \nu$ also[1], we know

$$R_{\alpha\alpha\mu\nu} = 0 \tag{48}$$
$$R_{\alpha\beta\mu\mu} = 0 \tag{49}$$

for all α, β, μ, ν. Using this we greatly reduce the number of components to investigate. We find

$$R_{11\mu\nu} = 0 \qquad R_{\alpha\beta 11} = 0 \tag{50}$$
$$R_{22\mu\nu} = 0 \qquad R_{\alpha\beta 22} = 0 \tag{51}$$

We also know the Riemann tensor is symmetric under exchange of pairs $\alpha\beta \leftrightarrow \mu\nu$, i.e.

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \tag{52}$$

Hence we find

$$R_{1212} = -R_{1221} = R_{2121} = -R_{2112} \tag{53}$$

We can calculate the component R_{1212} (and hence all can determine all other non-zero components) as

$$R_{1212} = \frac{1}{2}\left(g_{12,21} + g_{11,22} - g_{22,11}\right) \tag{54}$$
$$= \frac{1}{2} \tag{55}$$

[1]This can be easily proved from (47) by considering the symmetries of $g_{\alpha\beta,\mu\nu}$; however for brevity this proof is omitted.

(c) Prove that, in Klein's geometry, the Ricci tensor satisfies

$$R_{\alpha\beta} = -\frac{g_{\alpha\beta}}{a^2}, \tag{56}$$

and the Ricci scalar satifies

$$R = -\frac{2}{a^2}. \tag{57}$$

We can find the Ricci tensor by contracting the first and third indices of the Riemann tensor

$$\begin{aligned} R_{\alpha\beta} &= R^{\mu}{}_{\alpha\mu\beta} && (58)\\ &= g^{\mu\nu} R_{\nu\alpha\mu\beta} && (59) \end{aligned}$$

We shall calculate each component in Klein's geometry below.

$$\begin{aligned} R_{11} &= g^{22} R_{2121} && (60)\\ &= g^{22} R_{1212} && (61)\\ R_{12} &= g^{12} R_{2112} && (62)\\ &= -g^{12} R_{1212} = R_{21} && (63)\\ R_{22} &= g^{11} R_{1212} && (64) \end{aligned}$$

The Ricci scalar is formed by contracting the Ricci tensor:

$$\begin{aligned} R &= g^{\alpha\beta} R_{\alpha\beta} && (65)\\ &= -g^{\alpha\beta}\frac{g_{\alpha\beta}}{a^2} && (66)\\ &= -\frac{2}{a^2} && (67) \end{aligned}$$

since $g^{\alpha\beta} g_{\alpha\beta} = 2$ in Klein's geometry.

(d) Answer each of the following questions in one or two sentences.

i. In what fundamental way does Klein's geometry differ from a two-sphere?

ii. The hyperbola $x^2 - y^2 = 1$ is rotated around the y-axis to form a three-dimensional hyperboloid of revolution. Does it possess positive or negative curvature? Justify your answer physically with a diagram; do not attempt to calculate anything.

iii. The hyperbola $x^2 - y^2 = 1$ is now rotated around the x-axis. What is the sign of the curvature this time? Why?

iv. Setting aside their dimensionality, in what fundamental way do the hyperboloids of revolution in parts (d)(ii) and (d)(iii) differ from Klein's geometry? Justify your answer in words; don't try to calculate anything.

v. Identify a spacetime manifold, that resembles Klein's geometry. Don't worry too much about the precise mathematical meaning of "resembles", a qualitative justification is fine.

(e) Consider the triangle ΔABC, whose sides are "straight lines" (geodesics) joining the points A$(0, 0)$, B$(b, 0)$, and C$(0, b)$, with $b < 1$. It is easy to show (you don't need to!) that the sides AB and AC are just the curves $v = 0$ and $u = 0$ respectively.

i. What is the equation of the geodesic joining B and C?

ii. Prove that the sum of the interior angles of ΔABC is

$$\Sigma = \angle \mathrm{ABC} + \angle \mathrm{BCA} + \angle \mathrm{CAB} = \frac{\pi}{2} + 2\cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right). \tag{68}$$

The sum of the angles is less than 180 degrees!

iii. Triangles in Klein's geometry can have $\sum = 0$! Without proof, sketch what such a triangle might look like. Your sketch by necessity will be an incomplete representation; there is no way to draw a Klein triangle faithfully on a flat page.

(f) i. Write down a closed form expression for the area A of ΔABC as an integral over a subset of the (u, v) domain.

ii. By changing variables to $y = v + u$ and $z = v - u$, recast your integral in the form

$$A = 2a^2 \int_0^b \frac{dy\ y}{(2 - y^2)\sqrt{1 - y^2}}. \tag{69}$$

Hence show that one has

$$A = a^2(\pi - \Sigma). \tag{70}$$

iii. Explain briefly, in one or two sentences, why (70) guarantees the *nonexistence* of similar triangles in Klein's geometry.

(g) A vector $\vec{W}$ with equal components W^1 and W^2 at the point $A(0,0)$ is parallel transported along the geodesic AB. Show that its components, when it reaches the point B$(b,0)$, are in the ratio

$$\frac{W^1}{W^2} = (1-b^2)^{1/2} \tag{71}$$