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PHYC90012 General Relativity

Assignment 3

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3 Playing tennis in a four-dimensional "brane world"

Projectile motion in a constant gravitational field is a classic introductory problem in Newtonian mechanics. One way to make a uniform gravitational field is to assemble an infinite, flat sheet of matter. If the mass per unit area in the $x - y$ plane is σ, then the Newtonian gravitational potential is given by

$$\Phi_{\text{Newton}} = 2\pi G\sigma|z|, \tag{1}$$

where z is the Cartesian coordinate normal to the sheet. [Convince yourself that (1) is true by analogy with electrostatics or otherwise.] Below we ask whether it is possible to recreate this system in the strong-gravity regime in general relativity.

(a) By appealing to symmetry, argue that the most general metric of a plane-parallel, four-dimensional spacetime takes the form

$$ds^2 = -e^{2\Phi(z)}dt^2 + e^{2\Psi(z)}(dx^2 + dy^2) + dz^2, \tag{2}$$

where $\Phi(t, z)$ (not necessarily the same as Φ_{Newton}), $\Psi(t, z)$, and $\Lambda(t, z)$ are functions determined by Einstein's field equations. determined by Einstein's field equations.

An infinite flat sheet in the $x-y$ plane is is symmetric under interchange of x and y, so $g_{xx} = g_{yy}$. By symmetry of the sheet, under the transformations $x \to -x$, $y \to -y$, $z \to -z$, $t \to -t$ the spacetime is unchanged $\Rightarrow g_{\alpha\beta} = -g_{\beta\alpha} = 0$ (for $\alpha \neq \beta$).

Recall that the spacetime interval ds^2 can be written as

$$ds^2 = g_{00}dt^2 + g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2 \tag{3}$$

so that we have

$$ds^2 = -e^{2\Phi(z,t)}dt^2 + e^{2\Psi(z,t)}(dx^2 + dy^2) + e^{2\Lambda(z,t)}dz^2 \tag{4}$$

where the functions Φ, Ψ, Λ depend on z, t, and $g_{11}, g_{22} > 0, g_{00} < 0$ everywhere.

(b) i. If the spacetime is static (caution: such a solution may not necessarily exist), construct explicitly a coordinate transformation, that puts (2) into the form

$$ds^2 = -e^{2\Phi(z)}dt^2 + e^{2\Psi(z)}(dx^2 + dy^2) + dz^2, \tag{5}$$

where t, x, y, and z are suitably relabelled from (2).

A static spacetime has no time dependence, so we can rewrite the functions independent of time, i.e.

$$\begin{aligned}
e^{\Phi(z)}dt' &= e^{\Phi(z,t)}dt \\
e^{\Psi(z)}dx' &= e^{\Psi(z,t)}dx \\
e^{\Psi(z)}dy' &= e^{\Psi(z,t)}dy \\
dx' &= e^{\Lambda(z,t)}dz
\end{aligned}$$

Assuming the functions Φ, Ψ can be re-expressed in a time independent way, we will only need the transformation $dz' = e^{\Lambda(z,t)}dz$.

This leaves us with

$$ds^2 = -e^{2\Phi(z)}dt^2 + e^{2\Psi(z)}(dx^2 + dy^2) + dz^2 \tag{6}$$

as required.

(b) ii. Explain why it is impossible to get rid of the $e^{2\Lambda(r)}$ factor multiplying dr^2 in the "standard" Schwarzschild metric in the same way, such that the metric contains only one undetermined function $e^{2\Phi(r)}$.

Using the Schwarzchild metric $dr' = \left(1 - \frac{2M}{r}\right)^{-1/2} dr$, we find

$$r' = \sqrt{r^2 - 2Mr} + M\log\left(\sqrt{r^2 - 2Mr} + r - m\right) \tag{7}$$

At $r = 0$ the second term becomes $M\log(-m)$, which is undefined since $m > 0$.

(c) Nine of the Christoffel symbols associated with (5) are nonzero. Calculate them. Here and henceforth, please feel free to use a symbolic algebra package like *Mathematica* to make your life easier.

We can use the expression for the Christoffel symbols in terms of the metric,

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right) \tag{8}$$

where we have the metric components $g_{\alpha\alpha}$ as given by 6) (with all non-diagonal terms zero); i.e.

$$g_{\alpha\beta} = \mathrm{diag}\left(-e^{2\Phi(z)}, e^{2\Psi(z)}, e^{2\Psi(z)}, 1\right) \tag{9}$$

Now, using a mathematical calculator (specifically an *TI-89 Titanium* calculator) to make our lives easier, we calculate the Christoffel symbols and find the non-zero terms as

$$\Gamma^1_{14} = \Gamma^1_{41} = \Phi'(z) \tag{10}$$
$$\Gamma^2_{42} = \Gamma^2_{24} = \Psi'(z) \tag{11}$$
$$\Gamma^3_{43} = \Gamma^3_{34} = \Psi'(z) \tag{12}$$
$$\Gamma^4_{11} = e^{2\Phi}\Phi' \tag{13}$$
$$\Gamma^4_{22} = -e^{2\Psi}\Psi' \tag{14}$$
$$\Gamma^4_{33} = -e^{2\Psi}\Psi' \tag{15}$$

(d) Show that the nonzero components of the Ricci tensor $R_{\mu\nu}$ are

$$\begin{aligned} R_{tt} &= e^{2\Phi}[(\Phi')^2 + 2\Phi'\Psi' + \Phi''], && (16)\\ R_{xx} &= -e^{2\Phi}[2(\Psi')^2 + \Psi'\Phi' + \Psi''] && (17)\\ R_{yy} &= R_{xx}, && (18)\\ R_{zz} &= -(\Phi')^2 - 2(\Psi')^2 - \Phi'' - 2\Psi''. && (19) \end{aligned}$$

Primes denote derivatives with respect to z.

We recall the Ricci tensor $R_{\alpha\beta}$ is a contraction on the first and third indices of the Riemann tensor, written as

$$R_{\alpha\beta} = R^{i}{}_{\alpha i\beta} \tag{20}$$

We also recall that the Riemann tensor in the form $R^{\alpha}_{\beta\mu\nu}$ can be written as

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma} \tag{21}$$

We calculate the components of the Ricci tensor using our calculator, and find the non-zero components to be

$$\begin{aligned} R_{tt} &= e^{2\Phi}[(\Phi')^2 + 2\Phi'\Psi' + \Phi''], && (22)\\ R_{xx} &= -e^{2\Phi}[2(\Psi')^2 + \Psi'\Phi' + \Psi''] && (23)\\ R_{yy} &= R_{xx}, && (24)\\ R_{zz} &= -(\Phi')^2 - 2(\Psi')^2 - \Phi'' - 2\Psi'', && (25) \end{aligned}$$

as required.

(e) Show that the nonzero contravariant components of Einstein's field equations with cosmological constant Λ and stress-energy tensor $T_{\mu\nu}$ are

$$\begin{aligned} 8\pi T^{tt} &= -e^{-2\Phi}[3(\Psi')^2 + 2\Psi'' + \Lambda], && (26)\\ 8\pi T^{xx} &= e^{-2\Psi}[(\Phi')^2 + \Phi'\Psi' + (\Psi')^2 + \Phi'' + \Psi'' + \Lambda], && (27)\\ 8\pi T^{yy} &= 8\pi T^{xx}, && (28)\\ 8\pi T^{zz} &= \Psi'(2\Phi' + \Psi') + \Lambda. && (29) \end{aligned}$$

You can do this with pen and paper but will find *Mathematica* more soothing.

We know

$$\begin{aligned} 8\pi T_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda && (30)\\ \Rightarrow 8\pi T^{\alpha\beta} &= g^{\alpha\mu}g^{\beta\nu}8\pi T_{\mu\nu} && (31) \end{aligned}$$

From this, we can calculate the values of $8\pi T^{\alpha\alpha}$ using our calculator, with the results

$$8\pi T^{tt} = -e^{-2\Phi}[3(\Psi')^2 + 2\Psi'' + \Lambda], \tag{32}$$
$$8\pi T^{xx} = e^{-2\Psi}[(\Phi')^2 + \Phi'\Psi' + (\Psi')^2 + \Phi'' + \Psi'' + \Lambda], \tag{33}$$
$$8\pi T^{yy} = 8\pi T^{xx}, \tag{34}$$
$$8\pi T^{zz} = \Psi'(2\Phi' + \Psi') + \Lambda, \tag{35}$$

as required.

> **(f)** Einstein's field equations place strong constraints on the physical form of the stress-energy consistent with the metric (5).

> **i.** Explain why it is impossible to generate (5) with a sheet of cold matter (dust).

Starting with

$$8\pi T^{tt} = -e^{-2\Phi}\left(3(\Psi')^2 + 2\Psi'' + \Lambda\right) \tag{36}$$

we let $T^{tt} = \delta(z)$ (as we are confined to the sheet $z = 0$). As the LHS $\propto \delta(z)$, RHS $\propto \delta(z)$.

As a first derivative cannot be a delta-function (proof not shown), and as $\Lambda = 0$, we see

$$\Phi'' \propto \delta(z) \tag{37}$$

We also have

$$8\pi T^{xx} = 0 = \Phi'^2 + \Phi'\Psi' + \Psi'^2 + \Phi'' + \Psi'' + \Lambda \tag{38}$$
$$8\pi T^{zz} = 0 = \Psi'(2\Phi' + \Psi') + \Lambda \tag{39}$$

Subtracting (39) from (38) we find

$$\Phi'' = -\Psi'' \tag{40}$$

which implies that Φ'' must be a delta-function.

Similarly, subtracting (38) from (39) we find

$$0 = (\Phi')^2 - \Phi'\Psi' \tag{41}$$
$$\therefore \Psi' = 0 \text{ or } \Phi' = \Psi' \tag{42}$$

However, both of these possibilities contradict with our previous findings! Hence we cannot generate (5) with a sheet of cold matter.

> **ii.** By thinking about a suitable initial value problem qualitatively, speculate why this seemingly natural system - a static dust sheet - is impossible to assemble "from the ground up" in general relativity, even though there is no obstacle in a Newtonian context. What might "go wrong"? Do not try calculating anything, unless you are in the mood to win a Nobel Prize.

For each individual particle we add, a gravitational or electric force would act on the other particles currently in the "sheet" causing a distortion to the structure. This means that a flat sheet structure would be impossible to form!

(g) Consider the special case $\Psi = 0$, i.e. warped time, Euclidean space, no preferential warping of space in the z direction relative to the x and y directions.

i. In the bulk ($z = 0$), where there is zero stress-energy, show that $\Lambda = 0$ must hold to obtain a self-consistent solution of the form (5).

With zero stress-energy, we have

$$8\pi T^{tt} = 0 = \Psi'(2\Phi' + \Psi) + \Lambda = \Lambda \tag{43}$$

since $\Psi = 0 \Rightarrow \Psi' = 0$. Hence we see

$$\Lambda = 0 \tag{44}$$

as required.

ii. Hence show that the stress-energy must vanish on the sheet $z = 0$ too.

At $z = 0$, there will be no mass as $\Lambda = 0, \Psi = 0$. Hence no invariant mass $E_m \Rightarrow$ no energy density $\Rightarrow T^{tt} = 0$.

iii. Prove that (5) reduces to the Rindler metric for a uniformly accelerated frame. What is the implied acceleration?

In the special case $\Phi = 0$, a similar bulk-then-sheet approach yields the Minkowski metric.

We start with

$$ds^2 = -e^{2\Phi(z)}dt^2 + dx^2 + dy^2 + dz^2 \tag{45}$$

For small z, we can Taylor expand, so that we have

$$ds^2 = -\left(1 + 2\Phi'(z)\right) dt^2 + dx^2 + dy^2 + dz^2 \tag{46}$$

This is of the form of an accelerating frame with acceleration $2\Phi'(z)$!

(h) Consider the special case $\Psi = \Phi$. This corresponds to preferentially warping space in z relative to x and y, i.e. the "symmetric" coordinates (t, x, y) are treated as a Minkowski subspace with warping in the z coordinate. The system is the four dimensional analogue of the five-dimensional *Randall-Sundrum 1-brane* spacetime, which revolutionized the study of brane worlds

i. In the bulk, show that one obtains

$$\Phi = g|z|, \tag{47}$$

if $\Phi(z)$ vanishes at $z = 0$ without loss of generality. In (47), g is an integration constant with units of acceleration.

$$\begin{aligned} \Phi = \Psi \Rightarrow 0 &= 3(\Phi') + \Lambda && (48)\\ \Rightarrow \frac{d\Phi}{dz} &= \pm\sqrt{-\frac{\Lambda}{3}} && (49)\\ \Rightarrow \Phi &= \pm\sqrt{-\frac{\Lambda}{3}}|z|, \quad \text{as z is symmetric} && (50)\\ &= g|z| && (51) \end{aligned}$$

as required.

ii. Prove that g satisfies the fine-tuning condition

$$\Lambda = -3g^2. \tag{52}$$

Comment on the physical significance of the sign of Λ.

$$g = \pm\sqrt{-\frac{\Lambda}{3}} \therefore \Lambda = -3g^2 \tag{53}$$

Positive Λ gives an imaginary g, which is not physical. $\Rightarrow \Lambda$ is negative. Physically, this means that space contracts!

iii. Assume that the stress-energy is confined to the sheet, i.e. $T^{\mu\nu} \propto \delta(z)$. Prove that the stress-energy tensor takes the physically unusual form

$$T^{\mu\nu} = (2\pi)^{-1} g\delta(z)\ \mathrm{diag}(-1, 1, 1, 0). \tag{54}$$

Equation (54) implies fine tuning between $T^{\mu\nu}$ and Λ via g.

We have

$$\Phi = g|z| \Rightarrow \Phi'' = 2g\delta(z) \tag{55}$$

since

$$\begin{aligned} \frac{d\Phi}{dz} &= \mathrm{sgn}(z) && (56)\\ \frac{d}{dz}\mathrm{sgn}(z) &= z\delta(z) && (57) \end{aligned}$$

We also find that

$$3(\Phi')^2 = -\Lambda \tag{58}$$
$$\Rightarrow 8\pi T^{tt} = -e^{2g|z|}(4\Phi'') \tag{59}$$
$$\Rightarrow Ttt = -e^{2g|z|}(2\pi)^{-1}\delta(z) \tag{60}$$
$$8\pi T^{xx} = e^{2g|z|}(4\Phi'') \tag{61}$$
$$T^{xx} = (2\pi)^{-1}e^{2g|z|}\delta(z) \tag{62}$$
$$T^{yy} = T^{xx} \tag{63}$$
$$8\pi T^{zz} = 0 \tag{64}$$
$$\Rightarrow T^{zz} = 0 \tag{65}$$

which leaves us with

$$T^{\mu\nu} = (2\pi)^{-1}g\ \delta(z)\text{diag}(-1,1,1,0), \tag{66}$$

as required.

iv. Interpret physically the two cases $g < 0$ and $g > 0$.

In the case of $g < 0$, there is positive energy density, so the normal stress (pressure) is negative.

In the case of $g > 0$, there is negative energy density, so there normal stress (pressure) is positive.

v. Interpret physically the signs of T^{tt} and $T^{xx} = T^{yy}$ for $g > 0$.

For $g > 0$, T^{tt} is negative, $T^{xx} = T^{yy}$ is positive.

vi. Explain physically why $T^{zz} = 0$ makes sense in terms of a plausible microscopic ("particulate") model of the sheet.

T^{zz} is pressure in the $z-$direction. We note that pressure leads to motion in z of the sheet; this motion would cause the sheet to buckle. However, our sheet is flat which implies there must not be any pressure in $z \Rightarrow T^{zz} = 0$.

vii. Prove that it is impossible to build the sheet out of a perfect fluid or dust.

For a perfect fluid the stress-energy tensor is given by (see *Schutz* (4.36))

$$T^{\mu\nu} = \text{diag}(\rho, p, p, p) \tag{67}$$

However, our $T^{\mu\nu}$ is confined to the sheet,

$$T^{\mu\nu} \propto \text{diag}(-1,1,1,0) \tag{68}$$

In the perfect fluid (67), we see the last two elements of the tensor are identical. However, on the sheet (68) we see the last two elements are different ($1 \neq 0$) $\Rightarrow$ we cannot build a sheet out of a perfect fluid!

(i) A tennis player standing on the sheet hits a tennis ball into the $z > 0$ half-space with initial 4-velocity $u(0)$.

i. Starting from the geodesic equations in the form

$$\frac{du_\nu}{d\tau} = \frac{1}{2} g_{\alpha\beta,\nu} u^\alpha u^\beta, \tag{69}$$

identify three constants of the motion. In (69), τ denotes the proper time as measured in the ball's momentarily comoving reference frame.

The geodesic equation is given by

$$\frac{du_\nu}{d\tau} = \frac{1}{2} g_{\alpha\beta,\nu} u^\alpha u^\beta \tag{70}$$

where we have the metric $g_{\alpha\beta} = \text{diag}(-e^{2gz}, e^{2gz}, e^{2gz}, 1)$. The metric is dependent only for z, so we find

$$\frac{du_\nu}{d\tau} = 0 \quad \text{for } \nu = t, x, y \tag{71}$$

Hence u_t, u_x, u_y are constants of the motion.

ii. From the z component of (69), show that

$$\frac{du^z}{d\tau} = -g[1 + (u^z)^2]. \tag{72}$$

What is unusual physically about this acceleration?

Starting from the geodesic equation we have

$$\frac{du_z}{d\tau} = ge^{2gz}\left(-(u^t)^2 + (u^x)^2 + (u^y)^2\right) \Rightarrow \frac{du^z}{d\tau} = ge^{2gz}\left(-(u^t)^2 + (u^x)^2 + (u^y)^2\right) \tag{73}$$

where we raise the index ($u_z \to u^z$) by multiplying both sides by $g^{zz} = 1$.

As $\vec{u} \cdot \vec{u} = -1$, we find

$$e^{2gz}\left(-(u^t)^2 + (u^x)^2 + (u^y)^2\right) = -1 \tag{74}$$

$$\Rightarrow e^{2gz}\left(-(u^t)^2 + (u^x)^2 + (u^y)^2\right) = -\left(1 + (u^z)^2\right) \tag{75}$$

Hence we see

$$\frac{du^z}{d\tau} = -g\left[1 + (u^z)\right]^2 \tag{76}$$

as required!

We note that the acceleration in z depends on the velocity of the ball. In our own experience if we throw a ball up, we would expect the the acceleration on the ball to be a constant due to gravity. In this case, however, we see that this acceleration is dependent on the velocity! This is quite unsual.

iii. Similarly or otherwise, prove the equivalent result

$$\left(\frac{dz}{d\tau}\right)^2 = -1 + e^{-2gz}\{1 + [u^z(0)]^2\}. \tag{77}$$

We start with

$$\vec{u}\cdot\vec{u} = -1 \Rightarrow e^{-2gz}\left(-(u_t)^2 + (u_x)^2 + (u_y)^2\right) + (u_z)^2 = -1 \tag{78}$$

Using the initial condition $z = 0$ at $\tau = 0$, and the fact that u_x, u_y, u_t are constants of the motion:

$$-\left(u_t(0)\right)^2 + \left(u_x(0)\right)^2 + \left(u_y(0)\right)^2 + \left(u_z(0)\right)^2 = -1 \tag{79}$$

$$\Rightarrow e^{4gz}\left[\left(u^t(0)\right)^2 + \left(u^x(0)\right)^2 + \left(u^y\right)^2\right] = -\left[1 + \left(u^z(0)\right)^2\right] \tag{80}$$

$$\Rightarrow e^{2gz}\left[\left(u^t\right)^2 + \left(u^x\right)^2 + \left(u^y\right)^2\right] = -e^{-2gz}\left[1 + \left(u^z(0)\right)^2\right] \tag{81}$$

as u^t, u^x, u^y are constants of motion (e.g. $u^t = u^t(0)$).

Also, we note

$$\vec{u}\cdot\vec{u} = -1 \Rightarrow e^{2gz}\left(-(u^t)^2 + (u^x)^2 + (u^y)^2\right) = -\left(1 + (u^z)^2\right) \tag{82}$$

Equating the LHS's of (81) and (82) we find

$$-e^{-2gz}\left[1 + \left(u^z(0)\right)^2\right] = -\left(1 + (u^z)^2\right) \tag{83}$$

$$\Rightarrow \left(\frac{dz}{d\tau}\right)^2 = -1 + e^{-2gz}\left[1 + \left(u^z(0)\right)^2\right] \tag{84}$$

as required.

iv. Solve either (72) or (77) to obtain

$$z = g^{-1}\ln[\cos g\tau + u^z(0)\sin g\tau]. \tag{85}$$

Please note that there are many valid ways to prove (72)-(85).

We recall from (77) that

$$\frac{du^z}{d\tau} = -g\left(1 + (u^z)^2\right) \tag{86}$$

$$\Rightarrow \int \frac{du^z}{(1 + (u^z)^2)} = \int -g\ d\tau \tag{87}$$

$$\tan^{-1}(u^z) = -g\tau + c_1 \tag{88}$$

where c_1 is some constant.

$$u^z = \tan(-g\tau + c_1) \tag{89}$$

We can use the initial condition $z = 0, \tau = 0$ to determine the value of the constant

$$u^z(0) = \tan(c) \tag{90}$$

$$\Rightarrow c = \tan^{-1}\left(u^z(0)\right) \tag{91}$$

We shall substitute this value in later (leaving it as c_1 for now, for convenience.

$$u^z = \frac{dz}{dt} \tag{92}$$

$$\Rightarrow \int dz = \int d\tau \;\; \tan(-g\tau + c_1) \tag{93}$$

Evaluating this integral, we can now calculate z.

$$z = \frac{1}{g}\log\left[\cos(c_1 - g\tau)\right] + c_2 \tag{94}$$

where c_2 is some constant,

$$= \frac{1}{g}\log\left[\cos\Big(\tan^{-1}\left(u^z(0) - g\tau\right)\Big)\right] \tag{95}$$

$$= \frac{1}{g}\log\left[\cos\Big(\tan^{-1}\left(u^z(0)\right)\Big)\cos(g\tau) + \sin\Big(\tan^{-1}\left(u^z(0)\right)\Big)\sin(g\tau)\right] + c_2 \tag{96}$$

where we have used the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. We now also recall that

$$\cos\left[\tan^{-1}(x)\right] = \frac{1}{\sqrt{1+x^2}} \tag{97}$$

$$\sin\left[\tan^{-1}(x)\right] = \frac{x}{\sqrt{1+x^2}} \tag{98}$$

$$\Rightarrow z = \frac{1}{g}\log\left(\frac{\cos(g\tau) + u^z(0)\sin(g\tau)}{\sqrt{1 + (u^z(0))^2}}\right) + c_2 \tag{99}$$

$$= \frac{1}{g}\log\left[\cos(g\tau) + u^z(0)\sin(g\tau)\right] - \frac{1}{g}\log\left[\sqrt{1 + (u^z(0))^2}\right] + c_2 \tag{100}$$

We can determine c_2 using the initial conditions $z = 0, \tau = 0$,

$$\Rightarrow 0 = \underbrace{\frac{1}{g}\log(1)}_{=0} + \frac{1}{\log\left[\sqrt{1 + (u^z(0))^2}\right]} + c_2 \tag{101}$$

$$\Rightarrow c_2 = \frac{1}{g}\log\left[\sqrt{1 + (U^z(0))^2}\right] \tag{102}$$

Hence we conclude

$$z = g^{-1}\log[\cos g\tau + u^z(0)\sin g\tau]. \tag{103}$$

as required.

(j) Show that the ball reaches the apex of its trajectory at proper time

$$\tau_{\text{top}} = g^{-1} \tan^{-1} u^z(0), \tag{104}$$

and that the associated coordinate time is

$$t_{\text{top}} = \frac{u^z(0)\{1 + [u^x(0)]^2 + [u^z(0)]^2\}^{1/2}}{g\{1 + [u^z(0)]^2\}}, \tag{105}$$

in the coordinates in which (5) is written. Interestingly, τ_{top} is finite, no matter how hard the ball is struck initially. Does the same hold true for t_{top}?

We know there will be a maximum (or minimum) in z at $\frac{dz}{d\tau} = 0$; hence we can solve for τ at this point by

$$\frac{dz}{d\tau} = \tan\left[-g\tau + \tan^{-1}(u^z(0))\right] \tag{106}$$

$$\Rightarrow 0 = \tan\left[-g\tau_{\text{top}} + \tan^{-1}(u^z(0))\right] \tag{107}$$

$$g\tau_{\text{top}} = \tan^{-1}\left[u^z(0)\right] \tag{108}$$

$$\Rightarrow \tau_{\text{top}} = g^{-1}\tan^{-1}\left[u^z(0)\right] \tag{109}$$

as required. [1]

We have

$$\frac{du_t}{d\tau} = 0 \Rightarrow u_t = c \tag{110}$$

where c is some constant. By initial conditions $z = 0, \tau = 0$ we see

$$c = U_t(0) \tag{111}$$

Starting from u_t, we can calculate

$$u_t = g_{tt}u^t = -e^{2gz}u^t = c \tag{112}$$

$$\Rightarrow u^t = -e^{-2gz} \cdot c \tag{113}$$

$$\Rightarrow \frac{dt}{dz} = -u_t(0)e^{-2gz} \tag{114}$$

$$dt = -d\tau \cdot u_t(0)e^{-2gz} \tag{115}$$

$$= -d\tau \cdot u_t(0)e^{-2g\left\{g^{-1}\log[\cos(g\tau)+u^z(0)\sin(g\tau)]\right\}} \tag{116}$$

$$= -d\tau \cdot u_t(0)e^{\log\left\{[\cos(g\tau)+u^z(0)\sin(g\tau)]^{-2}\right\}} \tag{117}$$

$$= -d\tau \cdot u_t(0)\left[\cos(g\tau) + u^z(0)\sin(g\tau)\right]^{-2} \tag{118}$$

$$= -d\tau \cdot u_t(0)\left(1 + (u^z(0))^2\right)^{-1}\cos\left[-g\tau + \tan^{-1}(u^z(0))\right]^{-2} \tag{119}$$

where we have performed the opposite steps from (96).

$$\Rightarrow t = g^{-1}u_t(0)\left(1 + (U^z(0))^2\right)^{-1}\tan\left(-g\tau + \tan^{-1}(u^z(0))\right) + d \tag{120}$$

[1] We have not shown that z takes a maximum (rather than a minimum) at this τ, however it would not be be difficult to show this (e.g. by checking the sign of $\frac{d^2z}{d\tau^2}$).

We can determine the value of our constant d by considering the initial condition $t = 0, \tau = 0$

$$\Rightarrow d = -g^{-1} u_t(0) \left(1 + (u^z(0))^2\right)^{-1} u^z(0) \tag{121}$$

Since t_{top} will occur when $\tau = \tau_{\text{top}} = g^{-1} \tan^{-1}(u^z(0))$,

$$t_{\text{top}} = -g^{-1} u_t(0) \left(1 + (u^z(0))^2\right)^{-1} u^z(0) \tag{122}$$

Using $U_t(0) = u^t(0)$ and $-1 = \vec{u} \cdot \vec{u}$, we can see

$$U^t(0) = \pm\sqrt{1 + (u^x(0))^2 + (u^y(0))^2 + (u^z(0))^2} \tag{123}$$

Taking the negative square root gives us the expression for t_{top}

$$t_{\text{top}} = \frac{u^z(0)\{1 + [u^x(0)]^2 + [u^z(0)]^2\}^{1/2}}{g\{1 + [u^z(0)]^2\}}, \tag{124}$$

Since we can choose how fast we throw the ball, this can be infinite!

(k) Show that the horizontal distance traversed by the ball from its launch point to where it lands on the sheet, as measured in the coordinates associated with (5), is given by

$$x_{\text{land}} = \frac{2u^x(0)u^z(0)}{g\{1 + [u^z(0)]^2\}}. \tag{125}$$

How does (125) differ qualitatively from Newtonian projectile motion? Should we expect x_{land} to equal the x-component of the initial 3-velocity multiplied by $2t_{\text{top}}$, since the metric does not depend on x?

Starting from

$$\frac{du_x}{d\tau} = 0 \tag{126}$$

we find that u_x is constant; i.e.

$$u_x = c \tag{127}$$

From initial conditions, we have

$$c = u_x(0) \tag{128}$$

We then calculate

$$u_x = g_{xx} u^x = e^{2gz} u^x \tag{129}$$

$$u^x = e^{-2gz} u_x(0) \tag{130}$$

$$\frac{dx}{d\tau} = u_x(0) \left[\cos(g\tau) + u^z(0)\sin(g\tau)\right] \tag{131}$$

$$\Rightarrow x = -g^{-1} u^x(0) \left[1 + u^z(0)\right] \tan\left[-g\tau + \tan^{-1} u^z(0)\right] + d \tag{132}$$

similarly to what we have done previously (note, d is some constant).

Our initial conditions of $x = 0, \tau = 0$ allow us to calculate d as

$$d = g^{-1} U^x(0) \left[1 + (u^z(0))^2\right]^{-1} u^z(0) \tag{133}$$

We realise that x_{land} will occur at τ_{land}. We assume that $\tau_{\text{land}} = 2\tau_{\text{top}}$, but in any case we shall prove this below.

$$\begin{aligned}
z &= g^{-1} \log\left[\cos(g\tau) + u^z(0)\sin(g\tau)\right] && (134)\\
\Rightarrow 0 &= g^{-1} \log\left[\sqrt{1 + (u^z(0))^2}\cos(-g\tau_{\text{land}} + \tan^{-1}(u^z(0)))\right] && (135)\\
\Rightarrow \frac{1}{\sqrt{1 + (u^z(0))^2}} &= \cos\left(-g\tau_{\text{land}} + \tan^{-1}(u^z(0))\right) && (136)\\
\cos^{-1}\left(\frac{1}{\sqrt{1 + (u^z(0))^2}}\right) &= -g\tau_{\text{top}} + \tan^{-1}(u^z(0)) && (137)
\end{aligned}$$

Using the relation $\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \tan^{-1}(x)$, we have

$$\begin{aligned}
\tan^{-1}(u^z(0)) &= -g\tau_{\text{land}} + \tan^{-1}\left(U^z(0)\right) && (138)\\
\Rightarrow \tau_{\text{land}} &= 2g^{-1}\tan^{-1}\left(U^z(0)\right) && (139)
\end{aligned}$$

which is as we expected,

$$\tau_{\text{land}} = 2\tau_{\text{top}} \tag{140}$$

Using this value of τ, we can determine $x_{\max}$.

$$\begin{aligned}
x_{\max} &= -g^{-1}u^x(0)\left(1 + (u^z(0))^2\right)^{-1}\tan\left[-2\tan^{-1}(u^z(0)) + \tan^{-1}(u^z(0))\right] + c && (141)\\
&= g^{-1}u^x(0)\left[1 + (u^z(0))^2\right]^{-1}u^z(0) + g^{-1}u^x(0)\left[1 + (u^z(0))^2\right]^{-1}u^z(0) && (142)\\
\Rightarrow x_{\text{land}} &= \frac{2u^x(0)u^z(0)}{g\{1 + [u^z(0)]^2\}} && (143)
\end{aligned}$$

as required.

We should not expect $x_{\text{land}} =$ initial x-velocity $\times\, 2t_{\text{top}}$; we see from (143) that for large $u^z(0)$ we have

$$x_{\text{land}} \sim \frac{2u^x(0)}{gu^z(0)} \tag{144}$$

Since we are free to choose $u^x(0)$ and $u^z(0)$ we see that x_{land} can be whatever value we dictate (not just initial x-velocity $\times\, t_{\text{land}}$!).

(l) Finally, suppose that instead of a tennis ball we launch a laser beam into the $z > 0$ half-space with initial 4-momentum $p(0)$ per photon.

i. Let λ be an affine parameter tracing out the light ray. Starting from the geodesic equations or otherwise, solve for the photon's trajectory. You should find

$$z = g^{-1} \ln[1 + p^z(0) g\lambda]. \tag{145}$$

In other words, the photon never falls back to the sheet, no matter how much stress-energy the sheet contains.

Starting from the geodesic equations, we have that p_x, p_y, p_t are constants of the motion. We also consider $\vec{p} \cdot \vec{p}$. That is,

$$e^{2gz} \left[-(p^t)^2 + (p^x)^2 + (p^y)^2 \right] + (p^z)^2 = 0 \tag{146}$$

Now, using the initial condition $z = 0, \tau = 0$, we find

$$\left[-(p_t(0))^2 + (p_x(0))^2 + (p_y(0))^2 \right] + (p_z(0))^2 = 0 \tag{147}$$

$$e^{-2gz} \left[p^z(0) \right]^2 = -e^{2gz} \left[-(p^t)^2 + (p^x)^2 + (p^y)^2 \right] \tag{148}$$

where we use the inverse metric to raise indices. Combining equations (146) and (148) we find

$$\begin{aligned}
(p^z)^2 &= e^{-2gz} \left[p^z(0) \right]^2 && (149)\\
\Rightarrow \frac{dz}{d\lambda} &= e^{-gz} p^z(0) && (150)\\
\int dz \; e^{gz} &= \int d\lambda \; p^z(0) && (151)\\
g^{-1} e^{gz} &= p^z(0)\lambda + c && (152)\\
\Rightarrow e^{gz} &= g\lambda p^z(0) + c && (153)
\end{aligned}$$

where c is some constant. We can determine the value of c by considering the initial conditions $z = 0, \lambda = 0$

$$\begin{aligned}
\Rightarrow c &= 1 && (154)\\
\Rightarrow e^{gz} &= g\lambda p^z(0) + 1 && (155)\\
gz &= \log\left[g\lambda p^z(0) + 1 \right] && (156)\\
\Rightarrow z &= g^{-1} \log\left[1 + g\lambda p^z(0) \right] && (157)
\end{aligned}$$

as required.

ii. Consider an intergalactic variant of the Pound-Rebka experiment, in which we fly horizontally (in the x direction, say) in a rocket at constant speed V [as measured in the coordinates associated with (5)] while maintaining a constant altitude z_{rocket} above the sheet. Show that the laser frequency measured by the stationary emitter $\nu_{\rm em}$, and the frequency measured by the experimentalist in the rocket, $\nu_{\rm rec}$, are related according to

$$\frac{\nu_{\rm rec}}{\nu_{\rm em}} = \frac{e^{-gz_{\rm rocket}}}{(1-V^2)^{1/2}}\left[1 - \frac{Vp^x(0)}{\{[p^x(0)]^2 + [p^z(0)]^2\}^{1/2}}\right]. \tag{158}$$

Most of the physics in this question applies in five dimensions too. It is interesting to think about it in the context of Randall-Sundrum brane worlds.

For a photon, we have the energy $E = h\nu$, and its energy measured by an observer is given by $E = -\vec{p}\cdot\vec{u}$.

For the stationary emitter

$$U = (1,0,0,0) \tag{159}$$
$$\Rightarrow E = p^t e^{2gz} \tag{160}$$
$$= p^t(0) \tag{161}$$

as the emitter is at $z = 0$ (and we recall that p^t is a constant of the motion).

For the receiver, $\vec{u} = (Uut, Vu^t, 0, 0)$, as $\vec{u}\cdot\vec{u} = -1$.

$$-1 = (u^t)^2(-1+V^2)e^{2gz} \tag{162}$$
$$\Rightarrow u^t = \left[(1-V^2)^{-1}e^{-2gz}\right]^{1/2} \tag{163}$$
$$= (1-V^2)^{-1/2}e^{-gz_{\rm rocket}} \tag{164}$$

Therefore $-\vec{p}\cdot\vec{u}$ for the receiver is given by

$$-g_{\alpha\beta}p^\alpha u^\beta = u^t e^{2gz}p^t - u^t e^{2gz}Vp^x \tag{165}$$

So,

$$\frac{\nu_{\rm rec}}{\nu_{\rm emit}} = \frac{E_{\rm rec}}{E_{\rm emit}} = \frac{u^t e^{2gz}(p^t - Vp^x)}{p^t(0)} = \frac{e^{-gz}e^{2gz}(p^t - Vp^x}{(1-V^2)^{1/2}p^t(0)} \tag{166}$$

Also, we see

$$p^t = e^{-2gz}p_t = -e^{-2gt}p_t(0) = e^{-2gz}p^t(0) \tag{167}$$
$$p^x = e^{-2gz}p_x = e^{-2gz}p_x(0) = e^{-2gz}p^x(0) \tag{168}$$

$$\frac{e^{gz}\left[e^{-2gz}p^t(0) - Ve^{-2gz}p^x(0)\right]}{(1-V^2)^{1/2}p^t(0)} = \frac{e^{-gz}}{(1-V^2)^{1/2}} \tag{169}$$

$$-(p^t(0))^2 + (p^x(0))^2 + (p^y(0))^2 + (p^z(0))^2 = 0 \tag{170}$$

as $\vec{p} \cdot \vec{p} = 0$, with $z = \tau = 0$.

$$\Rightarrow p^t(0) = \sqrt{(p^x(0))^2 + (p^y(0))^2 + (p^z(0))^2} \tag{171}$$

$$\Rightarrow \frac{\nu_{\text{rec}}}{\nu_{\text{emit}}} = \frac{e^{-gz}}{(1-V^2)^{1/2}} \left(1 - \frac{Vp^x(0)}{\sqrt{(p^x(0))^2 + (p^y(0))^2 + (p^z(0))^2}}\right) \tag{172}$$

$$= \frac{e^{-gz}}{(1-V^2)^{1/2}} \left(1 - \frac{Vp^x(0)}{\sqrt{(p^x(0))^2 + (p^z(0))^2}}\right) \tag{173}$$

where because of symmetry, we have rescaled $p^{x'}(0)$ similar to t_{top} in Question (j).

This is the result as required!