General Relativity (Masters)
Assignment Four

1. **Detecting gravitational waves with lasers.** A Michelson laser interferometer lies in the plane $z = 0$. [1] Its arms are aligned with the x- and y-axes and have the same length, L, in the absence of a gravitational wave. A weak gravitational wave is incident normally on the interferometer. In the transverse, traceless gauge, the perturbation to the background Minkowski metric $\eta_{\mu\nu}$ is given by the plane-wave form

$$h_{\mu\nu} = H_{\mu\nu} \exp(iKz - i\Omega t) \ , \tag{1}$$

with

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2}$$

In this question, we assume that the gravitational wave consists purely of the plus polarisation, i.e. $h_\times = 0$.

(a) Traditionally we analyse the phase shift in a gravitational wave interferometer using ray optics, i.e. by considering photon geodesics.

i. In lectures we showed that the spatial coordinates of the interferometer's mirrors remain constant in the transverse, traceless gauge. Nevertheless the interferometer measures a phase shift between its two arms, when a gravitational wave interacts with the system. Explain in words why there is no contradiction.

ii. Let $[t(\lambda), x(\lambda), 0, 0]$ be the trajectory of a photon travelling along the x-axis, paramatrised by an affine parameter λ. Show that, along the trajectory, one has

$$\frac{dt}{dx} = \pm \left[1 + \frac{h_+}{2} \exp(-i\Omega t) \right] \tag{3}$$

to first order in the small dimensionless quantity h_+.

iii. Show that the round-trip travel time along the x-axis is

$$\Delta t = 2L + \frac{h_+}{\Omega} \sin(\Omega L) \exp(-i\Omega L) \tag{4}$$

to first order in h_+. Identify and justify any approximations you make.

iv. Without repeating the calculation, explain why the wave-induced perturbation to the round-trip travel time along the other arm (i.e. for photons travelling along the y-axis) is equal but opposite to the result in (4). Hence the total phase shift observed is given by

$$\Delta\phi = 2\omega(\Delta t - 2L) = \frac{2h_+\omega}{\Omega} \sin(\Omega L) \exp(-i\Omega L) \ , \tag{5}$$

where ω is the photon frequency.

[1] Here (t, x, y, z) are Minkowski coordinates describing the background spacetime in the absence of a gravitational wave.

(b) We now analyse the phase shift using physical optics, i.e. by solving for the electromagnetic fields of the laser in the curved spacetime of the gravitational wave. If we are "lucky", [2] we should get the same answer for $\Delta\phi$!

i. The inhomogeneous Maxwell equations (Gauss, Ampère) in vacuum take the exact form

$$F^{\mu\nu}{}_{,\nu} = 0 \tag{6}$$

in flat spacetime, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor, and $\vec{A}$ is the electromagnetic potential. Argue carefully that, in the curved spacetime of the gravitational wave, equation (6) takes the exact form

$$F^{\mu\nu}{}_{;\nu} = 0 \tag{7}$$

and hence that we have

$$F^{\mu\nu}{}_{,\nu} = -\Gamma^{\mu}{}_{\lambda\nu} F^{\lambda\nu} - \Gamma^{\nu}{}_{\lambda\nu} F^{\mu\lambda} \ , \tag{8}$$

where $\Gamma^{\mu}{}_{\lambda\nu}$ is a Christoffel symbol.

ii. Show that (8) reduces to

$$F^{\mu\nu}{}_{,\nu} = 0 \tag{9}$$

to first order in h_+. You may find it useful to apply the identity

$$\Gamma^{\nu}{}_{\lambda\nu} = \frac{(\sqrt{-g})_{,\lambda}}{\sqrt{-g}} \ , \tag{10}$$

where g is the metric determinant, but there are many other valid approaches as well.

iii. In flat space, equation (9) reduces to $\hat{D}A^\mu = 0$ in the electromagnetic Lorenz gauge, where $\hat{D} = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ symbolises the d'Alembertian (wave) operator. In the curved space of the gravitational wave, show that (9) reduces to

$$\hat{D}A^\mu = -h^{\mu\alpha}{}_{,\nu} F_\alpha{}^\nu - h^{\nu\alpha}{}_{,\nu} F^\mu{}_\alpha + h^{\mu\alpha} F_{\alpha\nu}{}^{,\nu} + h^{\nu\alpha} F^\mu{}_{\alpha,\nu} \tag{11}$$

up to first order in h_+, where all indices are raised and lowered with the Minkowski metric in (11) and henceforth.

(c) Consider a classical laser field propagating along the x-axis and polarised linearly along the z-axis, i.e. $\vec{A} = A^3(t,x)\vec{e}_3$.

i. Show that equation (11) reduces to

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right) A^3 = -h_+ \exp(-i\Omega t) \frac{\partial^2 A^3}{\partial x^2} \ . \tag{12}$$

ii. By inspecting the mathematical form of (12) and noting that h_+ is small, explain why the laser field can be written approximately as a zeroth-order carrier wave emitted by the laser (constant amplitude A_{in}, frequency ω, wavenumber k) plus a first-order correction generated by the interaction with the gravitational wave (slowly varying amplitude A_{out}, frequency $\omega + \Omega$, wavenumber k), viz.

$$A^3 = A_{\text{in}} \exp(-i\omega t + ikx) + A_{\text{out}}(t) \exp[-i(\omega+\Omega)t + ikx] \ . \tag{13}$$

[2] "Luck is what happens when preparation meets opportunity", as the Roman philosopher and dramatist Seneca the Younger (4 BC–AD 65) may or may not have said.

iii. Use (12), (13), and the fact that $A_{\text{out}}(t)$ varies slowly ($\Omega \ll \omega$) to obtain

$$\frac{dA_{\text{out}}}{dt} - i\Omega A_{\text{out}} = -\frac{ih_+ k A_{\text{in}}}{2} \; . \tag{14}$$

iv. Equation (14) has the simple solution (you don't need to prove this)

$$A_{\text{out}}(t) = -\frac{h_+ k A_{\text{in}}}{2\Omega}[\exp(i\Omega t) - 1] \tag{15}$$

for the initial condition $A_{\text{out}} = 0$ at $t = 0$. By writing out the laser field after one round-trip time in the approximate form $A^3 = A_{\text{in}} \exp(-i\omega t + ikx - i\phi)$, show that (15) introduces a phase correction

$$\phi = \frac{h_+ \omega}{\Omega} \sin(\Omega L) \exp(-i\Omega L) \; . \tag{16}$$

The result (16) applies to one arm of the interferometer (x-axis). The phase correction in the second arm is equal but opposite. Hence the total phase difference between the arms is twice (16), which is exactly the same as the ray optics result (5)!