SN: 587623

PHYC90012 General Relativity

Assignment 2

By

Braden Moore

Master of Science
The University of Melbourne

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2 Klein's geometry

A two-dimensional surface is covered by coordinates (u, v) in the domain $u^2 + v^2 = 1$. The independent components of the metric are given by

$$g_{11} = \frac{a^2(1 - v^2)}{(1 - u^2 - v^2)^2}, \tag{1}$$

$$g_{12} = \frac{a^2 uv}{(1 - u^2 - v^2)^2}, \tag{2}$$

$$g_{22} = \frac{a^2(1 - u^2)}{(1 - u^2 - v^2)^2}, \tag{3}$$

the independent components of the inverse metric are given by

$$g^{11} = a^{-2}(1 - u^2)(1 - u^2 - v^2), \tag{4}$$

$$g^{12} = -a^{-2}uv(1 - u^2 - v^2), \tag{5}$$

$$g^{22} = a^{-2}(1 - v^2)(1 - u^2 - v^2), \tag{6}$$

and the independent, non-zero Christoffel symbols are given by

$$\Gamma^1_{11} = \frac{2u}{1 - u^2 - v^2}, \tag{7}$$

$$\Gamma^1_{12} = \frac{v}{1 - u^2 - v^2}, \tag{8}$$

$$\Gamma^2_{12} = \frac{u}{1 - u^2 - v^2}, \tag{9}$$

$$\Gamma^2_{22} = \frac{2v}{1 - u^2 - v^2}. \tag{10}$$

Remember that $g_{\alpha\beta}$, $g^{\alpha\beta}$, and $\Gamma^\lambda_{\alpha\beta}$ are all symmetric in α and β.

(a) Starting from (1)-(6), derive the expression (7) for Γ^1_{11}.

We begin with the expression for the Christoffel symbols in terms of the metric

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} \left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu} \right) \tag{11}$$

We now calculate the values of $g_{\alpha\beta,\mu}$ from (1)-(3)

$$g_{11,1} = \frac{\partial g_{11}}{\partial x^1} = \frac{\partial\left(\frac{a^2(1-v^2)}{(1-u^2-v^2)^2}\right)}{\partial u} \tag{12}$$
$$= \frac{4a^2u(1-v^2)}{(1-u^2-v^2)^3} \tag{13}$$

$$g_{12,1} = \frac{\partial g_{12}}{\partial x^1} = \frac{\partial\left(\frac{a^2uv}{(1-u^2-v^2)^2}\right)}{\partial u} \tag{14}$$
$$= \frac{a^2v(3u^2-v^2+1)}{(1-u^2-v^2)^3} \tag{15}$$
$$= g_{21,1} \quad \text{by symmetry} \tag{16}$$
$$g_{22,1} = \frac{\partial g_{22}}{\partial x^1} = \frac{\partial\left(\frac{a^2(1-u^2)}{(1-u^2-v^2)^2}\right)}{\partial u} \tag{17}$$
$$= \frac{2a^2u(1-u^2+v^2)}{(1-u^2-v^2)^3} \tag{18}$$

By inspecting the components of the metric above, we see that $g_{11,1} \mapsto g_{22,2}$ with $u \leftrightarrow v$, similarly $g_{12,1} \mapsto g_{12,2}$ with $u \leftrightarrow v$, and $g_{11,2} \mapsto g_{22,1}$ with $u \leftrightarrow v$. Hence

$$g_{11,2} = \frac{2a^2v(1-v^2+u^2)}{(1-u^2-v^2)^3} \tag{19}$$
$$g_{12,2} = \frac{a^2u(3v^2-u^2+1)}{(1-u^2-v^2)^3} \tag{20}$$
$$= g_{21,2} \quad \text{by symmetry} \tag{21}$$
$$g_{22,2} = \frac{4a^2v(1-u^2)}{(1-u^2-v^2)^3} \tag{22}$$

So now we can evaluate the Christoffel symbols.

$$\Gamma^1_{11} = \frac{1}{2}g^{1\mu}\left(g_{\mu1,1} + g_{\mu1,1} - g_{11,\mu}\right) \tag{23}$$
$$= \frac{1}{2}\left[g^{11}g_{11,1} + g^{12}\left(2g_{21,1} - g_{11,2}\right)\right] \tag{24}$$
$$= \frac{1}{2}\left[a^{-2}(1-u^2)(1-u^2-v^2)\frac{4a^2u(1-v^2)}{(1-u^2-v^2)^3}\right.$$
$$\left. + -a^{-2}uv(1-u^2-v^2)\left(\frac{2a^2v(3u^2-v^2+1) - 2a^2v(1-v^2+u^2)}{(1-u^2-v^2)^3}\right)\right] \tag{25}$$
$$= \frac{1}{2}\left[\frac{(1-u^2)\cdot 4u(1-v^2) - uv\cdot 2v(3u^2-u^2)}{(1-u^2-v^2)^2}\right] \tag{26}$$
$$= \frac{1}{2}\left[\frac{4u(1-u^2)(1-v^2) - 4u^2v^2}{(1-u^2-v^2)^2}\right] \tag{27}$$
$$= 2u\left[\frac{1-u^2-v^2+\cancel{u^2v^2-u^2v^2}}{(1-u^2-v^2)^2}\right] \tag{28}$$
$$= \frac{2u}{1-u^2-v^2} \tag{29}$$

as required. As an exercise, I have further derived the remaining Christoffel symbols

$$
\begin{aligned}
\Gamma^1_{12} &= \frac{1}{2} g^{1\mu} \left(g_{\mu 1,2} + g_{\mu 2,1} - g_{12,\mu} \right) && (30)\\
&= \frac{1}{2} \left[g^{11} g_{11,2} + g^{12} g_{22,1} \right] && (31)\\
&= \frac{1}{2} \Bigg[a^{-2}(1-u^2)(1-u^2-v^2) \frac{2a^2 v(1-v^2+u^2)}{(1-u^2-v^2)^3} \\
&\quad + -a^{-2} uv(1-u^2-v^2) \frac{2a^2 u(1-u^2+v^2)}{(1-u^2-v^2)^3} \Bigg] && (32)\\
&= \frac{1}{2} \left[\frac{(1-u^2)\cdot 2v(1-v^2-u^2) - uv \cdot 2u(1-u^2+v^2)}{(1-u^2-v^2)^2} \right] && (33)\\
&= v \left[\frac{(1-u^2)(1-v^2+u^2) - u^2(1-u^2+v^2)}{(1-u^2-v^2)^2} \right] && (34)\\
&= v \left[\frac{\cancel{1-u^2-v^2}}{(1-u^2-v^2)^{\cancel{2}}} \right] && (35)\\
&= \frac{v}{1-u^2-v^2} = \Gamma^1_{21}, \quad \text{by symmetry} && (36)
\end{aligned}
$$

as given.

$$
\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2} g^{2\mu} \left(g_{\mu 1,2} + g_{\mu 2,1} - g_{12,\mu} \right) && (37)\\
&= \frac{1}{2} \left[g^{21} g_{11,2} + g^{22} g_{22,1} \right] && (38)
\end{aligned}
$$

Now, we deduced earlier that $g_{11,2} = g_{22,1}|_{u\leftrightarrow v}$, we see by inspection of (6) that $g^{22} = g^{11}|_{u\leftrightarrow v}$, and by symmetry of the metric we have $g^{21} = g^{12}$. We note also that g^{12} is symmetric under interchange of u and v. Combining these results we find

$$
\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2} \left[g^{12} g_{22,1} + g^{11} g_{11,2} \right] |_{u\leftrightarrow v} && (39)\\
&= \Gamma^1_{12}|_{u\leftrightarrow v} && (40)\\
&= \frac{u}{1-u^2-v^2} && (41)
\end{aligned}
$$

as given.

$$
\begin{aligned}
\Gamma^2_{22} &= \frac{1}{2} g^{2\mu} \left(g_{\mu 2,2} + g_{\mu 2,2} - g_{22,\mu} \right) && (42)\\
&= \frac{1}{2} \left[g^{21} \left(2g_{12,2} - g_{22,1} \right) + g^{22} g_{22,2} \right] && (43)
\end{aligned}
$$

Similar to our approach for Γ^2_{12}, we note by observation that $g_{12,2} = g_{21,1}|_{u\leftrightarrow v}$, $g_{22,1} = g_{11,2}|_{u\leftrightarrow v}$, $g_{22} = g_{11}|_{u\leftrightarrow v}$, and $g_{22,2} = g_{11,1}$, also noting that $g^{21} = g^{12} = g^{12}|_{u\leftrightarrow v}$. Thus we find

$$
\begin{aligned}
\Gamma^2_{22} &= \frac{1}{2} \left[g^{12}(2g_{21,2} - g_{11,2}) + g^{11} g_{11,1} \right] |_{u\leftrightarrow v} && (44)\\
&= \Gamma^1_{11}|_{u\leftrightarrow v} && (45)\\
&= \frac{2v}{1-u^2-v^2} = \Gamma^2_{21}, \quad \text{by symmetry} && (46)
\end{aligned}
$$

as given.

(b) Prove that the Riemann tensor with all indices lowered, $R_{\alpha\beta\gamma\delta}$, contains four nonzero elements, any three of which can be written in terms of the fourth.

We recall the Bianchi identities for the Riemann tensor

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu} + g_{\beta\mu,\alpha\nu} - g_{\alpha\mu,\beta\nu} - g_{\beta\nu,\alpha\mu}\right) \tag{47}$$

Since $R_{\alpha\beta\mu\nu}$ is anti-symmetric under exchange $\alpha \leftrightarrow \beta$, and $\mu \leftrightarrow \nu$ also[1], we know

$$R_{\alpha\alpha\mu\nu} = 0 \tag{48}$$
$$R_{\alpha\beta\mu\mu} = 0 \tag{49}$$

for all α, β, μ, ν. Using this we greatly reduce the number of components to investigate. We find

$$R_{11\mu\nu} = 0 \qquad R_{\alpha\beta 11} = 0 \tag{50}$$
$$R_{22\mu\nu} = 0 \qquad R_{\alpha\beta 22} = 0 \tag{51}$$

We also know the Riemann tensor is symmetric under exchange of pairs $\alpha\beta \leftrightarrow \mu\nu$, i.e.

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \tag{52}$$

Hence we find

$$R_{1212} = -R_{1221} = R_{2121} = -R_{2112} \tag{53}$$

(c) Prove that, in Kleins geometry, the Ricci tensor satisfies

$$R_{\alpha\beta} = -\frac{g_{\alpha\beta}}{a^2}, \tag{54}$$

and the Ricci scalar satifies

$$R = -\frac{2}{a^2}. \tag{55}$$

We can find the Ricci tensor by contracting the first and third indices of the Riemann tensor

$$\begin{aligned}
R_{\alpha\beta} &= R^{\mu}{}_{\alpha\mu\beta} && (56)\\
&= \Gamma^{\mu}{}_{\alpha\beta,\mu} - \Gamma^{\mu}{}_{\alpha\mu,\beta} + \Gamma^{\mu}{}_{\nu\mu}\Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\nu\beta}\Gamma^{\nu}{}_{\alpha\mu} && (57)\\
&= \Gamma^{1}{}_{\alpha\beta,1} + \Gamma^{2}{}_{\alpha\beta,2} - \Gamma^{1}{}_{\alpha1,\beta} - \Gamma^{2}{}_{\alpha2,\beta} + \Gamma^{1}{}_{\nu1}\Gamma^{\nu}{}_{\alpha\beta} + \Gamma^{2}{}_{\nu2}\Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{1}{}_{\nu\beta}\Gamma^{\nu}{}_{\alpha1} - \Gamma^{2}{}_{\nu\beta}\Gamma^{\nu}{}_{\alpha2} && (58)\\
&= \Gamma^{1}{}_{\alpha\beta,1} + \Gamma^{2}{}_{\alpha\beta,2} - \Gamma^{1}{}_{\alpha1,\beta} - \Gamma^{2}{}_{\alpha2,\beta} + \Gamma^{1}{}_{11}\Gamma^{1}{}_{\alpha\beta} + \Gamma^{1}{}_{21}\Gamma^{2}{}_{\alpha\beta} + \Gamma^{2}{}_{12}\Gamma^{1}{}_{\alpha\beta} + \Gamma^{2}{}_{22}\Gamma^{2}{}_{\alpha\beta}\\
&\quad - \Gamma^{1}{}_{1\beta}\Gamma^{1}{}_{\alpha1} - \Gamma^{1}{}_{2\beta}\Gamma^{2}{}_{\alpha1} - \Gamma^{2}{}_{1\beta}\Gamma^{1}{}_{\alpha2} - \Gamma^{2}{}_{2\beta}\Gamma^{2}{}_{\alpha2} && (59)
\end{aligned}$$

[1] This can be easily proved from (47) by considering the symmetries of $g_{\alpha\beta,\mu\nu}$; however for brevity this proof is omitted.

from Schutz (6.63).

We can easily calculate the values of $\Gamma^{\alpha}_{\beta\mu,\nu}$:

$$\Gamma^1_{11,1} = \frac{2(1-u^2+v^2)}{(1-u^2-v^2)^2} \qquad \Gamma^1_{11,2} = \frac{4uv}{(1-u^2-v^2)^2} \tag{60}$$

$$\Gamma^1_{12,1} = \frac{2uv}{(1-u^2-v^2)^2} \qquad \Gamma^1_{12,2} = \frac{1-u^2+v^2}{(1-u^2-v^2)^2} \tag{61}$$

$$\Gamma^2_{12,1} = \frac{1+u^2-v^2}{(1-u^2-v^2)^2} \qquad \Gamma^2_{12,2} = \frac{2uv}{(1-u^2-v^2)^2} \tag{62}$$

$$\Gamma^2_{22,1} = \frac{4uv}{(1-u^2-v^2)^2} \qquad \Gamma^2_{22,2} = \frac{2(1-u^2+v^2)}{(1-u^2-v^2)^2} \tag{63}$$

with all others zero.

We shall calculate each component of $R_{\alpha\beta}$ below using (59)

$$\begin{aligned} R_{11} = {} & \cancel{{\Gamma^1}_{11,1}} + {\Gamma^2}_{11,2} - \cancel{{\Gamma^1}_{11,1}} - {\Gamma^2}_{12,1} + \cancel{{\Gamma^1}_{11}{\Gamma^1}_{11}} + \cancel{{\Gamma^1}_{21}{\Gamma^2}_{11}} + {\Gamma^2}_{12}{\Gamma^1}_{11} + {\Gamma^2}_{22}{\Gamma^2}_{11} \\ & - \cancel{{\Gamma^1}_{11}{\Gamma^1}_{11}} - \cancel{{\Gamma^1}_{21}{\Gamma^2}_{11}} - {\Gamma^2}_{11}{\Gamma^1}_{12} - {\Gamma^2}_{21}{\Gamma^2}_{12} \end{aligned} \tag{64}$$

by cancellations,

$$= \cancel{{\Gamma^2}_{11,2}} - {\Gamma^2}_{12,1} + {\Gamma^2}_{12}{\Gamma^1}_{11} + \cancel{{\Gamma^2}_{22}{\Gamma^2}_{11}} - \cancel{{\Gamma^2}_{11}{\Gamma^1}_{12}} - {\Gamma^2}_{12}{\Gamma^2}_{12} \tag{65}$$

since ${\Gamma^2}_{11} = 0$,

$$= {\Gamma^2}_{12}{\Gamma^1}_{11} - {\Gamma^2}_{12}{\Gamma^2}_{12} - {\Gamma^2}_{12,1} \tag{66}$$

$$= \frac{1}{(1-u^2-v^2)}\left[u \cdot 2u - u \cdot u - (1+u^2-v^2)\right] \tag{67}$$

$$= \frac{1}{(1-u^2-v^2)^2}\left[2u^2 - u^2 - 1 - u^2 + v^2\right] \tag{68}$$

$$= -\frac{(1-v^2)}{(1-u^2-v^2)^2} \tag{69}$$

$$= -\frac{1}{a^2}\frac{a^2(1-v^2)}{(1-u^2-v^2)^2} \tag{70}$$

$$= -\frac{g_{11}}{a^2} \tag{71}$$

as required.

$$\begin{aligned} R_{12} = {} & {\Gamma^1}_{12,1} + \cancel{{\Gamma^2}_{12,2}} - {\Gamma^1}_{11,2} - \cancel{{\Gamma^2}_{12,2}} + \cancel{{\Gamma^1}_{11}{\Gamma^1}_{12}} + {\Gamma^1}_{21}{\Gamma^2}_{12} + \cancel{{\Gamma^2}_{12}{\Gamma^1}_{12}} + \cancel{{\Gamma^2}_{22}{\Gamma^2}_{12}} \\ & - \cancel{{\Gamma^1}_{12}{\Gamma^1}_{11}} - \cancel{{\Gamma^1}_{22}{\Gamma^2}_{11}} - \cancel{{\Gamma^2}_{12}{\Gamma^1}_{12}} - \cancel{{\Gamma^2}_{22}{\Gamma^2}_{12}} \end{aligned} \tag{72}$$

$$= {\Gamma^1}_{12,1} - {\Gamma^1}_{11,2} + {\Gamma^1}_{21}{\Gamma^2}_{12} \tag{73}$$

$$= \frac{1}{(1-u^2-v^2)}\left[2uv - 4uv + v \cdot u\right] \tag{74}$$

$$= -\frac{uv}{(1-u^2-v^2)^2} \tag{75}$$

$$= -\frac{1}{a^2}\frac{a^2 uv}{(1-u^2-v^2)^2} \tag{76}$$

$$= -\frac{g_{12}}{a^2} \tag{77}$$

as required (note by symmetry, $R_{12} = R_{21}$, so $R_{21} = -\frac{g_{21}}{a^2}$).

$$\begin{aligned}
R_{22} &= \cancel{{\Gamma^1}_{22,1}} + \cancel{{\Gamma^2}_{22,2}} - {\Gamma^1}_{21,2} - \cancel{{\Gamma^2}_{22,2}} + \cancel{{\Gamma^1}_{11}{\Gamma^1}_{22}} + {\Gamma^1}_{21}{\Gamma^2}_{22} + \cancel{{\Gamma^2}_{12}{\Gamma^1}_{22}} + \cancel{{\Gamma^2}_{22}{\Gamma^2}_{22}} \\
&\quad - {\Gamma^1}_{12}{\Gamma^1}_{21} - \cancel{{\Gamma^1}_{22}{\Gamma^2}_{21}} - \cancel{{\Gamma^2}_{12}{\Gamma^1}_{22}} - \cancel{{\Gamma^2}_{22}{\Gamma^2}_{22}} && (78) \\
&= {\Gamma^1}_{21}{\Gamma^2}_{22} - {\Gamma^1}_{12}{\Gamma^1}_{21} - {\Gamma^1}_{21,2} && (79) \\
&= \frac{1}{(1-u^2-v^2)^2}\left[v \cdot 2v - v \cdot v - (1 - u^2 + v^2)\right] && (80) \\
&= \frac{1}{(1-u^2-v^2)^2}\left[2v^2 - v^2 - 1 + u^2 - v^2)\right] && (81) \\
&= -\frac{(1-u^2)}{(1-u^2-v^2)^2} && (82) \\
&= -\frac{1}{a^2}\frac{a^2(1-u^2)}{(1-u^2-v^2)^2} && (83) \\
&= -\frac{g_{22}}{a^2} && (84)
\end{aligned}$$

as required.

$$ \tag{85}$$

Hence we have proved that, in Klen's geometry,

$$R_{\alpha\beta} = -\frac{g_{\alpha\beta}}{a^2} \tag{86}$$

The Ricci scalar is formed by contracting the Ricci tensor:

$$\begin{aligned}
R &= g^{\alpha\beta}R_{\alpha\beta} && (87) \\
&= -g^{\alpha\beta}\frac{g_{\alpha\beta}}{a^2} && (88) \\
&= -\frac{2}{a^2} && (89)
\end{aligned}$$

since $g^{\alpha\beta}g_{\alpha\beta} = g^{\alpha\beta}g_{\beta\alpha} = {\delta^\alpha}_\alpha = 2$ in Klein's geometry.

(d) Answer each of the following questions in one or two sentences.

i. In what fundamental way does Klein's geometry differ from a two-sphere?

???

ii. The hyperbola $x^2 - y^2 = 1$ is rotated around the y-axis to form a three-dimensional hyperboloid of revolution. Does it possess positive or negative curvature? Justify your answer physically with a diagram; do not attempt to calculate anything.

Along surfaces with positive curvature, geodesics will converge as they travel along the surface. By contrast, geodesics along surfaces of negative curvature will diverge.

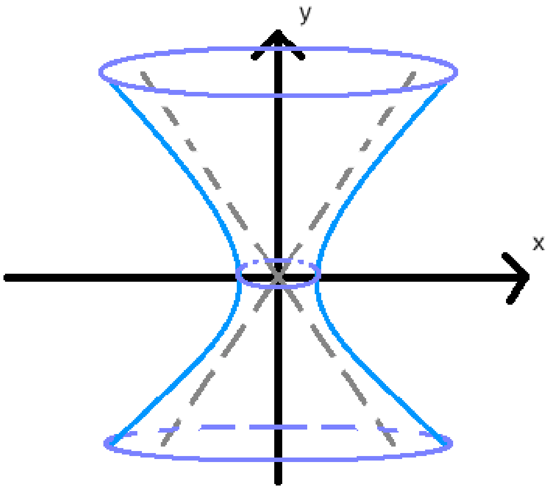

Figure 1: $x^2 - y^2 = 1$ rotated about the y-axis

We see that geodesics travelling along the surface would diverge; hence this hyperboloid possesses negative curvature. A triangle on the surface would look like FIGURE, with the sum of its angles $< 180°$.

iii. The hyperbola $x^2 - y^2 = 1$ is now rotated around the x-axis. What is the sign of the curvature this time? Why?

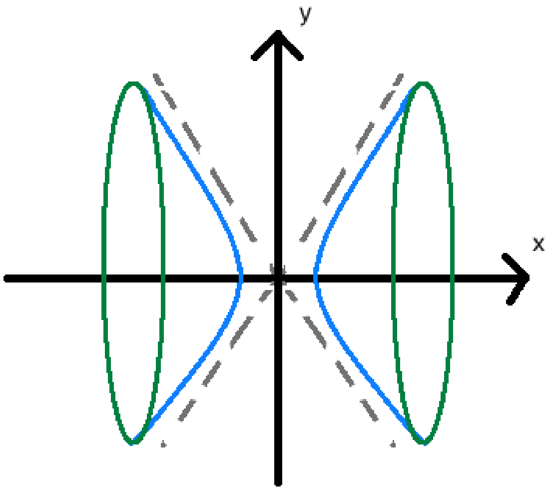

Figure 2: $x^2 - y^2 = 1$ rotated about the x-axis

We see that geodesics travelling along the surface would converge; hence this hyperboloid possesses positive curvature. A triangle on the surface would look like FIGURE, with the sum of its angles $> 180°$.

iv. Setting aside their dimensionality, in what fundamental way do the hyperboloids of revolution in parts (d)(ii) and (d)(iii) differ from Klein's geometry? Justify your answer in words; dont try to calculate anything.

v. Identify a spacetime manifold, that resembles Klein's geometry. Dont worry too much about the precise mathematical meaning of "resembles", a qualitative justification is fine.

A spacetime manifold resembling Klein's geometry is the interior surface of the paraboloid of rotation formed by rotating $z = x^2$ around the y-axis. After inspection of the (non-zero) Christoffel symbols, we note they become zero at $(u, v) = (0, 0)$; i.e. locally space is flat at the origin in Klein's geometry. Also, we note that the Christoffel symbols tend to infinity as $u^2 + v^2 \to 1$, i.e. space becomes more curved further from the origin. Both of these properties are seen in the aforementioned spacetime manifold.

NAH

Anti-de Sitter space.

(e) Consider the triangle ΔABC, whose sides are "straight lines" (geodesics) joining the points A$(0,0)$, B$(b,0)$, and C$(0,b)$, with $b < 1$. It is easy to show (you don't need to!) that the sides AB and AC are just the curves $v = 0$ and $u = 0$ respectively.

i. What is the equation of the geodesic joining B and C?

From Schutz (6.51), letting λ be the parameter of the geodesic with $x^1 = u$ and $x^2 = v$ we have the geodesic equation

$$\frac{d}{d\lambda}\left(\frac{dx^\alpha}{d\lambda}\right) + \Gamma^\alpha{}_{\mu\beta}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{90}$$

For $\alpha = 1$ we have

$$\frac{d}{d\lambda}\left(\frac{du}{d\lambda}\right) = -\Gamma^1{}_{\mu\beta}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda} \tag{91}$$

$$= -\Gamma^1{}_{11}\frac{dx^1}{d\lambda}\frac{dx^1}{d\lambda} - \Gamma^1{}_{12}\frac{dx^1}{d\lambda}\frac{dx^2}{d\lambda} - -\Gamma^1{}_{21}\frac{dx^2}{d\lambda}\frac{dx^1}{d\lambda} - \cancel{-\Gamma^1{}_{22}\frac{dx^2}{d\lambda}\frac{dx^2}{d\lambda}} \tag{92}$$

$$= -\Gamma^1{}_{11}\left(\frac{du}{d\lambda}\right)^2 - 2\Gamma^1{}_{12}\frac{du}{d\lambda}\frac{dv}{d\lambda} \tag{93}$$

which can be written as

$$\frac{d}{d\lambda}\left(\frac{du}{d\lambda}\right) + {\Gamma^1}_{11}\left(\frac{du}{d\lambda}\right)^2 + 2{\Gamma^1}_{12}\frac{du}{d\lambda}\frac{dv}{d\lambda} = 0 \tag{94}$$

$$\Rightarrow \frac{d}{d\lambda}\left(\frac{du}{d\lambda}\right) + 2u(1-u^2-v^2)^{-1}\left(\frac{du}{d\lambda}\right)^2 + 2v(1-u^2-v^2)^{-1}\frac{du}{d\lambda}\frac{dv}{d\lambda} = 0 \tag{95}$$

$$\Rightarrow \frac{d}{d\lambda}\left(\frac{du}{d\lambda}\right)(1-u^2-v^2)^{-1} + 2u(1-u^2-v^2)^{-2}\left(\frac{du}{d\lambda}\right)^2 + 2v(1-u^2-v^2)^{-2}\frac{du}{d\lambda}\frac{dv}{d\lambda} = 0 \tag{96}$$

$$\Rightarrow \frac{d}{d\lambda}\left[(1-u^2-v^2)^{-1}\frac{du}{d\lambda}\right] = 0 \tag{97}$$

We shall prove that going from (96) to (97) is correct.

$$\begin{aligned}
\frac{d}{d\lambda}\left[(1-u^2-v^2)^{-1}\frac{du}{d\lambda}\right] &= \frac{d}{d\lambda}\big[(1-u^2-v^2)^{-1}\big)\frac{du}{d\lambda} + (1-u^2-v^2)^{-1}\frac{d^2u}{d\lambda^2} && (98)\\
&= \left[-\frac{d}{d\lambda}\left(-u^2\right) + -\frac{d}{d\lambda}\left(-v^2\right)\right](1-u^2-v^2)^{-2}\frac{du}{d\lambda} + (1-u^2-v^2)^{-1}\frac{d^2u}{d\lambda^2} && (99)\\
&= \left[\frac{du}{d\lambda}\frac{d(u^2)}{du} + \frac{dv}{d\lambda}\frac{d(v^2)}{dv}\right](1-u^2-v^2)^{-2}\frac{du}{d\lambda} + (1-u^2-v^2)^{-1}\frac{d^2u}{d\lambda^2} && (100)\\
&= \left[2u\frac{du}{d\lambda} + 2v\frac{dv}{d\lambda}\right](1-u^2-v^2)^{-2}\frac{du}{d\lambda} + (1-u^2-v^2)^{-1}\frac{d^2u}{d\lambda^2} && (101)\\
&= \text{LHS of (96)} && (102)
\end{aligned}$$

Hence this is indeed correct.

For $\alpha = 2$ we have

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{dv}{d\lambda}\right) &= -{\Gamma^2}_{\mu\beta}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda} && (103)\\
&= \cancel{-{\Gamma^2}_{11}\frac{dx^1}{d\lambda}\frac{dx^1}{d\lambda}} - {\Gamma^2}_{12}\frac{dx^1}{d\lambda}\frac{dx^2}{d\lambda} - -{\Gamma^2}_{21}\frac{dx^2}{d\lambda}\frac{dx^1}{d\lambda} - -{\Gamma^2}_{22}\frac{dx^2}{d\lambda}\frac{dx^2}{d\lambda} && (104)\\
&= -{\Gamma^2}_{22}\left(\frac{dv}{d\lambda}\right)^2 - 2{\Gamma^2}_{12}\frac{du}{d\lambda}\frac{dv}{d\lambda} && (105)\\
&= -2v(1-u^2-v^2)^{-1}\left(\frac{dv}{d\lambda}\right)^2 - 2u(1-u^2-v^2)^{-1}\frac{du}{d\lambda}\frac{dv}{d\lambda} && (106)
\end{aligned}$$

By comparison with (95)-(97), we see that this implies

$$\frac{d}{d\lambda}\left[(1-u^2-v^2)^{-1}\frac{dv}{d\lambda}\right] = 0 \tag{107}$$

We now find

$$\begin{aligned}
\frac{d}{d\lambda}\left[(1-u^2-v^2)^{-1}\frac{du}{d\lambda}\right] = 0 \Rightarrow (1-u^2-v^2)^{-1}\frac{du}{d\lambda} = c_1 && (108)\\
\frac{d}{d\lambda}\left[(1-u^2-v^2)^{-1}\frac{dv}{d\lambda}\right] = 0 \Rightarrow (1-u^2-v^2)^{-1}\frac{dv}{d\lambda} = c_2 && (109)
\end{aligned}$$

Dividing (108) by (109) we find

$$\frac{du}{dv} = \frac{c_1}{c_2} \equiv c_3 \tag{110}$$

$$\Rightarrow v = c_3 u + c_4 \tag{111}$$

Since the point $(0, b)$ lies on the geodesic we have,

$$b = c_4 \tag{112}$$

$$\Rightarrow c_4 = b \tag{113}$$

Similarly using the point $(b, 0)$,

$$0 = bc_3 + b \tag{114}$$

$$\Rightarrow c_3 = -1 \tag{115}$$

Thus we conclude that equation of the geodesic joining B and C is

$$v = -u + b \tag{116}$$

ii. Prove that the sum of the interior angles of ΔABC is

$$\Sigma = \angle\text{ABC} + \angle\text{BCA} + \angle\text{CAB} = \frac{\pi}{2} + 2\cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right). \tag{117}$$

The sum of the angles is less than 180 degrees!

The cosine of the angle between two vectors $\vec{M}$ and $\vec{N}$ is given by

$$\cos\theta = \frac{\vec{M}\cdot\vec{N}}{|\vec{M}||\vec{N}|} \tag{118}$$

In component notation we have

$$\theta = \cos^{-1}\left(\frac{g_{\alpha\beta}M^\alpha N^\beta}{\sqrt{g_{ij}M^i M^j g_{kl}N^k N^l}}\right) \tag{119}$$

The angle between two curves at a point is equal to the angle between curves pointing in the same direction;

The equations bounding the triangle parametrised by λ are given as

$$\vec{a} = (\lambda, 0) \tag{120}$$

$$\vec{b} = (0, \lambda) \tag{121}$$

$$\vec{c} = (\lambda, b - \lambda) \tag{122}$$

where $\lambda \leq b$.

$$\Rightarrow \dot{\vec{a}} = (1, 0) \tag{123}$$

$$\dot{\vec{b}} = (0, 1) \tag{124}$$

$$\dot{\vec{c}} = (1, -1) \tag{125}$$

We shall use these direction vectors WHY????.

Calculating $\angle$ABC

We use $\vec{M} = \vec{a}$ and $\vec{N} = \vec{c}$. At the point B$(b, 0)$,

$$g_{11} = \frac{a^2}{(1-b^2)^2} \tag{126}$$

$$g_{12} = 0 \tag{127}$$

$$g_{22} = \frac{a^2}{1-b^2} \tag{128}$$

Hence we find

$$\angle\text{ABC} = \cos^{-1}\left(\frac{g_{11}}{\sqrt{g_{11}g_{11} + g_{11}g_{22}}}\right) \tag{129}$$

$$= \cos^{-1}\left(\frac{\frac{a^2}{(1-b^2)^2}}{\sqrt{\frac{a^4}{(1-b^2)^4} + \frac{a^4}{(1-b^2)^3}}}\right) \tag{130}$$

$$= \cos^{-1}\left(\frac{a^2}{(1-b^2)^2}\sqrt{\frac{1+(1-b^2)}{(1-b^2)^4}}\right) \tag{131}$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right) \tag{132}$$

Calculating $\angle$BCA

We use $\vec{M} = \vec{b}$ and $\vec{N} = \vec{c}$. At the point C$(0, b)$,

$$g_{11} = \frac{a^2}{1-b^2} \tag{133}$$

$$g_{12} = 0 \tag{134}$$

$$g_{22} = \frac{a^2}{(1-b^2)^2} \tag{135}$$

Hence we find

$$\angle\text{BCA} = \cos^{-1}\left(\frac{g_{22}}{\sqrt{g_{22}g_{11} + g_{22}g_{22}}}\right) \tag{136}$$

$$= \cos^{-1}\left(\frac{\frac{a^2}{(1-b^2)^2}}{\sqrt{\frac{a^4}{(1-b^2)^4} + \frac{a^4}{(1-b^2)^3}}}\right) \tag{137}$$

$$= \cos^{-1}\left(\frac{a^2}{(1-b^2)^2}\sqrt{\frac{1+(1-b^2)}{(1-b^2)^4}}\right) \tag{138}$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right) \tag{139}$$

Calculating $\angle$ABC

We use $\vec{M} = \vec{a}$ and $\vec{N} = \vec{c}$. At the point B$(b, 0)$,

$$g_{11} = \frac{a^2}{(1-b^2)^2} \tag{140}$$
$$g_{12} = 0 \tag{141}$$
$$g_{22} = \frac{a^2}{1-b^2} \tag{142}$$

Hence we find

$$\angle\text{ABC} = \cos^{-1}\left(\frac{g_{11}}{\sqrt{g_{11}g_{11} + g_{11}g_{22}}}\right) \tag{143}$$
$$= \cos^{-1}\left(\frac{\frac{a^2}{(1-b^2)^2}}{\sqrt{\frac{a^4}{(1-b^2)^4} + \frac{a^4}{(1-b^2)^3}}}\right) \tag{144}$$
$$= \cos^{-1}\left(\frac{a^2}{(1-b^2)^2}\sqrt{\frac{1+(1-b^2)}{(1-b^2)^4}}\right) \tag{145}$$
$$= \cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right) \tag{146}$$

Calculating $\angle$CAB

We use $\vec{M} = \vec{a}$ and $\vec{N} = \vec{b}$. At the point A$(0, 0)$,

$$g_{11} = a^2 \tag{147}$$
$$g_{12} = 0 \tag{148}$$
$$g_{22} = a^2 \tag{149}$$

Hence we find

$$\angle\text{BCA} = \cos^{-1}\left(\frac{\cancel{g_{12}}}{\sqrt{g_{11}g_{22}}}\right) \tag{150}$$
$$= \cos^{-1}(0) \tag{151}$$
$$= \frac{\pi}{2} \tag{152}$$

Summing these angles we indeed find

$$\Sigma = \angle\text{ABC} + \angle\text{BCA} + \angle\text{CAB} = \frac{\pi}{2} + 2\cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right) \tag{153}$$

as required.

iii. Triangles in Klein's geometry can have $\sum = 0$! Without proof, sketch what such a triangle might look like. Your sketch by necessity will be an incomplete representation; there is no way to draw a Klein triangle faithfully on a flat page.

(f) i. Write down a closed form expression for the area A of ΔABC as an integral over a subset of the (u, v) domain.

Surface area depends not on the parameterisation of the space, but only on the surface itself. We have an expression for area

$$A = \int_{\text{surface}} \sqrt{\det g}\ \mathrm{d}A \tag{154}$$

where g is the metric. We have

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{155}$$

$$\Rightarrow \det g = g_{11}g_{22} - g_{12}g_{21} \tag{156}$$

$$= \frac{a^4(1-v^2)(1-u^2) - a^4u^2v^2}{(1-u^2-v^2)^4} \tag{157}$$

$$= \frac{a^4}{(1-u^2-v^2)^4}\left[1 - u^2 - v^2 + \cancel{u^2v^2} - \cancel{u^2v^2}\right] \tag{158}$$

$$= \frac{a^4}{(1-u^2-v^2)^3} \tag{159}$$

We can now determine an expression for the area of ΔABC as

$$A = \iint \frac{a^4}{(1-u^2-v^2)^3}\ \mathrm{d}u\ \mathrm{d}v \tag{160}$$

$$= \int_{v=0}^{v=b}\int_{u=0}^{u=b-v} \frac{a^2}{(1-u^2-v^2)^{3/2}}\ \mathrm{d}u\ \mathrm{d}v \tag{161}$$

ii. By changing variables to $y = v + u$ and $z = v - u$, recast your integral in the form

$$A = 2a^2 \int_0^b \frac{\mathrm{d}y\ y}{(2-y^2)\sqrt{1-y^2}}. \tag{162}$$

Hence show that one has

$$A = a^2(\pi - \Sigma). \tag{163}$$

We begin by calculating the Jacobian,

$$J(y,z) = \begin{pmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{164}$$

$$\Rightarrow \det J = \frac{1}{2} \tag{165}$$

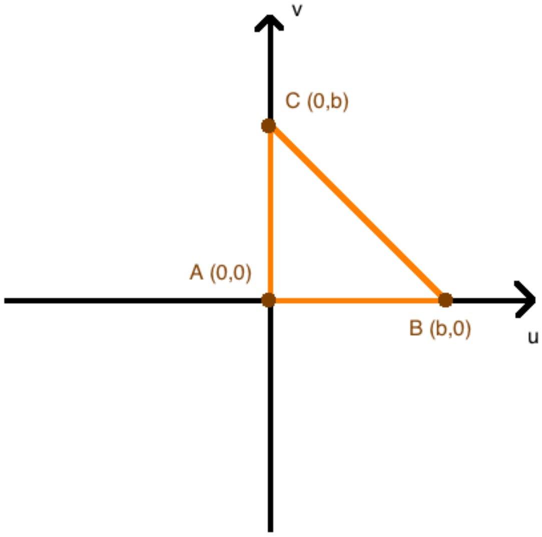

Figure 3: The triangle ΔABC on the $u - v$ axis

Next we consider the bounds of the integral; that is, the surface we are integrating over.

$$(u, v) = (0, 0) \Rightarrow (y, z) = (0, 0) \tag{166}$$
$$(u, v) = (b, 0) \Rightarrow (y, z) = (b, -b) \tag{167}$$
$$(u, v) = (0, b) \Rightarrow (y, z) = (b, b) \tag{168}$$

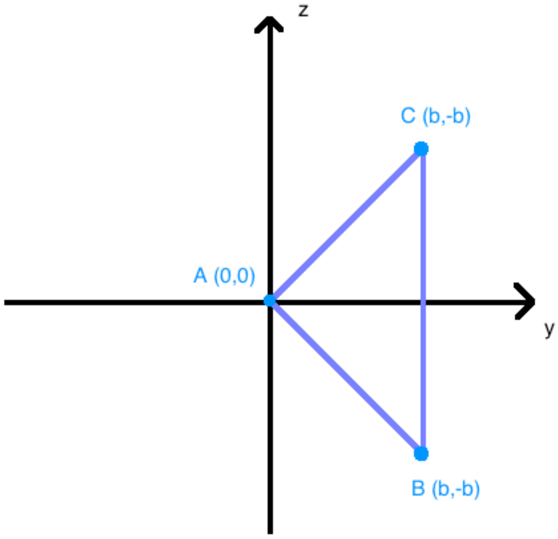

Figure 4: The triangle ΔABC on the $y - z$ axis

In $y - z$ space, our surface looks like Figure 4 above. Just as we integrated over first u in terms of v then integrated over v in (161), we shall integrate from $z = -y$ to $z = y$, then over $y = 0$ to $y = b$.

Also note that we can express u and v in terms of y and z as

$$u = \frac{1}{2}(y - z) \tag{169}$$
$$v = \frac{1}{2}(y + z) \tag{170}$$

Hence we can rewrite A as

$$A = 2a^2 \int_{y=0}^{y=b} \int_{z=-y}^{z=y} \left(1 - \left[\frac{1}{2}(y-z)\right]^2 - \left[\frac{1}{2}(y+z)\right]^2\right)^{-3/2} \frac{1}{4} \, \mathrm{d}y \, \mathrm{d}z \tag{171}$$
$$= \frac{a^2}{2} \int_{y=0}^{y=b} \int_{z=-y}^{z=y} \left(1 - \frac{1}{2}y^2 - \frac{1}{2}z^2\right)^{-3/2} \, \mathrm{d}y \, \mathrm{d}z \tag{172}$$

We now let $m = 1 - \frac{1}{2}y^2$, and let $z = \sqrt{2m} \sin r \Rightarrow \mathrm{d}z = \sqrt{2m} \cos r \, \mathrm{d}r$

$$A = \frac{a^2}{2} \int_{y=0}^{y=b} \int_{r=-\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)}^{r=\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)} \Bigg(\underbrace{m - m\sin^2 r}_{=m(\cos^2 r)}\Bigg)^{-3/2} \, \mathrm{d}y \, \sqrt{2m} \cos r \, \mathrm{d}r \tag{173}$$
$$= \frac{a^2}{2} \iint \sqrt{2} m^{-1} \cos^{-2} r \, \mathrm{d}y \, \mathrm{d}r \tag{174}$$
$$= \frac{a^2}{\sqrt{2}} \int_{y=0}^{y=b} m^{-1} \Big[\tan(r)\Big]_{-\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)}^{\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)} \, \mathrm{d}y \tag{175}$$
$$= \frac{a^2}{\sqrt{2}} \int m^{-1} \left[\frac{2y}{\sqrt{2 - 2y^2}}\right] \, \mathrm{d}y \tag{176}$$

In moving from (175) to (176) we use the identity

$$\tan\left[\sin^{-1}(ax)\right] = \frac{ax}{\sqrt{1 - a^2x^2}} \tag{177}$$
$$\Rightarrow \tan\left[\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)\right] = \frac{\frac{y}{\sqrt{2m}}}{\sqrt{1 - \frac{y^2}{2m}}} \tag{178}$$
$$= \frac{y}{\sqrt{2m - y^2}} \tag{179}$$

Since $m = 1 - \frac{1}{2}y^2$ we find

$$\tan\left[\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)\right] = \frac{y}{\sqrt{2 - 2y}} \tag{180}$$

Continuing from (176),

$$A = \frac{a^2}{\sqrt{2}} \int_0^b m^{-1} \left[\frac{2y}{\sqrt{2 - 2y^2}}\right] \, \mathrm{d}y \tag{181}$$
$$= a^2 \int_0^b \frac{1}{1 - \frac{1}{2}y^2} \frac{y}{\sqrt{1 - y^2}} \, \mathrm{d}y \tag{182}$$
$$= 2a^2 \int_0^b \frac{\mathrm{d}y \; y}{(2 - y^2)\sqrt{1 - y^2}} \tag{183}$$

as required.

iii. Explain briefly, in one or two sentences, why (163) guarantees the *nonexistence* of similar triangles in Kleins geometry.

(g) A vector $\vec{W}$ with equal components W^1 and W^2 at the point $A(0,0)$ is parallel transported along the geodesic AB. Show that its components, when it reaches the point B$(b,0)$, are in the ratio

$$\frac{W^1}{W^2} = (1 - b^2)^{1/2} \tag{184}$$