# General Relativity (Masters)
# Assignment Two

1. **Klein's geometry.** A two-dimensional surface is covered by coordinates $(u, v)$ in the domain $u^2 + v^2 < 1$. The independent components of the metric are given by

$$
\begin{aligned}
g_{11} &= \frac{a^2(1-v^2)}{(1-u^2-v^2)^2} \ , && (1)\\
g_{12} &= \frac{a^2uv}{(1-u^2-v^2)^2} \ , && (2)\\
g_{22} &= \frac{a^2(1-u^2)}{(1-u^2-v^2)^2} \ , && (3)
\end{aligned}
$$

the independent components of the inverse metric are given by

$$
\begin{aligned}
g^{11} &= a^{-2}(1-u^2)(1-u^2-v^2) \ , && (4)\\
g^{12} &= -a^{-2}uv(1-u^2-v^2) \ , && (5)\\
g^{22} &= a^{-2}(1-v^2)(1-u^2-v^2) \ , && (6)
\end{aligned}
$$

and the independent, nonzero Christoffel symbols are given by

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{2u}{1-u^2-v^2} \ , && (7)\\
\Gamma^1_{12} &= \frac{v}{1-u^2-v^2} \ , && (8)\\
\Gamma^2_{12} &= \frac{u}{1-u^2-v^2} \ , && (9)\\
\Gamma^2_{22} &= \frac{2v}{1-u^2-v^2} \ . && (10)
\end{aligned}
$$

Remember that $g_{\alpha\beta}$, $g^{\alpha\beta}$, and $\Gamma^\lambda_{\alpha\beta}$ are all symmetric in $\alpha$ and $\beta$.

(a) Starting from (1)–(6), derive the expression (7) for $\Gamma^1_{11}$.

(b) Prove that the Riemann tensor with all indices lowered, $R_{\alpha\beta\gamma\delta}$, contains four nonzero elements, any three of which can be written in terms of the fourth.

(c) Prove that, in Klein's geometry, the Ricci tensor satisfies

$$R_{\alpha\beta} = -\frac{g_{\alpha\beta}}{a^2} \ , \qquad (11)$$

and the Ricci scalar satisfies

$$R = -\frac{2}{a^2} \ . \qquad (12)$$

(d) Answer each of the following questions in one or two sentences.

i. In what fundamental way does Klein's geometry differ from a two-sphere?

ii. The hyperbola $x^2 - y^2 = 1$ is rotated around the $y$-axis to form a three-dimensional hyperboloid of revolution. Does it possess positive or negative curvature? Justify your answer physically with a diagram; do *not* attempt to calculate anything.

iii. The hyperbola $x^2 - y^2 = 1$ is now rotated around the $x$-axis. What is the sign of the curvature this time? Why?

iv. Setting aside their dimensionality, in what fundamental way do the hyperboloids of revolution in parts (d)(ii) and (d)(iii) differ from Klein's geometry? Justify your answer in words; don't try to calculate anything.

v. Identify a spacetime manifold, that resembles Klein's geometry. Don't worry too much about the precise mathematical meaning of "resembles", a qualitative justification is fine.

(e) Consider the triangle $\triangle$ABC, whose sides are "straight lines" (geodesics) joining the points A$(0,0)$, B$(b,0)$, and C$(0,b)$, with $b < 1$. It is easy to show (you don't need to!) that the sides AB and AC are just the curves $v = 0$ and $u = 0$ respectively.

i. What is the equation of the geodesic joining B and C?

ii. Prove that the sum of the interior angles of $\triangle$ABC is

$$\Sigma = \angle\mathrm{ABC} + \angle\mathrm{BCA} + \angle\mathrm{CAB} = \frac{\pi}{2} + 2\cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right) . \tag{13}$$

The sum of the angles is less than 180 degrees!

iii. Triangles in Klein's geometry can have $\Sigma = 0$! [1] Without proof, sketch what such a triangle might look like. Your sketch by necessity will be an incomplete representation; there is no way to draw a Klein triangle faithfully on a flat page.

(f) i. Write down a closed form expression for the area $A$ of $\triangle$ABC as an integral over a subset of the $(u, v)$ domain.

ii. By changing variables to $y = v + u$ and $z = v - u$, recast your integral in the form

$$A = 2a^2 \int_0^b \frac{dy\, y}{(2-y^2)\sqrt{1-y^2}} . \tag{14}$$

Hence show that one has

$$A = a^2(\pi - \Sigma) . \tag{15}$$

iii. Explain briefly, in one or two sentences, why (15) guarantees the *nonexistence* of similar triangles in Klein's geometry.

(g) A vector $\vec{W}$ with equal components $W^1$ and $W^2$ at the point A$(0,0)$ is parallel transported along the geodesic AB. Show that its components, when it reaches the point B$(b,0)$, are in the ratio

$$\frac{W^1}{W^2} = (1-b^2)^{1/2} . \tag{16}$$

[1] $\triangle$ABC is *not* such a triangle, as is obvious from (13), because $\angle$CAB is a right angle.