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PHYC90012 General Relativity

Assignment 2

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2 Klein's geometry

A two-dimensional surface is covered by coordinates (u, v) in the domain $u^2 + v^2 = 1$. The independent components of the metric are given by

$$g_{11} = \frac{a^2(1-v^2)}{(1-u^2-v^2)^2}, \tag{1}$$

$$g_{12} = \frac{a^2uv}{(1-u^2-v^2)^2}, \tag{2}$$

$$g_{22} = \frac{a^2(1-u^2)}{(1-u^2-v^2)^2}, \tag{3}$$

the independent components of the inverse metric are given by

$$g^{11} = a^{-2}(1-u^2)(1-u^2-v^2), \tag{4}$$

$$g^{12} = -a^{-2}uv(1-u^2-v^2), \tag{5}$$

$$g^{22} = a^{-2}(1-v^2)(1-u^2-v^2), \tag{6}$$

and the independent, non-zero Christoffel symbols are given by

$$\Gamma^1_{11} = \frac{2u}{1-u^2-v^2}, \tag{7}$$

$$\Gamma^1_{12} = \frac{v}{1-u^2-v^2}, \tag{8}$$

$$\Gamma^2_{12} = \frac{u}{1-u^2-v^2}, \tag{9}$$

$$\Gamma^2_{22} = \frac{2v}{1-u^2-v^2}. \tag{10}$$

Remember that $g_{\alpha\beta}$, $g^{\alpha\beta}$, and $\Gamma^\lambda_{\alpha\beta}$ are all symmetric in α and β.

(a) Starting from (1)-(6), derive the expression (7) for Γ^1_{11}.

We begin with the expression for the Christoffel symbols in terms of the metric

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2}g^{\lambda\mu}\left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right) \tag{11}$$

We now calculate the values of $g_{\alpha\beta,\mu}$ from (1)-(3)

$$g_{11,1} = \frac{\partial g_{11}}{\partial x^1} = \frac{\partial \left(\frac{a^2(1-v^2)}{(1-u^2-v^2)^2} \right)}{\partial u} \tag{12}$$
$$= \frac{4a^2u(1-v^2)}{(1-u^2-v^2)^3} \tag{13}$$

$$g_{12,1} = \frac{\partial g_{12}}{\partial x^1} = \frac{\partial \left(\frac{a^2uv}{(1-u^2-v^2)^2} \right)}{\partial u} \tag{14}$$
$$= \frac{a^2v(3u^2-v^2+1)}{(1-u^2-v^2)^3} \tag{15}$$
$$= g_{21,1} \quad \text{by symmetry} \tag{16}$$
$$g_{22,1} = \frac{\partial g_{22}}{\partial x^1} = \frac{\partial \left(\frac{a^2(1-u^2)}{(1-u^2-v^2)^2} \right)}{\partial u} \tag{17}$$
$$= \frac{2a^2u(1-u^2+v^2)}{(1-u^2-v^2)^3} \tag{18}$$

By inspecting the components of the metric above, we see that $g_{11,1} \mapsto g_{22,2}$ with $u \leftrightarrow v$, similarly $g_{12,1} \mapsto g_{12,2}$ with $u \leftrightarrow v$, and $g_{11,2} \mapsto g_{22,1}$ with $u \leftrightarrow v$. Hence

$$g_{11,2} = \frac{2a^2v(1-v^2+u^2)}{(1-u^2-v^2)^3} \tag{19}$$
$$g_{12,2} = \frac{a^2u(3v^2-u^2+1)}{(1-u^2-v^2)^3} \tag{20}$$
$$= g_{21,2} \quad \text{by symmetry} \tag{21}$$
$$g_{22,2} = \frac{4a^2v(1-u^2)}{(1-u^2-v^2)^3} \tag{22}$$

So now we can evaluate the Christoffel symbols.

$$\Gamma^1_{11} = \frac{1}{2} g^{1\mu} \left(g_{\mu 1,1} + g_{\mu 1,1} - g_{11,\mu} \right) \tag{23}$$
$$= \frac{1}{2} \left[g^{11} g_{11,1} + g^{12} \left(2g_{21,1} - g_{11,2} \right) \right] \tag{24}$$
$$= \frac{1}{2} \Bigg[a^{-2}(1-u^2)(1-u^2-v^2) \frac{4a^2u(1-v^2)}{(1-u^2-v^2)^3}$$
$$+ -a^{-2}uv(1-u^2-v^2) \left(\frac{2a^2v(3u^2-v^2+1) - 2a^2v(1-v^2+u^2)}{(1-u^2-v^2)^3} \right) \Bigg] \tag{25}$$
$$= \frac{1}{2} \left[\frac{(1-u^2) \cdot 4u(1-v^2) - uv \cdot 2v(3u^2-u^2)}{(1-u^2-v^2)^2} \right] \tag{26}$$
$$= \frac{1}{2} \left[\frac{4u(1-u^2)(1-v^2) - 4u^2v^2}{(1-u^2-v^2)^2} \right] \tag{27}$$
$$= 2u \left[\frac{1-u^2-v^2+\cancel{u^2v^2-u^2v^2}}{(1-u^2-v^2)^2} \right] \tag{28}$$
$$= \frac{2u}{1-u^2-v^2} \tag{29}$$

as required. As an exercise, I have further derived the remaining Christoffel symbols

$$
\begin{aligned}
\Gamma^1_{12} &= \frac{1}{2} g^{1\mu} \left(g_{\mu 1,2} + g_{\mu 2,1} - g_{12,\mu} \right) && (30) \\
&= \frac{1}{2} \left[g^{11} g_{11,2} + g^{12} g_{22,1} \right] && (31) \\
&= \frac{1}{2} \left[a^{-2}(1-u^2)(1-u^2-v^2) \frac{2a^2 v(1-v^2+u^2)}{(1-u^2-v^2)^3} \right. \\
&\quad \left. + -a^{-2} uv(1-u^2-v^2) \frac{2a^2 u(1-u^2+v^2)}{(1-u^2-v^2)^3} \right] && (32) \\
&= \frac{1}{2} \left[\frac{(1-u^2)\cdot 2v(1-v^2-u^2) - uv \cdot 2u(1-u^2+v^2)}{(1-u^2-v^2)^2} \right] && (33) \\
&= v \left[\frac{(1-u^2)(1-v^2+u^2) - u^2(1-u^2+v^2)}{(1-u^2-v^2)^2} \right] && (34) \\
&= v \left[\frac{\cancel{1-u^2-v^2}}{(1-u^2-v^2)^{\cancel{2}}} \right] && (35) \\
&= \frac{v}{1-u^2-v^2} = \Gamma^1_{21}, \quad \text{by symmetry} && (36)
\end{aligned}
$$

as given.

$$
\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2} g^{2\mu} \left(g_{\mu 1,2} + g_{\mu 2,1} - g_{12,\mu} \right) && (37) \\
&= \frac{1}{2} \left[g^{21} g_{11,2} + g^{22} g_{22,1} \right] && (38)
\end{aligned}
$$

Now, we deduced earlier that $g_{11,2} = g_{22,1}|_{u \leftrightarrow v}$, we see by inspection of (6) that $g^{22} = g^{11}|_{u \leftrightarrow v}$, and by symmetry of the metric we have $g^{21} = g^{12}$. We note also that g^{12} is symmetric under interchange of u and v. Combining these results we find

$$
\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2} \left[g^{12} g_{22,1} + g^{11} g_{11,2} \right] |_{u \leftrightarrow v} && (39) \\
&= \Gamma^1_{12}|_{u \leftrightarrow v} && (40) \\
&= \frac{u}{1-u^2-v^2} && (41)
\end{aligned}
$$

as given.

$$
\begin{aligned}
\Gamma^2_{22} &= \frac{1}{2} g^{2\mu} \left(g_{\mu 2,2} + g_{\mu 2,2} - g_{22,\mu} \right) && (42) \\
&= \frac{1}{2} \left[g^{21} \left(2 g_{12,2} - g_{22,1} \right) + g^{22} g_{22,2} \right] && (43)
\end{aligned}
$$

Similar to our approach for Γ^2_{12}, we note by observation that $g_{12,2} = g_{21,1}|_{u \leftrightarrow v}$, $g_{22,1} = g_{11,2}|_{u \leftrightarrow v}$, $g_{22} = g_{11}|_{u \leftrightarrow v}$, and $g_{22,2} = g_{11,1}$, also noting that $g^{21} = g^{12} = g^{12}|_{u \leftrightarrow v}$. Thus we find

$$
\begin{aligned}
\Gamma^2_{22} &= \frac{1}{2} \left[g^{12} (2 g_{21,2} - g_{11,2}) + g^{11} g_{11,1} \right] |_{u \leftrightarrow v} && (44) \\
&= \Gamma^1_{11}|_{u \leftrightarrow v} && (45) \\
&= \frac{2v}{1-u^2-v^2} = \Gamma^2_{21}, \quad \text{by symmetry} && (46)
\end{aligned}
$$

as given.

(b) Prove that the Riemann tensor with all indices lowered, $R_{\alpha\beta\gamma\delta}$, contains four nonzero elements, any three of which can be written in terms of the fourth.

We recall the Bianchi identities for the Riemann tensor

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu} + g_{\beta\mu,\alpha\nu} - g_{\alpha\mu,\beta\nu} - g_{\beta\nu,\alpha\mu}\right) \tag{47}$$

Since $R_{\alpha\beta\mu\nu}$ is anti-symmetric under exchange $\alpha \leftrightarrow \beta$, and $\mu \leftrightarrow \nu$ also[1], we know

$$R_{\alpha\alpha\mu\nu} = 0 \tag{48}$$
$$R_{\alpha\beta\mu\mu} = 0 \tag{49}$$

for all α, β, μ, ν. Using this we greatly reduce the number of components to investigate. We find

$$R_{11\mu\nu} = 0 \qquad R_{\alpha\beta 11} = 0 \tag{50}$$
$$R_{22\mu\nu} = 0 \qquad R_{\alpha\beta 22} = 0 \tag{51}$$

We also know the Riemann tensor is symmetric under exchange of pairs $\alpha\beta \leftrightarrow \mu\nu$, i.e.

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \tag{52}$$

Hence we find

$$R_{1212} = -R_{1221} = R_{2121} = -R_{2112} \tag{53}$$

We can calculate the component R_{1212} (and hence determine the remaining components) as

$$\begin{aligned}
R_{1212} &= \frac{1}{2}\left(g_{12,21} + g_{21,12} - g_{11,22} - g_{22,11}\right) && (54)\\
&= g_{12,12} && (55)\\
&= \frac{\partial}{\partial v}\left(g_{12,1}\right) && (56)\\
&= \frac{\partial}{\partial v}\left(\frac{a^2 v(3u^2 - v^2 + 1)}{(1 - u^2 - v^2)^3}\right) && (57)\\
&= \frac{-a^2\left[3v^2 - 2v^2(9u^2+1) + (u^2-1)(3u^2+1)\right]}{(1-u^2-v^2)^4} && (58)
\end{aligned}$$

which is, in general, non-zero (note that 58) is symmetric under interchange $u \leftrightarrow v$, as we would expect).

(c) Prove that, in Klein's geometry, the Ricci tensor satisfies

$$R_{\alpha\beta} = -\frac{g_{\alpha\beta}}{a^2}, \tag{59}$$

[1] This can be easily proved from (47) by considering the symmetries of $g_{\alpha\beta,\mu\nu}$; however for brevity this proof is omitted.

and the Ricci scalar satifies

$$R = -\frac{2}{a^2}. \tag{60}$$

We can find the Ricci tensor by contracting the first and third indices of the Riemann tensor

$$\begin{aligned}
R_{\alpha\beta} &= R^{\mu}{}_{\alpha\mu\beta} && (61)\\
&= \Gamma^{\mu}{}_{\alpha\beta,\mu} - \Gamma^{\mu}{}_{\alpha\mu,\beta} + \Gamma^{\mu}{}_{\nu\mu}\Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\nu\beta}\Gamma^{\nu}{}_{\alpha\mu} && (62)\\
&= \Gamma^{1}{}_{\alpha\beta,1} + \Gamma^{2}{}_{\alpha\beta,2} - \Gamma^{1}{}_{\alpha1,\beta} - \Gamma^{2}{}_{\alpha2,\beta} + \Gamma^{1}{}_{\nu1}\Gamma^{\nu}{}_{\alpha\beta} + \Gamma^{2}{}_{\nu2}\Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{1}{}_{\nu\beta}\Gamma^{\nu}{}_{\alpha1} - \Gamma^{2}{}_{\nu\beta}\Gamma^{\nu}{}_{\alpha2} && (63)\\
&= \Gamma^{1}{}_{\alpha\beta,1} + \Gamma^{2}{}_{\alpha\beta,2} - \Gamma^{1}{}_{\alpha1,\beta} - \Gamma^{2}{}_{\alpha2,\beta} + \Gamma^{1}{}_{11}\Gamma^{1}{}_{\alpha\beta} + \Gamma^{1}{}_{21}\Gamma^{2}{}_{\alpha\beta} + \Gamma^{2}{}_{12}\Gamma^{1}{}_{\alpha\beta} + \Gamma^{2}{}_{22}\Gamma^{2}{}_{\alpha\beta}\\
&\quad - \Gamma^{1}{}_{1\beta}\Gamma^{1}{}_{\alpha1} - \Gamma^{1}{}_{2\beta}\Gamma^{2}{}_{\alpha1} - \Gamma^{2}{}_{1\beta}\Gamma^{1}{}_{\alpha2} - \Gamma^{2}{}_{2\beta}\Gamma^{2}{}_{\alpha2} && (64)
\end{aligned}$$

from Schutz (6.63).

We can easily calculate the values of $\Gamma^{\alpha}_{\beta\mu,\nu}$:

$$\Gamma^{1}_{11,1} = \frac{2(1-u^2+v^2)}{(1-u^2-v^2)^2} \qquad \Gamma^{1}_{11,2} = \frac{4uv}{(1-u^2-v^2)^2} \tag{65}$$

$$\Gamma^{1}_{12,1} = \frac{2uv}{(1-u^2-v^2)^2} \qquad \Gamma^{1}_{12,2} = \frac{1-u^2+v^2}{(1-u^2-v^2)^2} \tag{66}$$

$$\Gamma^{2}_{12,1} = \frac{1+u^2-v^2}{(1-u^2-v^2)^2} \qquad \Gamma^{2}_{12,2} = \frac{2uv}{(1-u^2-v^2)^2} \tag{67}$$

$$\Gamma^{2}_{22,1} = \frac{4uv}{(1-u^2-v^2)^2} \qquad \Gamma^{2}_{22,2} = \frac{2(1-u^2+v^2)}{(1-u^2-v^2)^2} \tag{68}$$

with all others zero.

We shall calculate each component of $R_{\alpha\beta}$ below using (64)

$$\begin{aligned}
R_{11} &= \cancel{\Gamma^{1}{}_{11,1}} + \Gamma^{2}{}_{11,2} - \cancel{\Gamma^{1}{}_{11,1}} - \Gamma^{2}{}_{12,1} + \cancel{\Gamma^{1}{}_{11}\Gamma^{1}{}_{11}} + \cancel{\Gamma^{1}{}_{21}\Gamma^{2}{}_{11}} + \Gamma^{2}{}_{12}\Gamma^{1}{}_{11} + \Gamma^{2}{}_{22}\Gamma^{2}{}_{11}\\
&\quad - \cancel{\Gamma^{1}{}_{11}\Gamma^{1}{}_{11}} - \cancel{\Gamma^{1}{}_{21}\Gamma^{2}{}_{11}} - \Gamma^{2}{}_{11}\Gamma^{1}{}_{12} - \Gamma^{2}{}_{21}\Gamma^{2}{}_{12} && (69)
\end{aligned}$$

by cancellations,

$$= \cancel{\Gamma^{2}{}_{11,2}} - \Gamma^{2}{}_{12,1} + \Gamma^{2}{}_{12}\Gamma^{1}{}_{11} + \cancel{\Gamma^{2}{}_{22}\Gamma^{2}{}_{11}} - \cancel{\Gamma^{2}{}_{11}\Gamma^{1}{}_{12}} - \Gamma^{2}{}_{12}\Gamma^{2}{}_{12} \tag{70}$$

since $\Gamma^{2}{}_{11} = 0$,

$$\begin{aligned}
&= \Gamma^{2}{}_{12}\Gamma^{1}{}_{11} - \Gamma^{2}{}_{12}\Gamma^{2}{}_{12} - \Gamma^{2}{}_{12,1} && (71)\\
&= \frac{1}{(1-u^2-v^2)}\left[u\cdot 2u - u\cdot u - (1+u^2-v^2)\right] && (72)\\
&= \frac{1}{(1-u^2-v^2)^2}\left[2u^2 - u^2 - 1 - u^2 + v^2\right] && (73)\\
&= -\frac{(1-v^2)}{(1-u^2-v^2)^2} && (74)\\
&= -\frac{1}{a^2}\frac{a^2(1-v^2)}{(1-u^2-v^2)^2} && (75)\\
&= -\frac{g_{11}}{a^2} && (76)
\end{aligned}$$

as required.

$$\begin{aligned}
R_{12} &= \Gamma^1{}_{12,1} + \cancel{\Gamma^2{}_{12,2}} - \Gamma^1{}_{11,2} - \cancel{\Gamma^2{}_{12,2}} + \cancel{\Gamma^1{}_{11}\Gamma^1{}_{12}} + \Gamma^1{}_{21}\Gamma^2{}_{12} + \cancel{\Gamma^2{}_{12}\Gamma^1{}_{12}} + \cancel{\Gamma^2{}_{22}\Gamma^2{}_{12}} \\
&\quad - \cancel{\Gamma^1{}_{12}\Gamma^1{}_{11}} - \cancel{\Gamma^1{}_{22}\Gamma^2{}_{11}} - \cancel{\Gamma^2{}_{12}\Gamma^1{}_{12}} - \cancel{\Gamma^2{}_{22}\Gamma^2{}_{12}} && (77) \\
&= \Gamma^1{}_{12,1} - \Gamma^1{}_{11,2} + \Gamma^1{}_{21}\Gamma^2{}_{12} && (78) \\
&= \frac{1}{(1-u^2-v^2)}\left[2uv - 4uv + v \cdot u\right] && (79) \\
&= -\frac{uv}{(1-u^2-v^2)^2} && (80) \\
&= -\frac{1}{a^2}\frac{a^2uv}{(1-u^2-v^2)^2} && (81) \\
&= -\frac{g_{12}}{a^2} && (82)
\end{aligned}$$

as required (note by symmetry, $R_{12} = R_{21}$, so $R_{21} = -\frac{g_{21}}{a^2}$).

$$\begin{aligned}
R_{22} &= \cancel{\Gamma^1{}_{22,1}} + \cancel{\Gamma^2{}_{22,2}} - \Gamma^1{}_{21,2} - \cancel{\Gamma^2{}_{22,2}} + \cancel{\Gamma^1{}_{11}\Gamma^1{}_{22}} + \Gamma^1{}_{21}\Gamma^2{}_{22} + \cancel{\Gamma^2{}_{12}\Gamma^1{}_{22}} + \cancel{\Gamma^2{}_{22}\Gamma^2{}_{22}} \\
&\quad - \Gamma^1{}_{12}\Gamma^1{}_{21} - \cancel{\Gamma^1{}_{22}\Gamma^2{}_{21}} - \cancel{\Gamma^2{}_{12}\Gamma^1{}_{22}} - \cancel{\Gamma^2{}_{22}\Gamma^2{}_{22}} && (83) \\
&= \Gamma^1{}_{21}\Gamma^2{}_{22} - \Gamma^1{}_{12}\Gamma^1{}_{21} - \Gamma^1{}_{21,2} && (84) \\
&= \frac{1}{(1-u^2-v^2)^2}\left[v \cdot 2v - v \cdot v - (1-u^2+v^2)\right] && (85) \\
&= \frac{1}{(1-u^2-v^2)^2}\left[2v^2 - v^2 - 1 + u^2 - v^2)\right] && (86) \\
&= -\frac{(1-u^2)}{(1-u^2-v^2)^2} && (87) \\
&= -\frac{1}{a^2}\frac{a^2(1-u^2)}{(1-u^2-v^2)^2} && (88) \\
&= -\frac{g_{22}}{a^2} && (89)
\end{aligned}$$

as required.

$$(90)$$

Hence we have proved that, in Klen's geometry,

$$R_{\alpha\beta} = -\frac{g_{\alpha\beta}}{a^2} \tag{91}$$

The Ricci scalar is formed by contracting the Ricci tensor:

$$\begin{aligned}
R &= g^{\alpha\beta}R_{\alpha\beta} && (92) \\
&= -g^{\alpha\beta}\frac{g_{\alpha\beta}}{a^2} && (93) \\
&= -\frac{2}{a^2} && (94)
\end{aligned}$$

since $g^{\alpha\beta}g_{\alpha\beta} = g^{\alpha\beta}g_{\beta\alpha} = \delta^\alpha{}_\alpha = 2$ in Klein's geometry.

(d) Answer each of the following questions in one or two sentences.

i. In what fundamental way does Klein's geometry differ from a two-sphere?

A 2-sphere has positive curvature (the geodesics converge). By contrast, Klein's geometry has negative curvature!

ii. The hyperbola $x^2 - y^2 = 1$ is rotated around the y-axis to form a three-dimensional hyperboloid of revolution. Does it possess positive or negative curvature? Justify your answer physically with a diagram; do not attempt to calculate anything.

Along surfaces with positive curvature, geodesics will converge as they travel along the surface. By contrast, geodesics along surfaces of negative curvature will diverge.

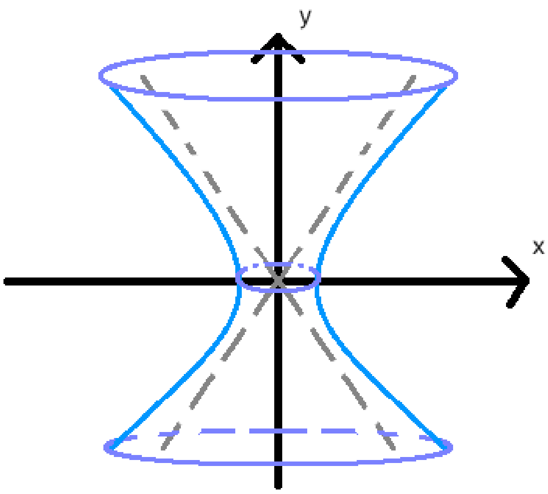

Figure 1: $x^2 - y^2 = 1$ rotated about the y-axis

We see that geodesics travelling along the surface would diverge; hence this hyperboloid possesses negative curvature. A triangle on the surface would look like Figure 2, with the sum of its angles $< 180°$.

iii. The hyperbola $x^2 - y^2 = 1$ is now rotated around the x-axis. What is the sign of the curvature this time? Why?

We see that geodesics travelling along the surface would converge; hence this hyperboloid possesses positive curvature. A triangle on the surface would look like Figure 4 below, with the sum of its angles $> 180°$.

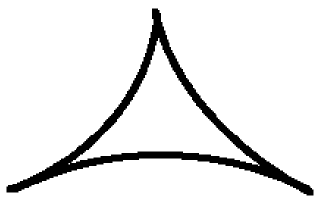

Figure 2: Triangle with $\Sigma < 180°$

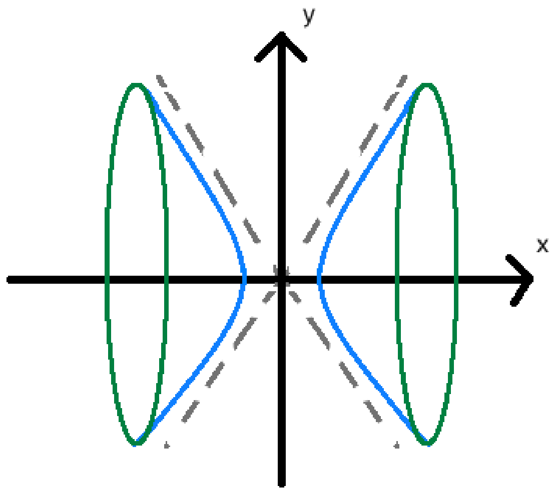

Figure 3: $x^2 - y^2 = 1$ rotated about the x-axis

iv. Setting aside their dimensionality, in what fundamental way do the hyperboloids of revolution in parts (d)(ii) and (d)(iii) differ from Klein's geometry? Justify your answer in words; don't try to calculate anything.

The hyperboloids differ from Klein's geometry in that geodesics can "traverse" around the space and return to their origin (for example, by travelling on a geodesic around the hyperboloid). Further, for these hyperboloids we have non-constant curvature.

In the hyperboloid in Figure 3, we note that there are obvious discontinuities. There are no discontinuities in Klein's geometry, which is an infinite plane.

v. Identify a spacetime manifold, that resembles Klein's geometry. Don't worry too much about the precise mathematical meaning of "resembles", a qualitative justification is fine.

We can consider anti-de Sitter space to resemble Klein's geometry. We would find that both have negative scalar curvature. The anti-de Sitter space is a maximally symmetric Lorenzian manifold (hence have a different signature metric), but "locally" the manifold would resemble Klein's geometry.

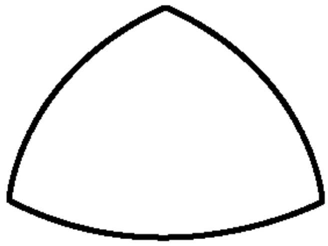

Figure 4: Triangle with $\Sigma < 180°$

(e) Consider the triangle ΔABC, whose sides are "straight lines" (geodesics) joining the points A$(0,0)$, B$(b,0)$, and C$(0,b)$, with $b < 1$. It is easy to show (you don't need to!) that the sides AB and AC are just the curves $v = 0$ and $u = 0$ respectively.

i. What is the equation of the geodesic joining B and C?

From Schutz (6.51), letting λ be the parameter of the geodesic with $x^1 = u$ and $x^2 = v$ we have the geodesic equation

$$\frac{d}{d\lambda}\left(\frac{dx^\alpha}{d\lambda}\right) + \Gamma^\alpha{}_{\mu\beta}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{95}$$

For $\alpha = 1$ we have

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{du}{d\lambda}\right) &= -\Gamma^1{}_{\mu\beta}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda} && (96)\\
&= -\Gamma^1{}_{11}\frac{dx^1}{d\lambda}\frac{dx^1}{d\lambda} - \Gamma^1{}_{12}\frac{dx^1}{d\lambda}\frac{dx^2}{d\lambda} - -\Gamma^1{}_{21}\frac{dx^2}{d\lambda}\frac{dx^1}{d\lambda} - \cancel{-\Gamma^1{}_{22}\frac{dx^2}{d\lambda}\frac{dx^2}{d\lambda}} && (97)\\
&= -\Gamma^1{}_{11}\left(\frac{du}{d\lambda}\right)^2 - 2\Gamma^1{}_{12}\frac{du}{d\lambda}\frac{dv}{d\lambda} && (98)
\end{aligned}$$

which can be written as

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{du}{d\lambda}\right) + \Gamma^1{}_{11}\left(\frac{du}{d\lambda}\right)^2 + 2\Gamma^1{}_{12}\frac{du}{d\lambda}\frac{dv}{d\lambda} &= 0 && (99)\\
\Rightarrow \frac{d}{d\lambda}\left(\frac{du}{d\lambda}\right) + 2u(1-u^2-v^2)^{-1}\left(\frac{du}{d\lambda}\right)^2 + 2v(1-u^2-v^2)^{-1}\frac{du}{d\lambda}\frac{dv}{d\lambda} &= 0 && (100)\\
\Rightarrow \frac{d}{d\lambda}\left(\frac{du}{d\lambda}\right)(1-u^2-v^2)^{-1} + 2u(1-u^2-v^2)^{-2}\left(\frac{du}{d\lambda}\right)^2 + 2v(1-u^2-v^2)^{-2}\frac{du}{d\lambda}\frac{dv}{d\lambda} &= 0 && (101)\\
\Rightarrow \frac{d}{d\lambda}\left[(1-u^2-v^2)^{-1}\frac{du}{d\lambda}\right] &= 0 && (102)
\end{aligned}$$

We shall prove that going from (101) to (102) is correct.

$$\begin{aligned}
\frac{d}{d\lambda}\left[(1-u^2-v^2)^{-1}\frac{du}{d\lambda}\right] &= \frac{d}{d\lambda}\left[(1-u^2-v^2)^{-1}\right)\frac{du}{d\lambda} + (1-u^2-v^2)^{-1}\frac{d^2u}{d\lambda^2} && (103)\\
&= \left[-\frac{d}{d\lambda}\left(-u^2\right) + -\frac{d}{d\lambda}\left(-v^2\right)\right](1-u^2-v^2)^{-2}\frac{du}{d\lambda} + (1-u^2-v^2)^{-1}\frac{d^2u}{d\lambda^2} && (104)\\
&= \left[\frac{du}{d\lambda}\frac{d(u^2)}{du} + \frac{dv}{d\lambda}\frac{d(v^2)}{dv}\right](1-u^2-v^2)^{-2}\frac{du}{d\lambda} + (1-u^2-v^2)^{-1}\frac{d^2u}{d\lambda^2} && (105)\\
&= \left[2u\frac{du}{d\lambda} + 2v\frac{dv}{d\lambda}\right](1-u^2-v^2)^{-2}\frac{du}{d\lambda} + (1-u^2-v^2)^{-1}\frac{d^2u}{d\lambda^2} && (106)\\
&= \text{LHS of (101)} && (107)
\end{aligned}$$

Hence this is indeed correct.

For $\alpha = 2$ we have

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{dv}{d\lambda}\right) &= -\Gamma^2{}_{\mu\beta}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda} && (108)\\
&= \cancel{-\Gamma^2{}_{11}\frac{dx^1}{d\lambda}\frac{dx^1}{d\lambda}} - \Gamma^2{}_{12}\frac{dx^1}{d\lambda}\frac{dx^2}{d\lambda} - -\Gamma^2{}_{21}\frac{dx^2}{d\lambda}\frac{dx^1}{d\lambda} - -\Gamma^2{}_{22}\frac{dx^2}{d\lambda}\frac{dx^2}{d\lambda} && (109)\\
&= -\Gamma^2{}_{22}\left(\frac{dv}{d\lambda}\right)^2 - 2\Gamma^2{}_{12}\frac{du}{d\lambda}\frac{dv}{d\lambda} && (110)\\
&= -2v(1-u^2-v^2)^{-1}\left(\frac{dv}{d\lambda}\right)^2 - 2u(1-u^2-v^2)^{-1}\frac{du}{d\lambda}\frac{dv}{d\lambda} && (111)
\end{aligned}$$

By comparison with (100)-(102), we see that this implies

$$\frac{d}{d\lambda}\left[(1-u^2-v^2)^{-1}\frac{dv}{d\lambda}\right] = 0 \tag{112}$$

We now find

$$\begin{aligned}
\frac{d}{d\lambda}\left[(1-u^2-v^2)^{-1}\frac{du}{d\lambda}\right] = 0 &\Rightarrow (1-u^2-v^2)^{-1}\frac{du}{d\lambda} = c_1 && (113)\\
\frac{d}{d\lambda}\left[(1-u^2-v^2)^{-1}\frac{dv}{d\lambda}\right] = 0 &\Rightarrow (1-u^2-v^2)^{-1}\frac{dv}{d\lambda} = c_2 && (114)
\end{aligned}$$

Dividing (113) by (114) we find

$$\begin{aligned}
\frac{du}{dv} &= \frac{c_1}{c_2} \equiv c_3 && (115)\\
\Rightarrow v &= c_3 u + c_4 && (116)
\end{aligned}$$

Since the point $(0, b)$ lies on the geodesic we have,

$$\begin{aligned}
b &= c_4 && (117)\\
\Rightarrow c_4 &= b && (118)
\end{aligned}$$

Similarly using the point $(b, 0)$,

$$0 = bc_3 + b \tag{119}$$
$$\Rightarrow c_3 = -1 \tag{120}$$

Thus we conclude that equation of the geodesic joining B and C is

$$v = -u + b \tag{121}$$

ii. Prove that the sum of the interior angles of ΔABC is

$$\Sigma = \angle\text{ABC} + \angle\text{BCA} + \angle\text{CAB} = \frac{\pi}{2} + 2\cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right). \tag{122}$$

The sum of the angles is less than 180 degrees!

The cosine of the angle between two vectors $\vec{M}$ and $\vec{N}$ is given by

$$\cos\theta = \frac{\vec{M}\cdot\vec{N}}{|\vec{M}||\vec{N}|} \tag{123}$$

In component notation we have

$$\theta = \cos^{-1}\left(\frac{g_{\alpha\beta}M^\alpha N^\beta}{\sqrt{g_{ij}M^i M^j g_{kl}N^k N^l}}\right) \tag{124}$$

The angle between two curves at a point is equal to the angle between curves pointing in the same direction;

The equations bounding the triangle parametrised by λ are given as

$$\vec{a} = (\lambda, 0) \tag{125}$$
$$\vec{b} = (0, \lambda) \tag{126}$$
$$\vec{c} = (\lambda, b - \lambda) \tag{127}$$

where $\lambda \leq b$.

$$\Rightarrow \dot{\vec{a}} = (1, 0) \tag{128}$$
$$\dot{\vec{b}} = (0, 1) \tag{129}$$
$$\dot{\vec{c}} = (1, -1) \tag{130}$$

We shall use these direction vectors WHY????.

Calculating $\angle$ABC

We use $\vec{M} = \vec{a}$ and $\vec{N} = \vec{c}$. At the point B$(b, 0)$,

$$g_{11} = \frac{a^2}{(1-b^2)^2} \tag{131}$$

$$g_{12} = 0 \tag{132}$$

$$g_{22} = \frac{a^2}{1-b^2} \tag{133}$$

Hence we find

$$\angle\text{ABC} = \cos^{-1}\left(\frac{g_{11}}{\sqrt{g_{11}g_{11} + g_{11}g_{22}}}\right) \tag{134}$$

$$= \cos^{-1}\left(\frac{\frac{a^2}{(1-b^2)^2}}{\sqrt{\frac{a^4}{(1-b^2)^4} + \frac{a^4}{(1-b^2)^3}}}\right) \tag{135}$$

$$= \cos^{-1}\left(\frac{a^2}{(1-b^2)^2}\sqrt{\frac{1+(1-b^2)}{(1-b^2)^4}}\right) \tag{136}$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right) \tag{137}$$

Calculating $\angle$BCA

We use $\vec{M} = \vec{b}$ and $\vec{N} = \vec{c}$. At the point C$(0, b)$,

$$g_{11} = \frac{a^2}{1-b^2} \tag{138}$$

$$g_{12} = 0 \tag{139}$$

$$g_{22} = \frac{a^2}{(1-b^2)^2} \tag{140}$$

Hence we find

$$\angle\text{BCA} = \cos^{-1}\left(\frac{g_{22}}{\sqrt{g_{22}g_{11} + g_{22}g_{22}}}\right) \tag{141}$$

$$= \cos^{-1}\left(\frac{\frac{a^2}{(1-b^2)^2}}{\sqrt{\frac{a^4}{(1-b^2)^4} + \frac{a^4}{(1-b^2)^3}}}\right) \tag{142}$$

$$= \cos^{-1}\left(\frac{a^2}{(1-b^2)^2}\sqrt{\frac{1+(1-b^2)}{(1-b^2)^4}}\right) \tag{143}$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right) \tag{144}$$

Calculating $\angle$ABC

We use $\vec{M} = \vec{a}$ and $\vec{N} = \vec{c}$. At the point B$(b, 0)$,

$$g_{11} = \frac{a^2}{(1-b^2)^2} \tag{145}$$

$$g_{12} = 0 \tag{146}$$

$$g_{22} = \frac{a^2}{1-b^2} \tag{147}$$

Hence we find

$$\angle\text{ABC} = \cos^{-1}\left(\frac{g_{11}}{\sqrt{g_{11}g_{11} + g_{11}g_{22}}}\right) \tag{148}$$

$$= \cos^{-1}\left(\frac{\frac{a^2}{(1-b^2)^2}}{\sqrt{\frac{a^4}{(1-b^2)^4} + \frac{a^4}{(1-b^2)^3}}}\right) \tag{149}$$

$$= \cos^{-1}\left(\frac{a^2}{(1-b^2)^2}\sqrt{\frac{1+(1-b^2)}{(1-b^2)^4}}\right) \tag{150}$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right) \tag{151}$$

Calculating $\angle$CAB

We use $\vec{M} = \vec{a}$ and $\vec{N} = \vec{b}$. At the point A$(0, 0)$,

$$g_{11} = a^2 \tag{152}$$

$$g_{12} = 0 \tag{153}$$

$$g_{22} = a^2 \tag{154}$$

Hence we find

$$\angle\text{BCA} = \cos^{-1}\left(\frac{\cancel{g_{12}}}{\sqrt{g_{11}g_{22}}}\right) \tag{155}$$

$$= \cos^{-1}(0) \tag{156}$$

$$= \frac{\pi}{2} \tag{157}$$

Summing these angles we indeed find

$$\Sigma = \angle\text{ABC} + \angle\text{BCA} + \angle\text{CAB} = \frac{\pi}{2} + 2\cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right) \tag{158}$$

as required.

iii. Triangles in Klein's geometry can have $\sum = 0$! Without proof, sketch what such a triangle might look like. Your sketch by necessity will be an incomplete representation; there is no way to draw a Klein triangle faithfully on a flat page.

A triangle in space with negative curvature will have $\Sigma < 180°$, as shown in Figure 2. One can imagine a triangle such as Figure 5 below.

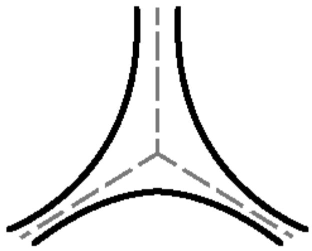

Figure 5: Triangle with $\Sigma = 0°$

If the dotted lines extended to infinity, we would have a triangle with $\Sigma = 0°$. In flat space, the lines would look somewhat parallel.

(f) i. Write down a closed form expression for the area A of ΔABC as an integral over a subset of the (u, v) domain.

Surface area depends not on the parameterisation of the space, but only on the surface itself. We have an expression for area

$$A = \int_{\text{surface}} \sqrt{\det g}\ \mathrm{d}A \tag{159}$$

where g is the metric. We have

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{160}$$

$$\Rightarrow \det g = g_{11}g_{22} - g_{12}g_{21} \tag{161}$$

$$= \frac{a^4(1-v^2)(1-u^2) - a^4u^2v^2}{(1-u^2-v^2)^4} \tag{162}$$

$$= \frac{a^4}{(1-u^2-v^2)^4}\left[1 - u^2 - v^2 + \cancel{u^2v^2} - \cancel{u^2v^2}\right] \tag{163}$$

$$= \frac{a^4}{(1-u^2-v^2)^3} \tag{164}$$

We can now determine an expression for the area of ΔABC as

$$A = \iint \frac{a^4}{(1-u^2-v^2)^3}\ \mathrm{d}u\ \mathrm{d}v \tag{165}$$

$$= \int_{v=0}^{v=b} \int_{u=0}^{u=b-v} \frac{a^2}{(1-u^2-v^2)^{3/2}}\ \mathrm{d}u\ \mathrm{d}v \tag{166}$$

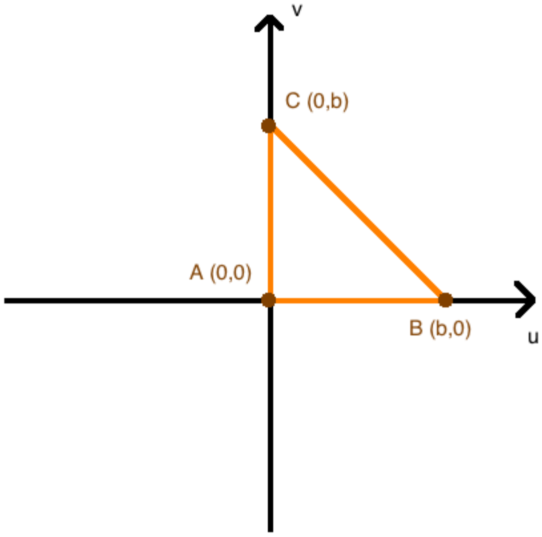

Figure 6: The triangle ΔABC on the $u - v$ axis

ii. By changing variables to $y = v + u$ and $z = v - u$, recast your integral in the form

$$A = 2a^2 \int_0^b \frac{\mathrm{d}y \; y}{(2 - y^2)\sqrt{1 - y^2}}. \tag{167}$$

Hence show that one has

$$A = a^2(\pi - \Sigma). \tag{168}$$

We begin by calculating the Jacobian,

$$J(y,z) = \begin{pmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{169}$$

$$\Rightarrow \det J = \frac{1}{2} \tag{170}$$

Next we consider the bounds of the integral; that is, the surface we are integrating over.

$$(u,v) = (0,0) \Rightarrow (y,z) = (0,0) \tag{171}$$

$$(u,v) = (b,0) \Rightarrow (y,z) = (b,-b) \tag{172}$$

$$(u,v) = (0,b) \Rightarrow (y,z) = (b,b) \tag{173}$$

In $y - z$ space, our surface looks like Figure 7 above. Just as we integrated over first u in terms of v then integrated over v in (166), we shall integrate from $z = -y$ to $z = y$, then over $y = 0$ to $y = b$.

Also note that we can express u and v in terms of y and z as

$$u = \frac{1}{2}(y - z) \tag{174}$$

$$v = \frac{1}{2}(y + z) \tag{175}$$

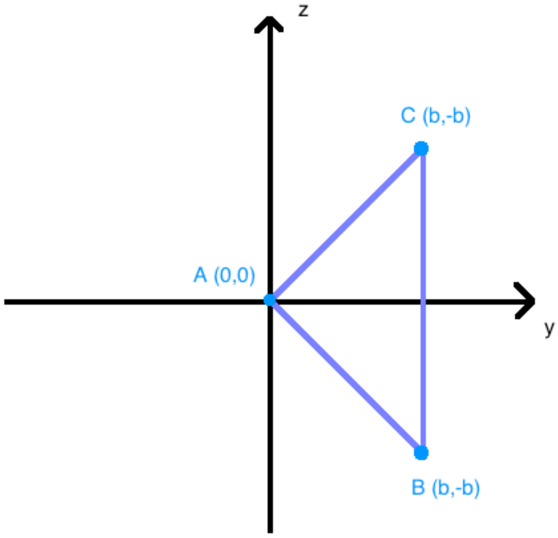

Figure 7: The triangle ΔABC on the $y - z$ axis

Hence we can rewrite A as

$$A = 2a^2 \int_{y=0}^{y=b} \int_{z=-y}^{z=y} \left(1 - \left[\frac{1}{2}(y-z)\right]^2 - \left[\frac{1}{2}(y+z)\right]^2\right)^{-3/2} \frac{1}{4}\,\mathrm{d}y\,\mathrm{d}z \tag{176}$$

$$= \frac{a^2}{2} \int_{y=0}^{y=b} \int_{z=-y}^{z=y} \left(1 - \frac{1}{2}y^2 - \frac{1}{2}z^2\right)^{-3/2} \mathrm{d}y\,\mathrm{d}z \tag{177}$$

We now let $m = 1 - \frac{1}{2}y^2$, and let $z = \sqrt{2m}\sin r \Rightarrow \mathrm{d}z = \sqrt{2m}\cos r\,\mathrm{d}r$

$$A = \frac{a^2}{2} \int_{y=0}^{y=b} \int_{r=-\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)}^{r=\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)} \Big(\underbrace{m - m\sin^2 r}_{=m(\cos^2 r)}\Big)^{-3/2} \mathrm{d}y\,\sqrt{2m}\cos r\,\mathrm{d}r \tag{178}$$

$$= \frac{a^2}{2} \iint \sqrt{2}m^{-1}\cos^{-2} r\,\mathrm{d}y\,\mathrm{d}r \tag{179}$$

$$= \frac{a^2}{\sqrt{2}} \int_{y=0}^{y=b} m^{-1} \Big[\tan(r)\Big]_{-\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)}^{\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)} \mathrm{d}y \tag{180}$$

$$= \frac{a^2}{\sqrt{2}} \int m^{-1} \left[\frac{2y}{\sqrt{2-2y^2}}\right] \mathrm{d}y \tag{181}$$

In moving from (180) to (181) we use the identity

$$\tan\left[\sin^{-1}(ax)\right] = \frac{ax}{\sqrt{1-a^2x^2}} \tag{182}$$

$$\Rightarrow \tan\left[\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)\right] = \frac{\frac{y}{\sqrt{2m}}}{\sqrt{1-\frac{y^2}{2m}}} \tag{183}$$

$$= \frac{y}{\sqrt{2m-y^2}} \tag{184}$$

Since $m = 1 - \frac{1}{2}y^2$ we find

$$\tan\left[\sin^{-1}\left(\frac{y}{\sqrt{2m}}\right)\right] = \frac{y}{\sqrt{2-2y}} \tag{185}$$

Continuing from (181),

$$\begin{aligned}
A &= \frac{a^2}{\sqrt{2}} \int_0^b m^{-1} \left[\frac{2y}{\sqrt{2-2y^2}}\right] \, \mathrm{d}y &(186)\\
&= a^2 \int_0^b \frac{1}{1-\frac{1}{2}y^2} \frac{y}{\sqrt{1-y^2}} \, \mathrm{d}y &(187)\\
&= 2a^2 \int_0^b \frac{\mathrm{d}y \; y}{(2-y^2)\sqrt{1-y^2}} &(188)
\end{aligned}$$

as required.

We will now show that $A = a^2(\pi - \Sigma)$.

$$A = 2a^2 \int_0^b \frac{\mathrm{d}y \; y}{(2-y^2)\sqrt{1-y^2}} \tag{189}$$

Let $p = \sqrt{1-y^2}$

$$\begin{aligned}
\Rightarrow \frac{dp}{dy} &= \frac{-2y \times \frac{1}{2}}{\sqrt{1-y^2}} = -\frac{y}{\sqrt{1-y^2}} = -\frac{y}{p} &(190)\\
\Rightarrow A &= 2a^2 \int_{p=1}^{p=\sqrt{1-b^2}} \frac{1}{1+p^2} \frac{1}{p} y \times \frac{-p}{y} \, \mathrm{d}p &(191)\\
&= -2a^2 \int_1^{\sqrt{1-b^2}} \frac{1}{1+p^2} \, \mathrm{d}p &(192)\\
&= 2a^2 \int_{\sqrt{1-b^2}}^1 \frac{1}{1+p^2} \, \mathrm{d}p &(193)\\
&= 2a^2 \left[\tan^{-1}(p)\right]_{\sqrt{1-b^2}}^1 &(194)\\
&= 2a^2 \left[\tan^{-1}(1) - \tan^{-1}\left(\sqrt{1-b^2}\right)\right] &(195)\\
&= 2a^2 \left[\frac{\pi}{4} - \tan^{-1}\left(\sqrt{1-b^2}\right)\right] &(196)
\end{aligned}$$

We now consider the remaining $\tan^{-1}$ term.

$$\tan^{-1}(x) = \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) \tag{197}$$

$$\Rightarrow \tan^{-1}\left(\sqrt{1-b^2}\right) = \sin^{-1}\left(\frac{\sqrt{1-b^2}}{\sqrt{2-b^2}}\right) \tag{198}$$

$$= \sin^{-1}\left(\sqrt{\frac{2-b^2-1}{2-b^2}}\right) \tag{199}$$

$$= \sin^{-1}\left(\sqrt{1-\frac{1}{2-b^2}}\right) \tag{200}$$

$$\cos^{-1}(x) = \sin^{-1}\left(\sqrt{1-x^2}\right), \quad \text{if } 0 \leq x \leq 1 \tag{201}$$

$$\Rightarrow \tan^{-1}\left(\sqrt{1-b^2}\right) = \sin^{-1}\left(\sqrt{1-\frac{1}{2-b^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right) \tag{202}$$

Substituting this into (196) we conclude

$$A = 2a^2\left[\frac{\pi}{4} - \cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right)\right] \tag{203}$$

$$= a^2\left[\frac{\pi}{2} - 2cos^{-1}\left(\frac{1}{\sqrt{2-b^2}}\right)\right] \tag{204}$$

$$= a^2\left(\frac{\pi}{2} - \Sigma\right) \tag{205}$$

as required.

iii. Explain briefly, in one or two sentences, why (168) guarantees the *nonexistence* of similar triangles in Klein's geometry.

The equation

$$A = a^2\left(\frac{\pi}{2} - \Sigma\right)$$

tells us that in Klein's geometry, the area of a triangle is linked directly to the sum of its internal angles. Similar triangles possess the same internal angles and hence the same sum of internal angles; however since equation (168) gives only a single value of area for any Σ, we see that the area is fixed, and hence similar triangles cannot exist in Klein's geometry.

(g) A vector $\vec{W}$ with equal components W^1 and W^2 at the point $A(0,0)$ is parallel transported along the geodesic AB. Show that its components, when it reaches the point B$(b,0)$, are in the ratio

$$\frac{W^1}{W^2} = (1-b^2)^{1/2} \tag{206}$$

Parallel transport of a vector $\vec{V}$ along a curve s is given by

$$U^\beta V^\alpha{}_{;\beta} = 0 \tag{207}$$

As V does not change ove the geodesic AB, we know $x^\beta = u = x^1$.

We have two geodesic equations for the transport of $\vec{W}$ along AB:

$$\frac{\partial W^1}{\partial u} + \Gamma^1{}_{11} W^1 + \Gamma^1_{ph121} W^2 = 0 \tag{208}$$
$$\frac{\partial W^2}{\partial u} + \Gamma^2{}_{21} W^2 = 0 \tag{209}$$

Solving the second equation gives us

$$\frac{\partial W^2}{\partial u} + \frac{u}{1-u^2-v^2} W^2 = 0 \tag{210}$$
$$\Rightarrow \frac{\mathrm{d}W^2}{W^2} = -\frac{u\ \mathrm{d}u}{1-u^2-v^2} \tag{211}$$
$$\log W^2 = \frac{1}{2}\log(1-u^2-v^2) + c_1 \tag{212}$$
$$W^2 = c_1\sqrt{1-u^2-v^2} \tag{213}$$

Now, substituting (213) into (208) we find

$$\frac{\partial W^1}{\partial u} + \frac{2u}{1-u^2-v^2} W^1 = \frac{c_1 v}{\sqrt{1-u^2-v^2}} = 0 \tag{214}$$
$$(1-u^2-v^2)^{-1}\frac{\mathrm{d}W^1}{\mathrm{d}u} + \frac{2u}{(1-u^2-v^2)^2} W^1 + \frac{c_1 v}{(1-u^2-v^2)^{3/2}} = 0 \tag{215}$$
$$\frac{\partial}{\partial u}\left(\frac{W^1}{1-u^2-v^2}\right) = -\frac{c_1 v}{(1-u^2-v^2)^{3/2}} \tag{216}$$
$$\frac{W^1}{1-u^2-v^2} = -c_1 \int \frac{v\ \mathrm{d}v}{(1-u^2-v^2)^{3/2}} \tag{217}$$

We shall solve this integral through some clever substitution. We let

$$k = 1-v^2, \tag{218}$$
$$m = \sqrt{k}\sin n \tag{219}$$

Then, we have

$$\Rightarrow \int \frac{v\ \mathrm{d}v}{(1-u^2-v^2)^{3/2}} = \frac{1}{(k-u^2)^{3/2}} \tag{220}$$
$$= k^{-3/2}\int \sec^2 n\ \mathrm{d}n \tag{221}$$
$$= k^{-3/2}\tan n + c_2 \tag{222}$$
$$= k^{-3/2}\frac{n}{\sqrt{k^2-n^2}} + c_2 \tag{223}$$
$$= \left(1-v^2)\right)^{-3/2}\frac{u}{\sqrt{1-u^2-v^2}} + c_2 \tag{224}$$

So now we have

$$\left(1-u^2-v^2\right)^{-1} W^1 = -c_1\left(1-v^2\right)^{-3/2} \frac{vu}{\sqrt{1-u^2-v^2}} + c_3 \tag{225}$$

$$\Rightarrow W^1 = -c_1\left(1-v^2\right)^{-3/2} vu\sqrt{1-u^2-v^2} + c_3\left(1-u^2-v^2\right) \tag{226}$$

We can now combine (226) and (213) to find the ratio $\frac{W^1}{W^2}$.

At $(u,v)=(0,0)$, we have $\vec{W}=(w,w)$ (where w is some constant; the components are equal at this point) so

$$W^1(0,0) = w = c_3 \tag{227}$$

$$W^2(0,0) = w = c_1 \tag{228}$$

Considering the point $(u,v)=(b,0)$ we find

$$\frac{W^1}{W^2} = \frac{c_3}{(1-b^2)} c\sqrt{1-b^2} \tag{229}$$

$$= \frac{w}{w}\left(1-b^2\right)^{-1/2} \tag{230}$$

$$= \left(1-b^2\right)^{-1/2} \tag{231}$$

as required.