General Relativity (Masters)
Assignment Three

1. **Playing tennis in a four-dimensional "brane world."** (Jones, P. et al. 2009, *Am. J. Phys.*, 76, 73.) [1] Projectile motion in a constant gravitational field is a classic introductory problem in Newtonian mechanics. One way to make a uniform gravitational field is to assemble an infinite, flat sheet of matter. If the mass per unit area in the x-y plane is σ, then the Newtonian gravitational potential is given by

$$\Phi_{\text{Newton}} = 2\pi G\sigma|z| \ , \tag{1}$$

where z is the Cartesian coordinate normal to the sheet. [Convince yourself that (1) is true by analogy with electrostatics or otherwise.] Below we ask whether it is possible to recreate this system in the strong-gravity regime in general relativity.

   (a) **Building the world.** By appealing to symmetry, argue that the most general metric of a plane-parallel, four-dimensional spacetime takes the form

$$ds^2 = -e^{2\Phi(t,z)}dt^2 + e^{2\Psi(t,z)}(dx^2 + dy^2) + e^{2\Lambda(t,z)}dz^2 \ , \tag{2}$$

   where $\Phi(t, z)$ (not necessarily the same as Φ_{Newton}), $\Psi(t, z)$, and $\Lambda(t, z)$ are functions determined by Einstein's field equations.

   (b) i. If the spacetime is *static* (caution: such a solution may not necessarily exist), construct explicitly a coordinate transformation, that puts (2) into the form

$$ds^2 = -e^{2\Phi(z)}dt^2 + e^{2\Psi(z)}(dx^2 + dy^2) + dz^2 \ , \tag{3}$$

   where t, x, y, and z are suitably relabelled from (2).

   ii. Explain why it is impossible to get rid of the $e^{2\Lambda(r)}$ factor multiplying dr^2 in the "standard" Schwarzschild metric in the same way, such that the metric contains only one undetermined function $e^{2\Phi(r)}$.

   (c) Nine of the Christoffel symbols associated with (3) are nonzero. Calculate them. Here and henceforth, please feel free to use a symbolic algebra package like *Mathematica* to make your life easier.

   (d) Show that the nonzero components of the Ricci tensor $R_{\mu\nu}$ are

$$\begin{aligned} R_{tt} &= e^{2\Phi}[(\Phi')^2 + 2\Phi'\Psi' + \Phi''] \ , &(4)\\ R_{xx} &= -e^{2\Psi}[2(\Psi')^2 + \Psi'\Phi' + \Psi''] \ , &(5)\\ R_{yy} &= R_{xx} \ , &(6)\\ R_{zz} &= -(\Phi')^2 - 2(\Psi')^2 - \Phi'' - 2\Psi'' \ . &(7) \end{aligned}$$

   Primes denote derivatives with respect to z.

   (e) Show that the nonzero contravariant components of Einstein's field equations with cosmological constant Λ and stress-energy tensor $T^{\mu\nu}$ are

$$\begin{aligned} 8\pi T^{tt} &= -e^{-2\Phi}[3(\Psi')^2 + 2\Psi'' + \Lambda] \ , &(8)\\ 8\pi T^{xx} &= e^{-2\Psi}[(\Phi')^2 + \Phi'\Psi' + (\Psi')^2 + \Phi'' + \Psi'' + \Lambda] \ , &(9)\\ 8\pi T^{yy} &= \text{same as for } T^{xx} \ , &(10)\\ 8\pi T^{zz} &= \Psi'(2\Phi' + \Psi') + \Lambda \ . &(11) \end{aligned}$$

   You can do this with pen and paper but will find *Mathematica* more soothing.

[1] Please note that this article contains minor mathematical errors (incorrect factors of two et cetera).

(f) Einstein's field equations place strong constraints on the physical form of the stress-energy consistent with the metric (3).

i. Explain why it is impossible to generate (3) with a sheet of cold matter (dust).

ii. By thinking about a suitable initial value problem *qualitatively*, speculate why this seemingly natural system — a static dust sheet — is impossible to assemble "from the ground up" in general relativity, even though there is no obstacle in a Newtonian context. What might "go wrong"? Do not try calculating anything, unless you are in the mood to win a Nobel Prize.

(g) Consider the special case $\Psi = 0$, i.e. warped time, Euclidean space, no preferential warping of space in the z direction relative to the x and y directions.

i. In the bulk ($z \neq 0$), where there is zero stress-energy, show that $\Lambda = 0$ must hold to obtain a self-consistent solution of the form (3).

ii. Hence show that the stress-energy must vanish on the sheet $z = 0$ too.

iii. Prove that (3) reduces to the Rindler metric for a uniformly accelerated frame. What is the implied acceleration?

In the special case $\Phi = 0$, a similar bulk-then-sheet approach yields the Minkowski metric.

(h) Consider the special case $\Psi = \Phi$. This corresponds to preferentially warping space in z relative to x and y, i.e. the "symmetric" coordinates (t, x, y) are treated as a Minkowski subspace with warping in the z coordinate. The system is the four-dimensional analogue of the five-dimensional *Randall-Sundrum 1-brane* spacetime, which revolutionized the study of brane worlds (see review article by R. Maartens & K. Koyama 2010, *Living Reviews of Relativity*, 13, 5).

i. In the bulk, show that one obtains

$$\Phi = g|z| \; , \tag{12}$$

if $\Phi(z)$ vanishes at $z = 0$ without loss of generality. In (12), g is an integration constant with units of acceleration.

ii. Prove that g satisfies the fine-tuning condition

$$\Lambda = -3g^2 \; . \tag{13}$$

Comment on the physical significance of the sign of Λ.

iii. Assume that the stress-energy is confined to the sheet, i.e. $T^{\mu\nu} \propto \delta(z)$. Prove that the stress-energy tensor takes the physically unusual form

$$T^{\mu\nu} = (2\pi)^{-1} g\delta(z)\, \mathrm{diag}(-1, 1, 1, 0) \; . \tag{14}$$

Equation (14) implies fine tuning between $T^{\mu\nu}$ and Λ via g.

iv. Interpret physically the two cases $g < 0$ and $g > 0$.

v. Interpret physically the signs of T^{tt} and $T^{xx} = T^{yy}$ for $g > 0$.

vi. Explain physically why $T^{zz} = 0$ makes sense in terms of a plausible microscopic ("particulate") model of the sheet.

vii. Prove that it is impossible to build the sheet out of a perfect fluid or dust.

(i) **Serving aces.** A tennis player standing on the sheet hits a tennis ball into the $z > 0$ half-space with initial 4-velocity $\vec{u}(0)$.

i. Starting from the geodesic equations in the form

$$\frac{du_\nu}{d\tau} = \frac{1}{2} g_{\alpha\beta,\nu} u^\alpha u^\beta \ , \tag{15}$$

identify three constants of the motion. In (15), τ denotes the proper time as measured in the ball's momentarily comoving reference frame.

ii. From the z component of (15), show that

$$\frac{du^z}{d\tau} = -g[1 + (u^z)^2] \ . \tag{16}$$

What is unusual physically about this acceleration?

iii. Similarly or otherwise, prove the *equivalent* result

$$\left(\frac{dz}{d\tau}\right)^2 = -1 + e^{-2gz}\{1 + [u^z(0)]^2\} \ . \tag{17}$$

iv. Solve either (16) or (17) to obtain

$$z = g^{-1} \ln[\cos g\tau + u^z(0) \sin g\tau] \ . \tag{18}$$

Please note that there are many valid ways to prove (16)–(18).

(j) Show that the ball reaches the apex of its trajectory at proper time

$$\tau_{\text{top}} = g^{-1} \tan^{-1} u^z(0) \ , \tag{19}$$

and that the associated coordinate time is

$$t_{\text{top}} = \frac{u^z(0)\{1 + [u^x(0)]^2 + [u^z(0)]^2\}^{1/2}}{g\{1 + [u^z(0)]^2\}} \ , \tag{20}$$

in the coordinates in which (3) is written. Interestingly, τ_{top} is finite, no matter how hard the ball is struck initially. [2] Does the same hold true for t_{top}?

(k) Show that the horizontal distance traversed by the ball from its launch point to where it lands on the sheet, as measured in the coordinates associated with (3), is given by

$$x_{\text{land}} = \frac{2u^x(0)u^z(0)}{g\{1 + [u^z(0)]^2\}} \ . \tag{21}$$

How does (21) differ *qualitatively* from Newtonian projectile motion? Should we expect x_{land} to equal the x-component of the initial 3-velocity multiplied by $2t_{\text{top}}$, since the metric does not depend on x?

(l) Finally, suppose that instead of a tennis ball we launch a laser beam into the $z > 0$ half-space with initial 4-momentum $\vec{p}(0)$ per photon.

[2] Convince yourself (optional; not for credit) that a particle starting from rest at infinity and falling vertically (i.e. in the z direction) hits the sheet after a finite proper time $\pi/(2g)$.

i. Let λ be an affine parameter tracing out the light ray. Starting from the geodesic equations or otherwise, solve for the photon's trajectory. You should find

$$z = g^{-1} \ln[1 + p^z(0) g\lambda] \ . \tag{22}$$

In other words, the photon never falls back to the sheet, no matter how much stress-energy the sheet contains.

ii. Consider an intergalactic variant of the Pound-Rebka experiment, in which we fly horizontally (in the x direction, say) in a rocket at constant speed V [as measured in the coordinates associated with (3)] while maintaining a constant altitude z_{rocket} above the sheet. Show that the laser frequency measured by the stationary emitter, ν_{em}, and the frequency measured by the experimentalist in the rocket, ν_{rec}, are related according to

$$\frac{\nu_{\text{rec}}}{\nu_{\text{em}}} = \frac{e^{-g z_{\text{rocket}}}}{(1 - V^2)^{1/2}} \left[1 - \frac{V p^x(0)}{\{[p^x(0)]^2 + [p^z(0)]^2\}^{1/2}} \right] \ . \tag{23}$$

Most of the physics in this question applies in five dimensions too. It is interesting to think about it in the context of Randall-Sundrum brane worlds.