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PHYC90012 General Relativity

Course Summary

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1 Syllabus

1.1 Part I

1. Introduction to gravity
   - Order of magnitude estimates
   - Small amount of quantum gravity
2. Equivalence principle + experimental foundations
3. Geometric objects
   - Need to understand geomtric components of GR
   - Vectors, metric, etc. that live on manifolds
   - Laws of nature do not depend on coordinates chosen
   - Hence can write laws of nature in terms of geometric objects w/o reference to coordinates
4. Kinematics
   - Time dilation, length contraction in GR framework
5. Calculus in curvilinear coordinates
   - Mass and energy curve spacetime
   - Hence geometric objects moved on curved manifolds
   - Distances are not only spatial but temporal; need to use mathematics of small change = calculus
   - Uses the covariant derivative (a geomtetric object; independent of basis/coordinate independent)
   - This point of the course we will not be considering curved space, but instead only curvilinear coords
     - A flat space can be covered (represented?) by curved coordinates, but an intrinsically curved surface cannot be covered by flat coordinates
6. Curved spaces
   - Manifolds
   - How to calculate lengths, volumes, angles in curved spaces
   - Introduces the idea of parallel transport ⇒ leads to curvature
   - Define the Riemann tensor, and its children etc. Ricci tensor, ...; these satisfy the Bianchi identities
7. Einsten's field equations
   - Stress-energy tensor
8. Weak-field limit
   - Gauge transformations

1.2 Part II - Applications

9. GR phenomena revisited
   - GPS, Mercury's orbit, gravitational lensing, gravitational redshift, ...
10. Gravitational waves
    - Propagation (phase speed, polarisation, ...)
    - Generation*
    - Detection*

      * = together these form the "antenna problem"
11. Relativisitic stars
    - neutron stars
    - equation of state (cannot study on Earth because largest nuclei only have 200 elements or so; need more density)
12. Black holes
    - Event horizons, singularities, ...
13. Cosmology
    - Friedman-Robertson-Walker (FRW) metric - describes a homogeneous, isotropic universe
      - We will derive this and the Friedman equations

2 Introduction to gravity

2.1 Strength of gravity

- Weak! Weakest of all fundamental forces
- Long-ranged force (like EM)
- Weakness determined by coupling constant
- Coupling constant = Newton's gravitational constant

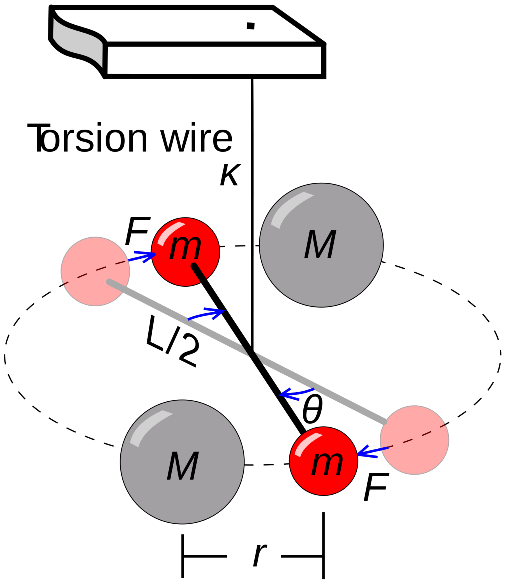
$$\vec{F} = \frac{Gm_1 m_2}{r_{12}^2}\hat{r} \tag{1}$$

- G is hard to measure; least well known of coupling constants

In 1797-98, Cavendish used torsion balls (1.8m torsion balance) with rod of big masses and rod of small masses.

- Spring constant of torsion balance was measured from free oscillation

- then introduced 158kg balls
- measured deflection angle of balance ⇒ can calculate force
  - using a mini-telescope against Vernier scale
- rearrange Newton's law to get G



Exercise: Show that Cavendish also measured density of Earth as a bonus at the same time.

Mass of Earth $M_\oplus = \rho V$ where $V = \frac{4}{3}\pi R^3$ assuming the Earth is a sphere. How does calculating G also calculate ρ? Well, we have $\vec{F} = \frac{Gm_1 m_2}{r_{12}^2}\hat{r}$. Let's take $m_1 = M_\oplus$ as the mass of the Earth, and $m_2 = m$ as some small object mass. Let's imagine the smaller object falling to the center of the Earth. We'll take r_{12} as the distance from the object to the Earth's center, which we can approximate as Earth's radius, i.e. $r_{12} = R$. This force should be equivalent to $F = ma$.

So we have

$$\begin{aligned}
\frac{GM_\oplus m}{R^2} &= mg \\
\frac{G\rho\frac{4}{3}\pi R^3}{R^2} &= g \\
\frac{4G\rho\pi R}{3} &= g \\
\Rightarrow \rho &= \frac{3g}{4\pi GR} \\
&= \frac{3 \times 9.8\,\mathrm{ms^{-2}}}{4\pi \times 6.673\,84 \times 10^{-11}\,\mathrm{kg^{-1}m^3s^{-2}} \times 6370\,\mathrm{km}} \\
&= \frac{3 \times 9.8}{4\pi \times 6.67384 \times 10^{-11} \times 6370 \times 10^3}\,\mathrm{kg\,m^{-3}} \\
&= 5503\,\mathrm{kg\,m^{-3}}
\end{aligned}$$

	$\frac{GM}{Rc^2} \lll 1$	$\frac{GM}{Rc^2} \geq 1$
$v \ll c$	Newtonian	CAN'T EXIST
$v \sim c$	special rel.	full GR (difficult)

- Modern $G = 6.67384(80) \times 10^{-11}\,\mathrm{Nm^{-2}\,kg^{-2}} = \mathrm{kg^{-1}\,m^3\,s^{-2}}$
- Product GM is known to 1 part in $\sim 10^{10}$ from astrophysics observations

  $\Rightarrow$ mass is hard to measure gravitationally
- We need a dimensionless number to characterise strength
- Newton: $\Phi = \frac{GM}{r}$ (potential)
- In free fall: $\frac{KE}{mass}$, $v^2 \sim \frac{GM}{r}$
- We claim gravity is strong is free-fall is relativistic, i.e. $v \sim c$
- This is an order of magnitude estimate

2.2 Strong vs. weak gravity

- Quasi-Newtonian:
  - characterticstic speed of body in free fall: $v^2 \sim \frac{GM}{r}$
- Strong gravity leads to relativistic free fall, i.e. $\frac{GM}{Rc^2} \geq 1$ where M is the total mass and R is th characteristic size

Example 1.1: $M = M_\odot$ (mass of the Sun)

$$\begin{aligned} R &\sim \frac{GM}{c^2} \quad \text{boundary of strong regime} \\ &\sim \frac{10^{-10}10^{30}}{10^{17}} \\ &\sim \text{km} \end{aligned}$$

cf. Schwarz radius of black hole = $\frac{2GM}{c^2}$

Example 1.2: Density of black hole with mass of $M_\odot$

$$\begin{aligned} &\sim \frac{M}{R^3} \sim \frac{10^{30}\,\mathrm{kg}}{(\mathrm{km})^3} \\ &\sim 10^{21} kgm^{-3} \end{aligned}$$

How does this density compare to maximum density of (say) nuclear matter? Let's compare.

$$\frac{m_n}{(1fm)^3} \sim \frac{10^{-27}kg}{10^{-45}m^3} \sim 10^{18}kgm^{-3}$$

We see a black hole is more dense that a nuclei. The characteristic size of a particle $1fm \sim \Delta x \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar}{m_n c}$, due to Heisenberg's uncertainty principle, and also the Pauli exclusion principle.

More generally: density of material that froms black hole $\sim \frac{M}{R^3}$, but note $M = \frac{c^2 R}{G}$ density $\rho \propto \frac{1}{R^2}$. This means that denser black holes are smaller.

Exercise: Estimate the strength of gravity $\frac{GM}{Rc^2}$ on Earth.

Example 2: The Universe is composed of 5% baryons + 25% dark matter + 70% dark energy. Estimate M and R.

$R \sim 10\,\text{Gpc}$

- Mass of baryons
  - 10^{11} stars in Milky Way
  - $(10^4)^3$ galaxies in Universe

  $\Rightarrow$ $M_{\text{baryons}} \sim 10^{23} M_\odot \sim 10^{53}\,\text{kg}$

$$\frac{GM_{tot}}{Rc^2} \sim \frac{10^{-10} \cdot 10^{53} \cdot 10}{10^{27} \cdot 10^{17}} \sim 1 \tag{2}$$

$$\begin{aligned}\text{Density } \rho \sim \frac{M_{tot}}{R_{tot}^3} &\sim \frac{c^2}{R^2 G}, \text{ use } \frac{GM}{Rc^2} \sim 1 \\ &\sim \frac{1}{G \times (\text{age of universe})^2}\end{aligned}$$

cf. critical density from Friedmann equations $\rho_{crit} = \frac{3H_0^2}{8\pi G}$.
Recall Hubble constant $H_0 \sim \frac{1}{\text{age}}$.

The critical density is the density of the universe at which expansion will asymptotically slow. Too dense leads to big crunch, too low leads to unbounded expansion.

Exercise: How do we reconcile a "flat" universe from critical density with the "curved" universe?

Important to remember: gravity is strong when $\frac{GM}{Rc^2} \sim 1$, which occurs around black holes, the universe at large. In a sense, cosmological results such as critical density, expansion of universe come from this.

2.3 Black hole oscillations

We can estimate the oscillation frequency of a "black hole" (i.e. something with $\frac{GM}{Rc^2} \sim 1$ as $\sim \frac{c}{R}$; that is, the time is takes light to travel the distance of the object. This is the natural frequency for this object. Using that ubioquitous expression we can express the oscillation frequency as $\sim \frac{c^3}{GM}$, e.g. $M = M_\odot \Rightarrow$ frequency $\sim 10\,\text{kHz}$.

Let's discuss charged black holes. It is difficult to astrophysically have charged black holes, because stars are not usually charged (due to the strength of the EM force, which would attract opposite charge and cancel out). So, these are artificial in nature. These have unusual geometry, and are called "Reissner-Nordstrom" black holes.

Example : What is the maximum charge on a black hole?

$$\frac{Q^2}{4\pi\epsilon_0 R} \leq \frac{GM^2}{R}$$
$$Q \leq (4\pi\epsilon_0 G)^{1/2} M$$

Above, we relate Coloumb force to gravitational force. The gravitational force holding a black hole together must overcome the Coloumb force pushing it apart.

2.4 Quantum Gravity

The problem wtih quantum gravity is that there is no theory... hence we must rely on numerology.

We consider a hypothetical elementry excitation of a "black hole" (again, we mean a *relativistic compact object*) of mass M. Hence the characteristic size, or "wavelength", of the excitation is $\frac{GM}{c^2}$ (fundamental excitation only). Introducing quantum mechanics: the Heisenberg uncertainty principle tells us that the zero-point motion associated with this excitation is

$$\lambda \sim \frac{\hbar}{\Delta p} \sim \underbrace{\frac{\hbar}{Mc}}_{\text{relativisitic}} \tag{3}$$

Equating length scales $\Rightarrow M_{pl} \approx \left(\frac{\hbar c}{G}\right)^{1/2}$; this is the Planck mass, about $10^{-8}\,\text{kg}$ (the mass below which quantum gravity is important.

Given M_{pl} we get $\lambda \sim \frac{\hbar}{M_{pl}c} \sim 10^{-33}\,\text{m}$; the Planck length - the length where quantum gravity is important (e.g. just after Big Bang).

2.4.1 Hawking Radiation

Let's return to our elementary excittion with $\lambda \sim \frac{GM}{c^2}$, i.e. frequency $\sim \frac{c^3}{GM} = \frac{c}{\lambda}$. Heisenberg tells us there is an associated energy fluctation $\Delta E \sim h \times$ frequency $\sim \frac{\hbar c^3}{GM}$. Suppose (note: this is a huge leap) energy fluctuation in the black hole sysytem is in thermal equilibrium with a bath at temperature T. Then $T \sim \frac{\Delta E}{k_B}$. This associates a temperature to a black hole.

We call a black hole a blackbody!

$$\begin{aligned}\text{Radiated power } &= k_B \times \text{area} \times T^4 \\ &= \sigma \times \underbrace{R^2}_{\left(\frac{GM}{c^2}\right)^2} \times \left(\frac{\hbar c^3}{GMk_b}\right)^4 \\ &\propto M^{-2}\end{aligned}$$

> **Exercise:** Plug in numbers to this!
>
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This shows that a black hole radiates energy $\Rightarrow$ eventually a black hole evaporates. We can estimate the time scale of this evaporation.

$$\text{time scale} \sim \frac{Mc^2}{\text{power}} \propto M^3 \tag{4}$$

$\Rightarrow$ small black holes evaporate fast!

As an aside, we could consider the rate of energy accretion. For system outside a black hole with uniform density ρ_{out}, we have

$$\begin{aligned}\text{rate of mass accretion} &\sim \rho_{\text{out}} \cdot c \cdot 4\pi R^2 \\ \text{rate of energy accretion} &\sim c^2 \cdot \text{rate of mass accretion}\end{aligned}$$