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PHYC90012 General Relativity

Course Summary

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    - Vectors, metric, etc. that live on manifolds
    - Laws of nature do not depend on coordinates chosen
    - Hence can write laws of nature in terms of geometric objects w/o reference to coordinates
4. Kinematics
    - Time dilation, length contraction in GR framework
5. Calculus in curvilinear coordinates
    - Mass and energy curve spacetime
    - Hence geometric objects moved on curved manifolds
    - Distances are not only spatial but temporal; need to use mathematics of small change = calculus
    - Uses the covariant derivative (a geomtetric object; independent of basis/coordinate independent)
    - This point of the course we will not be considering curved space, but instead only curvilinear coords
        - A flat space can be covered (represented?) by curved coordinates, but an intrinsically curved surface cannot be covered by flat coordinates
6. Curved spaces
    - Manifolds
    - How to calculate lengths, volumes, angles in curved spaces
    - Introduces the idea of parallel transport ⇒ leads to curvature
    - Define the Riemann tensor, and its children etc. Ricci tensor, ...; these satisfy the Bianchi identities
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      * = together these form the "antenna problem"
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      - We will derive this and the Friedman equations

1 Introduction to gravity

1.1 Strength of gravity

- Weak! Weakest of all fundamental forces
- Long-ranged force (like EM)
- Weakness determined by coupling constant
- Coupling constant = Newton's gravitational constant

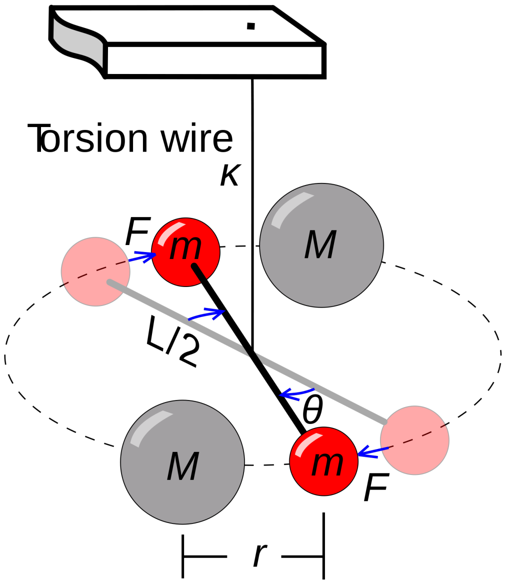
$$\vec{F} = \frac{Gm_1m_2}{r_{12}^2}\hat{r} \tag{1}$$

- G is hard to measure; least well known of coupling constants

In 1797-98, Cavendish used torsion balls (1.8m torsion balance) with rod of big masses and rod of small masses.

- Spring constant of torsion balance was measured from free oscillation

- then introduced 158kg balls
- measured deflection angle of balance $\Rightarrow$ can calculate force
  - using a mini-telescope against Vernier scale
- rearrange Newton's law to get G



Exercise: Show that Cavendish also measured density of Earth as a bonus at the same time.

Mass of Earth $M_\oplus = \rho V$ where $V = \frac{4}{3}\pi R^3$ assuming the Earth is a sphere. How does calculating G also calculate ρ? Well, we have $\vec{F} = \frac{Gm_1 m_2}{r_{12}^2}\hat{r}$. Let's take $m_1 = M_\oplus$ as the mass of the Earth, and $m_2 = m$ as some small object mass. Let's imagine the smaller object falling to the center of the Earth. We'll take r_{12} as the distance from the object to the Earth's center, which we can approximate as Earth's radius, i.e. $r_{12} = R$. This force should be equivalent to $F = ma$.

So we have

$$\begin{aligned}
\frac{GM_\oplus m}{R^2} &= mg \\
\frac{G\rho\frac{4}{3}\pi R^3}{R^2} &= g \\
\frac{4G\rho\pi R}{3} &= g \\
\Rightarrow \rho &= \frac{3g}{4\pi GR} \\
&= \frac{3 \times 9.8\,\mathrm{ms^{-2}}}{4\pi \times 6.673\,84 \times 10^{-11}\,\mathrm{kg^{-1}m^3s^{-2}} \times 6370\,\mathrm{km}} \\
&= \frac{3 \times 9.8}{4\pi \times 6.67384 \times 10^{-11} \times 6370 \times 10^3}\,\mathrm{kg\,m^{-3}} \\
&= 5503\,\mathrm{kg\,m^{-3}}
\end{aligned}$$

	$\frac{GM}{Rc^2} \lll 1$	$\frac{GM}{Rc^2} \geq 1$
$v \ll c$	Newtonian	CAN'T EXIST
$v \sim c$	special rel.	full GR (difficult)

- Modern $G = 6.67384(80) \times 10^{-11}\,\mathrm{Nm^{-2}\,kg^{-2}} = \mathrm{kg^{-1}\,m^3\,s^{-2}}$
- Product GM is known to 1 part in $\sim 10^{10}$ from astrophysics observations

  $\Rightarrow$ mass is hard to measure gravitationally
- We need a dimensionless number to characterise strength
- Newton: $\Phi = \frac{GM}{r}$ (potential)
- In free fall: $\frac{KE}{mass}$, $v^2 \sim \frac{GM}{r}$
- We claim gravity is strong is free-fall is relativistic, i.e. $v \sim c$
- This is an order of magnitude estimate

1.2 Strong vs. weak gravity

- Quasi-Newtonian:
  - characterticstic speed of body in free fall: $v^2 \sim \frac{GM}{r}$
- Strong gravity leads to relativistic free fall, i.e. $\frac{GM}{Rc^2} \geq 1$ where M is the total mass and R is th characteristic size

Example 1.1: $M = M_\odot$ (mass of the Sun)

$$\begin{aligned} R &\sim \frac{GM}{c^2} \quad \text{boundary of strong regime} \\ &\sim \frac{10^{-10}10^{30}}{10^{17}} \\ &\sim \text{km} \end{aligned}$$

cf. Schwarz radius of black hole $= \frac{2GM}{c^2}$

Example 1.2: Density of black hole with mass of $M_\odot$

$$\begin{aligned} &\sim \frac{M}{R^3} \sim \frac{10^{30}\,\text{kg}}{(\text{km})^3} \\ &\sim 10^{21} kgm^{-3} \end{aligned}$$

How does this density compare to maximum density of (say) nuclear matter? Let's compare.

$$\frac{m_n}{(1fm)^3} \sim \frac{10^{-27}kg}{10^{-45}m^3} \sim 10^{18}kgm^{-3}$$

We see a black hole is more dense that a nuclei. The characteristic size of a particle $1fm \sim \Delta x \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar}{m_n c}$, due to Heisenberg's uncertainty principle, and also the Pauli exclusion principle.

More generally: density of material that froms black hole $\sim \frac{M}{R^3}$, but note $M = \frac{c^2 R}{G}$ density $\rho \propto \frac{1}{R^2}$. This means that denser black holes are smaller.

Exercise: Estimate the strength of gravity $\frac{GM}{Rc^2}$ on Earth.

Example 2: The Universe is composed of 5% baryons + 25% dark matter + 70% dark energy. Estimate M and R.

$R \sim 10\,\text{Gpc}$

- Mass of baryons
  - 10^{11} stars in Milky Way
  - $(10^4)^3$ galaxies in Universe

  $\Rightarrow M_{\text{baryons}} \sim 10^{23} M_\odot \sim 10^{53}\,\text{kg}$

$$\frac{GM_{tot}}{Rc^2} \sim \frac{10^{-10}\cdot 10^{53}\cdot 10}{10^{27}\cdot 10^{17}} \sim 1 \tag{2}$$

$$\begin{aligned}\text{Density } \rho \sim \frac{M_{tot}}{R_{tot}^3} &\sim \frac{c^2}{R^2 G},\text{ use } \frac{GM}{Rc^2} \sim 1 \\ &\sim \frac{1}{G \times (\text{age of universe})^2}\end{aligned}$$

cf. critical density from Friedmann equations $\rho_{crit} = \frac{3H_0^2}{8\pi G}$.
Recall Hubble constant $H_0 \sim \frac{1}{\text{age}}$.

The critical density is the density of the universe at which expansion will asymptotically slow. Too dense leads to big crunch, too low leads to unbounded expansion.

Exercise: How do we reconcile a "flat" universe from critical density with the "curved" universe?

Important to remember: gravity is strong when $\frac{GM}{Rc^2} \sim 1$, which occurs around black holes, the universe at large. In a sense, cosmological results such as critical density, expansion of universe come from this.

1.3 Black hole oscillations

We can estimate the oscillation frequency of a "black hole" (i.e. something with $\frac{GM}{Rc^2} \sim 1$ as $\sim \frac{c}{R}$; that is, the time is takes light to travel the distance of the object. This is the natural frequency for this object. Using that ubioquitous expression we can express the oscillation frequency as $\sim \frac{c^3}{GM}$, e.g. $M = M_\odot \Rightarrow$ frequency $\sim 10\,\mathrm{kHz}$.

Let's discuss charged black holes. It is difficult to astrophysically have charged black holes, because stars are not usually charged (due to the strength of the EM force, which would attract opposite charge and cancel out). So, these are artificial in nature. These have unusual geometry, and are called "Reissner-Nordstrom" black holes.

Example : What is the maximum charge on a black hole?

$$\frac{Q^2}{4\pi\epsilon_0 R} \leq \frac{GM^2}{R}$$
$$Q \leq (4\pi\epsilon_0 G)^{1/2} M$$

Above, we relate Coloumb force to gravitational force. The gravitational force holding a black hole together must overcome the Coloumb force pushing it apart.

1.4 Quantum Gravity

The problem wtih quantum gravity is that there is no theory... hence we must rely on numerology.

We consider a hypothetical elementry excitation of a "black hole" (again, we mean a *relativistic compact object*) of mass M. Hence the characteristic size, or "wavelength", of the excitation is $\frac{GM}{c^2}$ (fundamental excitation only). Introducing quantum mechanics: the Heisenberg uncertainty principle tells us that the zero-point motion associated with this excitation is

$$\lambda \sim \frac{\hbar}{\Delta p} \sim \underbrace{\frac{\hbar}{Mc}}_{\text{relativisitic}} \tag{3}$$

Equating length scales $\Rightarrow M_{pl} \approx \left(\frac{\hbar c}{G}\right)^{1/2}$; this is the Planck mass, about $10^{-8}\,\mathrm{kg}$ (the mass below which quantum gravity is important.

Given M_{pl} we get $\lambda \sim \frac{\hbar}{M_{pl}c} \sim 10^{-33}\,\mathrm{m}$; the Planck length - the length where quantum gravity is important (e.g. just after Big Bang).

1.4.1 Hawking Radiation

Let's return to our elementary excittion with $\lambda \sim \frac{GM}{c^2}$, i.e. frequency $\sim \frac{c^3}{GM} = \frac{c}{\lambda}$. Heisenberg tells us there is an associated energy fluctation $\Delta E \sim h \times$ frequency $\sim \frac{\hbar c^3}{GM}$. Suppose (note: this is a huge leap) energy fluctuation in the black hole sysytem is in thermal equilibrium with a bath at temperature T. Then $T \sim \frac{\Delta E}{k_B}$. This associates a temperature to a black hole.

We call a black hole a blackbody!

$$\begin{aligned} \text{Radiated power } &= k_B \times \text{area} \times T^4 \\ &= \sigma \times \underbrace{R^2}_{\left(\frac{GM}{c^2}\right)^2} \times \left(\frac{\hbar c^3}{GMk_b}\right)^4 \\ &\propto M^{-2} \end{aligned}$$

Exercise: Plug in numbers to this!

This shows that a black hole radiates energy $\Rightarrow$ eventually a black hole evaporates. We can estimate the time scale of this evaporation.

$$\text{time scale} \sim \frac{Mc^2}{\text{power}} \propto M^3 \tag{4}$$

$\Rightarrow$ small black holes evaporate fast!

As an aside, we could consider the rate of energy accretion. For system outside a black hole with uniform density ρ_{out}, we have

$$\begin{aligned} \text{rate of mass accretion} &\sim \rho_{\text{out}} \cdot c \cdot 4\pi R^2 \\ \text{rate of energy accretion} &\sim c^2 \cdot \text{rate of mass accretion} \end{aligned}$$

A short review:

- there is no theory of quantum gravity
- we consider a relativisitic elementary oscillation $\lambda \sim \frac{GM}{c^2}$
- we use Heisenberg to relate this to an enegy flucttutation in the system
- energy fluctuation can be converted to a temperature $T_H = \frac{\hbar c^3}{GMk_B}$, the Hawking temperature
- we find small block holes evaportate more quickly than large black holes

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1.5 Black hole thermodynamics

1st law: $dS = \frac{dQ}{T}$ system constant volume

Hawking radiation: we lose a bit of heat dQ due to blackbody radiation.

$$dS = \frac{GMk_b}{\hbar c^3} d(Mc^2) \tag{5}$$

We see heat loss comes from rest energy

$$\frac{dS}{k_B} \approx \frac{1}{R^2_{\text{Planck}}} \underbrace{d(R^2)}_{R\sim\frac{GM}{c^2}} \tag{6}$$

This result relates the entropy of a black hole to its area; Bekenstein-Hawking entropy - $S_{\text{black hole}} \propto$ area of event horizon.

$$\frac{S}{k_B} = \frac{\text{area}}{4R^2_{\text{Planck}}}{}^{1} \tag{7}$$

However, there is a contradiction! Hawking radiation implies that $dA < 0 \Rightarrow dS < 0$.... this is bad.[2] One way to resolve this is by makung a generalised **2nd law**:

$$d\left(S_{\text{outside}} + \frac{\text{area}}{4R^2_{\text{Planck}}}\right) \geq 0 \tag{8}$$

Unfortunately this is not enough - there is still a contradiction. Consider a small box of radiation which we prepare far from a black hole. $S_{\text{box}} = \frac{4U_{\text{box}}}{3T_{\text{box}}}$. We can make photons very long wavelength, so that $U_{\text{box}} \approx 0$ byt $S_{\text{box}} \neq 0$. Then $dS_{\text{out}} = -S_{\text{box}} < 0$. $d(\text{area}) = 0$ because energy in box = 0, i.e. $d(Mc^2) = 0$.[3]

Exercise: Resolve the box paradox!

3rd law of BH thermodynamics: can't reduce T to zero.[4]

In heating a rubber band, it shrinks; this is because a shrunken arrangement of the molecules is a state of more entropy (more disordered)

BH

2 Einstein equivalence principle

We will define this (the weak and strong equivalence principles), and some of the tests been performed.

[1] In 1995, Maldacena also got this by counting microstates

[2] See Ted Jacobson's lecture at University of Utrecht for a discussion on this.

[3] Beware the Unruh radiation.

[4] If you are curious: Verlinde, arXiv:1001.0785, *On the Origin of Gravity and the Laws of Newton*, in which the idea of emphgravity is an entropic force is discussed.

2.1 Weak equivalence principle

Trajectory of body in free fall is independent of its mass and $\underbrace{\text{composition}}_{\text{binding energy}}$ (as per Galileo, feather vs. brick). Note that this is not equivalent for electric fields (the movement of a charged particle through an electric field depends on its charge).

2.2 Strong equivalence principle

1. weak equivalence principle is valid
2. results of any non-gravitational (e.g. EM, <u>not</u> Cavendish expt) experiment is independent of velocity fo freely falling frame
   - this is *local lorentz invariance*
3. results of non-gravitational experimental are indepent of where and when it is performed
   - this is *local position invariance*

The Einstein equivalence principle (EEP) $\equiv$ strong implies: existence of a "curved spacetime" with:

i. symmetric metric

ii. trajectories of free-falling bodies are geodesicas of metric

iii. the laws of physics in a freely-falling frame can be written in the lnaguage of special relativity

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A short review:

Einstein equivalence principle:

1. universality of free fall
2. local Lorentz invariance
3. local position invariance

Implications:

3) $\Rightarrow$ fundamental constant independent of $\vec{x}$, t

2) $\Rightarrow$ laws of non-gravitational physics <u>locally</u> independent of frame

1) $\Rightarrow$ space is <u>curved</u>

Why is space curved? Locally straight trajectory in free yet <u>yet</u> gravity producess curved trajectory (observed) $\Rightarrow$ only possible if coordinates change from one point to next

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3 Experimental tests

3.1 Experimental tests of free fall

Inertial mass $m_i = \frac{\text{applied non-gravitational force}}{\text{measured acceleration}}$

Gravitational mass $m_g =$ "passive" mass appearing in weight

Look for $m_i \neq m_g$

Write

$$m_g = m_i + \sum_{\text{interactions A in body}} \eta^A \frac{E^A}{c^2} \tag{9}$$

Here $E =$ "binding energy"/potential energy of interaction A.

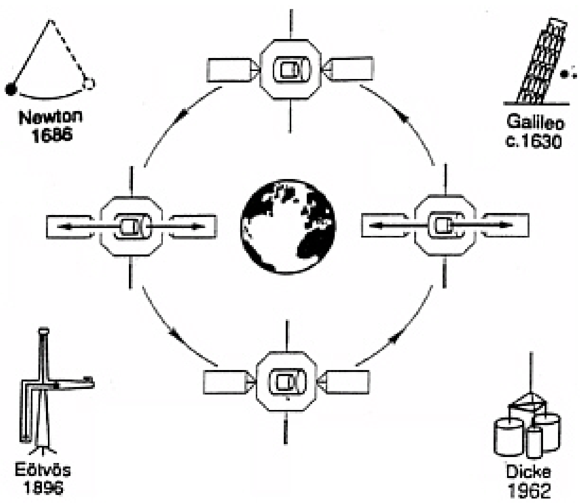
Tests:

1. Eötvös-type torsion balance experiements: two different materials may fall at difference rates (see Dicke, Braginsky)
2. Colorado: U, Cu laser interferometer $\Rightarrow relativiteacceleration$
3. Eöt-Wash experiements: fancy version of 1)

Result:

$$\frac{|m_g - m_i|}{m_i} \leq 10^{-13} \tag{10}$$

We test this, for eample, with aluminium (Al) and gold (Au) weights on a torsion balance. As the Sun moves from one side of the Earth to the other, if the gravitational mass of either differs from the other we will see diurnal oscillation in the balance.



3.2 Tests of local Lorentz invariance

- Michelson-Morley experiment (the aether)
- Rossi-Hall tests for lifetime of muons (time dilation $\Leftrightarrow$ LLI)
- Ives-Stiwell transverse Doppler shift
  - laser travelling some vector $\vec{v}$
  - we see perpendicular wave vector $\vec{k}$
  - measure frequency when laser intersects line of sight
  - Doppler shift arises due to time dilation

Mathematically:

$$\begin{bmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{bmatrix} = \begin{bmatrix} \gamma & \gamma v/c & 0 & 0 \\ \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega/c \\ 0 \\ k_y \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma\omega/c \\ \gamma\omega v^2/c \\ k_y \\ 0 \end{bmatrix} \tag{11}$$

Exercise: Compare with standard longitudinal Doppler:

$$\begin{pmatrix} \text{same} \\ \text{matrix} \end{pmatrix} \begin{pmatrix} \omega/c \\ k_x \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

A short review:

Last lecture:

- tests of LPI
- Schiff's thought experiement (gravitational redshift)

3.3 Gravitational redshift

$$\frac{h\nu' - h\nu}{h\nu} = \frac{(m_A g_A - m_B g_B)H}{(m_A - m_B)c^2} \tag{13}$$

If $g_A = g_B$ (universal free-fall), then gravitational redshift is gH/c^2

3.4 Observation of GR in experiments

There are situtations where GR makes measurable difference today

- Pound-Rebka experiment (gravitational redshift, also seen on white dwarf spectral lines)
- GPS
- 2015 discovery of gravitational waves from binary BH merger
- cosmological measurements (CMB, redshifts, H_0, ...)
- gravitational lensing (light bent by a mass) (see: Einstein cross, Abell clusters, stars behind Sun)
- precession of perihelion of Mercury
- orbital decay of Hulse-Taylor binary pulsar (can calculate to mm precision the decay of the orbit every 8 hours due to gravitational wave emission)
- Nordtredt/lunar ranging experiments
- Gravity Probe B - lense-thinning precession
- Shapiro time delay

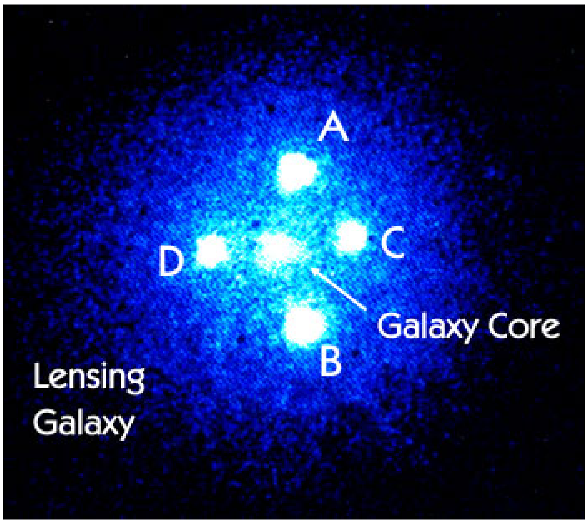


Figure 1: Einstein cross

3.4.1 Pound-Rebka experiment

Looks at gravitational redshift of 14.4 keV gamma rays from ^{57}Fe decay

^{57}Fe decays and emits photons directly down from a 23 m tall tower, to another box of ^{57}Fe below. Gravitational redshift occurs; time moves slower closer to the Earth, so gamma ray emitted will have a different energy than that required to be absorbed by the ^{57}Fe at the bottom.

Receiver box moves up/down at speed v. We adjust v at bottom so that kinematic Doppler shift $\propto \left(\frac{1-v/c}{1+v/c}\right)^{1/2}$ exactly cancels the gravitational redshift $\propto gH/c^2$

N.B.: recoil (in a random direction) when photon emitted/absorbed; energy $E_R = \frac{E_\gamma}{2M_{Fe}c^2}$

$$E_R \sim \frac{14.4\,\mathrm{keV}}{100\,\mathrm{GeV}} \sim 10^{-7} \tag{14}$$

We solve this problem through the Mossbauer effect: use whole crystal; whole crystal ($\sim 10^{23}$ Fe atoms) recoils

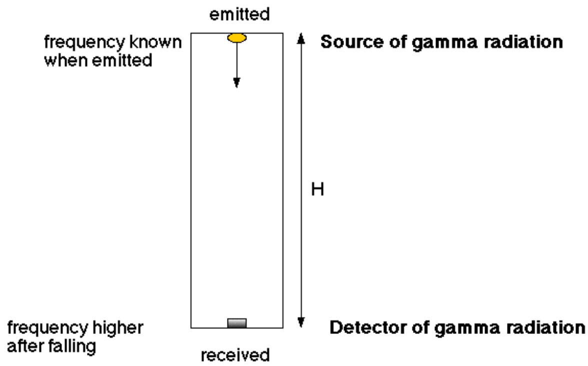


Figure 2: Pound-Rebka experiment

3.4.2 Lunar ranging

Williams + Dickey (2002)[5]

Moon not completely dormant

- fluid core (?)
- tidal dissipation internally
- etc ...

Multiple radar reflectors to improve accuracy. We search for an accurate Earth-Moon orbit versus time; we find

$$\frac{1}{G}\frac{dG}{dt} = (0.0 \pm 1.1) \times 10^{-12}\,\mathrm{yr}^{-1} \tag{15}$$

Uncertainty $\approx 0.02H_0$ - we don't see increasing separation between Earth and Moon due to expansion of universe. This is expected, however: gravitational bound objects do not separate with time, although the energy they require to stay bound will increase

[5] See also "Living Reviews" Relativity

3.4.3 Deflection of light (gravitational lensing)

In a three-body system with the Earth, the Sun and another star, the light from the star will bend due to the Sun before it reaches Earth (not taking a straight path). This causes the apparent position of the star to be different than the actual position.

Deflection angle $\delta\theta \propto \frac{GM}{c^2 d}$

GR deflection $= 2\times$ Newtonian deflection (Einstein 1911).

This effect is achromatic!

3.4.4 Shapiro delay

Roundtrip time from Earth to a distant mirror (in a three-body system including the Sun) is longer than if the Sun was not there.

$$\delta t \propto \frac{GM}{c^3} \ln(\text{geometric factors}) \tag{16}$$

Best measurements with Cassinin spacecraft (accuracy 1 in 10^5)