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PHYC90012 General Relativity

Course Summary

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March 4, 2016

1 Syllabus

1.1 Part I

1. Introduction to gravity
    - Order of magnitude estimates
    - Small amount of quantum gravity
2. Equivalence principle + experimental foundations
3. Geometric objects
    - Need to understand geomtric components of GR
    - Vectors, metric, etc. that live on manifolds
    - Laws of nature do not depend on coordinates chosen
    - Hence can write laws of nature in terms of geometric objects w/o reference to coordinates
4. Kinematics
    - Time dilation, length contraction in GR framework
5. Calculus in curvilinear coordinates
    - Mass and energy curve spacetime
    - Hence geometric objects moved on curved manifolds
    - Distances are not only spatial but temporal; need to use mathematics of small change = calculus
    - Uses the covariant derivative (a geomtetric object; independent of basis/coordinate independent)
    - This point of the course we will not be considering curved space, but instead only curvilinear coords
        - A flat space can be covered (represented?) by curved coordinates, but an intrinsically curved surface cannot be covered by flat coordinates
6. Curved spaces
    - Manifolds
    - How to calculate lengths, volumes, angles in curved spaces
    - Introduces the idea of parallel transport ⇒ leads to curvature
    - Define the Riemann tensor, and its children etc. Ricci tensor, ...; these satisfy the Bianchi identities
7. Einsten's field equations
    - Stress-energy tensor
8. Weak-field limit
    - Gauge transformations

1.2 Part II - Applications

9. GR phenomena revisited
   - GPS, Mercury's orbit, gravitational lensing, gravitational redshift, ...
10. Gravitational waves
    - Propagation (phase speed, polarisation, ...)
    - Generation*
    - Detection*

      * = together these form the "antenna problem"
11. Relativisitic stars
    - neutron stars
    - equation of state (cannot study on Earth because largest nuclei only have 200 elements or so; need more density)
12. Black holes
    - Event horizons, singularities, ...
13. Cosmology
    - Friedman-Robertson-Walker (FRW) metric - describes a homogeneous, isotropic universe
      - We will derive this and the Friedman equations

2 Introduction to gravity

2.1 Strength of gravity

- Weak! Weakest of all fundamental forces
- Long-ranged force (like EM)
- Weakness determined by coupling constant
- Coupling constant = Newton's gravitational constant

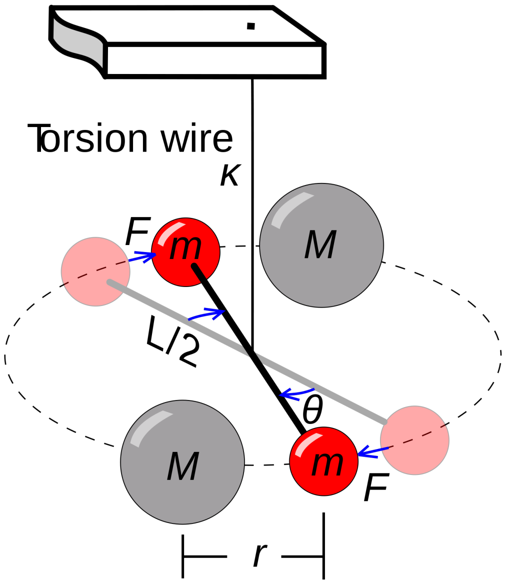
$$\vec{F} = \frac{Gm_1 m_2}{r_{12}^2}\hat{r} \tag{1}$$

- G is hard to measure; least well known of coupling constants

In 1797-98, Cavendish used torsion balls (1.8m torsion balance) with rod of big masses and rod of small masses.

- Spring constant of torsion balance was measured from free oscillation

- then introduced 158kg balls
- measured deflection angle of balance $\Rightarrow$ can calculate force
  - using a mini-telescope against Vernier scale
- rearrange Newton's law to get G



Exercise: Show that Cavendish also measured density of Earth as a bonus at the same time.

Mass of Earth $M_\oplus = \rho V$ where $V = \frac{4}{3}\pi R^3$ assuming the Earth is a sphere. How does calculating G also calculate ρ? Well, we have $\vec{F} = \frac{Gm_1 m_2}{r_{12}^2}\hat{r}$. Let's take $m_1 = M_\oplus$ as the mass of the Earth, and $m_2 = m$ as some small object mass. Let's imagine the smaller object falling to the center of the Earth. We'll take r_{12} as the distance from the object to the Earth's center, which we can approximate as Earth's radius, i.e. $r_{12} = R$. This force should be equivalent to $F = ma$.

So we have

$$\begin{aligned}
\frac{GM_\oplus m}{R^2} &= mg \\
\frac{G\rho\frac{4}{3}\pi R^3}{R^2} &= g \\
\frac{4G\rho\pi R}{3} &= g \\
\Rightarrow \rho &= \frac{3g}{4\pi GR} \\
&= \frac{3 \times 9.8\,\mathrm{ms^{-2}}}{4\pi \times 6.673\,84 \times 10^{-11}\,\mathrm{kg^{-1}m^3s^{-2}} \times 6370\,\mathrm{km}} \\
&= \frac{3 \times 9.8}{4\pi \times 6.67384 \times 10^{-11} \times 6370 \times 10^3}\mathrm{kg\,m^{-3}} \\
&= 5503\,\mathrm{kg\,m^{-3}}
\end{aligned}$$

	$\frac{GM}{Rc^2} \lll 1$	$\frac{GM}{Rc^2} \geq 1$
$v \ll c$	Newtonian	CAN'T EXIST
$v \sim c$	special rel.	full GR (difficult)

- Modern $G = 6.67384(80) \times 10^{-11}\,\mathrm{Nm^{-2}\,kg^{-2}} = \mathrm{kg^{-1}\,m^3\,s^{-2}}$
- Product GM is known to 1 part in $\sim 10^{10}$ from astrophysics observations

  $\Rightarrow$ mass is hard to measure gravitationally
- We need a dimensionless number to characterise strength
- Newton: $\Phi = \frac{GM}{r}$ (potential)
- In free fall: $\frac{KE}{mass}$, $v^2 \sim \frac{GM}{r}$
- We claim gravity is strong is free-fall is relativistic, i.e. $v \sim c$
- This is an order of magnitude estimate

2.2 Strong vs. weak gravity

- Quasi-Newtonian:
  - characterticstic speed of body in free fall: $v^2 \sim \frac{GM}{r}$
- Strong gravity leads to relativistic free fall, i.e. $\frac{GM}{Rc^2} \geq 1$ where M is the total mass and R is th characteristic size

Example 1.1: $M = M_\odot$ (mass of the Sun)

$$\begin{aligned} R &\sim \frac{GM}{c^2} \quad \text{boundary of strong regime} \\ &\sim \frac{10^{-10}10^{30}}{10^{17}} \\ &\sim \text{km} \end{aligned}$$

cf. Schwarz radius of black hole $= \frac{2GM}{c^2}$

Example 1.2: Density of black hole with mass of $M_\odot$

$$\begin{aligned} &\sim \frac{M}{R^3} \sim \frac{10^{30}\,\mathrm{kg}}{(\mathrm{km})^3} \\ &\sim 10^{21} kgm^{-3} \end{aligned}$$

How does this density compare to maximum density of (say) nuclear matter? Let's compare.

$$\frac{m_n}{(1fm)^3} \sim \frac{10^{-27}kg}{10^{-45}m^3} \sim 10^{18}kgm^{-3}$$

We see a black hole is more dense that a nuclei. The characteristic size of a particle $1fm \sim \Delta x \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar}{m_n c}$, due to Heisenberg's uncertainty principle, and also the Pauli exclusion principle.

More generally: density of material that froms black hole $\sim \frac{M}{R^3}$, but note $M = \frac{c^2 R}{G}$ density $\rho \propto \frac{1}{R^2}$. This means that denser black holes are smaller.

Exercise: Estimate the strength of gravity $\frac{GM}{Rc^2}$ on Earth.

Example 2: The Universe is composed of 5% baryons + 25% dark matter + 70% dark energy. Estimate M and R.

$R \sim 10\,\mathrm{Gpc}$

- Mass of baryons
  - 10^{11} stars in Milky Way
  - $(10^4)^3$ galaxies in Universe

  $\Rightarrow$ $M_{\text{baryons}} \sim 10^{23} M_\odot \sim 10^{53}\,\mathrm{kg}$

$$\frac{GM_{tot}}{Rc^2} \sim \frac{10^{-10} \cdot 10^{53} \cdot 10}{10^{27} \cdot 10^{17}} \sim 1 \tag{2}$$

$$\begin{aligned} \text{Density } \rho \sim \frac{M_{tot}}{R_{tot}^3} &\sim \frac{c^2}{R^2 G}, \text{ use } \frac{GM}{Rc^2} \sim 1 \\ &\sim \frac{1}{G \times (\text{age of universe})^2} \end{aligned}$$

cf. critical density from Friedmann equations $\rho_{crit} = \frac{3H_0^2}{8\pi G}$.
Recall Hubble constant $H_0 \sim \frac{1}{\text{age}}$.

The critical density is the density of the universe at which expansion will asymptotically slow. Too dense leads to big crunch, too low leads to unbounded expansion.

Exercise: How do we reconcile a "flat" universe from critical density with the "curved" universe?
