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PHYC90012 General Relativity

Course Summary

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6. Calculus in curvilinear coordinates
    - Mass and energy curve space time
    - Hence geometric objects moved on curved manifolds
    - Distances are not only spatial but temporal; need to use mathematics of small change = calculus
    - Uses the covariant derivative (a geometric object; independent of basis/coordinate independent)
    - This point of the course we will not be considering curved space, but instead only curvilinear coords
        - A flat space can be covered (represented?) by curved coordinates, but an intrinsically curved surface cannot be covered by flat coordinates
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    - Manifolds
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    - Define the Riemann tensor, and its children etc. Ricci tensor, ...; these satisfy the Bianchi identities
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      * = together these form the "antenna problem"
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      - We will derive this and the Friedman equations

Part I

Theoretical tools of the trade

1 Introduction to gravity

1.1 Strength of gravity

- Weak! Weakest of all fundamental forces
- Long-ranged force (like EM)
- Weakness determined by coupling constant
- Coupling constant = Newton's gravitational constant

$$\vec{F} = \frac{Gm_1m_2}{r_{12}^2}\hat{r} \tag{1}$$

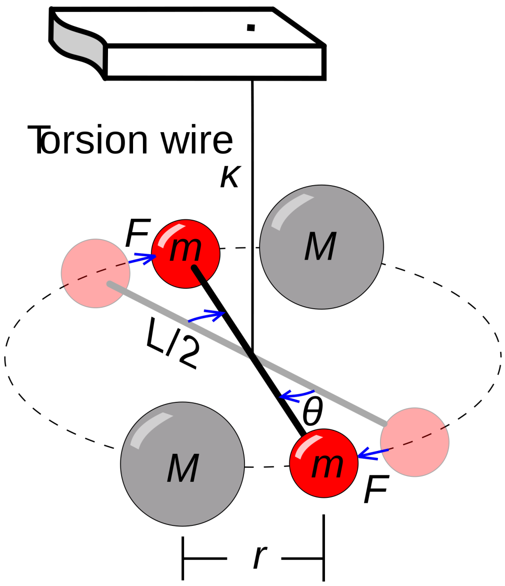
- G is hard to measure; least well known of coupling constants

In 1797-98, Cavendish used torsion balls (1.8m torsion balance) with rod of big masses and rod of small masses.

- Spring constant of torsion balance was measured from free oscillation
- then introduced 158kg balls
- measured deflection angle of balance ⇒ can calculate force
  - using a mini-telescope against Vernier scale
- rearrange Newton's law to get G

Exercise: Show that Cavendish also measured density of Earth as a bonus at the same time.

Mass of Earth $M_\oplus = \rho V$ where $V = \frac{4}{3}\pi R^3$ assuming the Earth is a sphere. How does calculating G also calculate ρ? Well, we have $\vec{F} = \frac{Gm_1m_2}{r_{12}^2}\hat{r}$. Let's take $m_1 = M_\oplus$ as the mass of the Earth, and $m_2 = m$ as some small object mass. Let's imagine the smaller object falling to the center of the Earth. We'll take r_{12} as the distance from the object to the Earth's center, which we can approximate as Earth's radius, i.e. $r_{12} = R$. This force should be equivalent to $F = ma$.



So we have

$$\begin{aligned}\frac{GM_{\oplus}m}{R^2} &= mg\\ \frac{G\rho\frac{4}{3}\pi R^3}{R^2} &= g\\ \frac{4G\rho\pi R}{3} &= g\\ \Rightarrow \rho &= \frac{3g}{4\pi GR}\\ &= \frac{3\times 9.8\,\mathrm{ms^{-2}}}{4\pi\times 6.673\,84\times 10^{-11}\,\mathrm{kg^{-1}m^3s^{-2}}\times 6370\,\mathrm{km}}\\ &= \frac{3\times 9.8}{4\pi\times 6.67384\times 10^{-11}\times 6370\times 10^3}\mathrm{kg\,m^{-3}}\\ &= 5503\,\mathrm{kg\,m^{-3}}\end{aligned}$$

- Modern $G = 6.67384(80)\times 10^{-11}\,\mathrm{Nm^{-2}\,kg^{-2}} = \mathrm{kg^{-1}\,m^3\,s^{-2}}$
- Product GM is known to 1 part in $\sim 10^{10}$ from astrophysics observations

  $\Rightarrow$ mass is hard to measure gravitationally
- We need a dimensionless number to characterise strength
- Newton: $\Phi = \frac{GM}{r}$ (potential)
- In free fall: $\frac{KE}{mass}$, $v^2 \sim \frac{GM}{r}$
- We claim gravity is strong is free-fall is relativistic, i.e. $v \sim c$
- This is an order of magnitude estimate

	$\frac{GM}{Rc^2} \lll 1$	$\frac{GM}{Rc^2} \geq 1$
$v \ll c$	Newtonian	CAN'T EXIST
$v \sim c$	special rel.	full GR (difficult)

1.2 Strong vs. weak gravity

- Quasi-Newtonian:
  - characteristic speed of body in free fall: $v^2 \sim \frac{GM}{r}$
- Strong gravity leads to relativistic free fall, i.e. $\frac{GM}{Rc^2} \geq 1$ where M is the total mass and R is th characteristic size

Example 1.1: $M = M_\odot$ (mass of the Sun)

$$\begin{aligned} R &\sim \frac{GM}{c^2} \quad \text{boundary of strong regime} \\ &\sim \frac{10^{-10}10^{30}}{10^{17}} \\ &\sim \text{km} \end{aligned}$$

cf. Schwarz radius of black hole $= \frac{2GM}{c^2}$

Example 1.2: Density of black hole with mass of $M_\odot$

$$\begin{aligned} &\sim \frac{M}{R^3} \sim \frac{10^{30}\,\text{kg}}{(\text{km})^3} \\ &\sim 10^{21} kgm^{-3} \end{aligned}$$

How does this density compare to maximum density of (say) nuclear matter? Let's compare.

$$\frac{m_n}{(1fm)^3} \sim \frac{10^{-27}kg}{10^{-45}m^3} \sim 10^{18}kgm^{-3}$$

We see a black hole is more dense that a nuclei. The characteristic size of a particle $1fm \sim \Delta x \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar}{m_n c}$, due to Heisenberg's uncertainty principle, and also the Pauli exclusion principle.

More generally: density of material that forms black hole $\sim \frac{M}{R^3}$, but note $M = \frac{c^2 R}{G}$ density $\rho \propto \frac{1}{R^2}$. This means that denser black holes are smaller.

Exercise: Estimate the strength of gravity $\frac{GM}{Rc^2}$ on Earth.

Example 2: The Universe is composed of 5% baryons + 25% dark matter + 70% dark energy. Estimate M and R.

$R \sim 10\,\text{Gpc}$

- Mass of baryons
  - 10^{11} stars in Milky Way
  - $(10^4)^3$ galaxies in Universe

  $\Rightarrow$ $M_{\text{baryons}} \sim 10^{23} M_\odot \sim 10^{53}\,\text{kg}$

$$\frac{GM_{tot}}{Rc^2} \sim \frac{10^{-10} \cdot 10^{53} \cdot 10}{10^{27} \cdot 10^{17}} \sim 1 \tag{2}$$

$$\begin{aligned}\text{Density } \rho \sim \frac{M_{tot}}{R_{tot}^3} &\sim \frac{c^2}{R^2 G}, \text{ use } \frac{GM}{Rc^2} \sim 1 \\ &\sim \frac{1}{G \times (\text{age of universe})^2}\end{aligned}$$

cf. critical density from Friedmann equations $\rho_{crit} = \frac{3H_0^2}{8\pi G}$.
Recall Hubble constant $H_0 \sim \frac{1}{\text{age}}$.

The critical density is the density of the universe at which expansion will asymptotically slow. Too dense leads to big crunch, too low leads to unbounded expansion.

Exercise: How do we reconcile a “flat” universe from critical density with the “curved” universe?

Important to remember: gravity is strong when $\frac{GM}{Rc^2} \sim 1$, which occurs around black holes, the universe at large. In a sense, cosmological results such as critical density, expansion of universe come from this.

1.3 Black hole oscillations

We can estimate the oscillation frequency of a “black hole” (i.e. something with $\frac{GM}{Rc^2} \sim 1$ as $\sim \frac{c}{R}$; that is, the time is takes light to travel the distance of the object. This is the natural frequency for this object. Using that ubiquitous expression we can express the oscillation frequency as $\sim \frac{c^3}{GM}$, e.g. $M = M_\odot \Rightarrow$ frequency $\sim 10\,\text{kHz}$.

Let’s discuss charged black holes. It is difficult to astrophysically have charged black holes, because stars are not usually charged (due to the strength of the EM force, which would attract opposite charge and cancel out). So, these are artificial in nature. These have unusual geometry, and are called “Reissner-Nordstrom” black holes.

Example : What is the maximum charge on a black hole?

$$\frac{Q^2}{4\pi\epsilon_0 R} \leq \frac{GM^2}{R}$$
$$Q \leq (4\pi\epsilon_0 G)^{1/2} M$$

Above, we relate Coloumb force to gravitational force. The gravitational force holding a black hole together must overcome the Coloumb force pushing it apart.

1.4 Quantum Gravity

The problem with quantum gravity is that there is no theory... hence we must rely on numerology.

We consider a hypothetical elementary excitation of a "black hole" (again, we mean a *relativistic compact object*) of mass M. Hence the characteristic size, or "wavelength", of the excitation is $\frac{GM}{c^2}$ (fundamental excitation only). Introducing quantum mechanics: the Heisenberg uncertainty principle tells us that the zero-point motion associated with this excitation is

$$\lambda \sim \frac{\hbar}{\Delta p} \sim \underbrace{\frac{\hbar}{Mc}}_{\text{relativistic}} \tag{3}$$

Equating length scales $\Rightarrow M_{pl} \approx \left(\frac{\hbar c}{G}\right)^{1/2}$; this is the Planck mass, about 10^{-8} kg (the mass below which quantum gravity is important.

Given M_{pl} we get $\lambda \sim \frac{\hbar}{M_{pl}c} \sim 10^{-33}$ m; the Planck length - the length where quantum gravity is important (e.g. just after Big Bang).

1.4.1 Hawking Radiation

Let's return to our elementary excitation with $\lambda \sim \frac{GM}{c^2}$, i.e. frequency $\sim \frac{c^3}{GM} = \frac{c}{\lambda}$. Heisenberg tells us there is an associated energy fluctuation $\Delta E \sim h \times$ frequency $\sim \frac{\hbar c^3}{GM}$. Suppose (note: this is a huge leap) energy fluctuation in the black hole system is in thermal equilibrium with a bath at temperature T. Then $T \sim \frac{\Delta E}{k_B}$. This associates a temperature to a black hole.

We call a black hole a blackbody!

$$\begin{aligned}\text{Radiated power } &= k_B \times \text{area} \times T^4 \\ &= \sigma \times \underbrace{R^2}_{\left(\frac{GM}{c^2}\right)^2} \times \left(\frac{\hbar c^3}{GMk_b}\right)^4 \\ &\propto M^{-2}\end{aligned}$$

Exercise: Plug in numbers to this!

This shows that a black hole radiates energy $\Rightarrow$ eventually a black hole evaporates. We can estimate the time scale of this evaporation.

$$\text{time scale} \sim \frac{Mc^2}{\text{power}} \propto M^3 \tag{4}$$

$\Rightarrow$ small black holes evaporate fast!

As an aside, we could consider the rate of energy accretion. For system outside a black hole with uniform density ρ_{out}, we have

$$\begin{aligned}\text{rate of mass accretion} &\sim \rho_{\text{out}} \cdot c \cdot 4\pi R^2 \\ \text{rate of energy accretion} &\sim c^2 \cdot \text{rate of mass accretion}\end{aligned}$$

A short review:

- there is no theory of quantum gravity
- we consider a relativistic elementary oscillation $\lambda \sim \frac{GM}{c^2}$
- we use Heisenberg to relate this to an energy fluctuation in the system
- energy fluctuation can be converted to a temperature $T_H = \frac{\hbar c^3}{GMk_B}$, the Hawking temperature
- we find small block holes evaporate more quickly than large black holes

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1.5 Black hole thermodynamics

1st law: $dS = \frac{dQ}{T}$ system constant volume

Hawking radiation: we lose a bit of heat dQ due to blackbody radiation.

$$dS = \frac{GMk_b}{\hbar c^3} d(Mc^2) \tag{5}$$

We see heat loss comes from rest energy

$$\frac{dS}{k_B} \approx \frac{1}{R_{\text{Planck}}^2} \underbrace{d(R^2)}_{R\sim\frac{GM}{c^2}} \tag{6}$$

This result relates the entropy of a black hole to its area; Bekenstein-Hawking entropy - $S_{\text{black hole}} \propto$ area of event horizon.

$$\frac{S}{k_B} = \frac{\text{area}}{4R_{\text{Planck}}^2}{}^{1} \tag{7}$$

However, there is a contradiction! Hawking radiation implies that $dA < 0 \Rightarrow dS < 0$.... this is bad.[2] One way to resolve this is by making a generalised **2nd law**:

$$d\left(S_{\text{outside}} + \frac{\text{area}}{4R_{\text{Planck}}^2}\right) \geq 0 \tag{8}$$

[1] In 1995, Maldacena also got this by counting microstates

[2] See Ted Jacobson's lecture at University of Utrecht for a discussion on this.

Unfortunately this is not enough - there is still a contradiction. Consider a small box of radiation which we prepare far from a black hole. $S_{\text{box}} = \frac{4U_{\text{box}}}{3T_{\text{box}}}$. We can make photons very long wavelength, so that $U_{\text{box}} \approx 0$ but $S_{\text{box}} \neq 0$. Then $dS_{\text{out}} = -S_{\text{box}} < 0$. $d(\text{area}) = 0$ because energy in box $= 0$, i.e. $d(Mc^2) = 0$.[3]

Exercise: Resolve the box paradox!

3rd law of BH thermodynamics: can't reduce T to zero.[4]

In heating a rubber band, it shrinks; this is because a shrunken arrangement of the molecules is a state of more entropy (more disordered)

BH

2 Einstein equivalence principle

We will define this (the weak and strong equivalence principles), and some of the tests been performed.

2.1 Weak equivalence principle

Trajectory of body in free fall is independent of its mass and $\underbrace{\text{composition}}_{\text{binding energy}}$ (as per Galileo, feather vs. brick). Note that this is not equivalent for electric fields (the movement of a charged particle through an electric field depends on its charge).

2.2 Strong equivalence principle

1. weak equivalence principle is valid
2. results of any non-gravitational (e.g. EM, <u>not</u> Cavendish expt) experiment is independent of velocity of freely falling frame
   - this is *local Lorentz invariance*
3. results of non-gravitational experimental are independent of where and when it is performed
   - this is *local position invariance*

The Einstein equivalence principle (EEP) $\equiv$ strong implies: existence of a "curved spacetime" with:

i. symmetric metric

[3]Beware the Unruh radiation.

[4]If you are curious: Verlinde, arXiv:1001.0785, *On the Origin of Gravity and the Laws of Newton*, in which the idea of emphgravity is an entropic force is discussed.

ii. trajectories of free-falling bodies are geodesics of metric

iii. the laws of physics in a freely-falling frame can be written in the language of special relativity

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A short review:

Einstein equivalence principle:

1. universality of free fall
2. local Lorentz invariance
3. local position invariance

Implications:

3) $\Rightarrow$ fundamental constant independent of $\vec{x}$, t

2) $\Rightarrow$ laws of non-gravitational physics locally independent of frame

1) $\Rightarrow$ space is curved

Why is space curved? Locally straight trajectory in free yet yet gravity produces curved trajectory (observed) $\Rightarrow$ only possible if coordinates change from one point to next

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3 Experimental tests

3.1 Experimental tests of free fall

Inertial mass $m_i = \frac{\text{applied non-gravitational force}}{\text{measured acceleration}}$

Gravitational mass $m_g =$ "passive" mass appearing in weight

Look for $m_i \neq m_g$

Write

$$m_g = m_i + \sum_{\text{interactions A in body}} \eta^A \frac{E^A}{c^2} \tag{9}$$

Here $E =$ "binding energy"/potential energy of interaction A.

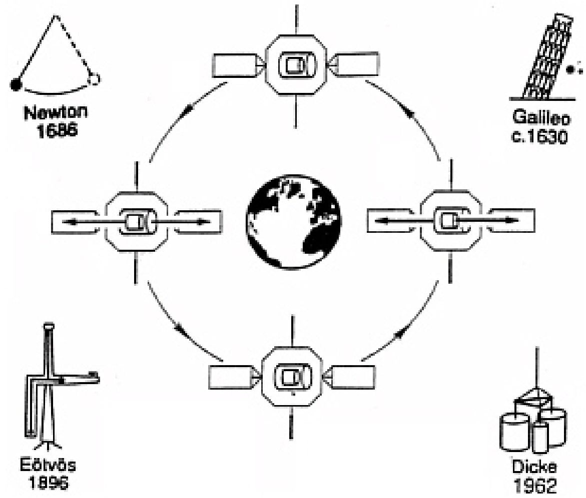
Tests:

1. Eötvös-type torsion balance experiments: two different materials may fall at difference rates (see Dicke, Braginsky)
2. Colorado: U, Cu laser interferometer $\Rightarrow relativiteacceleration$
3. Eöt-Wash experiments: fancy version of 1)

Result:

$$\frac{|m_g - m_i|}{m_i} \leq 10^{-13} \tag{10}$$

We test this, for example, with aluminium (Al) and gold (Au) weights on a torsion balance. As the Sun moves from one side of the Earth to the other, if the gravitational mass of either differs from the other we will see diurnal oscillation in the balance.



3.2 Tests of local Lorentz invariance

- Michelson-Morley experiment (the aether)
- Rossi-Hall tests for lifetime of muons (time dilation $\Leftrightarrow$ LLI)
- Ives-Stiwell transverse Doppler shift
  - laser travelling some vector $\vec{v}$
  - we see perpendicular wave vector $\vec{k}$
  - measure frequency when laser intersects line of sight
  - Doppler shift arises due to time dilation

Mathematically:

$$\begin{bmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{bmatrix} = \begin{bmatrix} \gamma & \gamma v/c & 0 & 0 \\ \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega/c \\ 0 \\ k_y \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma\omega/c \\ \gamma\omega v^2/c \\ k_y \\ 0 \end{bmatrix} \tag{11}$$

Exercise: Compare with standard longitudinal Doppler:

$$\begin{pmatrix} \text{same} \\ \text{matrix} \end{pmatrix} \begin{pmatrix} \omega/c \\ k_x \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

A short review:

Last lecture:

- tests of LPI
- Schiff's thought experiment (gravitational redshift)

3.3 Gravitational redshift

$$\frac{h\nu' - h\nu}{h\nu} = \frac{(m_A g_A - m_B g_B)H}{(m_A - m_B)c^2} \tag{13}$$

If $g_A = g_B$ (universal free-fall), then gravitational redshift is gH/c^2

3.4 Observation of GR in experiments

There are situations where GR makes measurable difference today

- Pound-Rebka experiment (gravitational redshift, also seen on white dwarf spectral lines)
- GPS
- 2015 discovery of gravitational waves from binary BH merger
- cosmological measurements (CMB, redshifts, H_0, ...)
- gravitational lensing (light bent by a mass) (see: Einstein cross, Abell clusters, stars behind Sun)
- precession of perihelion of Mercury
- orbital decay of Hulse-Taylor binary pulsar (can calculate to mm precision the decay of the orbit every 8 hours due to gravitational wave emission)
- Nordtredt/lunar ranging experiments
- Gravity Probe B - lense-thinning precession
- Shapiro time delay

3.4.1 Pound-Rebka experiment

Looks at gravitational redshift of 14.4 keV gamma rays from ^{57}Fe decay

^{57}Fe decays and emits photons directly down from a 23 m tall tower, to another box of ^{57}Fe below. Gravitational redshift occurs; time moves slower closer to the Earth, so gamma ray emitted will have a different energy than that required to be absorbed by the ^{57}Fe at the bottom.

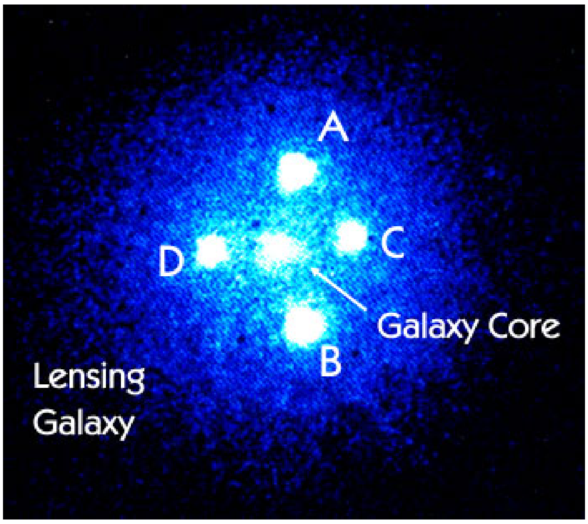


Figure 1: Einstein cross

Receiver box moves up/down at speed v. We adjust v at bottom so that kinematic Doppler shift $\propto \left(\frac{1-v/c}{1+v/c}\right)^{1/2}$ exactly cancels the gravitational redshift $\propto gH/c^2$

N.B.: recoil (in a random direction) when photon emitted/absorbed; energy $E_R = \frac{E_\gamma}{2M_{Fe}c^2}$

$$E_R \sim \frac{14.4\,\mathrm{keV}}{100\,\mathrm{GeV}} \sim 10^{-7} \tag{14}$$

We solve this problem through the Mossbauer effect: use whole crystal; whole crystal ($\sim 10^{23}$ Fe atoms) recoils

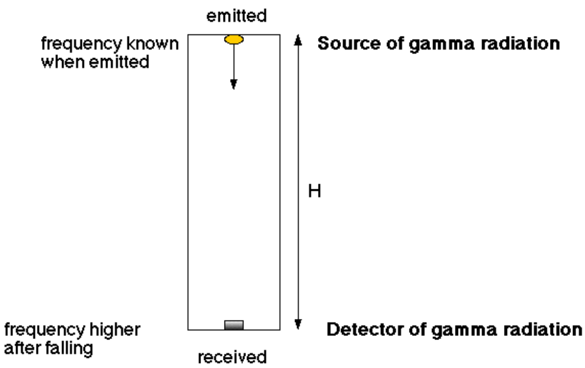


Figure 2: Pound-Rebka experiment

3.4.2 Lunar ranging

Williams + Dickey (2002)[5]

[5] See also "Living Reviews" Relativity

Moon not completely dormant

- fluid core (?)
- tidal dissipation internally
- etc ...

Multiple radar reflectors to improve accuracy. We search for an accurate Earth-Moon orbit versus time; we find

$$\frac{1}{G}\frac{dG}{dt} = (0.0 \pm 1.1) \times 10^{-12}\,\mathrm{yr}^{-1} \tag{15}$$

Uncertainty $\approx 0.02H_0$ - we don't see increasing separation between Earth and Moon due to expansion of universe. This is expected, however: gravitational bound objects do not separate with time, although the energy they require to stay bound will increase

3.4.3 Deflection of light (gravitational lensing)

In a three-body system with the Earth, the Sun and another star, the light from the star will bend due to the Sun before it reaches Earth (not taking a straight path). This causes the apparent position of the star to be different than the actual position.

Deflection angle $\delta\theta \propto \frac{GM}{c^2 d}$

GR deflection = $2\times$ Newtonian deflection (Einstein 1911).

This effect is achromatic!

3.4.4 Shapiro delay

Roundtrip time from Earth to a distant mirror (in a three-body system including the Sun) is longer than if the Sun was not there.

$$\delta t \propto \frac{GM}{c^3}\ln(\text{geometric factors}) \tag{16}$$

Best measurements with Cassinin spacecraft (accuracy 1 in 10^5)

4 Geometric Objects

4.1 Vectors

Invariants are measurable. Therefore, each coordinate representation is as valid as the other!

4.1.1 Definitions of a vector

Vector $\vec{A}$

- four numbers that are projections onto spacetime (can be dependent or independent of a metric)
  - Note: vectors can have meaning without a metric
- a geometric object with "length" (requires a metric) and a "direction", i.e. an arrow
- a geometric object which transforms from one coordinate system to another
- a linear function which takes a 1-form as an argument and returns a real number

$\underbrace{\text{Vector spaces}}_{\text{vectors}} \supseteq \underbrace{\text{normed vector spaces}}_{\text{length}} \supseteq \underbrace{\text{inner product spaces}}_{\text{angle}}$

$\Rightarrow$ you can have vector spaces without a norm or angle concept.

Exercise: $\vec{\Delta}x$ = displacement (in spacetime), is a vector

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Vector components, A^0, A^1, A^2, A^3; in another frame we have $A^{0'}, A^{1'}, A^{2'}, A^{3'}$.

4.1.2 What is a basis vector?

Projection requires a metric (for a dot product)! $\Rightarrow$ we have 4, special, linearly independent vectors which "point along" (no concept of length/angle) axes of coordinate system, $\{\vec{e}_\alpha\}$

$$\vec{A} = A^0\vec{e}_0 + A^1\vec{e}_1 + A^2\vec{e}_2 + A^3\vec{e}_3 \tag{17}$$

As $\vec{e}_0$ is itself a vector, using 17

$$\vec{e}_0 = (\vec{e}_0)^0\vec{e}_0 + (\vec{e}_0)^1\vec{e}_1 + (\vec{e}_0)^2\vec{e}_2 + \underbrace{(\vec{e}_0)^3}_{\text{3rd component of } \vec{e}_0}\vec{e}_3 \tag{18}$$

By linear independence, as $\vec{e}_0$ is linearly independent to $\vec{e}_1, \vec{e}_2, \vec{e}_3$

$$\Rightarrow (\vec{e}_0)^1 = (\vec{e}_0)^2 = (\vec{e}_0)^3 = 0 \tag{19}$$

$$\Rightarrow (\vec{e}_\alpha)^\beta = \delta^\beta_\alpha \tag{20}$$

which is *true in all coordinate systems*. Therefore in a primed frame,

$$\Rightarrow (\vec{e}_{\alpha'})^{\beta'} = \delta^{\beta'}_{\alpha'} \tag{21}$$

<u>But</u> $(\vec{e}_{\alpha'})^\beta \neq \delta^\beta_{\alpha'}$

- $\vec{e}_{\alpha'}$ is tied to the prime frame
- the geometric object is only defined with respect to the frame (where $\vec{A}$ could exist in all frames)
- you cannot measure a basis vector in another coordinate frame. You have to be in the coordinate frame to measure it.

4.1.3 Transformations

In general,

$$\left\{x^{\beta'}\right\} \mapsto \{x^\alpha\}$$
$$\text{from } \sigma' \to \sigma \quad \text{primed to unprimed}$$

$$\Lambda^{\alpha}_{\beta'} = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(x^{0'}, x^{1'}, x^{2'}, x^{3'})} = \frac{\partial x^\alpha}{x^{\beta'}} \quad \text{16 elements!} \tag{22}$$

If the transform is linear (e.g. Lorentz transform), we can consider

$$x^\alpha = \Lambda^{\alpha}_{\beta'} x^{\beta'} \tag{23}$$

where Λ^α is not necessarily a constant. But if the transform is non-linear, the left side is wrong. We instead have

$$dx^\alpha = \Lambda^{\alpha}_{\beta'}\ dx^{\beta'} \tag{24}$$

Why? If we define $\vec{x}$ as a displacement from origin to the point in a curved space, there are multiple different path. We would need more than 4 numbers to define this, therefor we no longer have a meaningful vector.

Vectors are defined localling an tangent space, **and** transformations of vectors are also restricted locally to tangent space. If $\vec{A}$ is a vector,

$$A^\alpha = \Lambda^{\alpha}_{\beta'} A^{\beta'} \tag{25}$$

A short review:

Vectors

$\vec{A}$: defined in tangent space

Components A^α: $\vec{A} = A^\alpha \vec{e}_\alpha$ where $\vec{e}_\alpha$ are basis vectors. These (unprimed) basis vectors only have meaning in unprimed coordinates

Transform like infinitesimal displacements:

$$\underbrace{A^{\alpha'}}_{\text{primed}} = \overbrace{\frac{\partial x^{\alpha'}}{\partial x^\alpha}}^{\text{transform matrix}} \underbrace{A^\alpha}_{\text{unprimed}} \tag{26}$$

How do basis vectors transform?

$$\begin{aligned} A^\alpha \vec{e}_\alpha = \vec{A} &= A^{\alpha'} \vec{e}_{\alpha'} \\ &= \frac{\partial x^{\alpha'}}{\partial x^\beta} A^\beta \vec{e}_{\alpha'} \\ &= \frac{\partial x^{\alpha'}}{\partial x^\alpha} A^\alpha \vec{e}_{\alpha'} \end{aligned}$$

This has to be true for all $\vec{A}$, i.e.

$$\vec{e}_\alpha = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \vec{e}_{\alpha'} \tag{27}$$

Or equivalently,

$$\vec{e}_{\alpha'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} \vec{e}_\alpha \tag{28}$$

Note this is the opposite of transformation law for vector components 26

Exercise: Try for Lorentz transformations.

Exercise: (later) Try for 2 types of Schwarz black hole coordinates.

4.2 1-forms

$\widetilde{p}$

- four numbers associated to four dimensions and associated to vectors in a specific way (see below)
- geometric object that transforms like a gradient
- tensor of type $\binom{0}{1}$, i.e. linear function which accepts vector as an argument and returns a real number

If we have a metric, a neat way to visualise a 1-form is as a contour.

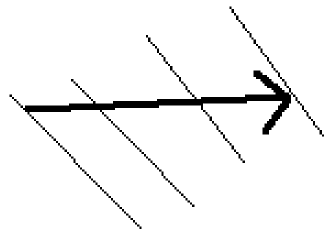
e.g. vector is an arrow (length, direction)

cf. 1-form is contours

Value of $\widetilde{p}(\vec{A}) = 4$; number of times $\vec{A}$ pierces surfaces of $\widetilde{p}$

Components defined to be

$$\begin{aligned} p_\alpha &= \widetilde{p}(\vec{e}_\alpha) \\ &= \langle \widetilde{p}, \vec{e}_\alpha \rangle \end{aligned}$$



From lines 1 to 2, we have equivalent notation. We are using an inner product. By convention we use a lowered index.

We cannot have curved contours, as they are only defined in tangent space.

We have a neat way to calculate the coordinate-independent quantity $\widetilde{p}(\vec{A})$

$$\begin{aligned}\widetilde{p}(\vec{A}) &= \widetilde{p}(A^\alpha \vec{e}_\alpha) \\ &= A^\alpha \widetilde{p}(\vec{e}_\alpha) \quad \text{linear} \\ &= A^\alpha p_\alpha\end{aligned}$$

This is a contraction. It is an inner product but <u>not</u> dot product (we require a metric, and for it to be between two vectors, for a dot product).

Vectors and 1-forms are "apples and oranges" - completely different geometric objects which cannot be compared even at the same point, e.g. $\widetilde{p} = \vec{A}$ is meaningless.

This is comparable to bras and kets in quantum mechanics; they form a dual space, but we cannot directly compare the objects of a pair

4.2.1 How do 1-forms transform?

We start with basis vectors

$$\vec{e}_\alpha = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \vec{e}_{\alpha'} \tag{29}$$

In component form:

$$\begin{aligned}p_{\alpha'} &= \widetilde{p}(\vec{e}_{\alpha'}) \\ &= \frac{\partial x^\alpha}{\partial x^{\alpha'}} \widetilde{p}(\vec{e}_\alpha) \quad \text{linear} \\ &= \frac{\partial x^\alpha}{\partial x^{\alpha'}} p_\alpha\end{aligned}$$

i.e. components of $\widetilde{p}$ transform like basis vectors (<u>not</u> components of basis vectors)

A short review:

Last lecture missing

1-forms and their relation to gradients defined

	Vectors $\vec{A} = A^\alpha \vec{e}_\alpha$	**1-forms** $p_\alpha = \widetilde{p}(\vec{e}_\alpha)$
Transformation for coordinates	$A^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} A^\alpha$	$p_{\alpha'} = \frac{\partial x_{\alpha'}}{\partial x_\alpha} p_\alpha$
Transformation for basis vectors	$\vec{e}_{\alpha'} = \frac{\partial x_{\alpha'}}{\partial x_\alpha} \vec{e}_\alpha$	$\widetilde{\omega}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \widetilde{\omega}^\alpha$

4.3 1-forms, basis 1-forms and gradients

A short review:

Given a scalar field $\phi(\vec{x})$ we define a 1-form $\widetilde{d\phi}$, with components $(\widetilde{d\phi})_\alpha = \frac{\partial \phi}{\partial x^\alpha}$

Notes:

1. If we take $\phi =$ one of the coordinates, e.g. $\phi = x^\alpha$ for a specific α, then $\widetilde{d\phi} = \delta^\alpha_\beta$ for $\phi = x^\alpha$
   - i.e. $\widetilde{d\phi}$, for $\phi = x^\alpha$ is just $\widetilde{\omega}^\alpha$, the basis 1-form
2. $\widetilde{d\phi}$ is the most natural definition of a normal in GR. We also has $\widetilde{d\phi}(\vec{t}) = 0$ if $\vec{t}$ is a tangent vector to the level surfaces of ϕ; don't need a metric
3. How do we construct components of vectors?
   - Components of 1-forms: $p_\alpha = \widetilde{p}(\vec{e}_\alpha)$
   - Dual: components of vectors: $A^\alpha = \vec{A}(\widetilde{\omega}^\alpha)$
   - Note that we do not need 1-forms to define vectors (and vice versa), but once we start discussing components, we do need them

---

4.4 Tensors

$\binom{M}{N}$ tensor is a linear function operating on M 1-forms and N vectors to return a real number.

For example if R is a $\binom{1}{1}$ tensor then its components are $R^\alpha_\beta = R(\widetilde{\omega}^\alpha, \vec{e}_\beta)$.

You can build big tensors out of little ones in several ways (and vice versa, e.g. via contraction). One important way is the outer product;

e.g. outer product of vectors $\vec{A}$ and $\vec{B} = \binom{2}{0}$ tensor; $T = \vec{A} \otimes \vec{B}$

Takes two 1-forms as arguments: $T(\widetilde{p}, \widetilde{q}) \equiv^{\text{def}} \vec{A}(\widetilde{p})\vec{B}(\widetilde{q})$ (this is multiplying two scalar numbers)

- Note: outer product not commutative.

$$\text{If } S = \vec{B} \otimes \vec{A} \text{ then } S(\widetilde{p}, \widetilde{q}) = \vec{B}(\widetilde{p})\vec{A}(\widetilde{q}) \neq T(\widetilde{p}, \widetilde{q}) \tag{30}$$

- Can't write a general $\binom{M}{N}$ tensor as an outer product necessarily, but can write as a linear combination of outer products

$$\underbrace{T}_{\text{geometric object}} = T^{\alpha_1,\ldots,\alpha_M}_{\beta_1,\ldots,\beta_N}\,\vec{e}_{\alpha_1}\otimes\ldots\otimes\vec{e}_{\alpha_M}\otimes\widetilde{\omega}^{\beta_1}\otimes\ldots\otimes\widetilde{\omega}^{\beta_N} \tag{31}$$

- even though we use M 1-forms and N vectors, we take the outer product over N 1-forms and M vectors

Exercise: Show that you get the components of T when you evaluate T for M basis 1-forms and N basis vectors.

Another way to generate new tensors:

$$\begin{aligned}\text{Symmetric part of T } (T^{(\alpha\beta)})\text{: } T^{(\widetilde{p},\widetilde{q})}_{(\text{sym})} &= \frac{1}{2}T(\widetilde{p},\widetilde{q}) + \frac{1}{2}T(\widetilde{q},\widetilde{p})\\ \text{Antisymmetric part of T } (T^{[\alpha\beta]})\text{: } T^{(\widetilde{p},\widetilde{q})}_{(\text{sym})} &= \frac{1}{2}T(\widetilde{p},\widetilde{q}) - \frac{1}{2}T(\widetilde{q},\widetilde{p})\end{aligned}$$

This example is for $\binom{2}{0}$ but can generalise to $\binom{M}{N}$

4.4.1 Metric tensor

$\binom{0}{2}$ tensor which accepts two vectors and returns a number which we call scalar or dot product.

$$g(\vec{A},\vec{B}) \equiv^{def} \vec{A}\cdot\vec{B} \qquad \text{this is \underline{not} contraction} \tag{32}$$

Clearly bilinear

What is g? Anything we like! It depends on the coordinates we choose to use! (And it tells us how lengths, angles, ... are measured in our favourite coordinates)

Components:

$$g_{\alpha\beta} = g(\vec{e}_\alpha,\vec{e}_\beta) = \vec{e}_\alpha\cdot\vec{e}_\beta \tag{33}$$

i.e. 16 numbers saying how the basis vectors relate to each other in any given coordinate system

What is $g(\Delta\vec{x},\Delta\vec{x})$ ($\Delta\vec{x}$ is a displacement vector)?

$$\begin{aligned}g(\Delta\vec{x},\Delta\vec{x}) &= g(\Delta x^\alpha\vec{e}_\alpha,\Delta x^\beta\vec{e}_\beta)\\ &= g_{\alpha\beta}\Delta x^\alpha\Delta x^\beta \qquad \text{linear}\end{aligned}$$

<u>This quantity is the spacetime interval in curved space. Invariant!</u>

A short review:

Metric: $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor $g(\vec{A}, \vec{B} \equiv \vec{A} \cdot \vec{B}$

Components:

$$g_{\alpha\beta} = g(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta \tag{34}$$

---

e.g. Minkowski metric

$$\{g_{\alpha\beta}\} = \text{diag}(-1, 1, 1, 1) \tag{35}$$

e.g. Schwarzschild black hole of mass M (we will prove this later)

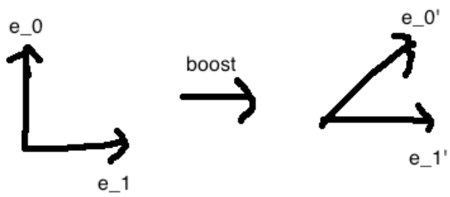
$$\{g_{\alpha\beta}\} = \left[-\left(1 - \frac{2M}{r}\right), \left(1 - \frac{2M}{r}\right)^{-1}, \ r^2, \ r^2 \sin^2\theta \right] \tag{36}$$

where r is the radial coordinate.

Vectors $\vec{A}$ and $\vec{B}$ are orthogonal vectors if $g(\vec{A}, \vec{B}) = 0$. But orthogonal doesn't necessarily mean perpendicular.

These include 4D vectors of "zero length":

- in Minkowski, momentum of photon $\vec{p} = \left(\frac{E}{c}, \widetilde{p}\right)$ in flat space, and we know $E = |\underline{\mathrm{p}}|c$ so $g(\vec{p}, \vec{p}) = -\frac{E^2}{c^2} + |\underline{\mathrm{p}}|^2 = 0$.
  - i.e. $\vec{p}$ is orthogonal with itself; has zero "length" but obviously <u>not</u> a zero vector, nor is it perpendicular to itself
- in Minkowski: boost between frames



  - the boost vectors are not perpendicular, but we still have $\vec{e}_{0'} \cdot \vec{e}_{1'} = 0$

4.5 Correspondence between vectors and 1-forms

Let g be a metric, and let $\vec{V}$ be an arbitrary vector. Then $g(\vec{V}, ...)$ has one argument free, and that argument is a vector, so $g(\vec{V}, ...)$ must be a 1-form. We call it $\widetilde{V}$ because it's the 1-form associated with $\vec{V}$. What are its components?

$$\begin{aligned} V_\alpha \equiv^{def} \widetilde{V}(\vec{e}_\alpha) &= g(\vec{V}, \vec{e}_\alpha) \\ &= g(V^\beta \vec{e}_\beta, \vec{e}_\alpha) \\ &= V^\beta g(\vec{e}_\beta, \vec{e}_\alpha) \quad \text{by linearity} \\ &= V^\beta g_{\beta\alpha} \quad \text{by definition} \end{aligned}$$

This is the "lowering the index" operation of yesteryear! The old language for this is: lower a contravariant index, get a covariant index.

This can go the other way as long as g is invertible. Define $g^{\alpha\beta}$ as 16 numbers you get if you treat $g_{\alpha\beta}$ as a matrix and invert it.

$$\text{i.e.,} \quad g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma} \tag{37}$$

Exercise: What is the geometric object whose components are $g^{\alpha\beta}$? Hint: $\binom{2}{0}$ tensor which takes two 1-forms as arguments.

We check that "raising the index" on V_α gets us back to $\vec{V}$.

$$V_\alpha \underbrace{g^{\alpha\gamma}}_{\text{inverse metric}} = V^\beta g_{\beta\alpha} g^{\alpha\beta} = V^\beta \delta^{\gamma}_{\beta} = V^\gamma \tag{38}$$

More generally:

$$g \text{ maps } \binom{M}{N} \text{ tensor to } \binom{M-1}{N+1} \text{ tensor}$$
$$g^{-1} \text{ maps } \binom{M}{N} \text{ tensor to } \binom{M+1}{N-1} \text{ tensor}$$

Physics example: consider a photon in curved (Schwarzschild) spacetime.

We know $\vec{p} \cdot \vec{p} = 0$ in the local freely falling frame(from weeks 1,2: Michelson-Morley experiment), and $\vec{p} \cdot \vec{p}$ is invariant $\Rightarrow \vec{p} \cdot \vec{p} = 0$ in all coordinate systems.

In Schwarzschild spacetime: radially in-falling photon $\vec{p} = \left(p^0, p^r, 0, 0\right)$

$$\begin{aligned} 0 &= g_{00}(p^0)^2 + g_{11}(p^1)^2 \\ &= -\left(1 - \frac{2M}{r}\right)(p^0)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(p^r)^2 \end{aligned}$$

i.e.

$$\frac{p^r}{p^0} = \left(1 - \frac{2M}{r}\right)^{-1} \neq 1$$
$$(\text{phase speed of light})^{-1} \neq 1$$

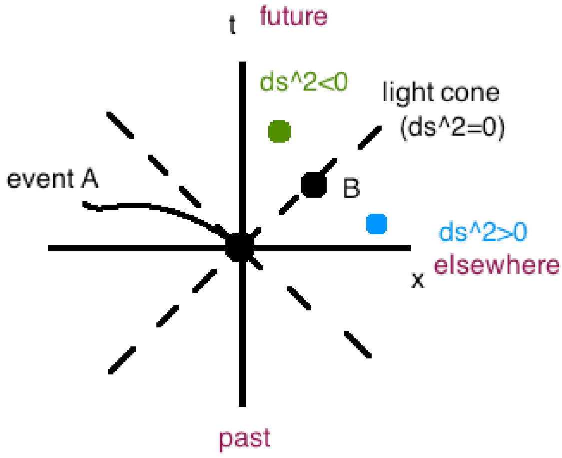
5 Kinematics

In general, for kinematics we do not care about the origin of the force, and only analyse the motion of the object. This is in contrast to dynamics, where we investigate where a force originates. In a GR context, for kinematics this means that we will look into the motion of an object while considering the spacetime, while for dynamics we also investigate what causes the curvature of spacetime.

Event: $\vec{x} = (t, x, y, z)$

Spacetime interval between 2 events: $ds^2 = g(d\vec{x}, d\vec{x})$

e.g. Minkowski



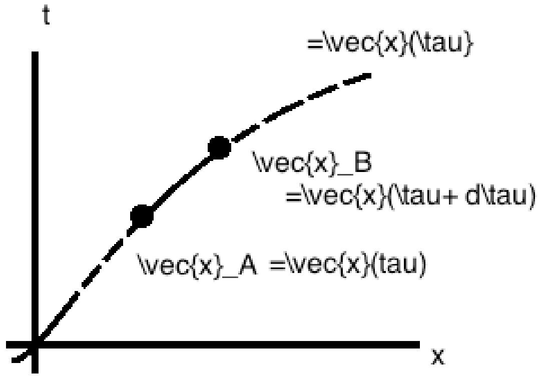
$$\begin{aligned}
ds^2 &= 0 && \text{null ray; defines light cone which joins events A and B} && (39)\\
ds^2 &> 0 && \text{events A and B and mutually spacelike; can't event get from A to B;} && (40)\\
& && \text{there always exists an inertial frame such that events occur at} &&\\
& && \text{different spacial locations at the same time} &&\\
ds^2 &< 0 && \text{events are timelike; can get from A to B;} && (41)\\
& && \text{frame exists where events occur at same spatial location at different times} &&
\end{aligned}$$

For $ds^2 > 0$, we will always be able to Lorentz transform into a frame where two events occur at the same time, but spatially separated. For $ds^2 < 0$, there will always be a Lorentz transform such that the two events occur at the same spatial location, but temporally separated.

Proper time: two events A and B separated infinitesimally along some 4D trajectory.

$$d\tau \equiv^{def} (-ds^2)^{1/2} \tag{42}$$

τ is an affine parameter (= label of path) with a particular normalisation, namely that τ tracks passage of time for an observer at rest with respect to the sequence of events defined by $\vec{x}(\tau)$.



5.1 4-velocity

From above:

$$-d\tau^2 = ds^2 = d\vec{x} \cdot d\vec{x}$$
$$\Rightarrow -1 = \frac{d\vec{x}}{d\tau}\frac{d\vec{x}}{d\tau}$$

where the dot product is with respect to the metric.

Call $\vec{u} = \dfrac{d\vec{x}}{d\tau}$ the 4-velocity. It is timelike, and $\vec{x} \propto d\vec{x}$. It relies on a geometric object (the sequence of events $\vec{x}(\tau)$) for its meaning. And if τ is the label, we have $\vec{u} \cdot \vec{u} = -1$ as normalisation.

5.1.1 Example 1: Momentarily comoving reference frame

For example, consider the momentarily comoving reference frame.

$$\frac{d\mathbf{x}}{dt} = 0 \qquad \therefore \vec{u} = (u^t, 0, 0, 0) \tag{43}$$

By definition $u^t = \frac{dt}{d\tau}$. But what is it? Normalisation gives

$$-1 = g_{tt}(u^t)^2 + 0$$
$$u^t = \left(-\frac{1}{g_{tt}}\right)^{1/2}$$

In flat space, $g_{tt} = -1$ and $u^t = 1$.

In Schwarzschild space, $g_{tt} = -\left(1 - \frac{2M}{r}\right)$ and $u^t = \left(1 - \frac{2M}{r}\right)^{-1/2}$. Recall that $u^t \equiv \frac{dt}{d\tau}$. So as $r \to 2M$, $\frac{dt}{d\tau} \to \infty$. Physically, this says that time slows down as we approach a black hole. $d\tau$ would be the clock nearby the black hole, while dt would be the clock on a faraway observer. They see an infinite amount of ticks on their clock, before the black hole clock ticks even once.

5.1.2 Example 2

Consider coordinates in which our body moves with speed $\mathbf{V} = \frac{d\mathbf{x}}{dt}$ (in, for example, the x-direction).

$$\begin{aligned} \vec{u} &= \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, 0, 0\right) \\ &= \left(\frac{dt}{d\tau}, V\frac{dt}{d\tau}, 0, 0\right) \quad \text{chain rule} \end{aligned}$$

By normalisation,

$$-1 = \vec{u}\cdot\vec{u} = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{tx}\left(\frac{dt}{d\tau}\right)^2 V + g_{xx}V^2\left(\frac{dt}{d\tau}\right)^2 \tag{44}$$

Special cases (note $c = 1$):

$$\begin{aligned} \text{(a) Minkowski:} \quad & g_{tt} = -1 \quad g_{tx} = 0 \quad g_{xx} = 1 \\ & \therefore -1 = -\left(\frac{dt}{d\tau}\right)^2 + V^2\left(\frac{dt}{d\tau}\right)^2 \\ & \therefore \frac{dt}{d\tau} = \frac{1}{\sqrt{1-V^2}} \\ & \text{and } \vec{u} = \left(\frac{1}{\sqrt{1-V^2}}, \frac{V}{\sqrt{1-V^2}}, 0, 0\right) \end{aligned}$$

$$\begin{aligned} \text{(b) Schwarz BH:} \quad & g_{tt} = -\left(1-\frac{2M}{r}\right) \qquad g_{tr} = 0 \qquad g_{rr} = \frac{1}{1-\frac{2M}{r}} \\ & \therefore -1 = -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1-\frac{2M}{r}}V^2\cdot\left(\frac{dt}{d\tau}\right)^2 \end{aligned}$$

Solve for $\frac{dt}{d\tau}$ again.

N.B. Self-consistently combines time dilation due to gravity (e.g. Pound-Rebka experiment) and motion (e.g. special relativity). It is <u>not</u> a simple addition!

A short review:

Last lecture: kinematics; $\vec{u}$ and time dilation

5.2 Energy measurements

Suppose an observer with 4-velocity $\vec{u}$ encounters a particle with 4-momentum $\vec{p}$. What is the energy E of the particle measured by the observer? (Note: $\vec{u}$ and $\vec{p}$ are geometric objects - they can be considered without reference to a frame (frame-independent). However, the energy measured by the observer is a frame-dependent quantity)

Equivalence principle: we can always "jump" into a *freely falling* frame at the location of the particle. In that frame, spacetime is flat $\Rightarrow$ we can describe the frame in Minkowski coordinates.

Local Lorentz invariance: we can boost ourselves so that we are instantaneously at rest with respect to the particle.

We now know:

$$\begin{aligned} g &= (-1, 1, 1, 1) \\ \vec{v} &= (1, 0, 0, 0) \\ \vec{p} &= (E, p^x, p^y, p^z) \quad \text{from special relativity} \end{aligned}$$

Can we express E as an invariant? Yes!

$$E = -\vec{u} \cdot \vec{p}, \tag{45}$$

recalling the dot product

$$\vec{u} \cdot \vec{p} = g_{\alpha\beta} u^\alpha p^\beta. \tag{46}$$

This is invariant, meaning it is independent of coordinates. We can always use this result even when we're given $\vec{u}$ and $\vec{p}$ in some other coordinates (from the principle of equivalence).

Example ?: Consider a Schwarz. BH with coordinates (t, r, θ, ϕ). Consider a radially infalling observer at some speed V, at some radius r. Then $\vec{u} = (u^t, Vu^t, 0, 0)$ and normalisation $\vec{u} \cdot \vec{u} = -1$ tells us u^t given the metric $g_{tt} = -(\left(1 - \frac{2M}{r}\right)$, $g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1}$. We dot product with $\vec{p}$ to get E.

Exercise: Measure velocity V as well as energy E, and package into a 4-vector with the form[6]

$$\vec{V} = \frac{\vec{p} + (\vec{p} \cdot \vec{u})\vec{u}}{-\vec{p} \cdot \vec{u}} \tag{47}$$

Exercise: What is the Doppler shift measured by a stationary observer who shines a laser at a moving mirror at speed V and measures reflected light?

5.3 4-acceleration

The definition $\vec{u} = \frac{d\vec{x}}{d\tau}$ is completely general in curved space. But acceleration is <u>not</u> $\vec{a} = \frac{d\vec{u}}{d\tau}$ in general. This is because the 2nd derivative $\frac{d^2\vec{x}}{d\tau^2}$ cares about curvature (while the 1st derivative does not).

Actually: $\vec{a} = \nabla_{\vec{u}} \vec{u}$ in general.

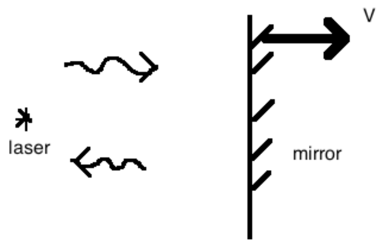


Figure 3: Mirror

In flat space: $\vec{a} = \frac{d\vec{u}}{d\tau}$, which is a special case.

$$\underbrace{\vec{u}\cdot\vec{u} = -1}_{\text{normalisation}} \Rightarrow \underbrace{\vec{u}\cdot\vec{a} = 0}_{\text{differentiate both sides w.r.t } \tau} \tag{48}$$

Equation (48) is true in curved space.

Let's consider a case of **Uniform acceleration**. Consider a rocket ship with constant *proper* acceleration g in the "1" direction (achieved by, for example, throwing bricks out the back of the rocket at a constant rate), in flat spacetime.

In momentarily comoving reference frame:

$$\begin{aligned} g &= (-1, 1, 1, 1) \\ \vec{u} &= (1, 0, 0, 0) \\ \vec{a} &= (0, \frac{d^2\vec{x}}{d\tau^2}, 0, 0) \end{aligned}$$

Above, a^0 is zero because

$$-a^t = \vec{u}\cdot\vec{a} = 0 \tag{49}$$

and a^1 is equal to g by definition. Hence $\vec{a}\cdot\vec{a} = g^2$. This invariant $\Rightarrow$ true in all coordinates, but ONLY true in *flat spacetime.*

Now consider motion in global Minkowski coordinates not tied to the rocket (which exists because spacetime is flat).

We want to solve for $u^0(\tau)$, $u^1(\tau)$, $a^0(\tau)$, $a^1(\tau)$ in general.

$$\begin{aligned} \vec{u}\cdot\vec{u} &= -1 & -(u^0)^2 + (u^1)^2 &= -1 \\ \vec{a}\cdot\vec{u} &= 0 & -a^0u^0 + u^1a^1 &= 0 \\ \vec{a}\cdot\vec{a} &= g^2 & -(a^0)^2 + (a^1)^2 &= g^2 \end{aligned}$$

Eliminating variables and solving (using straightforward algebra), we obtain

$$\frac{du^0}{d\tau} = a^0 = gu^1 \tag{50}$$

$$\frac{du^1}{d\tau} = a^1 = gu^0 \tag{51}$$

Integrating these gives

$$t = \frac{1}{g}\sinh g\tau \tag{52}$$

$$x = \frac{1}{g}\cosh g\tau \tag{53}$$

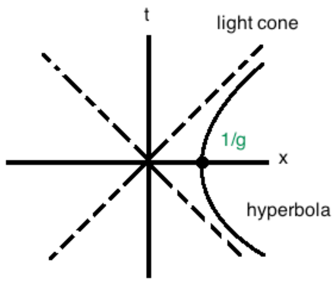


Figure 4: Rocket hyperbola

Remember this is in *flat spacetime* The interesting thing about this diagram is that is shows that ANY photon more than $1/g$ distance away from the rocket ship at $t = 0$ will never reach the rocket ship, even though the rocket ship is travelling less than c.

6 Calculus in curved space

6.1 Differentiation in curved space

Covariant derivatives → curvature → Einstein's field equations. Curvature is the second order change in space.

How does a tensor T of type $\binom{M}{N}$ change from spacetime point P_A to P_B? (we consider these points to be infinitesimally separated).

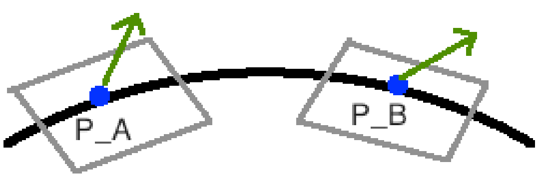


Figure 5: Tangent spaces

In general, we cannot compare geometric objects at P_A and P_B (they live in different tangent spaces) unless we have a recipe for "moving" geometric objects from P_A to P_B, e.g. parallel transport (which we will consider later).

For now we just consider <u>flat</u> space, where all points share a common tangent space.

Look for $\begin{pmatrix} M \\ N+1 \end{pmatrix}$ tensor ∇T which contracts with a vector $\vec{A}$ to give infinitesimal rate of change of T along $\vec{A}$. We call the rate of change $\nabla_{\vec{A}}T$ (type $\begin{pmatrix} M \\ N \end{pmatrix}$) the *covariant derivative*

$$\nabla_{\vec{A}}T = \langle \nabla T, \vec{A} \rangle \tag{54}$$

e.g. if T is a scalar $\underbrace{\phi}_{\begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ type}}$, then $\nabla T = \underbrace{\widetilde{d\phi}}_{\text{gradient } \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

e.g. if T is a vector, say $\vec{V}$: in general consider two points P_A, P_B joined by a world line $\vec{x}(t)$

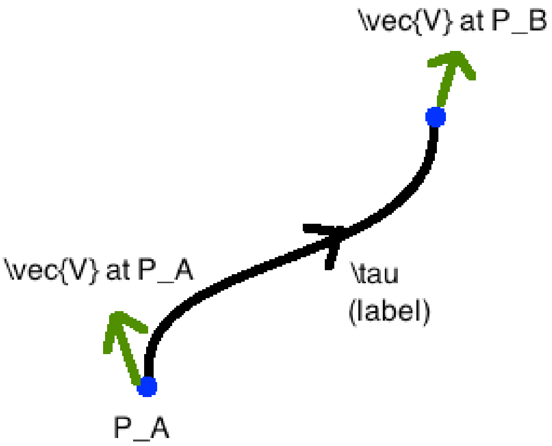


Figure 6: World line connecting points P_A and P_B

$$\begin{aligned}
\frac{d\vec{V}}{d\tau}(\vec{x}(\tau)) &= \frac{d}{d\tau}\left[V^{\alpha}(\vec{x}(\tau))\vec{e}_{\alpha}(\vec{x}(\tau))\right] \\
&= \frac{\partial V^{\alpha}}{\partial x^{\beta}}\frac{dx^{\beta}}{d\tau}\vec{e}_{\alpha}(\vec{x}(\tau)) + V^{\alpha}\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}}\frac{dx^{\beta}}{d\tau} \qquad \text{product rule} \\
&= \left(\frac{\partial V^{\alpha}}{dx^{\beta}}\vec{e}_{\alpha} + V^{\alpha}\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}}\right)u^{\beta} \qquad \text{4-velocity}
\end{aligned}$$

The parts inside the brackets of (**??**), denoted as $(\ldots)$, are contracted with $\vec{u}$ to give the LHS, which is a vector. So $(\ldots)$ is type $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}}$ are vectors, so can be written as a linear combination of basis vectors.

$$\text{Define: } \frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} = \Gamma^{\mu}_{\alpha\beta}\vec{e}_{\mu} \tag{55}$$

Then,

$$
\begin{aligned}
(\ldots) &= \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha \Gamma^\mu_{\alpha\beta}\vec{e}_\mu \\
&= \left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\beta}V^\mu\right)\vec{e}_\alpha
\end{aligned}
$$

The bracketed part above is, in index notation, $V^\alpha_{;\beta}$; the components of covariant derivation $\nabla\vec{V}$ of type $\binom{1}{1}$.

Notation: switch $\mu \leftrightarrow \alpha$ in dummy index (repeated).

Christoffel symbols Γ:

- $4 \times 4 \times 4 = 64$ numbers
- solve 64 equations given by (55) - linear equations (easy to solve!)
  - an alternative trick is to use the metric, which we will see later
- are Christoffel symbols components of a $\binom{1}{2}$ tensor?
  - No!
  - $\{\Gamma^\mu_{\alpha\beta}\vec{e}_\mu\}$ are components of a tensor, but the Γ's themselves are not
  - e.g. is Γ's are tensor, then
  $$\Gamma^{\mu'}_{\alpha'\beta'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}\Gamma^\mu_{\alpha\beta} \tag{56}$$
  - but from the definition of Γ's we have $\Gamma^\mu_{\alpha\beta} = 0$ for (say) Cartesian coordinates, which would then wrongly imply $\Gamma^{\mu'}_{\alpha'\beta'}$ from (56) too!

Physically: suppose you have a vector $\vec{V}$ which does not change from point to point. We want $\nabla\vec{V} = 0$.

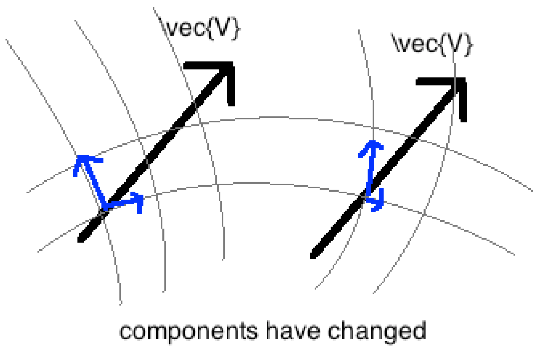


Figure 7: Vectors V

If the coordinates are curvilinear, then components "artificially" change. The Γ's "undo" the artificial change!

A short review:

$$T = \binom{M}{N} \text{ tensor}$$
$$\nabla T = \binom{M}{N+1} \text{ tensor}$$
$$\nabla_{\vec{A}} T = \text{covariant derivative of T along } \vec{A} = \langle \nabla T, \vec{A} \rangle = \text{type } \binom{M}{N}$$

e.g. if T of a scalar ϕ then $\nabla T = \widetilde{d\phi}$

e.g. if T is a vector $\vec{V}$ then $\nabla \vec{V}$ is a $\binom{1}{1}$ object

$$\underbrace{(\nabla \vec{V})^\alpha_\beta}_{\text{notation: } V^\alpha_{;\beta}} = \underbrace{\frac{\partial V^\alpha}{\partial x^\beta}}_{\text{notation: } V^\alpha_{,\beta}} + \underbrace{\Gamma^\alpha_{\lambda\beta}}_{\text{Christoffel}} V^\lambda \tag{57}$$

Christoffel symbol corrects for artificial change in components of $\vec{V}$ due to curvilinear coordinates (see Figure 7). Derived for curvilinear coordinates in flat space but remains true for curved spaces also!

Note: neither $V^\alpha_{,\beta}$ nor $\Gamma^\mu_{\alpha\beta}$ are tensors (Γ's are components of set of tensors $\{\nabla \vec{e}_\alpha\}$). But $V^\alpha_{;\beta}$

6.2 Covariant derivative of 1-form

What about 1-forms? Can we just say

$$p_{\alpha;\beta} = \frac{\partial p_\alpha}{\partial x^\beta} + \Gamma^\alpha_{\lambda\beta} p_\lambda \quad ??$$

No, we cannot - note the double lowered λ in the RHS.

$\nabla \widetilde{p}$ is $\binom{0}{2}$ type/

We redo rate-of-change calculation along world line $\vec{x}$

$$\frac{d\widetilde{p}}{d\tau} = \frac{d}{d\tau}\left[p_\alpha(\vec{x}(\tau))\widetilde{\omega}^\alpha(\vec{x}(\tau))\right] \tag{58}$$
$$= \frac{\partial p_\alpha}{dx^\beta}\frac{dx^\beta}{d\tau}\widetilde{\omega}^\alpha + p_\alpha \underbrace{\frac{\partial \widetilde{\omega}^\alpha}{\partial x^\beta}}_{\text{1-form}} \frac{dx^\beta}{d\tau} \tag{59}$$

We can write the 1-form as linear combination $x^\alpha_{\mu\beta}\widetilde{\omega}^\mu$

Are x's related to Γ;s? Yes - duality.

$$\delta^\alpha_\beta = \langle \widetilde{\omega}^\alpha, \vec{e}_\beta \rangle \qquad \text{from previous lectures} \tag{60}$$

Differentiate by x^γ

$$0 = \langle \frac{\partial \widetilde{\omega}^\alpha}{\partial x^\gamma}, \vec{e}_\beta \rangle \langle \widetilde{\omega}^\alpha, \frac{\partial \vec{e}_\beta}{\partial x^\gamma} \tag{61}$$
$$= \langle x^\alpha_{\mu\gamma} \widetilde{\omega}^\mu, \vec{e}_\beta \rangle + \langle \widetilde{\omega}^\alpha, \Gamma^\lambda_{\beta\gamma} \vec{e}_\lambda \rangle \tag{62}$$
$$= x^\alpha_{\mu\gamma} \delta^\mu_\beta + \Gamma^\lambda_{\beta\gamma} \delta^\alpha_\lambda \tag{63}$$
$$= x^\alpha_{\beta\gamma} + \Gamma^\alpha_{\beta\gamma} \tag{64}$$

i.e. opposites!

Components of $\nabla \widetilde{p}$, i.e. $(\nabla \widetilde{p})_{\alpha\beta}$ is notationally $p_{\alpha;\beta}$

$$p_{\alpha;\beta} = \frac{\partial p_\alpha}{\partial x^\beta} - \Gamma^\lambda_{\alpha\beta} p_\lambda \tag{65}$$

For general tensor: add correction term $\pm\Gamma$ tensor for each tensor index; + is index is up; − if index is down.

$$\text{e.g. } T^\alpha_{\beta;\gamma} = \frac{\partial T^\alpha_\beta}{\partial^\gamma} + \underbrace{\Gamma^\alpha_{\lambda\gamma} T^\lambda_\beta}_{\text{by analogy with vector}} - \underbrace{\Gamma^\lambda_{\beta\gamma} T^\alpha_\lambda}_{\text{by analogy with 1-form}} \tag{66}$$

6.3 Covariant derivative of metric

Recall

$$\nabla \vec{V} = \binom{1}{1} \text{ type} \tag{67}$$
$$\nabla \widetilde{V} = \binom{0}{2} \text{ type} \tag{68}$$

where $\widetilde{V}$ is the specific 1-form induced by the metric with $\vec{V}$ in the one slot ($\widetilde{V} = g(\vec{V}, \ldots)$).

We can go from $\binom{0}{2}$ to $\binom{1}{1}$ by contraction!

Consider $\nabla_{\vec{a}} \vec{V} = \binom{1}{0}$ object.

Then $g(\nabla_{\vec{A}} \vec{V}, \ldots)$ is a 1-form induced by g.

By definition this is $\nabla_{\vec{A}} \widetilde{V}$ which is a 1-form, i.e. $\nabla \widetilde{V}$ $\binom{0}{2}$ contracted with $\vec{A}$.

In index notation:

$$V_{\alpha;\beta} = g_{\alpha\gamma} V^\gamma_{;\beta} \qquad \text{as in previous lectures} \tag{69}$$

6.4 Christoffel Symbols and Metric and Acceleration

6.4.1 Covariant derivative

∇g is a $\binom{0}{3}$ tensor (with components $g_{\alpha\beta;\gamma}$).

From last time, given $\vec{V}$ there is a $\widetilde{V}$.

By definition,

$$V_{\alpha;\beta} = g_{\alpha\gamma} V^{\gamma}_{;\beta} \quad (\text{or } \nabla_{\vec{A}}\vec{V}) \tag{70}$$

Separately,

$$V_{\alpha} = g_{\alpha\beta} V^{\gamma} \tag{71}$$

Taking the covariant derivative of both sides of (71) with respect to x^{β}

$$\Rightarrow V_{\alpha;\beta} = g_{\alpha\gamma;\beta} V^{\gamma} + g_{\alpha\gamma} V^{\gamma}_{;\beta} \tag{72}$$

$$\Rightarrow g_{\alpha\gamma;\beta=0} \tag{73}$$

$$\Rightarrow \nabla g = 0 \tag{74}$$

We can use this to get a formula for Γ's (Γ's are coefficients in linear combination of basis vectors which give $\frac{\partial \vec{e}_{\alpha}}{\partial x_{\beta}}$)

An alternative derivation of $\nabla g = 0$:

In flat space, $g_{\alpha\beta,\gamma} = 0$ and Γ's are zero because $\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} = 0$

$$\Rightarrow g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - (\text{two terms involving Christoffel symbols}) \tag{75}$$

$$= 0 \quad \text{in flat space} \tag{76}$$

But $g_{\alpha\beta;\gamma} = 0$ involves only decent tensors, so it's free in ????

$$- = g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - \Gamma^{\lambda}_{\alpha\gamma} g_{\lambda\beta} \tag{77}$$

??? gives us 64 equations for 64 unknowns (Γ components) provided $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma}$ are given.

Simplification: symmetric in α, β; symmetric in lower indices of the Γ's

Using this and write three cyclic permutations of (*) and sum them (see Schutz p. 142):

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} (g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}) \tag{78}$$

You should memorise equation (78)!

- If Minkowski:

  - Γ's are 0

- If not:
  - Γ's $\neq 0$, coordinates are curved, but space may not be curved

Curvatures of manifold depend on 2nd derivatives of g.

6.5 The 4-acceleration

$\vec{u} = \frac{d\vec{x}}{d\tau}$ is a first derivative (i.e. this is calculated without reference to curvature of coordinates and/or manifold).

$\vec{a}$ is a second derivative. We need to refer to curvature of coordinates/manifold.

In general, $\vec{a} = \nabla_{\vec{u}}\vec{u}$, i.e. rate of change of $\vec{i}$ along itself.

In free fall: $\nabla_{\vec{u}}\vec{u} = 0$ (but with rocket engines $\nabla_{\vec{u}}\vec{u} \neq 0$)

Christoffel symbols contain curvature!

A short review:

Last lecture:

- $\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2}g^{\lambda\mu}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu})$
- $g_{\alpha\beta;\gamma} = 0$

Remember, for curved space, 4-acceleration $\vec{a} \neq \frac{d^2\vec{x}}{d\tau^2}$. This is the derivative of $x^{\alpha}(\vec{e}_{\alpha})$ not just coordinates. τ involves curvature.

For free fall, $\vec{a} = 0$ (compare with Newtonian 9.8 ms^{-2}

Example ?: Astronaut above Schwarz. black hole, at distance r (radial coordinate appearing in metric), with θ and ϕ also fixed. The Schwarzchild metric is:

$$g_{tt} = -\left(1 - \frac{2M}{r}\right) \tag{79}$$

$$g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1} \tag{80}$$

$$g_{\theta\theta} = \ldots \tag{81}$$

$$g_{\phi\phi} = \ldots \tag{82}$$

Note: rotating black hole is not diagonal! (but it will be symmetric because $A^{\alpha}\vec{e}_{\alpha} = A^{\beta}\vec{e}_{\beta}$; non-diagonal $\Leftarrow$ diagonal (one way!))

Question: what is the acceleration of this astronaut (globally not flat) at a fixed radius?

We need $\vec{u} = \left(\frac{dt}{d\tau}, 0, 0, 0\right)$

$$\vec{x} = (t, r, \theta, \phi) \tag{83}$$
$$\vec{u} = \frac{d\vec{x}}{d\tau} \tag{84}$$

What is u_t? We need to normalise.

$$\vec{u} \cdot \vec{u} = -1 \tag{85}$$
$$u^\beta g^{\alpha\beta} u_\alpha = -\left(1 - \frac{2M}{r}\right) = u_t^2 = -1 \tag{86}$$
$$\Rightarrow u_t = \left(1 - \frac{2M}{r}\right)^{-1/2} \tag{87}$$

$$a^\alpha = u^\beta u^\alpha_{;\beta} \quad \text{covariant derivative} \tag{88}$$
$$= u^\beta \left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\lambda\beta} u^\lambda\right) \tag{89}$$
$$= \underbrace{\frac{\partial u^\alpha}{\partial \tau}}_{\text{zero}} + \Gamma^\alpha_{\lambda\beta} u^\lambda u^\beta \tag{90}$$

Now,

$$a^t = a^\theta = a^\phi = 0 \tag{91}$$
$$\Rightarrow a^r = \Gamma^r_{tt} (u^t)^2 \tag{92}$$
$$= \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) \cdot \left[\left(1 - \frac{2M}{r}\right)^{-1/2}\right]^2 = \frac{M}{r^2} \tag{93}$$

$$\vec{a} = \nabla_{\vec{u}} \vec{u} = (0, \frac{M}{r^2}, 0, 0) = \frac{M}{r^2}\hat{r} \quad \text{(accelerating outwards!)} \tag{94}$$

We will prove that acceleration is always orthogonal to the 4-velocity.

$$\vec{u} \cdot \vec{u} = -1 = u^\alpha u_\alpha \tag{95}$$

Take the covariant derivative

$$\Rightarrow 0 = (u^\alpha_{;\beta} u_\alpha + u^\alpha u_{\alpha;\beta}) u^\beta \tag{96}$$
$$= (u^\alpha_{;\beta} u_\alpha + u_\gamma g^{\gamma\alpha} u_{\alpha;\beta}) u^\beta \quad \text{(note } g^{\gamma\alpha}_{;\beta} = 0) \tag{97}$$
$$0 = (u^\alpha_{;\beta} u_\alpha + u_\gamma u^\gamma_{;\beta}) u^\beta \tag{98}$$

where we have switched co- and contra-variant indices using the metric!

So $(u_\alpha u^\alpha_{;\beta})u^\beta = 0$ (both terms are equal).

$$\Rightarrow \vec{u}\cdot\vec{a} = 0 \tag{99}$$

6.6 Fermi-Walker Transport

What does the "natural basis" of an observer in a rocket ship ($\vec{a}$, $\vec{v}$) in curved space look like, compared to some external basis? (global coordinate system)

We let $\vec{v}$ be an arbitrary vector. We then solve

$$\nabla_{\vec{u}}\vec{v} = (\vec{a}\cdot\vec{v})\vec{u} - \vec{a}(\vec{u}\cdot\vec{v}) \tag{100}$$

to see what $\vec{v}$ and $\vec{u}$ look like in a global coordinate system.

Not unique apparently. However,

1. $$\nabla_{\vec{u}}(\vec{v}\cdot\vec{v}) = 2\vec{v}\cdot\nabla_{\vec{u}}\vec{v} = 0 \quad \text{lengths preserved} \tag{101}$$

2. $$\nabla_{\vec{u}}(\vec{v}\cdot\vec{w}) = \vec{v}\cdot\nabla_{\vec{u}}\vec{w} + \vec{w}\cdot\nabla_{\vec{u}}\vec{v} = 0 \quad \text{angles preserved (orthogonality)} \tag{102}$$

3. 4-velocity is Fermi-Walker transported automatically. (if $\vec{v} = \vec{u}$ then
$$\nabla_{\vec{u}}\vec{v} = (\vec{a}\cdot\vec{u})\vec{u} - \vec{a}(\vec{u}\cdot\vec{u}) \tag{103}$$
$$= \vec{a} \quad \text{identically satisfied} \tag{104}$$
You can treat your time axis to be $\vec{u}$ (along the path).

4. If $\vec{w}$ is spacelike (orthogonal to $\vec{a}$ and $\vec{u}$), then $\nabla_{\vec{u}}\vec{w} = 0$ ($\vec{w}$ does not rotate spatially relative to $\vec{u}$) (it's temporal (?))

6.7 Constants of motion

Approach #1: use Lagrangian methods

Approach #2: for free fall
$$\nabla_{\vec{u}}\vec{u} = 0 \tag{105}$$
(for a photon $\nabla_{\vec{p}}\vec{p} = 0$) We can also look at the 1-form version of equation (105)

$$\nabla_{\vec{u}}\widetilde{u} = 0 \tag{106}$$

$$\text{Components:} \quad u^{\beta}_{\alpha;\beta} = 0 \tag{107}$$

$$\underbrace{u^{\beta}}_{=\frac{\partial x^{\beta}}{d\tau}} \left(\frac{\partial u_{\alpha}}{\partial x^{\beta}} - \Gamma^{\lambda}_{\alpha\beta} u^{\lambda} \right) = 0 \tag{108}$$

$$\text{i.e.,} \quad \underbrace{\frac{\partial u_{\alpha}}{d\tau}}_{\text{chain rule}} = \Gamma^{\lambda}_{\alpha\beta} u_{\lambda} u^{\beta} \tag{109}$$

Recall:

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} (g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}) \tag{110}$$

$$\Gamma^{\lambda}_{\alpha\beta} u_{\lambda} u^{\beta} = \frac{1}{2} \underbrace{u^{\mu} u^{\beta}}_{\text{symm. } \mu\leftrightarrow\beta} (g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}) \tag{111}$$

Now, $g_{\mu\alpha,\beta}$ and $g_{\alpha\beta,\mu}$ are antisymmetric under exchange of $\mu \leftrightarrow \beta$

$$= \frac{1}{2} u^{\mu} u^{\beta} g_{\mu\beta,\alpha} \tag{112}$$

since contraction of symmetric and antisymmetric components vanishes.

i.e., if the metric is independent of coordinate x^{α} then u_{α} is constant along world line.

e.g., in Schwarz spacetime, a freely falling body has u_{ϕ} = constant and u_t = constant.

6.8 Polar coordinates

We shall explore polar coordinates through use of an example.

Polar coordinates (primed): (r, θ)

Cartesian (unprimed): (x, y)

$$x = r \cos \theta \tag{113}$$

$$y = r \sin \theta \tag{114}$$

Our plan is:

- get polar basis
- get polar metric
- get polar Christoffel symbol

To get **polar basis**:

Transformation matrix:

$$\frac{\partial x^{\alpha}}{\partial x^{\alpha'}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \tag{115}$$

$$\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \ldots \quad \text{exercise} \tag{116}$$

Basis vectors:

$$\vec{e}_\alpha = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\vec{e}_{\alpha'} \tag{117}$$
$$\vec{e}_r = \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y \tag{118}$$
$$= \cos\theta\vec{e}_x + \sin\theta\vec{e}_y \tag{119}$$
$$\vec{e}_\theta = \frac{\partial x}{\partial \theta}\vec{e}_x + \frac{\partial y}{\partial \theta}\vec{e}_y \tag{120}$$
$$= -r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y \tag{121}$$

[h]

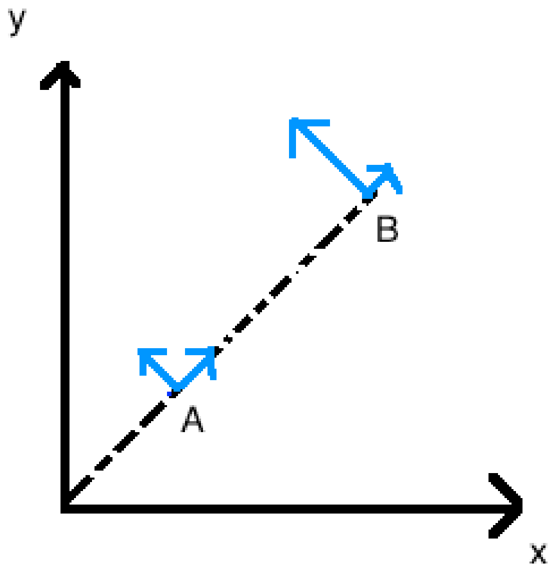


Figure 8: Polar coordinates?

In Figure 8 above we see the vector length at B is greater than at A; the length of $\vec{e}_\theta$ is proportional to r because we need to subtend a greater arc further out so as to produce unit change in θ.

Polar metric:

$$g_{\alpha'\beta'} = \vec{e}_{\alpha'}\vec{\beta'} \tag{122}$$
$$g_{rr} = \vec{e}_r \cdot \vec{e}_r = 1 \tag{123}$$
$$g_{r\theta} = \vec{e}_r \cdot \vec{e}_\theta = 0 \tag{124}$$
$$g_{\theta\theta} = \vec{e}_\theta \cdot \vec{e}_\theta = r^2 \tag{125}$$

Exercise: Check this this also follows from the transformation law

$$g_{\alpha'\beta'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}} \underbrace{g_{\alpha\beta}}_{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\text{Cartesian}} \tag{126}$$

Christoffel symbols describe derivatives of basis vectors in that basis itself

$$\frac{\partial \vec{e}_r}{\partial r} = 0 \tag{127}$$
$$\frac{\partial \vec{e}_r}{\partial \theta} = -\sin\theta \vec{e}_x + \cos\theta \vec{e}_y \tag{128}$$
$$\frac{\partial \vec{e}_\theta}{\partial r} = -\sin\theta \vec{e}_x + \cos\theta \vec{e}_y \tag{129}$$
$$\frac{\partial \vec{e}_\theta}{\partial \theta} = -r\cos\theta \vec{e}_x - r\sin\theta \vec{e}_y \tag{130}$$

Expressing these in the basis we get

$$\begin{array}{l|l|l}
\frac{\partial \vec{e}_r}{\partial r} = 0 & \Gamma^r_{rr} = 0 & \Gamma^\theta_{rr} = 0 \\
\frac{\partial \vec{e}_r}{\partial \theta} = \frac{1}{r}\vec{e}_\theta & \Gamma^r_{r\theta} = 0 & \Gamma^\theta_{r\theta} = \frac{1}{r} \\
\frac{\partial \vec{e}_\theta}{\partial r} = \frac{1}{r}\vec{e}_\theta & \Gamma^r_{\theta r} = 0 & \Gamma^\theta_{\theta r} = \frac{1}{r} \\
\frac{\partial \vec{e}_\theta}{\partial \theta} = -r\vec{e}_r & \Gamma^r_{\theta\theta} = -r & \Gamma^\theta_{\theta\theta} = 0
\end{array}$$

Important to recall:

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \vec{e}_\mu \Gamma^\mu_{\alpha\beta} \tag{131}$$

We should check this works when taking covariant derivative of $\vec{V}$.

Special case: $\vec{V} = \vec{e}_x =$ constant. We should find $\nabla_{\vec{A}}\vec{V} = 0$ for all $\vec{A}$.

To check: we write $\vec{V}$ in polar coordinates:

$$\vec{V} = \cos\theta \vec{e}_r - \frac{1}{r}\sin\theta \vec{e}_\theta \tag{132}$$

from transformation laws for basis vectors.

$$\begin{aligned}
\left(\nabla_{\vec{A}}\vec{V}\right)^{\alpha} &= A^{\beta}\left(\frac{\partial V^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\lambda\beta}V^{\lambda}\right) \\
\therefore \left(\nabla_{\vec{A}}\vec{V}\right)^{r} &= A^{\beta}\frac{\partial V^{r}}{\partial x^{\beta}} + \Gamma^{r}_{\lambda\beta}V^{\lambda}A^{\beta} \\
&= A^{\theta}\frac{\partial V^{r}}{\partial \theta} + \Gamma^{r}_{\theta\theta}V^{\theta}A^{\theta} \quad \text{other terms zero} \\
&= A^{\theta}(-\sin\theta) - r\cdot\left(-\frac{1}{r}\sin\theta\right)A^{\theta} \\
&= 0 \\
\left(\nabla_{\vec{A}}\vec{V}\right)^{\theta} &= A^{\beta}\frac{\partial V^{\theta}}{\partial x^{\beta}} + \Gamma^{\theta}_{\lambda\beta}V^{\lambda}A^{\beta} \\
&= A^{r}\frac{\partial V^{\theta}}{\partial r} + A^{\theta}\frac{\partial V^{\theta}}{\partial \theta} + \Gamma^{\theta}_{r\theta}V^{r}A^{\theta} + \Gamma^{\theta}_{\theta r}V^{\theta}A^{r} \\
&= A^{r}\cdot\frac{1}{r^2}\sin\theta + A\theta\cdot\left(-\frac{\cos\theta}{r}\right) + \frac{1}{r}\cdot\cos\theta\cdot A^{\theta} + \frac{1}{r}\left(-\frac{1}{r}\sin\theta\right)A^{r} \\
&= 0
\end{aligned}$$

Exercise: Metric $g = \mathrm{diag}(1, r^2)$ from previous lecture; check that $\nabla g = 0$ explicitly.

Exercise: Confirm that Γ's can also be derived from

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}) \tag{133}$$

Exercise: Prove divergence

$$V^{\alpha}_{;\alpha} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{\partial V_{\theta}}{\partial \theta} \tag{134}$$

as you would expect from undergrad.

Exercise: Derive $\chi^{\mu}_{\alpha\beta}$'s for basis one-forms

$$\widetilde{\omega}^{r} = \cos\theta\cdot\widetilde{\omega}^{x} + \sin\theta\cdot\widetilde{\omega}^{y} \tag{135}$$

$$\widetilde{\omega}^{\theta} = -\frac{\sin\theta}{r}\cdot\widetilde{\omega}^{x} + \frac{\cos\theta}{r}\cdot\widetilde{\omega}^{y} \tag{136}$$

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7 Curved space

In the presence of gravity, we cannot have a global inertial (Minkowski) frame.

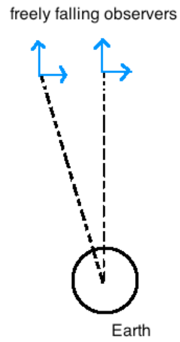


Figure 9: Freely falling observers

We see that the distance between the observers decreases! i.e. curved geodesics $\Rightarrow$ curved space!

7.1 Curved manifold

- Extrinsic curvature: is space curved with respect to the space in which it's embedded?

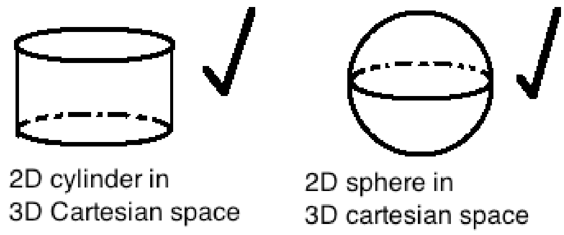


Figure 10: Extrinsic curvature

Yes!

- Intrinsic curvature: without refernce to an emedding. Do neighbouring geodesics ("free fall") diverge/converge? Note we need the idea of parallel transport to discuss geodesics.

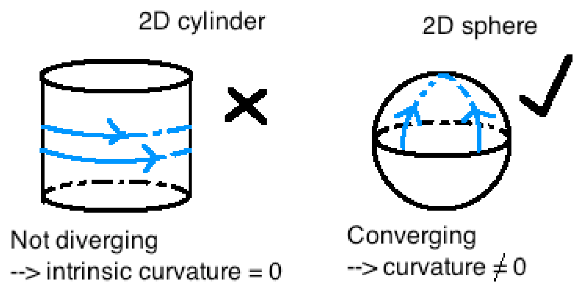


Figure 11: Intrinsic curvature

We see that a 2D cylinder does <u>not</u> have intrinsic curvature!

7.2 Local flatness

- Let $\{x^\alpha\}$ be some generic coordinates
- Make a transformation to $\{x^{\alpha'}\}$ coordinates
- See how for we can get in flattening the space near a point $\hat{x}_0$
- Can we choose $x^{\alpha'}(\vec{x})$ in such a way that the primed metric is Minkowski near $\vec{x}_0$?
  - exactly no, but approximately yes

At $\vec{x}_0$:

We are in a tangent space; first derivatives of the metric at the second derivatives disappear.

Away from $\vec{x}_0$:

Curvature contributes to $\vec{a}$ (we can set first derivatives to metric to zero)

To transform from primed to unprimed:

$$\Lambda^\alpha_{\mu'} = \frac{\partial x^\alpha}{\partial \mu'} \tag{137}$$

Aim: to compute transformed metric

$$g_{\mu'\nu'} = \Lambda^\alpha_{\mu'}\Lambda^\beta_{\nu'}g_{\alpha\beta} \tag{138}$$

We seek to cheek $x^{\alpha'}$ (and hence $\Lambda^\alpha_{\mu'}$) in order to make $g_{\mu'\nu'}$ as flat as possible.

Taylor expansion: small departures around $\vec{x}_0$; no cusps, space is differentiable, order of differentiation is unimportant.

$$\begin{aligned}
\Lambda^\alpha_{\mu'} &= \Lambda^\alpha_{\mu'}|_{\vec{x}_0} + \frac{\partial \Lambda^\alpha_{\mu'}}{\partial x^{\gamma'}}\left(x^{\gamma'} - x_0^{\gamma'} + \frac{1}{2}\frac{\partial^2 \Lambda^\alpha}{\partial x^{\gamma'}\,\partial x^{\delta'}}\left(x^{\gamma'} - x_0^{\gamma'}\right)(x^{\delta'} - x_0^{\delta'}\right) + \dots && (139)\\
&= \Lambda^\alpha_{\mu'}|_{\vec{x}_0} + \frac{\partial}{\partial x^{\gamma'}}\left(\frac{\partial x^\alpha}{\partial \mu'}\right)|^{(x^{\gamma'}-x_0^{\gamma'}}_{\vec{x}_0} + \frac{1}{2}\frac{\partial^3 x^\alpha}{\partial x^{\mu'}\,\partial x^{\gamma'}\,\partial x^{\delta'}}|_{\vec{x}_0}\left(x^{\gamma'} - x_0^{\gamma'}\right)\left(x^{\delta'} - x_0^{\delta'}\right) + \dots && (140)\\
g_{\alpha\beta} &= g_{\alpha\beta}|_{\vec{x}_0} + g_{\alpha\beta,\gamma'}\left(x^{\gamma'} - x_0^{\gamma'}\right) + \frac{1}{2}g_{\alpha\beta,\gamma'\delta'}\left(x^{\gamma'} - x_0^{\gamma'}\right)\left(x^{\delta'} - x_0^{\delta'}\right) + \dots && (141)
\end{aligned}$$

Therefore, at zeroth order

$$g_{\mu'\nu'} = \Lambda^\alpha_{\mu'}|_{\vec{x}_0}\ \Lambda^\beta_{\nu'}|_{\vec{x}_0}\ g_{\alpha\beta}|_{\vec{x}_0} \tag{142}$$

(Can we make this flat?)

7.3 Miscellaneous practical results

On curved space times.

7.3.1 Proper length

Also called the "geodesic length"

$$ds^2 = g_{\alpha\beta} \frac{dx^\alpha(\lambda)}{d\lambda} \cdot \frac{dx^\beta(\lambda)}{d\lambda} \tag{143}$$

$$\text{proper length} = \int_{\lambda_0}^{\lambda_1} dx \, \sqrt{\frac{d\vec{x}}{d\lambda} \cdot \frac{d\vec{x}}{d\lambda}} \tag{144}$$

Proper length is coordinate independent.

7.3.2 Volume

$$\begin{aligned} \text{Inifinitesimal volume} &= dx^{0'} \ dx^{1'} \ dx^{2'} \ dx^{3'} && (145)\\ &= \text{Jacobian} \times dx^0 \ dx^1 \ dx^2 \ dx^3 && (146)\\ &= \det \underbrace{\left(\frac{\partial x^{\alpha'}}{\partial x^\alpha}\right)}_{\Lambda^{\alpha'}_{\alpha}} dx^0 \ dx^1 \ dx^2 \ dx^3 && (147) \end{aligned}$$

For an infinitesimal volume: locally the manifold is flat. So there exists a transformation that maps g into Minkowski η.

In matrix notation:

$$g = \Lambda \eta \Lambda^\top \tag{148}$$

In component notation:

$$g_{\alpha\beta} = \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta} \eta_{\alpha'\beta'} \tag{149}$$

Determinant of both sides (in matrix notation)

$$\begin{aligned} \det g &= \det \Lambda \ \ \det \eta \ \ \det(\Lambda^\top) && (150)\\ &= -(\det \Lambda)^2 && (151) \end{aligned}$$

Therefore $\det \Lambda = (-\det g)^{1/2}$.

7.3.3 Divergence of a vector

Used in conservation laws.

$$\begin{aligned} V^\alpha_{;\beta} &= \frac{\partial V^\alpha}{\partial x^\beta} = \Gamma^\alpha_{\lambda\beta} V^\lambda && (152)\\ \text{div } \vec{V} &= V^\alpha_{;\alpha} && (153)\\ &= \frac{\partial V^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\lambda\alpha} V^\lambda && (154) \end{aligned}$$

$$\Gamma^{\alpha}_{\lambda\beta} = \frac{1}{2} g^{\alpha\mu} \left(g_{\mu\lambda,\beta} + g_{\mu\beta,\lambda} - g_{\lambda\beta,mu} \right) \tag{155}$$

$$\Rightarrow \Gamma^{\alpha}_{\lambda\alpha} + \frac{1}{2} g^{\alpha\mu} \left(g_{\mu\lambda,\alpha} + g_{\mu\alpha,\lambda} - g_{\lambda\alpha,\mu} \right) \tag{156}$$

$$= \frac{1}{2} g^{\alpha\mu} g_{\mu\alpha,\lambda} \tag{157}$$

An example: $g^{-1} g = 1$ and matrix algebra.

$$\rightarrow (\det\, g)_{,\mu} = \det\, g - g^{\alpha\beta} g_{\alpha\beta,\mu} \tag{158}$$

Hence

$$\Gamma^{\alpha}_{\lambda\alpha} = \frac{(\sqrt{-g})_{,\lambda}}{\sqrt{-g}} \tag{159}$$

where $g \equiv \det(g_{\alpha\beta})$.

So,

$$\text{div}\ \vec{V} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} V^{\alpha} \right) \tag{160}$$

Exercise: 3D spherical polar, $g = \det(g_{\alpha\beta}) = r^2 \sin\theta$.

7.3.4 Gauss Law in integral form

$$\int d^3\mathbf{x}'\ \text{div}\ '(\mathbf{V}) = \int \underbrace{d^2\mathbf{x}'}_{dA\ \hat{\mathbf{n}}} \cdot \mathbf{V} \tag{161}$$

This is 4D! let's measure conservation of the quantity $\vec{V}$ locally in flat space.

$$0 = V^{\alpha}_{,\alpha} \tag{162}$$

Locally flat $\Rightarrow$ Christoffel symbols for the unprimed flat coordinates are zero

$$V^{\alpha}_{,\alpha} = V^{\alpha}_{,\alpha} + \underbrace{\Gamma^{\alpha}_{\lambda\alpha} V^{\lambda}}_{\text{zero}} = V^{\alpha}_{;\alpha} \tag{163}$$

Integrate over small volume in flat space

$$\int d^4x\ V^{\alpha}_{,\alpha} = \int d^4x\ V^{\alpha}_{;\alpha}, \quad \text{as above} \tag{164}$$

transform into primed (non-flat) coordinates)

$$= \int d^4x' \, \underbrace{\sqrt{-g}}_{\text{Jacobian of transform}} \, \overbrace{V^{\alpha'}_{;\alpha'}}^{\text{unchanged; frame invariant}} \tag{165}$$

$$- \int d^4x' \, \sqrt{-g} \cdot \frac{1}{\sqrt{-g}} \cdot \left(\sqrt{-g} V^{\alpha'}\right)_{,\alpha'} \tag{166}$$

$$= \int d^4x' \, \left(\sqrt{-g} V^{\alpha'}\right)_{,\alpha'} \tag{167}$$

$$= \int d^3\mathbf{x}' \, n_{\alpha'} V^{\alpha'} \sqrt{-g} \tag{168}$$

where $n_{\alpha'}$ is normal to the 3-surface enclosing the 4-volume.

This is Gauss' law (also applies to the covariant derivative).

7.3.5 Angles

$$\frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \cos\theta \tag{169}$$

where $\vec{A} \cdot \vec{B} = g(\vec{A}, \vec{B})$

$$|\vec{A}| = \sqrt{g(\vec{A}, \vec{A})} \tag{170}$$

$$|\vec{B}| = \sqrt{g(\vec{B}, \vec{B})} \tag{171}$$

7.3.6 Geodesics

Curves that continue to progress in the same direction they were progressing in; i.e. the tangent is parallel to the tangent at the previous point.

$$\nabla_{\frac{d\vec{x}}{d\lambda}} d\vec{x} d\lambda = 0 \tag{172}$$

Covariant derivative of tangent along its own direction is zero.

In component notation: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\frac{dx^\beta}{d\lambda} \left(\frac{d\vec{x}}{d\lambda}\right)^\alpha_{;\beta} = 0 \tag{173}$$

$$\Rightarrow \frac{dx^\beta}{d\lambda} \left(\frac{d}{dx^\beta}\left(\frac{dx^\alpha}{d\lambda}\right) + \Gamma^\alpha_{\mu\beta}\frac{dx^\mu}{d\lambda}\right) = \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\beta}\frac{dx^\mu}{d\lambda} \cdot \frac{dx^\beta}{d\lambda} \tag{174}$$

Note $\vec{x}(\lambda = 0)$ and $\frac{d\vec{x}}{d\lambda}(\lambda = 0)$.

7.4 Curvature and Einstein's Field Equations

Parallel transport!

Generally, geometric objects like vectors as different points live in different tangent spaces, hence cannot be compared. We will investigate the looks for moving them.

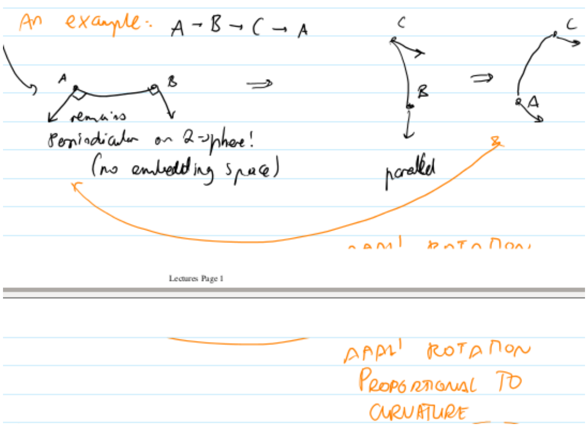
7.4.1 Rules

If T is a geometric object, then we can parallel transport along $\vec{x}(\lambda)$ by requiring that

$$\nabla \frac{d\vec{x}}{d\lambda} T = 0 \tag{175}$$

$\Rightarrow T$ remains constant along tangent vector (is zero ???)

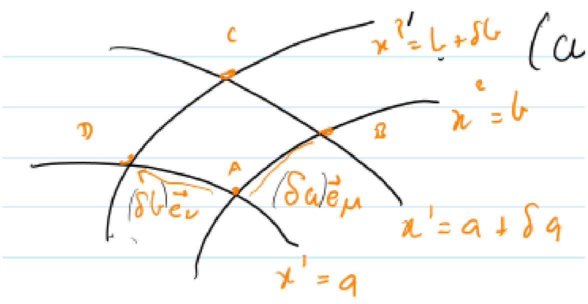
An example $A \to B \to C \to A$.



Theorem: if we parallel transport T around a <u>closed</u> curve, then the change in T is proportional to the intrinsic curvature of the manifold (proportional to the Riemann tensor). No curvature = flat $\Rightarrow$ no change.

What is $\Delta T = T_{\text{final}} - T_{\text{initial}}$ around a closed loop? (specialise to vector $\vec{V}$)

Consider a small patch (not flat) An example $A \to B \to C \to A$.



To parallel transport $\vec{V}$ around the loop $A \to B \to C \to D \to A$:

Consider AB:

$$\nabla_{\vec{e}_1}\vec{V} = 0 \quad \text{(the tangent to AB is } \vec{e}_1) \tag{176}$$
$$\Rightarrow V^\alpha_{;\beta} = 0 \quad (\beta = 1) \tag{177}$$
$$\Rightarrow 0 = \frac{\partial V^\alpha}{\partial x^1} + \Gamma^\alpha_{\lambda 1} V^\lambda \tag{178}$$

We integrate

$$V^\alpha(B) - V^\alpha(D) = \int_A^B \frac{\partial V^\alpha}{\partial x^1} dx^1 \tag{179}$$
$$= -\int_A^D dx^1 \; \Gamma^\alpha_{\lambda 1} V^\lambda \tag{180}$$

Along BC:

$$\nabla_{\vec{e}_2}\vec{V} = 0 \quad \text{(parallel transport!)} \tag{181}$$
$$\Rightarrow V^\alpha(C) - V^\alpha(B) = \int_B^C \frac{\partial V^\alpha}{\partial x^2} dx^2 \tag{182}$$
$$= -\int_B^C dx^2 \; \Gamma^\alpha_{\lambda 2} V^\lambda \tag{183}$$

Along CD:

$$\nabla_{-\vec{e}_2}\vec{V} = 0 \quad \text{(parallel transport!)} \tag{184}$$
$$\Rightarrow V^\alpha(D) - V^\alpha(C) = \int_C^D dx^1 \; \Gamma^\alpha_{\lambda 1} V^\lambda \tag{185}$$

Along DA:

$$\nabla_{-\vec{e}_2}\vec{V} = 0 \tag{186}$$
$$\Rightarrow V^\alpha(A) - V^\alpha(D) = \int_D^A dx^2 \; \Gamma^\alpha_{\lambda 2} V^\lambda \tag{187}$$

Note that $\Gamma^\alpha_{\lambda 1}$'s are not the smae at different positions! There is no zero.

The sum of the $A \to B \to C \to D \to A$ path is

$$-\int_A^B dx^1 \; \Gamma^\alpha_{\lambda 1} V^\lambda - \int_B^C \; dx^2 \Gamma^\alpha_{\lambda 2} V^\lambda + \int_C^D dx^1 \; \Gamma^\alpha_{\lambda 1} V^\lambda + \int_D^A dx^2 \; \Gamma^\alpha_{\lambda 2} V^\lambda \tag{188}$$

For small patches,

$$(\Gamma^\alpha_{\lambda 2})_{BC} = \left(\Gamma^\alpha_{\lambda 2} V^\lambda\right) AB + \delta_{a'???} \left(\Gamma^\alpha_{\lambda 2} V^\lambda\right) \tag{189}$$
$$\Rightarrow \left(\Gamma^\alpha_{\lambda 1} V^\lambda\right)_{CD} = \left(\Gamma^\alpha_{\lambda 1} V^\lambda\right)_{AB} + \delta b \frac{\partial}{\partial x^2} \left(\Gamma^\alpha_{\lambda 1} V^\lambda\right) \tag{190}$$

This is a Taylor expansion in x^2.

$$\Delta V^\alpha = -\delta a \frac{\partial}{\partial x^1}\left(\Gamma^\alpha_{\lambda 2} V^\lambda\right)\delta b \quad \text{(integral along BC)} \tag{191}$$

$$+ \delta b \frac{\partial}{\partial x^2}\left(\Gamma^\alpha_{\lambda 1} V^\lambda\right)\delta a \quad \text{(integral along CD)} \tag{192}$$

$$\Rightarrow \frac{\Delta V^\alpha}{\delta a\ \delta b} = -\frac{\partial \Gamma^\alpha_{\lambda 2}}{\partial x^1} V^\lambda - \Gamma^\alpha_{\lambda 2} \underbrace{\frac{\partial V^\lambda}{\partial x^1}}_{*} \tag{193}$$

$$+ \frac{\partial \Gamma^\alpha_{\lambda 1}}{\partial x^2} V^\lambda - \Gamma^\alpha_{\lambda 1} \underbrace{\frac{\partial V^\lambda}{\partial x^2}}_{**} \tag{194}$$

where the derivative * is the parallel transport condition $-\Gamma^\lambda_{\mu 1} V^\mu$, and the derivative ** is $-\Gamma^\lambda_{\mu 2} V^\mu$.

$$\Rightarrow \Delta V^\alpha \propto \delta a\ \delta b\ V^\alpha \tag{195}$$

where is proportional to constant; ???? and first derivaitives, Riemann tensor!! (????)

A short review:

Last lecture:

- parallel transport
- Riemann curvature tensor (20 independent numbers)

Today:

- symmetrics of Riemann

Parallel transport along $\vec{x}(\lambda)$: $\nabla_{\frac{d\vec{x}}{d\lambda}} T = 0$ where T is any geometric object.

If we parallel transport some object (say, vector $\vec{V}$) around a closed loop, then $\vec{V}$ does not return to its initial value if the spacetime is curved. We find $\Delta\vec{V} \propto$ area of curve $\times$ curvature.

In component notation:

$$\Delta V^\alpha = R^\alpha_{\beta\mu\nu} V^\beta \delta a\ \delta b \tag{196}$$

where $\delta a\ \delta b$ is thearea of quadrilaterial with sides δa and δb, and R is the Riemann tensor, contains 2nd derivatives of metric.

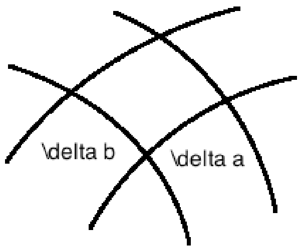


Figure 12: A quadrilateral in curved space

$$R^{\alpha}_{\beta\mu\nu} = 2 \times \text{Christoffel 1st derivative} + 2 \times \text{products of Christoffel} \tag{197}$$
$$= \Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\nu} + \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\mu} \tag{198}$$

7.4.2 Symmetrics of the Riemann tensor

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}_{\beta\nu\mu} \tag{199}$$

- $R_{\alpha\beta\mu\nu}$ is antisymmetric in $\mu \leftrightarrow \nu$ (see picture; this is eqivalent to circumnavigating the loop in opposit direction)
- $R_{\alpha\beta\mu\nu}$ is antisymmetric in $\alpha \leftrightarrow \beta$
- $R_{\alpha\beta\mu\nu}$ is symmetric under exchange of pairs; i.e. $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$
- "Symmetric" under cyclic permutation of last three indices (Jordan symmetry)

$$0 = R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} \tag{200}$$

After applying these symmetrics to the $4 \times 4 \times 4 \times 4 = 256$ components of the Riemann tensor, we reduce them down to only 20.

Proof: go into local Minkowski coordinates (i.e. flat patch). Here Γ's are zero, but their derivatives are not.

$$R_{\alpha\beta\mu\nu} = \Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\beta\nu,\mu} \tag{201}$$

Substitute

$$\Gamma^{\alpha}_{\beta\mu} = \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\beta,\mu} + g_{\lambda\mu,\beta} - g_{\beta\mu,\lambda} \right) \tag{202}$$

We lower the indices by

$$R_{\alpha\beta\mu\nu} = g_{\alpha\sigma} R^{\sigma}_{\beta\mu\nu} \tag{203}$$

We now derivates with respect to ν (as an exercise)

$$\Rightarrow R_{\alpha\beta\mu\nu} = \frac{1}{2} \left(g_{\alpha\nu,\beta\mu} + g_{\beta\mu,\alpha\nu} - g_{\alpha\mu,\beta\nu} - g_{\beta\nu,\alpha\mu} \right) \tag{204}$$

Let's start with last symmetry (12 terms).

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 12 \text{ terms} \tag{205}$$

Exercise: Check that these terms cancel pairwise.

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0. \tag{206}$$

which is derived in local falt. But it's a perfectly good tensor equation $\Rightarrow$ it is coordinate independent $\Rightarrow$ true in curved space, also.

A good tensor equation = something built out of tensors $\binom{M}{N}$ (which transform like they should) using valid operations like outer product, symmetrisation etc.

An example of a bad tensor equation:

$$R_{\alpha\beta\mu\nu,\lambda} = 0 \tag{207}$$

because ordinary derivatives **,** are "not good"; however

$$R_{\alpha\beta\mu\nu;\lambda} = 0 \tag{208}$$

is a good tensor equation, since covariant derivatives **;** are "good".

7.5 Constructing new tensors from Riemann on the way to Einstein's field equations

Aside: Riemann as a commutator

$$(V^{\mu}_{j\alpha})_{;\beta} - (V^{\mu}_{j\beta})_{;\alpha} = R^{\mu}_{\nu\alpha\beta}V^{\nu} \tag{209}$$

This is a good tensor equation: it is true in all reference frames.

Aim:

- Derive Bianchi identities, which contain infofrmation about symmetries of Riemann tensor's derivatives
- Contract Riemann in the hope of making a smaller tensor with 10 independent stress-energy tensor components, to match the 10 independent stress-energy components $T^{\mu\nu}$
- Combine the above 2 aims to "derive" Einstein's field equations in the form

$$(\text{curvature}) = (\text{stress energy}) \tag{210}$$

Bianchi identities

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu} + g_{\beta\mu,\alpha\nu} - g_{\alpha\mu,\beta\nu} - g_{\beta\nu,\alpha\mu}\right) \tag{211}$$

Note $R_{\alpha\beta\mu\nu}$ is antisymmetric under exchange $\alpha \leftrightarrow \beta$ and $\mu \leftrightarrow \nu$, and under $\alpha\beta \leftrightarrow \mu\nu$.

We differentiate with respect to x^{λ} and add

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}\left(g_{\alpha,\beta\mu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\alpha\mu,\beta\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}\right) \tag{212}$$

Add cyclic permutations

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\nu\lambda,\mu} + R_{\alpha\beta\lambda\mu,\nu} = 0 \tag{213}$$

(show this as an exercise)

Note that in flat space, Γ's are zero. So,

$$R_{\alpha\beta\mu\nu,\lambda} = R_{\alpha\beta\mu\nu,\lambda} + \underset{\text{correct for each index}}{\overbrace{\Gamma\text{'s} \times R}} \tag{214}$$
$$= R_{\alpha\beta\mu\nu;\lambda}, \quad \text{since } \Gamma\text{'s are zero} \tag{215}$$

Now we have a new good tensor equation valid in all coordinate systems.

$$0 = R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\nu\lambda;\mu} + R_{\alpha\beta\lambda\mu;\nu} \tag{216}$$

the Bianchi identity!

How does the field evolve due to a source? (; is coordinate independent)

Note: need derivatives of curvature is we hope to describe how curvature evolves in response to source $T^{\mu\nu}$ (and how it evolves).

7.5.1 Contractions of Riemann

How many independent ways can we do this? $R^{\nu}_{\beta\mu\nu}$

$$\Rightarrow R^{\alpha}_{\alpha\mu\nu} = \underbrace{g^{\alpha\lambda}}_{\text{sym } \alpha\leftrightarrow\lambda} \overbrace{R_{\lambda\alpha\mu\nu}}^{\text{anti-sym } \lambda\leftrightarrow\alpha} \tag{217}$$
$$= -g^{\lambda\alpha} R_{\alpha\lambda\mu\nu} \tag{218}$$
$$= -R^{\lambda}_{\lambda\mu\nu} \tag{219}$$
$$= 0 \tag{220}$$

$$R^{\alpha}_{\beta\alpha\nu} =? \tag{221}$$
$$R^{\alpha}_{\beta\mu\alpha} = g^{\alpha\lambda} R_{\lambda\beta\mu\alpha} \tag{222}$$
$$= -g^{\alpha\lambda} R_{\lambda\beta\alpha\mu} \tag{223}$$
$$= -R^{\alpha}_{\beta\alpha\mu} \tag{224}$$
$$= -R^{\alpha}_{\beta\alpha\nu} \tag{225}$$

(exercise: check the others)

The only independent contraction:

$$R_{\beta\nu} = R^{\alpha}_{\beta\alpha\nu} \tag{226}$$

Ricci tensor is symmetric!

$$R_{\nu\beta} = R^{\alpha}_{\nu\alpha\beta} = g^{\alpha\lambda} R_{\lambda\nu\alpha\beta} \tag{227}$$
$$= g^{\alpha\lambda} R_{\alpha\beta\lambda\nu} \tag{228}$$
$$= R^{\lambda}_{\beta\lambda\nu} \tag{229}$$
$$= R_{\beta\nu} \tag{230}$$

We can derive the Bianchi identities for the Ricci tensor from the Riemann tensor.

$$0 = R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\nu\lambda;\mu} + R_{\alpha\beta\lambda\mu;\nu} \tag{231}$$

Raise 1st index by contracting with metric (also $\nabla g = 0$). Contract α with μ

$$0 = \underbrace{R^{\alpha}_{\beta\alpha\nu;\lambda}}_{R_{\beta\nu;\lambda}} + R^{\alpha}_{\beta\nu\lambda;\alpha} + \underbrace{R^{\alpha}_{\beta\lambda\alpha;\nu}}_{-R^{\alpha}_{\beta\alpha\lambda;\nu} = -R_{\beta\lambda;\nu}} \tag{232}$$

$$0 = R_{\beta\nu;\lambda} + R^{\alpha}_{\beta\nu\lambda;\alpha} - R_{\beta\lambda;\nu} \tag{233}$$

We contract again (involves Ricci scalar)

$$R = R^{\beta}_{\beta} = g^{\beta\nu} R_{\beta\nu} \tag{234}$$

Contract Bianchi: raise β and contract with ν (use $\nabla g = 0$).

$$0 = \underbrace{R^{\beta}_{\beta;\lambda}}_{R_{;\lambda}} + \underbrace{R^{\alpha\beta}_{\beta\lambda;\alpha}}_{-R^{\beta\alpha}_{\beta\lambda;\alpha} = -R^{\alpha}_{\lambda;\alpha}} - R^{\beta}_{\lambda;\beta} \tag{235}$$

(note we have skipped a few steps to get to this point, incl. raising β) where $R_{;\lambda}$ is the Ricci scalar. It is important to remember that the antisymmetry properties of Riemann are only immediately true in the case where all indices are down; we cannot immediately say the second term has the antisymmetry shown (though it can be shown that it indeed does).

$$\text{i.e. } 0 = R_{;\lambda} - 2R^{\beta}_{\lambda;\beta} \tag{236}$$

We play with this to get it into the form $(\ldots)_{;\beta} = 0$

$$0 = \left(R\delta^{\beta}_{\lambda} - 2R^{\beta}_{\lambda}\right)_{;\beta} \tag{237}$$

We define the Einstein curvature tensor

$$G^{\beta\lambda} \equiv R^{\beta\lambda} - \frac{1}{2}Rg^{\beta\lambda} \tag{238}$$

Note this is just pretty notation, nothing exciting. Then,

$$G^{\beta}_{\lambda} = R^{\beta}_{\lambda} - \frac{1}{2}Rg^{\beta}_{\lambda} = \delta^{\beta}_{\lambda} \tag{239}$$

So we have that

$$G^{\beta}_{\lambda;\beta} = 0 \tag{240}$$

The diveregence of the Einstein curvature tensor vanishes!

7.6 Einstein's field equations

These cannot be derived; they are experimental laws. We shall motivate them heuristically.

1. Equivalence principle $\Rightarrow$ make sure our field equations are good tensor equations
2. In the weak field, we must recover Newtonian gravity
$$\nabla^2\Phi = 4\pi G\rho \tag{241}$$
3. Weak field suggests 2nd derivative of field (analogy: $g_{\mu\nu}$) proportional to energy density (ρc^2)
   - i.e., curvature (2nd derivatives of metric) $\propto T^{\mu\nu}$ (stress-energy tensor)
4. $T^{\mu\nu}$ has 10 independent components.
What curvature quantity matches this? Ricci (for example)! So should we postulate $R^{\mu\nu} \propto T^{\mu\nu}$?
5. No! It violates energy conservation. We must have $T^{\mu\nu}_{;\nu} = 0$. But then
$$R^{\mu\nu}_{;\nu} = -\text{Ricci scalar} \tag{242}$$
$$\neq 0 \text{ in general} \tag{243}$$
6. But we do have a nice curvature-related $\binom{0}{2}$ tensor (or $\binom{2}{0}$) whose divergence is zero, namely $G^{\beta\lambda}$. So we postulate
$$G^{\beta\lambda} \propto T^{\beta\lambda} \tag{244}$$

In fact we can be a tiny bit more general and ask: what symmetric tensor can we construct from Ricci and the metric on LHS?
$$R^{\alpha\beta} + \mu R g^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta} \tag{245}$$

1. Insist on $T^{\alpha\beta}_{;\beta} = 0$
$\Rightarrow \mu = -\frac{1}{2}$ as calculated above
2. Insist on $\nabla^2 = 4\pi G\rho$ in weak field (which we will prove later)
$$\Rightarrow k = 8\pi G = 8\pi \text{ in natural units} \tag{246}$$

Einstein's field equations:
$$\underbrace{R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}}_{G^{\alpha\beta}} + \Lambda g^{\alpha\beta} = 8\pi T^{\alpha\beta} \tag{247}$$

where Λ is the "cosmological" constant (air-quotes because we do not need to discuss it in a cosmological context). This term is sometimes missing from t-shirts.

This can also be written with indices down, as
$$\underbrace{R_{\alpha\beta}}_{\text{Ricci tensor}} - \frac{1}{2}\overbrace{R}^{\text{Ricci scalar}} g_{\alpha\beta} + \underbrace{\Lambda g_{\alpha\beta}}_{\text{cosmological constant}} = 8\pi T_{\alpha\beta} \tag{248}$$

Part II

Terrestrial and astronomical applications

Today we will discuss:

- what is $T_{\mu\nu}$?
- where did the 10 "missing" curvature degrees of freedom go?

8 Stress energy

The principle of equivalence says we work out T as a tensor in Minkowski space $\Rightarrow$ same tensor form in curved space. This is true for all geometric objects (when a new piece of phyiscs is introduced, we refer back to local flat space, because this is all we can do).

Components:

$$\begin{aligned}
& && \text{e.g. EM} \\
T^{00} &= \text{energy density} \equiv \text{energy/volume} && \frac{1}{2}\epsilon_0 E^2 + \frac{B^2}{2\mu_0} \qquad (249) \\
T^{0i} &= \text{energy flux} \equiv \text{energy/area/time} && \frac{(\mathbf{E}\times\mathbf{B})_i}{\mu_0} \qquad (250) \\
T^{ij} &= \text{momentum flux} && \epsilon_0 E_i E_j + \frac{1}{\mu_0} - \left(\frac{1}{2}\epsilon_0 E^2 + \frac{B^2}{2\mu_0}\right)\delta_{ij} \qquad (251)
\end{aligned}$$

T^{ij} is momentum/area/time in ith direction transported through area defined by normal along j direction.

It is easy (as an exercise; see textbooks) to write T in Minkowski coordinates as a tensor equation involving the Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Then this is true in curved space too.

For particles:

- "cold dust": at temperature = 0, all particles in local volume have a single bulk velocity
- perfect fluid/gas: particles have bulk motion plus random motions, i.e. pressure

Example ?: Considering dust,

$$T^{00} \sim \rho c^2, \quad T^{0i} \sim \rho v^i, \quad T^{ij} \sim \rho v^i v^j \quad \text{(heuristically)} \tag{252}$$

Formally, we can do this by letting n_0 = proper number density (in fluid bulk frame).

Then define a particle flux 4-vector $\vec{N} = n_0 \vec{u}$.

Note that divergence of $\vec{N}$ vanishes by particle conservation.

Then $T = \vec{p} \otimes \vec{N}$ where $\vec{p}$ is the bulk momentum.

If we consider a single species of aprticle, with mass m and proper density n_0,

$$T = m\vec{u} \otimes n_0 \vec{u} \tag{253}$$

with components

$$T^{\mu\nu} = mn_0 u^\mu u^\nu \tag{254}$$

Example ?: We now consider a perfect fluid.

Each particle (label it A) has 4-velocity

$$\vec{w}_A = \underbrace{\vec{u}_A}_{\text{organised bulk}} + \overbrace{\delta\vec{u}_A}^{\text{random}} \tag{255}$$

For the total fluid, we sum over the particle (i.e. take the average)

$$T = \sum_A mn_0 \left(\vec{u}_A \otimes \vec{u}_A + \underbrace{2\vec{u}_a \times \delta\vec{u}_A}_{\approx 0 \text{ (random vector } \delta\vec{u}_A)} + \overbrace{\delta\vec{u}_A \times \delta\vec{u}_A}^{\text{pressure}} \right) \tag{256}$$

$$= pg + (\rho + p)\ \vec{u} \otimes \underbrace{\vec{u}}_{\text{bulk 4-velocity}} \tag{257}$$

where ρ is the mass density. Note that in (256) the sum over the random vector will go to zero, as it averages to zero.

If there is no bulk motion then (Minkowski) $\vec{u} = (1, 0, 0, 0)$

$$T^{00} = pg^{00} + (\rho + p) \cdot 1 \qquad = \rho, \quad \text{rest energy density} \tag{258}$$

$$T^{11} = pg^{11} + (\rho + p) \cdot \underbrace{u^1 u^1}_{0} \qquad = p, \quad \underbrace{\text{isotropic pressure}}_{T^{11} = T^{22} = T^{33}} \tag{259}$$

Note that this section relates to Schutz Chapter 4; we should read it carefully and do some of the problems (also relating to assignment 3).

9 Gauges

We work towards a weak-field theory, which we need to treat solar system/terrestrial experiments (see earlier sections).

The Einstein field equations are 10 indepedent equations when viewed in isolation. But we know there are 4 degrees of freedom which we know physically to be dependent, because we can always choose 4 coordinates, i.e. 4 gauge choices, as we please (no matter how complicated T is).

$\therefore$ 6 truly independent field equations. We can compare this to electromagnetism (EM):

- 6 field variables **E**, **B**
- these reduce to 4 potentials Φ, **A**
- these reduce further to 3 equations via gauge choices

Recall in EM: Faraday field tensor (we will work in flat space)

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu && (260)\\ \text{Let } \bar{A}_\nu &= A_\nu + \partial_\nu \chi, \quad \text{where } \chi \text{ is an arbitrary scalar function} && (261)\\ \bar{F}_{\mu\nu} &= \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu && (262)\\ &= F_{\mu\nu} + \partial_\mu \partial_\nu \chi - \partial_\nu \partial_\mu \chi && (263)\\ &= F_{\mu\nu} && (264) \end{aligned}$$

i.e. Faraday tensor is unchanged under gauge transformation (as an exercise, show that this is also true in curved space).

Furthermore in flat space, the homogeneous Maxwell equations read

$$0 = F_{\mu\nu,\alpha} + F_{\nu\alpha,\mu} + F_{\alpha\mu,\nu} \tag{265}$$

and all the Christoffels are zero, so we can add zeroes to the RHS of the above equation to get in curved space

$$0 = F_{\mu\nu;\alpha} + F_{\nu\alpha;\mu} + F_{\alpha\mu;\nu} \tag{266}$$

This is of an identical form to the Bianchi identities (satisfied by the Riemann tensor in GR which is slightly larger; here it is satisfied by the Faraday tensor).

1. In EM, Biachi tells us we can write **E** and **B** in terms of **A** and Φ (via div $\mathbf{B} = 0$ + Faraday's law). Likewise, in GR Bianchi tells us $R_{\alpha\beta\gamma\delta}$ can be written in terms of "potentials" (see below).
2. By analogy with EM Bianchi, it is clear that GR also has an "electric" and "magnetic" subdivision of the metric satisfying the generalised Faraday's law. We often see talk of gravitoelectric and gravitomagnetic effects (even when $F^{\mu\nu} = 0$; i.e. when there are no electric and magnetic fields - these are not actually caused by EM, but rather appear analogously)

An example of a gravitomagnetic effect is Lense-Thirring precession.

What are the "potentials" in GR? Unfortunately it is not as simple as saying

$$R_{\alpha\beta\gamma\delta} = A_{\beta\gamma\delta;\alpha} - A_{\alpha\beta\gamma;\delta}, \quad \text{for some } A_{\alpha\beta\gamma}$$

This is clearly not true - the symmetry and anti-symmetry of the indices leads either to trivialities or contradictions

Luckily our maths ~~friends~~ frenemies :) have shown that there exists Ricci decomposition for Riemann

$$R_{\alpha\beta\gamma\delta} = \frac{R}{6}\underbrace{g_{\alpha[\gamma}g_{\delta]\beta}}_{\text{scalar}} + \underbrace{g_{\alpha[\gamma}S_{\delta]\beta} - g_{\beta[\gamma}S_{\delta]\alpha}}_{\text{semi-traceless}} + \underbrace{C_{\alpha\beta\gamma\delta}}_{\text{traceless}} \tag{267}$$

$$\text{where } [\ldots] = \text{antisymmetrisation} \tag{268}$$

$$S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{4}Rg_{\alpha\beta} \tag{269}$$

$$C_{\alpha\beta\gamma\delta} = \text{Weyl tensor} \tag{270}$$

See footnote for antisymmetrisation[7]

The Weyl tensor is a complicated thing involving derivatives of Lanczos tensor $H_{\alpha\beta\gamma}$; it is where the 10 "missing" degrees of freedom in Riemann went, i.e. the ones <u>not</u> in Eintein's field equation.

The Einstein field equations involve the trace of Riemann (to get the Ricci tensor), and since $C_{\alpha\beta\gamma\delta}$ is traceless and hence disappears (does not appear in the field equations.

Equation of motion for Weyl:

$$C_{\alpha\beta\gamma}{}^{\delta}{}_{;\delta} = \text{stuff involving } R_{\beta\nu} \text{ and its covariant} \tag{271}$$

By analogy, the EM inhomogeneous Maxwell:

$$F^{\mu\nu}{}_{;\nu} = \mu_0 J^\mu \tag{272}$$

GR case: invariant under gauge transform

$$\bar{H}_{\alpha\beta\gamma} = H_{\alpha\beta\gamma} + \chi_{[\alpha}g_{\beta]\gamma} \tag{273}$$

where χ_α is an arbitrary 1-form (4 functions because we are free to make 4 coordinate choices)

Nice gauge choices:

- algebraic
$$3\chi_\alpha = H_\alpha{}^\beta{}_\beta \tag{274}$$
- differential (like Lorenz in EM, $A^\mu{}_{;\mu} = 0$)
$$H_{\alpha\beta}{}^{\gamma}{}_{;\gamma} = 0 \tag{275}$$

9.1 Weak fields

Spacetime is nearly flat

$$g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{Minkowski}} + h_{\mu\nu} \tag{276}$$

[7] $g_{\alpha[\gamma}g_{\delta]\beta} = g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}$

with

$$|h_{\mu\nu}| \ll 1 \quad \text{(this is element-wise comparison)} \tag{277}$$

This implies a restricted choice of coordinates, e.g. spherical polars are not appropriate (r, θ, etc.)

We define and use two kinds of transformations. Our aim is to find a metric suitable for terrestrial and solar system applcations, e.g. Pound-Rekba experiment.

9.1.1 Background Lorentz transformations

$$dx^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta} dx^{\beta} \tag{278}$$

from special relativity with reference to a flat background. Applying this to the metric:

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\alpha}} \Lambda^{\beta}{}_{\bar{\beta}} \left(\eta_{\alpha\beta} + h_{\alpha\beta} \right) \tag{279}$$

$$= \eta_{\bar{\alpha}\bar{\beta}} + h_{\bar{\alpha}\bar{\beta}}, \quad \text{by definition} \tag{280}$$

where $h_{\bar{\alpha}\bar{\beta}}$ means "via background Lorentz".

9.1.2 Gauge transformation: small coordinate change

The gauge transformation we use is:

$$\vec{x} \mapsto \vec{x} + \underbrace{\vec{\zeta}(\vec{x})}_{\text{small}} \tag{281}$$

By small we mean $|\zeta| \ll |\vec{x}|$.

The transform matrix is then

$$x^{\alpha} \mapsto x^{\alpha'} = x^{\alpha} + \underbrace{\zeta^{\alpha}(\vec{x})}_{\text{small}} \tag{282}$$

where "small" is as previous, though evaluated element-wise.

So now, our metric in primed coordinates is:

$$g_{\alpha'\beta'} = \Lambda^{\alpha}{}_{\alpha'} \Lambda^{\beta}{}_{\beta'} (\eta_{\alpha\beta} + h_{\alpha\beta}) \tag{283}$$

$$= \delta^{\alpha}{}_{\alpha'} \delta^{\beta}{}_{\beta'} \eta_{\alpha\beta} - \frac{\partial \zeta^{\alpha}}{\partial x^{\alpha'}} \delta^{\beta}{}_{\beta'} \eta_{\alpha\beta} - \delta^{\alpha}{}_{\alpha'} \frac{\partial \zeta^{\beta}}{\partial x^{\beta'}} \eta_{\alpha\beta} + \delta^{\alpha}{}_{\alpha'} \delta^{\beta}{}_{\beta'} h_{\alpha\beta} + \text{2nd order corrections} \tag{284}$$

$$= \eta_{\alpha'\beta'} - \zeta_{\beta',\alpha'} - \zeta_{\alpha',\beta'} + h_{\alpha'\beta'} \tag{285}$$

where $\zeta_{\beta'} = \eta_{\beta'\gamma'} \zeta^{\gamma'}$, i.e. all raising and lowering is done with Minkowski metric.

This is like an EM gauge transform, cf. $A_{\mu} \mapsto A_{\mu} + \frac{\partial \chi}{\partial x^{\mu}}$ except that our $\vec{\zeta}$ has 4 degrees of freedom, not one (physically: we are free to choose 4 coordinates).

We can now calculate Christoffel symbols, Riemann tensor etc. from this metric. Algebra is left as an exercise.

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}{}_{\beta\mu\nu} = \frac{1}{2}\left(h_{\alpha\nu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta,nu} - h_{\beta\nu,\alpha,mu}\right) \tag{286}$$

noting that the derivatives of η vanishes.

Exercise: We can check that R is invariant under gauge transform

$$\bar{h}_{\alpha\nu} = h_{\alpha\nu} - \zeta_{\alpha,\nu} - \zeta_{\nu,\alpha} \tag{287}$$

We will get lots of pairwise cancellation because $\zeta_{\alpha,\beta\gamma\delta}$ is the same for all perumtations of $\{\beta, \gamma, \delta\}$. We can choose $\vec{\zeta}$ cleverly to simplify.

Considering the Ricci tensor,

$$R_{\beta\nu} = g^{\alpha\lambda} \underbrace{R_{\lambda\beta\alpha\nu}}_{\text{1st order only}} \cong \eta^{\alpha\lambda} R_{\lambda\beta\alpha\nu}, \quad \text{to 1st order} \tag{288}$$

The Ricci scalar:

$$R = g^{\beta\nu} \underbrace{R_{\beta\nu}}_{\text{also 1st order}} \cong \eta^{\beta\nu} R_{\beta\nu} + \text{quadratic bits} \tag{289}$$

Collect eveerything in Einstein tensor:

$$G_{\beta\nu} = R_{\beta\nu} - \frac{1}{2} R g_{\beta\nu} \tag{290}$$

The algebra is left as an exercise, but we get

$$G_{\beta\nu} = \frac{1}{2}\left(h^{\alpha}{}_{\nu,\beta\alpha} + h_{\beta\alpha,}{}^{\alpha}{}_{\nu} - h^{\alpha}{}_{\alpha,\beta\nu} - h_{\beta\nu,}{}^{\alpha}{}_{\alpha}\right) \tag{291}$$

$$-\frac{1}{2}\eta_{\beta\nu}\left(h^{\alpha}{}_{\lambda,}{}^{\lambda}{}_{\alpha} - h^{\lambda}{}_{\lambda,}{}^{\alpha}{}_{\alpha}\right) \tag{292}$$

Note, all raising and lowering is done with Minkowski metric.

$$\text{e.g. } h^{\alpha}{}_{\nu,\beta\alpha} = \eta^{\alpha\mu} \frac{\partial h_{\mu\nu}}{\partial x^{\beta} \partial x^{\alpha}} \tag{293}$$

$$h_{\beta\alpha,}{}^{\alpha}{}_{\nu} = \frac{\partial h_{\beta\alpha}}{\partial x^{\mu} \partial x^{\nu}} \eta^{\alpha\mu} \quad \text{etc.} \tag{294}$$

Terms (2nd derivatives, since curvature) are mainly of the form

$$\underbrace{(h_{\alpha\beta,}{}^{\alpha})}_{\text{div } h}, \text{ something} \tag{295}$$

$$\text{or } = \underbrace{(h^{\alpha})}_{\text{trace of } h}, \text{ two somethings} \tag{296}$$

This suggests a good gauge choice is something traceless and divergenceless (like Lorenz gauge in EM, divergenceless).

We define a trace-reveresed metric

$$\bar{h}_{\beta\nu} = h_{\beta\nu} - \frac{1}{2}\eta_{\beta\nu}h^{\lambda} \tag{297}$$

Also impose the gauge condition

$$\bar{h}_{\beta\nu,}{}^{\nu} = 0 \quad \text{(Lorenz)} \tag{298}$$

Substitute (as an exercise)

$$G_{\beta\nu} = -\frac{1}{2}\Box\bar{h}_{\beta\nu} \tag{299}$$

where $\Box$ is the D'Alembertian in flat space

$$\Box = \frac{\partial^2}{\partial t^2} + \nabla^2 \tag{300}$$

Einstein equation without cosmological constant is

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \quad \text{weak field} \tag{301}$$

This is a wave equation wih source on RHS.

A short review:

Recall, in the weak field limit we express the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{302}$$

where $h_{\mu\nu}$ is small. The trace-reversed version of h is

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\lambda}{}_{\lambda} \tag{303}$$

We choose our gauge so that

$$h_{\mu\nu,}{}^{\nu} = 0 \tag{304}$$

which is the Lorenz gauge. From this we get a wave equation,

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{305}$$

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10 Applications of Einstein's Field Equations

We will consider how we can apply the wave equation (305) to terrestrial situations (e.g. gravity on Earth).

To do this, we will have to neglect many components! (this is since we do not actually need 10 separate equations to describe, for example, the motion of a tennis ball).

10.1 Perfect fluid

For a perfect fluid, $T_{\mu\nu} = pg_{\mu\nu} + (p + \rho)u_\mu u_\nu$, so

$$T_{00} \sim \rho \tag{306}$$
$$T_{0i} \sim \rho \times \text{velocity} \tag{307}$$
$$T_{ij} \sim \rho \times (\text{velocity})^2 \tag{308}$$

where the last two are $\ll \rho$ if the velocity of the source is $\ll c$.

Consider only

$$\Box \bar{h}_{00} = -16\pi T_{00} = -16\pi\rho \tag{309}$$

where

$$\Box = -\frac{\partial^2}{\partial t^2} + \nabla^2 \approx \nabla^2 \tag{310}$$

because we want static gravity, not a wave

$$\tag{311}$$

Alternatively,

$$\frac{\partial/\partial t}{\nabla} \sim \frac{\text{length}}{\text{time}} \sim \text{velocity} \ll 1 \tag{312}$$
$$\text{i.e., } \nabla^2 \bar{h}_{00} = -16\pi\rho \tag{313}$$

In the Newtonian limit (experimental):

$$\nabla^2\Phi = 4\pi \underbrace{G}_{\text{we will set } G=1} \rho \tag{314}$$

That is,

$$\bar{h}_{00} = -4 \underbrace{\Phi}_{\text{Newtonian gravitational potential}} \tag{315}$$

and other $\bar{h}_{\mu\nu}$'s are tiny.

Going back to the original metric involving $h_{\mu\nu}$:

$$h_{00} = \bar{h}_{00} + \frac{1}{2}\eta_{00}h^\lambda{}_\lambda \tag{316}$$
$$= \bar{h}_{00} + \frac{1}{2} \underbrace{\eta_{00}}_{-1} \underbrace{\eta^{\lambda\alpha}}_{-1 \text{ for } \lambda=\alpha=0} \overbrace{h_{\lambda\alpha}}^{\text{only significant for } \lambda=0,\alpha=0} \tag{317}$$

NOTE: these is a minor error in the algebra above, according to Melatos.

$$\text{i.e., } h_{00} = \frac{1}{2}\bar{h}_{00} \tag{318}$$
$$= -2\Phi \tag{319}$$

Similarly for h_{11}, etc. (trace $\neq 0$).

Put together we get the weak-field "static" metric:

$$g_{\mu\nu} = \text{diag}\,(-1 - 2\Phi, 1 - 2\Phi, 1 - 2\Phi, 1 - 2\Phi) \tag{320}$$

10.2 Pound-Rebka

We consider the Pound-Rebka experiment (PUT FIGURE HERE). What is the photon frequency at the top/bottom of the tower?

$$E_{\text{photon}} = -\vec{u}\cdot\vec{p} \tag{321}$$
$$\vec{u}_{\text{top}}\cdot\vec{u}_{\text{top}} = -1 \tag{322}$$
$$g_{00}\left(u^0_{\text{top}}\right)^2 = -1 \tag{323}$$
$$\text{i.e., } u^0_{\text{top}} = \sqrt{1+2\Phi}, \quad \Phi \text{ is small on Earth} \tag{324}$$
$$\approx 1+\Phi \text{ (at top)} \tag{325}$$

Simialrly stat. observation at bottom:

$$u^0_{\text{bottom}} = 1+\Phi \text{ (at bottom)} \tag{326}$$

Also note that the metric is independent of time.

$$\text{i.e., } p_0 = \text{constant} \tag{327}$$
$$E_{\text{top}} = -u^0_{\text{top}}p_0 \quad \text{and} \quad E_{\text{bottom}} = -u^0_{\text{bottom}}p_0 \tag{328}$$
$$\frac{E_{\text{top}}}{E_{\text{bottom}}} = \frac{u^0_{\text{top}}}{u^0_{\text{bottom}}} = \frac{1+\Phi\text{ (top)}}{1+\Phi\text{ (bottom)}} \approx 1+gH \tag{329}$$

where H is the altitude difference, give $\Phi(z) = gz$.

10.3 Gravity Probe B/Lense-Thirring precession

L-T precession arises because inertial frames are dragged into rotation by a rotating object, e.g., a Kerr black hole!

Gravity Probe B is a test for precession of gyroscope (there are 4 probes) orbiting Earth (650 km radius orbit). Launched in 2004-2005.

Two types of precession where measured:

- geodetic [8] 6601 ± 18.3 mas yr^{-1}
  - cf. 6606 mas yr^{-1} predicted from GR (where mas = milli-arcsecond)
- Lense-Thirring [9] : 37.2 ± 7.2 mas yr^{-1}
  - cf. 39.2 mas yr^{-1} predicted from GR

Beware stray electrostatic torques! This was an issue that took about a decade to work out.

Analysis:

[8] Gyroscope parallel transported in free fall, but space is curved by mass of Earth (whether or not Earth rotates) $\Rightarrow$ spin doesn't return to original position

[9] We need Earth to rotate for Lense-Thirring precession. It is the angular momentum of Earth's contribution to $T^{\mu\nu}$

1. What is gyroscope spin in geometric language?
2. How does Earth's mass and rotation affect weak-field metric? (What we have done so far is not enough!)
3. How does spin evolve as it falls freely around Earth? (related to the geodesic equation)

10.4 Spin

$$T^{\mu\nu} = \text{energy momentum tensor} \tag{330}$$

Can we construct a geometric object containing angular momentum (AM) from $T^{\mu\nu}$? Recall 1st year:

$$\text{Angular momentum } I = \mathbf{x} \times \mathbf{p} \tag{331}$$

We propose in GR that angular momentum is some osrt of vector and tensor. Specifically try

$$M^{\alpha\beta\gamma} = x^\beta T^{\gamma\alpha} - x^\gamma T^{\beta\alpha} \tag{332}$$

Note this is antisymmetric in β, γ. Is it conserved?[10] Check divergence in local flatspace

$$\begin{aligned} M^{\alpha\beta\gamma}{}_{,\alpha} &= \delta^\beta{}_\alpha T^{\gamma\alpha} + x^\beta \underbrace{T^{\gamma\alpha}{}_{,\alpha}}_{=0 \text{ because } \propto \text{ energy-momentum conversation}} - \delta^\gamma{}_\alpha T^{\beta\alpha} - \underbrace{x^\gamma T^{\beta\alpha}}_{=0} && (333)\\ &= 0 && (334)\\ &= M^{\alpha\beta\gamma}{}_{;\alpha} && (335) \end{aligned}$$

hence this is also true in curved space.

Define

$$\begin{aligned} J^{\beta\gamma} &= \int \mathrm{d}^3\mathbf{x}\ M^{0\beta\gamma} && (336)\\ \Rightarrow \frac{\mathrm{d}J^{\beta\gamma}}{\mathrm{d}t} &= \int \mathbf{d}^3\mathbf{x}\ \frac{\partial M^{0\beta\gamma}}{\partial t} && (337)\\ &= -\int \mathrm{d}^3\mathbf{x} \frac{\partial M^{i\beta\gamma}}{\partial x^i} && (338) \end{aligned}$$

by conservation law, with $i = 1, 2, 3$

$$= -\int \mathrm{d}S\ n_i M^{i\beta\gamma} \tag{339}$$

by Gauss' law.

Physically:

- J^{0i} appears to depend on where you choose origin. We will ignore it.
- J^{ij} = angular momentum, (e.g. J^{12} is "3" component of AM)
- $J^{\beta\gamma}$ is not invariant under translation because it contains orbital AM.

[10] We have local energy conservation, but there is energy and momentum in global curvature, etc.

We pull out just the spin:

$$S_\alpha = \frac{1}{2} \overbrace{\epsilon_{\alpha\beta\gamma\delta}}^{\text{Levi-Civita (anti-symmetric)}} J^{\beta\gamma} \underbrace{u^\delta}_{4-\text{velocity}} \tag{340}$$

Check (as an exercise) physical meaning in components in MCRF (momentarily comoving reference frame) where $\vec{u} = (u^t, 0, 0, 0)$

$$S_0 = 0 \tag{341}$$
$$S_1 = J^{23} = \text{“1” component of angular momentum} \tag{342}$$
$$\text{etc.} \tag{343}$$

Note: $S_\alpha u^\alpha = 0$.

Step 2 $\rightarrow$ weak field metric (can also get as limit of Kerr BH solution, which we will see later)

Answer:

Post-Newtonian approximation

$$\Gamma^0{}_{i0}|_{\text{2nd order}} = \frac{\partial \Phi}{\partial x^i} \tag{344}$$

where Φ is the Newtonian potential.

$$\Phi = -\int \mathrm{d}^3\mathbf{x}' \ \frac{T^{00}|_{\text{0th order}}}{|\mathbf{x} - \mathbf{x}'|} \tag{345}$$
$$\Gamma^0{}_{ij}|_{\text{1st order}} = 0 \tag{346}$$
$$\Gamma^k{}_{i0}|_{\text{3rd order}} = \frac{1}{2}\left(\frac{\partial \zeta_i}{\partial x^k} - \frac{\partial \zeta^k}{\partial x^i}\right) - \delta^k{}_i \frac{\partial \Phi}{\partial t} \tag{347}$$

where

$$\zeta^i = -4 \int \mathrm{d}^3\mathbf{x}' \ \frac{T^{i0}|_{\text{1st order}}}{|\mathbf{x} - \mathbf{x}'|} \tag{348}$$

We get $T^{i0} \neq 0$ if rotating Earth, eg. $\propto M_{\text{Earth}} \cdot \text{rotational speed of Earth} \sim v^3$

$$\Gamma^k{}_{ij}|_{\text{2nd order}} = -\delta_{ij}\Phi^k{}_{,} - \delta^k{}_i \Phi_{,j} + \delta^k{}_j \Phi_{,i} \tag{349}$$

These orders are “leading order’ for each Christoffel symbol (i.e. all lower orders are zero). Also note $i, j = \{1, 2, 3\}$.

So... we now use the metric.

What are the equations of motion of S_α? We consider freefall: local Lorentz invariance then gives equations of motion of spin; $\vec{u}$ is the 4-velocity along the geodesic of freefall.

$$0 = \nabla_{\vec{u}} \widetilde{S} \tag{350}$$
$$u^\beta S_{\alpha;\beta} = u^\beta \left(S_{\alpha,\beta} - \Gamma^\lambda{}_{\alpha\beta} S_\lambda u^\beta\right) \tag{351}$$

We want $\frac{dS_\alpha}{d\tau}$, i.e. how the $\widetilde{S}$ changes along the geodesic parametrised by the proper time τ.

We simplify: eliminate S_0 using $S_0 u^0 = -S_i u^i$ (see Weinberg for the algebra; we don't need to know the algebra, only the result).

To leading order, we can define a 3-vector

$$\mathbf{L} = (1+\Phi)\mathbf{S} - \frac{1}{2}\mathbf{v}(\mathbf{v}\cdot\mathbf{S}) \tag{352}$$

So we find

$$\frac{d\mathbf{L}}{dt} = \left(- \underbrace{\frac{1}{2}\nabla\times\mathbf{S}}_{\text{Lense-Thirring}} \quad -\frac{3}{2}\mathbf{v}\times\nabla\Phi \right)\times\mathbf{L} \tag{353}$$

Geodetic $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ where $\mathbf{v}$ is the 3-velocity of the spacecraft.

Precession! c.f. (353) with

$$\frac{d\mathbf{L}}{dt} = \mathbf{\Omega}_{\text{precession}}\times\mathbf{L} \tag{354}$$

Geodesic happens because we are parallel transported around a closed loop.

11 Gravitational waves

There are 3 points to consider regarding grav. waves:

1. Propagation
2. Detection
3. Generation

11.1 Waves

Some examples:

- Scalar: sound (just need pressure, e.g. for space; or need density)
- Vector: photon in EM (two directions, need a vector)
- Tensor: grav. waves, elastic waves (displacements, not always same direction as force)

What do waves do? They transport energy, momentum, angular momentum.

$$\Rightarrow \text{merging black holes} \Rightarrow \text{gravitational waves} \tag{355}$$
$$\Rightarrow \text{rotating (off axis) magnet} \tag{356}$$

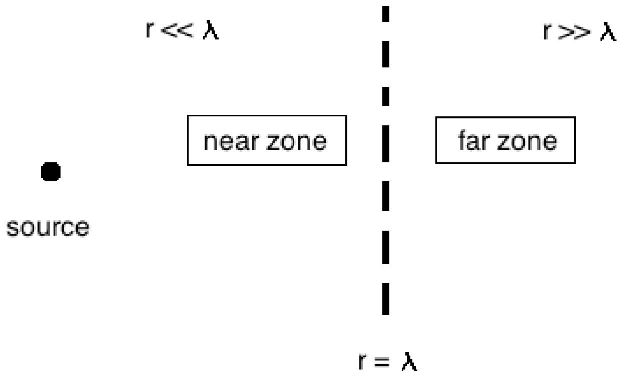

Figure 13: Near and far zones of a wave

Waves are launched by a localised source.

In the near zone, waves $\sim$ quasi-static (static field source) $e^{-i\omega t}$. Usually steeper than r^{-1}, but $E_{\text{flux}} \propto r^{-2}$.

In far zone, we approximate a plane wave $\propto e^{i(kr-\omega t)}$; amplitude $\propto r^{-1}$, usually transverse energy $E \propto r^{-2}$.

In general in GR, we need non-linear analysis in the near zone.

$$\text{Propagation :} \quad \underbrace{\Box \bar{h}_{\mu\nu}}_{\partial^\lambda \partial_\lambda;\text{ raise/lower } \bar{\omega} \text{ Minkowski for linear waves}} = -16\pi T_{\mu\nu} = 0 \quad \text{(in vacuum)} \tag{357}$$

$$\text{Plane wave solutions :} \quad \bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\alpha x^\alpha} \tag{358}$$

$$\Rightarrow \partial_\lambda \bar{h}_{\mu\nu} = A_{\mu\nu} \partial_\lambda (ik_\alpha x^\alpha) e^{ik_\alpha x^\alpha} \tag{359}$$

$$= A_{\mu\nu} ik_\alpha \delta^\alpha{}_\lambda e^{ik_\alpha x^\alpha} \tag{360}$$

$$= ik_\lambda \bar{h}_{\mu\nu} \tag{361}$$

$$\Box \bar{h}_{\mu\nu} = -k^\lambda k_\lambda \bar{h}_{\mu\nu} = 0 \quad \text{(in vacuum)} \tag{362}$$

We have two possibilities:

- no wave ($\bar{h}_{\mu\nu} = 0$)
- $k^\lambda k_\lambda = 0$, i.e. $\vec{k}$ null vector (or equivalently $\widetilde{k}$ is a null 1-form)

  $d\widetilde{\phi} = i\widetilde{k}$

Local Minkowski:

$$\vec{k} = \left(\frac{\omega}{c}, \mathbf{k}\right) \quad (c = 1) \tag{363}$$

i.e. $\omega^2 - |k|^2 = 0$ (use phase speed $= \frac{\omega}{k} = 1$)

i.e. grav. waves propagate at $c = 1$!

$$\Box\bar{h} = -16T \quad \text{in Lorentz (harmonic) gauge} \tag{364}$$

Harmonic $\Rightarrow 0 = \bar{h}_{\mu\nu,}{}^{\nu}$; i.e. $\partial^{\nu}\bar{h}_{\mu\nu} = 0$.

This implies a constraint on $A_{\mu\nu}$

Substitute in a plane wave,

$$0 = ik^{\nu}A_{\mu\nu}e^{ik_{\alpha}x^{\alpha}} \tag{365}$$

i.e. $ik^{\nu}A_{\mu\nu} = 0 \Rightarrow \vec{k}$ orthogonal to tensor A.

How many degrees of freedom in plane gravitational waves?

$$A_{\mu\nu} \text{ symmetric} \Rightarrow 10 \text{ independent variables} \tag{366}$$

By choosing coordinates wisely (gauge), we can reduce this by 4.

$$\Rightarrow 6 \text{ independent variables} \tag{367}$$

and $K^{\nu}A_{\mu\nu} = 0$, 4 equations $\Rightarrow$ 4 less independent variables

$$\Rightarrow 2 \text{ independent variables} \tag{368}$$

These two independent variables are for polarisation. Gravitational waves (linear, planar) have two independent polarisations called "???" and "cross" (this naming is explained later).

Some side notes on "good tensor equations":

For a good tensor equation, e.g. $T^{\alpha}{}_{\beta\gamma} =$ something, $T =$ something, we know

1. equations will hold for all coordinate systems
2. tensor will be the same for all coordinate systems
3. components of tensor won't necessarily be the same (i.e. representation may change)
4. if the LHS is scalar, its components won't change between coordinate systems

Linearised Einstein equations:

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} = 0 \quad \text{in a vacuum} \tag{369}$$

Plane waves:

$$\bar{h}_{\mu\nu} = A_{\mu\nu}e^{ik_{\alpha}x^{\alpha}} \tag{370}$$

This implies

$$k_\lambda k^\lambda = 0 \underbrace{\Rightarrow}_{\text{local Minkowski}} -\omega^2 + |\mathbf{k}|^2 = 0 \tag{371}$$

Light doesn't travel c in the large scale ????. We can omly compare the large scale speed of light on a global sense (can't compare vectors) $\Rightarrow$ want to get c globally in general.

Gauge is

$$\bar{h}_{\mu\nu,}{}^{\nu} = 0 \Rightarrow A_{\mu\nu} k^\nu = 0 \tag{372}$$

11.2 Gauge Freedom

This lets us impose any 4 constraints on $A_{\mu\nu}$.

We choose the following:

(1) A is traceless i.e. $A^\alpha_\alpha = 0$ How do we know we can do this? Gauage transformation

$$\bar{h}_{\mu\nu} \to \bar{h}^{\text{old}}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\lambda{}_{,\lambda} \tag{373}$$

Try $\vec{\xi} = \vec{B}e^{ik_\alpha x^\alpha}$.

Substitute (as an exercise):

$$A^{(\text{new})}_{\mu\nu} = A^{(\text{old})}_{\mu\nu} - ik_\nu B_\mu - ik_\mu B_\nu + \eta_{\mu\nu} \cdot ik_\lambda B^\lambda \tag{374}$$

If trace is zero

$$\begin{aligned} 0 &= A^{\mu(\text{new})}_{\mu} && (375)\\ &= A^{\mu(\text{old})}_{\mu} - ik^\mu B_\mu - ik_\mu B^\mu + 4ik_\lambda B^\lambda && (376)\\ &= A^{\mu(\text{old})}_{\mu} + 2ik_\lambda B^\lambda && (377)\\ &= 0, \quad \text{if } T^{\mu\nu} = 0 \text{ on boundary of volume} && (378) \end{aligned}$$

Given an $A^{(\text{old})}$ with non-zero trace, we can choose one component of $\vec{B}$ to remove trace.

(2) A is transverse to some vector $\vec{u}$ $\vec{u}$ is not necessarily $\vec{k}$.

We could

$$\Rightarrow A_{\mu\nu} k^\nu = 0 \tag{379}$$

similar to condition in Lorentz gauge. We could say that this is 4 equations. The above one is 1 equation. Now we have 5 choices! And we had 4 degrees of freedom. But we have that one of the equations is a linear combination of that from requirement (1) (not independent).

11.3 Specific implementation

Choose $\vec{u} = \vec{e}_0$ (time-like basis vector for Minkowski space.

Then $A_{\mu 0} = 0 \forall \mu$.

Now we orient $\vec{k}$ along $\vec{e}_3$ axis, i.e. $k^0 \neq 0$, $k^3 \neq 0$. Then,

$$A_{\mu\nu} k^0 = 0 \Rightarrow A_{\mu 0} k^0 + A_{\mu 3} k^3 = 0 \tag{380}$$

but $A_{\mu 0} = 0$ (from above) $\Rightarrow A_{\mu 3} = 0$ also!

So, A has the form

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{21} & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{381}$$

with $A_{12} = A_{21}$. Morever, $A^\alpha = 0 \Rightarrow A_{11} + A_{22} = 0$

$$\Rightarrow A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{382}$$

where we have defines

$$h_+ \equiv A_{11} \tag{383}$$

$$h_\times \equiv A_{12} \tag{384}$$

So we have two independent amplitudes $\Rightarrow$ two polarisations! (like in EM).

Note: stricly speaking $\vec{k} \leftrightarrow \vec{e}_3$; $\vec{k}$ depends on orientation of axis.

11.4 Gravitational wave detection

Motion of test particle in wave.

Consider a particle at rest in Mikowski spacetime before wave passes through. Orient our transverse traceless gauge such that $\vec{U} = \vec{e}_0$ =4-velocity of motionless particle initially. When wave passes through, particle is in free fall.

$$0 = \nabla_{\vec{u}} \vec{u} \tag{385}$$

$$= u^\beta \left(u^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\lambda\beta} u^\lambda \right) \tag{386}$$

$$= \frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\lambda\beta} u^\lambda u^\beta \tag{387}$$

At $\tau = 0$ (initially) we have $\vec{u} = (1, 0, 0, 0)$. Initially:

$$\frac{du^\alpha}{d\tau}|_{\tau=0} = -\Gamma^\alpha_{00} \tag{388}$$
$$= -\frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda 0,0} + g_{0\lambda,0} - g_{00,\lambda} \right) \tag{389}$$
$$= -\frac{1}{2} \eta^{\alpha\lambda} \left(h_{\lambda 0,0} + h_{0\lambda,0} - h_{00,\lambda} \right) \tag{390}$$

to leading order in h.

$$= 0, \quad \text{since } A_{\mu 0} = 0. \tag{391}$$

i.e., nothing "happens", meaning $\vec{u}$ doesn't change.

Lecture 9/5/16:

From last time, we recall

$$\nabla_{\vec{u}} \vec{u} = 0 \tag{392}$$
$$\frac{du^\alpha}{d\tau} = -{\Gamma^\alpha}_{\lambda\beta} u^\lambda u^\beta \tag{393}$$

Initially we have $\vec{u} = (1, 0, 0, 0)$,

$$\Rightarrow -{\Gamma^\alpha}_{00} = 0 \quad \text{in TT gauge} \tag{394}$$

Note: remember Artemis and Diana! Even though they were stationary with respect to each other, there was still a Doppler shift.

Let's consider neighbouring particles in freefall (geodesic trajectories)

FIGURE

Note that each particle satisfies

$$\frac{du^\alpha}{d\tau} = -{\Gamma^\alpha}_{\lambda\beta} u^\lambda u^\beta \tag{395}$$
$$\underline{\text{and}} \quad \vec{x}_B(\tau) = \vec{x}_A(\tau) + \vec{\xi}(\tau) \tag{396}$$
$$\therefore \vec{u}_B = \vec{u}_A + \underbrace{\frac{d\vec{\xi}}{d\tau}}_{\text{small}} \tag{397}$$

Geodesic derivation:

$$\frac{d^2 \xi^\alpha}{d\tau^2} = {R^\alpha}_{\beta\gamma\delta} u^\beta u^\gamma u^\delta \tag{398}$$

The Riemann tensor is just a derivative of Γ!

Curvature: is already 1st order in gravitational wave amplitude h. To leading leading, we only need $\vec{u}$ to 0th order, i.e.

$$\vec{u}_A = (1, 0, 0, 0) \tag{399}$$

We initially say that particles are separated by a small distance ϵ in the "1" direction. i.e.

$$\vec{\xi} = (0, \epsilon, 0, 0) \tag{400}$$

$$\Rightarrow \frac{d^2\xi}{d\tau^2} = R^{\alpha}{}_{001} \cdot \epsilon \tag{401}$$

IN TT gauge, $R^{\alpha}{}_{001} \neq 0$ only for $\alpha = 1, 2$.

$$\frac{d^2\xi^1}{d\tau^2} = \frac{1}{2}\epsilon \frac{d^2 h^{TT}_{xx}}{dt^2} \tag{402}$$

$$\frac{d^2\xi^2}{d\tau^2} = \frac{1}{2}\epsilon \underbrace{\frac{d^2 h^{TT}_{xy}}{dt^2}}_{\text{initially } t = \tau \text{, Minkowski intially}} \tag{403}$$

If initially $\vec{\xi} = (0, 0, \epsilon, 0)$,

$$\frac{d^2\xi^1}{dt^2} = \frac{1}{2}\epsilon \frac{\partial^2 h^{TT}_{xy}}{\partial t^2} \tag{404}$$

$$\frac{d^2\xi^2}{dt^2} = -\frac{1}{2}\epsilon \frac{\partial h^{TT}_{xx}}{\partial t^2} \tag{405}$$

There are two scenarioes:

$$h^{TT}_{xx} = 0 \quad (x \text{ separation}) \tag{406}$$

or

$$h^{TT}_{xy} = 0 \quad (y \text{ separation}) \tag{407}$$

(and other linear combinations).

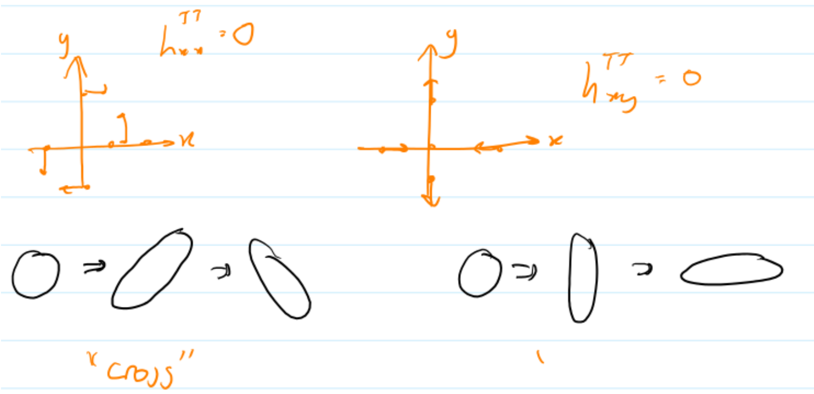

Figure 14: Polarisations

Lecture 11/5/16:

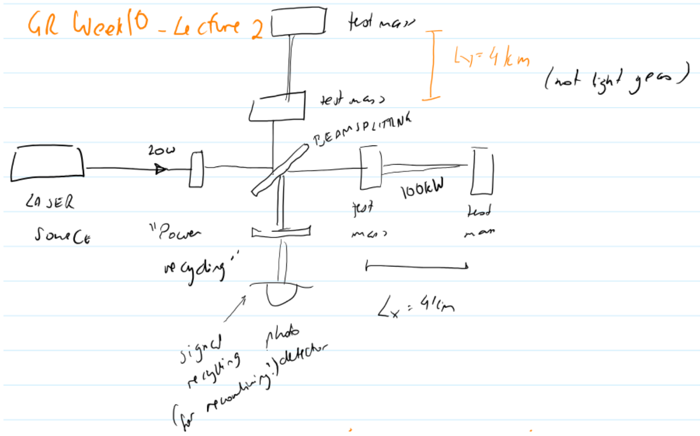

Figure 15: LIGO arms

12 LIGO and gravitational wave detection

Strain is given by

$$\frac{\Delta L}{L} = h \tag{408}$$

which is unitless (note: L is the length of an arm). This h is a component of $h_{\mu\nu}$; it is a linear combination of h_+ and $h_\times$, "cross-polarisation".

Take $h(t)$ = strain as a function of time;

$$h(t) \xrightarrow{\text{Fourier}} h(f) = \int e^{-2\pi i f(t)} h(t)\ dt \tag{409}$$

$$\underbrace{\text{SNR}}_{\text{signal-to-noise ratio}} = \int_{-\infty}^{\infty} df\ \frac{|h(f)|^2}{S_n(f)} \tag{410}$$

where $|h(f)|^2$ is the power of signal, and $S_n(f)$ is the power of signal noise.

12.1 Noise spectral density

An alternative:

$$S_n(f) = (\text{strain noise})^2 \tag{411}$$

Alternative: let $n(t)$ be noise amplitude vs. time.

Autocorrelation function: $S(\tau) =< n(f) \cap (t+\tau) >$

Then $S_n(f)$ is Fourier transform of $S(\tau)$!

12.2 LIGO noise sources

- Low f - seismic/Newtonian

- Mid f - thermal
- High f - shot noise
- Spikes - resonances

Lecture 12-5-16 missing

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13 Relativistic stars

We consider relativistic stars to be perfect fluids, spherical and spherically symmetric, static and isotropic. Recall that ds^2 is given in terms of g as

$$ds^2 = g(d\vec{x}, d\vec{x}) \tag{412}$$

- function of "radius" $\mathbf{x} \cdot \mathbf{x} = r^2$
- function of $d\vec{x}$ in an isotropic sense (independent of rotation)

i.e.,

$$ds^2 = -F(r)dt^2 + 2E(r)dt\ \mathbf{x} \cdot d\mathbf{x}' + D(r)(\mathbf{x} \cdot d\mathbf{x})^2 + C(r)\ d\mathbf{x} \cdot \mathbf{x} \tag{413}$$

where we find

$$d\mathbf{x} \cdot d\mathbf{x} = dr^2 + r^2\ d\theta^2 + r^2 \sin^2\theta \sin^{\phi} \tag{414}$$

Simplify by transformations:

1. Redefine clocks so $t' = t + \Phi(r)$ (tangle t and r tp get rid of off-diagonal terms)
2. Remove the $dt\ dr$ term; choose $\Phi(r) = -\int_0^r \frac{dr'\ r'C(r')}{F(r')}$
3. Redefine radius $r'^2 = r^2 C(r)$

We should check that our spacetime interval becomes

$$ds^2 = -\underbrace{B(r')}_{e^{2\Phi(r)}} dt'^2 + \underbrace{A(r')}_{e^{2\Lambda(r)}} dr'^2 + r'^2 \left(d\theta^2 + \sin^2\theta\ d\phi^2\right) \tag{415}$$

We can drop the primes, since only r' terms exist (no r terms).

Substituting into the Einstein field equations:

$$G_{tt} = \frac{1}{r^2} e^{2\Phi} \frac{d}{dr} \left[r \left(1 - e^{-2\Lambda} \right) \right] \tag{416}$$

$$G_{rr} = -\frac{1}{r^2} e^{2\Lambda} \left(1 - e^{-2\Lambda} \right) - \frac{2}{r} \Phi' \tag{417}$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda} \left(\Phi'' + (\Phi')^2 + \frac{\Phi'}{r} - \Phi'\Lambda - \frac{\Lambda'}{r} \right) \tag{418}$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta} \tag{419}$$

where the primes represent derivatives with respect to r (i.e. $\frac{d}{dr}$).

The RHS is stress-energy. Assume a perfect fluid,

$$T = pg + (p + \rho)\vec{u} \otimes \vec{u} \tag{420}$$

If star's fluid is static, then $\vec{u} = (u^t, 0, 0, 0)$ and hence $\vec{u} \cdot \vec{u} = -1$ implies $-e^{2\Phi}(u^t)^2 = -1$, $u^t = e^{-\Phi}$.

$$T_{tt} = pg_{tt} + (p + \rho)u_t u_t \tag{421}$$

$$= -pe^{2\Phi} + (p + \rho)e^{\lambda\Phi} \tag{422}$$

$$= \rho e^{2\Phi} \tag{423}$$

note ρ is only present in velocity terms. Remember $\vec{u} = (u^t, 0, 0, 0)$.

$$T_{rr} = rg_{rr} \tag{424}$$

$$= pe^{2\lambda} \tag{425}$$

$$T_{\theta\theta} = r^2 p \tag{426}$$

$$T_{\phi\phi} = r^2 \sin^2\theta \cdot p \tag{427}$$

Lecture 16-5-16 missing

A short review:

Last lecture: Einstein's equations for relativistic star (spherical ball of perfect fluid) hence energy-momentum is of a physical sort (has pressure and density)

Today:

- we solve this
- then consider exterior solution if star has finite extent

13.1 Orbits

13.1.1 Metric

Relativistic stars have the metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin\theta d\phi^2 \tag{428}$$

Note $\Phi(r)$ is <u>not</u> the Newtonian gravitational potential, but it does reduce to it in the weak field limit.

Recall Einstein's equations[11] gave 3 independent equations: tt, rr, $\theta\theta$. We will evaluate these.

$$tt: \quad \frac{1}{r^2}e^{2\Phi}\frac{d}{dr}\left[r(1-e^{-2\Lambda})\right] = 8\pi\rho e^{2\Phi} \tag{429}$$

For convenience we replace $\Lambda(r)$ with $m(r) = \frac{r}{2}(1-e^{-2\Lambda})$.

$$\Rightarrow \frac{dm}{dr} = 4\pi\rho r^2 \tag{430}$$

$$rr: \quad -\frac{1}{r}e^{2\Lambda}(1-e^{-2\Lambda}) + \frac{2}{r}\Phi' = 8\pi p e^{2\Lambda} \tag{431}$$

After some substitution and rearrangement,

$$\frac{d\Phi}{dr} = \frac{m + 4\pi p r^3}{r(r-2m)} \tag{432}$$

The $\theta\theta$ term doesn't give anything extra.

We can use the equation of motion for matter (given the fields, the matter responds and "moves" (this also includes motion in the t direction; our star is static); this motion itself affects the fields!):

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{433}$$

i.e. energy-momentum conservation.

Radial component: after algevra, with covariant derivative given assumed metric,

$$(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr} \tag{434}$$

The other components tell us nothing extra.

We count the unknowns from equations (430), (432) and (434): $m(r)$, $\Phi(r)$, $\rho(r)$, $p(r)$.

So we have 4 unknowns but only 3 equations. However, we also need an equation of state relating p and ρ - this is neither from GR or energy conservation; it is a statistical mechanics property of the fluid.

[11] These are the equations of motions for the field ("curvature").

13.1.2 The physics

(430): "Mass" continuity relation, i.e.

$$\begin{aligned} m(r) &= \text{mass enclosed in radius } r \text{ if } \rho \text{ is density.} \qquad (435)\\ &= \int_0^r dr'\ 4\pi r'^2 \rho(r') \qquad (436) \end{aligned}$$

However,

$$\begin{aligned} \text{true mass enclosed} &= \int_V d^3\mathbf{x}\ \rho\sqrt{-g} \qquad (437)\\ &= \int_0^r dr'\ 4\pi r'^2 \rho(r') e^{\Phi(r')+\Lambda(r')} \qquad (438) \end{aligned}$$

where g is the determinant of the metric. This correction is because we must integrate over the proper volume.

The correction adjusts for gravitational binding energy.

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(432): $\frac{d\Phi}{dr} \sim \frac{m}{r^2}$; just like Newton's law for gravitational acceleration, i.e. $\nabla^2\Phi = 4\pi\rho$, and $\Phi \sim \frac{GM}{r^2}$

However, pressure is also a source of intertia and hence curvature $\Rightarrow$ curvature of spacetime modifies r^{-2} law of gravity.

---

(434): Hydrostatic equilibrium, e.g. Earth's atmosphere etc., $\rho \times \underbrace{(\text{gravitational acceleration})}_{\text{vector}} = -\nabla p$

However, pressure also gravitates, i.e. inertia of fluid includes its pressure.

We recall

$$\frac{dm}{dr} = 4\pi r^2 \rho \qquad (439)$$

$$\frac{d\Phi}{dr} = \frac{m + 4\pi\rho r^3}{r(r-2m)} \qquad (440)$$

Suppose star has fininte extent: $\rho = p = 0$ for $r \geq R$.

Exterior:

- $m =$ constant (call it M)
- $\frac{d\Phi}{dr} = \frac{m}{r(r-2m)}$

Integrate:

$$2\Phi = \ln\left(1 - \frac{2m}{r}\right) \qquad (441)$$

Substitute for Φ and Λ in original metric:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \tag{442}$$

This is the Schwarzchild metric! (which we have now derived). It is the exterior metric for <u>any</u> spherically symmetric mass distribution (Birkhoff's therorem).

What happens to an astronaut orbiting a BH?

Assume free fall: solve geodesic equations $\nabla_{\vec{u}}\vec{u} = 0$. [12]

Note that $\nabla_{\vec{u}}\vec{u} = 0$ also implies $\nabla_{\vec{u}}\widetilde{u} = 0$.

We already proved: $\frac{du_\alpha}{d\tau} = \frac{1}{2} g_{\mu\nu,\alpha} u^\mu u^\nu$.

Schwarz: metric independent of t and $\phi \Rightarrow u_t$ and u_ϕ are constants.

We call

- $u_t = -E$ energy/mass at infinity
- $u_\phi = L$ angular momentum/mass at infinity (component along z-axis)

We can also prove (see various books) orbit of particle is confined to a plane (like Newtonian).

i.e.,

$$u^\theta = \frac{d\theta}{d\tau} = 0 \text{ and hence } u_\theta = g_{\theta\theta} u^\theta = 0 \tag{443}$$

Take $\theta = \frac{\pi}{2}$ (equatorial)

$$u^t = g^{tt} u_t = E\left(1 - \frac{2m}{r}\right)^{-1} \tag{444}$$

$$u^\phi = g^{\phi\phi} u_\phi = \frac{L}{r^2} \tag{445}$$

Equation of motion for u^r: equivalently take r component of $\nabla_{\vec{u}}\vec{u} = 0$ <u>or</u> recall normalisation $\vec{u} \cdot \vec{u} = -1$.

$$-1 = g_{tt}(u^t)^2 + g_{rr}(u^r)^2 + 0 + g_{\phi\phi}(u^\phi)^2 \tag{446}$$

$$= -\left(1 - \frac{2m}{r}\right) \cdot E^2 \left(1 - \frac{2m}{r}\right)^{-2} + \left(1 - \frac{2m}{r}\right)^{-1} (u^r)^2 + r^2 \cdot \left(\frac{L}{r^2}\right)^2 \tag{447}$$

$$\text{i.e., } \left(\frac{dr}{d\tau}\right)^2 = E^2 - \underbrace{\left(\frac{L^2}{r^2} + 1\right)\left(1 - \frac{2M}{r}\right)}_{V_{\text{eff}}(r)^2} \tag{448}$$

Orbit only exists if $E \geq V_{\text{eff}}$

[12] If not in free fall, i.e. rockets turned on, we solve $\nabla_{\vec{u}}\vec{u} = \vec{a}$ and infer $\vec{a}$ in terms of proper acceleration. We will not consider this case.

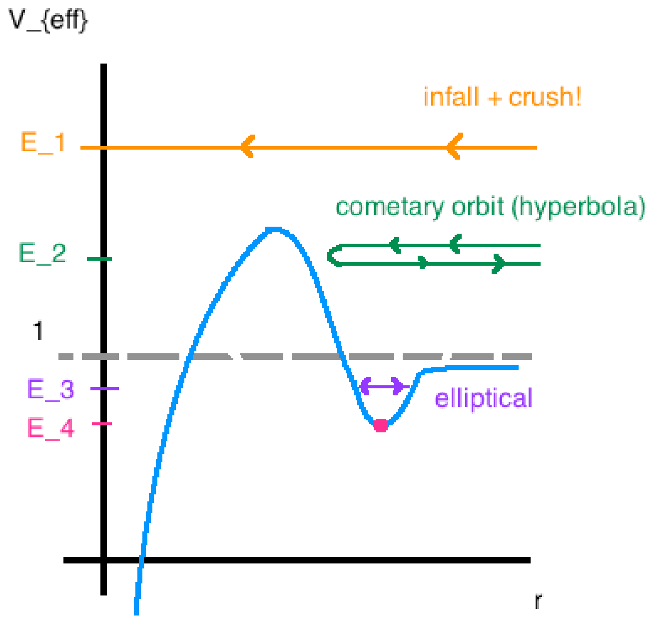

Figure 16: Various orbit types

Circular: $\frac{dV_{\text{eff}}}{dr} = 0$.

$$r = \frac{L^2}{2M}\left(1 + \sqrt{1 - \frac{12M^2}{L^2}}\right) \tag{449}$$

Unlike Newtonian mechniacs, a circular orbit only exists for $L^2 12M^2$. Small circular orbit:

$$r = \frac{12M^2}{2M} = 6M. \tag{450}$$

$\Rightarrow$ ISCO = innermost stable circular orbit.

Lecture 19-5-16:

14 Photons

Consider the 4-momentum of a photon

$$\vec{p} \propto \frac{d\vec{x}(\lambda)}{d\lambda} \tag{451}$$

where λ is the affine parameter.

$$p_\tau = -E, \quad p_\phi = L, \quad p_\theta = 0 \tag{452}$$

all constants.

We have the normalisation $\vec{p} \cdot \vec{p} = 0$ for a null ray.

Consider:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \underbrace{\frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)}_{V_{\text{eff}}^2} \tag{453}$$

where L is angular momentum. To find R, we maximise V_{eff} then can solve for R.

14.1 Perihelion advance of Mercury

"You may not know this, but Mercury really is a potato."

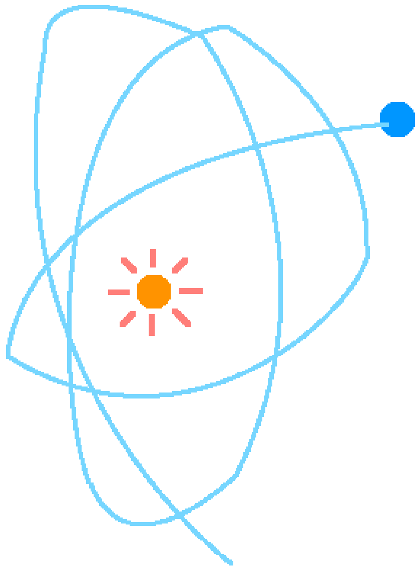

Figure 17: Perihelion precession

Where could this come from?

- torques from other planets
- spin-orbit coupling (Mercury is a potato; 2:3 resonance
- change in 43" per century; unexplainined in early 1900's (a planet "Vulcan" was predicted)

GR (Einstein) predicted: gravitational acceleration around mass M is <u>not</u> simply $-\frac{M}{r^2}$; correction like $-\frac{M}{r^2}\frac{1}{1-\frac{2M}{r}}$ (approx.).

Special case: potential $\propto r^2$ <u>or</u> r^{-1} leads to <u>closed</u> bound orbits. All other potentials do not!

i.e., GR corrections leads to precession.

To calculate this, we need $r(\phi)$ or equivalently $\phi(r)$. From previous lectures we have $\frac{dr}{d\tau}$, $\frac{d\phi}{d\tau}$.

$$\Rightarrow \frac{dr}{d\phi} = \frac{dr}{d\tau} \times \left(\frac{d\phi}{d\tau}\right)^{-1} \tag{454}$$

$$\text{Let } u = \frac{1}{r} \quad \text{(just for convenience)} \tag{455}$$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2}{L^2} - (1 - 2Mu(\left(\frac{1}{L^2} + u^2\right) \tag{456}$$

In the Newtonian case: ignore y^3 term (GR).

Circular orbit has $u = \frac{M}{L^2} = \text{constant}$.

So we have deviation from circularity as:

$$y = u - \frac{M}{L^2} \tag{457}$$

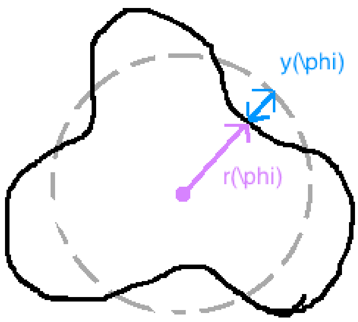

Figure 18: Deviation from circularity of an orbit

Substituting this into (453);

$$\left(\frac{dy}{d\phi}\right)^2 = \frac{E^2 - 1}{L^2} + \frac{M^2}{L^4} - y^2 + \underbrace{\frac{2M^4}{L^6} + \frac{6M^3 y}{L^2} + \frac{6M^2}{L^2}y^2}_{GR} \tag{458}$$

The non-GR bit has a colution

$$y = A\cos(\phi + \phi_0) \tag{459}$$

This has period 2π.

GR bit has solution:

$$y = A\cos\left[\left(1 - \frac{6M^2}{L^2}\right)\phi + \phi_0\right] \tag{460}$$

There is a mismatch between the period of y and orbitial period (2π).

$$\text{Perihelion shift per orbit} = 2\pi - \frac{2\pi}{1 - \frac{6M^2}{L^2}} \tag{461}$$

Exercise: Show that this equals the expected 43" per century.

15 Black holes

15.1 Kerr black holes

A Kerr blackhole is a source with mass M and angular momentum $J = aM$ $(a < 1)$.

Metric is given in Schutz. It is distinguished by $g_{t\phi} = g_{\phi t} \neq 0$ (off-diagonal element). The metric is independent of t and $\phi \Rightarrow u_t$ and u_ϕ are constants on any free-fall orbit.

$$u^t = g^{tt}u_t + g^{t\phi}u_\phi \tag{462}$$
$$u^\phi = g^{t\phi}u_t + g^{\phi\phi}u_\phi \tag{463}$$

Suppose we drop a particle from ∞ directly (radially) into the centre of a Kerr BH (free-fall). Zero angular momentum at infinity $\Rightarrow u_\phi = 0 =$ constant.

At radius $r < \infty$:

$$\text{Angular velocity in global coordinates} = \frac{d\phi}{dt} = \frac{u^\phi}{u^t} = \frac{g^{t\phi}u_t}{g^{tt}u_t} = \frac{g^{t\phi}}{g^{tt}} \neq 0 \tag{464}$$

i.e., a particle is dragged into rotation by the rotating spacetime around a BH. This **frame dragging** leads to Lense-Thirring precession of a gyroscope.

15.1.1 Kerr ergosphere and horizon

Suppose we have a photon on a circular orbit in equatorial plane (in a Schwarzchild, and also Kerr, space: circular orbit is unstable for photons; we need mirrors!).

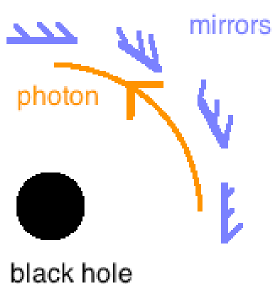

Figure 19: Mirrors around a BH to maintain photon orbit

$$ds^2 = 0 \quad \text{(null ray)} \tag{465}$$
$$= g_{tt}dt^2 + 2g_{t\phi}dt\ d\phi + g_{\phi\phi}d\phi^2 \tag{466}$$

noting that $dr = d\theta = 0$ on a null ray.

$$\Rightarrow \frac{d\phi}{dt} = \frac{-2g_{t\phi} \pm \sqrt{4g_{t\phi}^2 - 4g_{tt}g_{\phi\phi}}}{2} \tag{467}$$

This means that for our photon, two angular velocities (in global coordinates) are possible. This is due to frame dragging (prograde - rotating with BH - and retrograde - rotating against the BH - are not symmetric).

Like with Schwarzchild: a radius exists where $g_{tt} = 0$. This radius is the <u>horizon</u>. At this radius:

$$\frac{d\phi}{dt} = -2g_{t\phi}\ \underline{\text{or}}\ \text{zero!} \tag{468}$$

depending on whether motion is with or against BH rotation. A retrograde photon makes no "headway" against frame dragging (while a prograde photon would be "helped along")!

Schutz gives the radius of the ergosphere as

$$r_{\text{ergo}} = M + \sqrt{M^2 + a^2 \cos^2\theta} \tag{469}$$

Note: this is all in global coordinates! If instead we were in a local reference frame, the photon would be travel at c.

15.2 Black hole coordinates

We will use a Schwarzchild example, using the standard (t, r, θ, ϕ).

Define new coordinates: stretch radial coordinates

$$(r, t) \mapsto (r, \widetilde{V}) \tag{470}$$

with

$$\widetilde{V} = t + \underbrace{r^*}_{\text{tortoise coordinate}} = t + r + 2M \ln\left|\frac{r}{2M} - 1\right| \tag{471}$$

Why is this form useful?

$$\frac{dr^*}{d\tau} = \frac{dr}{d\tau} + \frac{2M}{\frac{r}{2M} - 1} \cdot \frac{1}{2M}\frac{dr}{d\tau} \tag{472}$$
$$\text{i.e.,} \quad dr^* = \frac{dr}{1 - \frac{2M}{r}} \tag{473}$$

This is useful for "killing" (removing) the divergent factor (as $r \to 2M$) in standard Schwarzchild.

We also transform the metric (left as exercise):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\widetilde{V}^2 + 2d\widetilde{V}\ dr + r^2\ d\Omega^2 \tag{474}$$

called Eddington-Finkelstein coordinates.

Considering the light-cone now: $ds^2 = 0$. Fix $d\Omega^2 = 0$.

$$\therefore \frac{d\widetilde{V}}{dr} = 0 \quad \text{and} \quad \frac{2}{1 - \frac{2M}{r}} \tag{475}$$

Two edges of the light cone.

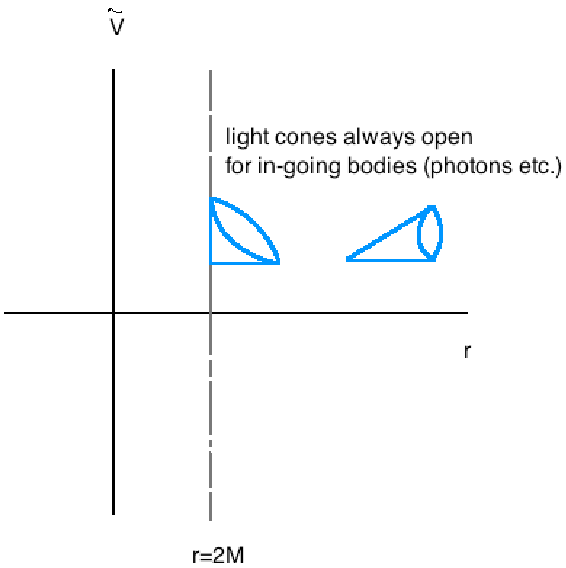

Figure 20: EF lightcone

Also exists: Eddington-Finkelstein outgoing

$$(r, t) \mapsto (r, \widetilde{U}) \quad \text{with } \widetilde{U} = t - r^* \tag{476}$$

Exercise: ?

Calculate the light cone

.

Can we combine these so that the event horizon is regularised for all (both going in or out) particles?

Let's try:

$$(t, r) \mapsto (\widetilde{U}, \widetilde{V}) \tag{477}$$
$$\widetilde{V} - \widetilde{U} = 2r^* \tag{478}$$
$$\widetilde{V} + \widetilde{U} = 2t \tag{479}$$

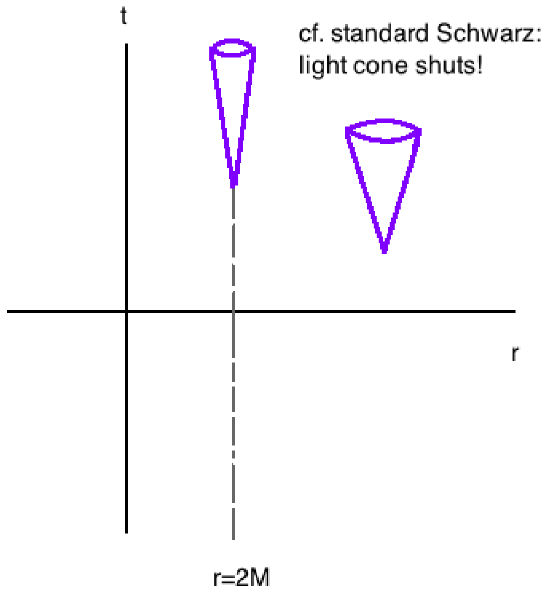

Figure 21: Schwarzchild lightcone

We transform the metric as usual (do this as an exercise):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\widetilde{U}\ d\widetilde{V} + r^2\ d\Omega^2 \tag{480}$$

Sadly this still leads to an event horizon at $r = 2m$. e.g. consider a stationary observer and use $\vec{u} \cdot \vec{u} = -1$ to show that the component blows up.

But luckily, we see form the definition:

$$e^{r^*/2M} = e^{r/2M}\left(\frac{r}{2M} - 1\right) \tag{481}$$

$$\text{i.e.,}\quad e^{(\widetilde{V}-\widetilde{U})/4M} = e^{r/2M}\left(\frac{r}{2M} - 1\right) \tag{482}$$

We try relabelling:

$$\widetilde{u} = -e^{-\widetilde{U}/4M} \quad \text{and} \quad \widetilde{v} = e^{\widetilde{V}/4M} \tag{483}$$

Transform metric (as usual):

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\ d\widetilde{u}\ d\widetilde{v} + r^2\ d\Omega^2 \tag{484}$$

i.e.

$$g_{\widetilde{u}\widetilde{u}} = 0 = g_{\widetilde{v}\widetilde{v}} \tag{485}$$

Now looking at this, we have no event horizon at $r = 2M$! Only the essential singularity is left! (at $r = 0$; this is singularity is unavoidable and cannot be removed by clever methods)

The "singularity" we had at the event horizon is not actually physical - an astronaut passing through it would not feel anything "singular"!

15.2.1 Kruskal-Szekeres

$$\text{Let } u = \frac{1}{2}(\widetilde{v} - \widetilde{u}) \tag{486}$$

$$v = \frac{1}{2}(\widetilde{v} + \widetilde{u}) \tag{487}$$

$$ds^2 = -\frac{32M^3}{r}e^{-r(u,v)/2M)}(dv^2 - du^2) + r^2\, d\Omega^2 \tag{488}$$

In these coordinates, the lightcone is: $v = \pm u$, which is just like in Minkowski!

The central singularity ($r = 0$) corresponds to $v^2 - u^2 = 1$.

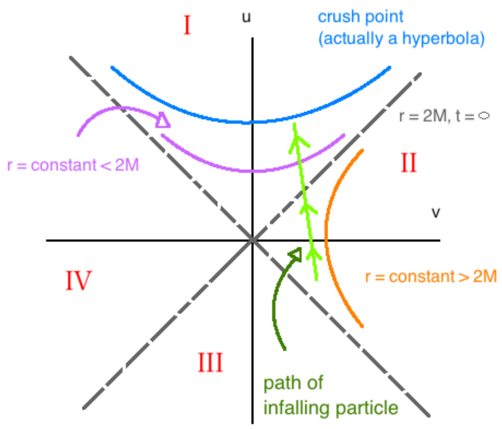

Figure 22: Kruskal-Szekeres coordinates

Standard Schwarz = I union II, or III union IV.

i.e. Schwarzchild standfard coordinatec over only half the manifold!! (cf. twin paradox).

16 Global methods

MTW ch. 34 as starting point.

Aim: "compactify" the manifold, i.e. introduce coordinatrs with fininte values at ∞.

e.g., flat:

$$ds^2 = -dt^2 + dr^2 + r^2\, d\Omega^2 \tag{489}$$

$$\text{Let } t + r = \tan \frac{\psi + \xi}{2} \tag{490}$$

$$t - r = \tan \frac{\psi - \xi}{2} \tag{491}$$

So that the transformed metric becomes

$$ds^2 = \frac{-d\psi^2 + d\xi^2}{4 \cos^2 \frac{\psi+\xi}{2} \cos^2 \frac{\psi-\xi}{2}} + r^2 \, d\Omega^2 \tag{492}$$

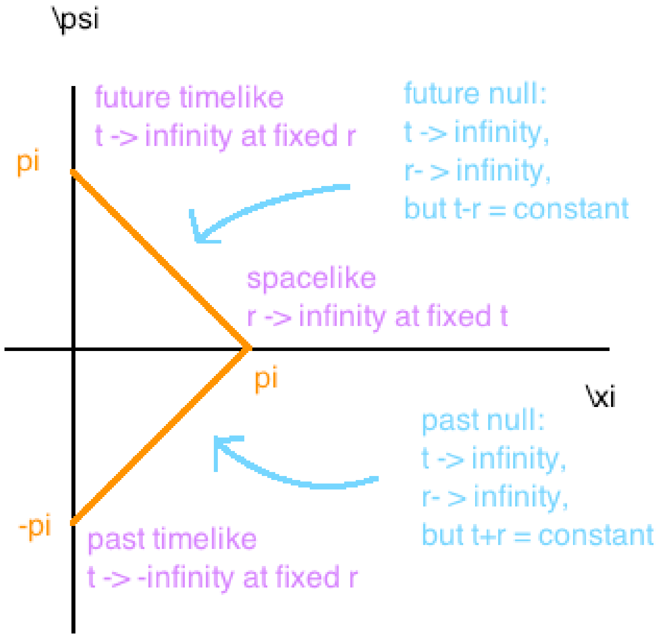

Figure 23: Global methods

Part III

Appendix

Indices up = vector-like

Indices down = 1-form-like

16.1 Covariant derviative notation

16.1.1 Vectors

We realise that the component form of $\nabla\vec{V}$ can be written as $(\nabla\vec{V})^\alpha_\beta$.

$$\underbrace{(\nabla\vec{V})^\alpha_\beta}_{\text{notation: } V^\alpha_{;\beta}} = \underbrace{\frac{\partial V^\alpha}{\partial x^\beta}}_{\text{notation: } V^\alpha_{,\beta}} + \Gamma^\alpha_{\lambda\beta}V^\lambda \tag{493}$$

That is:

$$V^\alpha_{,\beta} := \frac{\partial V^\alpha}{\partial x^\beta} \tag{494}$$

$$(\nabla\vec{V})^\alpha_\beta \equiv V^\alpha_{;\beta} := V^\alpha_{,\beta} + \Gamma^\alpha_{\lambda\beta}V^\lambda \tag{495}$$

16.1.2 One-forms

Components of $\nabla\widetilde{p}$, i.e. $(\nabla\widetilde{p})_{\alpha\beta}$ is notationally $p_{\alpha;\beta}$

$$p_{\alpha;\beta} = \frac{\partial p_\alpha}{\partial x^\beta} - \Gamma^\lambda_{\alpha\beta}p_\lambda \tag{496}$$

16.1.3 General tensors

For general tensor: add correction term $\pm\Gamma$ tensor for each tensor index; + is index is up; − if index is down.

$$\text{e.g.} \quad T^\alpha_{\beta;\gamma} = \frac{\partial T^\alpha_\beta}{\partial x^\gamma} + \underbrace{\Gamma^\alpha_{\lambda\gamma}T^\lambda_\beta}_{\text{by analogy with vector}} - \underbrace{\Gamma^\lambda_{\beta\gamma}T^\alpha_\lambda}_{\text{by analogy with 1-form}} \tag{497}$$

$$\nabla_{\vec{u}}\vec{v} = (\nabla\vec{v})\cdot\vec{u} \tag{498}$$

...probably?

16.1.4 Metric

In flat space, $g_{\alpha\beta,\gamma} = 0$ and Γ's are zero because $\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = 0$

$$\begin{aligned} \Rightarrow g_{\alpha\beta;\gamma} &= g_{\alpha\beta,\gamma} - \text{(two terms involving Christoffel symbols)} \quad (499) \\ &= 0 \quad \text{in flat space} \quad (500) \end{aligned}$$