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PHYC90012 General Relativity

Course Summary

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1 Syllabus

1.1 Part I

1. Introduction to gravity
 - Order of magnitude estimates
 - Small amount of quantum gravity
2. Equivalence principle + experimental foundations
3. Geometric objects
 - Need to understand geometric components of GR
 - Vectors, metric, etc. that live on manifolds
 - Laws of nature do not depend on coordinates chosen
 - Hence can write laws of nature in terms of geometric objects w/o reference to coordinates
4. Kinematics
 - Time dilation, length contraction in GR framework
5. Calculus in curvilinear coordinates
 - Mass and energy curve spacetime
 - Hence geometric objects moved on curved manifolds
 - Distances are not only spatial but temporal; need to use mathematics of small change = calculus
 - Uses the covariant derivative (a geometric object; independent of basis/coordinate independent)
 - This point of the course we will not be considering curved space, but instead only curvilinear coords
 - A flat space can be covered (represented?) by curved coordinates, but an intrinsically curved surface cannot be covered by flat coordinates
6. Curved spaces
 - Manifolds
 - How to calculate lengths, volumes, angles in curved spaces
 - Introduces the idea of parallel transport \Rightarrow leads to curvature
 - Define the Riemann tensor, and its children etc. Ricci tensor, ...; these satisfy the Bianchi identities
7. Einstein's field equations
 - Stress-energy tensor
8. Weak-field limit
 - Gauge transformations

1.2 Part II - Applications

9. GR phenomena revisited

- GPS, Mercury's orbit, gravitational lensing, gravitational redshift, ...

10. Gravitational waves

- Propagation (phase speed, polarisation, ...)
- Generation*
- Detection*
 - * = “antenna problem”

11. Relativistic stars

- neutron stars
- equation of state (cannot study on Earth because largest nuclei only have 200 elements or so; need more density)

12. Black holes

- Event horizons, singularities, ...

13. Cosmology

- Friedman-Robertson-Walker (FRW) metric - describes a homogeneous, isotropic universe
 - We will derive this and the Friedman equations

2 Introduction to gravity

2.1 Strength of gravity

- Weak! Weakest of all fundamental forces
- Long-ranged force (like EM)
- Weakness determined by coupling constant
- Coupling constant = Newton's gravitational constant

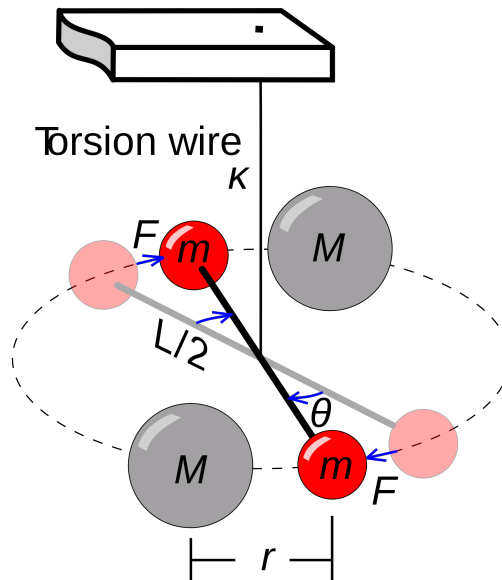
$$\vec{F} = \frac{Gm_1m_2}{r_{12}^2}\hat{r} \quad (1)$$

- G is hard to measure; least well known of coupling constants

In 1797-98, Cavendish used torsion balls (1.8m torsion balance) with rod of big masses and rod of small masses.

- Spring constant of torsion balance was measured from free oscillation

- ten introduced 158kg balls
- measured deflection angle of balance \Rightarrow force
 - using a mini-telescope against Vernier scale
- rearrange Newton's law to get G



Exercise: Show that Cavendish also measured density of Earth as a bonus at the same time

- Modern $G = 6.67384(80) \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} = \text{kg}^{-1} \text{ m}^3 \text{ s}^{-2}$
- Product GM is known to 1 part in 10^{10} from astrophysics observations
 - \Rightarrow mass is hard to measure gravitationally
- We need a dimensionless number to characterise strength
- Newton: $\Phi = \frac{GM}{r}$ (potential)
- In free fall: $\frac{KE}{mass}, v^2 \frac{GM}{r}$
- We claim gravity is strong if free-fall is relativistic, i.e. $v \sim c$
- This is an order of magnitude estimate

2.2 Strong vs. weak gravity

- Quasi-Newtonian:

	$\frac{GM}{Rc^2} \ll 1$	$\frac{GM}{Rc^2} \geq 1$
$v \ll c$	Newtonian	CAN'T EXIST
$v \sim c$	special rel.	full GR (difficult)

– characteristic speed of body in free fall: $v^2 \sim \frac{GM}{r}$

- Strong gravity leads to relativistic free fall, i.e. $\frac{GM}{Rc^2} \geq 1$ where M is the total mass and R is the characteristic size

Example 1.1: $M = M_\odot$ (mass of the Sun)

$$\begin{aligned}
 R &\sim \frac{GM}{c^2} \quad \text{boundary of strong regime} \\
 &\sim \frac{10^{-10} 10^{30}}{10^{17}} \\
 &\sim \text{km}
 \end{aligned}$$

cf. Schwarz radius of black hole = $\frac{2GM}{c^2}$

Example 1.2: Density of black hole with mass of M_\odot

$$\begin{aligned}
 &\sim \frac{M}{R^3} \sim \frac{10^{30} \text{kg}}{(1 \text{km})^3} \\
 &\sim 10^{21} \text{kgm}^{-3}
 \end{aligned}$$

How does this density compare to maximum density of (say) nuclear matter? Let's compare.

$$\frac{m_n}{(1 \text{fm})^3} \sim \frac{10^{-27} \text{kg}}{10^{-45} \text{m}^3} \sim 10^{18} \text{kgm}^{-3}$$

We see a black hole is more dense than a nuclei. The characteristic size of a particle $1 \text{fm} \sim \Delta x \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar}{m_n c}$, due to Heisenberg's uncertainty principle, and also the Pauli exclusion principle.

More generally: density of material that forms black hole $\sim \frac{M}{R^3}$, but note $M = \frac{c^2 R}{2G}$ density $\rho \propto \frac{1}{R^2}$. This means that denser black holes are smaller.

Exercise: Estimate the strength of gravity $\frac{GM}{Rc^2}$ on Earth.

Example 2: The Universe is composed of 5% baryons + 25% dark matter + 70% dark energy. Estimate M and R.

$R \sim 10 \text{ Gpc}$

- Mass of baryons
 - 10^{11} stars in Milky Way

- $(10^4)^3$ galaxies in Universe
- $\Rightarrow M_{\text{baryons}} \sim 10^{23} M_{\odot} \sim 10^{53} kg$

$$\frac{GM_{tot}}{Rc^2} \sim \frac{10^{-10} 10^{53} 10}{10^{27} 10^{17}} \sim 1 \quad (2)$$

$$\begin{aligned} \text{Density } \rho &\sim \frac{M_{tot}}{R_{tot}^3} \sim \frac{c^2}{R^2 G}, \text{ use } \frac{GM}{Rc^2} \sim 1 \\ &\sim \frac{1}{G \times (\text{age of universe})^2} \end{aligned}$$

cf. critical density from Friedmann equations $\rho_{crit} = \frac{3H_0^2}{8\pi G}$.
Recall Hubble constant $H_0 \sim \frac{1}{\text{age}}$.

The critical density is the density of the universe at which expansion will asymptotically slow. Too dense leads to big crunch, too low leads to unbounded expansion.

Exercise: How do we reconcile a “flat” universe from critical density with the “curved” universe?