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PHYC90012 General Relativity

Course Summary

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   - Hence geometric objects moved on curved manifolds
   - Distances are not only spatial but temporal; need to use mathematics of small change = calculus
   - Uses the covariant derivative (a geometric object; independent of basis/coordinate independent)
   - This point of the course we will not be considering curved space, but instead only curvilinear coords
     - A flat space can be covered (represented?) by curved coordinates, but an intrinsically curved surface cannot be covered by flat coordinates
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   - Manifolds
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    - GPS, Mercury's orbit, gravitational lensing, gravitational redshift, ...
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    - Generation*
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      * = together these form the "antenna problem"
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      - We will derive this and the Friedman equations

1 Introduction to gravity

1.1 Strength of gravity

- Weak! Weakest of all fundamental forces
- Long-ranged force (like EM)
- Weakness determined by coupling constant
- Coupling constant = Newton's gravitational constant

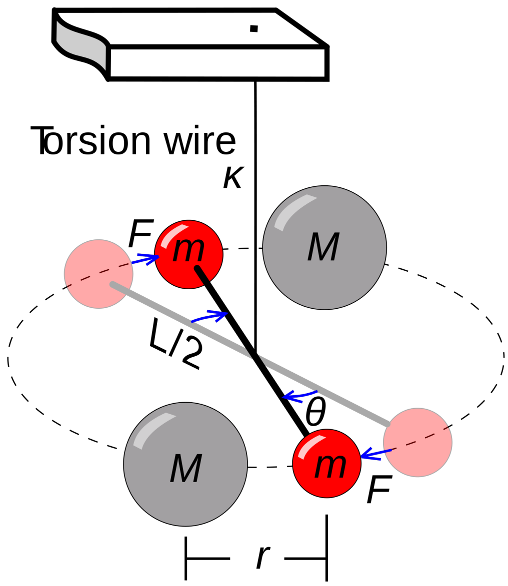
$$\vec{F} = \frac{Gm_1m_2}{r_{12}^2}\hat{r} \tag{1}$$

- G is hard to measure; least well known of coupling constants

In 1797-98, Cavendish used torsion balls (1.8m torsion balance) with rod of big masses and rod of small masses.

- Spring constant of torsion balance was measured from free oscillation

- then introduced 158kg balls
- measured deflection angle of balance ⇒ can calculate force
  - using a mini-telescope against Vernier scale
- rearrange Newton's law to get G



Exercise: Show that Cavendish also measured density of Earth as a bonus at the same time.

Mass of Earth $M_\oplus = \rho V$ where $V = \frac{4}{3}\pi R^3$ assuming the Earth is a sphere. How does calculating G also calculate ρ? Well, we have $\vec{F} = \frac{Gm_1 m_2}{r_{12}^2}\hat{r}$. Let's take $m_1 = M_\oplus$ as the mass of the Earth, and $m_2 = m$ as some small object mass. Let's imagine the smaller object falling to the center of the Earth. We'll take r_{12} as the distance from the object to the Earth's center, which we can approximate as Earth's radius, i.e. $r_{12} = R$. This force should be equivalent to $F = ma$.

So we have

$$\begin{aligned}
\frac{GM_\oplus m}{R^2} &= mg \\
\frac{G\rho\frac{4}{3}\pi R^3}{R^2} &= g \\
\frac{4G\rho\pi R}{3} &= g \\
\Rightarrow \rho &= \frac{3g}{4\pi GR} \\
&= \frac{3 \times 9.8\,\mathrm{ms^{-2}}}{4\pi \times 6.673\,84 \times 10^{-11}\,\mathrm{kg^{-1}m^3s^{-2}} \times 6370\,\mathrm{km}} \\
&= \frac{3 \times 9.8}{4\pi \times 6.67384 \times 10^{-11} \times 6370 \times 10^3}\mathrm{kg\,m^{-3}} \\
&= 5503\,\mathrm{kg\,m^{-3}}
\end{aligned}$$

	$\frac{GM}{Rc^2} \lll 1$	$\frac{GM}{Rc^2} \geq 1$
$v \ll c$	Newtonian	CAN'T EXIST
$v \sim c$	special rel.	full GR (difficult)

- Modern $G = 6.67384(80) \times 10^{-11}\,\mathrm{Nm^{-2}\,kg^{-2}} = \mathrm{kg^{-1}\,m^3\,s^{-2}}$
- Product GM is known to 1 part in $\sim 10^{10}$ from astrophysics observations

  $\Rightarrow$ mass is hard to measure gravitationally
- We need a dimensionless number to characterise strength
- Newton: $\Phi = \frac{GM}{r}$ (potential)
- In free fall: $\frac{KE}{mass}$, $v^2 \sim \frac{GM}{r}$
- We claim gravity is strong is free-fall is relativistic, i.e. $v \sim c$
- This is an order of magnitude estimate

1.2 Strong vs. weak gravity

- Quasi-Newtonian:
  - characteristic speed of body in free fall: $v^2 \sim \frac{GM}{r}$
- Strong gravity leads to relativistic free fall, i.e. $\frac{GM}{Rc^2} \geq 1$ where M is the total mass and R is th characteristic size

Example 1.1: $M = M_\odot$ (mass of the Sun)

$$\begin{aligned} R &\sim \frac{GM}{c^2} \quad \text{boundary of strong regime} \\ &\sim \frac{10^{-10}10^{30}}{10^{17}} \\ &\sim \text{km} \end{aligned}$$

cf. Schwarz radius of black hole = $\frac{2GM}{c^2}$

Example 1.2: Density of black hole with mass of $M_\odot$

$$\begin{aligned} &\sim \frac{M}{R^3} \sim \frac{10^{30}\,\text{kg}}{(\text{km})^3} \\ &\sim 10^{21} kgm^{-3} \end{aligned}$$

How does this density compare to maximum density of (say) nuclear matter? Let's compare.

$$\frac{m_n}{(1fm)^3} \sim \frac{10^{-27}kg}{10^{-45}m^3} \sim 10^{18}kgm^{-3}$$

We see a black hole is more dense that a nuclei. The characteristic size of a particle $1fm \sim \Delta x \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar}{m_n c}$, due to Heisenberg's uncertainty principle, and also the Pauli exclusion principle.

More generally: density of material that forms black hole $\sim \frac{M}{R^3}$, but note $M = \frac{c^2 R}{G}$ density $\rho \propto \frac{1}{R^2}$. This means that denser black holes are smaller.

Exercise: Estimate the strength of gravity $\frac{GM}{Rc^2}$ on Earth.

Example 2: The Universe is composed of 5% baryons + 25% dark matter + 70% dark energy. Estimate M and R.

$R \sim 10\,\text{Gpc}$

- Mass of baryons
  - 10^{11} stars in Milky Way
  - $(10^4)^3$ galaxies in Universe

  $\Rightarrow$ $M_{\text{baryons}} \sim 10^{23} M_\odot \sim 10^{53}\,\text{kg}$

$$\frac{GM_{tot}}{Rc^2} \sim \frac{10^{-10} \cdot 10^{53} \cdot 10}{10^{27} \cdot 10^{17}} \sim 1 \tag{2}$$

$$\begin{aligned}\text{Density } \rho &\sim \frac{M_{tot}}{R_{tot}^3} \sim \frac{c^2}{R^2 G}, \text{ use } \frac{GM}{Rc^2} \sim 1 \\ &\sim \frac{1}{G \times (\text{age of universe})^2}\end{aligned}$$

cf. critical density from Friedmann equations $\rho_{crit} = \frac{3H_0^2}{8\pi G}$.
Recall Hubble constant $H_0 \sim \frac{1}{\text{age}}$.

The critical density is the density of the universe at which expansion will asymptotically slow. Too dense leads to big crunch, too low leads to unbounded expansion.

Exercise: How do we reconcile a "flat" universe from critical density with the "curved" universe?

Important to remember: gravity is strong when $\frac{GM}{Rc^2} \sim 1$, which occurs around black holes, the universe at large. In a sense, cosmological results such as critical density, expansion of universe come from this.

1.3 Black hole oscillations

We can estimate the oscillation frequency of a "black hole" (i.e. something with $\frac{GM}{Rc^2} \sim 1$ as $\sim \frac{c}{R}$; that is, the time is takes light to travel the distance of the object. This is the natural frequency for this object. Using that ubiquitous expression we can express the oscillation frequency as $\sim \frac{c^3}{GM}$, e.g. $M = M_\odot \Rightarrow$ frequency $\sim 10\,\text{kHz}$.

Let's discuss charged black holes. It is difficult to astrophysically have charged black holes, because stars are not usually charged (due to the strength of the EM force, which would attract opposite charge and cancel out). So, these are artificial in nature. These have unusual geometry, and are called "Reissner-Nordstrom" black holes.

Example : What is the maximum charge on a black hole?

$$\frac{Q^2}{4\pi\epsilon_0 R} \leq \frac{GM^2}{R}$$
$$Q \leq (4\pi\epsilon_0 G)^{1/2} M$$

Above, we relate Coloumb force to gravitational force. The gravitational force holding a black hole together must overcome the Coloumb force pushing it apart.

1.4 Quantum Gravity

The problem with quantum gravity is that there is no theory... hence we must rely on numerology.

We consider a hypothetical elementary excitation of a "black hole" (again, we mean a *relativistic compact object*) of mass M. Hence the characteristic size, or "wavelength", of the excitation is $\frac{GM}{c^2}$ (fundamental excitation only). Introducing quantum mechanics: the Heisenberg uncertainty principle tells us that the zero-point motion associated with this excitation is

$$\lambda \sim \frac{\hbar}{\Delta p} \sim \underbrace{\frac{\hbar}{Mc}}_{\text{relativistic}} \tag{3}$$

Equating length scales $\Rightarrow M_{pl} \approx \left(\frac{\hbar c}{G}\right)^{1/2}$; this is the Planck mass, about $10^{-8}\,\text{kg}$ (the mass below which quantum gravity is important.

Given M_{pl} we get $\lambda \sim \frac{\hbar}{M_{pl}c} \sim 10^{-33}\,\text{m}$; the Planck length - the length where quantum gravity is important (e.g. just after Big Bang).

1.4.1 Hawking Radiation

Let's return to our elementary excitation with $\lambda \sim \frac{GM}{c^2}$, i.e. frequency $\sim \frac{c^3}{GM} = \frac{c}{\lambda}$. Heisenberg tells us there is an associated energy fluctuation $\Delta E \sim h \times \text{frequency} \sim \frac{\hbar c^3}{GM}$. Suppose (note: this is a huge leap) energy fluctuation in the black hole system is in thermal equilibrium with a bath at temperature T. Then $T \sim \frac{\Delta E}{k_B}$. This associates a temperature to a black hole.

We call a black hole a blackbody!

$$\begin{aligned}
\text{Radiated power } &= k_B \times \text{area} \times T^4 \\
&= \sigma \times \underbrace{R^2}_{\left(\frac{GM}{c^2}\right)^2} \times \left(\frac{\hbar c^3}{GMk_b}\right)^4 \\
&\propto M^{-2}
\end{aligned}$$

Exercise: Plug in numbers to this!

This shows that a black hole radiates energy $\Rightarrow$ eventually a black hole evaporates. We can estimate the time scale of this evaporation.

$$\text{time scale} \sim \frac{Mc^2}{\text{power}} \propto M^3 \tag{4}$$

$\Rightarrow$ small black holes evaporate fast!

As an aside, we could consider the rate of energy accretion. For system outside a black hole with uniform density ρ_{out}, we have

$$\begin{aligned}
\text{rate of mass accretion} &\sim \rho_{\text{out}} \cdot c \cdot 4\pi R^2 \\
\text{rate of energy accretion} &\sim c^2 \cdot \text{rate of mass accretion}
\end{aligned}$$

A short review:

- there is no theory of quantum gravity
- we consider a relativistic elementary oscillation $\lambda \sim \frac{GM}{c^2}$
- we use Heisenberg to relate this to an energy fluctuation in the system
- energy fluctuation can be converted to a temperature $T_H = \frac{\hbar c^3}{GMk_B}$, the Hawking temperature
- we find small block holes evaporate more quickly than large black holes

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1.5 Black hole thermodynamics

1st law: $dS = \frac{dQ}{T}$ system constant volume

Hawking radiation: we lose a bit of heat dQ due to blackbody radiation.

$$dS = \frac{GMk_b}{\hbar c^3} d(Mc^2) \tag{5}$$

We see heat loss comes from rest energy

$$\frac{dS}{k_B} \approx \frac{1}{R^2_{\text{Planck}}} \underbrace{d(R^2)}_{R\sim\frac{GM}{c^2}} \tag{6}$$

This result relates the entropy of a black hole to its area; Bekenstein-Hawking entropy - $S_{\text{black hole}} \propto$ area of event horizon.

$$\frac{S}{k_B} = \frac{\text{area}}{4R^2_{\text{Planck}}}{}^1 \tag{7}$$

However, there is a contradiction! Hawking radiation implies that $dA < 0 \Rightarrow dS < 0$.... this is bad.[2] One way to resolve this is by making a generalised **2nd law**:

$$d\left(S_{\text{outside}} + \frac{\text{area}}{4R^2_{\text{Planck}}}\right) \geq 0 \tag{8}$$

Unfortunately this is not enough - there is still a contradiction. Consider a small box of radiation which we prepare far from a black hole. $S_{\text{box}} = \frac{4U_{\text{box}}}{3T_{\text{box}}}$. We can make photons very long wavelength, so that $U_{\text{box}} \approx 0$ but $S_{\text{box}} \neq 0$. Then $dS_{\text{out}} = -S_{\text{box}} < 0$. $d(\text{area}) = 0$ because energy in box $= 0$, i.e. $d(Mc^2) = 0$.[3]

Exercise: Resolve the box paradox!

3rd law of BH thermodynamics: can't reduce T to zero.[4]

In heating a rubber band, it shrinks; this is because a shrunken arrangement of the molecules is a state of more entropy (more disordered)

BH

2 Einstein equivalence principle

We will define this (the weak and strong equivalence principles), and some of the tests been performed.

[1]In 1995, Maldacena also got this by counting microstates

[2]See Ted Jacobson's lecture at University of Utrecht for a discussion on this.

[3]Beware the Unruh radiation.

[4]If you are curious: Verlinde, arXiv:1001.0785, *On the Origin of Gravity and the Laws of Newton*, in which the idea of emphgravity is an entropic force is discussed.

2.1 Weak equivalence principle

Trajectory of body in free fall is independent of its mass and $\underbrace{\text{composition}}_{\text{binding energy}}$ (as per Galileo, feather vs. brick). Note that this is not equivalent for electric fields (the movement of a charged particle through an electric field depends on its charge).

2.2 Strong equivalence principle

1. weak equivalence principle is valid
2. results of any non-gravitational (e.g. EM, not Cavendish expt) experiment is independent of velocity of freely falling frame
   - this is *local Lorentz invariance*
3. results of non-gravitational experimental are independent of where and when it is performed
   - this is *local position invariance*

The Einstein equivalence principle (EEP) $\equiv$ strong implies: existence of a "curved spacetime" with:

i. symmetric metric

ii. trajectories of free-falling bodies are geodesics of metric

iii. the laws of physics in a freely-falling frame can be written in the language of special relativity

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A short review:

Einstein equivalence principle:

1. universality of free fall
2. local Lorentz invariance
3. local position invariance

Implications:

3) $\Rightarrow$ fundamental constant independent of $\vec{x}$, t

2) $\Rightarrow$ laws of non-gravitational physics locally independent of frame

1) $\Rightarrow$ space is curved

Why is space curved? Locally straight trajectory in free yet yet gravity produces curved trajectory (observed) $\Rightarrow$ only possible if coordinates change from one point to next

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3 Experimental tests

3.1 Experimental tests of free fall

Inertial mass $m_i = \frac{\text{applied non-gravitational force}}{\text{measured acceleration}}$

Gravitational mass $m_g =$ "passive" mass appearing in weight

Look for $m_i \neq m_g$

Write

$$m_g = m_i + \sum_{\text{interactions A in body}} \eta^A \frac{E^A}{c^2} \tag{9}$$

Here $E =$ "binding energy"/potential energy of interaction A.

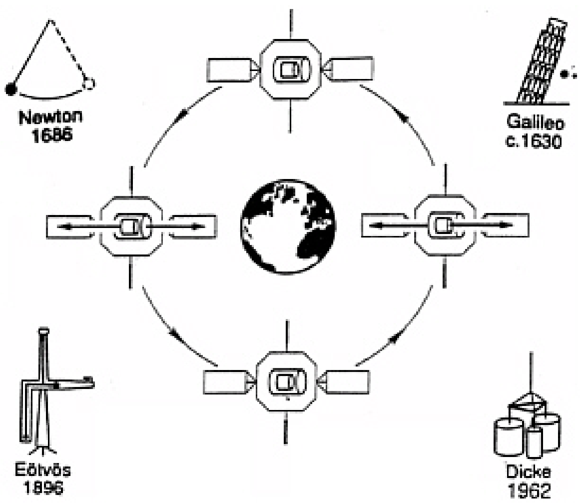
Tests:

1. Eötvös-type torsion balance experiments: two different materials may fall at difference rates (see Dicke, Braginsky)
2. Colorado: U, Cu laser interferometer $\Rightarrow relativiteacceleration$
3. Eöt-Wash experiments: fancy version of 1)

Result:

$$\frac{|m_g - m_i|}{m_i} \leq 10^{-13} \tag{10}$$

We test this, for example, with aluminium (Al) and gold (Au) weights on a torsion balance. As the Sun moves from one side of the Earth to the other, if the gravitational mass of either differs from the other we will see diurnal oscillation in the balance.



3.2 Tests of local Lorentz invariance

- Michelson-Morley experiment (the aether)
- Rossi-Hall tests for lifetime of muons (time dilation $\Leftrightarrow$ LLI)
- Ives-Stiwell transverse Doppler shift
  - laser travelling some vector $\vec{v}$
  - we see perpendicular wave vector $\vec{k}$
  - measure frequency when laser intersects line of sight
  - Doppler shift arises due to time dilation

Mathematically:

$$\begin{bmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{bmatrix} = \begin{bmatrix} \gamma & \gamma v/c & 0 & 0 \\ \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega/c \\ 0 \\ k_y \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma\omega/c \\ \gamma\omega v^2/c \\ k_y \\ 0 \end{bmatrix} \tag{11}$$

Exercise: Compare with standard longitudinal Doppler:

$$\begin{pmatrix} \text{same} \\ \text{matrix} \end{pmatrix} \begin{pmatrix} \omega/c \\ k_x \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

A short review:

Last lecture:

- tests of LPI
- Schiff's thought experiment (gravitational redshift)

3.3 Gravitational redshift

$$\frac{h\nu' - h\nu}{h\nu} = \frac{(m_A g_A - m_B g_B)H}{(m_A - m_B)c^2} \tag{13}$$

If $g_A = g_B$ (universal free-fall), then gravitational redshift is gH/c^2

3.4 Observation of GR in experiments

There are situations where GR makes measurable difference today

- Pound-Rebka experiment (gravitational redshift, also seen on white dwarf spectral lines)
- GPS
- 2015 discovery of gravitational waves from binary BH merger
- cosmological measurements (CMB, redshifts, H_0, ...)
- gravitational lensing (light bent by a mass) (see: Einstein cross, Abell clusters, stars behind Sun)
- precession of perihelion of Mercury
- orbital decay of Hulse-Taylor binary pulsar (can calculate to mm precision the decay of the orbit every 8 hours due to gravitational wave emission)
- Nordtredt/lunar ranging experiments
- Gravity Probe B - lense-thinning precession
- Shapiro time delay

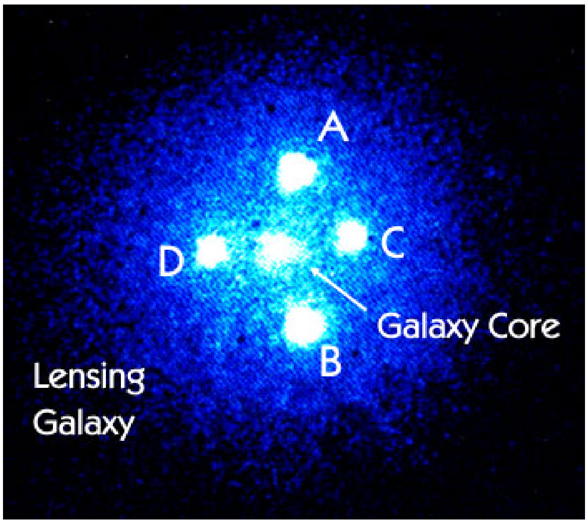


Figure 1: Einstein cross

3.4.1 Pound-Rebka experiment

Looks at gravitational redshift of 14.4 keV gamma rays from ^{57}Fe decay

^{57}Fe decays and emits photons directly down from a 23 m tall tower, to another box of ^{57}Fe below. Gravitational redshift occurs; time moves slower closer to the Earth, so gamma ray emitted will have a different energy than that required to be absorbed by the ^{57}Fe at the bottom.

Receiver box moves up/down at speed v. We adjust v at bottom so that kinematic Doppler shift $\propto \left(\frac{1-v/c}{1+v/c}\right)^{1/2}$ exactly cancels the gravitational redshift $\propto gH/c^2$

N.B.: recoil (in a random direction) when photon emitted/absorbed; energy $E_R = \frac{E_\gamma}{2M_{Fe}c^2}$

$$E_R \sim \frac{14.4\,\mathrm{keV}}{100\,\mathrm{GeV}} \sim 10^{-7} \tag{14}$$

We solve this problem through the Mossbauer effect: use whole crystal; whole crystal ($\sim 10^{23}$ Fe atoms) recoils

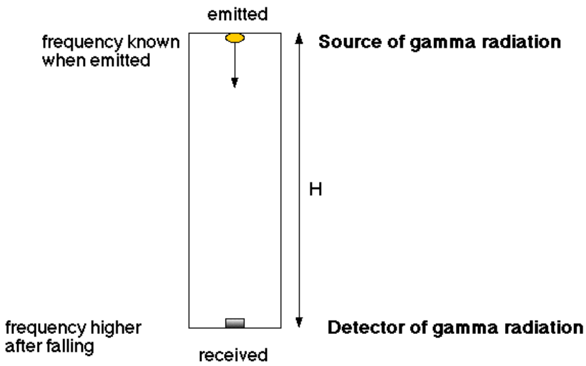


Figure 2: Pound-Rebka experiment

3.4.2 Lunar ranging

Williams + Dickey (2002)[5]

Moon not completely dormant

- fluid core (?)
- tidal dissipation internally
- etc ...

Multiple radar reflectors to improve accuracy. We search for an accurate Earth-Moon orbit versus time; we find

$$\frac{1}{G}\frac{dG}{dt} = (0.0 \pm 1.1) \times 10^{-12}\,\mathrm{yr}^{-1} \tag{15}$$

Uncertainty $\approx 0.02H_0$ - we don't see increasing separation between Earth and Moon due to expansion of universe. This is expected, however: gravitational bound objects do not separate with time, although the energy they require to stay bound will increase

[5]See also "Living Reviews" Relativity

3.4.3 Deflection of light (gravitational lensing)

In a three-body system with the Earth, the Sun and another star, the light from the star will bend due to the Sun before it reaches Earth (not taking a straight path). This causes the apparent position of the star to be different than the actual position.

Deflection angle $\delta\theta \propto \frac{GM}{c^2 d}$

GR deflection = $2\times$ Newtonian deflection (Einstein 1911).

This effect is achromatic!

3.4.4 Shapiro delay

Roundtrip time from Earth to a distant mirror (in a three-body system including the Sun) is longer than if the Sun was not there.

$$\delta t \propto \frac{GM}{c^3} \ln(\text{geometric factors}) \tag{16}$$

Best measurements with Cassinin spacecraft (accuracy 1 in 10^5)

4 Geometric Objects

4.1 Vectors

Invariants are measurable. Therefore, each coordinate representation is as valid as the other!

4.1.1 Definitions of a vector

Vector $\vec{A}$

- four numbers that are projections onto spacetime (can be dependent or independent of a metric)
  - Note: vectors can have meaning without a metric
- a geometric object with "length" (requires a metric) and a "direction", i.e. an arrow
- a geometric object which transforms from one coordinate system to another
- a linear function which takes a 1-form as an argument and returns a real number

$\underbrace{\text{Vector spaces}}_{\text{vectors}} \supseteq \underbrace{\text{normed vector spaces}}_{\text{length}} \supseteq \underbrace{\text{inner product spaces}}_{\text{angle}}$

$\Rightarrow$ you can have vector spaces without a norm or angle concept.

Exercise: $\vec{\Delta}x$ = displacement (in spacetime), is a vector

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Vector components, A^0, A^1, A^2, A^3; in another frame we have $A^{0'}, A^{1'}, A^{2'}, A^{3'}$.

4.1.2 What is a basis vector?

Projection requires a metric (for a dot product)! $\Rightarrow$ we have 4, special, linearly independent vectors which "point along" (no concept of length/angle) axes of coordinate system, $\{\vec{e}_\alpha\}$

$$\vec{A} = A^0\vec{e}_0 + A^1\vec{e}_1 + A^2\vec{e}_2 + A^3\vec{e}_3 \tag{17}$$

As $\vec{e}_0$ is itself a vector, using 17

$$\vec{e}_0 = (\vec{e}_0)^0\vec{e}_0 + (\vec{e}_0)^1\vec{e}_1 + (\vec{e}_0)^2\vec{e}_2 + \underbrace{(\vec{e}_0)^3}_{\text{3rd component of } \vec{e}_0}\vec{e}_3 \tag{18}$$

By linear independence, as $\vec{e}_0$ is linearly independent to $\vec{e}_1, \vec{e}_2, \vec{e}_3$

$$\Rightarrow (\vec{e}_0)^1 = (\vec{e}_0)^2 = (\vec{e}_0)^3 = 0 \tag{19}$$

$$\Rightarrow (\vec{e}_\alpha)^\beta = \delta^\beta_\alpha \tag{20}$$

which is *true in all coordinate systems.* Therefore in a primed frame,

$$\Rightarrow (\vec{e}_{\alpha'})^{\beta'} = \delta^{\beta'}_{\alpha'} \tag{21}$$

<u>But</u> $(\vec{e}_{\alpha'})^\beta \neq \delta^\beta_{\alpha'}$

- $\vec{e}_{\alpha'}$ is tied to the prime frame
- the geometric object is only defined with respect to the frame (where $\vec{A}$ could exist in all frames)
- you cannot measure a basis vector in another coordinate frame. You have to be in the coordinate frame to measure it.

4.1.3 Transformations

In general,

$$\left\{x^{\beta'}\right\} \mapsto \{x^\alpha\}$$
$$\text{from } \sigma' \to \sigma \quad \text{primed to unprimed}$$

$$\Lambda^\alpha_{\beta'} = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(x^{0'}, x^{1'}, x^{2'}, x^{3'})} = \frac{\partial x^\alpha}{x^{\beta'}} \quad \text{16 elements!} \tag{22}$$

If the transform is linear (e.g. Lorentz transform), we can consider

$$x^\alpha = \Lambda^\alpha_{\beta'} x^{\beta'} \tag{23}$$

where Λ^α is not necessarily a constant. But if the transform is non-linear, the left side is wrong. We instead have

$$dx^\alpha = \Lambda^\alpha_{\beta'}\ dx^{\beta'} \tag{24}$$

Why? If we define $\vec{x}$ as a displacement from origin to the point in a curved space, there are multiple different path. We would need more than 4 numbers to define this, therefor we no longer have a meaningful vector.

Vectors are defined localling an tangent space, **and** transformations of vectors are also restricted locally to tangent space. If $\vec{A}$ is a vector,

$$A^\alpha = \Lambda^\alpha_{\beta'} A^{\beta'} \tag{25}$$

A short review:

Vectors

$\vec{A}$: defined in tangent space

Components A^α: $\vec{A} = A^\alpha \vec{e}_\alpha$ where $\vec{e}_\alpha$ are basis vectors. These (unprimed) basis vectors only have meaning in unprimed coordinates

Transform like infinitesimal displacements:

$$\underbrace{A^{\alpha'}}_{\text{primed}} = \overbrace{\frac{\partial x^{\alpha'}}{\partial x^\alpha}}^{\text{transform matrix}} \underbrace{A^\alpha}_{\text{unprimed}} \tag{26}$$

How do basis vectors transform?

$$\begin{aligned} A^\alpha \vec{e}_\alpha = \vec{A} &= A^{\alpha'} \vec{e}_{\alpha'} \\ &= \frac{\partial x^{\alpha'}}{\partial x^\beta} A^\beta \vec{e}_{\alpha'} \\ &= \frac{\partial x^{\alpha'}}{\partial x^\alpha} A^\alpha \vec{e}_{\alpha'} \end{aligned}$$

This has to be true for all $\vec{A}$, i.e.

$$\vec{e}_\alpha = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \vec{e}_{\alpha'} \tag{27}$$

Or equivalently,

$$\vec{e}_{\alpha'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} \vec{e}_\alpha \tag{28}$$

Note this is the opposite of transformation law for vector components 26

Exercise: Try for Lorentz transformations.

Exercise: (later) Try for 2 types of Schwarz black hole coordinates.

4.2 1-forms

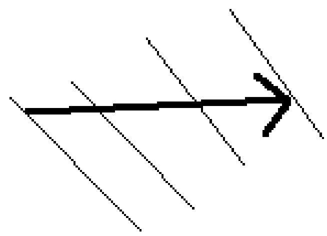
$\widetilde{p}$

- four numbers associated to four dimensions and associated to vectors in a specific way (see below)
- geometric object that transforms like a gradient
- tensor of type $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, i.e. linear function which accepts vector as an argument and returns a real number

If we have a metric, a neat way to visualise a 1-form is as a contour.

e.g. vector is an arrow (length, direction)

cf. 1-form is contours



Value of $\widetilde{p}(\vec{A}) = 4$; number of times $\vec{A}$ pierces surfaces of $\widetilde{p}$

Components defined to be

$$\begin{aligned} p_\alpha &= \widetilde{p}(\vec{e}_\alpha) \\ &= \langle \widetilde{p}, \vec{e}_\alpha \rangle \end{aligned}$$

From lines 1 to 2, we have equivalent notation. We are using an inner product. By convention we use a lowered index.

We cannot have curved contours, as they are only defined in tangent space.

	Vectors $\vec{A} = A^\alpha \vec{e}_\alpha$	**1-forms** $p_\alpha = \widetilde{p}(\vec{e}_\alpha)$
Transformation for coordinates	$A^{\alpha'} = \dfrac{\partial x^{\alpha'}}{\partial x^\alpha} A^\alpha$	$p_{\alpha'} = \dfrac{\partial x_{\alpha'}}{\partial x_\alpha} p_\alpha$
Transformation for basis vectors	$\vec{e}_{\alpha'} = \dfrac{\partial x_{\alpha'}}{\partial x_\alpha} \vec{e}_\alpha$	$\widetilde{\omega}^{\alpha'} = \dfrac{\partial x^{\alpha'}}{\partial x^\alpha} \widetilde{\omega}^\alpha$

We have a neat way to calculate the coordinate-independent quantity $\widetilde{p}(\vec{A})$

$$\begin{aligned} \widetilde{p}(\vec{A}) &= \widetilde{p}(A^\alpha \vec{e}_\alpha) \\ &= A^\alpha \widetilde{p}(\vec{e}_\alpha) \quad \text{linear} \\ &= A^\alpha p_\alpha \end{aligned}$$

This is a contraction. It is an inner product but not dot product (we require a metric, and for it to be between two vectors, for a dot product).

Vectors and 1-forms are "apples and oranges" - completely different geometric objects which cannot be compared even at the same point, e.g. $\widetilde{p} = \vec{A}$ is meaningless.

This is comparable to bras and kets in quantum mechanics; they form a dual space, but we cannot directly compare the objects of a pair

4.2.1 How do 1-forms transform?

We start with basis vectors

$$\vec{e}_\alpha = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \vec{e}_{\alpha'} \tag{29}$$

In component form:

$$\begin{aligned} p_{\alpha'} &= \widetilde{p}(\vec{e}_{\alpha'}) \\ &= \frac{\partial x^\alpha}{\partial x^{\alpha'}} \widetilde{p}(\vec{e}_\alpha) \quad \text{linear} \\ &= \frac{\partial x^\alpha}{\partial x^{\alpha'}} p_\alpha \end{aligned}$$

i.e. components of $\widetilde{p}$ transform like basis vectors (not components of basis vectors)

A short review:

Last lecture missing

1-forms and their relation to gradients defined

4.3 1-forms, basis 1-forms and gradients

A short review:

Given a scalar field $\phi(\vec{x})$ we define a 1-form $\widetilde{d\phi}$, with components $(\widetilde{d\phi})_\alpha = \frac{\partial \phi}{\partial x^\alpha}$

Notes:

1. If we take $\phi =$ one of the coordinates, e.g. $\phi = x^\alpha$ for a specific α, then $\widetilde{d\phi} = \delta^\alpha_\beta$ for $\phi = x^\alpha$
   - i.e. $\widetilde{d\phi}$, for $\phi = x^\alpha$ is just $\widetilde{\omega}^\alpha$, the basis 1-form
2. $\widetilde{d\phi}$ is the most natural definition of a normal in GR. We also has $\widetilde{d\phi}(\vec{t}) = 0$ if $\vec{t}$ is a tangent vector to the level surfaces of ϕ; don't need a metric
3. How do we construct components of vectors?
   - Components of 1-forms: $p_\alpha = \widetilde{p}(\vec{e}_\alpha)$
   - Dual: components of vectors: $A^\alpha = \vec{A}(\widetilde{\omega}^\alpha)$
   - Note that we do not need 1-forms to define vectors (and vice versa), but once we start discussing components, we do need them

---

4.4 Tensors

$\binom{M}{N}$ tensor is a linear function operating on M 1-forms and N vectors to return a real number.

For example if R is a $\binom{1}{1}$ tensor then its components are $R^\alpha_\beta = R(\widetilde{\omega}^\alpha, \vec{e}_\beta)$.

You can build big tensors out of little ones in several ways (and vice versa, e.g. via contraction). One important way is the outer product;

e.g. outer product of vectors $\vec{A}$ and $\vec{B} = \binom{2}{0}$ tensor; $T = \vec{A} \otimes \vec{B}$

Takes two 1-forms as arguments: $T(\widetilde{p}, \widetilde{q}) \equiv^{\text{def}} \vec{A}(\widetilde{p})\vec{B}(\widetilde{q})$ (this is multiplying two scalar numbers)

- Note: outer product not commutative.

$$\text{If } S = \vec{B} \otimes \vec{A} \text{ then } S(\widetilde{p}, \widetilde{q}) = \vec{B}(\widetilde{p})\vec{A}(\widetilde{q}) \neq T(\widetilde{p}, \widetilde{q}) \tag{30}$$

- Can't write a general $\binom{M}{N}$ tensor as an outer product necessarily, but can write as a linear combination of outer products

$$\underbrace{T}_{\text{geometric object}} = T^{\alpha_1,\ldots,\alpha_M}_{\beta_1,\ldots,\beta_N} \vec{e}_{\alpha_1} \otimes \ldots \otimes \vec{e}_{\alpha_M} \otimes \widetilde{\omega}^{\beta_1} \otimes \ldots \otimes \widetilde{\omega}^{\beta_N} \tag{31}$$

- even though we use M 1-forms and N vectors, we take the outer product over N 1-forms and M vectors

Exercise: Show that you get the components of T when you evaluate T for M basis 1-forms and N basis vectors.

Another way to generate new tensors:

$$\text{Symmetric part of T } (T^{(\alpha\beta)})\text{: } T^{(\widetilde{p},\widetilde{q})}_{(\text{sym})} = \frac{1}{2}T(\widetilde{p},\widetilde{q}) + \frac{1}{2}T(\widetilde{q},\widetilde{p})$$
$$\text{Antisymmetric part of T } (T^{[\alpha\beta]})\text{: } T^{(\widetilde{p},\widetilde{q})}_{(\text{sym})} = \frac{1}{2}T(\widetilde{p},\widetilde{q}) - \frac{1}{2}T(\widetilde{q},\widetilde{p})$$

This example is for $\binom{2}{0}$ but can generalise to $\binom{M}{N}$

4.4.1 Metric tensor

$\binom{0}{2}$ tensor which accepts two vectors and returns a number which we call scalar or dot product.

$$g(\vec{A},\vec{B}) \equiv^{def} \vec{A}\cdot\vec{B} \qquad \text{this is \underline{not} contraction} \tag{32}$$

Clearly bilinear

What is g? Anything we like! It depends on the coordinates we choose to use! (And it tells us how lengths, angles, ... are measured in our favourite coordinates)

Components:

$$g_{\alpha\beta} = g(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta \tag{33}$$

i.e. 16 numbers saying how the basis vectors relate to each other in any given coordinate system

What is $g(\Delta\vec{x},\Delta\vec{x})$ ($\Delta\vec{x}$ is a displacement vector)?

$$\begin{aligned} g(\Delta\vec{x},\Delta\vec{x}) &= g(\Delta x^\alpha \vec{e}_\alpha, \Delta x^\beta \vec{e}_\beta) \\ &= g_{\alpha\beta}\Delta x^\alpha \Delta x^\beta \qquad \text{linear} \end{aligned}$$

<u>This quantity is the spacetime interval in curved space. Invariant!</u>

A short review:

Metric: $\binom{0}{2}$ tensor $g(\vec{A},\vec{B} \equiv \vec{A}\cdot\vec{B}$

Components:

$$g_{\alpha\beta} = g(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta \tag{34}$$

---

e.g. Minkowski metric

$$\{g_{\alpha\beta}\} = \text{diag}(-1,1,1,1) \tag{35}$$

e.g. Schwarzschild black hole of mass M (we will prove this later)

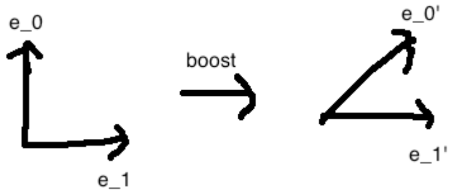
$$\{g_{\alpha\beta}\} = \left[-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, \ r^2, \ r^2\sin^2\theta\right] \tag{36}$$

where r is the radial coordinate.

Vectors $\vec{A}$ and $\vec{B}$ are orthogonal vectors if $g(\vec{A}, \vec{B}) = 0$. But orthogonal doesn't necessarily mean perpendicular.

These include 4D vectors of "zero length":

- in Minkowski, momentum of photon $\vec{p} = \left(\frac{E}{c}, \widetilde{p}\right)$ in flat space, and we know $E = |\underline{\mathrm{p}}|c$ so $g(\vec{p}, \vec{p}) = -\frac{E^2}{c^2} + |\underline{\mathrm{p}}|^2 = 0$.
  - i.e. $\vec{p}$ is orthogonal with itself; has zero "length" but obviously <u>not</u> a zero vector, nor is it perpendicular to itself
- in Minkowski: boost between frames



  - the boost vectors are not perpendicular, but we still have $\vec{e}_{0'} \cdot \vec{e}_{1'} = 0$

4.5 Correspondence between vectors and 1-forms

Let g be a metric, and let $\vec{V}$ be an arbitrary vector. Then $g(\vec{V}, ...)$ has one argument free, and that argument is a vector, so $g(\vec{V}, ...)$ must be a 1-form. We call it $\widetilde{V}$ because it's the 1-form associated with $\vec{V}$. What are its components?

$$\begin{aligned} V_\alpha \equiv^{def} \widetilde{V}(\vec{e}_\alpha) &= g(\vec{V}, \vec{e}_\alpha) \\ &= g(V^\beta \vec{e}_\beta, \vec{e}_\alpha) \\ &= V^\beta g(\vec{e}_\beta, \vec{e}_\alpha) \quad \text{by linearity} \\ &= V^\beta g_{\beta\alpha} \quad \text{by definition} \end{aligned}$$

This is the "lowering the index" operation of yesteryear! The old language for this is: lower a contravariant index, get a covariant index.

This can go the other way as long as g is invertible. <u>Define</u> $g^{\alpha\beta}$ as 16 numbers you get if you treat $g_{\alpha\beta}$ as a matrix and invert it.

$$\text{i.e.,} \quad g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma \tag{37}$$

Exercise: What is the geometric object whose components are $g^{\alpha\beta}$? Hint: $\binom{2}{0}$ tensor which takes two 1-forms as arguments.

We check that “raising the index” on V_α gets us back to $\vec{V}$.

$$V_\alpha \underbrace{g^{\alpha\gamma}}_{\text{inverse metric}} = V^\beta g_{\beta\alpha} g^{\alpha\beta} = V^\beta \delta^\gamma_\beta = V^\gamma \tag{38}$$

More generally:

$$g \text{ maps } \binom{M}{N} \text{ tensor to } \binom{M-1}{N+1} \text{ tensor}$$
$$g^{-1} \text{ maps } \binom{M}{N} \text{ tensor to } \binom{M+1}{N-1} \text{ tensor}$$

Physics example: consider a photon in curved (Schwarzschild) spacetime.

We know $\vec{p}\cdot\vec{p} = 0$ in the local freely falling frame(from weeks 1,2: Michelson-Morley experiment), and $\vec{p}\cdot\vec{p}$ is invariant $\Rightarrow \vec{p}\cdot\vec{p} = 0$ in all coordinate systems.

In Schwarzschild spacetime: radially in-falling photon $\vec{p} = \left(p^0, p^r, 0, 0\right)$

$$\begin{aligned} 0 &= g_{00}(p^0)^2 + g_{11}(p^1)^2 \\ &= -\left(1 - \frac{2M}{r}\right)(p^0)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(p^r)^2 \end{aligned}$$

i.e.

$$\frac{p^r}{p^0} = \left(1 - \frac{2M}{r}\right)^{-1} \neq 1$$
$$(\text{phase speed of light})^{-1} \neq 1$$

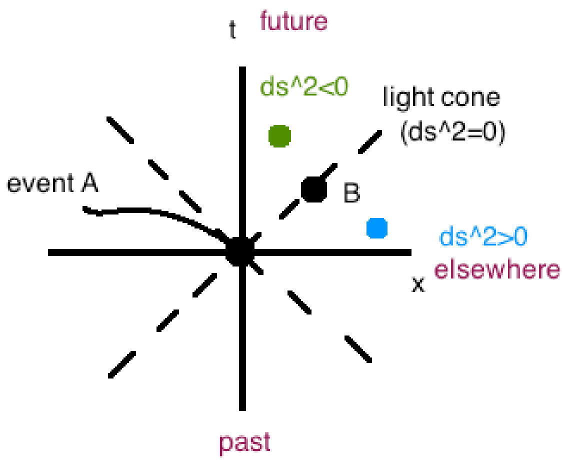
5 Kinematics

In general, for kinematics we do not care about the origin of the force, and only analyse the motion of the object. This is in contrast to dynamics, where we investigate where a force originates. In a GR context, for kinematics this means that we will look into the motion of an object while considering the spacetime, while for dynamics we also investigate what causes the curvature of spacetime.

Event: $\vec{x} = (t, x, y, z)$

Spacetime interval between 2 events: $ds^2 = g(d\vec{x}, d\vec{x})$

e.g. Minkowski

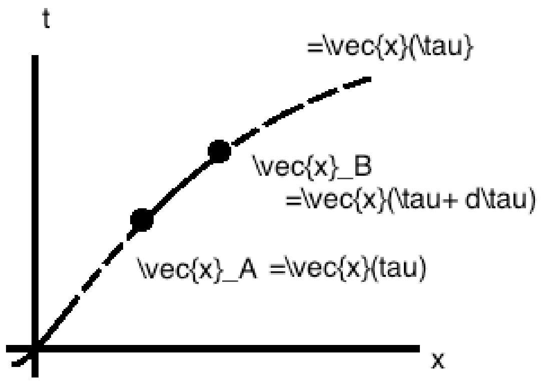


$ds^2 = 0$	null ray; defines light cone which joins events A and B	(39)
$ds^2 > 0$	events A and B and mutually spacelike; can't event get from A to B; there always exists an inertial frame such that events occur at different spacial locations at the same time	(40)
$ds^2 < 0$	events are timelike; can get from A to B; frame exists where events occur at same spatial location at different times	(41)

For $ds^2 > 0$, we will always be able to Lorentz transform into a frame where two events occur at the same time, but spatially separated. For $ds^2 < 0$, there will always be a Lorentz transform such that the two events occur at the same spatial location, but temporally separated.

Proper time: two events A and B separated infinitesimally along some 4D trajectory.

$$d\tau \equiv^{def} (-ds^2)^{1/2} \tag{42}$$



τ is an affine parameter (= label of path) with a particular normalisation, namely that τ tracks passage of time for an observer at rest with respect to the sequence of events defined by $\vec{x}(\tau)$.

5.1 4-velocity

From above:

$$-d\tau^2 = ds^2 = d\vec{x} \cdot d\vec{x}$$
$$\Rightarrow -1 = \frac{d\vec{x}}{d\tau}\frac{d\vec{x}}{d\tau}$$

where the dot product is with respect to the metric.

Call $\vec{u} = \dfrac{d\vec{x}}{d\tau}$ the 4-velocity. It is timelike, and $\vec{x} \propto d\vec{x}$. It relies on a geometric object (the sequence of events $\vec{x}(\tau)$) for its meaning. And if τ is the label, we have $\vec{u} \cdot \vec{u} = -1$ as normalisation.

5.1.1 Example 1: Momentarily comoving reference frame

For example, consider the momentarily comoving reference frame.

$$\frac{d\mathbf{x}}{dt} = 0 \qquad \therefore \vec{u} = (u^t, 0, 0, 0) \tag{43}$$

By definition $u^t = \frac{dt}{d\tau}$. But what is it? Normalisation gives

$$-1 = g_{tt}(u^t)^2 + 0$$
$$u^t = \left(-\frac{1}{g_{tt}}\right)^{1/2}$$

In flat space, $g_{tt} = -1$ and $u^t = 1$.

In Schwarzschild space, $g_{tt} = -\left(1 - \frac{2M}{r}\right)$ and $u^t = \left(1 - \frac{2M}{r}\right)^{-1/2}$. Recall that $u^t \equiv \frac{dt}{d\tau}$. So as $r \to 2M$, $\frac{dt}{d\tau} \to \infty$. Physically, this says that time slows down as we approach a black hole. $d\tau$ would be the clock nearby the black hole, while dt would be the clock on a faraway observer. They see an infinite amount of ticks on their clock, before the black hole clock ticks even once.

5.1.2 Example 2

Consider coordinates in which our body moves with speed $\mathbf{V} = \frac{d\mathbf{x}}{dt}$ (in, for example, the x-direction).

$$\vec{u} = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, 0, 0\right)$$
$$= \left(\frac{dt}{d\tau}, V\frac{dt}{d\tau}, 0, 0\right) \quad \text{chain rule}$$

By normalisation,

$$-1 = \vec{u} \cdot \vec{u} = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{tx}\left(\frac{dt}{d\tau}\right)^2 V + g_{xx}V^2\left(\frac{dt}{d\tau}\right)^2 \tag{44}$$

Special cases (note $c = 1$):

$$\begin{aligned}
\text{(a) Minkowski:}\quad & g_{tt} = -1 \quad g_{tx} = 0 \quad g_{xx} = 1 \\
& \therefore -1 = -\left(\frac{dt}{d\tau}\right)^2 + V^2\left(\frac{dt}{d\tau}\right)^2 \\
& \therefore \frac{dt}{d\tau} = \frac{1}{\sqrt{1-V^2}} \\
& \text{and } \vec{u} = \left(\frac{1}{\sqrt{1-V^2}}, \frac{V}{\sqrt{1-V^2}}, 0, 0\right)
\end{aligned}$$

$$\begin{aligned}
\text{(b) Schwarz BH:}\quad & g_{tt} = -\left(1 - \frac{2M}{r}\right) \qquad g_{tr} = 0 \qquad g_{rr} = \frac{1}{1-\frac{2M}{r}} \\
& \therefore -1 = -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1-\frac{2M}{r}}V^2 \cdot \left(\frac{dt}{d\tau}\right)^2
\end{aligned}$$

Solve for $\dfrac{dt}{d\tau}$ again.

N.B. Self-consistently combines time dilation due to gravity (e.g. Pound-Rebka experiment) and motion (e.g. special relativity). It is not a simple addition!

A short review:

Last lecture: kinematics; $\vec{u}$ and time dilation

5.2 Energy measurements

Suppose an observer with 4-velocity $\vec{u}$ encounters a particle with 4-momentum $\vec{p}$. What is the energy E of the particle measured by the observer? (Note: $\vec{u}$ and $\vec{p}$ are geometric objects - they can be considered without reference to a frame (frame-independent). However, the energy measured by the observer is a frame-dependent quantity)

Equivalence principle: we can always "jump" into a *freely falling* frame at the location of the particle. In that frame, spacetime is flat $\Rightarrow$ we can describe the frame in Minkowski coordinates.

Local Lorentz invariance: we can boost ourselves so that we are instantaneously at rest with respect to the particle.

We now know:

$$\begin{aligned}
g &= (-1, 1, 1, 1) \\
\vec{v} &= (1, 0, 0, 0) \\
\vec{p} &= (E, p^x, p^y, p^z) \quad \text{from special relativity}
\end{aligned}$$

Can we express E as an invariant? Yes!

$$E = -\vec{u}\cdot\vec{p}, \tag{45}$$

recalling the dot product

$$\vec{u}\cdot\vec{p} = g_{\alpha\beta}u^\alpha p^\beta. \tag{46}$$

This is invariant, meaning it is independent of coordinates. We can always use this result even when we're given $\vec{u}$ and $\vec{p}$ in some other coordinates (from the principle of equivalence).

Example ?: Consider a Schwarz. BH with coordinates (t, r, θ, ϕ). Consider a radially infalling observer at some speed V, at some radius r. Then $\vec{u} = (u^t, Vu^t, 0, 0)$ and normalisation $\vec{u}\cdot\vec{u} = -1$ tells us u^t given the metric $g_{tt} = -((1 - \frac{2M}{r}),\ g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1}$. We dot product with $\vec{p}$ to get E.

Exercise: Measure velocity V as well as energy E, and package into a 4-vector with the form[6]

$$\vec{V} = \frac{\vec{p} + (\vec{p}\cdot\vec{u})\vec{u}}{-\vec{p}\cdot\vec{u}} \tag{47}$$

Exercise: What is the Doppler shift measured by a stationary observer who shines a laser at a moving mirror at speed V and measures reflected light?

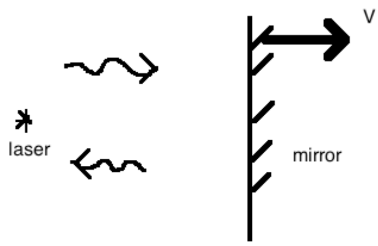


Figure 3: Mirror

5.3 4-acceleration

The definition $\vec{u} = \frac{d\vec{x}}{d\tau}$ is completely general in curved space. But acceleration is <u>not</u> $\vec{a} = \frac{d\vec{u}}{d\tau}$ in general. This is because the 2nd derivative $\frac{d^2\vec{x}}{d\tau^2}$ cares about curvature (while the 1st derivative does not).

Actually: $\vec{a} = \nabla_{\vec{u}}\vec{u}$ in general.

In flat space: $\vec{a} = \frac{d\vec{u}}{d\tau}$, which is a special case.

$$\underbrace{\vec{u}\cdot\vec{u} = -1}_{\text{normalisation}} \Rightarrow \underbrace{\vec{u}\cdot\vec{a} = 0}_{\text{differentiate both sides w.r.t } \tau} \tag{48}$$

Equation (48) is true in curved space.

Let's consider a case of **Uniform acceleration**. Consider a rocket ship with constant *proper* acceleration g in the "1" direction (achieved by, for example, throwing bricks out the back of the rocket at a constant rate), in flat spacetime.

In momentarily comoving reference frame:

$$g = (-1, 1, 1, 1)$$
$$\vec{u} = (1, 0, 0, 0)$$
$$\vec{a} = (0, \frac{d^2\vec{x}}{d\tau^2}, 0, 0)$$

Above, a^0 is zero because

$$-a^t = \vec{u} \cdot \vec{a} = 0 \tag{49}$$

and a^1 is equal to g by definition. Hence $\vec{a} \cdot \vec{a} = g^2$. This invariant $\Rightarrow$ true in all coordinates, but ONLY true in *flat spacetime.*

Now consider motion in global Minkowski coordinates not tied to the rocket (which exists because spacetime is flat).

We want to solve for $u^0(\tau)$, $u^1(\tau)$, $a^0(\tau)$, $a^1(\tau)$ in general.

$$\begin{aligned} \vec{u} \cdot \vec{u} &= -1 & \qquad -(u^0)^2 + (u^1)^2 &= -1 \\ \vec{a} \cdot \vec{u} &= 0 & -a^0u^0 + u^1a^1 &= 0 \\ \vec{a} \cdot \vec{a} &= g^2 & -(a^0)^2 + (a^1)^2 &= g^2 \end{aligned}$$

Eliminating variables and solving (using straightforward algebra), we obtain

$$\frac{du^0}{d\tau} = a^0 = gu^1 \tag{50}$$
$$\frac{du^1}{d\tau} = a^1 = gu^0 \tag{51}$$

Integrating these gives

$$t = \frac{1}{g} \sinh g\tau \tag{52}$$
$$x = \frac{1}{g} \cosh g\tau \tag{53}$$

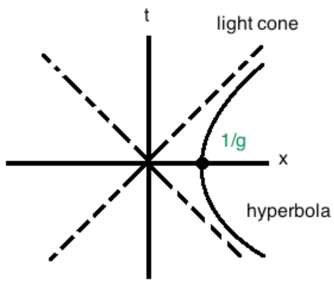


Figure 4: Rocket hyperbola

Remember this is in *flat spacetime* The interesting thing about this diagram is that is shows that ANY photon more than $1/g$ distance away from the rocket ship at $t = 0$ will never reach the rocket ship, even though the rocket ship is travelling less than c.

6 Calculus in curved space

6.1 Differentiation in curved space

Covariant derivatives → curvature → Einstein's field equations. Curvature is the second order change in space.

How does a tensor T of type $\binom{M}{N}$ change from spacetime point P_A to P_B? (we consider these points to be infinitesimally separated).

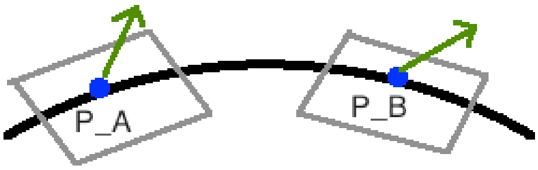


Figure 5: Tangent spaces

In general, we cannot compare geometric objects at P_A and P_B (they live in different tangent spaces) unless we have a recipe for "moving" geometric objects from P_A to P_B, e.g. parallel transport (which we will consider later).

For now we just consider flat space, where all points share a common tangent space.

Look for $\binom{M}{N+1}$ tensor ∇T which contracts with a vector $\vec{A}$ to give infinitesimal rate of change of T along $\vec{A}$. We call the rate of change $\nabla_{\vec{A}} T$ (type $\binom{M}{N}$) the *covariant derivative*

$$\nabla_{\vec{A}} T = \langle \nabla T, \vec{A} \rangle \tag{54}$$

e.g. if T is a scalar $\underbrace{\phi}_{\binom{0}{0} \text{ type}}$, then $\nabla T = \underbrace{\widetilde{d\phi}}_{\text{gradient } \binom{0}{1}}$

e.g. if T is a vector, say $\vec{V}$: in general consider two points P_A, P_B joined by a world line $\vec{x}(t)$

$$\begin{aligned}
\frac{d\vec{V}}{d\tau}(\vec{x}(\tau)) &= \frac{d}{d\tau}\left[V^\alpha(\vec{x}(\tau))\vec{e}_\alpha(\vec{x}(\tau))\right] \\
&= \frac{\partial V^\alpha}{\partial x^\beta}\frac{dx^\beta}{d\tau}\vec{e}_\alpha(\vec{x}(\tau)) + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta}\frac{dx^\beta}{d\tau} \qquad \text{product rule} \\
&= \left(\frac{\partial V^\alpha}{dx^\beta}\vec{e}_\alpha + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta}\right) u^\beta \qquad \text{4-velocity}
\end{aligned}$$

The parts inside the brackets of (**??**), denoted as $(\ldots)$, are contracted with $\vec{u}$ to give the LHS, which

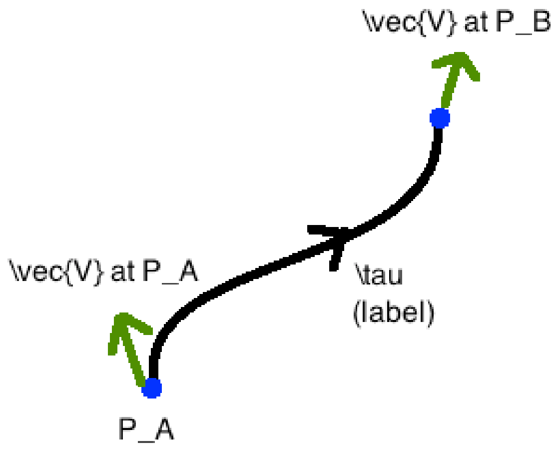


Figure 6: World line connecting points P_A and P_B

is a vector. So $(\ldots)$ is type $\begin{pmatrix}1\\1\end{pmatrix}$.

$\frac{\partial \vec{e}_\alpha}{\partial x^\beta}$ are vectors, so can be written as a linear combination of basis vectors.

$$\text{Define: } \frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta}\vec{e}_\mu \tag{55}$$

Then,

$$\begin{aligned}(\ldots) &= \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha\Gamma^\mu_{\alpha\beta}\vec{e}_\mu \\ &= \left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\beta}V^\mu\right)\vec{e}_\alpha\end{aligned}$$

The bracketed part above is, in index notation, $V^\alpha_{;\beta}$; the components of covariant derivation $\nabla\vec{V}$ of type $\begin{pmatrix}1\\1\end{pmatrix}$.

Notation: switch $\mu \leftrightarrow \alpha$ in dummy index (repeated).

Christoffel symbols Γ:

- $4 \times 4 \times 4 = 64$ numbers
- solve 64 equations given by (55) - linear equations (easy to solve!)
  - an alternative trick is to use the metric, which we will see later
- are Christoffel symbols components of a $\begin{pmatrix}1\\2\end{pmatrix}$ tensor?
  - No!

- $\{\Gamma^{\mu}_{\alpha\beta}\vec{e}_{\mu}\}$ are components of a tensor, but the Γ's themselves are not
- e.g. is Γ's are tensor, then

$$\Gamma^{\mu'}_{\alpha'\beta'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \Gamma^{\mu}_{\alpha\beta} \tag{56}$$

- but from the definition of Γ's we have $\Gamma^{\mu}_{\alpha\beta} = 0$ for (say) Cartesian coordinates, which would then wrongly imply $\Gamma^{\mu'}_{\alpha'\beta'}$ from (56) too!

Physically: suppose you have a vector $\vec{V}$ which does not change from point to point. We want $\nabla\vec{V} = 0$.

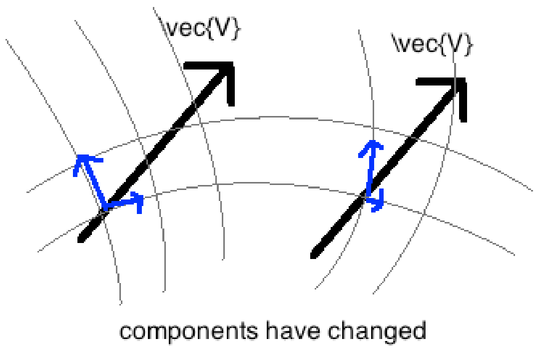


Figure 7: Vectors V

If the coordinates are curvilinear, then components "artificially" change. The Γ's "undo" the artificial change!

A short review:

$$T = \binom{M}{N} \text{ tensor}$$

$$\nabla T = \binom{M}{N+1} \text{ tensor}$$

$$\nabla_{\vec{A}} T = \text{covariant derivative of T along } \vec{A} = \langle \nabla T, \vec{A} \rangle = \text{type } \binom{M}{N}$$

e.g. if T of a scalar ϕ then $\nabla T = \widetilde{d\phi}$

e.g. if T is a vector $\vec{V}$ then $\nabla\vec{V}$ is a $\binom{1}{1}$ object

$$\underbrace{(\nabla\vec{V})^{\alpha}_{\beta}}_{\text{notation: } V^{\alpha}_{;\beta}} = \underbrace{\frac{\partial V^{\alpha}}{\partial x^{\beta}}}_{\text{notation: } V^{\alpha}_{,\beta}} + \underbrace{\Gamma^{\alpha}_{\lambda\beta}}_{\text{Christoffel}} V^{\lambda} \tag{57}$$

Christoffel symbol corrects for artificial change in components of $\vec{V}$ due to curvilinear coordinates (see Figure 7). Derived for curvilinear coordinates in flat space but remains true for curved spaces also!

Note: neither $V^\alpha_{,\beta}$ nor $\Gamma^\mu_{\alpha\beta}$ are tensors (Γ's are components of set of tensors $\{\nabla\vec{e}_\alpha\}$). But $V^\alpha_{;\beta}$

6.2 Covariant derivative of 1-form

What about 1-forms? Can we just say

$$p_{\alpha;\beta} = \frac{\partial p_\alpha}{\partial x^\beta} + \Gamma^\alpha_{\lambda\beta} p_\lambda \quad ??$$

No, we cannot - note the double lowered λ in the RHS.

$\nabla\widetilde{p}$ is $\binom{0}{2}$ type/

We redo rate-of-change calculation along world line $\vec{x}$

$$\frac{d\widetilde{p}}{d\tau} = \frac{d}{d\tau}\left[p_\alpha(\vec{x}(\tau))\widetilde{\omega}^\alpha(\vec{x}(\tau))\right] \tag{58}$$

$$= \frac{\partial p_\alpha}{dx^\beta}\frac{dx^\beta}{d\tau}\widetilde{\omega}^\alpha + p_\alpha \underbrace{\frac{\partial\widetilde{\omega}^\alpha}{\partial x^\beta}}_{\text{1-form}}\frac{dx^\beta}{d\tau} \tag{59}$$

We can write the 1-form as linear combination $x^\alpha_{\mu\beta}\widetilde{\omega}^\mu$

Are x's related to Γ;s? Yes - duality.

$$\delta^\alpha_\beta = \langle\widetilde{\omega}^\alpha, \vec{e}_\beta\rangle \qquad \text{from previous lectures} \tag{60}$$

Differentiate by x^γ

$$0 = \langle\frac{\partial\widetilde{\omega}^\alpha}{\partial x^\gamma}, \vec{e}_\beta\rangle\langle\widetilde{\omega}^\alpha, \frac{\partial\vec{e}_\beta}{\partial x^\gamma} \tag{61}$$

$$= \langle x^\alpha_{\mu\gamma}\widetilde{\omega}^\mu, \vec{e}_\beta\rangle + \langle\widetilde{\omega}^\alpha, \Gamma^\lambda_{\beta\gamma}\vec{e}_\lambda\rangle \tag{62}$$

$$= x^\alpha_{\mu\gamma}\delta^\mu_\beta + \Gamma^\lambda_{\beta\gamma}\delta^\alpha_\lambda \tag{63}$$

$$= x^\alpha_{\beta\gamma} + \Gamma^\alpha_{\beta\gamma} \tag{64}$$

i.e. opposites!

Components of $\nabla\widetilde{p}$, i.e. $(\nabla\widetilde{p})_{\alpha\beta}$ is notationally $p_{\alpha;\beta}$

$$p_{\alpha;\beta} = \frac{\partial p_\alpha}{\partial x^\beta} - \Gamma^\lambda_{\alpha\beta}p_\lambda \tag{65}$$

For general tensor: add correction term $\pm\Gamma$ tensor for each tensor index; + is index is up; − if index is down.

$$\text{e.g. } T^\alpha_{\beta;\gamma} = \frac{\partial T^\alpha_\beta}{\partial^\gamma} + \underbrace{\Gamma^\alpha_{\lambda\gamma}T^\lambda_\beta}_{\text{by analogy with vector}} - \underbrace{\Gamma^\lambda_{\beta\gamma}T^\alpha_\lambda}_{\text{by analogy with 1-form}} \tag{66}$$

6.3 Covariant derivative of metric

Recall

$$\nabla\vec{V} = \binom{1}{1} \text{ type} \tag{67}$$

$$\nabla\widetilde{V} = \binom{0}{2} \text{ type} \tag{68}$$

where $\widetilde{V}$ is the specific 1-form induced by the metric with $\vec{V}$ in the one slot ($\widetilde{V} = g(\vec{V}, \ldots)$).

We can go from $\binom{0}{2}$ to $\binom{1}{1}$ by contraction!

Consider $\nabla_{\vec{a}}\vec{V} = \binom{1}{0}$ object.

Then $g(\nabla_{\vec{A}}\vec{V}, \ldots)$ is a 1-form induced by g.

By definition this is $\nabla_{\vec{A}}\widetilde{V}$ which is a 1-form, i.e. $\nabla\widetilde{V}\binom{0}{2}$ contracted with $\vec{A}$.

In index notation:

$$V_{\alpha;\beta} = g_{\alpha\gamma}V^{\gamma}_{;\beta} \qquad \text{as in previous lectures} \tag{69}$$

6.4 Christoffel Symbols and Metric and Acceleration

6.4.1 Covariant derivative

∇g is a $\binom{0}{3}$ tensor (with components $g_{\alpha\beta;\gamma}$).

From last time, given $\vec{V}$ there is a $\widetilde{V}$.

By definition,

$$V_{\alpha;\beta} = g_{\alpha\gamma}V^{\gamma}_{;\beta} \quad (\text{or } \nabla_{\vec{A}}\vec{V}) \tag{70}$$

Separately,

$$V_{\alpha} = g_{\alpha\beta}V^{\gamma} \tag{71}$$

Taking the covariant derivative of both sides of (71) with respect to x^{β}

$$\Rightarrow V_{\alpha;\beta} = g_{\alpha\gamma;\beta}V^{\gamma} + g_{\alpha\gamma}V^{\gamma}_{;\beta} \tag{72}$$

$$\Rightarrow g_{\alpha\gamma;\beta=0} \tag{73}$$

$$\Rightarrow \nabla g = 0 \tag{74}$$

We can use this to get a formula for Γ's (Γ's are coefficients in linear combination of basis vectors which give $\frac{\partial\vec{e}_{\alpha}}{\partial x_{\beta}}$)

An alternative derivation of $\nabla g = 0$:

In flat space, $g_{\alpha\beta,\gamma} = 0$ and Γ's are zero because $\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = 0$

$$\Rightarrow g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - \text{(two terms involving Christoffel symbols)} \tag{75}$$
$$= 0 \quad \text{in flat space} \tag{76}$$

But $g_{\alpha\beta;\gamma} = 0$ involves only decent tensors, so it's free in ????

$$- = g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - \Gamma^\lambda_{\alpha\gamma} g_{\lambda\beta} \tag{77}$$

??? gives us 64 equations for 64 unknowns (Γ components) provided $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma}$ are given.

Simplification: symmetric in α, β; symmetric in lower indices of the Γ's

Using this and write three cyclic permutations of (*) and sum them (see Schutz p. 142):

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} (g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}) \tag{78}$$

You should memorise equation (78)!

- If Minkowski:
  - Γ's are 0
- If not:
  - Γ's $\neq 0$, coordinates are curved, but space may not be curved

Curvatures of manifold depend on 2nd derivatives of g.

6.5 The 4-acceleration

$\vec{u} = \frac{d\vec{x}}{d\tau}$ is a first derivative (i.e. this is calculated without reference to curvature of coordinates and/or manifold).

$\vec{a}$ is a second derivative. We need to refer to curvature of coordinates/manifold.

In general, $\vec{a} = \nabla_{\vec{u}} \vec{u}$, i.e. rate of change of $\vec{i}$ along itself.

In free fall: $\nabla_{\vec{u}} \vec{u} = 0$ (but with rocket engines $\nabla_{\vec{u}} \vec{u} \neq 0$)

Christoffel symbols contain curvature!

A short review:

Last lecture:

- $\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} (g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu})$

- $g_{\alpha\beta;\gamma} = 0$

Remember, for curved space, 4-acceleration $\vec{a} \neq \frac{d^2\vec{x}}{d\tau^2}$. This is the derivative of $x^\alpha(\vec{e}_\alpha)$ not just coordinates. τ involves curvature.

For free fall, $\vec{a} = 0$ (compare with Newtonian 9.8 ms^{-2}

Example ?: Astronaut above Schwarz. black hole, at distance r (radial coordinate appearing in metric), with θ and ϕ also fixed. The Schwarzchild metric is:

$$g_{tt} = -\left(1 - \frac{2M}{r}\right) \tag{79}$$
$$g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1} \tag{80}$$
$$g_{\theta\theta} = ... \tag{81}$$
$$g_{\phi\phi} = ... \tag{82}$$

Note: rotating black hole is not diagonal! (but it will be symmetric because $A^\alpha\vec{e}_\alpha = A^\beta\vec{e}_\beta$; non-diagonal $\Leftarrow$ diagonal (one way!))

Question: what is the acceleration of this astronaut (globally not flat) at a fixed radius?

We need $\vec{u} = \left(\frac{dt}{d\tau}, 0, 0, 0\right)$

$$\vec{x} = (t, r, \theta, \phi) \tag{83}$$
$$\vec{u} = \frac{d\vec{x}}{d\tau} \tag{84}$$

What is u_t? We need to normalise.

$$\vec{u} \cdot \vec{u} = -1 \tag{85}$$
$$u^\beta g^{\alpha\beta} u_\alpha = -\left(1 - \frac{2M}{r}\right) = u_t^2 = -1 \tag{86}$$
$$\Rightarrow u_t = \left(1 - \frac{2M}{r}\right)^{-1/2} \tag{87}$$

$$a^\alpha = u^\beta u^\alpha_{;\beta} \quad \text{covariant derivative} \tag{88}$$
$$= u^\beta \left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\lambda\beta} u^\lambda\right) \tag{89}$$
$$= \underbrace{\frac{\partial u^\alpha}{\partial \tau}}_{\text{zero}} + \Gamma^\alpha_{\lambda\beta} u^\lambda u^\beta \tag{90}$$

Now,

$$a^t = a^\theta = a^\phi = 0 \tag{91}$$
$$\Rightarrow a^r = \Gamma^r_{tt}(u^t)^2 \tag{92}$$
$$= \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) \cdot \left[\left(1 - \frac{2M}{r}\right)^{-1/2}\right]^2 = \frac{M}{r^2} \tag{93}$$

$$\vec{a} = \nabla_{\vec{u}}\vec{u} = (0, \frac{M}{r^2}, 0, 0) = \frac{M}{r^2}\hat{r} \quad \text{(accelerating outwards!)} \tag{94}$$

We will prove that acceleration is always orthogonal to the 4-velocity.

$$\vec{u} \cdot \vec{u} = -1 = u^\alpha u_\alpha \tag{95}$$

Take the covariant derivative

$$\Rightarrow 0 = (u^\alpha_{;\beta} u_\alpha + u^\alpha u_{\alpha;\beta})u^\beta \tag{96}$$
$$= (u^\alpha_{;\beta} u_\alpha + u_\gamma g^{\gamma\alpha} u_{\alpha;\beta})u^\beta \quad (\text{note } g^{\gamma\alpha}_{;\beta} = 0) \tag{97}$$
$$0 = (u^\alpha_{;\beta} u_\alpha + u_\gamma u^\gamma_{;\beta})u^\beta \tag{98}$$

where we have switched co- and contra-variant indices using the metric!

So $(u_\alpha u^\alpha_{;\beta})u^\beta = 0$ (both terms are equal).

$$\Rightarrow \vec{u} \cdot \vec{a} = 0 \tag{99}$$

6.6 Fermi-Walker Transport

What does the "natural basis" of an observer in a rocket ship ($\vec{a}$, $\vec{v}$) in curved space look like, compared to some external basis? (global coordinate system)

We let $\vec{v}$ be an arbitrary vector. We then solve

$$\nabla_{\vec{u}}\vec{v} = (\vec{a} \cdot \vec{v})\vec{u} - \vec{a}(\vec{u} \cdot \vec{v}) \tag{100}$$

to see what $\vec{v}$ and $\vec{u}$ look like in a global coordinate system.

Not unique apparently. However,

1. $$\nabla_{\vec{u}}(\vec{v} \cdot \vec{v}) = 2\vec{v} \cdot \nabla_{\vec{u}}\vec{v} = 0 \quad \text{lengths preserved} \tag{101}$$

2. $$\nabla_{\vec{u}}(\vec{v} \cdot \vec{w}) = \vec{v} \cdot \nabla_{\vec{u}}\vec{w} + \vec{w} \cdot \nabla_{\vec{u}}\vec{v} = 0 \quad \text{angles preserved (orthogonality)} \tag{102}$$

3. 4-velocity is Fermi-Walker transported automatically. (if $\vec{v} = \vec{u}$ then

$$\nabla_{\vec{u}}\vec{v} = (\vec{a}\cdot\vec{u})\vec{u} - \vec{a}(\vec{u}\cdot\vec{u}) \tag{103}$$
$$= \vec{a} \quad \text{identically satisfied} \tag{104}$$

You can treat your time axis to be $\vec{u}$ (along the path).

4. If $\vec{w}$ is spacelike (orthogonal to $\vec{a}$ and $\vec{u}$), then $\nabla_{\vec{u}}\vec{w} = 0$ ($\vec{w}$ does not rotate spatially relative to $\vec{u}$) (it's temporal (?))

6.7 Constants of motion

Approach #1: use Lagrangian methods

Approach #2: for free fall

$$\nabla_{\vec{u}}\vec{u} = 0 \tag{105}$$

(for a photon $\nabla_{\vec{p}}\vec{p} = 0$) We can also look at the 1-form version of equation (105)

$$\nabla_{\vec{u}}\widetilde{u} = 0 \tag{106}$$

$$\text{Components:} \quad u^{\beta}_{\alpha;\beta} = 0 \tag{107}$$
$$\underbrace{u^{\beta}}_{=\frac{\partial x^{\beta}}{d\tau}}\left(\frac{\partial u_{\alpha}}{\partial x^{\beta}} - \Gamma^{\lambda}_{\alpha\beta}u^{\lambda}\right) = 0 \tag{108}$$

$$\text{i.e.,} \quad \underbrace{\frac{\partial u_{\alpha}}{d\tau}}_{\text{chain rule}} = \Gamma^{\lambda}_{\alpha\beta}u_{\lambda}u^{\beta} \tag{109}$$

Recall:

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2}g^{\lambda\mu}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}) \tag{110}$$
$$\Gamma^{\lambda}_{\alpha\beta}u_{\lambda}u^{\beta} = \frac{1}{2}\underbrace{u^{\mu}u^{\beta}}_{\text{symm. } \mu\leftrightarrow\beta}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}) \tag{111}$$

Now, $g_{\mu\alpha,\beta}$ and $g_{\alpha\beta,\mu}$ are antisymmetric under exchange of $\mu \leftrightarrow \beta$

$$= \frac{1}{2}u^{\mu}u^{\beta}g_{\mu\beta,\alpha} \tag{112}$$

since contraction of symmetric and antisymmetric components vanishes.

i.e., if the metric is independent of coordinate x^{α} then u_{α} is constant along world line.

e.g., in Schwarz spacetime, a freely falling body has u_{ϕ} = constant and u_t = constant.

6.8 Polar coordinates

We shall explore polar coordinates through use of an example.

Polar coordinates (primed): (r, θ)

Cartesian (unprimed): (x, y)

$$x = r\cos\theta \tag{113}$$
$$y = r\sin\theta \tag{114}$$

Our plan is:

- get polar basis
- get polar metric
- get polar Christoffel symbol

To get **polar basis**:

Transformation matrix:

$$\frac{\partial x^\alpha}{\partial x^{\alpha'}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \tag{115}$$
$$\frac{\partial x^{\alpha'}}{\partial x^\alpha} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \ldots \quad \text{exercise} \tag{116}$$

Basis vectors:

$$\vec{e}_\alpha = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\vec{e}_{\alpha'} \tag{117}$$
$$\vec{e}_r = \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y \tag{118}$$
$$= \cos\theta\vec{e}_x + \sin\theta\vec{e}_y \tag{119}$$
$$\vec{e}_\theta = \frac{\partial x}{\partial \theta}\vec{e}_x + \frac{\partial y}{\partial \theta}\vec{e}_y \tag{120}$$
$$= -r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y \tag{121}$$

[h]

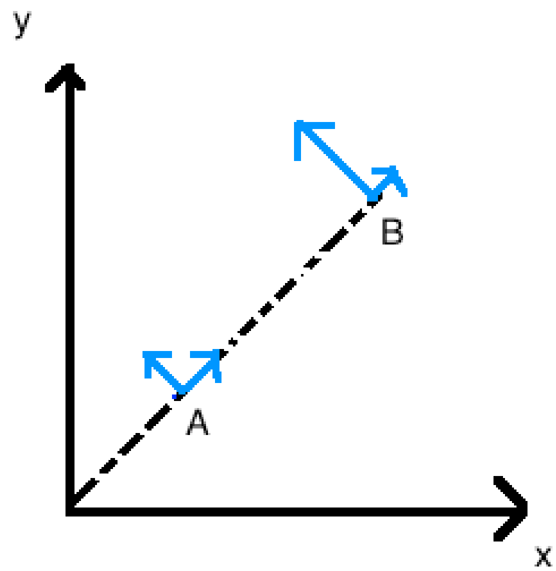


Figure 8: Polar coordinates?

In Figure 8 above we see the vector length at B is greater than at A; the length of $\vec{e}_\theta$ is proportional to r because we need to subtend a greater arc further out so as to produce unit change in θ.

Polar metric:

$$g_{\alpha'\beta'} = \vec{e}_{\alpha'}\vec{\beta'} \tag{122}$$

$$g_{rr} = \vec{e}_r \cdot \vec{e}_r = 1 \tag{123}$$

$$g_{r\theta} = \vec{e}_r \cdot \vec{e}_\theta = 0 \tag{124}$$

$$g_{\theta\theta} = \vec{e}_\theta \cdot \vec{e}_\theta = r^2 \tag{125}$$

Exercise: Check this this also follows from the transformation law

$$g_{\alpha'\beta'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\beta'}} \underbrace{g_{\alpha\beta}}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{Cartesian}} \tag{126}$$

Christoffel symbols describe derivatives of basis vectors in that basis itself

$$\frac{\partial \vec{e}_r}{\partial r} = 0 \tag{127}$$
$$\frac{\partial \vec{e}_r}{\partial \theta} = -\sin\theta \vec{e}_x + \cos\theta \vec{e}_y \tag{128}$$
$$\frac{\partial \vec{e}_\theta}{\partial r} = -\sin\theta \vec{e}_x + \cos\theta \vec{e}_y \tag{129}$$
$$\frac{\partial \vec{e}_\theta}{\partial \theta} = -r\cos\theta \vec{e}_x - r\sin\theta \vec{e}_y \tag{130}$$

Expressing these in the basis we get

$$\begin{array}{l|l|l} \frac{\partial \vec{e}_r}{\partial r} = 0 & \Gamma^r_{rr} = 0 & \Gamma^\theta_{rr} = 0 \\ \frac{\partial \vec{e}_r}{\partial \theta} = \frac{1}{r}\vec{e}_\theta & \Gamma^r_{r\theta} = 0 & \Gamma^\theta_{r\theta} = \frac{1}{r} \\ \frac{\partial \vec{e}_\theta}{\partial r} = \frac{1}{r}\vec{e}_\theta & \Gamma^r_{\theta r} = 0 & \Gamma^\theta_{\theta r} = \frac{1}{r} \\ \frac{\partial \vec{e}_\theta}{\partial \theta} = -r\vec{e}_r & \Gamma^r_{\theta\theta} = -r & \Gamma^\theta_{\theta\theta} = 0 \end{array}$$

Important to recall:

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \vec{e}_\mu \Gamma^\mu_{\alpha\beta} \tag{131}$$

We should check this works when taking covariant derivative of $\vec{V}$.

Special case: $\vec{V} = \vec{e}_x =$ constant. We should find $\nabla_{\vec{A}}\vec{V} = 0$ for all $\vec{A}$.

To check: we write $\vec{V}$ in polar coordinates:

$$\vec{V} = \cos\theta \vec{e}_r - \frac{1}{r}\sin\theta \vec{e}_\theta \tag{132}$$

from transformation laws for basis vectors.

$$\begin{aligned}
\left(\nabla_{\vec{A}}\vec{V}\right)^\alpha &= A^\beta \left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\lambda\beta} V^\lambda\right) \\
\therefore \left(\nabla_{\vec{A}}\vec{V}\right)^r &= A^\beta \frac{\partial V^r}{\partial x^\beta} + \Gamma^r_{\lambda\beta} V^\lambda A^\beta \\
&= A^\theta \frac{\partial V^r}{\partial \theta} + \Gamma^r_{\theta\theta} V^\theta A^\theta \quad \text{other terms zero} \\
&= A^\theta(-\sin\theta) - r\cdot\left(-\frac{1}{r}\sin\theta\right) A^\theta \\
&= 0 \\
\left(\nabla_{\vec{A}}\vec{V}\right)^\theta &= A^\beta \frac{\partial V^\theta}{\partial x^\beta} + \Gamma^\theta_{\lambda\beta} V^\lambda A^\beta \\
&= A^r \frac{\partial V^\theta}{\partial r} + A^\theta \frac{\partial V^\theta}{\partial \theta} + \Gamma^\theta_{r\theta} V^r A^\theta + \Gamma^\theta_{\theta r} V^\theta A^r \\
&= A^r \cdot \frac{1}{r^2}\sin\theta + A\theta\cdot\left(-\frac{\cos\theta}{r}\right) + \frac{1}{r}\cdot\cos\theta\cdot A^\theta + \frac{1}{r}\left(-\frac{1}{r}\sin\theta\right) A^r \\
&= 0
\end{aligned}$$

Exercise: Metric $g = \text{diag}(1, r^2)$ from previous lecture; check that $\nabla g = 0$ explicitly.

Exercise: Confirm that Γ's can also be derived from

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}) \tag{133}$$

Exercise: Prove divergence

$$V^{\alpha}_{;\alpha} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{\partial V_\theta}{\partial \theta} \tag{134}$$

as you would expect from undergrad.

Exercise: Derive $\chi^{\mu}_{\alpha\beta}$'s for basis one-forms

$$\widetilde{\omega}^r = \cos\theta \cdot \widetilde{\omega}^x + \sin\theta \cdot \widetilde{\omega}^y \tag{135}$$

$$\widetilde{\omega}^\theta = -\frac{\sin\theta}{r} \cdot \widetilde{\omega}^x + \frac{\cos\theta}{r} \cdot \widetilde{\omega}^y \tag{136}$$

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7 Curved space

In the presence of gravity, we cannot have a global inertial (Minkowski) frame.

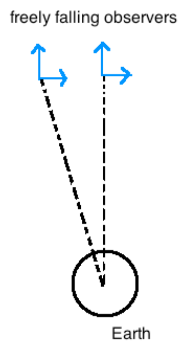


Figure 9: Freely falling observers

We see that the distance between the observers decreases! i.e. curved geodesics ⇒ curved space!

7.1 Curved manifold

- Extrinsic curvature: is space curved with respect to the space in which it's embedded?

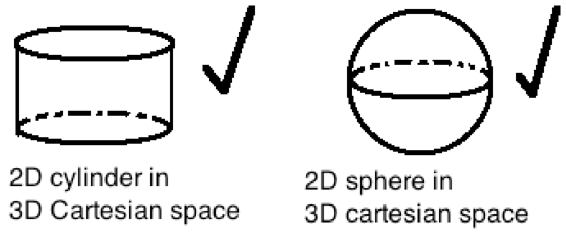


Figure 10: Extrinsic curvature

Yes!

- Intrinsic curvature: without refernce to an emedding. Do neighbouring geodesics ("free fall") diverge/converge? Note we need the idea of parallel transport to discuss geodesics.

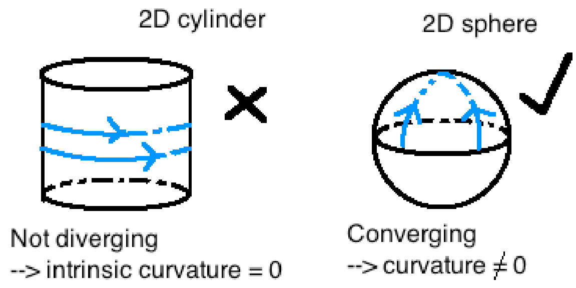


Figure 11: Intrinsic curvature

We see that a 2D cylinder does <u>not</u> have intrinsic curvature!

7.2 Local flatness

- Let $\{x^\alpha\}$ be some generic coordinates
- Make a transformation to $\{x^{\alpha'}\}$ coordinates
- See how for we can get in flattening the space near a point $\hat{x}_0$
- Can we choose $x^{\alpha'}(\vec{x})$ in such a way that the primed metric is Minkowski near $\vec{x}_0$?
  - exactly no, but approximately yes

<u>At $\vec{x}_0$</u>:

We are in a tangent space; first derivatives of the metric at the second derivatives disappear.

<u>Away from $\vec{x}_0$</u>:

Curvature contributes to $\vec{a}$ (we can set first derivatives to metric to zero)

To transform from primed to unprimed:

$$\Lambda^{\alpha}_{\mu'} = \frac{\partial x^{\alpha}}{\partial \mu'} \tag{137}$$

Aim: to compute transformed metric

$$g_{\mu'\nu'} = \Lambda^{\alpha}_{\mu'}\Lambda^{\beta}_{\nu'}g_{\alpha\beta} \tag{138}$$

We seek to cheek $x^{\alpha'}$ (and hence $\Lambda^{\alpha}_{\mu'}$) in order to make $g_{\mu'\nu'}$ as flat as possible.

Taylor expansion: small departures around $\vec{x}_0$; no cusps, space is differentiable, order of differentiation is unimportant.

$$\Lambda^{\alpha}_{\mu'} = \Lambda^{\alpha}_{\mu'}|_{\vec{x}_0} + \frac{\partial \Lambda^{\alpha}_{\mu'}}{\partial x^{\gamma'}}\left(x^{\gamma'} - x_0^{\gamma'} + \frac{1}{2}\frac{\partial^2 \Lambda^{\alpha}}{\partial x^{\gamma'}\,\partial x^{\delta'}}\left(x^{\gamma'} - x_0^{\gamma'}\right)(x^{\delta'} - x_0^{\delta'}\right) + \dots \tag{139}$$

$$= \Lambda^{\alpha}_{\mu'}|_{\vec{x}_0} + \frac{\partial}{\partial x^{\gamma'}}\left(\frac{\partial x^{\alpha}}{\partial \mu'}\right)|_{\vec{x}_0}^{(x^{\gamma'}-x_0^{\gamma'}} + \frac{1}{2}\frac{\partial^3 x^{\alpha}}{\partial x^{\mu'}\,\partial x^{\gamma'}\,\partial x^{\delta'}}|_{\vec{x}_0}\left(x^{\gamma'} - x_0^{\gamma'}\right)\left(x^{\delta'} - x_0^{\delta'}\right) + \dots \tag{140}$$

$$g_{\alpha\beta} = g_{\alpha\beta}|_{\vec{x}_0} + g_{\alpha\beta,\gamma'}\left(x^{\gamma'} - x_0^{\gamma'}\right) + \frac{1}{2}g_{\alpha\beta,\gamma'\delta'}\left(x^{\gamma'} - x_0^{\gamma'}\right)\left(x^{\delta'} - x_0^{\delta'}\right) + \dots \tag{141}$$

Therefore, at zeroth order

$$g_{\mu'\nu'} = \Lambda^{\alpha}_{\mu'}|_{\vec{x}_0}\;\Lambda^{\beta}_{\nu'}|_{\vec{x}_0}\;g_{\alpha\beta}|_{\vec{x}_0} \tag{142}$$

(Can we make this flat?)

7.3 Miscellaneous practical results

On curved space times.

7.3.1 Proper length

Also called the "geodesic length"

$$ds^2 = g_{\alpha\beta}\frac{dx^{\alpha}(\lambda)}{d\lambda}\cdot\frac{dx^{\beta}(\lambda)}{d\lambda} \tag{143}$$

$$\text{proper length} = \int_{\lambda_0}^{\lambda_1} dx\,\sqrt{\frac{d\vec{x}}{d\lambda}\cdot\frac{d\vec{x}}{d\lambda}} \tag{144}$$

Proper length is coordinate independent.

7.3.2 Volume

$$\text{Inifinitesimal volume} = dx^{0'}\;dx^{1'}\;dx^{2'}\;dx^{3'} \tag{145}$$

$$= \text{Jacobian} \times dx^0\;dx^1\;dx^2\;dx^3 \tag{146}$$

$$= \det\underbrace{\left(\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right)}_{\Lambda^{\alpha'}_{\alpha}} dx^0\;dx^1\;dx^2\;dx^3 \tag{147}$$

For an infinitesimal volume: locally the manifold is flat. So there exists a transformation that maps g into Minkowski η.

In matrix notation:

$$g = \Lambda \eta \Lambda^{\top} \tag{148}$$

In component notation:

$$g_{\alpha\beta} = \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta} \eta_{\alpha'\beta'} \tag{149}$$

Determinant of both sides (in matrix notation)

$$\det g = \det \Lambda \ \det \eta \ \det(\Lambda^{\top}) \tag{150}$$

$$= -(\det \Lambda)^2 \tag{151}$$

Therefore $\det \Lambda = (-\det g)^{1/2}$.

7.3.3 Divergence of a vector

Used in conservation laws.

$$V^{\alpha}_{;\beta} = \frac{\partial V^{\alpha}}{\partial x^{\beta}} = \Gamma^{\alpha}_{\lambda\beta} V^{\lambda} \tag{152}$$

$$\text{div } \vec{V} = V^{\alpha}_{;\alpha} \tag{153}$$

$$= \frac{\partial V^{\alpha}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\lambda\alpha} V^{\lambda} \tag{154}$$

$$\Gamma^{\alpha}_{\lambda\beta} = \frac{1}{2} g^{\alpha\mu} \left(g_{\mu\lambda,\beta} + g_{\mu\beta,\lambda} - g_{\lambda\beta,mu} \right) \tag{155}$$

$$\Rightarrow \Gamma^{\alpha}_{\lambda\alpha} + \frac{1}{2} g^{\alpha\mu} \left(g_{\mu\lambda,\alpha} + g_{\mu\alpha,\lambda} - g_{\lambda\alpha,\mu} \right) \tag{156}$$

$$= \frac{1}{2} g^{\alpha\mu} g_{\mu\alpha,\lambda} \tag{157}$$

An example: $g^{-1} g = 1$ and matrix algebra.

$$\rightarrow (\det g)_{,\mu} = \det g - g^{\alpha\beta} g_{\alpha\beta,\mu} \tag{158}$$

Hence

$$\Gamma^{\alpha}_{\lambda\alpha} = \frac{(\sqrt{-g})_{,\lambda}}{\sqrt{-g}} \tag{159}$$

where $g \equiv \det(g_{\alpha\beta})$.

So,

$$\text{div } \vec{V} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} V^{\alpha} \right) \tag{160}$$

Exercise: 3D spherical polar, $g = \det(g_{\alpha\beta}) = r^2 \sin\theta$.

7.3.4 Gauss Law in integral form

$$\int d^3\mathbf{x}' \text{ div }'(\mathbf{V}) = \int \underbrace{d^2\mathbf{x}'}_{dA\ \hat{\mathbf{n}}} \cdot \mathbf{V} \tag{161}$$

This is 4D! let's measure conservation of the quantity $\vec{V}$ locally in flat space.

$$0 = V^{\alpha}_{,\alpha} \tag{162}$$

Locally flat $\Rightarrow$ Christoffel symbols for the unprimed flat coordinates are zero

$$V^{\alpha}_{,\alpha} = V^{\alpha}_{,\alpha} + \underbrace{\Gamma^{\alpha}_{\lambda\alpha} V^{\lambda}}_{\text{zero}} = V^{\alpha}_{;\alpha} \tag{163}$$

Integrate over small volume in flat space

$$\int d^4x\ V^{\alpha}_{,\alpha} = \int d^4x\ V^{\alpha}_{;\alpha}, \quad \text{as above} \tag{164}$$

transform into primed (non-flat) coordinates)

$$= \int d^4x'\ \underbrace{\sqrt{-g}}_{\text{Jacobian of transform}}\ \overbrace{V^{\alpha'}_{;\alpha'}}^{\text{unchanged; frame invariant}} \tag{165}$$

$$- \int d^4x'\ \sqrt{-g} \cdot \frac{1}{\sqrt{-g}} \cdot \left(\sqrt{-g} V^{\alpha'}\right)_{,\alpha'} \tag{166}$$

$$= \int d^4x'\ \left(\sqrt{-g} V^{\alpha'}\right)_{,\alpha'} \tag{167}$$

$$= \int d^3\mathbf{x}'\ n_{\alpha'} V^{\alpha'} \sqrt{-g} \tag{168}$$

where $n_{\alpha'}$ is normal to the 3-surface enclosing the 4-volume.

This is Gauss' law (also applies to the covariant derivative).

7.3.5 Angles

$$\frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \cos\theta \tag{169}$$

where $\vec{A} \cdot \vec{B} = g(\vec{A}, \vec{B})$

$$|\vec{A}| = \sqrt{g(\vec{A}, \vec{A})} \tag{170}$$

$$|\vec{B}| = \sqrt{g(\vec{B}, \vec{B})} \tag{171}$$

7.3.6 Geodesics

Curves that continue to progress in the same direction they were progressing in; i.e. the tangent is parallel to the tangent at the previous point.

$$\nabla_{\frac{d\vec{x}}{d\lambda}} d\vec{x} d\lambda = 0 \tag{172}$$

Covariant derivative of tangent along its own direction is zero.

In component notation: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\frac{dx^\beta}{d\lambda}\left(\frac{d\vec{x}}{d\lambda}\right)^\alpha_{;\beta} = 0 \tag{173}$$

$$\Rightarrow \frac{dx^\beta}{d\lambda}\left(\frac{d}{dx^\beta}\left(\frac{dx^\alpha}{d\lambda}\right) + \Gamma^\alpha_{\mu\beta}\frac{dx^\mu}{d\lambda}\right) = \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\beta}\frac{dx^\mu}{d\lambda}\cdot\frac{dx^\beta}{d\lambda} \tag{174}$$

Note $\vec{x}(\lambda = 0)$ and $\frac{d\vec{x}}{d\lambda}(\lambda = 0)$.

7.4 Curvature and Einstein's Field Equations

Parallel transport!

Generally, geometric objects like vectors as different points live in different tangent spaces, hence cannot be compared. We will investigate the looks for moving them.

7.4.1 Rules

If T is a geometric object, then we can parallel transport along $\vec{x}(\lambda)$ by requiring that

$$\nabla\frac{d\vec{x}}{d\lambda}T = 0 \tag{175}$$

$\Rightarrow T$ remains constant along tangent vector (is zero ???)

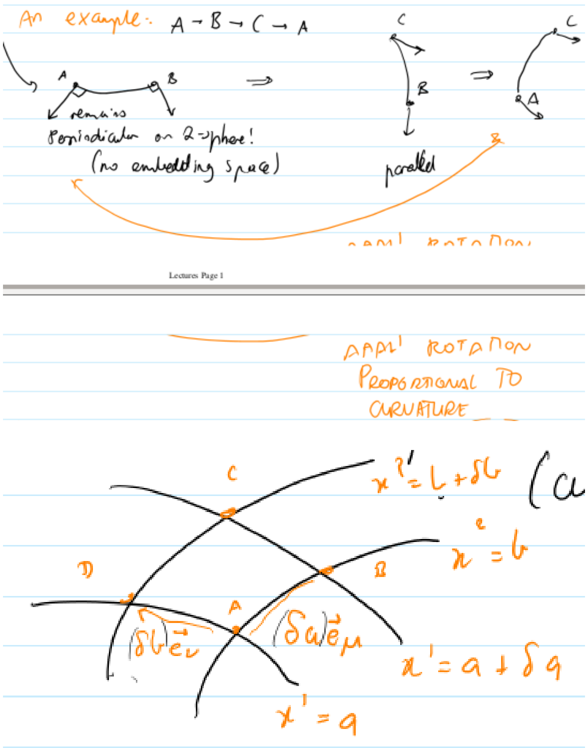
An example $A \to B \to C \to A$.

Theorem: if we parallel transport T around a closed curve, then the change in T is proportional to the intrinsic curvature of the manifold (proportional to the Riemann tensor). No curvature = flat $\Rightarrow$ no change.

What is $\Delta T = T_{\text{final}} - T_{\text{initial}}$ around a closed loop? (specialise to vector $\vec{V}$)

Consider a small patch (not flat) An example $A \to B \to C \to A$.

To parallel transport $\vec{V}$ around the loop $A \to B \to C \to D \to A$:



Consider AB:

$$\nabla_{\vec{e}_1}\vec{V} = 0 \quad \text{(the tangent to AB is } \vec{e}_1) \tag{176}$$

$$\Rightarrow V^\alpha_{;\beta} = 0 \quad (\beta = 1) \tag{177}$$

$$\Rightarrow 0 = \frac{\partial V^\alpha}{\partial x^1} + \Gamma^\alpha_{\lambda 1} V^\lambda \tag{178}$$

We integrate

$$V^\alpha(B) - V^\alpha(D) = \int_A^B \frac{\partial V^\alpha}{\partial x^1} dx^1 \tag{179}$$

$$= -\int_A^D dx^1\, \Gamma^\alpha_{\lambda 1} V^\lambda \tag{180}$$

Along BC:

$$\nabla_{\vec{e}_2}\vec{V} = 0 \quad \text{(parallel transport!)} \tag{181}$$

$$\Rightarrow V^\alpha(C) - V^\alpha(B) = \int_B^C \frac{\partial V^\alpha}{\partial x^2} dx^2 \tag{182}$$

$$= -\int_B^C dx^2\, \Gamma^\alpha_{\lambda 2} V^\lambda \tag{183}$$

Along CD:

$$\nabla_{-\vec{e}_2}\vec{V} = 0 \quad \text{(parallel transport!)} \tag{184}$$

$$\Rightarrow V^\alpha(D) - V^\alpha(C) = \int_C^D dx^1\, \Gamma^\alpha_{\lambda 1} V^\lambda \tag{185}$$

Along DA:

$$\nabla_{-\vec{e}_2}\vec{V} = 0 \tag{186}$$
$$\Rightarrow V^\alpha(A) - V^\alpha(D) = \int_D^A dx^2\ \Gamma^\alpha_{\lambda 2}V^\lambda \tag{187}$$

Note that $\Gamma^\alpha_{\lambda 1}$'s are not the smae at different positions! There is no zero.

The sum of the $A \to B \to C \to D \to A$ path is

$$-\int_A^B dx^1\ \Gamma^\alpha_{\lambda 1}V^\lambda - \int_B^C\ dx^2\Gamma^\alpha_{\lambda 2}V^\lambda + \int_C^D\ dx^1\ \Gamma^\alpha_{\lambda 1}V^\lambda + \int_D^A dx^2\ \Gamma^\alpha_{\lambda 2}V^\lambda \tag{188}$$

For small patches,

$$(\Gamma^\alpha_{\lambda 2})_{BC} = \left(\Gamma^\alpha_{\lambda 2}V^\lambda\right)\ AB + \delta_{a'???}\left(\Gamma^\alpha_{\lambda 2}V^\lambda\right) \tag{189}$$
$$\Rightarrow \left(\Gamma^\alpha_{\lambda 1}V^\lambda\right)_{CD} = \left(\Gamma^\alpha_{\lambda 1}V^\lambda\right)_{AB} + \delta b\frac{\partial}{\partial x^2}\left(\Gamma^\alpha_{\lambda 1}V^\lambda\right) \tag{190}$$

This is a Taylor expansion in x^2.

$$\Delta V^\alpha = -\delta a\frac{\partial}{\partial x^1}\left(\Gamma^\alpha_{\lambda 2}V^\lambda\right)\delta b \quad \text{(integral along BC)} \tag{191}$$
$$+\,\delta b\frac{\partial}{\partial x^2}\left(\Gamma^\alpha_{\lambda 1}V^\lambda\right)\delta a \quad \text{(integral along CD)} \tag{192}$$
$$\Rightarrow \frac{\Delta V^\alpha}{\delta a\ \delta b} = -\frac{\partial\Gamma^\alpha_{\lambda 2}}{\partial x^1}V^\lambda - \Gamma^\alpha_{\lambda 2}\underbrace{\frac{\partial V^\lambda}{\partial x^1}}_{*} \tag{193}$$
$$+\frac{\partial\Gamma^\alpha_{\lambda 1}}{\partial x^2}V^\lambda - \Gamma^\alpha_{\lambda 1}\underbrace{\frac{\partial V^\lambda}{\partial x^2}}_{**} \tag{194}$$

where the derivative * is the parallel transport condition $-\Gamma^\lambda_{\mu 1}V^\mu$, and the derivative ** is $-\Gamma^\lambda_{\mu 2}V^\mu$.

$$\Rightarrow \Delta V^\alpha \propto \delta a\ \delta b\ V^\alpha \tag{195}$$

where is proportional to constant; ???? and first derivaitives, Riemann tensor!! (????)

A short review:

Last lecture:

- parallel transport
- Riemann curvature tensor (20 independent numbers)

Today:

- symmetrics of Riemann

Parallel transport along $\vec{x}(\lambda)$: $\nabla_{\frac{d\vec{x}}{d\lambda}}T = 0$ where T is any geometric object.

If we parallel transport some object (say, vector $\vec{V}$) around a closed loop, then $\vec{V}$ does not return to its initial value if the spacetime is curved. We find $\Delta\vec{V} \propto$ area of curve $\times$ curvature.

In component notation:

$$\Delta V^{\alpha} = R^{\alpha}_{\beta\mu\nu} V^{\beta} \delta a \; \delta b \tag{196}$$

where $\delta a \; \delta b$ is thearea of quadrilaterial with sides δa and δb, and R is the Riemann tensor, contains 2nd derivatives of metric.

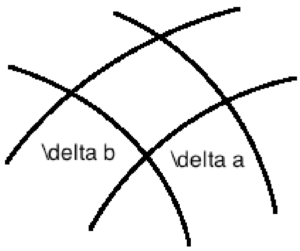


Figure 12: A quadrilateral in curved space

$$\begin{aligned} R^{\alpha}_{\beta\mu\nu} &= 2 \times \text{Christoffel 1st derivative} + 2 \times \text{products of Christoffel} && (197) \\ &= \Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\nu} + \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\mu} && (198) \end{aligned}$$

7.4.2 Symmetrics of the Riemann tensor

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}_{\beta\nu\mu} \tag{199}$$

- $R_{\alpha\beta\mu\nu}$ is antisymmetric in $\mu \leftrightarrow \nu$ (see picture; this is eqivalent to circumnavigating the loop in opposit direction)
- $R_{\alpha\beta\mu\nu}$ is antisymmetric in $\alpha \leftrightarrow \beta$
- $R_{\alpha\beta\mu\nu}$ is symmetric under exchange of pairs; i.e. $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$
- "Symmetric" under cyclic permutation of last three indices (Jordan symmetry)

$$0 = R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} \tag{200}$$

After applying these symmetrics to the $4 \times 4 \times 4 \times 4 = 256$ components of the Riemann tensor, we reduce them down to only 20.

Proof: go into local Minkowski coordinates (i.e. flat patch). Here Γ's are zero, but their derivatives are not.

$$R_{\alpha\beta\mu\nu} = \Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\beta\nu,\mu} \tag{201}$$

Substitute

$$\Gamma^{\alpha}_{\beta\mu} = \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\beta,\mu} + g_{\lambda\mu,\beta} - g_{\beta\mu,\lambda} \right) \tag{202}$$

We lower the indices by

$$R_{\alpha\beta\mu\nu} = g_{\alpha\sigma} R^{\sigma}_{\beta\mu\nu} \tag{203}$$

We now derivates with respect to ν (as an exercise)

$$\Rightarrow R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu} + g_{\beta\mu,\alpha\nu} - g_{\alpha\mu,\beta\nu} - g_{\beta\nu,\alpha\mu}\right) \tag{204}$$

Let's start with last symmetry (12 terms).

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 12 \text{ terms} \tag{205}$$

> **Exercise:** Check that these terms cancel pairwise.
>
> ---

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0. \tag{206}$$

which is derived in local falt. But it's a perfectly good tensor equation $\Rightarrow$ it is coordinate independent $\Rightarrow$ true in curved space, also.

A good tensor equation = something built out of tensors $\binom{M}{N}$ (which transform like they should) using valid operations like outer product, symmetrisation etc.

An example of a bad tensor equation:

$$R_{\alpha\beta\mu\nu,\lambda} = 0 \tag{207}$$

because ordinary derivatives **,** are "not good"; however

$$R_{\alpha\beta\mu\nu;\lambda} = 0 \tag{208}$$

is a good tensor equation, since covariant derivatives **;** are "good".

7.5 Constructing new tensors from Riemann on the way to Einstein's field equations

Aside: Riemann as a commutator

$$(V^{\mu}_{j\alpha})_{;\beta} - (V^{\mu}_{j\beta})_{;\alpha} = R^{\mu}_{\nu\alpha\beta} V^{\nu} \tag{209}$$

This is a good tensor equation: it is true in all reference frames.

Aim:

- Derive Bianchi identities, which contain infofrmation about symmetries of Riemann tensor's derivatives
- Contract Riemann in the hope of making a smaller tensor with 10 independent stress-energy tensor components, to match the 10 independent stress-energy components $T^{\mu\nu}$

- Combine the above 2 aims to "derive" Einstein's field equations in the form

$$\text{(curvature)} = \text{(stress energy)} \tag{210}$$

Bianchi identities

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(g_{\alpha\nu,\beta\mu} + g_{\beta\mu,\alpha\nu} - g_{\alpha\mu,\beta\nu} - g_{\beta\nu,\alpha\mu}\right) \tag{211}$$

Note $R_{\alpha\beta\mu\nu}$ is antisymmetric under exchange $\alpha \leftrightarrow \beta$ and $\mu \leftrightarrow \nu$, and under $\alpha\beta \leftrightarrow \mu\nu$.

We differentiate with respect to x^λ and add

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}\left(g_{\alpha,\beta\mu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\alpha\mu,\beta\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}\right) \tag{212}$$

Add cyclic permutations

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\nu\lambda,\mu} + R_{\alpha\beta\lambda\mu,\nu} = 0 \tag{213}$$

(show this as an exercise)

Note that in flat space, Γ's are zero. So,

$$R_{\alpha\beta\mu\nu,\lambda} = R_{\alpha\beta\mu\nu,\lambda} + \overbrace{\Gamma\text{'s} \times R}_{\text{correct for each index}} \tag{214}$$

$$= R_{\alpha\beta\mu\nu;\lambda}, \quad \text{since } \Gamma\text{'s are zero} \tag{215}$$

Now we have a new good tensor equation valid in all coordinate systems.

$$0 = R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\nu\lambda;\mu} + R_{\alpha\beta\lambda\mu;\nu} \tag{216}$$

the Bianchi identity!

How does the field evolve due to a source? (; is coordinate independent)

Note: need derivatives of curvature is we hope to describe how curvature evolves in response to source $T^{\mu\nu}$ (and how it evolves).

7.5.1 Contractions of Riemann

How many independent ways can we do this? $R^{\nu}_{\beta\mu\nu}$

$$\Rightarrow R^{\alpha}_{\alpha\mu\nu} = \underbrace{g^{\alpha\lambda}}_{\text{sym } \alpha\leftrightarrow\lambda} \overbrace{R_{\lambda\alpha\mu\nu}}^{\text{anti-sym } \lambda\leftrightarrow\alpha} \tag{217}$$

$$= -g^{\lambda\alpha} R_{\alpha\lambda\mu\nu} \tag{218}$$

$$= -R^{\lambda}_{\lambda\mu\nu} \tag{219}$$

$$= 0 \tag{220}$$

$$
\begin{aligned}
R^{\alpha}_{\beta\alpha\nu} &= ? && (221)\\
R^{\alpha}_{\beta\mu\alpha} &= g^{\alpha\lambda} R_{\lambda\beta\mu\alpha} && (222)\\
&= -g^{\alpha\lambda} R_{\lambda\beta\alpha\mu} && (223)\\
&= -R^{\alpha}_{\beta\alpha\mu} && (224)\\
&= -R^{\alpha}_{\beta\alpha\nu} && (225)
\end{aligned}
$$

(exercise: check the others)

The only independent contraction:

$$R_{\beta\nu} = R^{\alpha}_{\beta\alpha\nu} \tag{226}$$

Ricci tensor is symmetric!

$$
\begin{aligned}
R_{\nu\beta} = R^{\alpha}_{\nu\alpha\beta} &= g^{\alpha\lambda} R_{\lambda\nu\alpha\beta} && (227)\\
&= g^{\alpha\lambda} R_{\alpha\beta\lambda\nu} && (228)\\
&= R^{\lambda}_{\beta\lambda\nu} && (229)\\
&= R_{\beta\nu} && (230)
\end{aligned}
$$

We can derive the Bianchi identities for the Ricci tensor from the Riemann tensor.

$$0 = R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\nu\lambda;\mu} + R_{\alpha\beta\lambda\mu;\nu} \tag{231}$$

Raise 1st index by contracting with metric (also $\nabla g = 0$). Contract α with μ

$$0 = \underbrace{R^{\alpha}_{\beta\alpha\nu;\lambda}}_{R_{\beta\nu;\lambda}} + R^{\alpha}_{\beta\nu\lambda;\alpha} + \underbrace{R^{\alpha}_{\beta\lambda\alpha;\nu}}_{-R^{\alpha}_{\beta\alpha\lambda;\nu} = -R_{\beta\lambda;\nu}} \tag{232}$$

$$0 = R_{\beta\nu;\lambda} + R^{\alpha}_{\beta\nu\lambda;\alpha} - R_{\beta\lambda;\nu} \tag{233}$$

We contract again (involves Ricci scalar)

$$R = R^{\beta}_{\beta} = g^{\beta\nu} R_{\beta\nu} \tag{234}$$

Contract Bianchi: raise β and contract with ν (use $\nabla g = 0$).

$$0 = \underbrace{R^{\beta}_{\beta;\lambda}}_{R_{;\lambda}} + \underbrace{R^{\alpha\beta}_{\beta\lambda;\alpha}}_{-R^{\beta\alpha}_{\beta\lambda;\alpha} = -R^{\alpha}_{\lambda;\alpha}} - R^{\beta}_{\lambda;\beta} \tag{235}$$

(note we have skipped a few steps to get to this point, incl. raising β) where $R_{;\lambda}$ is the Ricci scalar. It is important to remember that the antisymmetry properties of Riemann are only immediately true in the case where all indices are down; we cannot immediately say the second term has the antisymmetry shown (though it can be shown that it indeed does).

$$\text{i.e. } 0 = R_{;\lambda} - 2R^{\beta}_{\lambda;\beta} \tag{236}$$

We play with this to get it into the form $(\ldots)_{;\beta} = 0$

$$0 = \left(R\delta^{\beta}_{\lambda} - 2R^{\beta}_{\lambda}\right)_{;\beta} \tag{237}$$

We define the Einstein curvature tensor

$$G^{\beta\lambda} \equiv R^{\beta\lambda} - \frac{1}{2}Rg^{\beta\lambda} \tag{238}$$

Note this is just pretty notation, nothing exciting. Then,

$$G^{\beta}_{\lambda} = R^{\beta}_{\lambda} - \frac{1}{2}Rg^{\beta}_{\lambda} = \delta^{\beta}_{\lambda} \tag{239}$$

So we have that

$$G^{\beta}_{\lambda;\beta} = 0 \tag{240}$$

The diveregence of the Einstein curvature tensor vanishes!

7.6 Einstein's field equations

These cannot be derived; they are experimental laws. We shall motivate them heuristically.

1. Equivalence principle $\Rightarrow$ make sure our field equations are good tensor equations
2. In the weak field, we must recover Newtonian gravity
$$\nabla^2\Phi = 4\pi G\rho \tag{241}$$
3. Weak field suggests 2nd derivative of field (analogy: $g_{\mu\nu}$) proportional to energy density (ρc^2)
   - i.e., curvature (2nd derivatives of metric) $\propto T^{\mu\nu}$ (stress-energy tensor)
4. $T^{\mu\nu}$ has 10 independent components.
   What curvature quantity matches this? Ricci (for example)! So should we postulate $R^{\mu\nu} \propto T^{\mu\nu}$?
5. No! It violates energy conservation. We must have $T^{\mu\nu}_{;\nu} = 0$. But then
$$R^{\mu\nu}_{;\nu} = -\text{Ricci scalar} \tag{242}$$
$$\neq 0 \text{ in general} \tag{243}$$
6. But we do have a nice curvature-related $\binom{0}{2}$ tensor (or $\binom{2}{0}$) whose divergence is zero, namely $G^{\beta\lambda}$. So we postulate
$$G^{\beta\lambda} \propto T^{\beta\lambda} \tag{244}$$

In fact we can be a tiny bit more general and ask: what symmetric tensor can we construct from Ricci and the metric on LHS?

$$R^{\alpha\beta} + \mu Rg^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta} \tag{245}$$

1. Insist on $T^{\alpha\beta}_{;\beta} = 0$
   $\Rightarrow \mu = -\frac{1}{2}$ as calculated above

2. Insist on $\nabla^2 = 4\pi G\rho$ in weak field (which we will prove later)

$$\Rightarrow k = 8\pi G = 8\pi \text{ in natural units} \tag{246}$$

Einstein's field equations:

$$\underbrace{R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}}_{G^{\alpha\beta}} + \Lambda g^{\alpha\beta} = 8\pi T^{\alpha\beta} \tag{247}$$

where Λ is the "cosmological" constant (air-quotes because we do not need to discuss it in a cosmological context). This term is sometimes missing from t-shirts.

8 Appendix

Indices up = vector-like

Indices down = 1-form-like

8.1 Covariant derviative notation

8.1.1 Vectors

We realise that the component form of $\nabla\vec{V}$ can be written as $(\nabla\vec{V})^\alpha_\beta$.

$$\underbrace{(\nabla\vec{V})^\alpha_\beta}_{\text{notation: } V^\alpha_{;\beta}} = \underbrace{\frac{\partial V^\alpha}{\partial x^\beta}}_{\text{notation: } V^\alpha_{,\beta}} + \Gamma^\alpha_{\lambda\beta}V^\lambda \tag{248}$$

That is:

$$V^\alpha_{,\beta} := \frac{\partial V^\alpha}{\partial x^\beta} \tag{249}$$

$$(\nabla\vec{V})^\alpha_\beta \equiv V^\alpha_{;\beta} := V^\alpha_{,\beta} + \Gamma^\alpha_{\lambda\beta}V^\lambda \tag{250}$$

8.1.2 One-forms

Components of $\nabla\widetilde{p}$, i.e. $(\nabla\widetilde{p})_{\alpha\beta}$ is notationally $p_{\alpha;\beta}$

$$p_{\alpha;\beta} = \frac{\partial p_\alpha}{\partial x^\beta} - \Gamma^\lambda_{\alpha\beta}p_\lambda \tag{251}$$

8.1.3 General tensors

For general tensor: add correction term $\pm\Gamma$ tensor for each tensor index; + is index is up; − if index is down.

$$\text{e.g.} \quad T^\alpha_{\beta;\gamma} = \frac{\partial T^\alpha_\beta}{\partial x^\gamma} + \underbrace{\Gamma^\alpha_{\lambda\gamma}T^\lambda_\beta}_{\text{by analogy with vector}} - \underbrace{\Gamma^\lambda_{\beta\gamma}T^\alpha_\lambda}_{\text{by analogy with 1-form}} \tag{252}$$

$$\nabla_{\vec{u}}\vec{v} = (\nabla\vec{v})\cdot\vec{u} \tag{253}$$

...probably?

8.1.4 Metric

In flat space, $g_{\alpha\beta,\gamma} = 0$ and Γ's are zero because $\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = 0$

$$\Rightarrow g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - \text{(two terms involving Christoffel symbols)} \tag{254}$$

$$= 0 \quad \text{in flat space} \tag{255}$$