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Subspace Arrangements in Vision and Learning

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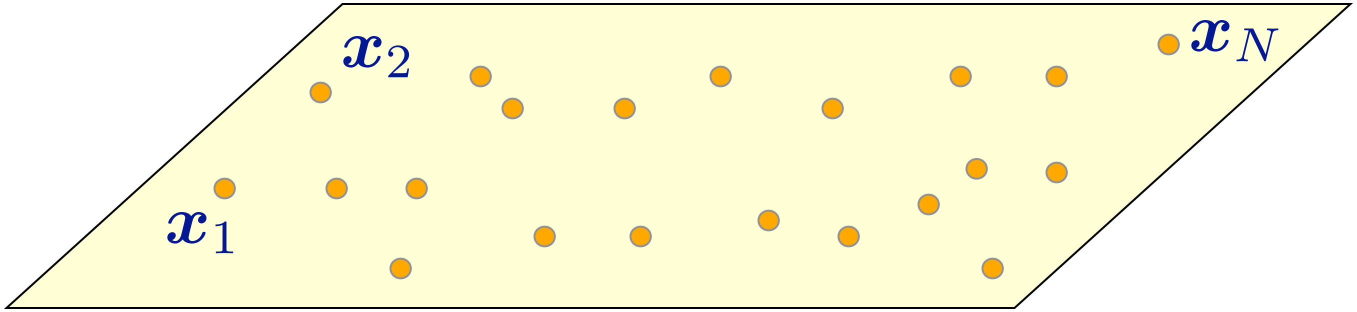
Principal Component Analysis (PCA)

- Given a set of points lying in one subspace, identify
  - Geometric PCA: find a subspace *S* passing through them
  - Statistical PCA: find projection directions that maximize the variance



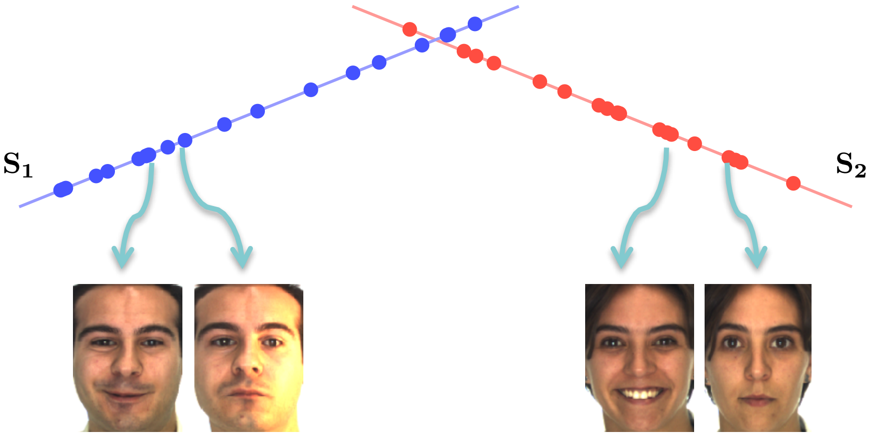
- Solution (Beltrami'1873, Jordan'1874, Hotelling'33, Eckart-Householder-Young'36)

$$U \Sigma V^\top = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_N \end{bmatrix} \in \mathbb{R}^{D \times N}$$

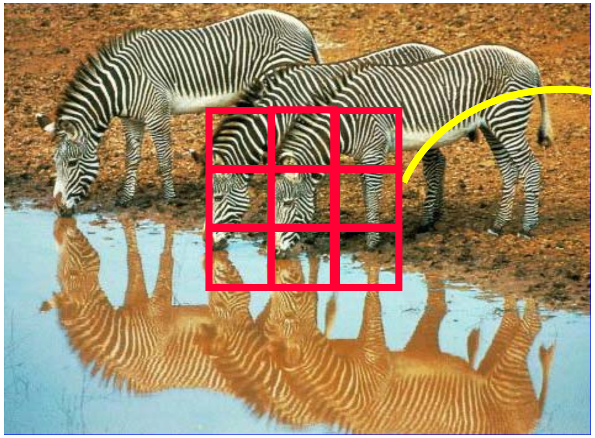
- Applications:
  - Signal/image processing, computer vision (eigenfaces), machine learning, genomics, neuroscience (multi-channel neural recordings)

Vision Applications Need Multiple Subspaces

- Face clustering and classification

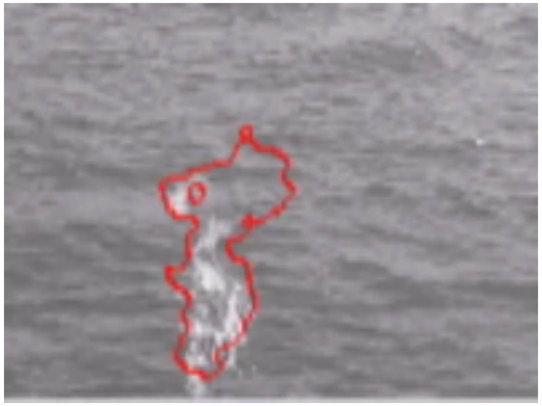


- Lossy image representation



- Motion segmentation

- DT segmentation



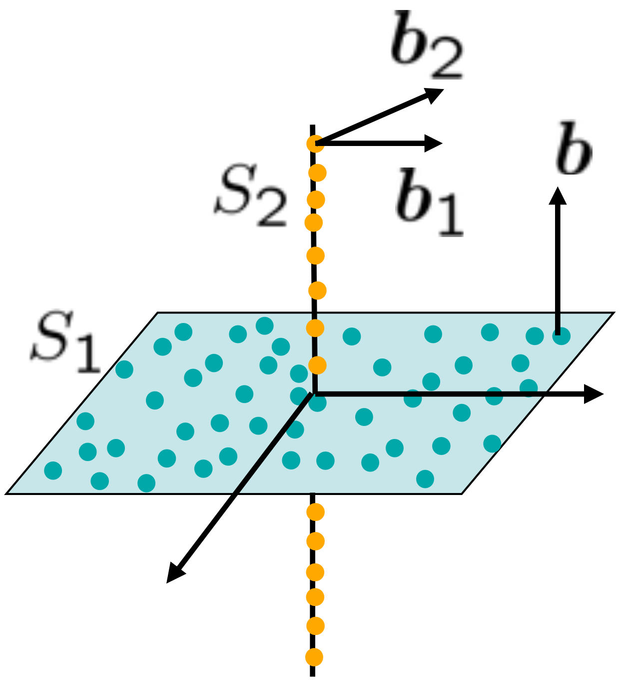
- Video segmentation

Subspace Clustering Problem

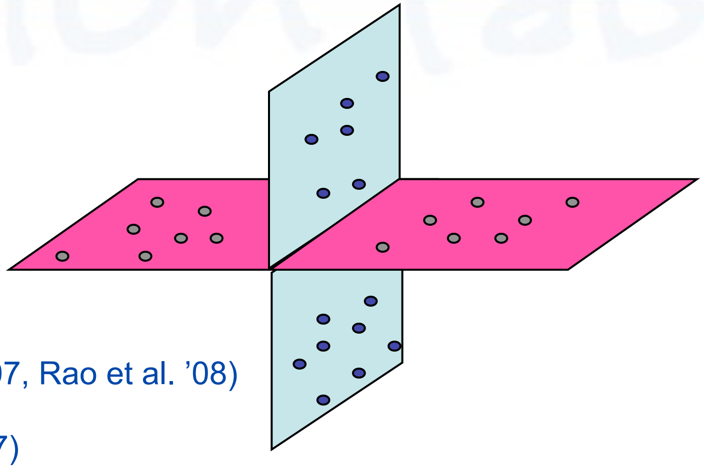
- Given a set of points lying in multiple subspaces, identify
  - The number of subspaces and their dimensions
  - A basis for each subspace
  - The segmentation of the data points

- Challenges
  - Model selection
  - Nonconvex
  - Combinatorial

- More challenges
  - Noise
  - Outliers
  - Missing entries



Prior Work on Subspace Clustering



- Iterative-Probabilistic Methods
  - K-subspaces (Bradley-Mangasarian ’00, Kambhatla-Leen ’94, Tseng’00, Agarwal-Mustafa ’04, Zhang et al. ’09, Aldroubi et al. ’09)
  - Mixtures of PPCA (Tipping-Bishop ’99, Grubber-Weiss ’04, Kanatani ’04, Archambeau et al. ’08, Chen ’11)
  - Agglomerative Lossy Compression (Ma et al. ’07, Rao et al. ’08)
  - RANSAC (Leonardis et al.’02, Yang et al. ’06, Haralik-Harpaz ’07)
- Algebraic-Geometric Methods
  - Factorization (Boult-Brown ’91, Costeira-Kanade ’98, Gear ’98, Kanatani et al. ’01, Wu ea ’01, Sekmen ‘13)
  - Generalized Principal Component Analysis (GPCA) (Shizawa-Maze ’91, Vidal et al. ’03 ’04 ’05, Huang et al. ’05, Yang et al. ’05, Derksen ’07, Ma et al. ’08, Ozay et al. ’10, Tsakiris-Vidal ’14 ‘15)
- Spectral Clustering-Based Methods
  - Local (Zelnik-Manor ’03, Yan-Pollefeys ’06, Fan-Wu ’06, Goh-Vidal ’07, Sekmen et al. ’12)
  - Global (Govindu ’05, Agarwal et al. ’05, Chen-Lerman ’08, Lauer-Schnorr ’09, Zhang et al. ’10)
  - Sparse and Low-Rank (Elhamifar-Vidal ’09 ’10 ‘13, Candes et al. ’12 ’13, Liu ’10 ‘13, Favaro-Vidal ’11 ’13, Wang et al.’13)

Prior Work on Subspace Clustering

Prior Work on Subspace Clustering

Interdisciplinary Applied Mathematics

René Vidal · Yi Ma · Shankar Sastry

Generalized Principal Component Analysis

This book provides a comprehensive introduction to the latest advances in the mathematical theory and computational tools for modeling highdimensional data drawn from one or multiple low-dimensional subspaces (or manifolds) and potentially corrupted by noise, gross errors, or outliers. This challenging task requires the development of new algebraic, geometric, statistical, and computational methods for efficient and robust estimation and segmentation of one or multiple subspaces. The book also presents interesting real-world applications of these new methods in image processing, image and video segmentation, face recognition and clustering, and hybrid system identification etc.

This book is intended to serve as a textbook for graduate students and beginning researchers in data science, machine learning, computer vision, image and signal processing, and systems theory. It contains ample illustrations, examples, and exercises and is made largely self-contained with three Appendices which survey basic concepts and principles from statistics, optimization, and algebraic-geometry used in this book.

Mathematics

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Generalized Principal Component Analysis

Interdisciplinary Applied Mathematics 40

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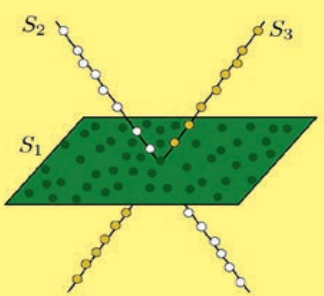
Yi Ma
Shankar Sastry

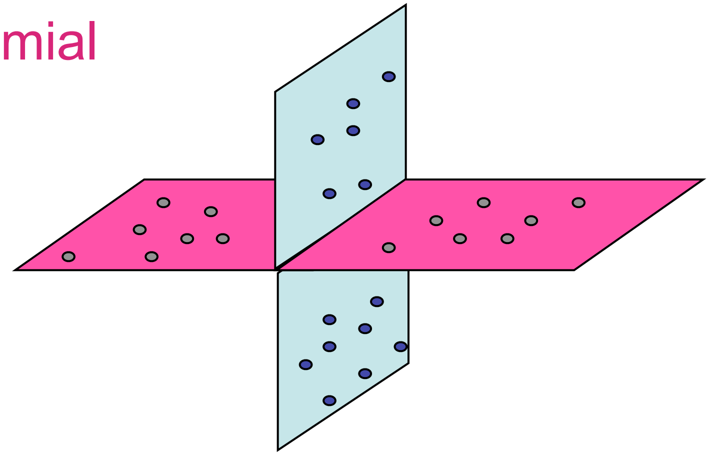
Generalized Principal Component Analysis



Springer

Algebraic-Geometric Subspace Clustering

- Key ideas
  - Union of subspaces = zero set of collection of polynomials
  - Number of subspaces = degree of polynomial
  - Subspaces = factors of polynomial

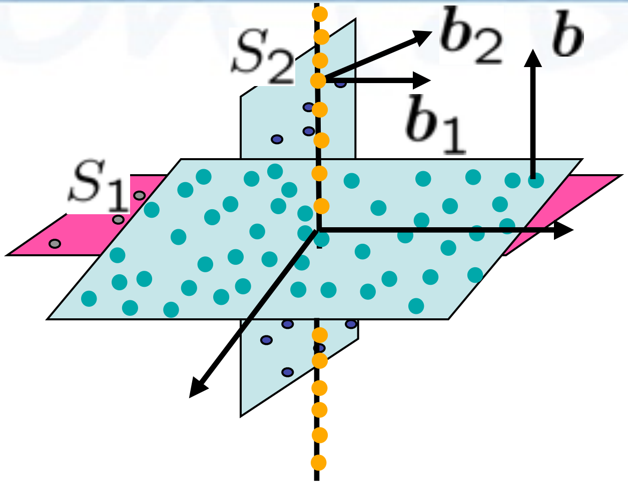


- Methods
  - Algebraic (Shizawa-Maze '91, Vidal et al. '03 '04 '05, Derksen '07, Ma et al. '08, Tsakiris and Vidal '14)
  - Robust algebraic (Huang et al. '05, Yang et al. '05, Ozay et al. '10, Tsakiris and Vidal '15)

Advantages	**Disadvantages / Open Problems**
Closed form solution	Computational complexity
Arbitrary dimensions	Sensitive to noise, outliers, missing entries

Talk Outline

- Algebraic Hyperplane Clustering (2003)
  - Polynomial fitting and factorization
- Algebraic Subspace Clustering (2004)
  - Polynomial fitting and differentiation
- Abstract and Robust Algebraic Subspace Clustering (2015)
  - Descending subspace filtrations + spectral clustering
- Extensions to Multiple View Geometry
- Applications
  - Motion segmentation
  - Handwritten digit clustering



R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA). CVPR 2003, CVPR 2004, PAMI 2005.
M. Tsakiris and R. Vidal. Abstract Algebraic Subspace Clustering. Asilomar 14, arxiv 15

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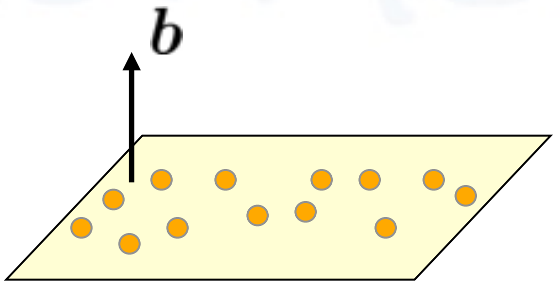
Algebraic Hyperplane Clustering (2003)

René Vidal, Yi Ma and Shankar Sastry

Representing a Union of Hyperplanes

- One hyperplane

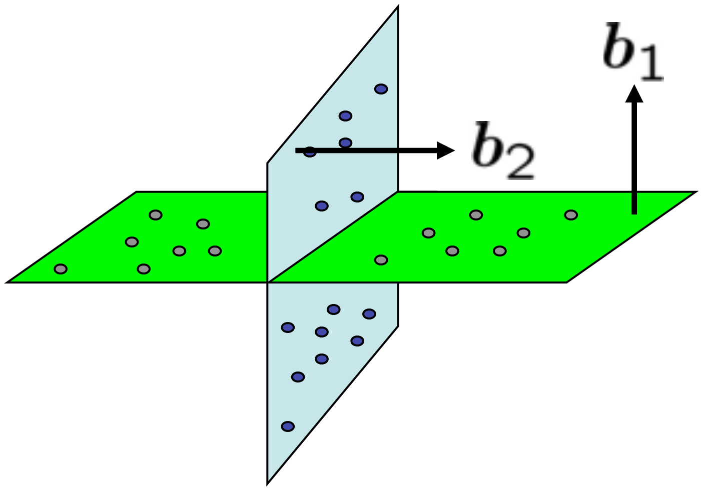
$$\boldsymbol{b}^\top \boldsymbol{x} = b_1 x_1 + b_2 x_2 + b_3 x_3 = 0$$



- Two hyperplanes

$$(\boldsymbol{b}_1^T \boldsymbol{x} = 0)\ \boxed{\text{or}}\ (\boldsymbol{b}_2^T \boldsymbol{x} = 0)$$

$$\boxed{p_2(\boldsymbol{x}) = (\boldsymbol{b}_1^T \boldsymbol{x})(\boldsymbol{b}_2^T \boldsymbol{x}) = 0}$$

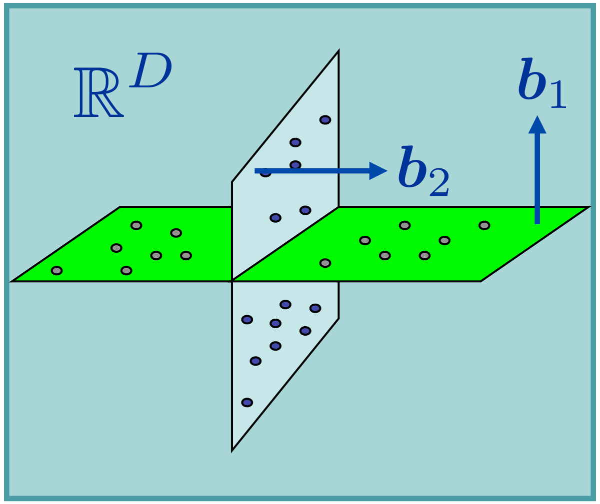


- A union of *n* hyperplanes can be represented as the zero set of one homogeneous polynomial of degree *n*

R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA), CVPR 2003.

Fitting Polynomial to Data Points

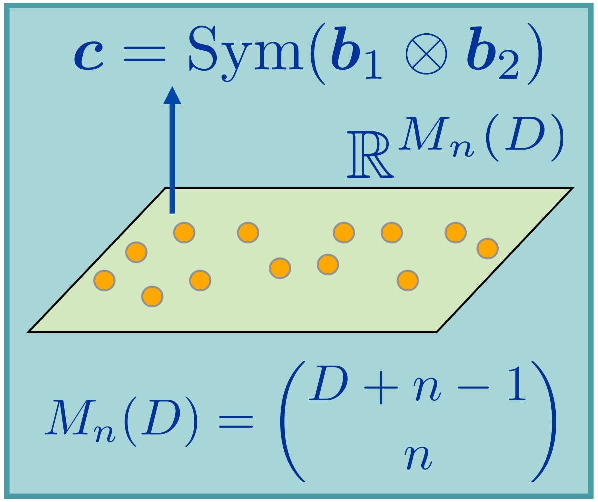
$$p_2(x) = (\boldsymbol{b}_1^\top \boldsymbol{x})(\boldsymbol{b}_2^\top \boldsymbol{x}) = c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2 = \boldsymbol{c}^\top \nu_2(\boldsymbol{x}) = 0$$



Veronese map

$$\nu_n : \mathbb{R}^D \to \mathbb{R}^{M_n(D)}$$

$$\boldsymbol{x} \to [x_1^{n_1} x_2^{n_2} \dots x_D^{n_D}]$$



- Coefficients can be found from null space of embedded data

$$L_n \boldsymbol{c} = \begin{bmatrix} \nu_n(\boldsymbol{x}_1) & \cdots & \nu_n(\boldsymbol{x}_N) \end{bmatrix}^\top \boldsymbol{c} = \boldsymbol{0}$$

- Number of hyperplanes from rank of embedded data matrix

$$n = \min\{i : L_i \text{ drops rank}\}$$

R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA), CVPR 2003.

Polynomial Factorization Algorithm

- Given polynomial, how to find normals?

$$\boldsymbol{c} \in \mathbb{R}^{M_n}$$

$$\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \cdots \quad \boldsymbol{b}_n$$

$$p_n(\boldsymbol{x}) = \boldsymbol{c}^\top \nu_n(\boldsymbol{x}) = (\boldsymbol{b}_1^\top \boldsymbol{x}) \cdots (\boldsymbol{b}_n^\top \boldsymbol{x})$$

- Polynomial Factorization Algorithm
  - Find roots of polynomial of degree n in one variable
  - Solve D-2 linear systems in n variables

- Challenges
  - Computing roots may be sensitive to noise
  - The estimated polynomial may not perfectly factor with noisy data

R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA), CVPR 2003.

Polynomial Differentiation Algorithm

$$\boldsymbol{c} \in \mathbb{R}^{M_n}$$

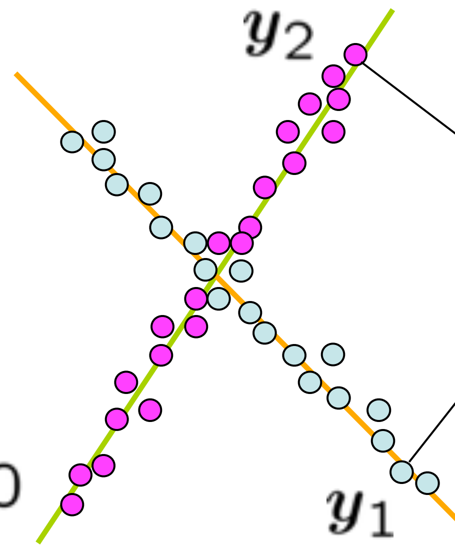
$$\boldsymbol{b}_1 \qquad \boldsymbol{b}_2 \qquad \dots \qquad \boldsymbol{b}_n$$

$$\boldsymbol{b}_i = Dp_n(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{y}_i} \qquad \boldsymbol{y}_i \in S_i$$

$$\boxed{p_n(\boldsymbol{x}) = 0}$$

$$\boldsymbol{b}_1^T \boldsymbol{x} = 0$$

$$\boldsymbol{b}_2^T \boldsymbol{x} = 0$$



$$\boldsymbol{b}_2 \sim Dp_n(\boldsymbol{y}_2)$$

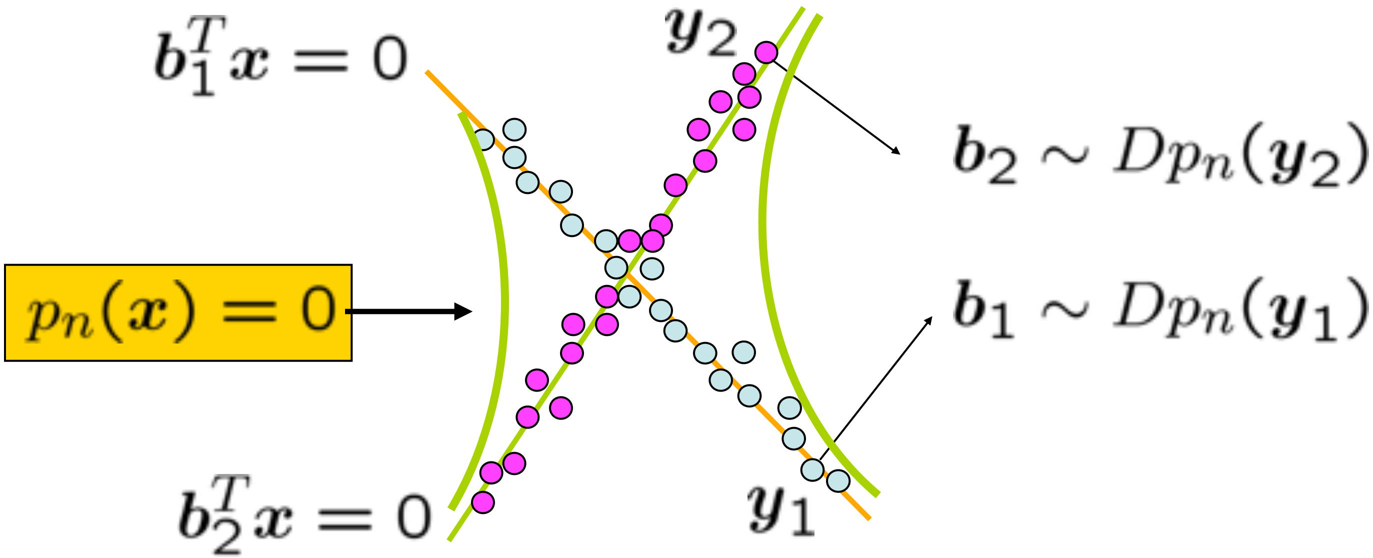
$$\boldsymbol{b}_1 \sim Dp_n(\boldsymbol{y}_1)$$

- To learn a union of hyperplanes we just need one positive example per class

Vidal, Ma, Piazzi, Sastry. A new GPCA Algorithm for Clustering Subspaces by Fitting, Differentiating and Dividing Polynomials, CVPR 04.
R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA), PAMI 2005.

Polynomial Differentiation Algorithm

- With noise and outliers
  - Polynomials may not be a perfect union of subspaces



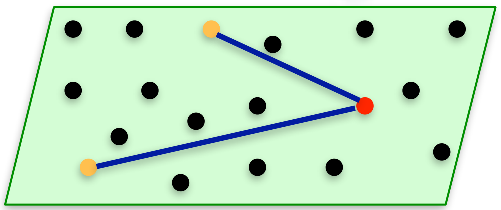
  - Normals can estimated correctly by choosing points optimally
- Distance to closest subspace without knowing segmentation?

$$\|x - \tilde{x}\| = \sqrt{\frac{|p_n(x)|}{\|Dp_n(x)\|}} + O\left(\|x - \tilde{x}\|^2\right)$$

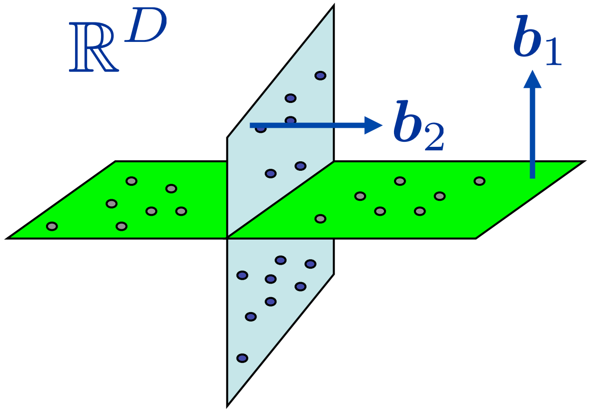
Vidal, Ma, Piazzi, Sastry. A new GPCA Algorithm for Clustering Subspaces by Fitting, Differentiating and Dividing Polynomials, CVPR 04.
R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA), PAMI 2005..

Spectral Algebraic Hyperplane Clustering

- Spectral clustering
  - Represent data points as nodes in graph G
  - Connect nodes i and j with weight c_{ij}
  - Infer clusters from Laplacian of G



- How to define a good affinity matrix A for hyperplanes?
  - points in the same subspace: $A_{ij} \approx 1$
  - points in different subspaces: $A_{ij} \approx 0$



- Use cosine of angle between normals at data points *i* and *j*

$$A_{ij} = |\langle \boldsymbol{b}_i, \boldsymbol{b}_j \rangle|$$

Vidal, Ma, Piazzi, Sastry. A new GPCA Algorithm for Clustering Subspaces by Fitting, Differentiating and Dividing Polynomials, CVPR 04.
R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA), PAMI 2005.

Spectral Algebraic Hyperplane Clustering

- Polynomial coefficients from null space of embedded data matrix
  - Sensitive to outliers
  - Computationally intensive

$$L_n \boldsymbol{c} = \begin{bmatrix} \nu_n(\boldsymbol{x}_1)^\top \\ \vdots \\ \nu_n(\boldsymbol{x}_N)^\top \end{bmatrix} \boldsymbol{c} = \boldsymbol{0}$$

- Number of hyperplanes from rank of embedded data matrix
  - Sensitivity to noise and outliers

$$n = \min\{i : L_i \text{ drops rank}\}$$

- Hyperplane normals $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_N$ from gradients of the polynomial

$$\boldsymbol{b}_j = \nabla p_n(\boldsymbol{x}_j)$$

- Clustering using angles between normals to construct an affinity

$$A_{ij} = |\langle \boldsymbol{b}_i, \boldsymbol{b}_j \rangle|$$

Vidal, Ma, Piazzi, Sastry. A new GPCA Algorithm for Clustering Subspaces by Fitting, Differentiating and Dividing Polynomials, CVPR 04.
R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA), PAMI 2005.

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Algebraic Subspace Clustering (2004)

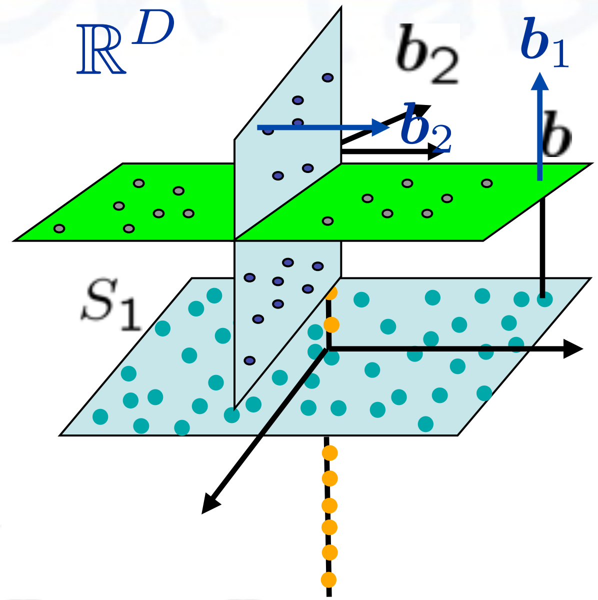
René Vidal, Yi Ma and Shankar Sastry

Representing a Union of *n* Subspaces

- Two planes $(b_1^T x = 0)$ $\boxed{\text{or}}$ $(b_2^T x = 0)$

$$\boxed{p_2(x) = (b_1^T x)(b_2^T x) = 0}$$

- One plane and one line
  - Plane: $S_1 = \{x : b^T x = 0\}$
  - Line: $S_2 = \{x : b_1^T x = b_2^T x = 0\}$



$$S_1 \cup S_2 = \{x : (b^T x = 0) \ \boxed{\text{or}} \ (b_1^T x = b_2^T x = 0\}$$

De Morgan's rule

$$S_1 \cup S_2 = \{x : \boxed{(b^T x)(b_1^T x) = 0} \ \boxed{\text{and}} \ \boxed{(b^T x)(b_2^T x) = 0\}}$$

- A union of n subspaces can be represented as the zero set of a collection of homogeneous polynomials of degree n

Vidal, Ma, Piazzi, Sastry. A new GPCA Algorithm for Clustering Subspaces by Fitting, Differentiating and Dividing Polynomials, CVPR 04.
R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA), PAMI 2005.

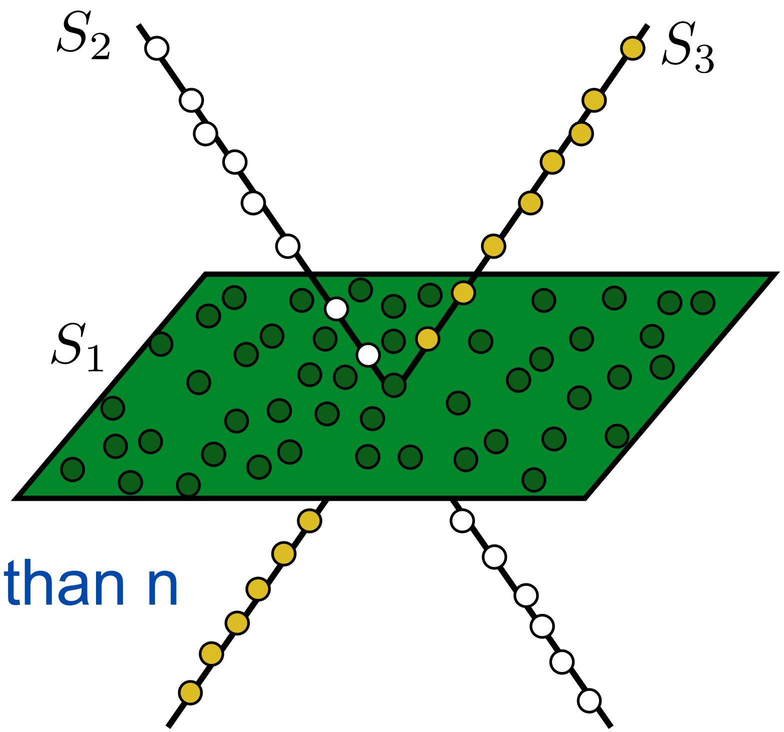
Fitting Polynomials to Data Points

- # polynomials depends on # subspaces and their dimensions

$$\boxed{L_n \boldsymbol{c} = \begin{bmatrix} \nu_n(\boldsymbol{x}_1) & \cdots & \nu_n(\boldsymbol{x}_N) \end{bmatrix}^\top \boldsymbol{c} = \boldsymbol{0}}$$

- Polynomials may not factorize

$$\begin{aligned} p_1 &= x_1(x_1 + x_2) \\ p_2 &= x_2(x_2 - x_1) \\ p_1 + p_2 &= x_1^2 + x_2^2 \end{aligned}$$



- Polynomials may be of degree less than n
  - 1 polynomial of degree 2
  - 4 polynomials of degree 3

Derksen. Hilbert series of subspace arrangements, Journal of Pure and Applied Algebra, 2007
Vidal, Ma, Piazzi, Sastry. A new GPCA Algorithm for Clustering Subspaces by Fitting, Differentiating and Dividing Polynomials, CVPR 04.
R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA), PAMI 2005.

Polynomial Differentiation Algorithm

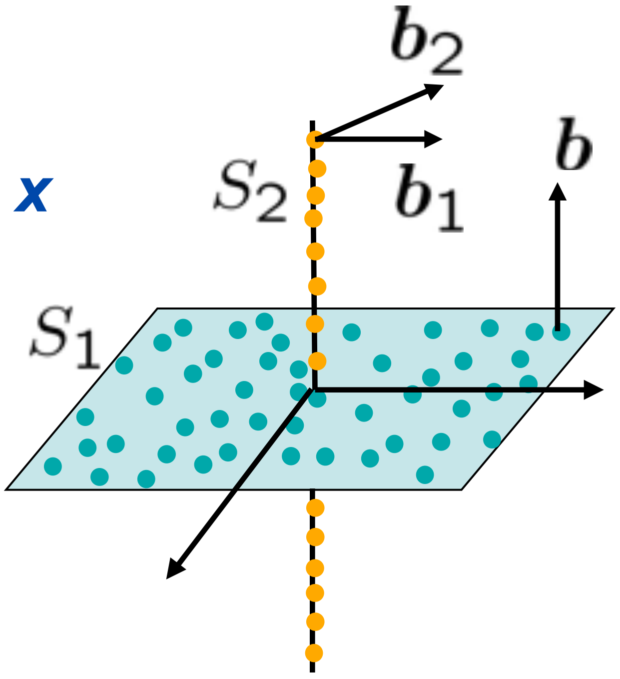
- **Theorem 1**: Gradients at ***x*** of any vanishing polynomial are normals to the subspace passing through ***x***

$$\forall \boldsymbol{x} \in S_i, \quad \nabla p_n(\boldsymbol{x}) \perp S_i$$

- **Theorem 2**: Gradients at ***x*** span orthogonal complement to subspace at ***x***

$$\forall \boldsymbol{x} \in S_i, \boldsymbol{x} \notin S_k, k \neq i$$

$$S_i = \operatorname{span}\{\nabla p_n(\boldsymbol{x})\}^{\perp}$$



Vidal, Ma, Piazzi, Sastry. A new GPCA Algorithm for Clustering Subspaces by Fitting, Differentiating and Dividing Polynomials, CVPR 04.
R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA), PAMI 2005.

Spectral Algebraic Subspace Clustering

- Polynomial coefficients from null space of embedded data matrix
  - Need upper bound on *n*
  - Hard to know # polynomials

$$L_n \boldsymbol{c} = \begin{bmatrix} \nu_n(\boldsymbol{x}_1)^\top \\ \vdots \\ \nu_n(\boldsymbol{x}_N)^\top \end{bmatrix} \boldsymbol{c} = \boldsymbol{0}$$

- Subspace normals from gradients of the polynomial
  - Need subspace dimension?

$$B_j = \mathrm{PCA}(\mathrm{span}\{\nabla p_n(\boldsymbol{x}_j)\})$$

- Clustering using affinity based on
  - Subspace angles
  - Distance from points to subspaces

$$A_{ij} = |\langle B_i, B_j \rangle|$$

$$A_{ij} = 1 - \frac{1}{2}\mathrm{dist}(\boldsymbol{x}_i, B_j) - \frac{1}{2}\mathrm{dist}(\boldsymbol{x}_j, B_i)$$

Vidal, Ma, Piazzi, Sastry. A new GPCA Algorithm for Clustering Subspaces by Fitting, Differentiating and Dividing Polynomials, CVPR 04.
R. Vidal, Y. Ma, S. Sastry. Generalized Principal Component Analysis (GPCA), PAMI 2005.
Tsakiris and Vidal. Filtrated Spectral Algebraic Subspace Clustering. ICCV Workshop on Robust Subspace Learning and Computer Vision, 2015.

Dealing with High-Dimensional Data

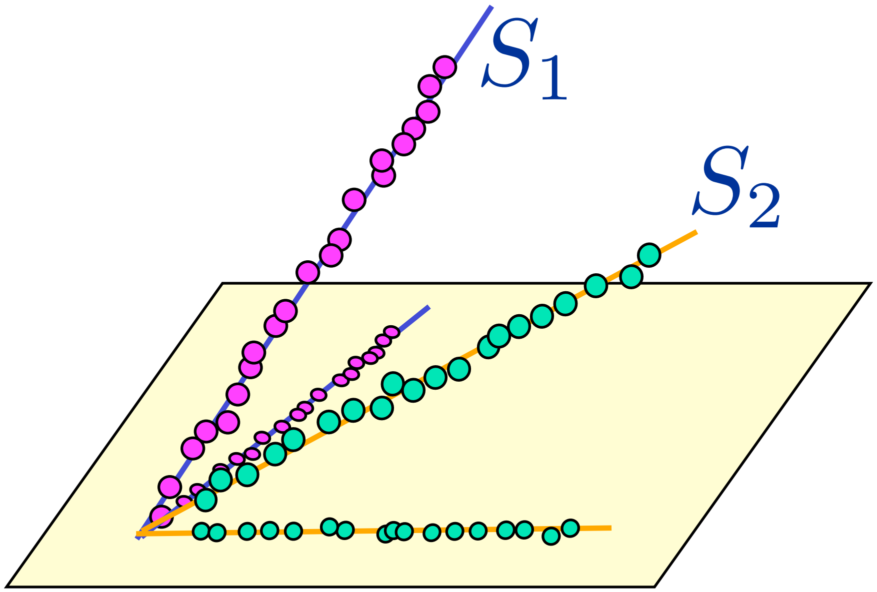
- Minimum number of points
  - D = dimension of the data
  - n = number of subspaces

$$M_n(D) = \binom{D+n-1}{n}$$

- When dimension of each subspace is small

$$d_i \ll D$$

  - # subspaces and their dimensions are preserved by a random linear projection



- Apply method to data projected onto subspace of dimension

$$d_{\max} = \max_i d_i + 1$$

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Abstract and Robust Algebraic Subspace Clustering (2015)

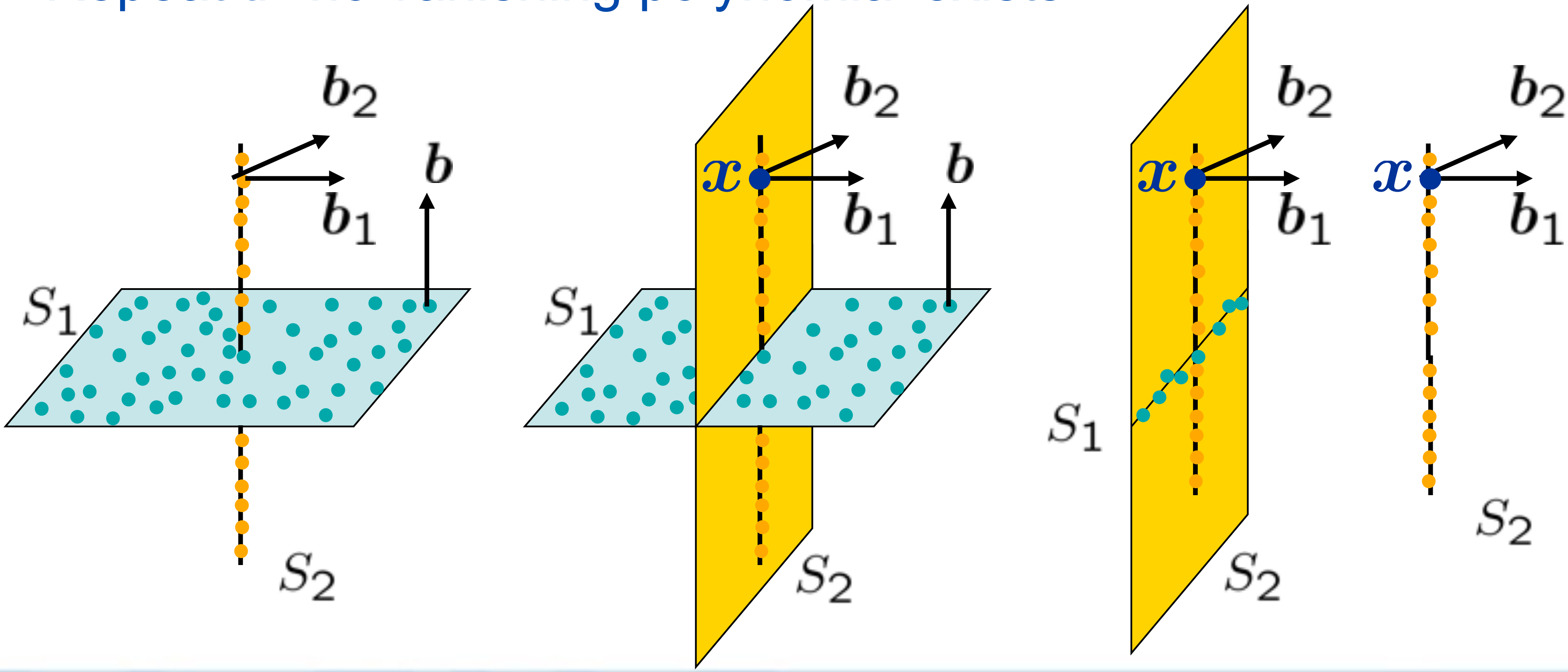
Manolis Tsakiris and René Vidal

So Far So Good?

- Hyperplanes are easy
  - single polynomial
- Subspaces of equal dimensions
  - Can be reduced to the case of hyperplanes
- Subspaces of arbitrary dimensions:
  - Need a basis of polynomials of degree n
  - In practice, we need to know n
- How to proceed if *n* is unknown?

Descending Filtration at Reference Point *x*

- Find polynomial *p* vanishing on (projected) data: $L_n \boldsymbol{c} = 0$
- Find hyperplane *H* passing through ***x***: $\boldsymbol{b} = \nabla p(\boldsymbol{x})$
- Project data onto *H*
- Repeat till no vanishing polynomial exists



Tsakiris and Vidal. Abstract Algebraic-Geometric Subspace Clustering. Asilomar Conference on Signals, Systems and Computers, 2014
Tsakiris and R. Vidal. Abstract Algebraic Subspace Clustering. ArXiv, 2015.

Abstract Algebraic Subspace Clustering

- **Abstract Algebraic Subspace Clustering Algorithm**
  - Choose any data point x
  - Find descending filtration at x to identify subspace S passing through x
  - Remove points in S
  - Repeat until all points are assigned to a subspace

- **Theorem 3**: A union of an unknown number of subspaces in general position and of arbitrary and unknown dimensions, can be decomposed to the list of its constituent subspaces in a recursive fashion, using the filtrations described above.

Tsakiris and Vidal. Abstract Algebraic-Geometric Subspace Clustering. Asilomar Conference on Signals, Systems and Computers, 2014
Tsakiris and R. Vidal. Abstract Algebraic Subspace Clustering. ArXiv, 2015.

Filtrated Spectral Algebraic Subspace Clustering

- **Problem 1**: two points approximately in the same subspace will produce different hyperplanes. Which point to choose?
- **Problem 2**: given a hyperplane passing approximately through a point x, how do we determine other points in the same hyperplane? Choose a threshold, but which one?
- **Problem 3**: how many steps in the filtration should be taken?
- **Solution**
  - Compute filtration at each data point
  - Define affinity by comparing their filtrations

$$A_{ij}^k = \begin{cases} \|\pi_k^i \circ \cdots \circ \pi_1^i(x_j)\| & \text{if } \boldsymbol{x}_j \text{ remains} \\ 0 & \text{else} \end{cases}$$

Tsakiris and Vidal. Filtrated Spectral Algebraic Subspace Clustering. ICCV Workshop on Robust Subspace Learning and Computer Vision, 2015.

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Extensions to Nonlinear Manifolds in Multiple View Geometry

René Vidal, Yi Ma, Stefano Soatto, Shankar Sastry, Richard Hartley

Extensions to Nonlinear Manifolds

- Segmentation of quadratic forms (Rao-Yang-Ma '05, '06)

$$\prod_{i=1}^{n}(\boldsymbol{x}^\top A\boldsymbol{x}) = \nu_n(\boldsymbol{x})^\top \mathcal{A}\nu_n(\boldsymbol{x}) = 0$$

- Segmentation of bilinear surfaces (Vidal-Ma-Soatto-Sastry '03, '06)

$$\prod_{i=1}^{n}(\boldsymbol{x}_2^\top F_i\boldsymbol{x}_1) = \nu_n(\boldsymbol{x}_2)\mathcal{F}\nu_n(\boldsymbol{x}_1) = 0 \qquad F_i \in so(3) \times SO(3)$$

- Segmentation of mixed linear and bilinear (Singaraju-Vidal '06)

$$\prod_{i=1}^{n}(\boldsymbol{b}_i^\top \boldsymbol{x}_1)\prod_{j=1}^{m}(\boldsymbol{x}_2^\top A_j\boldsymbol{x}_1) = \nu_n(\boldsymbol{x}_2)^\top \mathcal{A}\nu_{n+m}(\boldsymbol{x}_1) = 0$$

- Segmentation of trilinear surfaces (Hartley-Vidal '05, '08)

$$\prod_{i=1}^{n}(\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3 T_i) = \nu_n(\boldsymbol{x}_1)\nu_n(\boldsymbol{x}_2)\nu_n(\boldsymbol{x}_3)\mathcal{T} = 0$$

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Simulation Experiments and Applications in Computer Vision

THE DEPARTMENT OF BIOMEDICAL ENGINEERING
The Whitaker Institute at Johns Hopkins

Center for IMAGING SCIENCE

Experiments on Synthetic Data

Table 1. Mean clustering error in % over 500 independent experiments on synthetic data for 3 subspaces of $\mathbb{R}^5$ of varying dimensions (d_1, d_2, d_3) and varying levels of noise $\sigma \in \{0, 1, 3, 5\}\%$.

method	$(1,1,1)$	$(2,2,2)$	$(3,3,3)$	$(4,4,4)$	$(1,2,3)$	$(2,3,4)$
$\sigma = 0\%$						
FSASC	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
SASC-D	0.76	0.96	0.03	**0.00**	0.42	0.13
SASC-A	**0.00**	34.2	23.3	**0.00**	20.8	10.7
SSC	**0.00**	0.16	6.18	43.2	1.38	47.1
LRR	**0.00**	1.37	25.5	48.7	1.84	25.9
LRR-H	**0.00**	1.15	18.1	48.4	1.53	20.8
LRSC	**0.00**	1.32	25.6	48.7	2.27	26.5
LSR	**0.00**	7.36	34.9	48.8	5.42	28.8
LSR-H	**0.00**	1.31	28.8	29.3	1.46	32.7

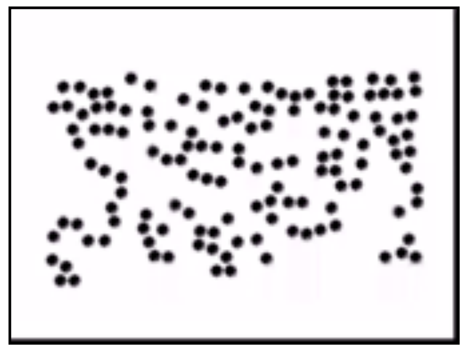
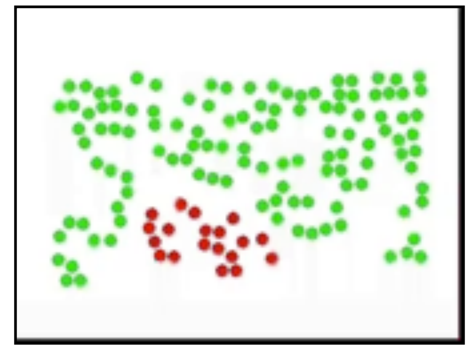
Experiments on Synthetic Data

Table 1. Mean clustering error in % over 500 independent experiments on synthetic data for 3 subspaces of $\mathbb{R}^5$ of varying dimensions (d_1, d_2, d_3) and varying levels of noise $\sigma \in \{0, 1, 3, 5\}\%$.

method	$(1,1,1)$	$(2,2,2)$	$(3,3,3)$	$(4,4,4)$	$(1,2,3)$	$(2,3,4)$
$\sigma = 1\%$						
FSASC	1.70	**0.20**	**0.22**	3.17	**0.94**	**0.81**
SASC-D	2.91	0.58	0.96	**2.39**	1.78	1.89
SSC	**1.51**	1.27	6.24	42.5	8.71	44.2
$\sigma = 3\%$						
FSASC	4.39	**1.16**	**1.40**	7.67	**2.82**	**2.88**
SASC-D	8.41	2.68	4.05	**6.15**	5.40	6.46
SSC	**3.90**	5.17	7.35	44.7	9.50	33.2

Experiments on 3D Motion Segmentation

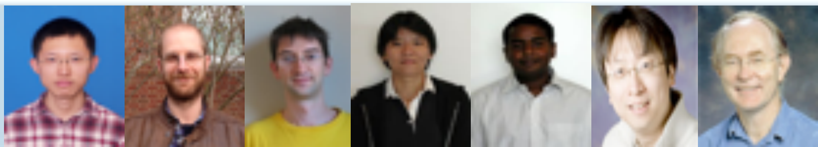
- Motion segmentation problem
  - Input: multiple images of a scene with multiple rigid-body motions
  - Output: number of motions, motion model parameters, segmentation

- Motion of a rigid-body: 4D subspace (Boult and Brown '91, Tomasi and Kanade '92)
  - P = #points
  - F = #frames

$$\underbrace{\begin{bmatrix} \boldsymbol{x}_{11} & \cdots & \boldsymbol{x}_{1P} \\ \vdots & \ddots & \vdots \\ \boldsymbol{x}_{F1} & \cdots & \boldsymbol{x}_{FP}\end{bmatrix}}_{2F\times P} = \underbrace{\begin{bmatrix} \boldsymbol{A}_1 \\ \vdots \\ \boldsymbol{A}_F\end{bmatrix}}_{2F\times 4} \underbrace{\begin{bmatrix} \boldsymbol{X}_1 & \cdots & \boldsymbol{X}_P\end{bmatrix}}_{4\times P}$$

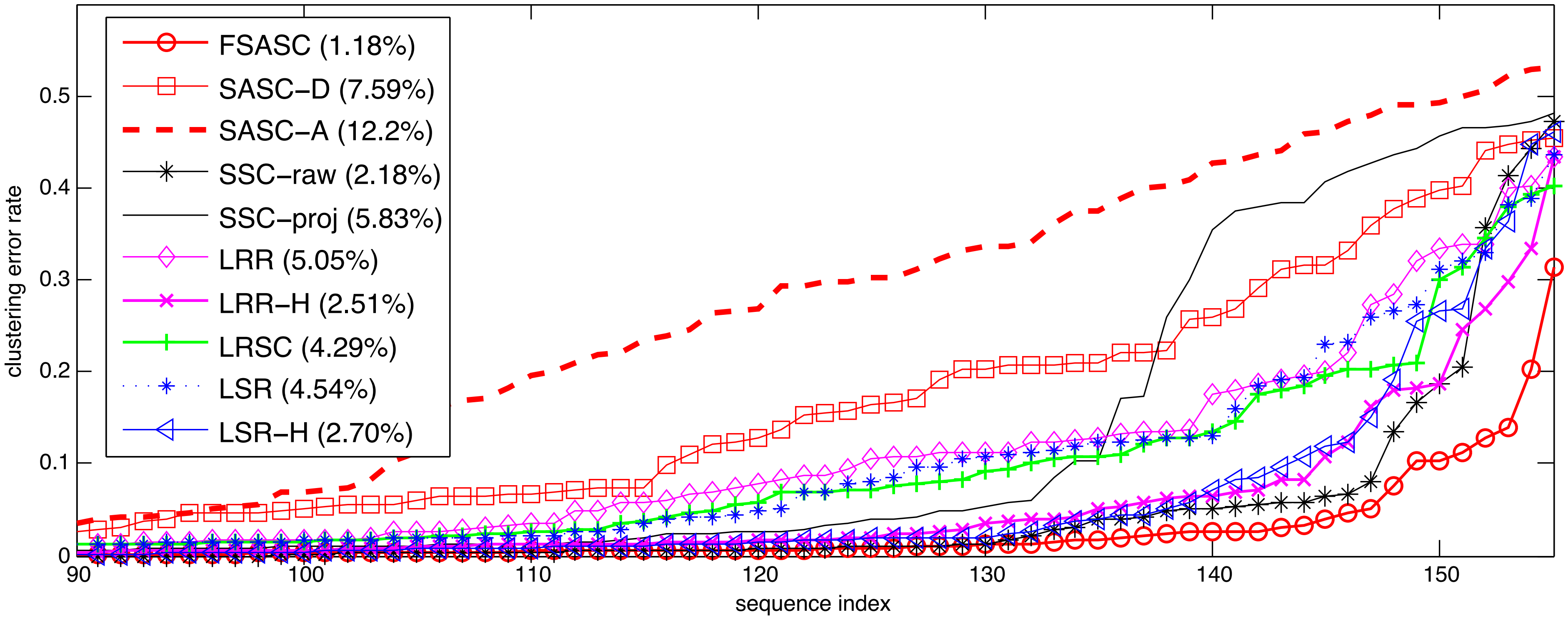
Vidal et al., ECCV02, IJCV06; Vidal, Ma and Sastry CVPR03, PAMI05; Vidal and Sastry CVPR03; Vidal and Ma ECCV04, JMIV06; Vidal and Hartley, CVPR04; Tron and Vidal, CVPR07; Li et al. CVPR07; Goh and Vidal CVPR07; Vidal and Hartley, PAMI08; Vidal, Tron and Hartley IJCV08; Rao et al. CVPR 08, PAMI 09; Elhamifar and Vidal, CVPR 09, TPAMI 13; Vidal SPM11; Tsakiris '15



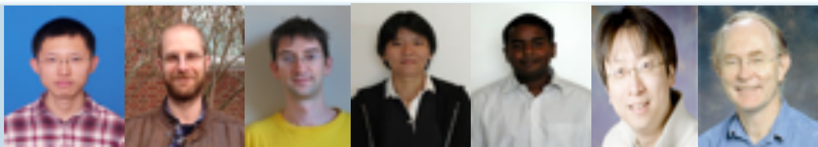
Experiments on 3D Motion Segmentation

- Misclassification rates on Hopkins 155 database

R. Tron and R. Vidal. A Benchmark for the Comparison of 3-D Motion Segmentation Algorithms. CVPR 2007.

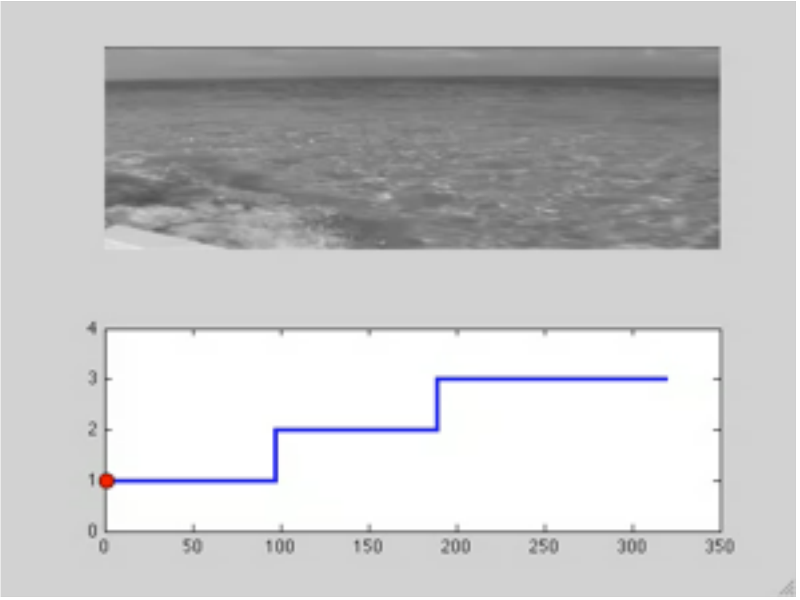


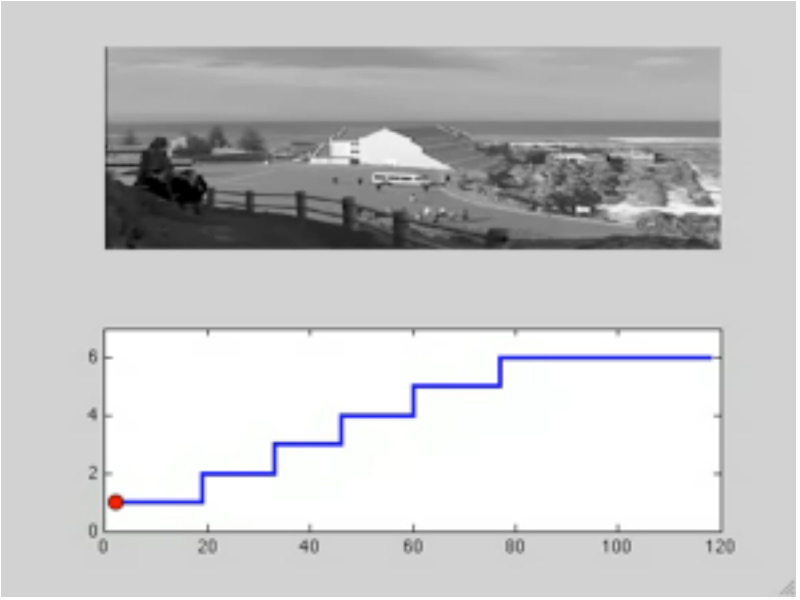
Vidal et al., ECCV02, IJCV06; Vidal, Ma and Sastry CVPR03, PAMI05; Vidal and Sastry CVPR03; Vidal and Ma ECCV04, JMIV06; Vidal and Hartley, CVPR04; Tron and Vidal, CVPR07; Li et al. CVPR07; Goh and Vidal CVPR07; Vidal and Hartley, PAMI08; Vidal, Tron and Hartley IJCV08; Rao et al. CVPR 08, PAMI 09; Elhamifar and Vidal, CVPR 09, TPAMI 13; Vidal SPM11; Tsakiris ‘15



Experiments on Video Segmentation

- Model each video segment as a low-dimensional subspace
- Cluster video frames into multiple segments





- Advantages
  - SSC easily detects sharp transitions in the video
  - SSC can handle camera motion and scene variations

Handwritten Digit Clustering

- MNIST Dataset
  - 200 images per class
  - 100 trials
  - Apply ASC methods to 13 principal components

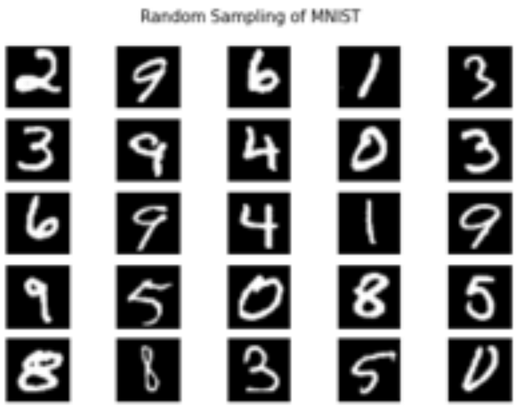


Table 5. Clustering error (%) for two digits $(1, i)$, $i = 0, 2, \ldots, 9$ in the MNIST dataset.

	Digits Pair								
method	$(1,0)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$	$(1,7)$	$(1,8)$	$(1,9)$
FSASC	**0.50**	4.67	1.55	3.31	1.11	1.62	2.27	4.88	1.81
SASC-D	4.91	14.2	10.3	23.9	8.55	13.1	10.2	21.5	17.5
SSC-raw	1.12	9.15	2.66	5.77	2.78	1.87	2.90	13.3	2.00
LSR	1.03	5.26	2.13	19.1	1.29	1.50	5.40	15.3	5.90
LSR-H	0.74	**1.35**	**1.12**	**3.15**	**0.88**	**0.90**	**1.33**	**4.38**	**1.11**

Conclusions and Open Problems

- Algebraic-geometric algorithms for clustering data lying in an arbitrary union of subspaces of unknown and different dimensions
  - Projecting data onto a low-dimensional subspace
  - Fitting polynomials to projected subspaces
  - Differentiating polynomials to obtain a basis
  - Using descending filtrations to compute an affinity for spectral clustering
- **Robust and efficient polynomial fitting methods**
  - Characterizing the set of homogeneous polynomials that factorize as a product of linear forms (Brill's equations)
  - Using such characterization to estimate a "factorizable" vanishing polynomial from data corrupted by noise, outliers and missing entries
  - Avoid computing the Veronese map?
- **Robust and efficient polynomial factorization methods**
  - Methods for factorizing polynomials in 1000+ variables in < 1ms

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  - Shankar Sastry
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  - ONR Young Investigator Award
  - Sloan Research Fellowship
  - NSF BIGDATA Award 1447822

- More information/code
  - Vision Lab @ Johns Hopkins University http://www.vision.jhu.edu

Thank You!

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