A TWO DIMENSIONAL EXPLICIT FINITE ELEMENT APPROXIMATION TO SHALLOW WATER EQUATIONS

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1 ABSTRACT

This paper shows the application of a finite element method for solving shallow water flow problems as well as a comparison between the use of lumped mass SUPG and consistent mass SUPG method. An explicit scheme has been employed for the integration in the temporal layer. Several examples based on the one-dimensional shallow-water equations illustrate the accuracy and efficiency obtained with such methods. This work is the first stage of a project oriented to parallel computing to solve turbulent shallow water equations.

KEY WORDS: computational fluid dinamics; explicit scheme; shallow water; SUPG method

2 INTRODUCTION

Applications of the shallow water equation include a wide variety of coastal phenomena such as drift and tidal current, pollutant dispersion, storm surge, tsunami wave propagation, drifts and transport. A great number of civil engineering projects in river hydraulics, coastal water and estuaries require predictive models of the flow. The trend is towards computational methods based on the one, or two dimensional shallow water equations. The use of a fine spatial grid is often required in practical applications thereby necessitating a very small integration time step if an explicit scheme is employed, having as a consequence, an unacceptable computational cost.

Due to the current state of parallel computation, explicit methods have found a new interest because of their great adaptation to the parallel programming A.Di Filippo (1995),C.E.Baumann and S.R.Idelsohn (1992),Kawahara (1980),Matsuto Kawahara (1978),Mutsuto Kawahara and Inagaki (1982),R.Lohner (1984).

In this paper an application of an explicit scheme in time using both Lumped SUPG mass matrix or Consistent SUPG mass matrix in combination with the use of a stabilized spatial discretization for solving shallow water flow problems has been examinedC.E.Baumann and S.R.Idelsohn (1992),Hughes and Tezduyar (1985). For simplicity we began adopting the lumped SUPG mass matrix scheme but our results show a very diffusive behavior in the description of a typical test problem like a solitary wave propagation along a one-dimensional channel with uniform bottom slope. On the other hand and thinking about to be consistent with the stabilized SUPG formulation we found that this scheme results to be unstable for the planar shallow water one-dimensional equation. These facts have been the main motivation to the introduction of a β-parameter in the mass matrix of the consistent SUPG method in order to recover part of its stability taking advantage of its reasonable accuracy. We have confirmed this hypothesis by numerical analysis arguments taking as a model equation the clasical unsteady one-dimensional advective equation. Moreover, several numerical examples based on the one-dimensional shallow-water equations illustrate the accuracy and efficiency obtained with such methods.

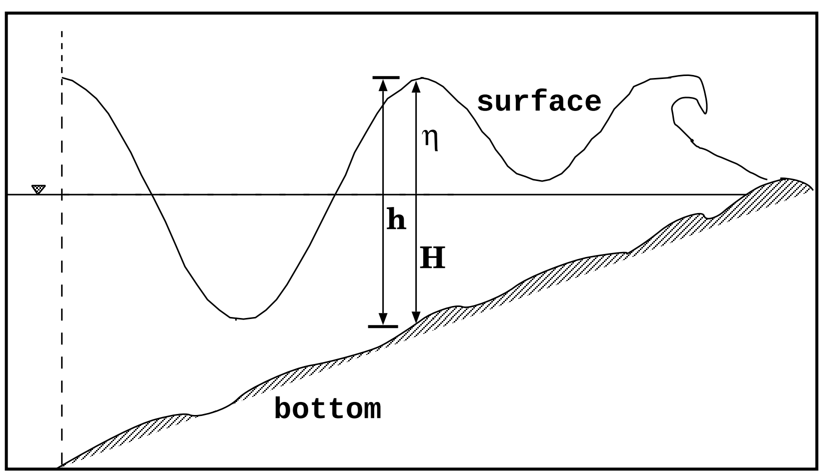


Figure 1: Coordinate system for shallow water equations.

3 Multidimensional shallow water equations

The shallow water equations can be written in the following form:

$$\mathbf{U}_{,t} + \mathbf{F}_{j,j}(\mathbf{U}) = \mathbf{B} \quad on \quad \Omega \subset \mathbb{R}^2 \tag{1}$$

where

$$\mathbf{U} = \begin{bmatrix} h\mathbf{u_1} \\ h\mathbf{u_2} \\ h \end{bmatrix} \tag{2}$$

$$\mathbf{F}_j = \begin{bmatrix} hu_1u_j + \delta_{1j}\frac{1}{2}gh^2 \\ \\ hu_2u_j + \delta_{2j}\frac{1}{2}gh^2 \\ \\ h\mathbf{u_j} \end{bmatrix} \tag{3}$$

$$\mathbf{B} = \begin{bmatrix} gh\frac{\partial H}{\partial x} \\ \\ gh\frac{\partial H}{\partial y} \\ \\ 0 \end{bmatrix} \tag{4}$$

The flux vector $\mathbf{F}_j(\mathbf{U})$ is a homogeneous function of degree one in the conservative variables $\mathbf{U}$; is follows that

$$\mathbf{F}_j(\mathbf{U}) = \mathbf{A}_j\mathbf{U}$$
$$\mathbf{F}_{j,j} = \mathbf{A}_j\mathbf{U}_{,j}$$

Let $\mathbf{n} = (n_1, n_2)$ be the outward unit normal vector on the boundaries and let $\mathbf{F}_j$ be split in the following way:

$$\mathbf{F}_j = \mathbf{F}_j^{(1)} + \mathbf{F}_j^{(2)} = \begin{bmatrix} \delta_{1j}\frac{1}{2}gh^2 \\ \\ \delta_{2j}\frac{1}{2}gh^2 \\ \\ 0 \end{bmatrix} + \begin{bmatrix} hu_1u_j \\ \\ hu_2u_j \\ \\ h\mathbf{u_j} \end{bmatrix}$$

Later we will make reference to

$$\mathbf{F}_n^{(2)} = \mathbf{F}_j^{(2)}\mathbf{n}_j = \begin{bmatrix} hu_1u_{\mathbf{n}} \\ hu_2u_{\mathbf{n}} \\ h\mathbf{u}_{\mathbf{n}} \end{bmatrix}$$

4 Finite Element Formulation

Consider a discretization of Ω into element subdomains $\Omega^e, e = 1, \cdots, nel$ where nel is the number of elements. We assume

$$\overline{\Omega} = \overset{nel}{\underset{e=1}{\cup}} \overline{\Omega^e}, \quad \emptyset = \overset{nel}{\underset{e=1}{\cap}} \Omega^e$$

Also let Γ^e be the whole boundary of element e, Γ the boundary of Ω and Γ_{int} the following set:

$$\Gamma_{int} = (\overset{nel}{\underset{e=1}{\cup}} \Gamma^e) - \Gamma$$

The SUPG formulation is employed for the spatial discretization and the flow field to be analyzed is divided into small regions called finite elements Ω^e. The finite approximation leads to

$$\mathbf{U} = \sum_{j=1}^{numnp} \mathbf{N}^j \mathbf{U}^j, \quad \mathbf{F}_i = \sum_{j=1}^{numnp} \mathbf{N}^j \mathbf{F}_i{}^j$$

where $numnp$ denotes the total number of nodes in the discretization, $\mathbf{N}^j$ are the global piecewise bilinear basis functions and $\mathbf{U}^j, \mathbf{F}_i^j$ are the values of $\mathbf{U}, \mathbf{F}_i$ at node j. In the SUPG formulation the weighting functions are modified by the addition of a given perturbation function $\widetilde{\mathbf{P}}^j$, and now, the modified weighting function takes the form:

$$\widetilde{\mathbf{N}}^j = \mathbf{N}^j + \widetilde{\mathbf{P}}^j$$

where $\widetilde{\mathbf{P}}^j$ is written in the following form:

$$\widetilde{\mathbf{P}}^j = \alpha \frac{u_i}{\| u \|} \widetilde{\mathbf{N}}^j{}_{,i}$$

where α is the upwind parameter determined by the element Peclet number. Using the weighted residual method based on the SUPG formulation we may write

$$\sum_e \int_{\Omega^e} \widetilde{\mathbf{N}}^j (\mathbf{U},_t + \mathbf{F}_j,_j (\mathbf{U}) - \mathbf{B}) d\Omega^e - \int_{\Gamma_{slip}} \mathbf{N}^j \mathbf{F}_n^{(2)} d\Gamma - \int_{\Gamma_{int}} \mathbf{N}^j [\mathbf{F}_n] d\Gamma = 0 \tag{5}$$

in which the multidimensional shallow water equations are the following

$$\begin{array}{rlll} \mathbf{U},_t + \mathbf{F}_j,_j (\mathbf{U}) = \mathbf{B} & on & \Omega & governing \quad equation \\ \mathbf{F}_n^{(2)} = 0 & on & \Gamma_{slip} & null \quad flux \quad condition \\ [\mathbf{F}_n] = 0 & on & \Gamma_{int} & continuity \quad condition \end{array}$$

In the latest equation, the square brackets represents the jump of $\mathbf{F}_n$ across the interelement boundary. In fact, this equation is automatically verified because $\mathbf{F}_n$ is a continuous function. Integrating by parts, we obtain the weak form of the weighted residual equation

$$\begin{aligned} &\sum_e \int_{\Omega^e} \widetilde{\mathbf{P}}^j (\mathbf{U},_t + \mathbf{F}_j,_j (\mathbf{U}) - \mathbf{B}) d\Omega^e + \sum_e \int_{\Omega^e} (\mathbf{N}^j \mathbf{U},_t - \mathbf{N}^j,_i \mathbf{F}_i - \mathbf{N}^j \mathbf{B}) d\Omega^e + \\ &+ \sum_e \int_{\Gamma^e} \widetilde{\mathbf{N}}^j \mathbf{F}_n d\Gamma - \int_{\Gamma_{slip}} \mathbf{N}^j \mathbf{F}_n^{(2)} d\Gamma - \int_{\Gamma_{int}} \mathbf{N}^j [\mathbf{F}_n] d\Gamma = 0 \end{aligned} \tag{6}$$

Using the following splitting,

$$\sum_e \int_{\Gamma^e} \widetilde{\mathbf{N}}^j \mathbf{F}_n d\Gamma = \int_{\Gamma_{int}} \mathbf{N}^j [\mathbf{F}_n] d\Gamma + \int_{\Gamma_{int/outflow}} \mathbf{N}^j \mathbf{F}_n d\Gamma + \int_{\Gamma_{slip}} \mathbf{N}^j \mathbf{F}_n d\Gamma \tag{7}$$

We can write

$$\begin{aligned} &\sum_e \int_{\Omega^e} \widetilde{\mathbf{N}}^j \mathbf{U},_t d\Omega^e + \sum_e \int_{\Omega^e} (\widetilde{\mathbf{P}}^j \mathbf{F}_{i,i} - \mathbf{N}^j,_i \mathbf{F}_i - \widetilde{\mathbf{N}}^j \mathbf{B}) d\Omega^e + \int_{\Gamma_{int/outflow}} \mathbf{N}^j \mathbf{F}_n d\Gamma + \\ &\int_{\Gamma_{slip}} \mathbf{N}^j \mathbf{F}_n^{(1)} d\Gamma = 0 \end{aligned} \tag{8}$$

Making use of the forward Euler scheme in the time discretization, we can write the complete formulation in matrix form as follows:

$$\mathbf{M}\Delta\mathbf{b} - \Delta t \mathbf{R} = 0$$

where $\mathbf{M}$ is the consistent mass matrix, $\mathbf{R}$ the residue and $\Delta\mathbf{b}$ the vector of nodal variations of the conservation variables.(More details see reference C.E.Baumann and S.R.Idelsohn (1992)).

In this paper a variant of $SUPG$ method is presented by means of the introduction of a β factor in the perturbation function to calculate the consistent mass matrix beside the temporal term. In this case the perturbation function can be different in the temporal and spatial terms, and the procedure is called $\beta-$ SUPG method. An analysis of the influence of the β factor is analized in the following section.

5 STABILITY ANALYSIS OF β-SUPG METHOD

In order to analize the stability of numerical schemes obtained by the application of the β-SUPG procedure in combination with an explicit scheme, let us consider one of the most representative equations for modeling transport phenomena, the hyperbolic unsteady advection equation, written as follows:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad 0 \leq x \leq L,\ t \geq 0$$

where u is the unknown function of (x,t) and a is the convection speed $(a > 0)$. When linear elements are used, global matrices $\mathbf{M}$ and $\mathbf{K}$ will be obtained by assembling the element matrices. Matrix $\mathbf{M}^e$ may be diagonalized by using the row-sum lumping technique(see Hughes and Tezduyar (1984) for different choices of M^e arising from numerical integration). When this matrix is not diagonalized and the element matrices are assembled, a typical algorithmic equation for an internal node m (consistent mass SUPG method) may be written as:

$$\frac{1}{2}(\frac{1}{3}+\frac{\alpha}{2})(u_m^{n+1}-u_m^n)+\frac{4}{3}(u_m^{n+1}-u_m^n)+\frac{1}{2}(\frac{1}{3}-\frac{\alpha}{2})(u_m^{n+1}-u_m^n) =$$
$$\Delta t[\frac{\alpha a}{2h}(u_{m+1}^n-2u_m^n+u_{m-1}^n)-\frac{a}{2h}(u_{m+1}^n-u_{m-1}^n)]$$

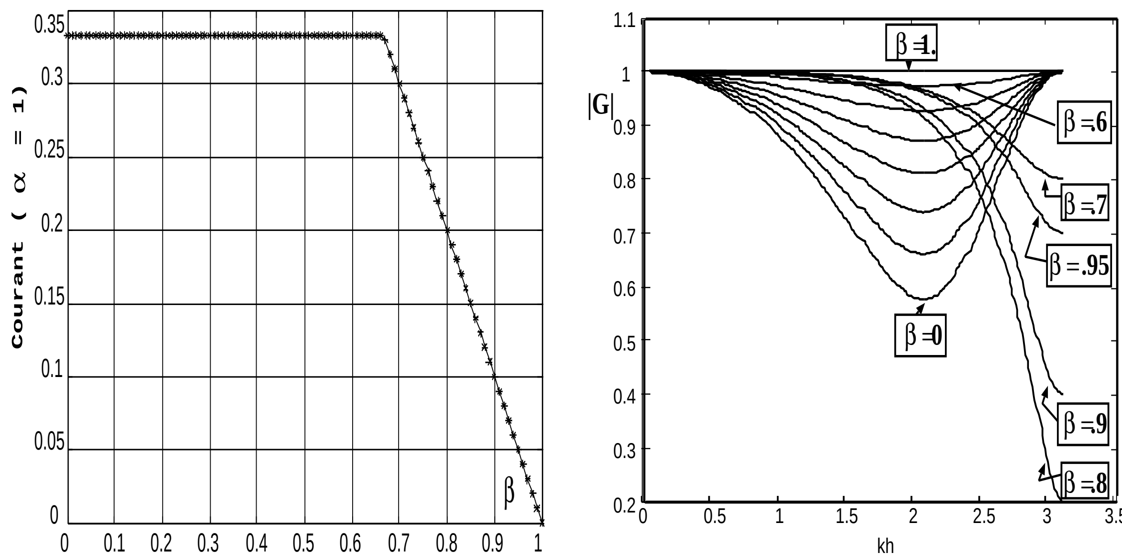
In the practice we have used $\alpha = 1$ and we have a compromise between the accuracy and stability of this procedure. One of the way to guarantee the stability property of this scheme is introducing a β parameter in the mass matrix corresponding to the temporal term. Based on this idea, using the β -SUPG discretization, we obtain an explicit scheme in the following form:

$$\frac{1}{2}(\frac{1}{3}+\beta\frac{\alpha}{2})(u_m^{n+1}-u_m^n)+\frac{4}{3}(u_m^{n+1}-u_m^n)+\frac{1}{2}(\frac{1}{3}-\beta\frac{\alpha}{2})(u_m^{n+1}-u_m^n) =$$
$$\Delta t[\frac{\alpha a}{2h}(u_{m+1}^n-2u_m^n+u_{m-1}^n)-\frac{a}{2h}(u_{m+1}^n-u_{m-1}^n)]$$

Replacing in the above equation the following field $u_m^n = e^{\mathrm{i(kmh-\omega n\Delta t)}}$, where i is the imaginary unit and k is the wave number in the x direction, we have an equation for the interior nodes in which the function G is the amplification factor

$$G = 1 + C\frac{\alpha(\cos{(kh)} - 1) - \mathrm{i}\sin{(\mathrm{kh})}}{\frac{\cos{(kh)}+2}{3} - \mathrm{i}\beta\frac{\alpha\sin{(\mathrm{kh})}}{2}}$$

where C is the Courant number. A scale of diffusivity has been obtained by means of the introduction of β parameter. By means of this scale the critical Courant number has been determined. In the Figure 2(a) the critical Courant values respect to the β parameter for $\alpha = 1$ are shown. For $\beta \geq .95$, we obtain a critical Courant value less than 0.05. We know that the "Consistent mass" SUPG method is obtained for $\beta = 1$ value, and their representative curve is not included in the family of allowed curves that characterize the β- SUPG method.



(a) optimal β-values

(b) Amplification Factor $\mid G \mid$

Figure 2: β-SUPG method.

Figure 2(b) shows the absolute values of the amplification factor G. On this figure the stability property for several β values is represented. We can appreciate that β- SUPG method is only unstable for $\beta = 1$ (the full consistent SUPG method).

6 NUMERICAL RESULTS

A computational code, PETSC-FEM, has been developed for the solution of large-scale nonlinear system resulting from the finite element discretization of multi-disciplinary problems. One, of applications developed in this numerical code is the computation of solution of shallow water problems. To illustrate the adaptability of the present finite element method, several numerical examples are discussed in this section. Through of them, several numerical schemes has been used for the finite element discretization, such as: lumped mass SUPG method, full consistent SUPG method and β-SUPG method.

A solitary wave propagated along a one-dimensional channel with uniform bottom slope

The first example is the analysis of a solitary wave propagated along a one-dimensional channel with uniform bottom slope. The problem is purely one-dimensional, however, to seek the adaptability of the method, the two-dimensional finite element computer program is applied based on equations ((1)). The channel has a length of $200m$ where the water depth is $1m$.

Several matematical models based on the numerical solution steady have been examined using linear elements in the spatial discretization.

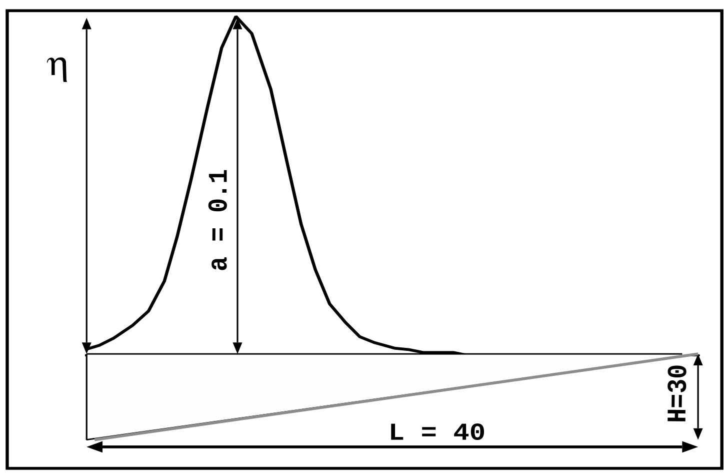


Figure 3: Coordinate system for shallow water equations.

The initial configuration of the solitary wave and the variation of the depth is shown in Figure 3. The initial conditions are given by:

$$\eta = a\ \mathrm{sech}^2 \frac{1}{2}\sqrt{3a}(x - \frac{1}{\alpha})$$

$$u = -(1 + a/2)\eta/(\alpha x + \eta)$$

where $a = 0.1, g = 1.0, \alpha = 1/30$. Figure 4 shows the computed results by the explicit scheme using "Lumped mass" SUPG method at left and β - "Consistent mass" SUPG method at right. For this problem the exact solution has a peak value of 1.2 times the initial peak value. This example allows to examine the numerical diffusivity property of several schemesMutsuto Kawahara and Inagaki (1982),R.Lohner (1984). For the "Lumped mass" SUPG method the computed results seem to include a significant damping effect. This is contrary to the expected behaviour, as the number of subdivisions is increased(e.g. $N = 160$ or 320) the results seem to improve. It is very noticeable that the β - "Consistent mass" SUPG method produces better solutions even for relatively coarse grids (see figure 4(right) where the peak value increased up to 1.2 times the initial peak value). In the above computations, the time increment Δt is chosen to be as long as possible but within the limit where stable computations are obtained.

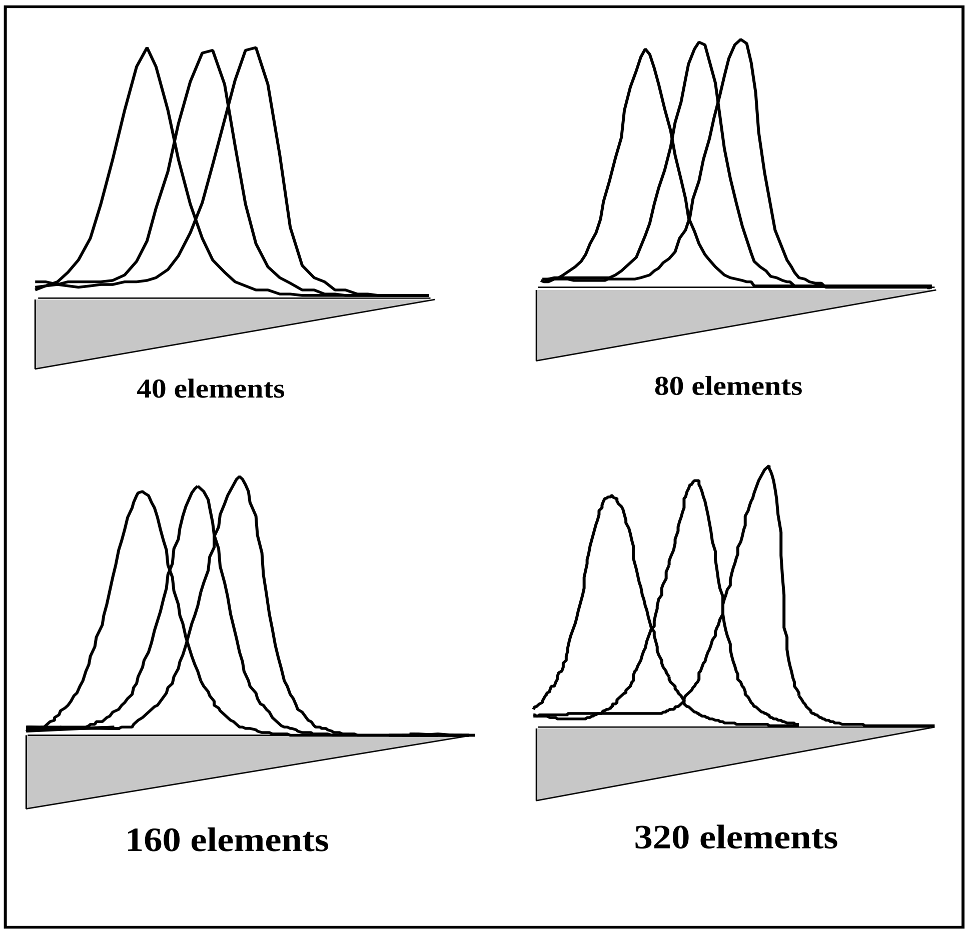


Figure 4: Shoaling of a wave. Solution for 40, $80, 160 and 320$ elements by $\beta = .95$-SUPG method

Flow past a circular cylinder

The next example considered was the classical problem of two-dimensional flow over a circular cylinder crosswise to the flow. The physical situation is that of an infinitely long circular cylinder of unit diameter inmersed in a fluid medium of infinite extent. The flow far upstream is uniform.R.H.Gallagher, J.T.Oden, C.Taylor and O.C.Zienkiewicz (1975) The problem considers steady uniform flow with free velocity $U_\infty = .35$ passing through a cylinder of one unit radius placed along the centerline $x = 0$. The domain field is divided with a polar partition by a quadrangular extensive annulus with outer radius = 11× inner radius. A mixed boundary-value problem formulation is used with natural boundary condition automatically satisfied on the vertical centerline, $x = 0$ and uniform horizontal flow U_∞ on the outer radius. For the numerical computations we impose a slip boundary condition on the inner radius and a first order absorbent boundary condition on the outer radius, and a Courant number equal to 0.6.

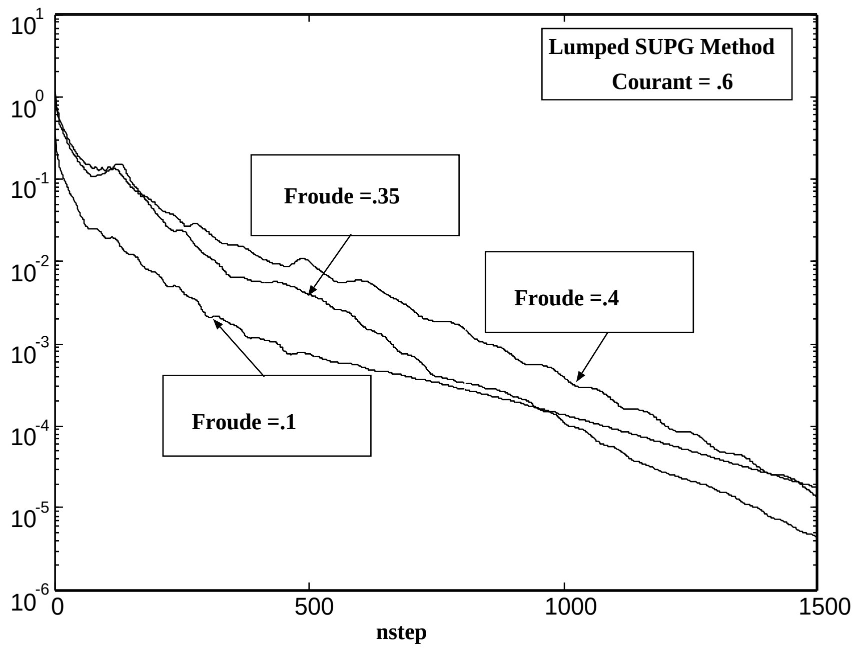
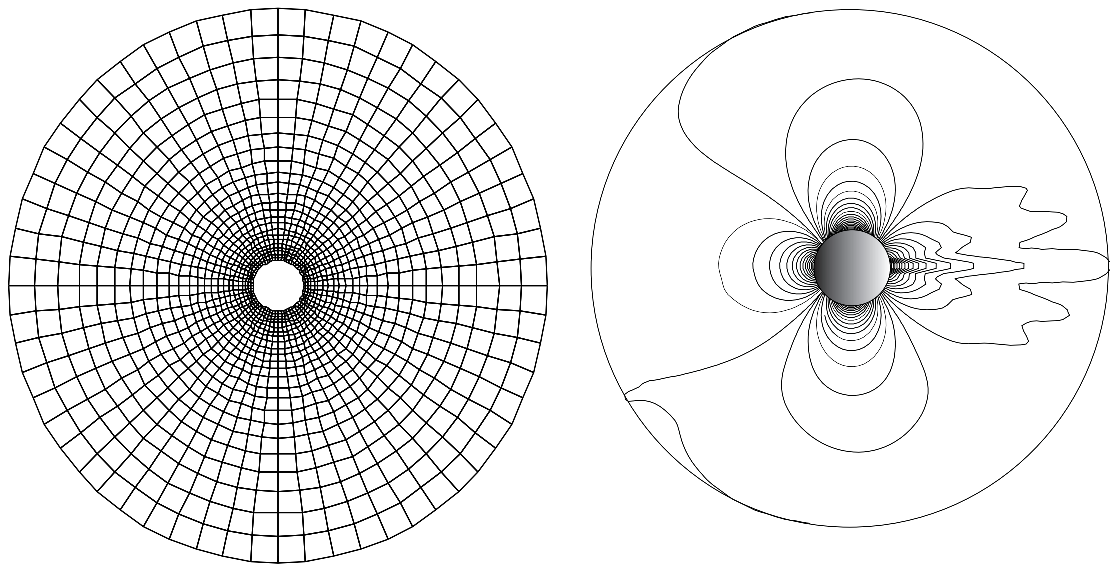


Figure 5: Convergence History for several Froude number.

Figure 5 shows the convergence history for several subcritical flux. We can appreciate that the convergence decays with increments of the Froude number. In particular the behaviour of the flow, for a Froude number equal to .35 has been examined.



(a) Element mesh

(b) Contours of typical absolute values of velocity

Figure 6: Flux past a cylinder(steady state).

Figure 6 shows the finite element mesh used for the computation and the contours of absloute value of velocity for a Froude number equal .35. We can appreciate the symmetry of flow field around a circular cylinder, as expected.

Shock-capturing operator

In this section a shock-capturing operator is used in order to solve fine details of shock discontinuities. This operator is built to satisfy a few design conditions: in order to control oscillations, this operator should act in the direction of the gradient; for consistency it should be proportional to the residual term; and for accuracy it should vanish quickly in regions where the solution is smooth. F.L.Ribeiro and L.Landau (1998)

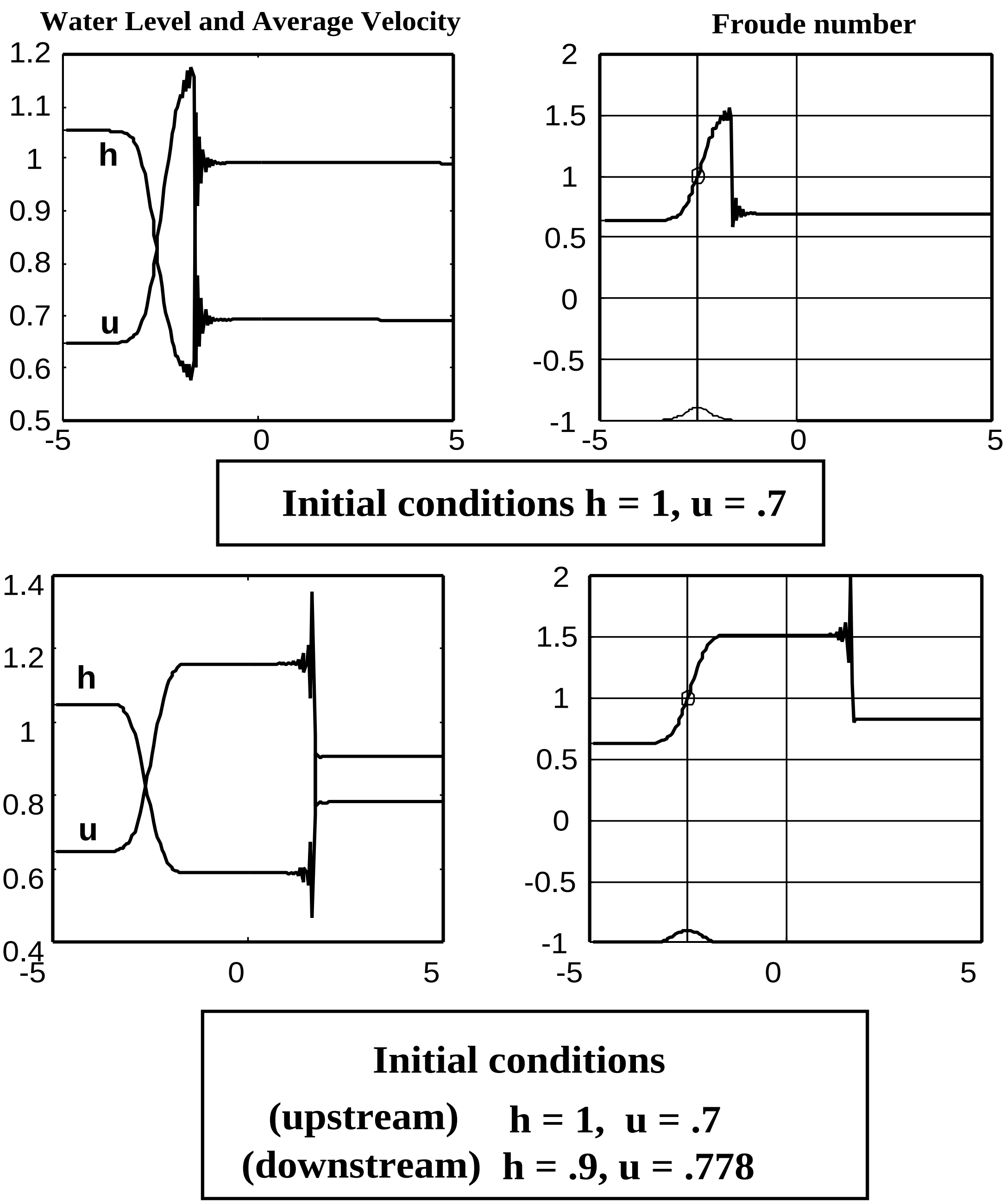


Figure 7: Flow over the shoal(without shock-capturing) using "Lumped mass" SUPG method.

Now, some numerical results obtained with this methodology are presented. The following example is the well known problem of a flow over a shoal at the depth. In the present test the influence of a localized perturbation of the water depth H in the water level. The water depth is given by

$$H = Hmax * e^{\left(\frac{-(x-xc)^2}{stretch}\right)}$$

with $Hmax = .1$; $xc = -2.5$; $stretch = .2$. A spatial discretization of 200 linear one-dimensional element mesh is used. The "Lumped mass" SUPG method has been used for obtaining computational results with several initial conditions.

Figure 7 describes the steady state numerical solution of the water flux over the shoal. On this figure, we can appreciate the hidraulic jump when the flow passed over the shoal and the shock wave generated after this. In this process the Froude number is 1 at the crest of the shoal and then a shock wave front is generated ,which, in virtue of the initial values of velocity and water elevation level the shock wave front advances by the channel. Figure 7 shows the advance of shock wave front for different initial conditions: in the first case the initial conditions at the upstream end and downstream end are equal to $h = 1,\ u = .7$, and in the second case, the initial conditions at downstream end take the values $h = .9,\ u = .778$. In the last case, the shock wave front has a major advances. On this figure the presence of the oscillations on the near regions to the shock discontinuities is noticeable. In virtue of this, a shock-capturing operator is introduced on the numerical scheme. In the PETSC-FEM code, a parameter that caracterize the capture of discontinuities is a threshold. If the threshold decreases, the high frequency oscillations may be elimінted. Figure 8 shows the convergence history for several values of the shock-capturing parameter. We can appreciate that the oscillations desappear when the parameter takes values near to 0.001.

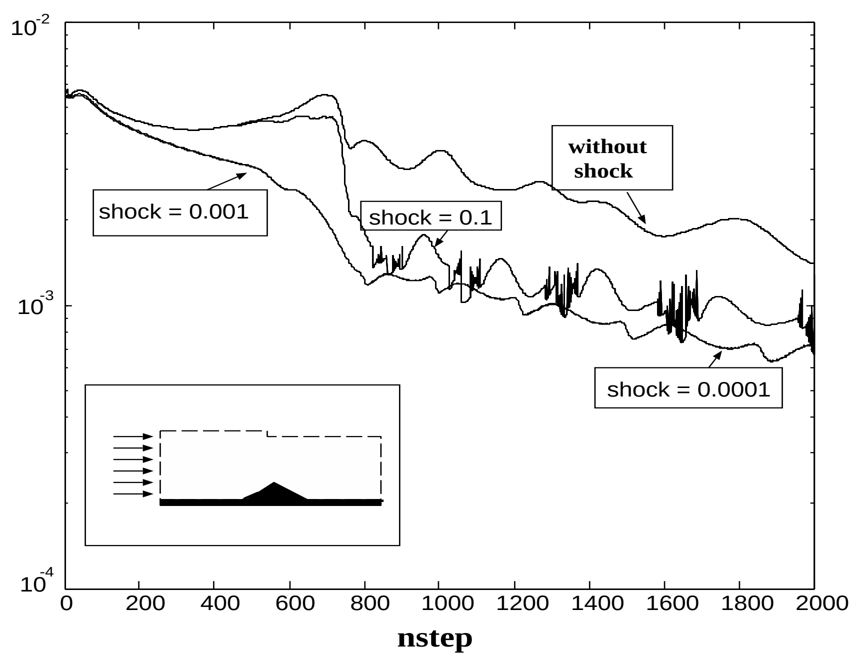


Figure 8: Convergence History with shock-capturing.

Having chosen this shock capturing threshold the water flux over the shoal solution was obtained and it was included in figure 9.

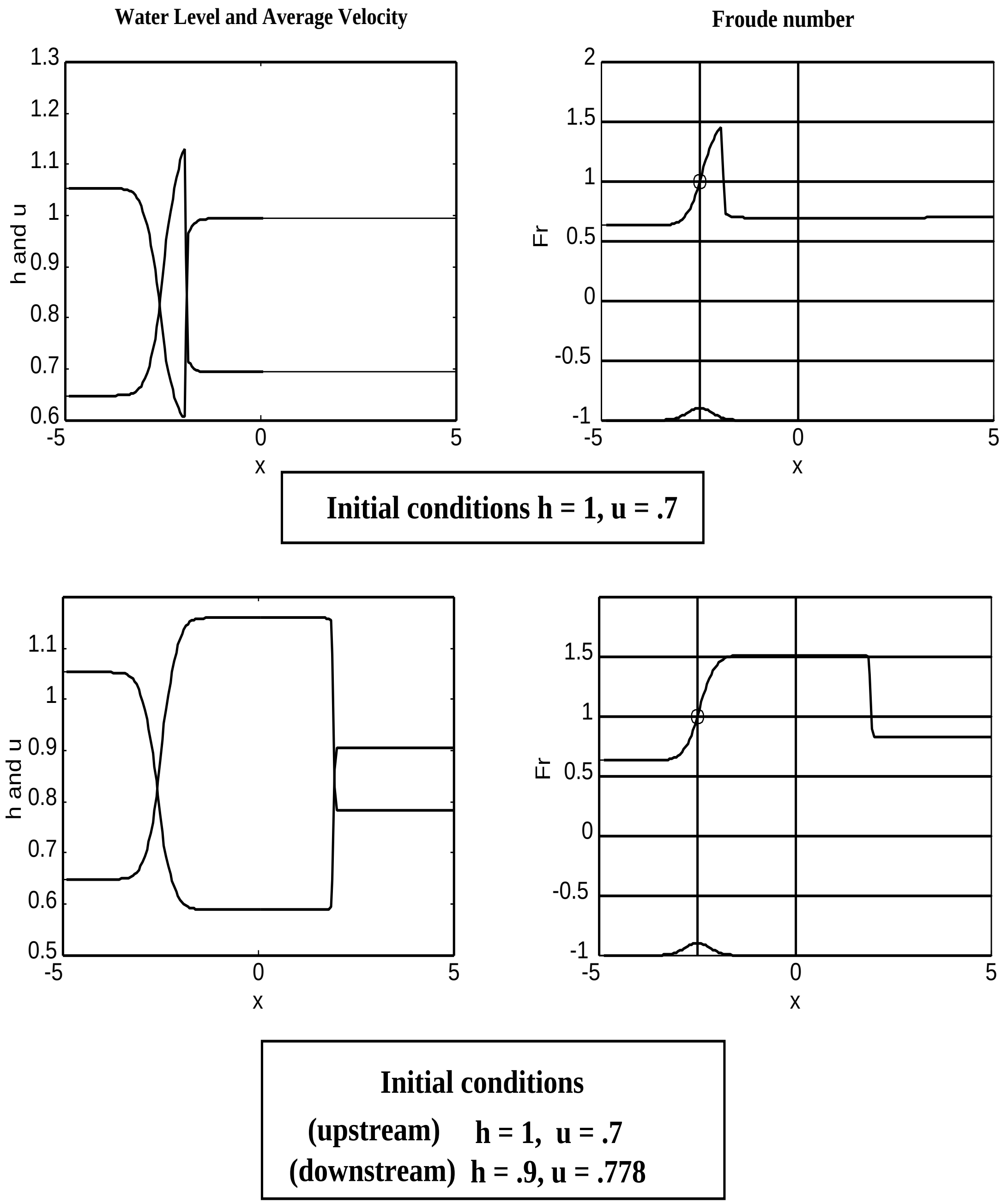


Figure 9: Flow over the shoal(with shock-capturing) using "Lumped mass" SUPG method.

Hydraulic jump in a diverging rectangular channel

A hydraulic jump is generally used to dissipate energy in water flowing over dams, weirs and other hydraulic structures to prevent scouring of the downstream channel. These types of structures are called stilling basins. Most of the stilling basins are rectangular, but several have diverging side walls, e.g. St. Anthony Falls Stilling basin and W.Darcy McKeough Dam (Khalifa and McCorquodale,1979)Muhammad Younus (1994). A hydraulic jump in a diverging channel is also encountered in critical flow measuring devices, e.g., the Parshall and Cut-throat flumes.

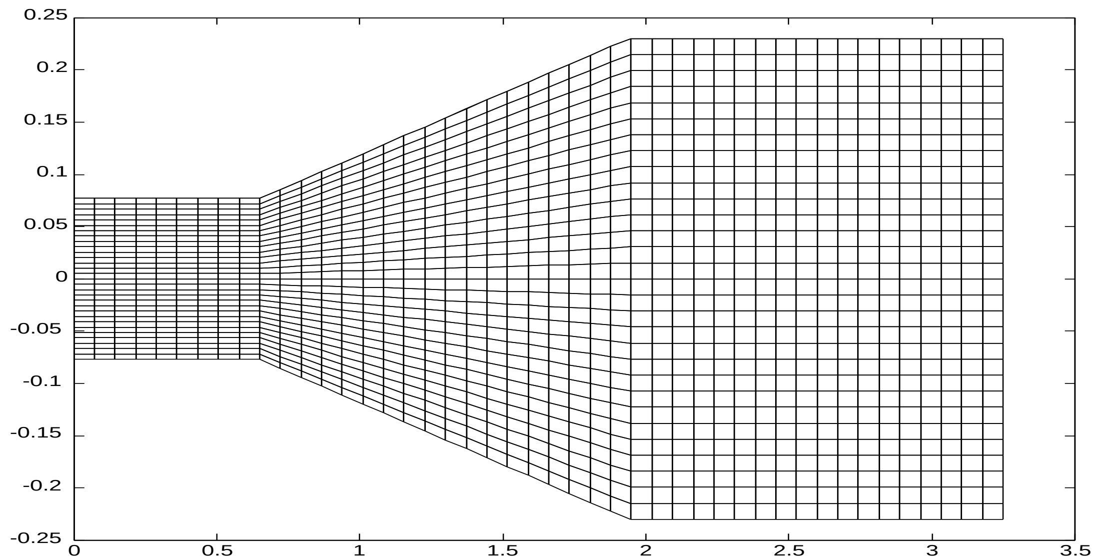


Figure 10: Grid to simulated hydraulic jump in a diverging channel.

In this study, we use an explicit method with a lumped mass spatial discretization scheme to simulate the hydraulic jump in a diverging channel, by means of the solution of the two-dimensional depth-averaged equations. As, Bhallamudi and Chaudhry(1992) we have neglected the effective stresses and we have not compared the computed results with any observed data.

A computational grid system of (212×31) nodes was used in the simulations, as shown in Figure 10. In this problem, a supercritical flow at the upstream end and the subcritical flow at the downstream end is considered. In virtue of these assumptions the initial conditions were the following: (upstream) $h = 0.0976m$, $u = 1.94m/s$, $v = 0m/s$, and (downstream) $h = 0.2632$, $u = 0.7193$, $v = 0$. The computed results were obtained for a Courant number equal to 0.25. The model was run until the solution was converged to a steady state. Figure 11 shows that the model describes adequately the hydraulic jump. The numerical results obtained have a good agreement with Chaudhry(1994) for artificial viscosity coefficient equal to zero.

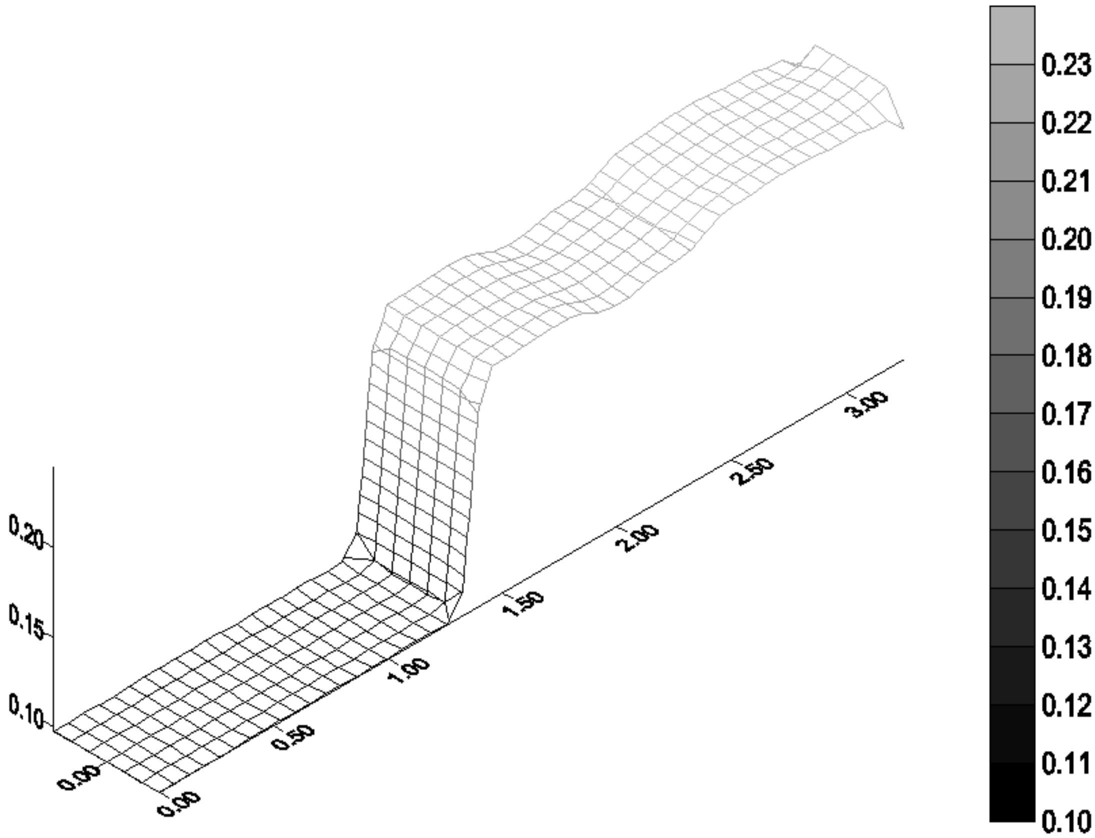


Figure 11: Hydraulic jump by explicit method, with Courant number $= .25$

7 CONCLUSIONS

In this paper a technique to solve shallow water equations system has been examined. The use of the SUPG method and an explicit scheme in the time discretization allows to describe several physical phenomena for inviscid flow. An analysis about the stability of this method for these equations system shows the convenience of using β-SUPG method in the solution of problems where others methods become very diffusive. A comparison between several methods has been developed. The numerical tests show that the numerical code describes adequately the physical phenomenon. The future work is oriented towards large scale simulation of a shallow water model with turbulent effects.

8 ACKNOWLEDGEMENT

This work has received financial support from Consejo Nacional de Investigaciones Cientficas y Tcnicas (CONICET, Argentina, grants PIP 0198/98, PIP 0266/98), Universidad Nacional del Litoral (Argentina, grants CAI+D 96-004-024, CAI+D 2000/43) and ANPCyT (Argentina, grants PICT 12-0051, PICT 12-06973, PID-99/74, PID-99/76). We made extensive use of software distributed by the Free Software Foundation / GNU-Project : Linux ELF-OS, Octave, Tgif,MPI, PETSc, gcc compilers among many others.

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