A PARAMETRIC METHOD FOR THE SHALLOW WATER EQUATIONS

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ABSTRACT

This paper report progress on a technique to stabilize the consistent mass SUPG method, when it is used in combination with an explicit scheme to solve shallow water flow problems to steady solutions. According to this a comparison between the use of lumped mass SUPG and consistent mass SUPG method is done. A scale of diffusivity in this scheme is developed by means of the introduction of β parameter in the mass matrix of the consistent SUPG method and a stability analysis of β-SUPG method is presented. Several examples based on the one-dimensional shallow-water equations illustrate the accuracy and efficiency obtained with such methods.

1 INTRODUCTION

A great number of civil engineering projects in river hydraulics, coastal water and estuaries require predictive models of the flow. The trend is towards computational methods based on the one, or two dimensional shallow water equations. The use of a fine spatial grid is often required in practical applications thereby necessitating a very small integration time step if an explicit scheme is employed, having as a consequence, an unacceptable computing cost specially for steady state problems [1],[3],[7],[9],[10],[12]. Stability considerations are crucial for explicit numerical schemes of the shallow water equations. The stability of several early FEM models was examined by many investigators, e.g. [5],[6], [4], [3], [9], [7]. Due to the current state of parallel computation, explicit methods have found a new interest because of their great adaptation to the parallel programming.

The use of upwind differencing or more generally of Petrov-Galerkin methods in steady-state convection-diffusion problems is generally accepted to improve the accuracy of computation. Special forms of such Petrov-Galerkin processes using so-called streamline diffusion are particularly effective[3],[19],[20],[14],[15],[16],[17],[18],[21],[11],[13],[2],[8]

On the other hand to remove numerical diffusion through a full consistent SUPG formulation we found that this scheme results to be unstable for the planar shallow water one-dimensional equation. These facts have been the main motivation to the introduction of a β-parameter in the mass matrix of the consistent SUPG method in order to recover part of its stability taking advantage of its reasonable accuracy. We have confirmed this hypotesis by numerical analysis arguments taking as a model equation the clasical unsteady one-dimensional advective equation.

In this paper an application of an explicit scheme in time using both Lumped SUPG mass matrix or Consistent SUPG mass matrix in combination with the use of a stabilized spatial discretization for solving shallow water flow problems has been examined[3],[20].

For simplicity we began adopting the lumped SUPG mass matrix scheme but our results show a very diffusive behavior in the description of a typical test problem like a solitary wave propagation along a one-dimensional channel with uniform bottom slope.

Moreover, several numerical examples based on the one-dimensional shallow-water equations illustrate the accuracy and efficiency obtained with such methods. These results are valid for adevective equations system in general.

2 Stability analysis of β-SUPG method

In order to analize the stability of numerical schemes obtained by the application of the β-SUPG procedure in combination with an explicit scheme, let us consider one of the most representative equations for modeling transport phenomena, the convective, hyperbolic equation, written here as follows:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad 0 \leq x \leq L,\ t \geq 0 \tag{1}$$

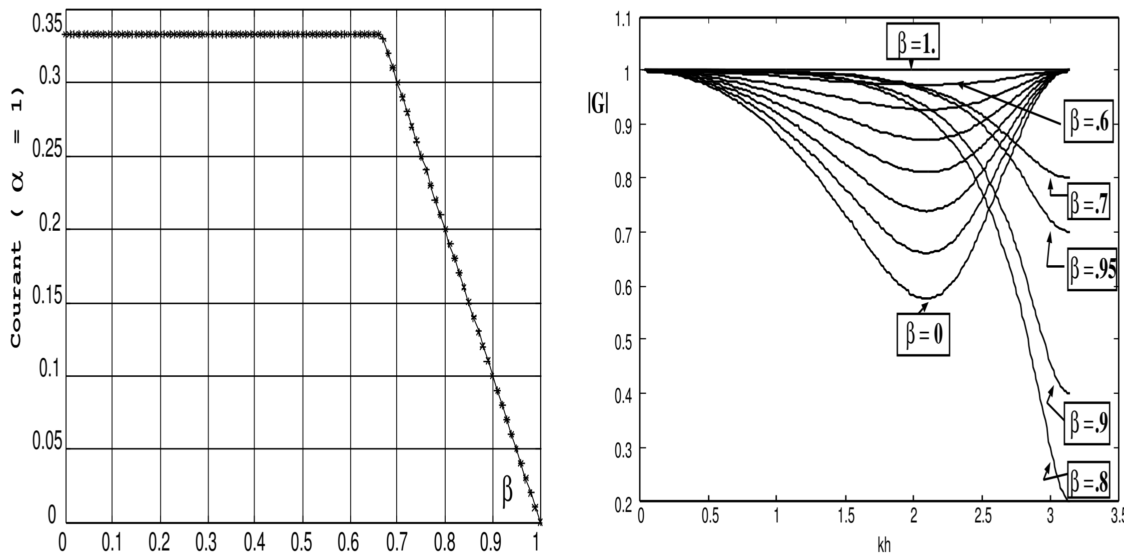
where u is the unknown function of (x, t) and a is the convection speed ($a > 0$). When linear elements are used, global matrices $\mathbf{M}$ and $\mathbf{K}$ will be obtained by assembling the element matrices $\mathbf{M}^e$ and $\mathbf{K}^e$. Matrix $\mathbf{M}^e$ may be diagonalized by using the row-sum lumping technique (see [19] for different choices of M^e arising from numerical integration). Otherwise a consistent matrix appear from the formulation and we can select between a "Consitent mass" Galerkin matrix or a "Consistent mass" SUPG depending on the inclusion of the weighting function chosen. When this matrix is not diagonalized and the element matrices are assembled, a typical algorithmic equation for an internal node m (consistent mass SUPG method) may be written as:[3]

$$\frac{1}{2}(\frac{1}{3} + \frac{\alpha}{2})(u_m^{n+1} - u_m^n) + \frac{4}{3}(u_m^{n+1} - u_m^n) + \frac{1}{2}(\frac{1}{3} - \frac{\alpha}{2})(u_m^{n+1} - u_m^n) =$$
$$\Delta t[\frac{\alpha a}{2h}(u_{m+1}^n - 2u_m^n + u_{m-1}^n) - \frac{a}{2h}(u_{m+1}^n - u_{m-1}^n)]$$

where α is the upwind parameter determined by the Peclet number of element. In the practice we have used $\alpha = 1$ and we have a compromise between the accuracy and stability of this procedure. One of the way to guarantee the stability property of this scheme is introducing a β parameter in the mass matrix. Based on this idea, using the β -SUPG discretization, we obtain an explicit scheme in the following form:

$$\frac{1}{2}(\frac{1}{3} + \beta\frac{\alpha}{2})(u_m^{n+1} - u_m^n) + \frac{4}{3}(u_m^{n+1} - u_m^n) + \frac{1}{2}(\frac{1}{3} - \beta\frac{\alpha}{2})(u_m^{n+1} - u_m^n) =$$
$$\Delta t[\frac{\alpha a}{2h}(u_{m+1}^n - 2u_m^n + u_{m-1}^n) - \frac{a}{2h}(u_{m+1}^n - u_{m-1}^n)]$$

Replacing in the above equation the following field $u_m^n = e^{\mathrm{i(kmh-\omega n\Delta t)}}$, where i is the imaginary unit and k is the wave number in the x direction, we have an equation for the interior nodes in which the function G is the amplification factor



(a) optimal β-values

(b) Amplification Factor $\mid G \mid$

Figure 1: β-SUPG method.

$$G = 1 + C\frac{\alpha(\cos{(kh)} - 1) - \mathrm{i}\sin{(\mathrm{kh})}}{\frac{\cos{(kh)}+2}{3} - \mathrm{i}\beta\frac{\alpha\sin{(\mathrm{kh})}}{2}}$$

where C is the Courant number. A scale of diffusivity has been obtained by means of the introduction of β parameter and based on this scale the critical Courant number has been determined. In the Figure 1(a) the critical Courant values respect to the β parameter for $\alpha = 1$ are shown. For $\beta \geq .95$, we obtain a critical Courant value less than 0.05. We remark that the "Consistent mass" SUPG method is obtained for $\beta = 1$ value, and their representative curve is not included in the family of allowed curves that characterize the β-SUPG method because it is unconditionally unstable.

Figure 1(b) shows the absolute values of amplification factor G. On this figure the stability property for several β values is represented. We can appreciate that β-SUPG method is unstable only for $\beta = 1$. A comparison between several methods can be appreciated in Figure 2. On this figure the stability region of each method is represented. We can note that the stability region of β-SUPG method is less than the "Lumped mass" SUPG method, therefore, the diffusion goes to the minimum value while the β value increase to values near to 1.

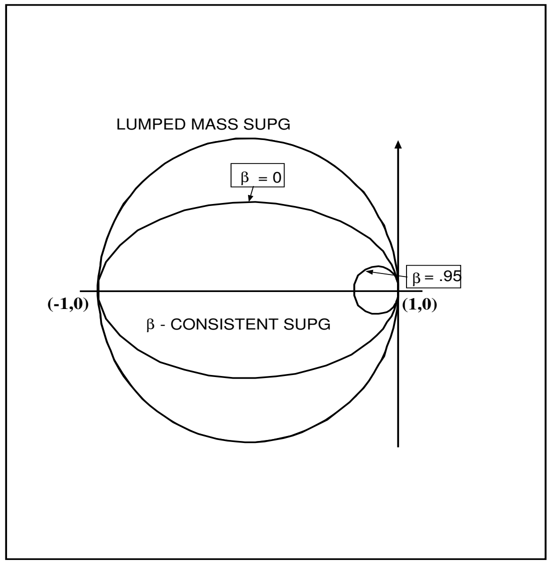
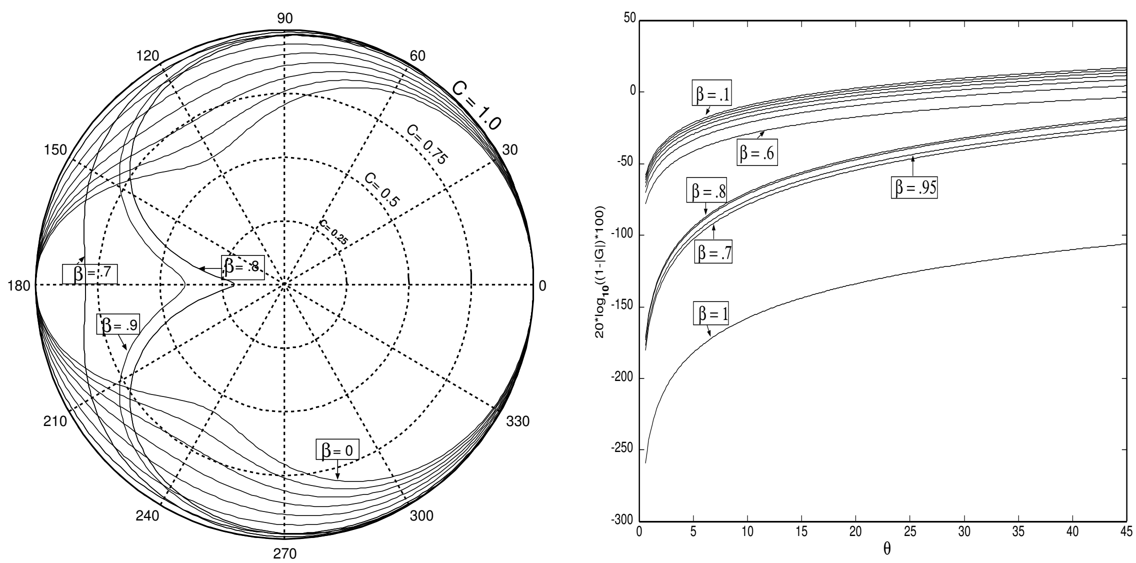


Figure 2: Stability region of several methods.



(a) Disipation error values($\alpha = 1$)

(b) Disipation error ($0 < kh < \pi/4$)

Figure 3: Disipation error of β-SUPG method ($\alpha = 1$)

Let us to examine the diffusion and dispersion properties of this scheme and to define the accuracy of the scheme as a function of the dimensionless wave number kh. To calculate the dissipation error, we use the expresion

$$\epsilon_D = \frac{\mid G \mid}{e^{\bar{\eta}\Delta t}}$$

where G is the amplification factor and $\bar{\eta}$ is the evanescent term. For the pure advective equation $\eta = 0$, then we have

$$\epsilon_D = \mid G \mid = \sqrt[2]{[1 - \frac{C}{2}(kh)^2(1 + \alpha(1-\beta))]^2 + C^2(kh)^2}$$

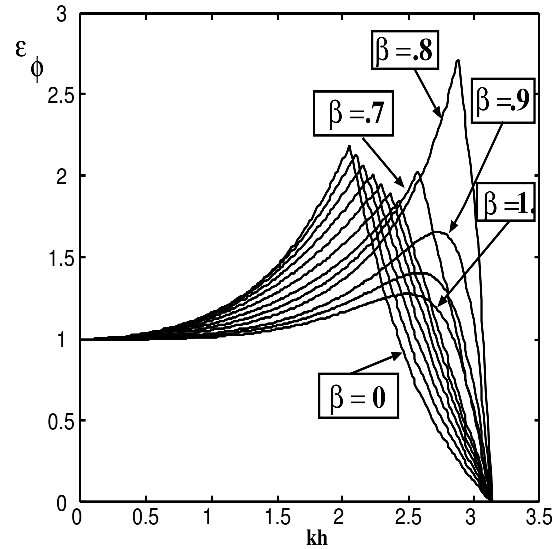


Figure 4: Dispersion error for the β-SUPG method.

Figure 3 shows the error disipation curves for several values of β with $\alpha = 1$. We can appreciate that the error dissipation is very dependent of β parameter values, having a non-monotone behavior. For β values between .1 and .6 we can find a monotone behavior, however, the curves obtained for $\beta \geq .7$ have different behavior.

To find the dispersion error it is neccesary to calculate the phase angle of the exact and the numerical solutions, with the dispersion error given by

$$\epsilon_\phi = \frac{-\arctan(\frac{ImG}{ReG})}{C \times kh}$$

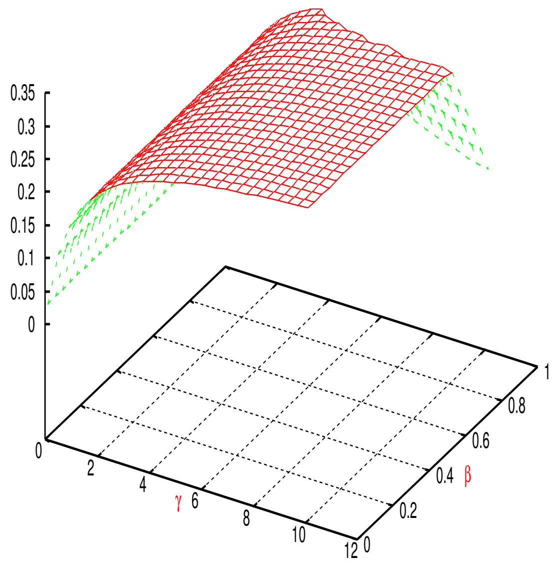
Figure 4 shows the dispersion error for several β parameter values. We note that the dispersion error decreases while the β parameter values increase.

3 Influence of the diffusive term

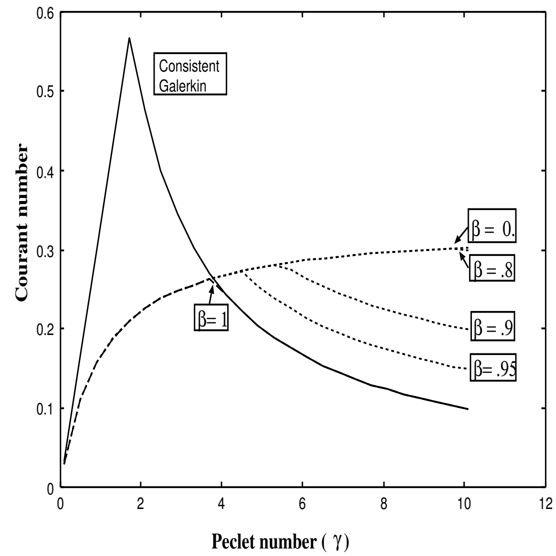
An stability analysis about the influence of a diffusive term is developed in the present section. For this the equation (1) is modified to the following convection-advection equation by the addition of the second order term $k\frac{\partial^2 u}{\partial x^2}$. Then, it takes the form

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - k\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \leq x \leq L,\ t \geq 0$$

Let us call $\gamma = \frac{ah_e}{2k}$ the element Peclet number of a given partition, such that h_e is the element size for this partition. This dimensionless number gives an idea of the relative importance of convection and diffusion. Convection will be dominant when $\mid \gamma \mid$ is large, whereas diffusive effects will predominate for small values of $\mid \gamma \mid$. In virtue of this, using the β-SUPG discretization, we obtain an explicit scheme in the following form:



(a) Critical Courant number surface



(b) Stability limits

Figure 5: Stability limits for the convection-diffusion equation using linear elements.

$$\frac{1}{2}(\frac{1}{3}+\beta\frac{\alpha}{2})(u_m^{n+1}-u_m^n)+\frac{4}{3}(u_m^{n+1}-u_m^n)+\frac{1}{2}(\frac{1}{3}-\beta\frac{\alpha}{2})(u_m^{n+1}-u_m^n)=$$
$$\Delta t[(\frac{k}{h^2}+\frac{\alpha a}{2h})(u_{m+1}^n-2u_m^n+u_{m-1}^n)-\frac{a}{2h}(u_{m+1}^n-u_{m-1}^n)]$$

Replacing in the above equation the following field $u_m^n = e^{\mathrm{i(kmh-\omega n\Delta t)}}$, where i is the imaginary unit and k is the wave number in the x direction, we have an equation for the interior nodes in which the function G is the amplification factor

$$G = 1 + C\frac{(\frac{1}{\gamma}+\alpha)(\cos(kh)-1)-\mathrm{i}\sin(\mathrm{kh})}{\frac{\cos(kh)+2}{3}-\mathrm{i}\beta\frac{\alpha\sin(\mathrm{kh})}{2}}$$

Figure 5(a) shows the stability region respect to the Peclet number γ and the parameter β in the case $\alpha = 1$. We can observe that the Critical Courant number less than 0.35. A comparison between the Consistent Galerkin and β-SUPG method is shown in Figure 5(b). We can appreciate that the stability region obtained by the β-SUPG method increases for Peclet number values great than 0.4

Figure 6 shows for several values of γ the error dissipation curves respect to the β values, with $\alpha = 1$. We can appreciate that the error dissipation decreases when the γ values increase (for the same β parameter values), having a non-monotone behavior.

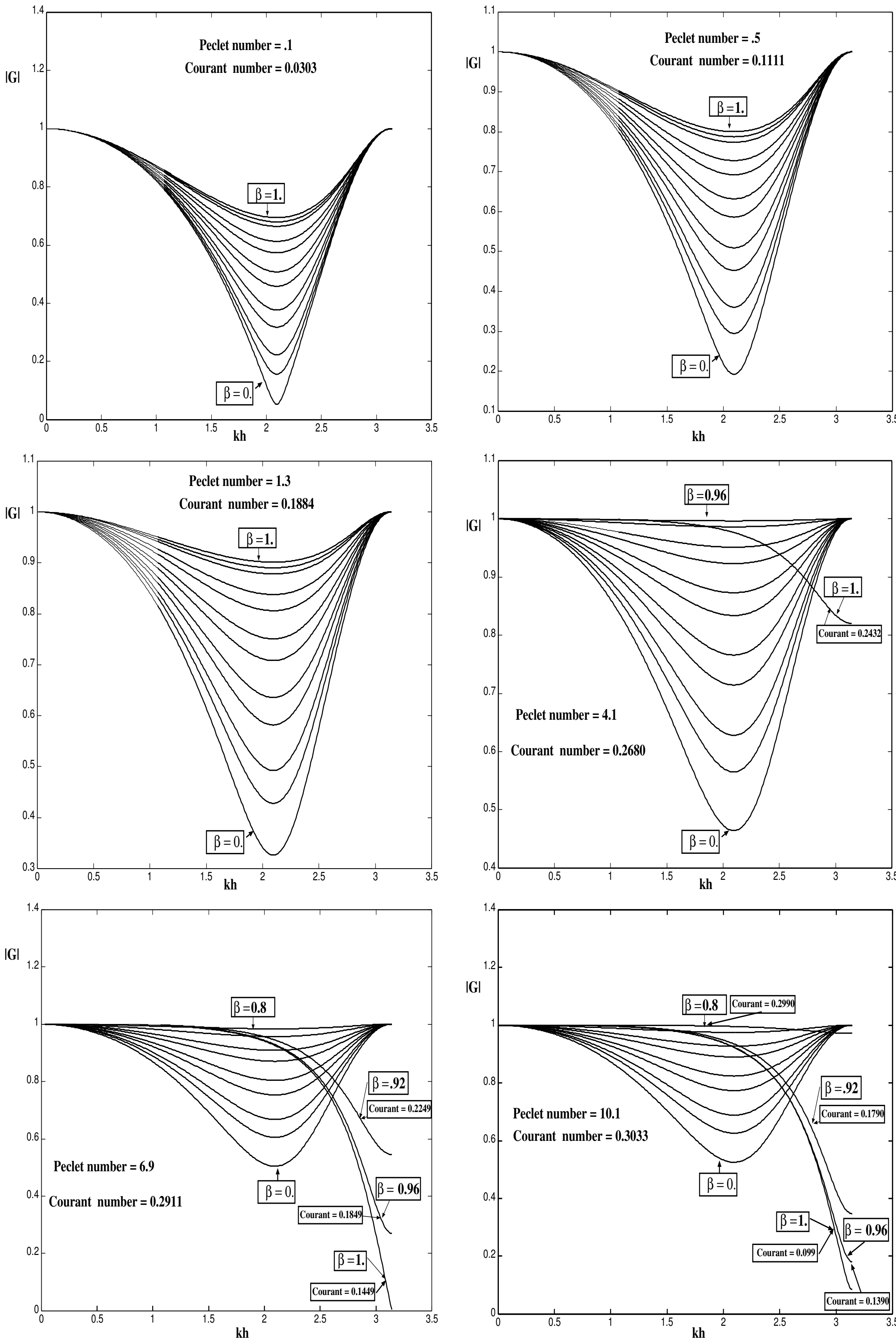


Figure 6: Amplification factor for the convection-diffusion equation using linear elements

In the limit case $\gamma \to 0$,the dispersion error ϵ_ϕ takes the form

$$\lim_{\gamma \to 0} \epsilon_\phi = \frac{-\arctan(\frac{\beta\alpha \sin(kh)}{\cos(kh)-1})}{C \times kh}$$

Figure 7 shows the behavior of the dispersion error in the limit case respect to several β values. We can appreciate that this error increases when β values increase too

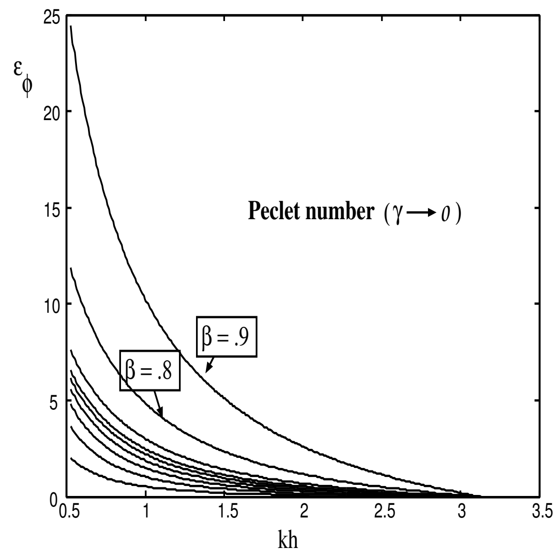
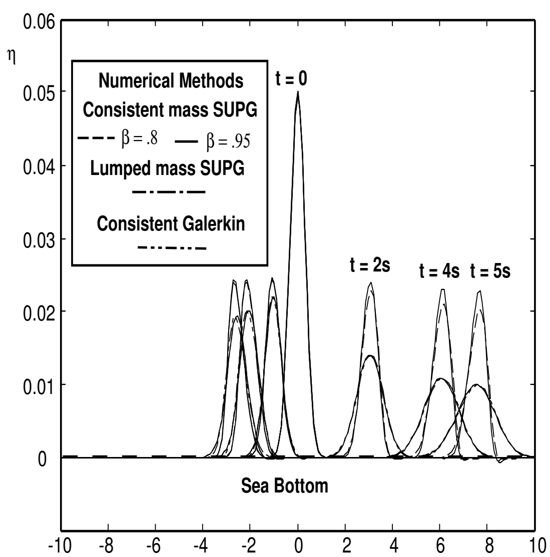


Figure 7: Dispersion error respect to the parameter β (for $\gamma \to 0$ and $\alpha = 1$)

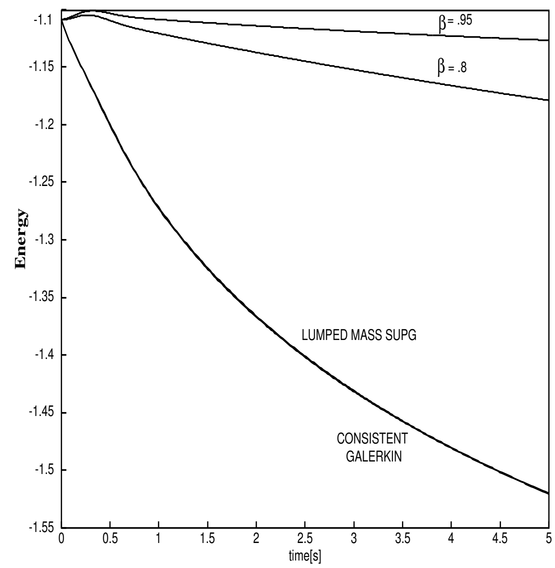
4. Numerical Results

One-dimensional flow along a channel of uniform width

The first example is a one-dimensional flow along a channel of uniform width, with uniform depth and without friction. The problem is purely one-dimensional, however, to seek the adaptability of the method, the two-dimensional finite element computer program is applied based on multidimensional shallow water equations[10]. The channel has a length of $200m$ where the water depth is $1m$. The Figure 8(a) shows the water elevation η for several time instants $t = 0.2s$, $4s$, y $5s$. A comparison between the different methods has been done, and we notice that the β-SUPG method (for $\beta = .95$, $\alpha = 1$) becomes less diffusive than the other methods. Figure 8(b) despicts curves on the energy conservation of several methods versus time t using as measure $\log_{20}(||res||)$. While the diffusivity of these methods decreases the convergence rate of them decreases too.



(a) Water Elevation profiles



(b) Energy Conservation Test

Figure 8: Water elevation profiles in subcritical regime. A comparison of several methods.

A solitary wave propagated along a one-dimensional channel with uniform bottom slope.

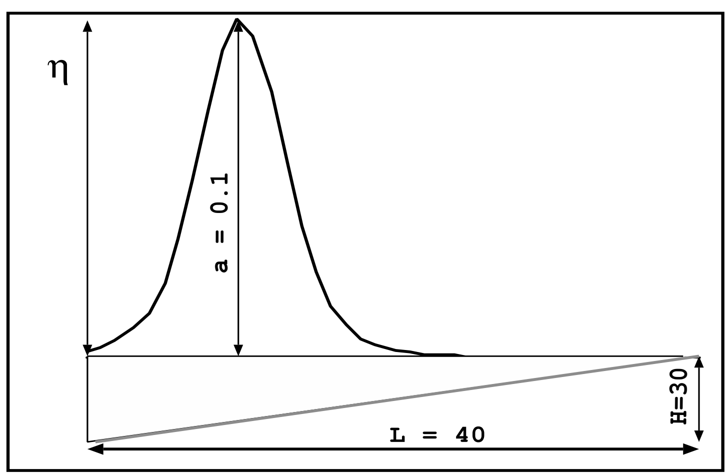


Figure 9: Coordinate system for shallow water equations.

The following example is the analysis of a solitary wave propagated along a one-dimensional channel with uniform bottom slope. The initial configuration of the solitary wave and the variation of the depth is shown in Figure 9. The initial conditions are given by:

$$\eta = a\ \mathrm{sech}^2 \frac{1}{2}\sqrt{3a}(x - \frac{1}{\alpha})$$
$$u = -(1 + a/2)\eta/(\alpha x + \eta)$$

where $a = 0.1, g = 1.0, \alpha = 1/30$. Figure 10 shows the computed results by "Lumped mass" SUPG method at left and β - "Consistent mass" SUPG method at right. For this problem the exact solution has a peak value of 1.2 times the initial peak value. This example allows to examine the numerical diffusivity property of several schemes[10],[12]. For the "Lumped mass" SUPG method the computed results seem to include a significant damping effect, which, as the number of subdivisions is increased(e.g. $N = 160$ or 320) the results seem to improve. It is very noticeable that the β - "Consistent mass" SUPG method produces better solutions even for relatively coarse grids (see Figure 10(right) where the peak value increased up to 1.2 times the initial peak value).

5. CONCLUSIONS

In this paper a technique to solve shallow water equations system has been examined. The use of the β-SUPG method with an explicit scheme in the time discretization to improves the accuracy of the solution with no damage in the stability condition. A detailed numerical analysis of this method for the advective equation shows the convenience of using β-SUPG method where others methods become very diffusive. A comparison among several methods has been developed and numerical tests confirm our hypothesis. The future work is oriented towards large scale simulation of a shallow water model with turbulent effects.

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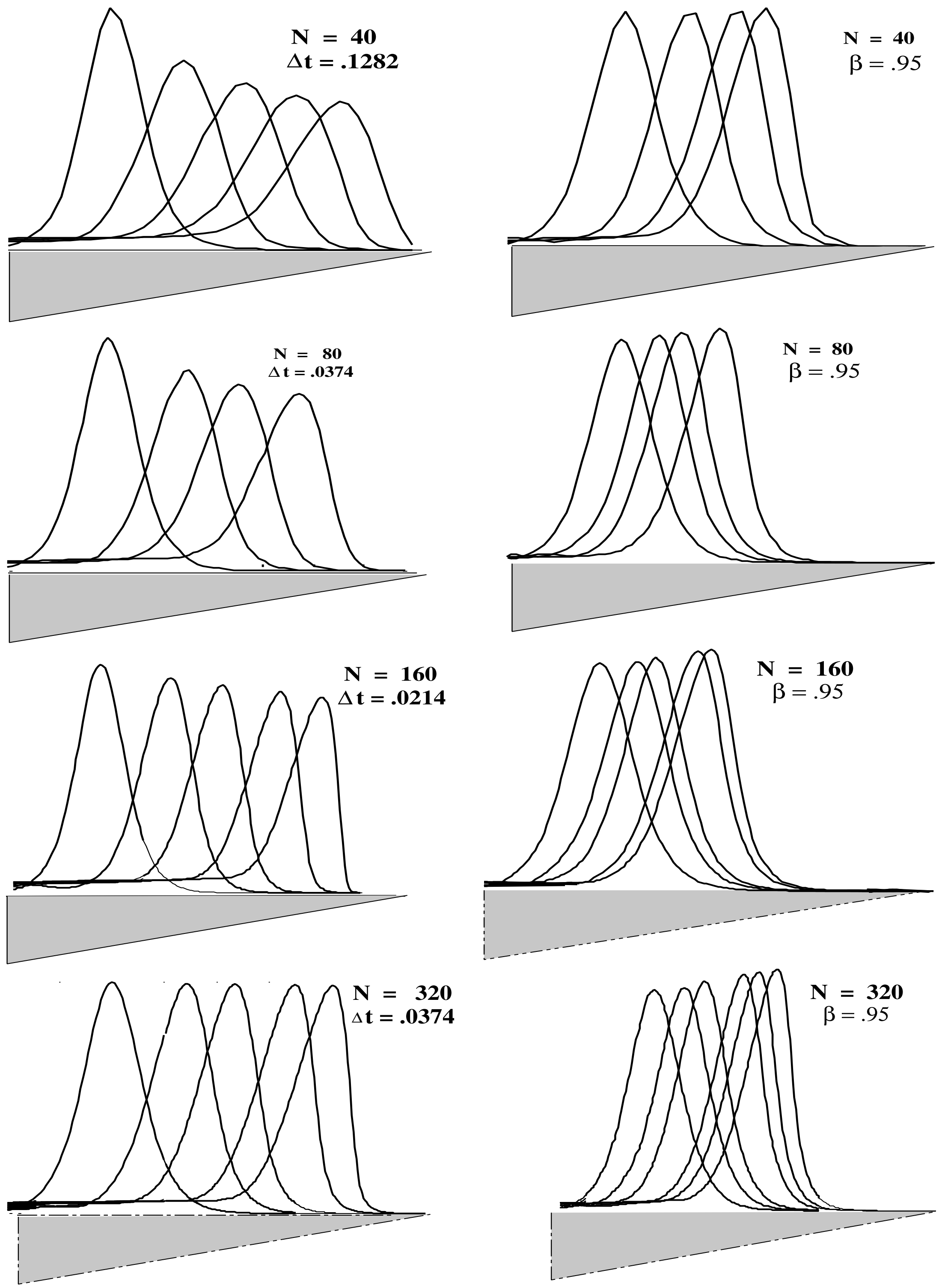

Figure 10: Shoaling of a wave. Solution for 40, 80 and 160 elements.(left)Lumped mass matrix (right)$\beta = .95$-SUPG method