Inviscid flow in open channels with lateral contraction

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1 ABSTRACT

In this paper a 2D dimensional shallow water equation model for the study of theinviscid flow in an open channel with contraction lateral has been implemented.

2 INTRODUCTION

The fundamental behavior of steady flows in an open channel with a lateral contraction is investigated experimentally and numerically to clarify the the mass-exchange mechanism near to lateral contraction. Typical flow features around ofthe lateral contraction such as acceleration of the flow, or generation of a vortex due to the contraction in this region are shown by findings of laboratory tests. The flow near a lateral contraction is characterized by formation of a vortex and behind of this by a separation zone. These have a marked effect on the mass exchange before and after the contracted zone. The influence of the bottom variations, in suchs phenomena are examined also.

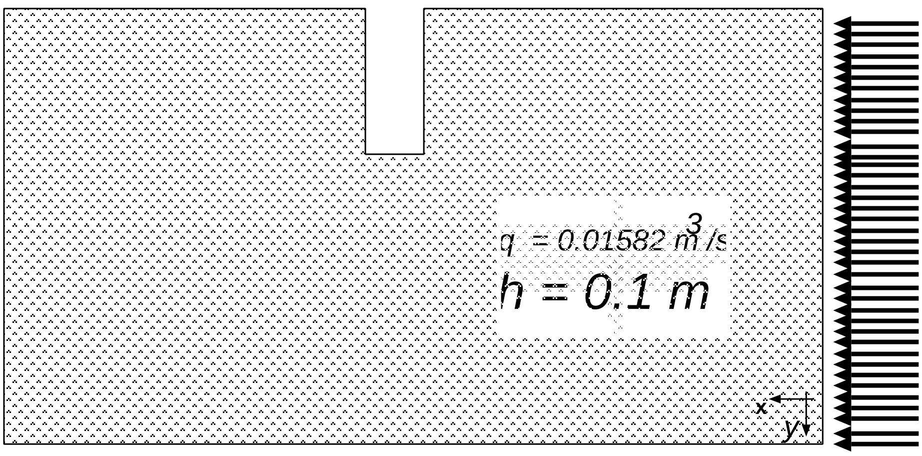


Figure 1: Diagrams of Flow Domain.

These flow features are examined numerically by plane two-dimensional (2D) shallow water flow equations. Two numerical models have been used: SMS code and petscfem code. The numerical results can be obtained with the petscfem code corresponding to the inviscid flow or Euler equation system. The SMS code describes the actual behavior of the flux. The calculated values for temporal and spatial variations in velocity and depth are compared with observed values.

The numerical results show a furth transverse velocity gradient in the contracted zone, feature more noteacible in the case of inviscid flux. A good correspondence between the numerical results of them is obtained downstream, where there aren't experimental results.

An estimate of downstream water depth has been done by means of the implementation of several downstream boundary conditions (with the petscfem code).In this code an explicit scheme in time using both Lumped SUPG mass matrix or Consistent SUPG mass matrix in combination with the use of a stabilized spatial discretization for solving shallow water flow problems has been implemented. Such numerical results show a adequate physic behavior of the flux.

3 2D Shallow Water Equation

The shallow water equations can be written in the following form:

$$\mathbf{U}_{,t} + \mathbf{F}_{j,j}(\mathbf{U}) = \mathbf{B} \quad on \quad \Omega \subset \mathbb{R}^2 \tag{1}$$

where

$$\mathbf{U} = \begin{bmatrix} h\mathbf{u_1} \\ h\mathbf{u_2} \\ h \end{bmatrix} \tag{2}$$

$$\mathbf{F}_j = \begin{bmatrix} hu_1u_j + \delta_{1j}\frac{1}{2}gh^2 \\ \\ hu_2u_j + \delta_{2j}\frac{1}{2}gh^2 \\ \\ h\mathbf{u_j} \end{bmatrix} \tag{3}$$

$$\mathbf{B} = \begin{bmatrix} gh\frac{\partial H}{\partial x} \\ \\ gh\frac{\partial H}{\partial y} \\ \\ 0 \end{bmatrix} \tag{4}$$

The flux vector $\mathbf{F}_j(\mathbf{U})$ is a homogeneous function of degree one in the conservative variables $\mathbf{U}$; is follows that

$$\begin{aligned} \mathbf{F}_j(\mathbf{U}) &= \mathbf{A}_j\mathbf{U} \\ \mathbf{F}_{j,j} &= \mathbf{A}_j\mathbf{U}_{,j} \end{aligned}$$

Let $\mathbf{n} = (n_1, n_2)$ be the outward unit normal vector on the boundaries and let $\mathbf{F}_j$ be split in the following way:

$$\mathbf{F}_j = \mathbf{F}_j^{(1)} + \mathbf{F}_j^{(2)} = \begin{bmatrix} \delta_{1j}\frac{1}{2}gh^2 \\ \\ \delta_{2j}\frac{1}{2}gh^2 \\ \\ 0 \end{bmatrix} + \begin{bmatrix} hu_1u_j \\ \\ hu_2u_j \\ \\ h\mathbf{u_j} \end{bmatrix}$$

Later we will make reference to

$$\mathbf{F}_n^{(2)} = \mathbf{F}_j^{(2)}\mathbf{n}_j = \begin{bmatrix} hu_1u_\mathbf{n} \\ hu_2u_\mathbf{n} \\ h\mathbf{u_n} \end{bmatrix}$$

4 Finite Element Formulation

Consider a discretization of Ω into element subdomains $\Omega^e, e = 1, \cdots, nel$ where nel is the number of elements. We assume

$$\overline{\Omega} = \overset{nel}{\underset{e=1}{\cup}} \overline{\Omega^e}, \quad \emptyset = \overset{nel}{\underset{e=1}{\cap}} \Omega^e$$

Also let Γ^e be the whole boundary of element e, Γ the boundary of Ω and Γ_{int} the following set:

$$\Gamma_{int} = (\overset{nel}{\underset{e=1}{\cup}} \Gamma^e) - \Gamma$$

The SUPG formulation is employed for the spatial discretization and the flow field to be analyzed is divided into small regions called finite elements Ω^e. The finite approximation leads to

$$\mathbf{U} = \sum_{j=1}^{numnp} \mathbf{N}^j\mathbf{U}^j, \quad \mathbf{F}_i = \sum_{j=1}^{numnp} \mathbf{N}^j{\mathbf{F}_i}^j$$

where $numnp$ denotes the total number of nodes in the discretization, $\mathbf{N}^j$ are the global piecewise bilinear basis functions and $\mathbf{U}^j, \mathbf{F}_i^j$ are the values of $\mathbf{U}, \mathbf{F}_i$ at node j. In the SUPG formulation the weighting functions are modified by the addition of a given perturbation function $\widetilde{\mathbf{P}}^j$, and now, the modified weighting function takes the form:

$$\widetilde{\mathbf{N}}^j = \mathbf{N}^j + \widetilde{\mathbf{P}}^j$$

where $\widetilde{\mathbf{P}}^j$ is written in the following form:

$$\widetilde{\mathbf{P}}^j = \alpha\frac{u_i}{|| u ||}\widetilde{\mathbf{N}}^j,_i$$

where α is the upwind parameter determined by the element Peclet number. Using the weighted residual method based on the SUPG formulation we may write

$$\sum_e \int_{\Omega^e} \widetilde{\mathbf{N}}^j(\mathbf{U},_t + \mathbf{F}_j,_j(\mathbf{U}) - \mathbf{B})d\Omega^e - \int_{\Gamma_{slip}} \mathbf{N}^j\mathbf{F}_n^{(2)}d\Gamma - \int_{\Gamma_{int}} \mathbf{N}^j[\mathbf{F}_n]d\Gamma = 0 \tag{5}$$

in which the multidimensional shallow water equations are the following

$$
\begin{aligned}
\mathbf{U}_{,t} + \mathbf{F}_{j,j}(\mathbf{U}) = \mathbf{B} \qquad & on \quad \Omega \qquad && governing \quad equation \\
\mathbf{F}_n^{(2)} = 0 \qquad & on \quad \Gamma_{slip} \qquad && null \quad flux \quad condition \\
[\mathbf{F}_n] = 0 \qquad & on \quad \Gamma_{int} \qquad && continuity \quad condition
\end{aligned}
$$

In the latest equation, the square brackets represents the jump of $\mathbf{F}_n$ across the interelement boundary. In fact, this equation is automatically verified because $\mathbf{F}_n$ is a continuous function. Integrating by parts, we obtain the weak form of the weighted residual equation

$$
\begin{aligned}
&\sum_e \int_{\Omega^e} \widetilde{\mathbf{P}}^j (\mathbf{U}_{,t} + \mathbf{F}_{j,j}(\mathbf{U}) - \mathbf{B}) d\Omega^e + \sum_e \int_{\Omega^e} (\mathbf{N}^j \mathbf{U}_{,t} - \mathbf{N}^j{}_{,i} \mathbf{F}_i - \mathbf{N}^j \mathbf{B}) d\Omega^e + \\
&+ \sum_e \int_{\Gamma^e} \widetilde{\mathbf{N}}^j \mathbf{F}_n d\Gamma - \int_{\Gamma_{slip}} \mathbf{N}^j \mathbf{F}_n^{(2)} d\Gamma - \int_{\Gamma_{int}} \mathbf{N}^j [\mathbf{F}_n] d\Gamma = 0
\end{aligned} \tag{6}
$$

Using the following splitting,

$$
\sum_e \int_{\Gamma^e} \widetilde{\mathbf{N}}^j \mathbf{F}_n d\Gamma = \int_{\Gamma_{int}} \mathbf{N}^j [\mathbf{F}_n] d\Gamma + \int_{\Gamma_{int/outflow}} \mathbf{N}^j \mathbf{F}_n d\Gamma + \int_{\Gamma_{slip}} \mathbf{N}^j \mathbf{F}_n d\Gamma \tag{7}
$$

We can write

$$
\begin{aligned}
&\sum_e \int_{\Omega^e} \widetilde{\mathbf{N}}^j \mathbf{U}_{,t} \, d\Omega^e + \sum_e \int_{\Omega^e} (\widetilde{\mathbf{P}}^j \mathbf{F}_{i,i} - \mathbf{N}^j{}_{,i} \mathbf{F}_i - \widetilde{\mathbf{N}}^j \mathbf{B}) d\Omega^e + \int_{\Gamma_{int/outflow}} \mathbf{N}^j \mathbf{F}_n d\Gamma + \\
&\int_{\Gamma_{slip}} \mathbf{N}^j \mathbf{F}_n^{(1)} d\Gamma = 0
\end{aligned} \tag{8}
$$

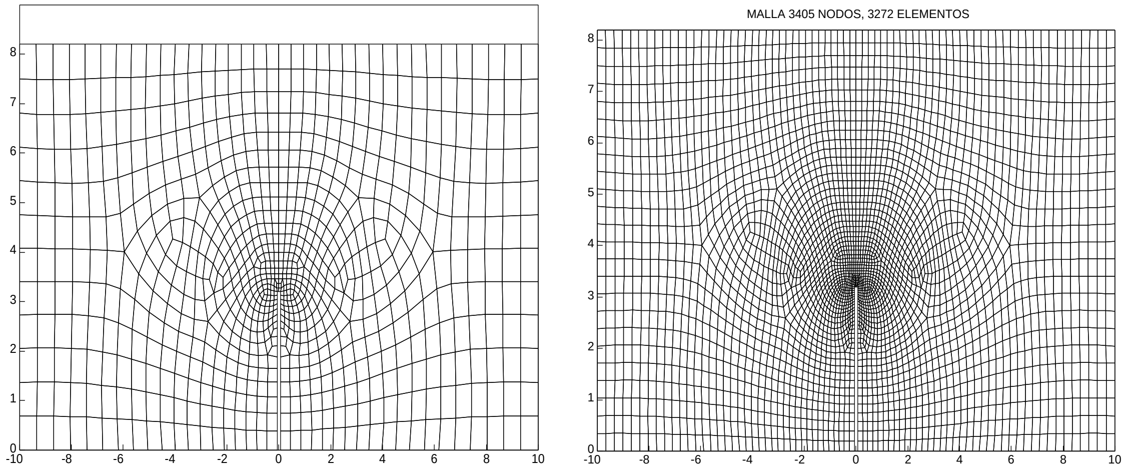
Making use of the forward Euler scheme in the time discretization, we can write the complete formulation in matrix form as follows:

$$
\mathbf{M} \Delta \mathbf{b} - \Delta t \mathbf{R} = 0
$$

where $\mathbf{M}$ is the consistent mass matrix, $\mathbf{R}$ the residue and $\Delta \mathbf{b}$ the vector of nodal variations of the conservation variables.(More details see reference[1]). In this paper a procedure based on row-sum lumping technique will be implemented.

5 Numerical Examples

Numerical simulation with the basic equations is done under the ideal conditions for the flow domain1. Two finite element meshes has been employed in the computation of flow, one with 818 elements and other, with 3272 elements, which can be observed in Figure 2



(a) Mesh with 818 elements

(b) Mesh with 3272 elements

Figure 2: Computational Domain.

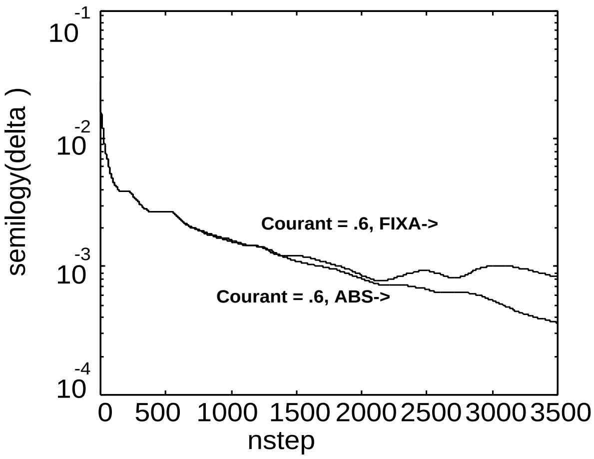


Figure 3: Convergence history , mesh with 3272 elements

A computational code, PETSC-FEM, has been employed for the solution of this problems. A row-sum lumped mass SUPG method for the spatial finite element discretization and an explicit method for the discretization in the temporal direction technique has been used. To illustrate the adaptability of the present finite element method, we show the convergence history for several downstream boundary conditions. Figure 3 shows such results for the Courant number equal to .6.

Figures 4 and 5 show the velocity field around the contraction lateral with abobe two finite element meshes mentioned. In both cases we note a recirculation zone after the lateral contraction. In this zone the water depth is very small.

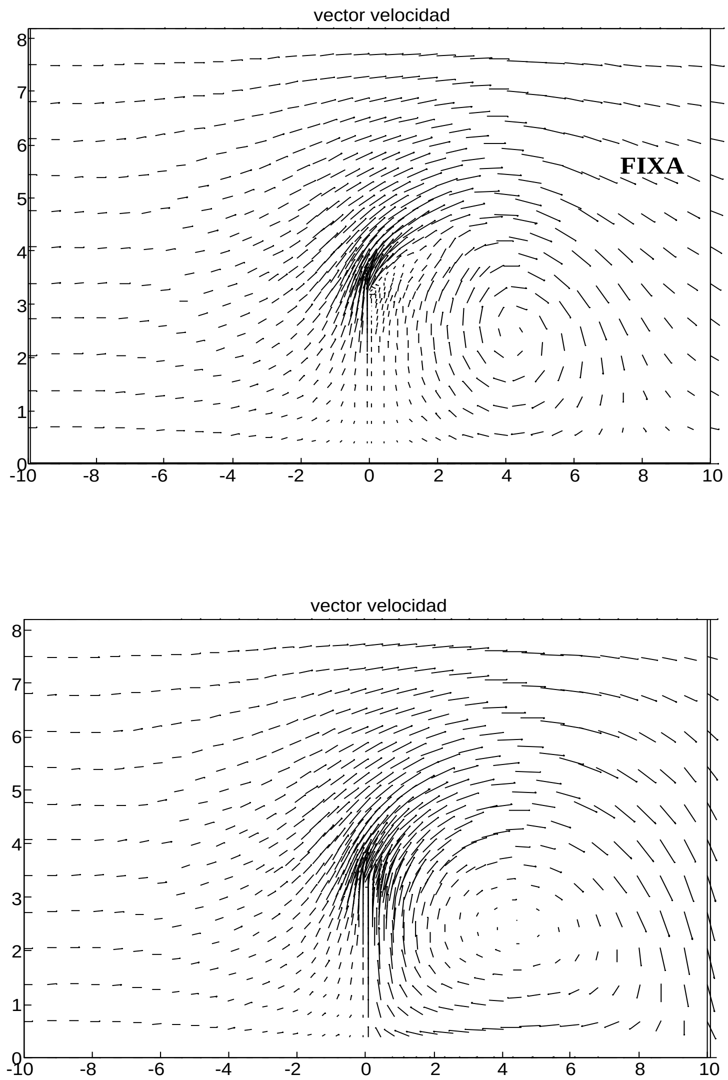
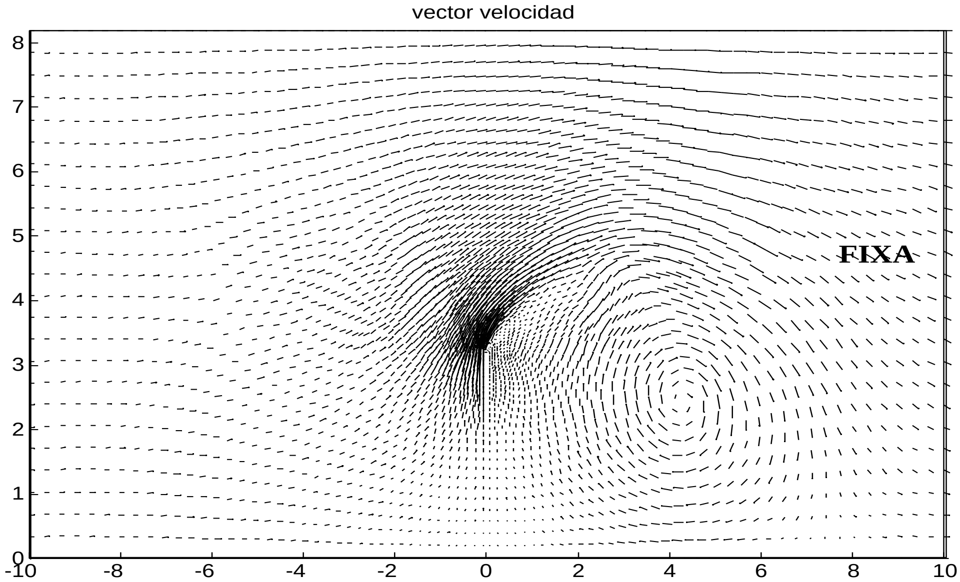


Figure 4: Magnitud of velocity



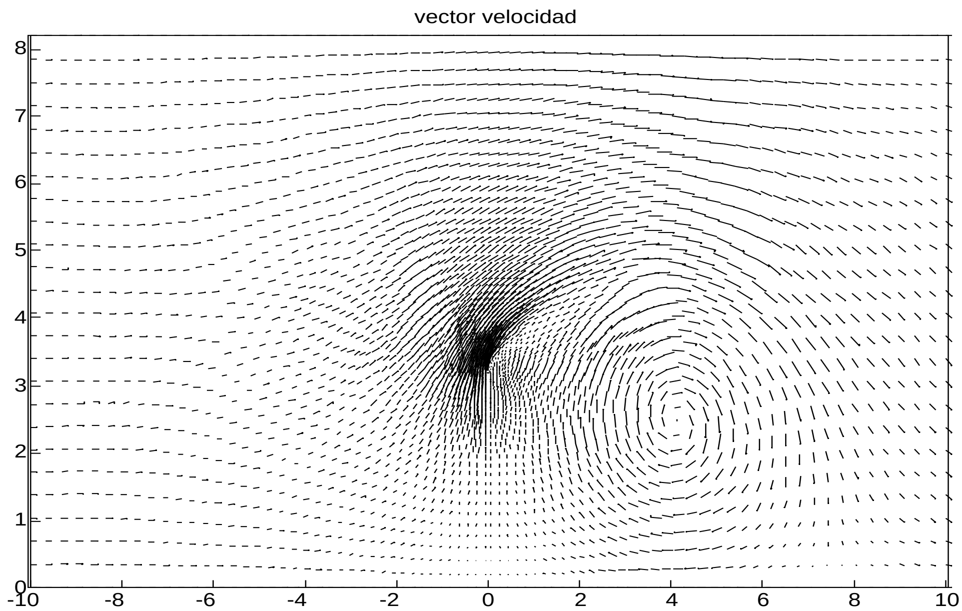
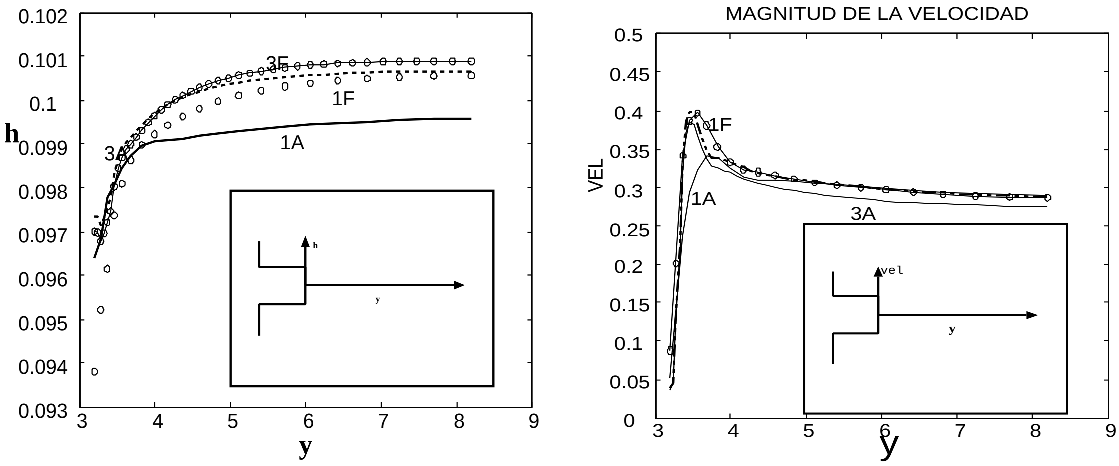


Figure 5: Magnitud of velocity

A comparison between the numerical results along the transversal direction to the contraction lateral are presented in Figure 6. We note that the petscfem code arises numerical results a little higher respect to SMS code.



(a) Water depth

(b) Velocity Field

Figure 6: Transversal section of flow.

A tridimensional surface of water depth and froude number can be observed in last figures. We recognize that the subcriticle flow in all the domain , with the froude nuber increases near the contraction lateral.

6 CONCLUSIONS

In this paper an application of the numerical method for the description of the behaviour of the flow in the open channel with contraction lateral has been examined. we appreciate that the petscfem code arises results quite different to SMS code. Such questions will be investigued in future works.

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References

[1] C.E.Baumann, M.A. Storti and S.R.Idelsohn. “Improving the convergence rate of the petrov-galerkin techniques for the solution of transonic and supersonic flows”. *IJNME*, 34:543–568, 1992.