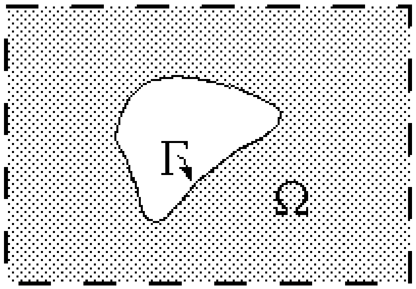
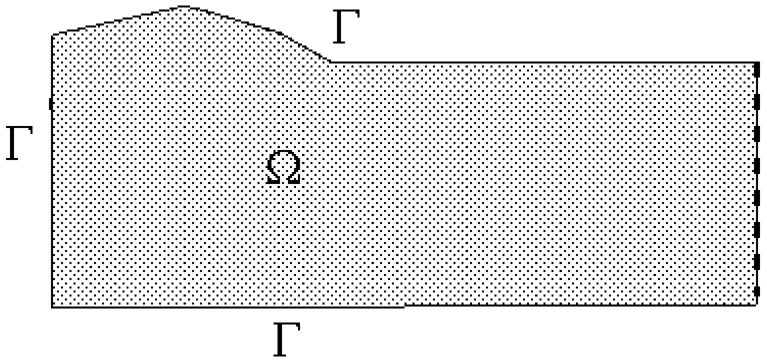
OPEN BOUNDARY CONDITIONS TO HELMHOLTZ EQUATION

Ruperto P. Bonet

ruperto.bonet@upc.edu

AN UNBOUNDED PROBLEM GOVERNED BY HELMHOLTZ EQUATION





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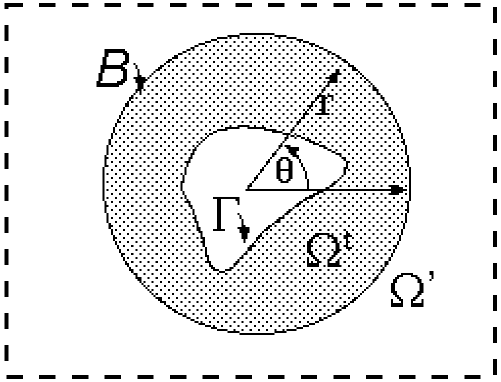
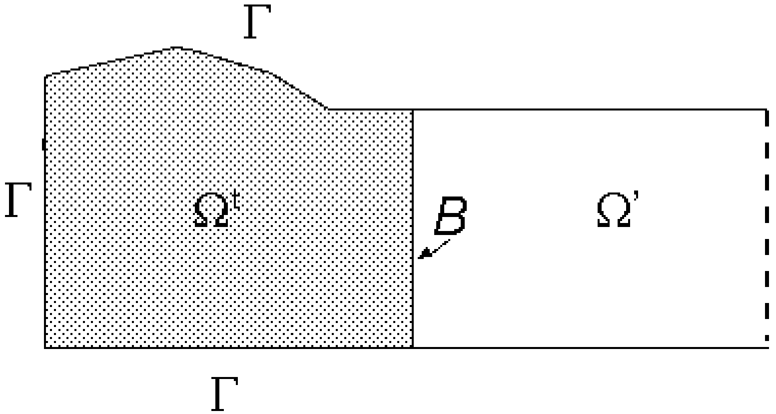
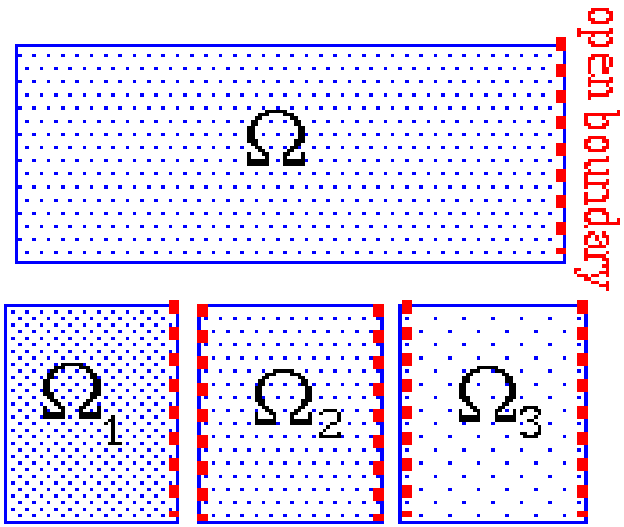
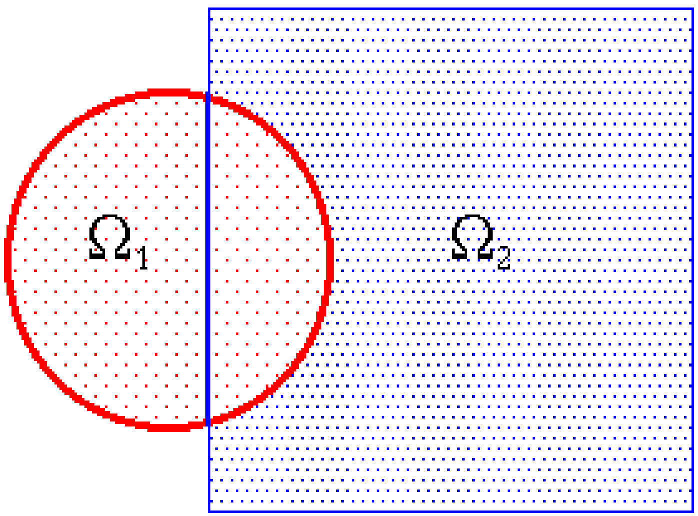




INTERFACE CONNECTIONS IN DOMAIN DECOMPOSITION METHODS

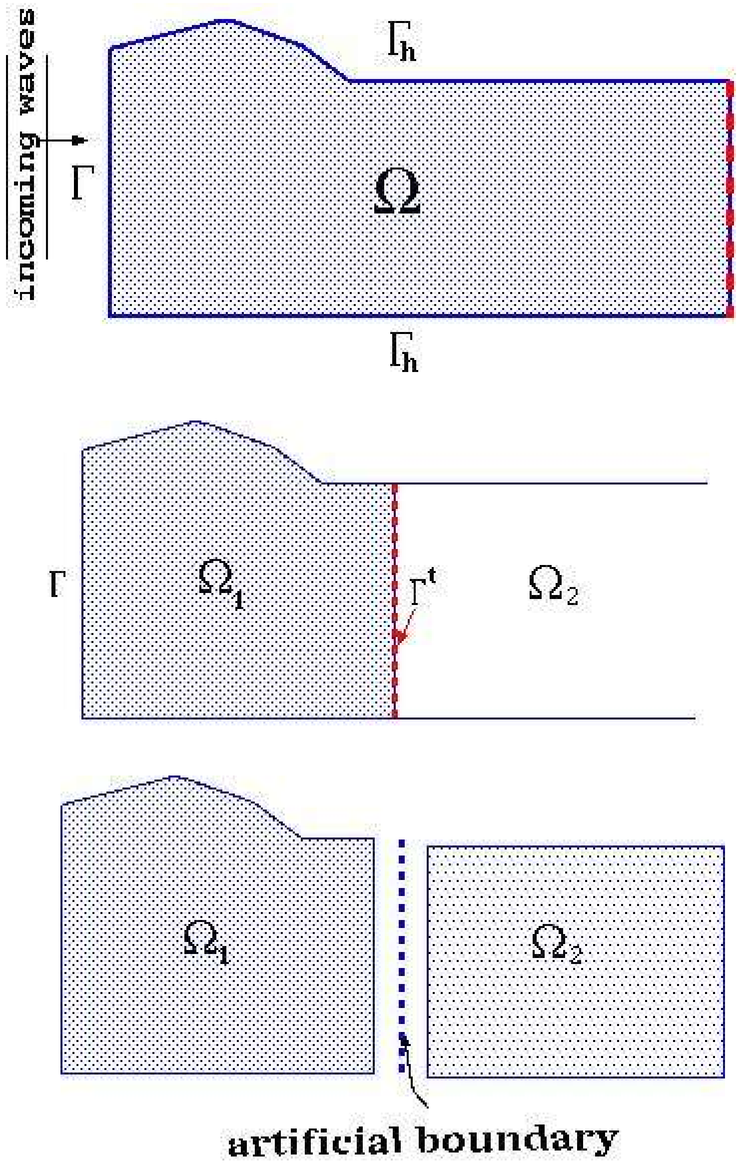


Non-overlapping



Overlapping

STATEMENT OF A WAVE GUIDE PROBLEM



$$\Delta\phi + k^2\phi = 0 \qquad in\ \Omega$$

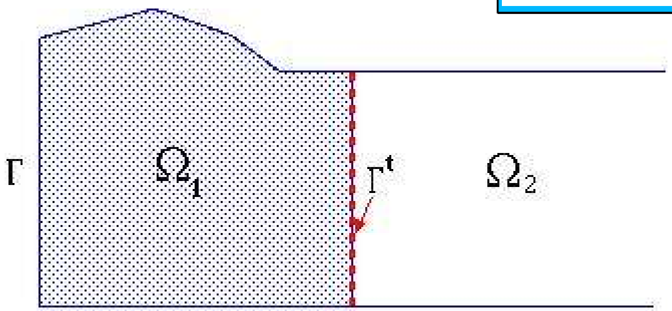
$$\phi = g \qquad in\ \Gamma$$

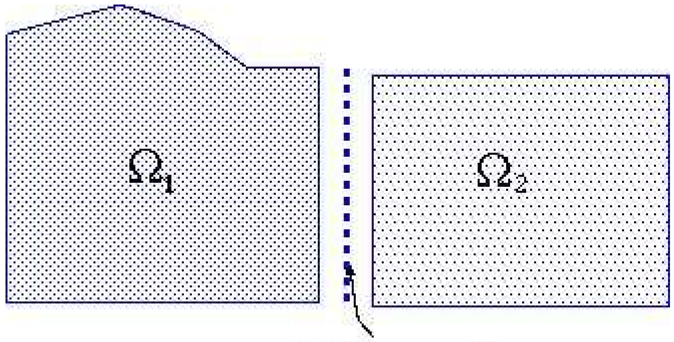
$$\frac{\partial\phi}{\partial \mathbf{n}} = 0 \qquad on\ \Gamma_h$$

$$+\ some\ b.c. \qquad \text{at infinity}$$

Steklov – Poincaré Operator

$$S_i : u_{\Gamma_t} \longrightarrow \frac{\partial u}{\partial \mathbf{n}_i}\Big|_{\Gamma_t}$$





$$\begin{aligned}
\Delta\phi_1 + k^2\phi_1 &= 0 && \mathit{in}\ \Omega_1\\
\phi_1 &= g && \mathit{in}\ \Gamma\\
\frac{\partial\phi_1}{\partial\mathbf{n}} &= 0 && \mathit{on}\ \Gamma_h\\
\phi_1 &= u && \mathit{on}\ \Gamma^t
\end{aligned}$$

$$\begin{aligned}
\Delta\phi_2 + k^2\phi_2 &= 0 && \mathit{in}\ \Omega_2\\
\frac{\partial\phi_2}{\partial\mathbf{n}} &= 0 && \mathit{on}\ \Gamma_h\\
\phi_2 &= u && \mathit{on}\ \Gamma^t\\
+\ \mathit{some\ b.c.} & && \text{at infinity}
\end{aligned}$$

Iterative scheme with domain decomposition

$$\frac{d^2\phi}{dx^2} + k^2\phi = 0 \quad (x \geq 0)$$
$$\phi(0) = 1\ ,$$
$$\lim_{x\to+\infty} \left(\frac{d}{dx}\phi - ik\phi\right) = 0$$

- 1) $\phi_1^0 \equiv 0,\ \phi_2^0 \equiv 0,\ n = 0$
- 2) n $\leftarrow$ n+1
- 3) Get $\phi_{1,2}^n$
- 4) Goto step 2 (or end)

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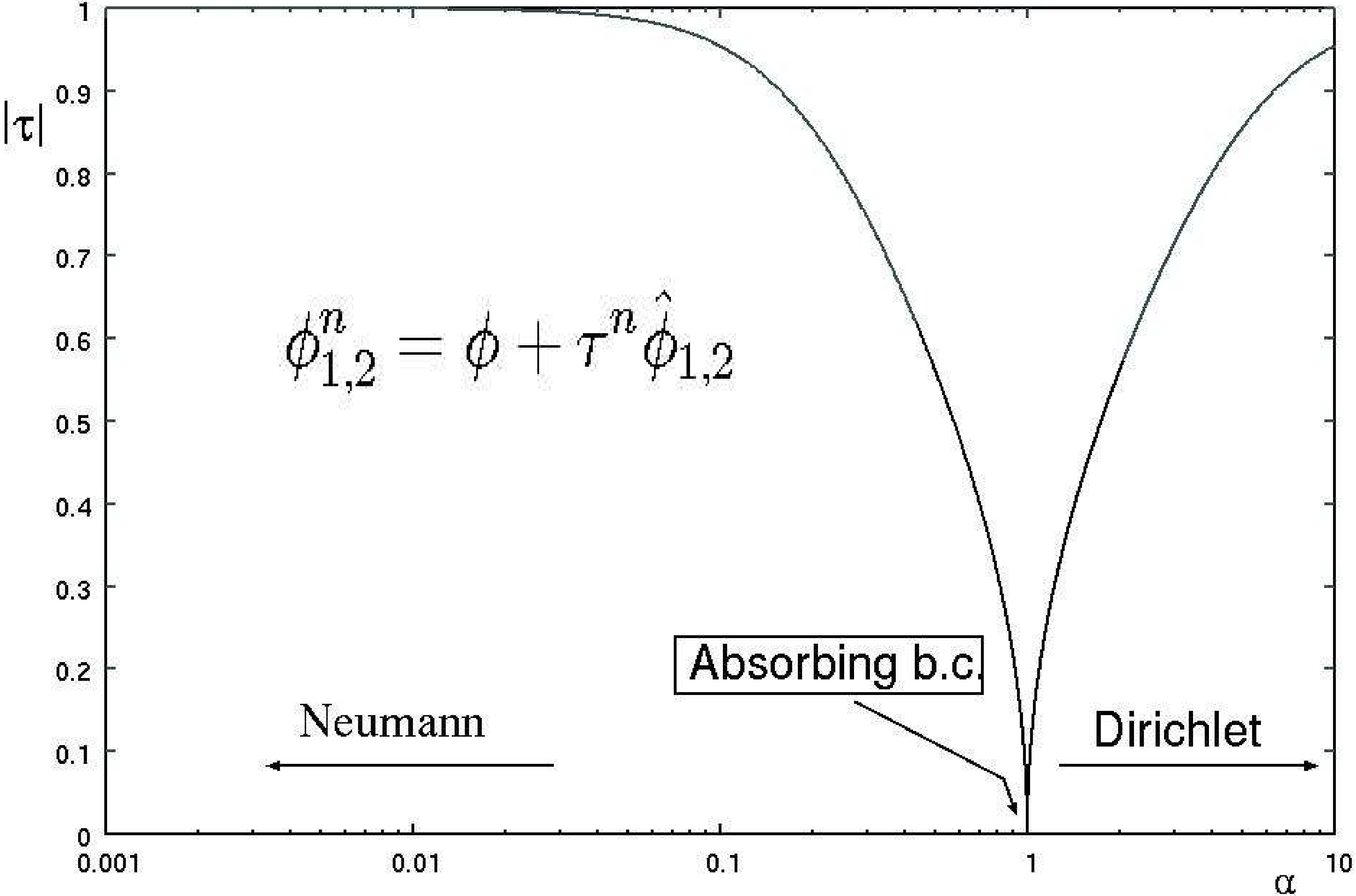
in Ω_1

$$\begin{aligned} \phi_{1,xx}^n + k^2\phi_1^n &= 0 \\ \phi_1^n(0) &= 1 \\ \left(\phi_{1,x}^n + \mathrm{ik}\alpha\phi_1^n\right)\Big|_{x=L^-} &= \left(\phi_{2,x}^{n-1} + \mathrm{ik}\alpha\phi_2^{n-1}\right)\Big|_{x=L^+} \end{aligned}$$

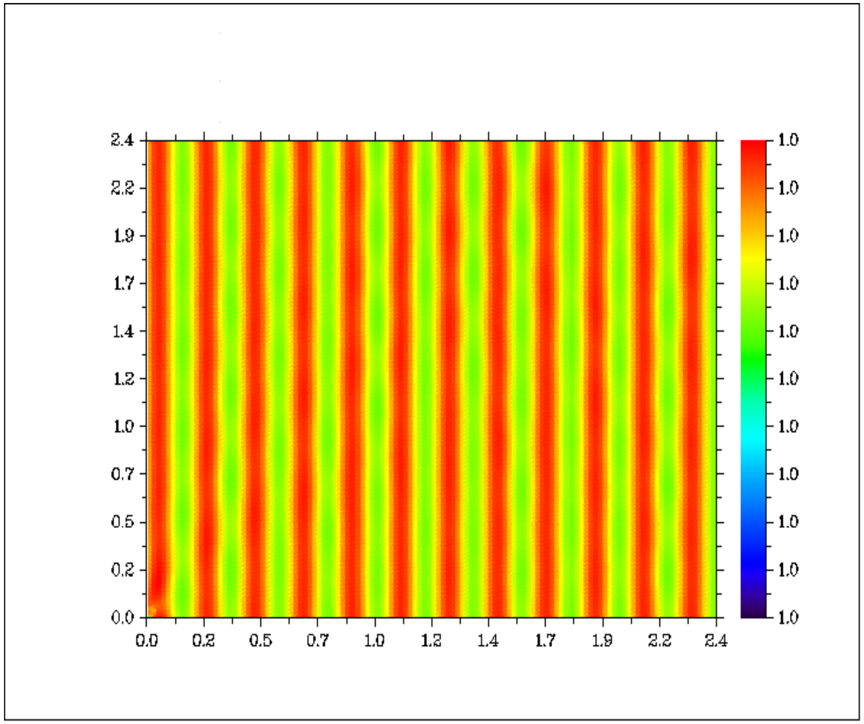
in Ω_2

$$\begin{aligned} \phi_{2,xx}^n + k^2\phi_2^n &= 0 \\ \left(\phi_{2,x}^n - \mathrm{ik}\alpha\phi_2^n\right)\Big|_{x=L^+} &= \left(\phi_{1,x}^{n-1} - \mathrm{ik}\alpha\phi_1^{n-1}\right)\Big|_{x=L^-} \\ \left(\phi_{2,x}^n + \mathrm{ik}\alpha\phi_2^n\right)\Big|_{x=2L^-} &= 0 \end{aligned}$$

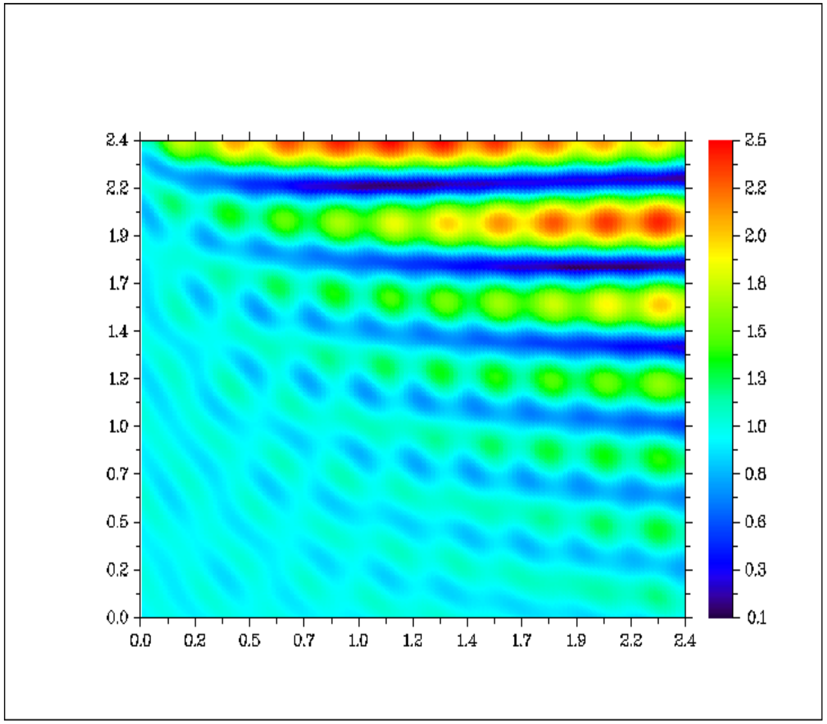
RATE OF CONVERGENCE



Numerical Example with k= cte



a) g = exp(ikx)



b) g = exp(ikcos(π /6)x)

PRINCIPAL MATHEMATICAL PROBLEM

Given an elliptic differential operator L
to get the DtN operator

$$\frac{\partial}{\partial \mathbf{n_i}} + DtN$$

- With constant coefficients
- With variable coefficients in the normal direction
- With variable coefficients

The Continuous Case

$$\mathcal{L}(\phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2 \phi = 0 \quad on \quad \Omega$$

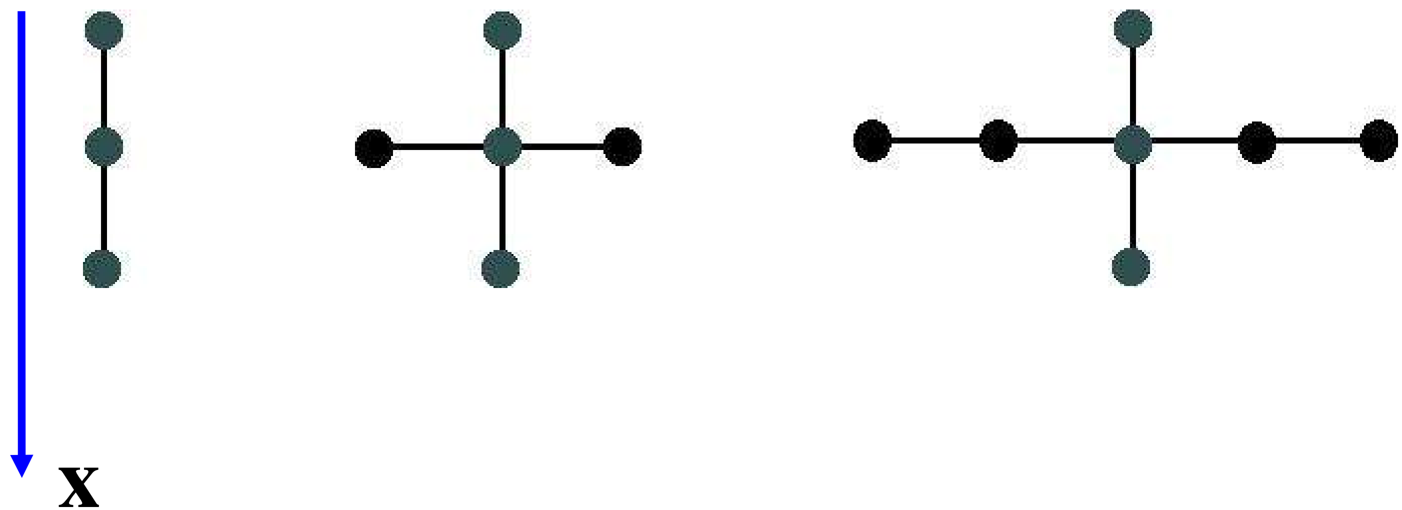
$$\begin{aligned} \frac{\partial \phi^+}{\partial x} &= +\mathrm{i}\mathcal{R}\phi^+ + G \\ \frac{\partial \phi^-}{\partial x} &= -\mathrm{i}\mathcal{R}\phi^- - G \\ \phi &= \phi^+ + \phi^- \end{aligned}$$

$$\mathcal{R}^2 = k^2 (1 + \frac{1}{k^2} \frac{\partial^2}{\partial y^2})$$

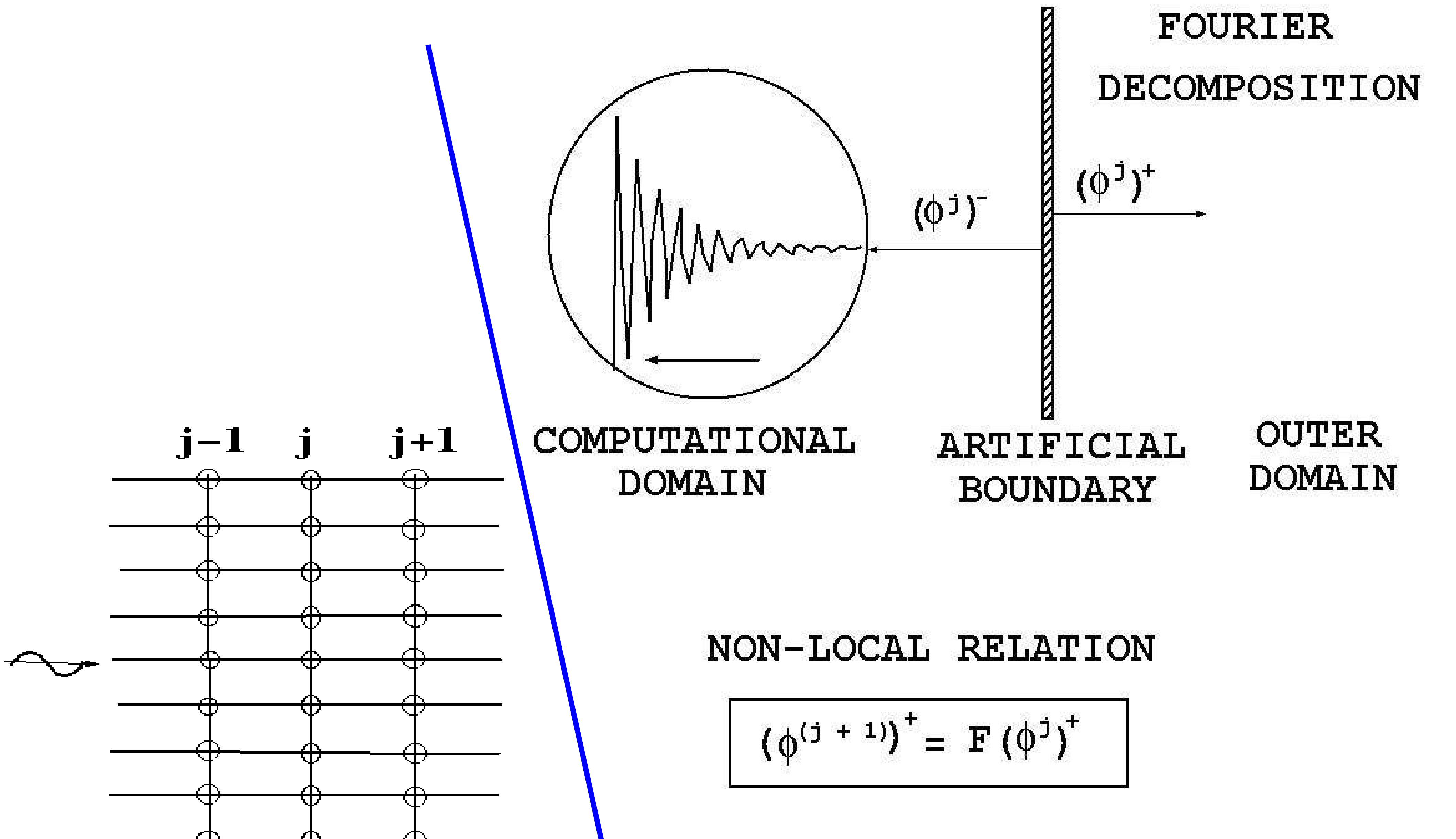
G = ?

Remarks

$$DtN = -\sum_{m=0}^{N} \frac{\partial^m}{\partial y^m}\left(\beta_m(x,y)\frac{\partial^m}{\partial y^m}\right)$$

N	**0**	**1**	**2**
scheme			
Band matrix	1	3	5

DISCRETE NON-LOCAL (DNL) METHOD



HELMHOLTZ EQUATION

$$\Delta\phi + k_o^2\phi = 0$$

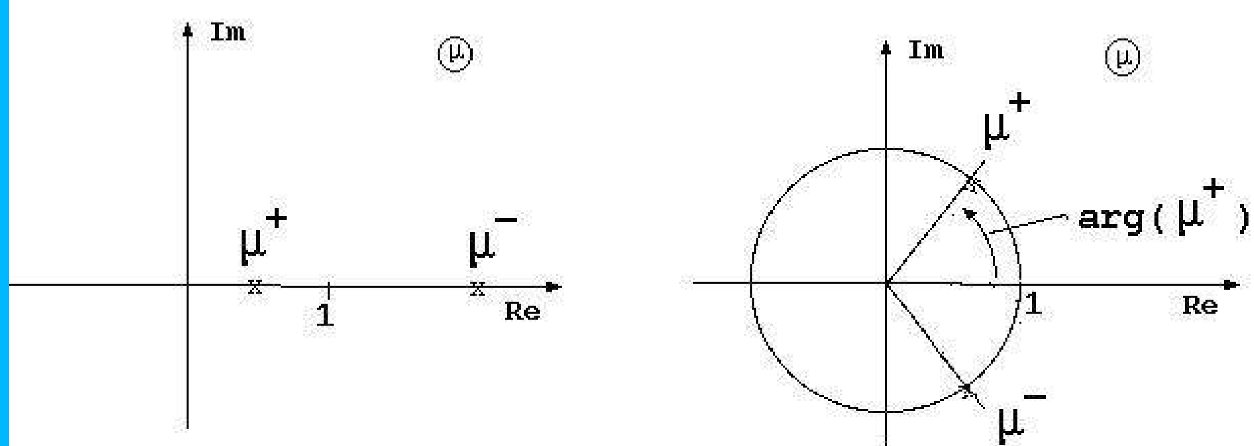
FEM

$$A\phi^{j-1} + B\phi^{j} + A\phi^{j+1} = 0$$

$j = 1, 2, \dots, M,$

$$\mu^2 + \lambda\mu + 1 = 0$$

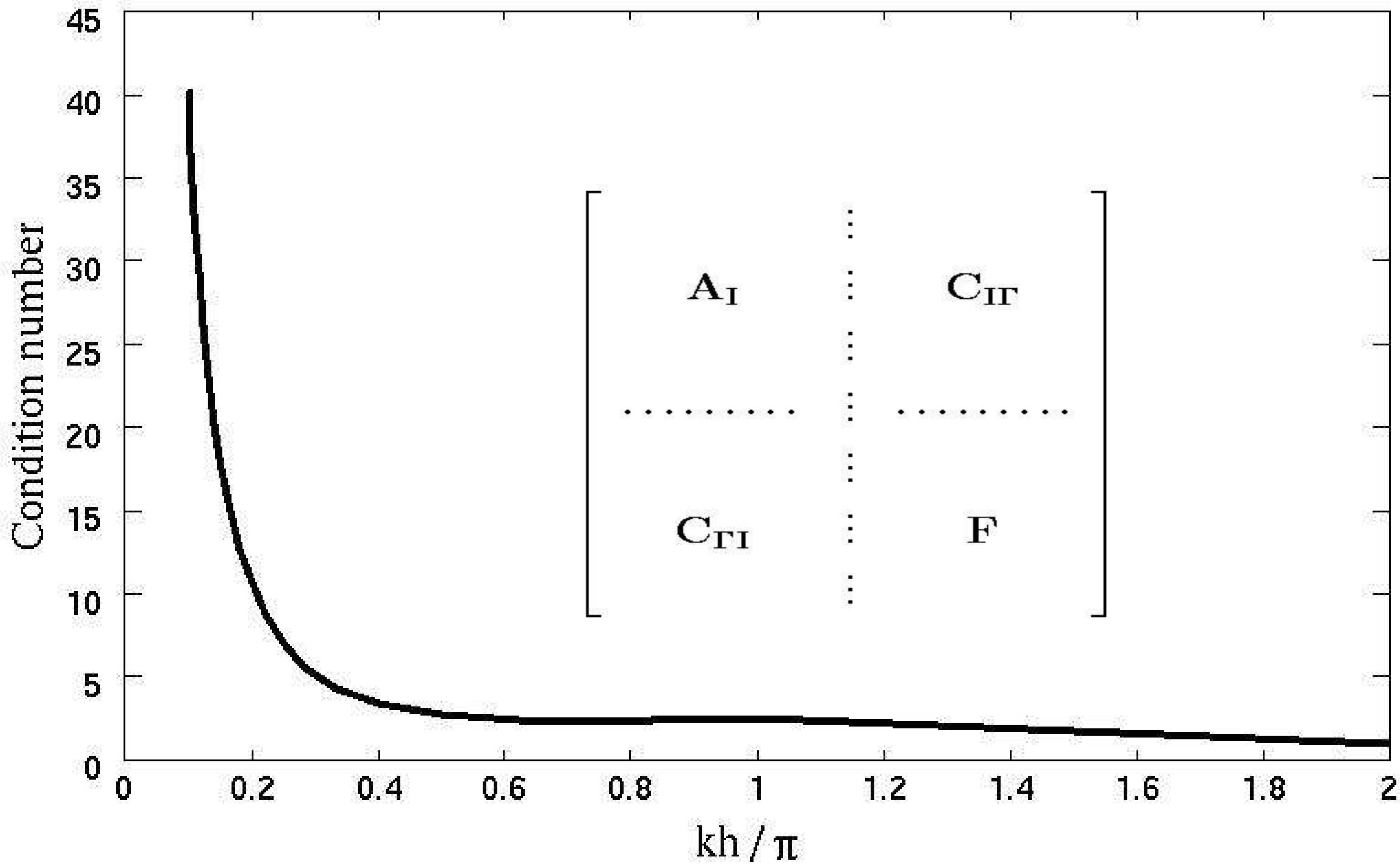
OPERATIONAL EQUATION



$|\lambda| > 2$ case
evanescent waves

$|\lambda| < 2$ case
progressive waves

DNL Matrix Condition Number



DNL GAUSS FILTER METHOD

$$\frac{\partial^2 \hat{\psi}}{\partial x^2} + l^2 \hat{\psi} = 0 \qquad in\ x_0 \leq x \leq \delta$$

$$\hat{\psi} = g \qquad on\ x = x_0$$

$$\int_{x_0}^{\delta} \sigma(x) \psi e^{ikx} dx = 0$$

$$\sigma(x_j) = \begin{cases} e^{\frac{-s^2(x_j - x_c)^2}{2}} & \text{for } 0 \leq j \leq n_j, \\ 0 & \text{for } j < 0 \end{cases}$$

OPTIMIZATION OF DISCRETE FILTERING LAYER

$$\min \frac{1}{(M+1)} \sum_{m=0}^{M} | R_m(\theta_m, s) |^2 \cos(\theta_m)$$

layer thick (λ)	optimal s value	Average R
1	3.1430	2.7775e-03
2	2.4905	1.3249e-03
3	1.9199	2.7755e-04
4	1.7823	2.8770e-04
5	1.5157	9.6061e-05
6	1.4585	1.1577e-04
7	1.2935	4.5070e-05
8	1.2638	5.8650e-05
9	1.1472	2.4434e-05
10	1.1229	3.0625e-05

$$s \sim 3.2\, \delta^{-0.45}$$

Optimal Numerical Reflection

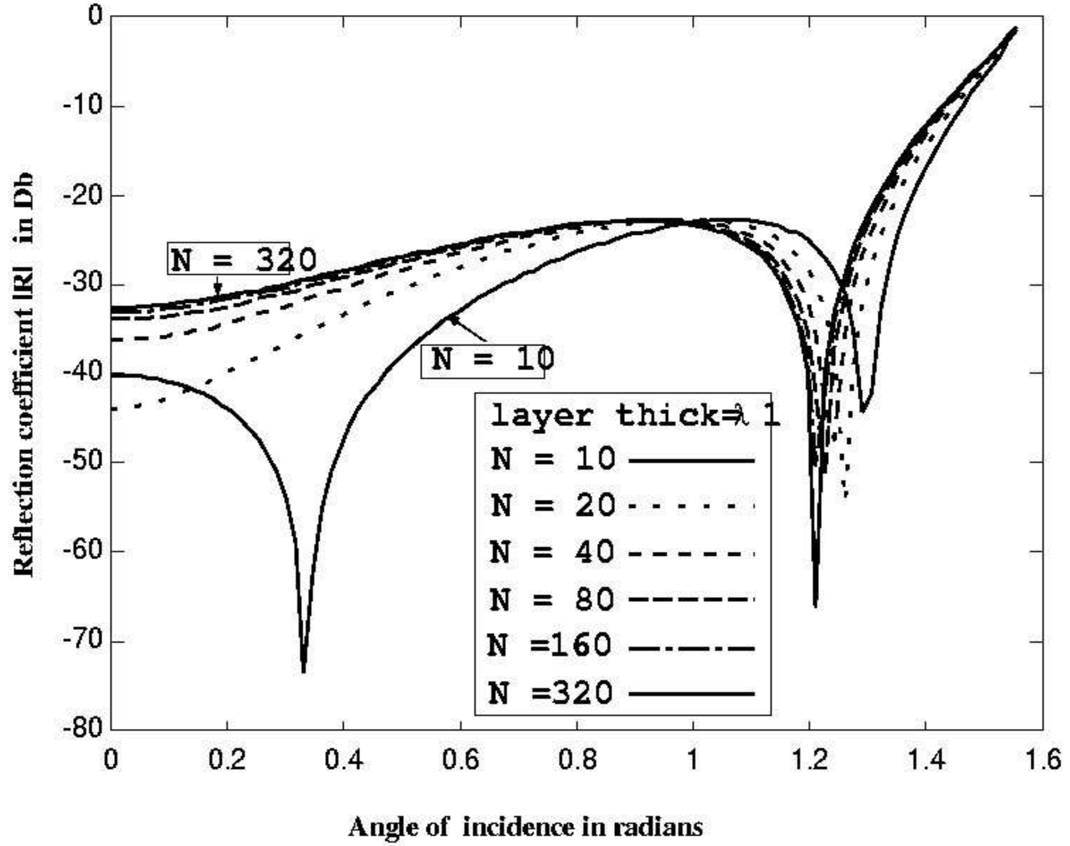


Figure 1: Optimal numerical reflection coefficient (in Db, i.e. $20log_{10}|R|$) versus angle of incidence in radians, for several number of points per wavelenghts.

Optimal Numerical Reflection

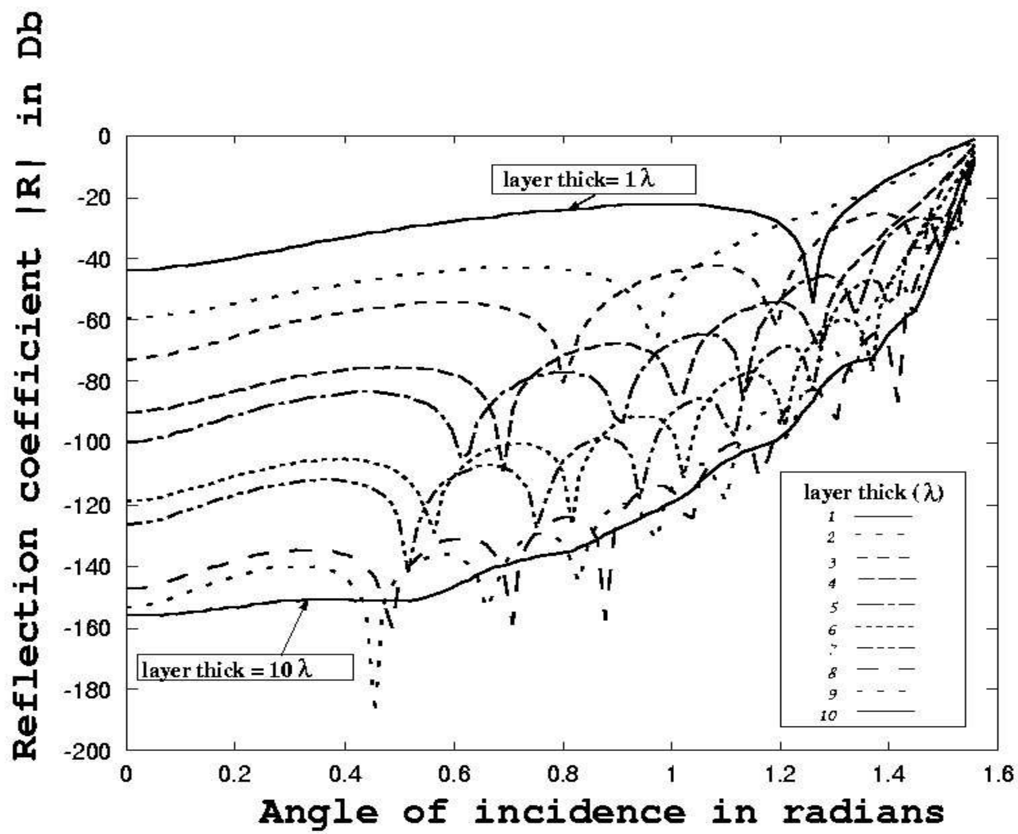


Figure 1: Optimal numerical reflection coefficient (in Db, i.e. $20log_{10}|R|$) versus angle o: incidence in radians, for several layer thickness (in wavelenghts)

NUMERICAL EXAMPLES

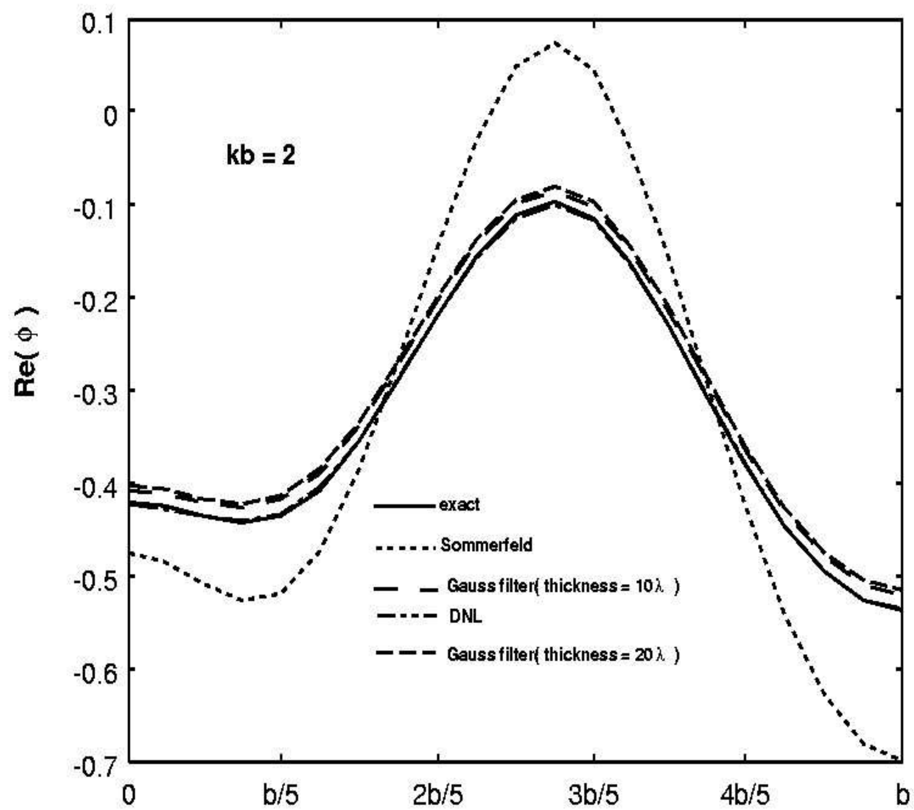


Figure 1: Comparison of boundary conditions along the artificial boundary, for $kb = 2$

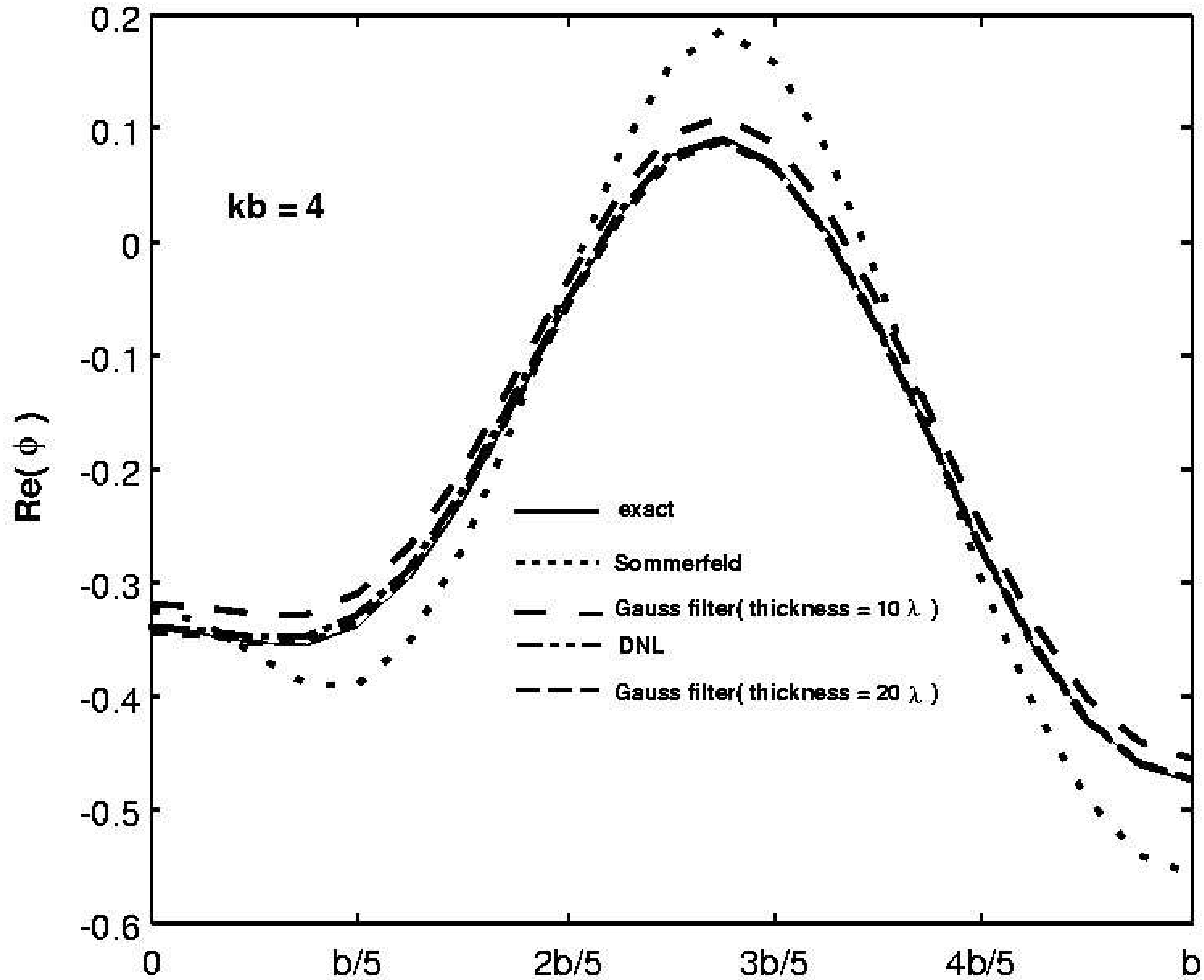


Figure 2: Comparison of boundary conditions along the artificial boundary, for $kb = 4$

RELATIVE ERROR ON THE OPEN BOUNDARY

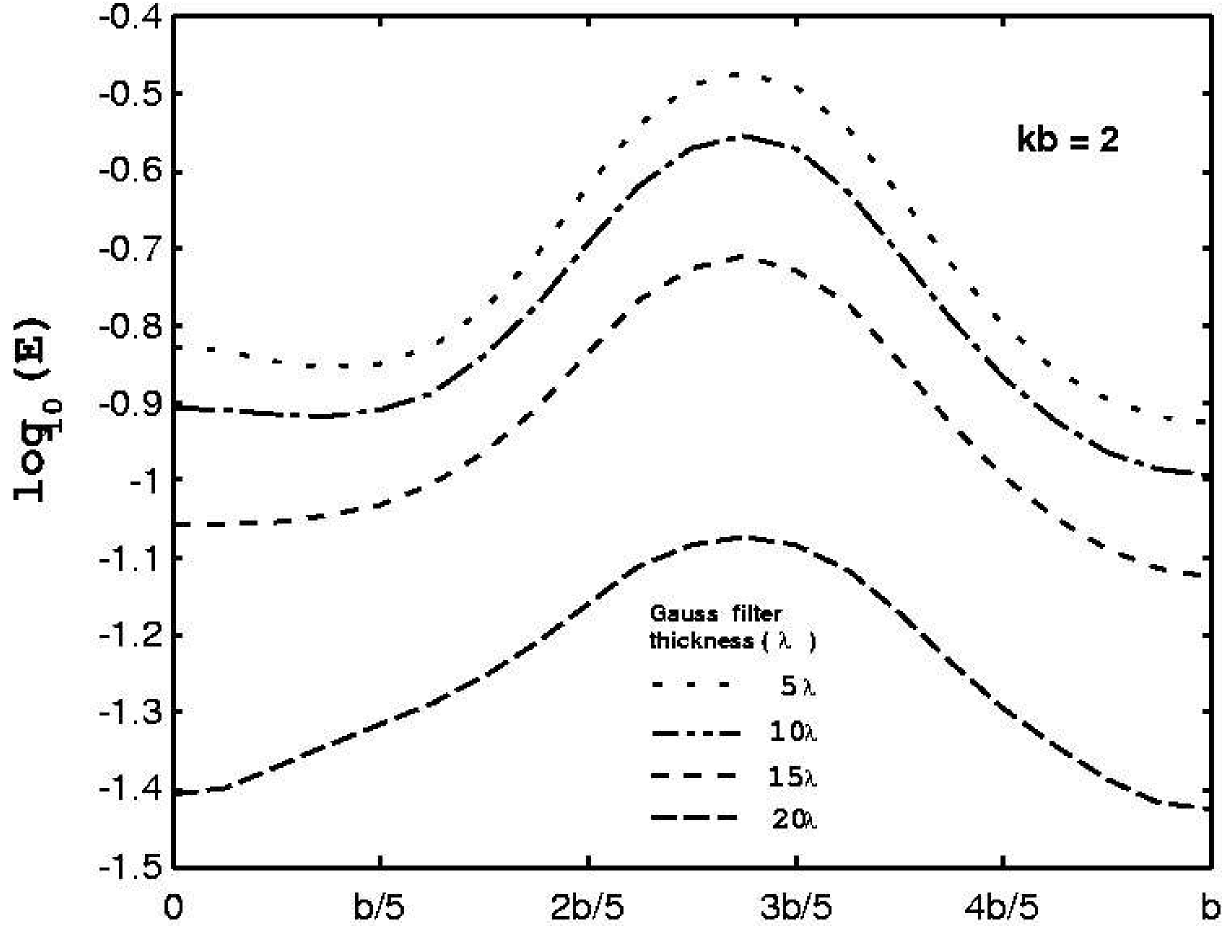


Figure 1: Dependence of the relative error on the layer thickness, for $kb = 2$

RELATIVE ERROR ON THE OPEN BOUNDARY

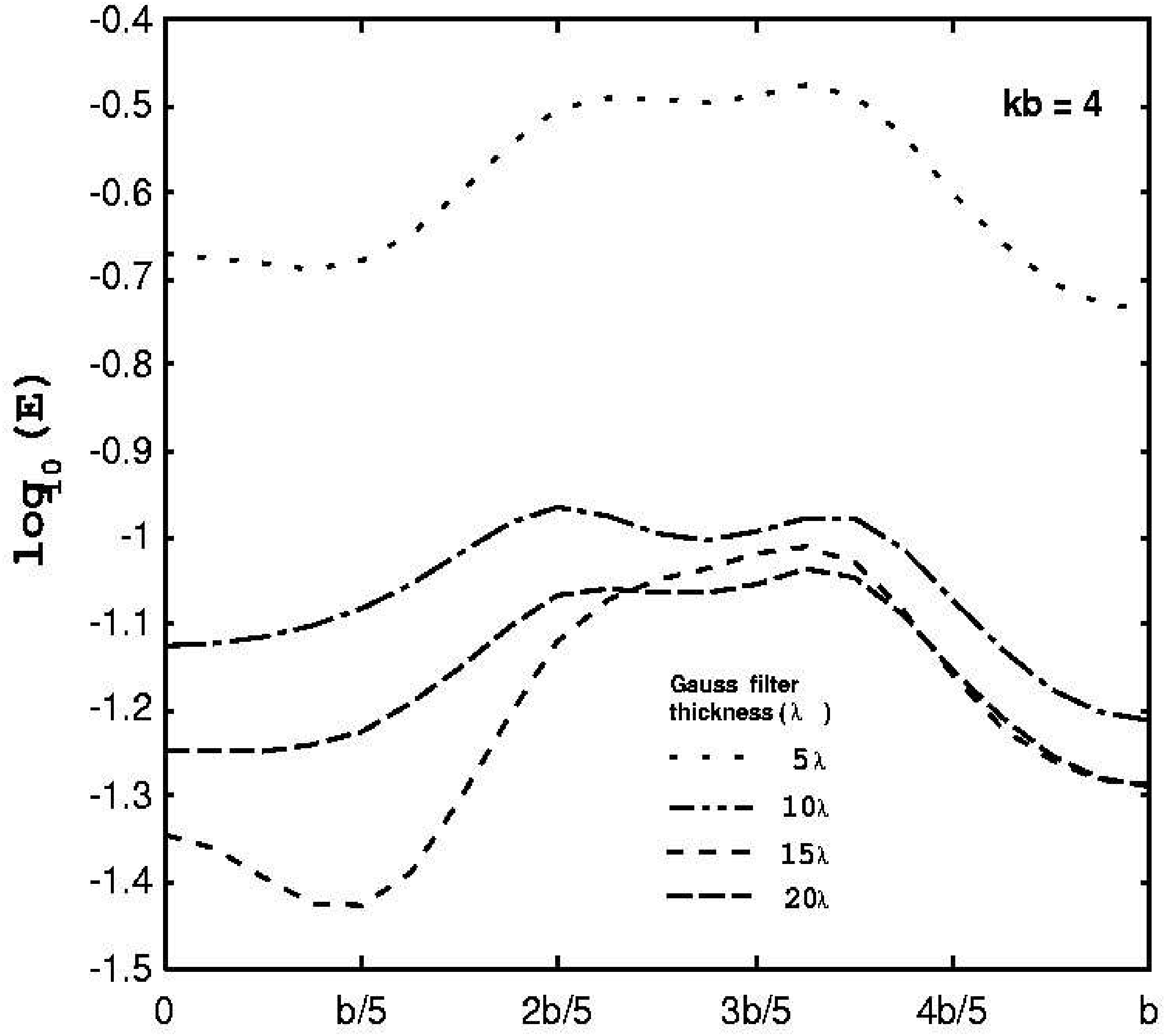


Figure 1: Dependence of the relative error on the layer thickness, for $kb = 4$

CONCLUSIONS

- The DNL method is suitable way to develop discrete absorbing boundary conditions

- This methodology allows investigate this subject as a particular case of interface connections in DDM philosophy