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The Experience of Discovery in Mathematics

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Abstract: We discuss experience of discovery against experience of creation. Then we study experience of discovery in different layers of abstractions and in different cognitive styles. Then we compare experience of discovery within the subject of discovery and within the cognitive structure. Finally, we discuss the effect of discovery of truth on changing the truth.

Introduction

There are many mathematicians longing to experience the taste of discovery of other great mathematicians. This means that, the experience of discovery is diverse and is not unique. Particularly, not everybody could claim that I fully understand experience of discovery and have tasted it thoroughly. For example, the experience of discovery is different for verbal and pictorial cognitive types and it is not the same for holistic and analytic cognitions. Before we compare experience of discovery for different cognitive structures, we shall understand what is meant by "discovery". This is why you compare discovery against creation trying to define both of them by making explicit borders between them. Then we study problem solving and theorization as faces of discovery and creation respectively. At the final section of the paper, we discuss the deepest experience of discovery and call it discovery of Truth. Then we extrapolate if the experience of discovery could change the truth or not.

On the difference of discovery and creation

We will define the concept of discovery against the concept of creation. When you create, you make something new. But when you discover, you will find something created by others. In creation, you create what you desire. But in discovery, you should find out something desired by another person creating the subject of discovery. Although discovery of a truth could affect the truth or even change it, but there are many predetermined aspects in the subject of discovery which are not known to the discoverer. Even in many ways, the possible changes to the discovered truth are predetermined and to yet be discovered. But creation is free of all these limitations. Then comes the main question and that is: Is it possible for humans to create a mathematical truth? Is mathematical truth pre-engineered in the creation of our brain? Or even if what we create is pre-engineered in the genes we present to our children? Many of these questions have been already discussed in contemporary philosophy of cognition. We do not deal with these questions, since we concentrate on the concept of discovery. Discovery has many incarnations in practice of mathematics. One of the most important incarnation is problem solving. A problem solver is in the process of discovery of solution rather than creating it. Although a problem solver may have to create an appropriate setting or language to do computations needed for solving a problem. Anyhow the solution to the problem must be predetermined by definition.

Problem solving as a face of discovery

Although the first experience of discovery is problem solving in elementary mathematics, by no means it is the case that experience of discovery is as simple as discovery in elementary mathematics. Many great mathematicians are longing to taste experience of discovery through the eyes of other fellow mathematicians. Of course, the experience of discovering the truth is not limited to problem solving situations. For example, when you read a textbook, you may experience discovery in many senses. But, for sure, problem solving could be regarded as one of the faces of discovery which different aspects of discovery incarnate in this process. The question is, if concentrating on problem solving, is it possible to miss some of the main aspects of discovery. There is a difference when you read mathematics and when you do mathematics. When you do mathematics, problem solving is a very nice representative of different aspects of discovery, and in fact these are the aspects people are interested in. These aspects are indeed more advanced, since learning mathematics is happening in earlier steps of doing mathematics. Also, if it could be that learning mathematics could make tasting the experience of discovery possible, it would not be the case that mathematicians find the experience of other mathematician inaccessible to them. Theorizing is another face of doing mathematics which has more to do with creation rather than discovery. So, we should study now theorization as a face of creation.

Theorization as a face of creation

A theorizer is more of a creator rather than a discoverer. The path of a theorizer is not predetermined, and it is not like the path of a discovery. This means that the path of a theorizer is not unique. Which means that theorizer makes some choices in how to build his pass. Although there are choices made by a problem solver also, but the choices of a problem solver are predetermined and choices of a theorizer are not. This doesn't mean that there are no limitations on theorizer on how to theorize. We shall demonstrate examples where theorization is not unique but in these examples, certain computations as special cases, serve for important examples of different theories developed from them. For example, there are several geometric theories based on the same set of objects in Euclidean geometry. Examples include, projective geometry against analytic geometry, and there are several methods of computational supporting the same object, like vectors, complex numbers, Cartesian coordinates, pole and polar trigonometry and metric equalities. Sometimes a theory can be built on several similar objects like hyperbolic trigonometry and spherical trigonometry and Euclidean trigonometry. This shows that the experience of creation by theorizer is diverse and should be subject of a study.

The experience of creation in theorization

Are analogies discovery or creation? For example, is hyperbolic trigonometry discovered or created? Or even spherical trigonometry which is almost as old as triangle trigonometry discovered or created. To me, Pythagoras theorem is a discovery and formulating similar theories of computation called Euclidean trigonometry, and spherical trigonometry, and hyperbolic trigonometry is creation. Although there are analogies between these creation and although they lead to discovery of hyperbolic and spherical Pythagoras theorem. The trichotomy of elliptic, parabolic and hyperbolic is discovery as well as hyperbolic, spherical and Euclidean geometry, but building several models for hyperbolic geometry is theorization and is creation, not discovery. Now the question is: do we create models only or can we also create truth. For example, did Thales create geometry or discovered it. Is the concept of a circle and its center creation or discovery? To me the object of a circle is a mathematical model for many similar experiences in everyday life, and there is a truth behind it. Is this truth created by Thales? I say yes, although it could be that truth is pre-engineered in human brain. But I still call it a creation, not a discovery. Now, that I made myself clear about what I mean by creation and what I mean by truth, it is time to extrapolate the experience of discovery.

The experience of discovery in problem solving

The element of surprise is the most elementary experience of a problem solver. Even a child has experienced the element of surprise in problem-solving. Most important part of the experience of discovery is observing the cognitive structure evolving under the influence of discovery. This evolution is different for different facts of discovery and different subjects of discovery. Discovery of objects, concepts, way of deforming concepts, phenomena, structures, eliminations and discovery of truth could be considered different subjects of discovery, and discovery of different cognitive structure could be considered as different acts of discovery. For example, discovery of a verbal cognition against the discovery of a pictorial cognition, discovery of a holistic cognition against discovery of an analytic cognition could be compared and understood. The experience of discovery could affect the relation between the cognitive structure and subject of discovery. There is also metacognitive aspects which are related to experience of discovery from outside the cognitive structure. Experiences of discovery is so diverse that many strong mathematicians long their whole life for tasting the experience of another fellow mathematician and they find it inaccessible to their cognitive structure. Many others are lead to taste of experiences they could have never dreamed of. We start by exploring the different subjects of discovery While problem solving.

On the discovery of objects

Although circle is not discovered as intersection of plane and a cone but conic sections most probability are. This is how the process of generalization leads to discovery of new objects. You concentrate here, on the experience of discovery of objects while solving problems. Indeed, generalization is a process you need for solving many problems. Some objects are made by accumulating information. An example of such discoveries is defining the Galois group as a limit of finite groups. Sometimes objects are used as models for an abstract mathematical structure. Like Poincare half-plain or Klein disc as models for hyperbolic geometry discovered before by Gauss, Bolyai, and Lobachevski. Discovery of hyperbolic geometry is an example of discovery of a structure which we will study in upcoming sections. Sometimes the objects on the consideration are hidden in the mathematics which preceded there discovery. But we should note that some objects are created and not discovered. The objects of “scheme”, particularly “Spec” and many cohomology theories, particularly “Etale cohomology” are all inventions of human mind and one cannot say that they were hidden in the mathematics surrounding then before their discovery. Therefore dichotomy of discovery and creation exists even at the level of mathematical objects. The same is true for mathematical concepts. Namely, concepts could be discovered and also created by the mind of mathematicians.

On the discovery of concepts

A concept could be embedded in a problem, and be discovered by problem solver attempting to solve the problem. If this concept tends to show up in all solutions presented for the problem you could say that the concept is discovered and not created. For example, the concept of segments is hidden in everyday experiences but the concept of infinite line as the limit of segments is created by human mind. Or the concept of circle is hidden in many everyday problems but the concept of center and the geometric definition of the circle is created by human mind. You may complain that infinite line and circle are objects not concepts. But I say that the property of line being infinite and circle being the moduli space of points of given distance to a given point are concepts which define objects. This calls for a clear distinction between mathematical objects and mathematical concepts. You could define objects as sets and concepts as properties that these sets satisfy. Therefore you need concept to define objects but not all concepts are associated objects. For example, some concepts are about relations between objects. Like similarity of triangles. It is hard to imagine a situation where concept come to mind and no objects are referred to. This is one reason why it is difficult to distinguish between objects and concepts. But discovering concepts is a different experience from discovering objects. Concepts deform in ways different from deforming objects. We will discuss this in the next section.

On the discovery of ways to deform concepts

Let us start with some examples of deformation of concepts. The concept of line has gone through some changes along history of mathematics. The concept of line in Euclidean geometry, the concept of line in linear algebra and the concept of line in finite geometries are not the same things. One is led to the concept of geodesic and the other to the concept of sub vector space and third ends up in finite mathematics. Of course, there are intersections between these three paths. For example, a line in vector space over finite field is a combinatorial object. Or, a line is also a geodesic within the natural Riemannian metric. This example shows that the path of deformation of a mathematical concept is not unique. It is difficult to distinguish between discovery and creation in deforming concepts. Each path of deformation goes through some new concept which should be individually studied. There could be both created concepts and discovered concepts on the same path. Actually process of discovery of new concepts in is often of the form of generalizing or deforming previously known concepts along solving a problem. In the process of deforming previously known concept, one often uses analogies between phenomena happening between similarly relative concepts. For example in the Klein model and Poincare model for hyperbolic geometry, there are similarities between the Euclidian line and the Kleinian line. Indeed, they are both straight. Also, they are analogies between the Kleinian line and the Poincare line, which lead to the idea that line could bend and eventually to the concept of geodesics.

On the discovery of a phenomena

An analogy between the structure of relations of a few concepts and another set of concepts is called a phenomena. For example the rise of elliptic, parabolic and hyperbolic geometries in analogy with ellipse, hyperbola and parabola is called the EPH-phenomena. EPH stands for elliptic, parabolic, hyperbolic. A common feature could be spherical geometry tending to Euclidian geometry in limit, and same for hyperbolic geometry, which is a feature similar to ellipse and hyperbola tending to parabola in the limit. Or, the algebraic structure of Euclidean geometry is similar to elliptic curves, which should actually be called parabolic curves, since they have Euclidian universal covers. Discovering analogy is a much wider concept than the discovery of phenomena. For example, there could be an analogy between theories discovered, which is not necessarily done by problem solvers. Theoreticians are more likely to discover such analogies. Sometimes the analogies are more explicit than being about relations between concepts. For example, discovery of mathematical structures is such a discovery. Although you can say that mathematical structures are particularly mathematical concepts also, but it is not the case that any mathematical concept is a mathematical structure. For example, in a EPH-phenomena, being elliptic, parabolic or hyperbolic is in terms of analogy of concepts but not according to the same mathematical structure. We shall find a way to distinguish between mathematical concepts and mathematical structures.

On the discovery of a mathematical structure

For the purpose of this paper, we assume that mathematical structures consist of sets and a structure on them. Like the concept of group discovered by Galois. Abel had the same understanding of degree five equations, but did not extract concepts like field, finite field, group of symmetries and so on. Therefore, solving the same mathematical problem does not necessarily lead to discovery of the hidden mathematical structures in those problem. Also, discovering mathematical structure is a surprising procedure even if it is done by a problem solver. This is why we think of Galois as a theoretician although he was trying to solve the problem of solving quintics with radicals. Abel did not bring out the structure of abelian group, or abelian variety, although he solved many problems using these concepts. He actually solve the same problem as Galois without extracting the related mathematical structure. This is why you consider Abel as a problem solver not a theoretician. What Makes Galois see the mathematical structures and Abel not, is the concept of illumination. Illumination is a light by which one discovers new worlds. These new worlds, contain new objects, new concepts and the new mathematical structures. But, which light give Galois this illumination? We shall first, try to give a simple example of illumination and study the experience of discovery when illuminated by new light and then go back to the illumination of Galois!

On the discovery of illumination

The most basic example of illumination I know is the discovery of spherical trigonometry, and hyperbolic trigonometry and their analogy with Euclidian trigonometry. This discovery of analogy happened very early for spherical geometry and very late in 19 th Century in hyperbolic geometry. The light helping to make this discovery was the parallel postulate that even the question of how many parallel lines pass through a point with respect to given line illuminated the new world of trigonometry which is common between spherical, Euclidian, and hyperbolic geometry. But what was the experience of discovery of illumination by Galois? What question or what concept motivated Galois to see the hidden mathematical structures. We believe the concept of morphism between mathematical structures illuminated many mathematical structures to him. Analogy between the structural relations between concepts in different examples, were indeed by morphisms between mathematical structures. The concept of morphism simply didn't exist before Galois. This eventually lead to category theory and then Grothendieck mathematics. The concept of functor was the cause for discovery of many other mathematical concepts and structure safter the rise of category theory. The notion of analogy goes even deeper in the roots of Mathematics. Sometimes there is a common truth manifesting into different mathematical schemes. This brings up the experience of discovery of truth.

On the discovery of Truth

Is there a truth behind the EPH-phenomena appearing in so many different places in mathematics? Such a point of view would mean that there is a source of mathematical truth outside human mind. At least there's a common realm of human consciousness that truth created by someone could incarnate in the mind of other mathematicians. These Force us to assume many philosophical assumptions on the nature of mathematics. Is it the case that whatever we discover is created before by another human or another intelligent being? Is it the case that our brain is programmed by our creator to create certain predetermined concepts and we called that discovery? Are both options partially true? Are other intelligent beings connected to human global consciousness? You see that these questions quickly take us out of the realm which we can confidently speak of. So for now, we assume that realm of human consciousness is nothing but the union of human minds. Everything in there is created by a human mind or deformed by another human mind. This brings us to the question that does discovery of truth could change the truth or if we could change a truth other than by discovering it? Let us postpone answering these questions to the end of paper. Now we shall study the experience of discovery according to different cognitive styles.

On the nature of discovery in verbal cognition

A. Wiles Describes his experience of Discovery as walking in a dark room trying to understand the things located in the room by touching objects one by one until he discovers the electric key and illuminate the room and suddenly understands where every single object is exactly located in the room lots of details show off when one compares this experience, with the experience of discovery of a person with pictorial cognitive style. The location of objects in the room could be arranged from before. Origin of light is not seen. The means of cognition is primarily not visual. Vision appears very suddenly and at the end. There are a lot of cases about what could be seen through the experience of discovery. There is experience of touching objects and hearing a piano, and feeling the pain of bumping to a chair or table but absolutely no vision until the final scene. There is absolute darkness before that. When the light comes vision replaces all other means of cognition and only that gives a relief feeling that one completely understand the material discovered by means of visualization. Many of this descriptions are quite different from the report of a discovery made by a person with visual cognitive style. As you will see, the main difference is the fact that the scene of discovery for visual cognition is not made by human. But verbal cognition discovers a world which is human made. The scene of discovery by visual cognition is chosen from nature.

On the nature of discovery in pictorial cognition

M. Mirzakhani describes her experience of discovery as being lost in a jungle, one tries to go uphill to find a view, suddenly one can see the sun and the geometry of the jungle and the path one should take to escape. This experience is not only fully visual from the beginning, but also the material being discovered is part of nature and is not made by human hand. Through all the steps of discovery vision is present but only when the source of light is discovered the vision becomes illuminating. Illumination is in the direction of understanding the global geometry of the truth, versus the verbal cognition that visualizes a local phenomena, happening in a room where every object could be touched and everything is human made. Other scenes partially replace the discovery made by a verbal cognition versus the geometric experience of discovery being irreplaceable for a visual cognition. We shall emphasize that many of the differences between the two experiences is not only related to verbal or pictorial cognition structures, but partially related to being holistic or analytic which is another aspect of cognitive personality. Although statistically most of verbal cognitions are analytic and most of pictorial cognitions are holistic, the experience of analytic pictorials and holistic verbals is also very interesting and different from the above.

On the nature of Discovery in holistic verbal cognition

Discovery of a holistic verbals is similar to discovery of big city at night with artificial light. Street by street you have another view, but a global view of the city is nothing but a bunch of light points. You see some streets if you are standing above a hill and boundaries of lakes and parks and city center which are global understanding of the city, but none of these regions can be replaced by touching or closer contact. May be the lights of parts of the city are out, and if we are in the streets partially some experiences of discovery could be touchable. These experiences are of course not that much holistic. So they are of no value to a holistic verbal cognition. Parts of what such a cognition understands is the boundaries of light which speak of natural barriers like a lake or mountain. Of course there could be partial light on the lake or on the mountains but the lights cannot misguide the vision of cognition of natural boundaries. After a while many of the lights go off as people go to sleep and the cognitive structure loses its vision. The vision of a holistic verbal cognition depends on millions of lights turned on by millions of people. He can have vision in a city which has many inhabitants. Some is in fact true for an analytic verbal cognition. The room needs many builders and so much infra structure for production of the light.

On the nature of discovery in analytic pictorial cognition

The experience of an analytic pictorial cognition is similar to the example of the room with the natural light. This is not much different from the experience of an analytic verbal cognition except that vision is always there and you don't try to touch objects for knowing them, but you touch objects for the purpose of creating new objects. Also, you have a view of outside from the window, but how much you depends on the perspective. In general, the experience of an analytic pictorial cognition is very similar to experience of an analytic variable cognition. So analyticality is more influential than being pictorial to verbal. But the experience of a holistic pictorial and holistic verbal are more similar, and in fact again being holistic say more about the experience of discovery from being verbal or pictorial. The next step would be to study the cognitive classification which relates to the structure of the subject of studying, and then relate to the relation between cognitive structure and the subject of study. Also the example of discovery within the cognitive structure and the experience from the point of view of meta- cognition should be studied. Examples of analytic and holistic and pictorial and verbal cognitions all fit into the experience of discovery within the cognitive structure.

Does discovery of the truth affect the truth?

The right question is to ask is if the change which discovery of truth forces on the truth is noticeable? It is surely not noticeable. Otherwise, discovering everything one should see the signature of people who have made experience of discovery under discovered truth. That would make the discovery of the truth very difficult and complicated. At least we can claim, the concept of truth is defined in a manner that discovery of the truth does not change the truth. Of course if you enter a room you leave some DNA data of yourself but that could hardly be noticeable or even discoverable. So, if you change something on purpose, like the place of an object, only few people know that room good enough to be able to notice the change. In fact noticeable change in the material world and also in the world of ideas is such a difficult job, since you have to convince many people to reconstruct and use what you have created already. That is what is difficult with creating a new truth or changing an existing truth. It is difficult to make people accept your way of doing things. Creating a new idea or a new object is much easier than convincing people to use your idea or your objects in their everyday life. Creating new ideas or new objects is not the only way you could change the truth. Let us go by the way of analogy to understand the possible changes which could be made in the truth by means of encounter of human mind.

Does truth change or is it solid?

Does the experience of turning on the lights in a room change the room? Does the experience of walking up the hill in a jungle change the jungle? Hardly ever it does. But human can change the jungle by much harder work than changing the room. You can change the order of objects in a room or even build up something in room. What is clear is that the process of changing the truth is analytic and local to Global. Human constructs and creates from parts to whole, from details to global construction. So the truth Changes, but very slowly and to change it through completely takes the life of many people, or at least occupies them for a long time. To make the process of creation easy, human has built tools and these tools are also human creations in some other ways. Discovery of the truth hardly changes the truth, or at least the changes its makes to the truth are very insignificant. Hardly anybody understands if you have been turning on the light of a room or you have moved up here in the jungle to discover global vision of the map of jungle. But other than discovering the truth what sort of changes you can make into the geometry of the truth? Can you move the moon or planets?

What could change the truth other than discovering it?

Let us speak by way of analogy. People have changed many things on Earth at places which are habitable and all the changes has been purposeful. The same hold in the world of imagination. One can change the truth with many efforts and hard work but this should be purposeful and worth the efforts. The importance of change in truth is not change for the sake of change itself, but we change truth to improve the life of people and for that we have to convince them to use the change proposed to us or so to say created by us. To propose such changes, we get ideas from nature and life of animals and from metaphysics, and also other people's idea which have the same sources. Therefore changing the truth is a collective process in which many human thoughts share ideas. Among them some are leaders and some are executive members. Different people have different roles in changing the truth, and in fact that is what consists the mathematical life of people. Some names become famous and some remain in history, but the important fact is that history is made by everybody and this is a collective process. Although people who write history cannot record the names of everybody, this does not mean that their role is insignificant.

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