Ay 7B Worksheet
Week 1

1. **Beginnings**

   What are Einstein's Postulates, which lead to the Theory of Special Relativity?

   (a) *Nothing travels faster than the speed of light.*

   (b) *The laws of physics are the same in all inertial reference frames (an inertial reference frame is a frame moving with constant velocity no accelerations).*

2. **Relativistic Cookies**

   It's Girl Scout Cookie season, and those girl scouts need a lot of cookies to sell. In order to help out, you decide to start up a bakery that will produce all the cookies they need. In order to meet the large demand, you're making your cookies on an automated production line that moves the dough continuously through from mixing to baked and packaged product. And also to do it fast enough, you've made the conveyor belt move everything through at a speed of $0.8c = 240,000,000$ m s^{-1} $= 540,000,000$ mph! At one station, the dough is cut into individual cookies by a circular cutter that is 5 cm in diameter.

   Version 1 Examining some of the finished cookies, you notice a problem: they're not circular! You measure that the cookies are still 5 cm wide, as they should be, but they are the wrong length (where width is the direction across the conveyor belt and length is the direction along it). Finally you realize that relative to the cutter, the dough is length contracted due to its motion. What length of dough is actually getting cut? Are the cookies too long or too short?

   The cutter sees the dough to be length contracted. Given the speed of $v = 0.8c$, the relevant contraction factor is $\gamma = (1 - v^2/c^2)^{-1/2} = 5/3$. Thus if the (moving) length of dough that is cut is $L_{moving} = 5cm$ in the cutter frame, this corresponds to a length of $L_{rest} = \gamma L_{moving} = (5/3)(5cm) = 8.3cm$. Thus the cookies are too long.

   Version 2 Examining some of the finished cookies, you notice a problem: theyre not circular! You measure that the cookies are still 5 cm wide, as they should be, but they are the wrong length (where width is the direction across the conveyor belt and length is the direction along it). Finally you think about how the cookies are seeing things and realize that as the cutter appears to be rushing over it (from the doughs point of view), it is length contracted. What length of dough is actually getting cut? Are the cookies too long or too short?

   From the doughs point of view, its just sitting there while the rest of the bakery rushes by around it, including the cutter. Thus it says the cutter is moving, and thus is length contracted. Although the cutter thinks it is $L_{rest} = 5cm$ when at rest, the dough says it is $L_{moving} = L_{rest}/\gamma = (5cm)/(5/3) = 3$ cm long (γ has the same value as above since the speed is the same). Thus only 3 cm are actually cut, and the cookies are too short.

   But wait! Weve just said the cookies are both too long and too short! Thats not possible. At the end you pick up a cookies and its either one or the other. It cant be both. That means someone above is wrong. But we used the same laws of physics for both people, and relativity is founded on the assumption that physics must be the same for both.

In the frame of the cutter, the dough is contracted and thus the cutter must be cutting it to be too long. In the frame of the dough, the cutter is contracted. The problem is that we assumed the cutter cut the dough all at once. We assumed that the front and back ends of the cutter hit the dough at the same time. This, in fact, is our error. Although the two ends do hit simultaneously in the cutters rest frame, they dont do so for the dough. Whether or not two events are simultaneous depends on who observes them.

In this case, the dough sees the front edge of the cutter hit first, then later the back edge, with the cutter appearing to roll along the dough in between. Now the cutter is indeed only 3 cm long in the doughs frame, but during the delay between the front and back hitting, the dough has moved forward more. Thus more dough has gotten past the back of the cutter by the time it finally hits. Showing correctly how much got through in the frame of the dough takes a bit more math than were getting into in this class, but low and behold the answer is that the dough says 8.3 cm got through by the time it was cut. Thus the cookies are definitively too long.

3. **Traveling to the Stars**

Sammy Spaceman and Earthly Earl are twin brothers who have always loved the idea of traveling to the stars. When one day NASA offered them both the chance to go to τ Ceti (the nearest single star that is similar to the Sun), Sammy Spaceman jumped at the opportunity. Earthly Earl, on the other hand, turned it down due to not wanting to be away from his family for the very long time the trip would take. After all, τ Ceti is 11.7 lightyears away.

Version 1 Wishing your boring brother farewell, you, Sammy Spaceman, hop into the rocket and blast off. In practically no time at all you've reached your cruising speed of $0.95c$. As you sit still in your rocket, the rest of the universe passes you by rather rapidly. Since this causes the distance to τ Ceti to be length contracted, how long will it take for you to get there? Presuming that once you arrive, you then quickly turn around and head back to Earth at the same speed, how long will the total round trip take you? (Hint: Ignore just how lame and anti-climactic it would be to travel that far just to immediately go back home.)

The relativistic factor for this problem is $\gamma = (1-v^2/c^2)^{-1/2} = 3.2$. Thus Sammy Spaceman sees the distance to τ Ceti as being contracted by this factor to $d = 11.7/3.2 = 3.66$ lightyears. Traveling at $v = 0.95c = 0.95$ lightyears per year, this takes $t = d/v = 3.85 years$. Once you turn around, the same is true for the return trip. Thus in total it takes you 7.7 years to get there and back.

All the while as you've been traveling, from your point of view Earth flew quickly away from you and then quickly back while you've just sat in your rocket. This means that since they were moving, their clocks were time dilated. How long has Earthly Earl been sitting around on Earth waiting for you to return (total time since you left)? Are you now older or younger than your twin brother?

Throughout this time, Earthly Earl appeared to be moving relative to you (at the same speed), and thus his clock ran slow from your point of view. This includes his biological clock that controls how much he ages. Since his clock ran slow, less time elapsed for him (by a factor of γ). Thus while Sammy Spaceman aged 7.7 years, he says Earthly Earl aged only 7.7/3.2 = 2.4 years. Thus you (Sammy Spaceman) are now older than your brother (Earthly Earl).

Version 2 Wishing your antisocial brother farewell, you, Earthly Earl, sit back and relax as you watch your brother set off on a lonely mission all by himself. In practically no time at all he reaches his cruising speed of $0.95c$. How long (as measured by you) will it take for Sammy Spaceman to get to τ Ceti? Presuming that once he arrives, he then quickly turns around and heads back to Earth at the same speed, how long will the total round trip take him? (Hint: Ignore just how lame and anti-climactic it would be to travel that far just to immediately go back home.)

Traveling at 0.95 lightyears per year, it takes $t = d/v = 11.7/0.95 = 12.3$ years for Sammy Spaceman to get there, and the same to get back. Thus the total trip would take 24.6 years as measured by Earthly Earl on Earth.

All the while as he was traveling, however, his clock was time dilated due to his motion. How long will he have measured the total trip to take? Are you now older or younger than your twin brother?

Since Sammy Spacemans clock was time dilated, his ran slower, and thus ticked off less time. So Sammy Spaceman should only have aged $t = 24.6/3.2 = 7.7$ years. Thus you (Earthly Earl) are now older than your brother (Sammy Spaceman).

But wait! Weve again come to contradictory conclusions. Although both agree Sammy Spaceman aged 7.7 years, they differ on how old Earthly Earl must be. Now that Sammy Spaceman has returned to Earth, he can easily stand next to Earthly Earl.

Only one of them can be older than the other. They cant be both. So which point of view is correct? Here again, the problem is that weve naively applied special relativity without really thinking it through first. SR is derived assuming that all observers are in inertial reference frames, i.e. ones that are moving at constant speed relatively to each other. For the effects here, the direction of travel isnt important, but it is important that it changed. When Sammy Spaceman turned around, he had to accelerate in order to come to a stop and get back up to full speed in the new direction. This might have been a negligibly short time for Sammy Spaceman, but was it so for Earthly Earl?

SR no longer applies to Sammy Spaceman while he was accelerating due to him no longer being an inertial observer. Earthly Earl, on the other hand, was always an inertial observer since he just sat on Earth. Thus Earthly Earls analysis must be correct, and Earthly Earl must actually be the older twin. One way to see this is that according to the equivalence principle, Sammy Spacemans acceleration is equivalent to him having been standing in a gravitational field during that time. Thus he experienced a gravitational time dilation (We'll get to this next week when we talk about General Relativity). Since he was the one feeling the gravity, his clock ran slow. In order to turn around so quickly, the acceleration must have been huge, which in turn means a very strong GR time dilation. Thus that short time for him was actually a very long time for Earthly Earl (22.2 years, in fact).

Note that although we normally think of GR time dilation as gravitational time dilation, it applies to all accelerations, even if no gravity is involved. Note also that this is due to his rockets acceleration, not due to the gravitational field of τ Ceti (it took would cause a time dilation, but would be negligibly small).

4. **Barnyard Olympics** Training to be an Olympic athlete, Polly the Pole Runner asks her friend Barny the Farmer to use his farm as a training ground. In the Space Olympics (the year 3022), one of the events is to run fast enough carrying an 80 foot pole so that it shrinks to half its size according to an outside observer. How fast will Polly need to run?

   From length contraction, we know that $L_{rest} = \gamma L_{moving}$, where we know $L_{rest} = 80$ m and we want $L_{moving} = 0.5 L_{rest} = 40$ m. Thus we see that for this to occur, we need $\gamma = 2$. Now, since $\gamma = (1 - v^2/c^2)^{-1/2}$, we can solve for v to get $v = c\sqrt{1 - \gamma^{-2}} = c\sqrt{0.75} = 0.866c$. She must be a really fast runner!

   As added incentive to have her achieve the necessary speed, Barny has her run through his 40 foot barn. At the instant that all 40 feet of the pole (from Barny's point of view) are inside his barn, he flips a switch that closes the front and back door of the barn simultaneously (and then instantly opens them back up again). So for a brief moment, Barny sees that he has Polly and her entire pole confined in his barn.

   However, consider the problem from Polly's Frame. To her, the pole is stationary (and 80 feet long), and the barn is speeding towards her. How large is the barn from her point of view? Clearly her 80 foot pole cannot fit completely inside the shrunken barn, so how do you rectify this with Barny's observations? Is Polly really safe?

Polly will regard the pole as stationary, and thus as 80 meters long, and the barn as approaching at high speed. Thus in this frame, using the same math as above ($\gamma = 2$) the barn is only 20 meters long from Polly's point of view. Surely the runner is in trouble if the doors close while she is inside. The pole is sure to get caught, right?

Well does the pole get caught in the door or doesn't it? You can't have it both ways. This is the "Barn-pole paradox." The answer is buried in the misuse of the word "simultaneously" back in the earlier part of the story. In SR, that events separated in space that appear simultaneous in one frame of reference (say, Barny's) need not appear simultaneous in another frame of reference (Polly's). The closing doors are two such separate events.

SR explains that the two doors are never closed at the same time in the runner's frame of reference. (This is a similar explanation to the cookie cutter problem above!). So there is always room for the pole. In fact, the Lorentz transformation for time is $t' = (t - vx/c^2)/\sqrt{(1 - v^2/c^2)}$ It's the vx term in the numerator that causes the mischief here. In the runner's frame the more distant event (larger x) happens earlier. The far door is closed first. It opens right as she arrives there, and the near door closes behind her. Safe again – either way you look at it, provided you remember that simultaneity is not a constant of physics.