Ay 7b – Spring 2012
Section Worksheet 13
Cosmology

1. **The evolution of the universe**
   We've been talking about the universe in many (three) different regimes. Let's put all the eras together and see what the universe has been doing these last 13.7 billion years.

   (a) Write down a COMPLETE version of the Friedmann equation that includes contributions from radiation, matter, and dark energy. Explicitly include the dependence of each density on the scale factor R. What happens to the Friedmann equation when one component dominates?

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\left(\rho_m + \rho_r + \rho_\Lambda\right)\right] R^2 = -kc^2$$

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\left(\frac{\rho_{m,0}}{R^3} + \frac{\rho_{r,0}}{R^4} + \rho_{\Lambda,0}\right)\right] R^2 = -kc^2$$

$$\text{and since} \quad \rho_c = \frac{3H^2}{8\pi G},$$

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G}, \text{ so multiply the } \rho_{\mathrm{i}} \text{ terms by } 1 = \frac{3H_0^2/(8\pi G)}{\rho_{c,0}}:$$

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8\pi G}{3}\left(\frac{3H_0^2}{8\pi G}\right)\left(\frac{\rho_{m,0}}{\rho_{c,0}R^3} + \frac{\rho_{r,0}}{\rho_{c,0}R^4} + \frac{\rho_{\Lambda,0}}{\rho_{c,0}}\right)\right] R^2 = -kc^2$$

$$\left(\frac{dR}{dt}\right)^2 - H_0^2\left(\frac{\Omega_{m,0}}{R} + \frac{\Omega_{r,0}}{R^2} + \Omega_{\Lambda,0}R^2\right) = -kc^2$$

   When one component dominates, we can simply ignore the other two components to solve the Friedmann equation for $R(t)$.

   (b) The Baby Universe: Find how R scales with t in the radiation-dominated era.

   For this and all following parts, assume the universe is FLAT! ($k = 0$)

   Radiation-dominated era $\rightarrow$ drop the mass and dark energy terms, yielding (assuming a flat universe so that $k = 0$)

$$\left(\frac{dR}{dt}\right)^2 - H_0^2\left(\frac{\Omega_{r,0}}{R^2}\right) = 0$$

   separating variables, this becomes

$$RdR = H_0\sqrt{\Omega_{r,0}}dt$$

   so, integrating, we find

$$\frac{1}{2}R^2 = H_0\sqrt{\Omega_{r,0}}t$$

   so we find

$$R \propto t^{1/2}$$

(c) The Teenage Universe: Find how R scales with t in the matter-dominated era.

Using the same procedure as above,

$$\left(\frac{dR}{dt}\right)^2 - H_0^2\left(\frac{\Omega_{m,0}}{R}\right) = 0$$

$$R^{1/2}dR = H_0\sqrt{\Omega_{m,0}}dt$$

$$\frac{2}{3}R^{3/2} = H_0\sqrt{\Omega_{m,0}}t$$

so we find

$$R \propto t^{2/3}$$

(d) The Adult Universe: Find how R scales with t in the Λ-dominated era.

Using the same procedure as above,

$$\left(\frac{dR}{dt}\right)^2 - H_0^2\left(\Omega_{\Lambda,0}R^2\right) = 0$$

$$\frac{dR}{R} = H_0\sqrt{\Omega_{\Lambda,0}}dt$$

$$\ln(R) = H_0\sqrt{\Omega_{\Lambda,0}}t$$

so we find

$$R \propto e^t$$

(e) The Universe Hits Puberty: Find the value of the scale factor and the redshift z when radiation and matter contribute equally to the evolution of the universe. This corresponds to a time of $t_{r-m} \approx 47000$ yr. Why is this value of t tricky to calculate?

Simply set ρ_m equal to ρ_r:

$$\rho_m = \rho_r \rightarrow \frac{\rho_{m,0}}{R^3} = \frac{\rho_{r,0}}{R^4}$$

$$R = \frac{1}{1+z} = \frac{\rho_{r,0}}{\rho_{m,0}} = \frac{\Omega_{r,0}}{\Omega_{m,0}}$$

Using the values from C&O, $\Omega_{rel,0} = 8.24 \times 10^{-5}$ and $\Omega_{m,0} = 0.27$ (pg 1193), we find

$$R \approx 3.1 \times 10^{-4}$$

$$z \approx 3300$$

The value of t is tricky to calculate because you have to integrate back from the present day or forward from the big bang over the different eras (dominated by different components) so it becomes a messy integral.

(f) The Universe Leaves Home: Find the value of the scale factor and the redshift z when matter and dark energy contribute equally to the evolution of the universe. This corresponds to a time of $t_{m-\Lambda} \approx 9.5 \times 10^9$ yr.

Set ρ_m equal to ρ_Λ:

$$\rho_m = \rho_\Lambda \rightarrow \frac{\rho_{m,0}}{R^3} = \rho_{\Lambda,0}$$

$$R = \frac{1}{1+z} = \left(\frac{\rho_{m,0}}{\rho_{\Lambda,0}}\right)^{1/3} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}$$

Using the values from C&O, $\Omega_{\Lambda,0} = 0.73$ and $\Omega_{m,0} = 0.27$ (pg 1193), we find

$$R \approx 0.72$$

$$z \approx 0.4$$

(g) A Photo Album of Our Universe—A Retrospective: Put everything you've calculated above together in a sketch of R vs. t. Use a log-log scale and indicate the equality times.

I'm too lazy to make this plot, but the idea is that in log log space, in the radiation dominated era (early time), it is a straight line with slope of $1/2$, in the matter dominated era, it is a straight line with slope of $2/3$ and in the dark energy dominated era, the exponential dependence on t causes the line to curve upwards.