Ay 7b – Spring 2012
Week 2
Special Relativity

1. Minkowski Diagram

Suppose that a light bulb is suspended from the back wall of a freight car of length L, which is moving at a constant velocity u along a straight track. A clock on board the train and a clock at the station the train is passing synchronize as the light bulb turns on and the origins of the two frames of reference align($t = t' = 0$ when $x = x' = 0$). The hobo standing at the front of the freight car turns on his green laser pointer at time $t' = 0$ in his rest frame. (Let primed indicate the rest frame of the freight car and hobo, and unprimed indicate the frame of the station - let the light bulb be at $x' = 0$, and the hobo is standing at $x' = L$)

(a) Draw the Minkowski diagram for hobos rest frame, including the worldlines of the light from the bulb at the back of the train , the hobos green laser pointer, and the front and back walls of the freight car. Label the following events:
A - the light bulb turns on
B - the hobo turns on his laser pointer
C - the light from the bulb hits the front wall
D - the light from the laser pointer hits the back wall
Are events C and D simultaneous in this frame?

In the frame of the hobo, the front and back of the the train are stationary which implies that their worldlines are simply vertical lines. Since we also know that photons always travel an a 45 degree angle, it is easy to see that the photons from the hobos laser reach the back of the train at the same time the photons from the light at the back of the train reach the front of the train.
So $\boxed{\text{A \& B and C \& D are simultaneous}}$. (See Figure 1)

(b) Now draw the Minkowski diagram for frame of the station. Include the worldlines and events listed above. (HINT: be careful about placing event B – you don't need to be quantitative though, just get the right idea.)
Are events A and B simultaneous in the stations frame? Are C and D?

Now were in the frame of an observer watching these events transpire in a freight car that is moving at speed u. In this frame, the front and back of the train are moving away with us with speed u which will translate into lines of slope c/u on our diagram. Now, the lights will not go on at the same time! How do we know this? Lets go back to our Lorentz transform equations. In the hobo's frame, $t'_A = t'_B$. Now lets calculate $t_B t_A$ using the inverse transform from C&O (4.24):

$$t_B - t_A = \frac{(x'_B - x'_A)u/c^2}{\sqrt{1-(u/c)^2}} \tag{1}$$

This is a positive number which implies the hobo fires his laser AFTER the light from the back of the train turns on. Drawing in the paths of the photons, we also see that they don't arrive at the same time either!
So now, $\boxed{\text{neither A \& B or C \& D are simultaneous}}$.

(c) Sketch the axes of the S frame (rest frame of the hobo) on the Minkowski diagram of the S frame (station's frame).

Now we want to overlay the axes of the hobo on top of our diagram from (b). In the hobos frame A & B are simultaneous which implies the x' axis must pass through events A & B. The ct' axis will lie on top of the worldline for the back of the train.

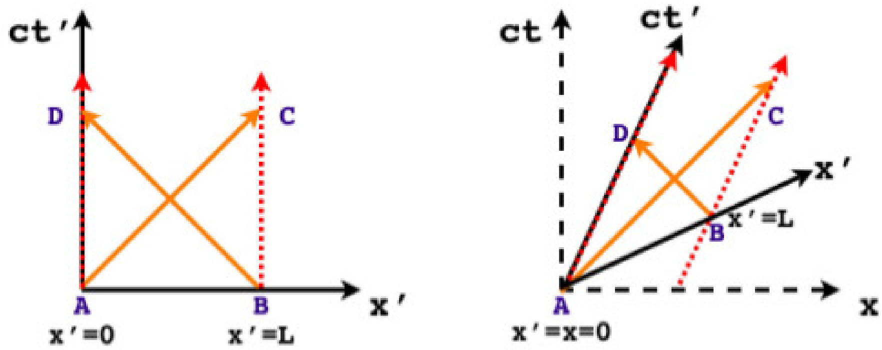

Figure 1: The left figure is the Minkowski diagram in the frame of the hobo (the so-called S' frame) and the right figure is the Minkowski diagram of an observer on the platform (the unprimed frame)(dashed axes) watching the train pass by with speed u. Overlaid in the right frame are the axes from the hobos frame in solid lines. Photons are orange lines. Red dotted lines represent the worldlines of the front and back of the train. Note that the angles between x and x' axes and between t and t' axes are $\tan^{-1}(u/c)$.

2. Energy & Momentum Conservation

(a) How much energy would it take to accelerate an electron to the speed of light according to classical physics (that is, before introducing any concepts from special relativity)?

By conservation of energy, the energy put in to accelerate the electron must equal the change in kinetic energy. Assuming the electron started from rest, this is (classically):

$$E = \frac{1}{2}m_e v^2 = \frac{1}{2}m_e v^2 \tag{2}$$

With this energy, what would its actual velocity be, now using special relativity?

To find the actual velocity, we again use the fact that the change in energy is equal to the energy put in. The total relativisitc energy is given by $E = \gamma mc^2$, so the change in energy is

$$E_f - E_i = \gamma_f mc^2 - \gamma_i mc^2 = mc^2(\gamma - 1) \tag{3}$$

Note this is the expression for relativistic kinetic energy. So we have

$$E_f - E_i = \frac{1}{2}m_e c^2 = m_e c^2(\gamma_f - 1) = m_e c^2 \left(\frac{1}{\sqrt{1 - v_f^2/c^2}} - 1 \right) \tag{4}$$

Solving for v_f, we find $\boxed{v = \sqrt{5}c/3}$.

(b) An isolated photon cannot be converted into an electron-positron pair, $\gamma \to e^- + e^+$. (The conservation laws allow this to happen only near another object.) Why is that?

Suppose a photon traveling in the x-direction is converted into an e^- and e^+ as shown in Figure 2. From the energy conservation, we have $p_\gamma c = 2E_e$ where p_γ is momentum of the photon and E_e is the total energy of e^- which is equal to the total energy of e^+. Note that the masses of e^- and e^+ are same and the momentum conservation implies their kinetic energies are same, so the total energies of e^- and e^+ are equal.

Conservation of momentum in y-direction gives $p_\gamma \sin(0) = \mathbf{p}_{e^+} \sin\theta + \mathbf{p}_{e^-} \sin\theta$. That is, $|\mathbf{p}_{e^+}| = p_{e^+} = |\mathbf{p}_{e^-}| = p_{e^-} = p_e$. Also, conservation of the momentum in x-direction gives $p_\gamma = 2p_e \cos\theta$. Dividing the equations obtained by energy and momentum conservation, we have $c = E_e/(p_e \cos\theta)$. By taking squares, the equation becomes

$$p_e^2 c^2 \cos^2\theta = E_e^2 \tag{5}$$

However, $E_e^2 = p_e^2 c^2 + m_e^2 c^4 > p_e^2 c^2$, so the above equation cannot be satisfied for $\cos^2\theta \leqslant 1$. This result can also be seen by transforming to a frame where $p_x = 0$ after the collision. But, before the collision, $p_x = p_\gamma c \neq 0$ in any frame moving along the x-axis. So, without another object nearby, momentum cannot be conserved; thus, the process cannot take place.

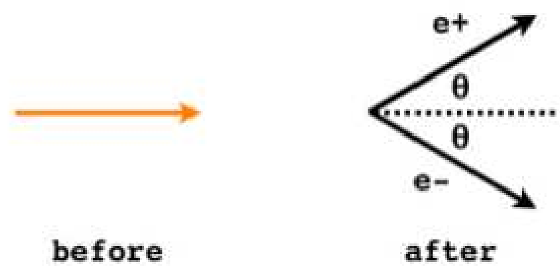

Figure 2: An isolated photon cannot be converted into an electron-positron pair. Orange arrow represents a photon and two black arrows represent electron and positron, respectively.

(c) A neutral pion π^0 moving at speed $v = 0.98c$ decays in flight into two photons. If the two photons emerge on each side of the pions direction with equal angles, θ, then find the angle θ and energies of the photons. The rest energy of π^0 is 135 MeV. (Hint: Use energy conservation and momentum conservation.)

Conservation of energy gives $E_\pi = 2E_\gamma$, where E_π is the total energy of π^0 and E_γ is the total energy of each photon. Again, conservation of momentum in y-direction implies that the photons have the same energy. Thus, $\gamma E_{\pi,\,rest} = 2E_\gamma$. Plugging in the numbers, we get the energy of each photon,

$$E_\gamma = \frac{\gamma E_{\pi,\,rest}}{2} = \frac{135\text{ MeV}}{2\sqrt{1-0.98^2}} = \boxed{339\text{ MeV}} \tag{6}$$

To get the angle, we use the momentum conservation in x-direction, $\gamma m_\pi v = 2p_\gamma \cos\theta$ where m_π and v are the mass and the speed of each pion, respectively, and p_γ is the momentum of each photon. The mass of a pion is $m_\pi c^2 = 135\text{ MeV}$ and the momentum of the photon is $E_\gamma = 339\text{ MeV} = p_\gamma c$. Thus

$$\cos\theta = \frac{\gamma m_\pi v}{2p_\gamma} = \frac{1}{\sqrt{1-0.98^2}} \frac{(135\text{ MeV}/c^2)(0.98c)}{2(399\text{ MeV}/c)} = 0.98 \tag{7}$$

Finally, we get $\boxed{\theta = \cos^{-1}(0.98) = 11.3\ \text{degrees}}$.

3. Doppler Effect

A star is known to be moving away from the Earth at a speed of 4×10^4 m/s. This speed is determined by measuring the shift of the H_α line ($\lambda = 656.3$nm). By how much and in what direction is the shift of the wavelength of the H_α line? Use both classical physics and special relativity and compare the results.

In the star frame(the source frame, unprimed), the frequency of each emitted photon is ν and in the Earth frame(the observer frame, primed), the frequency of each received photon is ν' . Using the doppler formula in special relativity, we get

$$\nu' = \sqrt{\frac{1-\beta}{1+\beta}}\,\nu \tag{8}$$

or since $\lambda = c/\nu$,

$$\lambda' = \sqrt{\frac{1+\beta}{1-\beta}}\,\lambda \tag{9}$$

With $\lambda = 656.3$ nm and $\beta = \frac{4\times10^4}{3\times10^8}$, we get $\boxed{\lambda = 656.3087 \text{ nm}}$.

$\boxed{\text{So the shift is } 0.087 \text{ nm towards the red (longer wavelengths).}}$

If we don't use special relativity, the Doppler formula is given by

$$\frac{\lambda_{\text{obs}} - \lambda_{\text{rest}}}{\lambda_{\text{rest}}} = \frac{v}{c} \tag{10}$$

Then $\boxed{\text{we would have a } 0.087 \text{ nm shift towards the red,}}$ which is the same result as that obtained from special relativity. This is because the speed $v = 4\times10^4$ m/s is much smaller than the speed of light, so the Doppler formula from special relativity can be reduced to the classical Doppler formula.

4. Orbit of a Satellite (*Following C&O, although there are some integral nuances that are glossed over here, and are treated properly in the C&O discussion.*)

In General Relativity, we've told you that particles move along worldlines that are *geodesics* of the spacetime they inhabit (that is to say, their path is an extremum of $\int ds$). It's not intuative to see how more familiar Newtonian mechanics solutions come out of the weak field limit for GR, so let's look at the case of a satellite orbiting the Earth.

For a satellite orbiting the Earth, we should use the Schwarzschild metric (which describes the warped spacetime around a central point mass M). We'll ignore any effects of the satellite's mass on spacetime (since those effects would be very small, as $m_{satellite} << M_\oplus$). The Schwarzschild metric is

$$(ds)^2 = \left(c\,dt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2 \tag{11}$$

(a) Let's make this problem more simple: assume the satellite has a constant angular speed $\omega = v/r$, goes in a circular orbit, and orbits around the equator. How do these simplifications change the metric? Write out the modified metric.

Above the equator implies $\theta = 90°$. Circular orbit implies $dr = 0$ and $d\theta = 0$ (fixed above the equator). Constant angular speed implies $d\phi = \omega dt$. So the metric simplifies to

$$\boxed{(ds)^2 = \left[\left(c\sqrt{1-2GM/rc^2}\right) - r^2\omega^2\right]dt^2 = \left(c^2 - \frac{2GM}{r} - r^2\omega^2\right)dt^2} \tag{12}$$

(b) Using the simplified metric, what is the spacetime interval Δs over one orbit?

Say our orbit runs from time $t = 0$ to time $t = 2\pi/\omega$. Then we just integrate ds over one orbit to find Δs:

$$\boxed{\Delta s = \int_0^{2\pi/\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\, dt} \tag{13}$$

(c) What is the value of r such that Δs is an extremum?
(Recall extrema are found where $d/dx[f(x)] = 0$.)
(Aside: this is where we've been a bit sloppy as far as rigor.)

Extrema is where $d/dr(\Delta s) = 0$. For this to be true in general, it must be true of the integrand, so we bring the derivative inside the integral and solve

$$\frac{d}{dr}\sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2} = 0 \tag{14}$$

So we find that

$$\boxed{\frac{2GM}{r^2} - 2r\omega^2 = 0} \tag{15}$$

gives the value of r that makes Δs an extremum.

(d) What is the orbital velocity $v = \omega r$ corresponding to this value of r? How does this GR-determined orbital velocity compare with the result from Newtonian mechanics?

Using equation (15), we find that

$$\boxed{v = \omega r = \sqrt{\frac{GM}{r}}} \tag{16}$$

This, in fact, looks exactly like the result from Newtonian mechanics, which uses centripetal acceleration to describe the acceleration due to gravity:

$$\frac{v^2}{r} = \frac{GM}{r^2}, \tag{17}$$

which gives

$$v = \sqrt{\frac{GM}{r}} \tag{18}$$