Ay 7B – Spring 2010
Section Worksheet 9
Propagating up the Distance Ladder

First let's refresh your memory a little bit on error propagation. Recall that if you have a function $f(x_1, x_2, ...)$ and errors in the x_n's $\delta x_1, \delta x_2, ...$, then the error in f is given by:

$$(\delta f)^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 (\delta x_1)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 (\delta x_2)^2 + ...$$

1. So for $f(x) = x^3$, what is the error in f for a given error in x, δx? Or perhaps more useful would be what is the percent error in f, $\frac{\delta f}{f}$ in terms of the percent error in x, $\frac{\delta x}{x}$?

   Solution:
   $$(\delta f)^2 = \left(\frac{\partial f}{\partial x}\right)^2 (\delta x)^2 = (3x^2)^2(\delta x)^2 = \left(\frac{3f}{x}\right)^2 (\delta x)^2$$

   The trick to getting percent errors is that we rewrite $\left(\frac{\partial f}{\partial x}\right)$ in terms of f and x. Then rewritten in terms of percent errors we have:
   $$\frac{\delta f}{f} = 3\frac{\delta x}{x}$$

2. Now that you've got that down, let's look at how error might propagate up the distance ladder. The Tully-Fisher relation tells you that the luminosity of a galaxy goes as the fourth power of the maximum rotational velocity, v_{max}: i.e., $L = Cv_{max}^4$ where C is a constant (c.f. C&O page 955). Since this is an empirical relation, we need to determine the value of C from observations.

   Good news! You observe a Cepheid variable star in Galaxy A! Using the Cepheid you calculate that Galaxy A is at a distance d_A. You also observe that Galaxy A has a flux F_A and a maximum rotational velocity $v_{max,A}$. Write an expression for the value of C.

   Solution: $L_A = Cv_{max,A}^4$ and $F_A = \frac{L_A}{4\pi d_A^2}$, so we have $C = \frac{4\pi d_A^2 F_A}{v_{max,A}^4}$.

3. Hooray! Now you can figure out the distance to super-mysterious Galaxy X, which you have observed to have flux F_X and $v_{max,X}$. Since you know C (from part 2), what is the distance to Galaxy X, d_X, in terms of C?

   Solution: $d_X = \sqrt{\frac{Cv_{max,X}^4}{4\pi F_X}}$

4. Bummer! Your good luck has come to an end - it turns out we didn't understand Cepheids very well and the distance you calculated using the Cepheid in Galaxy A has a 10% error (i.e., $\frac{\delta d_a}{d_A} = 0.10$). What percent error does this mean for C?

   Solution: $(\delta C)^2 = \left(\frac{\partial C}{\partial d_A}\right)^2 (\delta d_A)^2 = \left(\frac{8\pi d_A F_A}{v_{max,A}^4}\right)^2 (\delta d_A)^2 = \left(\frac{2C}{d_A}\right)^2 (\delta d_A)^2$

   So we can write it in terms of percent errors: $\frac{\delta C}{C} = 2\frac{\delta d_A}{d_A} = 2(0.10) = 0.20$.

5. Ouch! What percent error does this mean for d_X?

   Solution:

$$(\delta d_X)^2 = \left(\frac{\partial d_X}{\partial C}\right)^2 (\delta C)^2 = \left(\sqrt{\frac{v_{max,X}^4}{4\pi F_X}}\frac{1}{2\sqrt{C}}\right)^2 (\delta C)^2 = \left(\frac{d_X}{2C}\right)^2 (\delta C)^2$$

   In terms of percent errors: $\frac{\delta d_X}{d_X} = \frac{1}{2}\frac{\delta C}{C} = \frac{1}{2}(0.20) = 0.10 \Rightarrow 10\%$ error in d_X.

6. ¡Qué terriblé! Just when you thought your luck might change, you find out that Galaxy X was observed on a hazy night and there is a 5% error in your value for $v_{max,X}$. Now what is the percent error in d_X?

   Solution: Now we have to do the error propagation with two variables.

$$(\delta d_X)^2 = \left(\frac{\partial d_X}{\partial C}\right)^2 (\delta C)^2 + \left(\frac{\partial d_X}{\partial v_{max,X}}\right)^2 (\delta v_{max,X})^2$$

$$(\delta d_X)^2 = \left(\sqrt{\frac{v_{max,X}^4}{4\pi F_X}}\frac{1}{2\sqrt{C}}\right)^2 (\delta C)^2 + \left(2v_{max,X}\sqrt{\frac{C}{4\pi F_X}}\right)^2 (\delta v_{max,X})^2$$

$$(\delta d_X)^2 = \left(\frac{d_X}{2C}\right)^2 (\delta C)^2 + \left(\frac{2d_X}{v_{max,X}}\right)^2 (\delta v_{max,X})^2$$

   In terms of percent errors:

$$\frac{\delta d_X}{d_X} = \sqrt{\frac{1}{4}\left(\frac{\delta C}{C}\right)^2 + 4\left(\frac{\delta v_{max,X}}{v_{max,X}}\right)^2} = \sqrt{\frac{1}{4}(0.20)^2 + 4(0.05)^2} \approx 0.14$$

   So there is a 14% error in d_X.