Ay 7B – Spring 2012
Section Worksheet 7
Rotation Curve and Density Profile

1. **Model the Galaxy**[1 2]
   By making some assumptions, we can model the mass distribution and the velocity curve of the inner Milky Way Galaxy. Observational evidence indicates that at the center of our Galaxy is a black hole of mass $M_0 = 4 \times 10^6\ M_\odot$. For the rest of the Galaxy's mass, let's assume it has an isothermal density distribution that goes as r^{-2} (to see where this dependence comes from, refer to pages 917-918 in the book).

   (a) Write down the density profile, $\rho(r)$ in terms of r and a constant C.

   $$\rho(r) = \frac{C}{r^2}$$

   (b) Assuming the mass distribution in the Galaxy is spherically symmetric and integrating the density profile over the radius, show that the mass contained within a distance r from the center of the Galaxy can be written as $M_r = kr + M_0$, where k is a constant that we will calculate later on. Note that there is a black hole at the center.
   `The isothermal component has a density of` $\rho(r) = C/r^2$ `as we wrote above. Then the interior mass due to that component is`

   $$M_{r,i} = \int_0^r \rho(r) 4\pi r^2 dr = \int_0^r \frac{C}{r^2} 4\pi r^2 dr = 4\pi C r.$$

   `Adding a single point mass` M_0 `at the center gives`

   $$M_r = M_{r,i} + M_0 = kr + M_0,$$

   `where` $k = 4\pi C$.

   (c) Now for another assumption: assume that masses are in perfectly circular orbits about the center of the Galaxy. Using Newtonian gravity, show that the orbital velocity curve is

   $$v = \sqrt{G\left(k + \frac{M_0}{r}\right)}. \qquad (1)$$

   For small r, how does v scale with r? `Assume centripetal acceleration, so` $F = mv^2/r$. `Also, Newton says` $F = GM_r m/r^2$. `Set the two equal, plug in` M_r `from part (b), and solve for v.`
   `For small` r, $v \approx \sqrt{GM_0/r} \propto r^{-1/2}$ `which is Keplerian (ie what is expected for a single central massive object)`
   .

   (d) If a mass located 2 pc from the center of the Galaxy has an orbital velocity of 110 km/s, calculate the constant k.
   `Plug in` $r = 2\text{ pc} = 6 \times 10^18$ `cm and` $v = 110\text{ km/s} = 1.1 \times 10^7$ `cm s`$^{-1}$ `to the result from part (c) to get` $k = 4.8 \times 10^{20}\text{ g cm}^{-1} = 7 \times 10^5\ \mathrm{M}_\odot\ \mathrm{pc}^{-1}$.

[1] This problem is based on Caroll and Ostlie 24.36.
[2] This is not necessarily the *best* model of the MW. See problem 2.

2. **Galaxy Rotation Curve Fitting**

We've talked about rotation curves and that they imply density distributions and from those density distributions we can infer that more mass exists in a galaxy than is simply accounted for my luminous (stellar) material. Well, now we'll really drive this home by deriving a general form of the density distribution for a given rotation curve. Figure 1 is a plot of three rotation curves all generated from the functional form $v = \alpha r^{\beta}$ where r is in pc, v is in km/s, and α and β are constants.

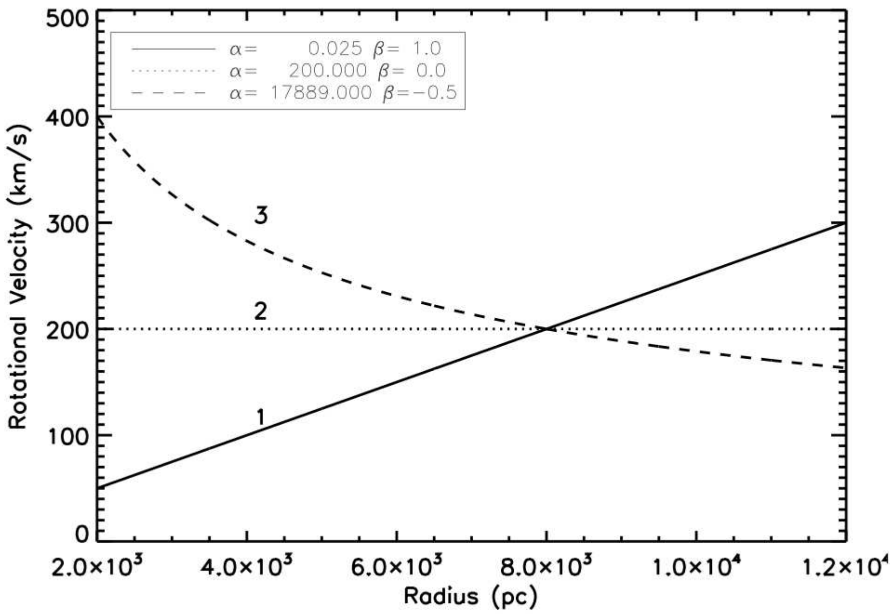

Figure 1: Different rotation curves

These rotation curves can be thought of as simple power law fits to observed rotating mass distributions. We want to use these rotation curves to generate their accompanying density distributions, $\rho(r)$. Before we get too involved in the math, let's recall that a Keplerian rotation curve goes like $r^{-0.5}$, a rigid body rotation curve goes like r^1, and a flat rotation curve goes like r^0 (constant).

(a) Using the power law form for a rotation curve, $v = \alpha r^{\beta}$, solve for density $\rho(r)$ and enclosed mass $M(r)$ in terms of r, α, and β.

$$\frac{v^2}{r} = \frac{GM}{r^2} \Rightarrow v^2 = \frac{GM}{r}.$$

$$v = \alpha r^{\beta} \Rightarrow v^2 = \alpha^2 r^{2\beta} = \frac{GM}{r}.$$

`Solve for` $M(r) = \frac{\alpha^2}{G} r^{2\beta+1}$. `Then plug in` $M = \rho \times \frac{4\pi}{3} r^3$ `to find` $\rho(r) = \frac{3\alpha^2}{4\pi G} r^{2(\beta-1)}$.

(b) How do density and enclosed mass scale with radius for each of the three rotation curves depicted in the plot above? Sketch density profiles.

i. `Rigid body rotation curve:` $\beta = 1 \Rightarrow \rho \propto r^0$ and $M \propto r^3$.

ii. `Flat rotation curve:` $\beta = 0 \Rightarrow \rho \propto r^{-2}$ and $M \propto r^1$.

iii. `Keplerian rotation curve:` $\beta = -0.5 \Rightarrow \rho \propto r^{-3}$ and $M \propto r^0$.

(c) Let's put this all together into a realistic (albeit simplified) view of the Milky Way's rotation curve. The Milky Way's rotation curve, when looked at over larger scales (ie not super close to the central SMBH), look roughly like the curve below.

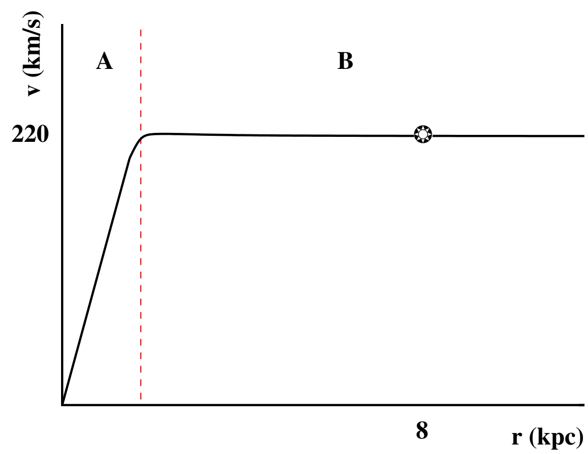

Figure 2: Simplified MW rotation curve

i. In region (B), where the rotation curve is flat, how does v scale with r? How do M and ρ scale with r?
`In region (B),` v `is constant with` r`, so we find`

$$\frac{v^2}{r} = \frac{GM}{r^2} \quad \Rightarrow \quad M = \frac{rv^2}{G}$$

`so` $\boxed{M \propto r \text{ and } \rho \propto r^{-2}}$.

ii. In region (A), where the rotation curve is rising, roughly how does v appear to be scaling with r? What does this v say about M and ρ at these small radii?
`In region (A),` v `appears to be linearly rising with` r `so` $v \propto r$. `So`

$$\frac{v^2}{r} = \frac{GM}{r^2} \quad \Rightarrow \quad \frac{v^2}{r} \propto r \propto \frac{GM}{r^2}$$

`so` $\boxed{M \propto r^3 \text{ and } \rho \propto const}$.

iii. We've said that the NFW profile,

$$\rho_{NFW}(r) = \frac{\rho_0}{(r/a)(1+r/a)^2} \tag{2}$$

describes the dark matter halo of galaxies well. How does the NFW profile predict that ρ will scale with r for $r \ll a$?
`For` $r \ll a$, $(1 + r/a) \approx 1$, `so` $\boxed{\rho \propto r^{-1}}$.

iv. Does the NFW ρ for small r agree with your answer to (ii)? If they don't, what might be different between the cases?

`The densities don't agree. However, for` $\rho_{NFW} \propto r^{-1}$ `for small` r`, this would give` $M \propto r^2$`, so` $GM/r^2 \propto const$`, so` $v \propto r^{1/2}$`. So maybe we can't tell the difference between a linear and a square root rising with the amount of ''twiddle astro'' we've done here.`

`Another possibility is that the NFW profile is designed to model the DM distribution. We know that towards the center of a galaxy there is lots of other stuff besides DM: stars, gas, the central SMBH, ... These more normal masses will modify the total graviational potential, which will affect the rotation curve. So perhaps this is the appropriate answer. Honestly, I'm not 100% sure. For the most part I've used NFW profiles to describe` $r \approx a$ `or` $r \gg a$`.`

(d) **(Bonus)**[3] The real Milky Way rotation curve looks less flat than Figure 2 shows. Rather, it looks more like the figure below.

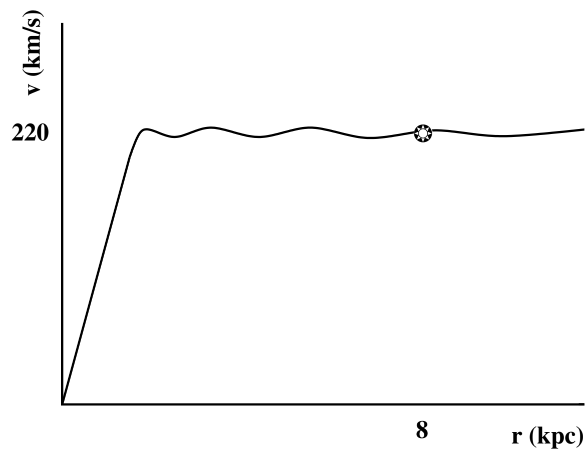

Figure 3: More realistic MW rotation curve

It is true that for large radii, the rotation curve is mostly flat but it definitely has waves in it. Can you think of a reason for why these waves exist?

`Spiral arms! The Milky Way doesn't have a perfect flat disk, but instead has a more complicated spiral structure. You can imagine that by introducing spaces of no matter and then lots of matter (in the arms), you would expect the rotation curve to not be perfectly flat. And it turns out, if you do the physical modelling, this is in fact the case!`

[3]No really, this is outside the scope of Ay7b. I just think it's kinda neat.