Ay 7b – Spring 2012
Section Worksheet 12
Expanding Universe

1. Imagine a flat universe which expanded such that the scale factor evolved with time as

$$\frac{dR}{dt} = \frac{A}{R},$$

where A is some constant. Note that this is not true for our universe now (this describes the universe in the past when radiation was a dominant component of the energy of the universe), but everything below would be computed in exactly the same way for our universe. You'd just use the real expression for dR/dt (which we will do later in the class). Consider also light rays traveling radially in the universe ($d\theta = d\phi = 0$). The metric for this universe is

$$ds^2 = c^2 dt^2 - R^2(d\varpi^2 + \varpi^2 d\theta^2 + \varpi^2 \sin^2\theta d\phi^2).$$

(a) Consider a light ray emitted a comoving distance ϖ away at a time t_e which is then received by us at the origin at some later time t_r (not necessarily today, allow it to possibly have been way in the past). Using the metric, show that $d\varpi = -cdt/R$, including arguing why it is negative.

`Light always travels on paths such that` $ds^2 = 0$. `Thus taking a radial path and using this, the metric tell us` $d\varpi^2 = c^2dt^2/R^2$. `This tells us that` $d\varpi = \pm cdt/R$. `In this case we want the negative root so that light is traveling towards us;` ϖ `gets smaller( closer to us at the origin) as time goes on.`

(b) Integrate this result to find a general expression for ϖ as a function of R_e and R_r , the scale factor at the time the photon is emitted and received, respectively.

$$\begin{aligned}
\int_{\varpi}^{0} d\varpi &= -c\int_{t_e}^{t_r} \frac{dt}{R} \\
-\varpi &= -c\int_{R_e}^{R_r} \frac{1}{R}\frac{dt}{dR}dR \\
\varpi &= \frac{c}{A}\int_{R_e}^{R_r} dR \\
&= \frac{c}{A}(R_r - R_e) \qquad (1)
\end{aligned}$$

(c) Now consider photons observed today which are observed to have a cosmological redshift of z. In this case, you are finding the comoving distance (ϖ) to the object which emitted the light. Since the proper distance (or, physical distance) is given by $\mathcal{L} = R\varpi$, and $R = 1$ today, the comoving distance today is also the proper distance to the object now. (But note that this is different from the proper distance the object was when the light was emitted, see below.) Thus either way in this case, ϖ is what we would normally think of as the distance. Give a general expression for this comoving distances as a function of redshift. Show also that if $z \ll 1$, $cz = A\varpi$. This is just Hubble's Law, and it means that $A = H_0$.

`If it is received today, then` $R_r = 1$. `We also use the fact that` $R = \frac{1}{1+z}$ `to replace` R_e. `Thus we have`

$$\varpi = \frac{c}{A}\left(1 - \frac{1}{1+z}\right).$$

`If` $z \ll 1$, `then we can use the binomial expansion` $(1/(1+z) \approx 1 - z)$ `to get`

$$A\varpi \approx c[1 - (1 - z)] = cz.$$

(d) Given the object is at the above comoving distance today, what was its comoving distance when it emitted the light? What was the proper distance to the object when the light was emitted?

The comoving coordinate of an object moves along with the object as the universe expands (hence it's name). Thus it does not change with time. So the comoving distance was the same when it was emitted as it is today. The proper distance is always related to the comoving distance as $\mathcal{L} = R\varpi$. Since the light has a redshift z, the scale factor when it was emitted was $R = 1/(1+z)$. Thus

$$\mathcal{L} = \frac{c}{H_0(1+z)}\left(1 - \frac{1}{1+z}\right).$$

(e) What you have found is comoving and proper distances to an object with redshift z. This relation is how astronomers use redshift as a measure of distance at cosmological scales. Think about your expressions for $\varpi(z)$ and $\mathcal{L}(z)$. Consider them in the limit where the redshift is vary large and very small. Do they make sense?

As z goes to 0, ϖ and $\mathcal{L}$ both also go to 0. Thus very small redshifts come from things that are very close. Since they are close, the proper distance now and when they emitted are practically the same because not much time has passed in between. This makes sense. As z goes to ∞, ϖ approaches the constant c/H_0 but $\mathcal{L}$ goes to 0 again. This means that things so far away that they are innitely redshifted are only a finite comoving distance away! But they were 0 proper distance away when the light was emitted? This is because this is light form the big bang. At that first moment, everything was at the same place. But they didn't stay there, and it took light a long while to catch up with the expansion and get to us. Stuff from comoving distances even farther away have not yet had enough time to reach us today.

(f) Go back to your original general formula in terms of R_e and R_r (the answer you got in(b)). Consider now light emitted at the time of the big bang, from $t = 0$, and which are received at some later time when the scale factor is R_r. This is the maximum distance light could have traveled up to that time, and thus you can't see beyond it. For this reason it is called the cosmic horizon, and it marks the edge of the visible universe. Give general expressions for both the comoving and proper distances to this cosmic horizon in terms of the scale factor. What is the value of the horizon today?

At $t = 0$, $R_e = 0$ and we'll call $R_r = R$. Thus

$$\varpi = \frac{cR}{H_0} \quad \text{and} \quad \mathcal{L} = R\varpi = \frac{cR^2}{H_0}.$$

Today, $R = 1$, so both of these distances are $\boxed{c/H_0}$.

(g) Imagine a photon emitted at time te when the scale factor was R_e and received at a time t_r when the scale factor was R_r. Given our expression for how the scale factor evolves, find and expression for $\Delta t = t_r - t_e$ as a function of the scale factor at those times.

This just requires taking our given equation for dR/dt and integrating it through separation of variables.

$$\int_{t_e}^{t_r} H_0 dt = \int_{R_e}^{R_r} R dR$$

$$t_r - t_e = \Delta t = \frac{1}{2H_0}(R_r^2 - R_e^2).$$

(h) Just as before, consider such photons received today and observed to have a redshift of z. Find and expression for $\Delta t(z)$ for this case. This is the lookback time for that object, i.e. how long ago its light was emitted.

In this case, $R_r = 1$ and $R_e = 1/(1+z)$. Thus

$$\Delta t = \frac{1}{2H_0}\left(1 - \frac{1}{(1+z)^2}\right).$$

(i) Now consider if it was emitted at the time of the big bang and observed at some later time when the scale factor was R such that photons emitted at this later time would be observed today to have a redshift of z. Find an expression for the received time, t, as a function of the redshift at that time. This is how old the universe was when light observed today to have a redshift of z was emitted.

Using $t_e = 0$ and $R_e = 0$ and $R_r = 1/(1+z)$ gives us

$$t_r = t = \frac{R^2}{2H_0} = \frac{1}{2H_0(1+z)^2}.$$

(j) What you have found above are the lookback time to an object with redshift z and how old the universe was at that time. This relation is how astronomers use redshift as a measure of time at cosmological scales. Thus when we say that galaxy is at a redshift of "3", that is a statement about how far away it is, how long ago the light was emitted, and how old the universe was at the time. Finally, what do your results say the age of the universe is today, t_0? What is the distance to the cosmic horizon today, d_h ? Give an expression for the horizon distance in terms of the age and argue why the result doesn't seem to make sense. Then argue why it actually is ok.

Plugging in that $R = 1$ and $z = 0$ today give that $t_0 = 1/(2H_0)$and that $d_h = c/H_0$ This means that $dh = 2ct_0$. This seems to suggest that the horizon is twice as far away as light could have traveled in the entire age of the universe! But we derived dh as the distance light could have traveled in that time. How is this possible? The answer is that d_h is the distance to the horizon today, but is not the actual distance traveled by the light. The expansion means the horizon has been getting farther and farther away as that light from it was coming it us. Thus it is now farther away today than it was when the light was emitted.