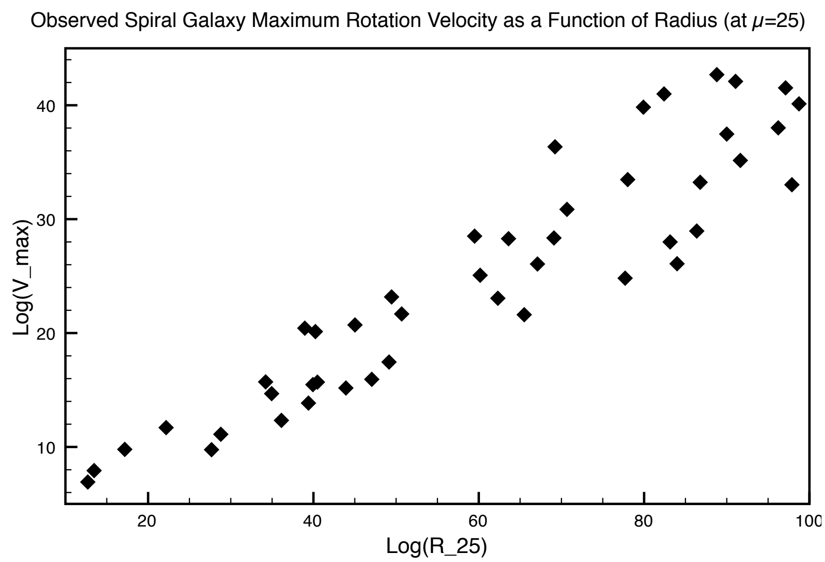
Ay 7B – Spring 2010
Section Worksheet 8
Relations, Relations, Relations.. & Dynamical Friction

1. **Everything's Related through Relations**
   The Tully-Fisher and Radius-Luminosity relations for spiral galaxies imply the existence of a third relation, the V_{max}-R_{25} relation between the maximum rotation velocity of a galaxy (in km/s) and the disk radius (in kpc) corresponding to a surface-brightness level of 25 B-mag arcsec^{-2}.

   (a) Some clever observational astronomers, expecting to discover this relation, plotted observed V_{max} with R_{25} as shown in the following figure.



   What is approximately the slope of this observed V_{max}-R_{25} relation? (In other words, if the relation is expressed as a linear equation $\log V_{max} = m \times \log R_{25} + b$, what is m? It might be helpful to draw a line to represent the linear relation then estimate m from that line.)

   `Simple rise over run yields` $m = \frac{40-7}{95-12} = 0.4$.

   (b) Use the Tully-Fisher and Radius-Luminosity relations to eliminate M_B and solve for a V_{max}-R_{25} relation. Keep the logarithms such that your final equation should be of the form $\log V_{max} = m \times \log R_{25} + b$. Use the Tully-Fisher relation for Sa spiral galaxies and find m and b.

   `Re-writing the Radius-Luminosity relation to solve for` M_B `yields` $M_B = -4.02 \log R_{25} - 16.06$. `Then setting this equal to the Tully-Fisher relations and solving for` $\log V_{max}$ `results in the answers:`
   `Sa` $\Rightarrow$ $\log V_{max} = 0.40 \times \log R_{25} + 1.93$.

   (c) How does your derived relation compare with the slope of the observed data? Explain what might account for any differences.

   `The slope in the derived relations should be close to the slope estimated from the observational plot. Any discrepancy should be less than 0.15. The plot depicts a` `great deal of scatter(`$\sim \pm 20\%$`), indicating that it is not reasonable to infer spiral galaxy type from simple observations of rotation velocity and radius.`

2. **Dynamical Friction - Order of Magnitude Estimation** [1]

   Imagine a eld of little objects (each with mass m and a uniform number density of n) hanging around all sitting still. Now imagine a larger object of mass M flying through this sea of things at a speed v_M. Many astrophysical situations can be described like this, such as (but not limited to): A) a planet moving around a protoplanetary disk full of bits of debris, B) a large star moving within a cluster of smaller stars, C) a globular cluster moving through amongst the stars in a galaxy, or D) a large galaxy moving through a cluster of smaller galaxies.

   As the interloper moves through, it will pull the other objects down towards it. These will build up and collect behind it, making a wake that has a higher density than the surrounding background. That higher density will then gravitationally pull back on the interloper, slowing it down. Thus there is a force slowing down the object due to it's motion: a kind of friction! But unlike classic friction or normal air drag, there is no actual collisions here causing it.

   Here, we will derive the basic result through our typical order of magnitude estimations. We will take scenerio C above but the results will be fully general for any generic system described above. For clarity, we will *call the background objects stars and the interloping object a cluster* . The mass and the velocity of an individual background star is m and v_m, respectively, and those of the cluster is M and v_M, respectively. Also, we assume that the background objects were initially stationary, i.e. $v_{m,i} = 0$. Note that subscripts i and f represent before the encounter and after the encounter, respectively.

   (a) First, consider an interaction between the cluster and one single background star.

   i. The center of mass velocity v_{cm} cannot change due to the interaction. i.e. $\Delta v_{cm} = 0$. (Hint: $v_{cm} = (mv_m + Mv_M)/(m + M)$.) From this, derive the relation between Δv_m and Δv_M.

   `Since the center of mass velocity can't change, we have`

$$\Delta v_{cm} = \frac{m\Delta v_m + M\Delta v_M}{m + M} = 0,$$

   `which gives that`

$$\Delta v_m = -\frac{M}{m}\Delta v_M. \tag{1}$$

   ii. Let V be the relative velocity between the two stars, $V_i = v_{m,i} - v_{M,i} = -v_{M,i}$ since we said the stars are all stationary. Any change in this caused by the interaction is thus

$$\Delta V = V_f - V_i = v_{m,f} - v_{M,f} - (v_{m,i} - v_{M,i}) = \Delta v_m - \Delta v_M.$$

   Using the result above and the information stated here and assuming that $M \gg m$, show that

$$\Delta v_M = -\frac{m}{M}\Delta V.$$

[1] Your text book says a derivation of this effect is beyond their scope, and so just quotes the result. A fully rigorous treatment is indeed beyond what we need to worry about here (the brave can find a copy of Galactic Dynamics by Binney and Tremaine and see section 7.1 for a full treatment of the topic).

Combining eq.(??) with $\Delta V = \Delta v_m - \Delta v_M$, we get

$$\begin{aligned} \Delta V + \Delta v_M &= -\frac{M}{m}\Delta v_M \\ \Delta V &= -(1+\frac{M}{m})\Delta v_M \\ \Delta V &\approx -\frac{M}{m}\Delta v_M. \end{aligned} \quad (2)$$

Rearranging this,

$$\boxed{\Delta v_M \approx -\frac{m}{M}\Delta V}. \quad (3)$$

(b) Obviously every star will be at a different distance from the cluster. A small few will be very close and thus very strongly effect it. Very many will be very far and have only a very small effect on it. The interactions that cause most of the inuence are those for which the relative kinetic energy is on the order of the (absolute value of the) potential energy of the interaction.

i. Write down the relative kinetic energy of the star (i.e. The kinetic energy of the star from the point of view of the cluster).

From the point of view of the cluster is at rest relative to itself, so it has no kinetic energy. The mass of a background star is m and the velocity is $V_i \equiv V$, relative velocity before the encounter, because the cluster is at rest in this frame, thus, the kinetic energy of the star is $\boxed{\frac{1}{2}mV^2}$.

ii. Write down the potential energy of the star. Consider only a star and the cluster and the distance between the star and the cluster is r.

The potential energy between the star and the cluster is $\boxed{U = -\frac{GMm}{r}}$.

iii. By equating the kinetic energy and the potential energy equations, calculate the typical interaction distance, r. Note that for the potential energy, use the absolute value.

$$\frac{1}{2}mV^2 = \frac{GMm}{r}$$

$$\boxed{r = \frac{2GM}{V^2}} \quad (4)$$

(c) For a typical strong encounter such as this, the star's trajectory relative to the cluster will be deflected by 90 degrees, i.e. the relative velocity after the encounter will be $V_{||,f} = 0$ parallel to the initial incoming velocity, and $V_{\perp,f} = V_i$ perpendicular to it. Note that we are working back in the outside frame and hereafter we will use $v_{M,i} \equiv v_M$ and $V_i \equiv V$.

i. What is the change in the parallel component of the relative velocity?

$$\boxed{\Delta V_{||}} = V_{||,f} - V_{||,i} = 0 - V_i = \boxed{-V}.$$

ii. Using above result and the relation derived in (a), show that

$$\Delta v_{M,||} = -\frac{m}{M} v_M,$$

the negative sign showing the force is acting to show down the cluster.

From eq.?? and combining the above result,

$$\Delta v_{M,||} = -\frac{m}{M} \Delta V_{||} = \frac{m}{M} V.$$

Also, as mentioned in (a), we know $V_i = -v_{M,i} \Rightarrow V = -v_M$. Thus,

$$\boxed{\Delta v_{M,||} = -\frac{m}{M} v_M}.$$

(d) The result we got above is the change in the cluster's velocity due to just one star. But there are many stars. Here we will get the force on the cluster parallel to the original direction of motion inserted by many stars.

i. Write down the interaction rate(number of interactions per second) equation. (Hint: you may write it in terms of number density, cross section and velocity.)

The interaction rate is given by $n\sigma v$ where n is the number density, σ is the cross-section area and v is the velocity of the object.

ii. Considering the typical interaction distance found above, show that the rate at which the cluster encounters the stars is

$$\alpha = \frac{4\pi G^2 M^2 n}{v_M^3},$$

where α is the interaction rate.

In this problem, n is the number density of background stars, σ is a circle with radius equal to the typical interaction length, and v is the relative velocity between the cluster and background stars. So, $\sigma = \pi r^2$ and $v = V$. Also, r is given by eq.(??) Thus,

$$\alpha = n\sigma v = n\pi r^2 V = n\frac{4\pi G^2 M^2}{V^4} V = \boxed{\frac{4\pi G^2 M^2 n}{v_M^3}}.$$

iii. What is the acceleration parallel to the initial incoming direction felt by the cluster? (Hint: Acceleration means the change in velocity per unit time.)

The acceleration felt by the cluster is the total of al l the changes in velocity in some time i.e. $\Delta v/\Delta t$. α is the number of interactions per time so the acceleration is just the change in the velocity of the cluster times the interaction rate, $\boxed{\Delta v_{M,||}\alpha}$.

iv. Using above equations, show that the force on the cluster parallel to the original direction of motion is

$$F_{||} = -\frac{4\pi G^2 M^2 \rho}{v_M^2}.$$

The force just mass times acceleration. Thus, the force felt by cluster is the cluster mass, M times the acceleration given above,

$$F_{||} = \Delta v_{M,||}\alpha = -\frac{m}{M}v_M\frac{4\pi G^2M^2n}{v_M^3}M = -\frac{4\pi G^2M^2\rho}{v_M^2}.$$

This is the equation we saw in the book.

(e) So far, we obtained the parallel force. What about the force perpendicular to the motion? Since the cluster pulled the stars down towards it, it must have been pulled up as well. Give an argument why, despite this, there is no overall perpendicular force on the cluster.

While it is true that any individual encounter must pull the cluster towards the star, it is just as likely to be pulled up towards one above it as down towards one below (or similarly left or right). Thus over many encounters, all of those tugs in different directions will cancel out. But all stars encountered will produce a net backwards force against the velocity.

(f) Now, we want to calculate the time scale for the cluster to lose all of its velocity due to the dynamical friction.

i. The velocity of a globular cluster is orbiting a distance R from a galaxy of uniform stellar density ρ. Assuming its motion is circular, show that it's orbital velocity would be

$$v_M^2 = \frac{4}{3}\pi G\rho R^2.$$

In general, the orbital velocity is

$$\frac{mv^2}{r} = \frac{GmM_{\text{enclosed}}}{r^2}$$

. Assuming uniform stellar density, we have ρ, $M_{\text{enclosed}} = 4\pi R^3\rho/3$. r is the distance between a star and the cluster, R and v is the encounter velocity of the cluster, v_M. Thus,

$$v_M^2 = \frac{4}{3}\pi G\rho R^2.$$

ii. The time scale for the cluster to lose all of its velocity is $\tau = v/a$ where v is the velocity and a is acceleration. Express τ for the cluster in terms of v_M, R, M and fundamental constants.

Using our dynamical friction force, the timescale is then

$$\tau = \frac{v}{a} = \frac{v_M}{F/M} = \frac{v_M^3}{4\pi G^2M\rho}.$$

Plugging in expression for v_M^2 as a function of R then gives

$$\tau = \left(\frac{4}{3}\pi G\rho R^2\right)^{3/2}\left(\frac{1}{4\pi G^2M\rho}\right).$$