Ay 7b – Spring 2012

Section Worksheet 3

Tidal Forces

1. **Tides on Earth**
   On February 1st, there was a full Moon over a beautiful beach in Pago Pago. Below is a plot of the water level over the course of that day for the beautiful beach.

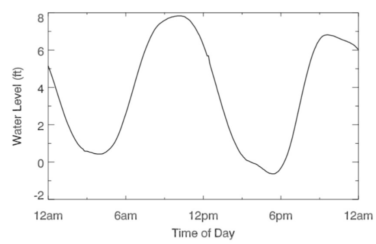

Figure 1: Phase of tides on the beach on February 1st

(a) Sketch a similar plot of the water level for February 15th. Nothing fancy here, but be sure to explain how you came up with the maximum and minimum heights and at what times they occur in your plot.

On February 1st, there was a full the Moon over the beautiful beaches of Pago Pago. The Earth, Moon, and Sun were co-linear. When this is the case, the tidal forces from the Sun and the Moon are both acting in the same direction creating large tidal bulges. This would explain why the plot for February 1st showed a fairly dramatic difference ($\sim$8 ft) in water level between high and low tides. Two weeks later, a New Moon would be in the sky once again making the Earth, Moon, and Sun co-linear. As a result, the tides will be at roughly the same times and heights (give or take an hour or so).

(b) Imagine that the Moon suddenly disappeared. (Don't fret about the beauty of the beach. It's still pretty without the moon.) Then how would the water level change? How often would tides happen? (For the level of water change, calculate the ratio of the height of tides between when moon exists and when moon doesn't exist.) Sketch a plot of the water level on the figure.

The tidal force exerted by the Moon is

$$F_M = \frac{2GM_M M_\oplus R_\oplus}{r^3_{M\oplus}}$$

where M_M is the mass of the Moon, $M_\oplus$ is the mass of the Earth, $R_\oplus$ is the radius of the Earth $r_{M\oplus}$ is the distance between the Moon and the Earth. The tidal force exerted by the Sun is

$$F_\odot = \frac{2GM_\odot M_\oplus R_\oplus}{r^3_{\odot\oplus}}$$

where $M_\odot$ and $r_{\odot\oplus}$ are the mass of the Sun and the distance between the Sun and the Earth, respectively. Since the height of tides is proportional to the tidal force (think about the analogy with the work done by gravitational force, i.e. $\int F ds = mgh$.), we get the ratio between the height of tides by the Moon, h_M and that by the Sun, $h_\odot$,

$$\frac{h_M}{h_\odot} = \left(\frac{2GM_M M_\oplus R_\oplus}{r_{M\oplus}^3}\right) / \left(\frac{2GM_\odot M_\oplus R_\oplus}{r_{\odot\oplus}^3}\right) = \frac{M_M}{M_\odot}\left(\frac{r_{\odot\oplus}}{r_{M\oplus}}\right)^3.$$

Two high tides and two low tides occur because of the Earth's rotation, even if the Moon didn't exist, two high tides and two low tides would occur. However, before there was a total heigth $h_M + h_\odot$, and now there will just be height $h_\odot$, so the total height will be decreased by a multiplicative factor of $\boxed{\frac{h_\odot}{h_M + h_\odot} = \left[1 + \frac{M_M}{M_\odot}\left(\frac{r_{\odot\oplus}}{r_{M\oplus}}\right)^3\right]^{-1}}$.

So, by plugging in the numbers, $r_{M\oplus} \approx 384.4 \times 10^3$km, $r_{\odot\oplus} \approx 1.5 \times 10^8$km, $M_M \approx 7 \times 10^{22}$kg and $M_\odot \approx 2 \times 10^{30}$kg, and an initial tide height of ≈ 8 ft, we get about $\boxed{2.6\text{ feet}}$, which is smaller than the height caused by both the Sun and the Moon.

2. **Tidal Forces and Spaghettification**[1]

(a) Prof. Chiang is so sad that Ay 7A is over that he dives out of his spaceship head first and floats toward a black hole. What is the gravitational force on his head (F_h) due to a black hole of mass M when his head is a distance r from the center of the black hole? Assume Prof. Chiang has a mass m.

$$F_h = \frac{GMm}{r^2}$$

(b) Now let's write down an expression for the force on his feet (F_f), assuming that the professor's height is x.

$$F_f = \frac{GMm}{(r+x)^2}$$

(c) Subtract these two values to get the *difference* in forces across Prof. Chiang's body. This is called the "tidal force" on the professor. (You calculated this in class on Thursday, more or less).

$$\Delta F = F_h - F_f = \frac{GMm}{r^2} - \frac{GMm}{(r+x)^2}$$

$$\Delta F = GMm\left(\frac{1}{r^2} - \frac{1}{(r+x)^2}\right)$$

(d) Now assume that $r \gg x$ (i.e., the distance to the center of the black hole is much, much larger than the professor's height) and Taylor expand the above expression for the tidal force (as usual, anything $\ll 1$ to the second – or larger – power can be dropped).

$$\Delta F = GMm\left(\frac{1}{r^2} - \frac{1}{r^2(1+\frac{x}{r})^2}\right)$$

$$\Delta F = \frac{GMm}{r^2}\left(1 - \frac{1}{(1+\frac{x}{r})^2}\right)$$

[1]**Yes, that is a technical term.**

$$\Delta F \approx \frac{GMm}{r^2}\left(1-\left[1-2\frac{x}{r}\right]\right)$$

$$\Delta F \approx \frac{GMm}{r^2}\left(2\frac{x}{r}\right)$$

$$\Delta F \approx \frac{2GMmx}{r^3}$$

(e) In class you calculted the tensile strength of a person based on the assumption they were held together by their skin. In this problem, let's assume Prof. Chiang is instead held together by his own *self-gravity.* Model him as two identical spheres of mass $0.5m$ gravitationally bound to each other. Go ahead. It's OK. He won't mind.[2]

What is the maximum tidal force Prof. Chiang can withstand?

$$F_G = \frac{G(\frac{m}{2})(\frac{m}{2})}{x^2}$$

$$F_G = \frac{Gm^2}{4x^2}$$

(f) If the tidal force outside a black hole (ΔF) is larger than the maximum force human bones can withstand ($F_{\max}$) then we say that a person will get "spaghettified" by the black hole (i.e., they will be gravitationally ripped apart before plunging into the black hole). By "outside" we mean beyond the "event horizon" (aka the edge, or the "point of no return") of the black hole. Anything (including light) that gets closer than this distance to the center of a black hole *must* free-fall to the center of the black hole (known as the "singularity" — a single point of infinite density).

However, if the tidal force from a black hole is low enough, then a human can plunge directly into a black hole in one piece. For this to be true, the tidal force from a black hole at its event horizon must be smaller than the maximum force human bones can withstand. The event horizon is, by definition, one Schwarzschild radius away from the center of a black hole. The Schwarzschild radius is given by:

$$r_S = \frac{2GM}{c^2}$$

Using this equation, derive an expression for the tidal force of a black hole *at its event horizon.*

$$\Delta F = \frac{2GMmx}{r_S^3}$$

$$\Delta F = 2GMmx\left(\frac{c^2}{2GM}\right)^3$$

$$\Delta F = \frac{mxc^6}{4G^2M^2}$$

[2] If you didn't take 7A with Prof. Chiang, this probably doesn't seem very funny to you. Trust me, it is.

(g) Set the above tidal force equal to the maximum force Prof. "Two Spheres" Chiang can withstand and solve for M. This represents the *minimum* black hole mass that will not spaghettify Prof. Chiang and thus allow him to float right through the event horizon in one piece.

$$\Delta F = \frac{mxc^6}{4G^2M^2} = \frac{Gm^2}{4x^2}$$

$$M = \left(\frac{x^3c^6}{mG^3}\right)^{1/2}$$

(h) To order of magnitude, let's say the professor is about 1 m tall and 100 kg.[3] Calculate an actual value for the M you just solved for, in solar masses.

$$M = \left(\frac{(1m)^3(3\times 10^8 m/s)^6}{100kg\,(6.67\times 10^{-11}m^3/kgs^2)^3}\right)^{1/2} = 4.96\times 10^{40}kg = 2.5\times 10^{10}M_\odot$$

`There are some SMBH in the center of other galaxies that are as big as` $10^{10}M_\odot$`, so we can even find a black hole relatively nearby for a gravitationally bound Professor to safely fall into!`

(i) Calculate the tidal forces this gravitationally bound model of Prof. Chiang experiences every day on earth. Can people possibly be gravitationally bound?

`The tidal forces we feel standing on the surface of the earth are:`

$$\Delta F = \frac{2GM_Emx}{r_E^2} = \frac{2(6.67\times 10^{-11}m^3/kgs^2)(6\times 10^{24}kg)(100kg)(1m)}{(6400\times 10^3m)^2} = 1954N$$

`Numerically, our force of self gravity is:`

$$F = \frac{Gm^2}{4x^2} = \frac{(6.67\times 10^{-11}m^3/kgs^2)(100kg)^2}{4(1m)^2} = 1.67\times 10^{-7}N$$

`Thankfully we aren't held together by self gravity, or we would be torn apart trying to stand on the surface of the earth!`

[3] Actually, these numbers would be more accurate for an Oompa-Loompa version of Prof. Chiang. Now come on, that's funny.