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Probability

(a) Exercise 1.6

Show that if two variables x and y are independent, then their covariance is zero.

We can set:

$E(X) = \mu, E(y) = \upsilon.$

According to the rule of covariance, $Cov(x, y) = E[E(x - \mu)E(y - \upsilon)] = E(x * y) - \mu\upsilon$

because variables x and y are independent then E(x*y)=E(x)E(y)=$\mu\upsilon$

So Cov(x,y)=0

(b)

the total number of people is 158 and set N = 158

i) from the table, the probability of test being positive given the patient has HIV, $P(+|HIV) = \frac{72}{75} = 96\%$

ii) from the table, the probability of test being negative given the patient has HIV, $P(-|HIV) = \frac{3}{75} = 4\%$

iii)$P(HIV) = 12\%, P(H\bar{I}V) = 1 - P(HIV) = 88\%$

$P(+) = P(+|HIV) * P(HIV) + P(+|\neg HIV) * P(\neg HIV) = 24.24\%$

$P(HIV|+) = \frac{P(+|HIV)*P(HIV)}{P(+)} = 47.52\%$

$P(HIV|-) = \frac{P(-|HIV)*P(HIV)}{P(-|HIV)*P(HIV)+P(-|\neg HIV)P(\neg HIV)} = 0.63\%$

Bayes Theorem

a)

set A,B,C,D are the events that there is a big prize behind the door then$2P(A) = 2P(B) = 2P(D) = P(C)$ and $P(A) + P(B) + P(D) + P(C) = 1$

so $P(C|\bar{B}andopen) = \frac{P(\bar{B}andopen|C)*P(C)}{P(\bar{B}andopen)}$

$P(\bar{B}andopen) = P(\bar{B}andopen|\bar{B}andC\bar{A}\bar{D}) * P(\bar{B}andC\bar{A}\bar{D}) + P(B|\bar{A}\bar{D}and\bar{C}) * P(BopenB\bar{A}\bar{D}and\bar{C}) + 2 * P(openB|A\bar{B}\bar{C}\bar{D}) * P(A\bar{B}\bar{C}\bar{D}) = 33.3\%$

so $P(C|\bar{B}andopen) = 40\%$

then $P(A|\bar{B}andopen) = P(D|\bar{B}andopen) = \frac{P(\bar{B}andopen|A)*P(A)}{P(\bar{B}andopen)}$

$P(A|\bar{B}andopen) = P(D|\bar{B}andopen) = 30\%$

the probability of choosing the door C to get the big prize is larger than choosing door A or door D

Michael doesn't need to change his selection

b)

Set T is that Professor chin identifies that the student as a cheater, then $\bar{T}$ is the opposite event

Set S is that the student is cheating,then $\bar{S}$ is the opposite event

then $P(T|S) = 90\%$

$P(T|\bar{S}) = 20\%$
$P(T) = P(T|S) * P(S) + P(T|\bar{S}) * P(\bar{S}) = 0.9 * \frac{2}{75} + 0.2 * \frac{72}{75} = 0.219$ $P(S|T) = \frac{P(TS)}{P(T)} = \frac{P(T|S)*P(S)}{P(T)} = 11\%$

Linear Algebra
a)
Set A is a symmetric matrix and A^{-1} is existent
Because A is a symmetric matrix, then $A^T = A$.
$(A^{-1})^T = (A^T)^{-1} = A^{-1}$

b)
A = $\begin{bmatrix} 3 & 4 & -1 \\ -1 & -2 & 1 \\ 3 & 9 & 0 \end{bmatrix}$
Set x is a non-zero 3-by-1 $x^T = [x_1, x_2 x_3]$
I=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
then $Ax = \lambda x$.
$Ax - I\lambda x = 0$
$(A - I\lambda)x = 0$
A-I=A = $\begin{bmatrix} 3-\lambda & 4 & -1 \\ -1 & -2-\lambda & 1 \\ 3 & 9 & -\lambda \end{bmatrix}$
compute determinant A-I
eigenvalues is $\lambda_1 = -3$,$\lambda_2 = 2$,$\lambda_3 = 2$
compute eigenvectors
$Ax = \lambda_1 x$,$Ax = \lambda_2 x$, $Ax = \lambda_2 x$
eigenvectors is $x_1^T = [1, -1, 2]$
$x_2^T = [-1, 1, 3]$
$x_3^T = [-1, 1, 3]$
it is not a positive definite

Probability Distributions Exercise2.2
$P(x|\mu) = (\frac{1-\mu}{2})^{\frac{1-x}{2}}(\frac{1+\mu}{2})^{\frac{1+x}{2}}$
when x = -1, $P(-1|\mu) = \frac{1-\mu}{2}$
when x= 1,$P(1|\mu) = \frac{1+\mu}{2}$
$P(-1|\mu) + P(1|\mu) = 1$
So the distribution is normalized
$E(X) = \sum x * P(x|\mu) = -1 * \frac{1-\mu}{2} + 1 * \frac{1+\mu}{2} = \mu$
$Var(x) = E(x^2) - [E(x)]^2 = \sum x^2 * P(x|\mu) - \mu^2 = 1 - \mu^2$
entropy
$H(x) = -\sum P(x|\mu) * ln(P(x|\mu)) = -(\frac{1-\mu}{2} * ln(\frac{1-\mu}{2}) + \frac{1+\mu}{2} * ln(\frac{1+\mu}{2}))$

Exercise2.10

Using the fact that the Dirichlet distribution is normalized

$\int \prod_{k=1}^{M} \mu_k^{a_k - 1} d\mu = \frac{\Gamma(a_1)...\Gamma(a_M)}{\Gamma(a_0)}$

where $0 \leq \mu_k \leq 1$ and $\sum \mu_k = 1$ now consider μ_j

$E[\mu_j] = \frac{\Gamma(a_1)...\Gamma(a_M)}{\Gamma(a_0)} \int \mu_j \prod_{k=1}^{M} \mu_k^{a_k - 1} d\mu = \frac{\Gamma(a_0)}{\Gamma(a_1)...\Gamma(a_M)} * \frac{\Gamma(a_1).\Gamma(a_j+1)..\Gamma(a_M)}{\Gamma(a_0+1)} = \frac{a_j}{a_0}$

because $\Gamma(x+1) = x\Gamma(x)$

then $Var[\mu_j] = E[\mu_j^2] - E[\mu_j]^2 = \frac{a_j(a_j+1)}{a_0(a_0+1)} - \frac{a_j^2}{a_0^2} = \frac{a_j(a_0 - a_j)}{a_0^2(a_0+1)}$

$Cov(\mu_j, \mu_l) = E[\mu_j \mu_l] - E[\mu_j]E[\mu_l] = \frac{a_j a_l}{a_0(a_0+1)} - \frac{a_j}{a_0}\frac{a_l}{a_0} = -\frac{a_j a_l}{a_0^2(a_0+1)}$

Exercise2.12

$U(x|a,b) = \frac{1}{b-a} \int_a^b \frac{1}{b-a} dx = \frac{x}{b-a}|_a^b = \frac{b}{b-a} - \frac{a}{b-a} = 1$

this distribution is normalized

E(x)=$\int_a^b x \frac{1}{b-a} dx = \frac{x^2}{b-a}|_a^b = \frac{b+a}{2}$

$Var(x) = E(x^2) - [E(x)]^2 = \int_a^b x^2 \frac{1}{b-a} dx - (\frac{b+a}{2})^2 = \frac{(b-a)^2}{12}$

Exercise2.15

$H[X] = \int N(X|\mu, \sum) \ \ln N(X|\mu, \sum) \ dx$

$= \int N(X|\mu, \sum) \ \frac{1}{2}(D \ln(2\pi) + \ln |\sum| + (x-\mu)^T \sum^{-1} (x-\mu)) \ dx$

$= \frac{1}{2}(D \ln(2\pi) + \ln |\sum| + Tr[\sum^{-1} \sum])$

$= \frac{1}{2}(D \ln(2\pi) + \ln |\sum| + D)$