Waveform Reconstruction of Bandwidth-Limited Signal by Compressed Sensing

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Abstract—**The inevitable bandwidth limitation is seen in various fields of science. Some researches tackled on this problem from the perspective of detecting some information outside the band, but direct reconstruction of waveform have not been sufficiently discussed so far. We analyse system with bandwidth limiting subsystem and write down the reconstruction problem as linear equation. This equation is revealed to have the large condition number and essentially unsolvable. Then we propose the framework to reconstruct the original signal from the output signal. This can be done with frequency conversion by mixing with random signal and compressed sensing technique. We demonstrate our framework by the experiment using some electronic devices and show that we can achieve above purpose. Our framework are sufficiently work for reconstructing high intensity coefficients of sparse basis.**

Index Terms—**discrete time systems, LTI system analysis, frequency conversion, compressed sensing**

I. Introduction

When one thinks about developing communication devices, designing circuits or measuring physical phenomena, which frequency band to use must be carefully concerned. Once these bands are properly chosen to achieve a certain purpose, the performance outside the band is not guaranteed, namely, bandwidth is *limited* for certain purpose.

This inevitable limitation of bandwidth is seen in various fields of science. One example is low temporal resolution of scanning tunneling microscopy [1]. The measurement with high spatial and temporal resolution is greatly desired in the field of condensed matter physics. Scanning tunneling microscopy, which images sample surface by measuring tunneling current, achives atomic-scale spatial resolution. The tunneling current is required to be amplified for improving the signal-to-noise ratio since this current is quite small. This amplification decreases high frequency component of the signal, and it causes poor temporal resolution. Scientists have been worked on this problem and have achieved to detect atomic scale and high frequency signal by refining equipments [2], directly measuring the tunnel junction resistance with radio frequency resonator [3] and using pump probe method [4], [5]. But these approaches are mainly aiming to detect the specific frequency in their interest and not to reconstruct the waveform of tunneling current.

Another example is the bandwidth limitation in voice telecommunications. Telephone, which is invented by Alexander Graham Bell, started to be served for public in 1870's. The conventional telecommunication network restricted the voice signal to the range of 300 Hz-3 kHz which is called as voice frequency. This restriction is too strict and the voice over the telephone network is heard different from one's normal voice. Improved audio coding technique like adaptive multi-rate wideband [6] and the establishment of voice over IP protocols [7] led to wideband audio transmission. However, it is still far from human hearing range and innumerable researches for overcomig bandwidth limitation is eagerly investigated.

Under such an inevitable limitation of course we never be able to observe original signal but be able to estimate the signal if we have known the feature of the signal well. Compressed sensing [8], [9] is a framework for reconstructing signal based on the assumption that all signals are redundant and have the sparsity on the certain basis. This assumption is considered reasonable in various fields, and compressed sensing is applied to medical image processing [10], [11], image denoising [12], surface science [13], and other fields.

For the bandwidth limitation issue, however, it is insufficient to use compressed sensing because it is also obvious that we can't even estimate the signal with complete absence of the information about outside the band. So we must convert the signal outside the band to the signal inside the band. To do this frequency cooonversion, it is useful to multiply another signal and generate sum frequency and difference frequency. This technique have been widely used in physical measurement as heterodyne measurement. In our framework, we multiply original signal and random signal to randomly mix all signal outside the band into the band, from which someone may be recall spread spectrum.

Even if we can get information of high frequency components, it is insufficient to achieve our purpose because information about high frequency component mixed to low frequency component are not able to be apart each other. This difficulty can be attacked by using prior information about original signal, which is called compressed sensing.

The rest of this paper is organized as follows; We first describe our model in subsection II-A, and see that the naive method fails to reconstruct in II-B. Then we describe our new framework in II-C and show experimental results in III.

II. Theory

A. System Description

Our purpose is to reconstruct the original signal from bandwidth-limited output signal. Let us consider about the system drawn in Fig. 1. Assume that original signal is the discrete sequence $\{x_t\}_{t=0,1,\cdots,n-1}$. It also be represented as

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$\mathbf{x} \in \mathbb{R}^n$ by aranging the all elements in the order of time. This signal is modified to continuous signal by zero-order holder (ZOH, [14]) like

$$x(t) = \sum_{k=0}^{n-1} x_k \left\{ u(t-kT) - u(t-(k+1)T) \right\}, \tag{1}$$

where $u(t)$ is the unit step function

$$u(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t \geq 0) \end{cases} \tag{2}$$

and $T > 0$ is the holding time. This original signal is then processed by linear and time invariant analog subsystem whose transfer function is written as $G(s)$. After that, it is sampled by time bins of T and is observed as $\{y_t\}_{t=0,1,\cdots n-1}$, which also represented as $\mathbf{y} \in \mathbb{R}^n$. Then we can obtain the transfer function of total system as

$$\begin{aligned} H(z) =& \mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right]\Big|_{t=0,T,\cdots,(n-1)T} \right] \\ &- z^{-1}\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right]\Big|_{t=0,T,\cdots,(n-1)T} \right] \qquad (3) \\ =:& \mathcal{Z}\left[\frac{G(s)}{s}\right] - z^{-1}\mathcal{Z}\left[\frac{G(s)}{s}\right], \qquad (4) \end{aligned}$$

where $\mathcal{Z}$ and $\mathcal{L}$ represent z transform and Laplace transform respectively (see appendix for derivation).

$\mathcal{Z}\left[\frac{G(s)}{s}\right]$ corresponds to the discretized step response function of subsystem because $\mathcal{Z}\left[\frac{1}{s}\right]$ corresponds to the step function. That means, with letting $\{p_t\}$ as discretized (sampled by time bins of T) step response function, $h_t = p_t - p_{t-1}$ holds. Now we can get

$$y_t = \mathcal{Z}^{-1}\left[X(z)H(z)\right] = x_t * h_t = \sum_{k=0}^{t} x_{t-k} h_k, \tag{5}$$

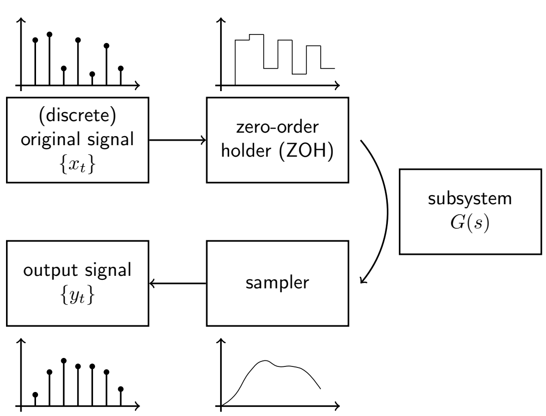

Fig. 1. The diagram which describe system. The discrete original signal $\{x_t\}$ is modified to continuous signal by ZOH, processed by subsystem, sampled, and the discrete signal $\{y_t\}$ is outputed.

which is written by matrix multiplication form as

$$\begin{aligned} \begin{pmatrix} \\ \mathbf{y} \\ \\ \end{pmatrix} &= \begin{pmatrix} h_0 & & & \\ h_1 & h_0 & & 0 \\ h_2 & h_1 & h_0 & \\ \vdots & & & \ddots \\ h_{n-1} & & \cdots & & h_0 \end{pmatrix} \begin{pmatrix} \\ \mathbf{x} \\ \\ \end{pmatrix} \qquad (6) \\ &=: H\mathbf{x}. \qquad (7) \end{aligned}$$

Since $h_t = p_t - p_{t-1}$, we can calculate output signal by using step response of subsystem. We call $\{h_t\}$ as discretized impulse response because $\{h_t\}$ is the output for the discretized unit impulse $\{x_t\} = \{1, 0, \cdots\}$ and the output signal is calculated by convoluting an original signal with h_t.

B. Analysis of Naive Method

It is valuable to consider whether it is possible to reconstruct $\mathbf{x}$, the original signal when we observe the output signal $\mathbf{y}$. If subsystem $G(s)$ limites the bandwidth, it is obviously impossible. Below, we review this and see how to tackle this.

The problem is, for given $\mathbf{y} \in \mathbb{R}^n$ and $H \in \mathbb{R}^{n \times n}$, to find the $\mathbf{x} \in \mathbb{R}^n$ which satisfies $\mathbf{y} = H\mathbf{x}$. It is obvious that if H is invertible, $\mathbf{x} = H^{-1}\mathbf{y}$ is the unique solution of this problem. Since H is lower triangular matrix, this condition is equivalent to $h_0 \neq 0$ or $p_0 \neq 0$, which indicates the infinitely fast rising of step response function. If $p_0 = 0$, all eigenvalues of H are equal to 0. So, above equation never has the unique solution, it means that we are not able to determin original signal $\mathbf{x}$ from observed signal $\mathbf{y}$.

If p_0 is not exactly equal to 0, however, above equation still remaines unsolvable due to its large condition number. The matrix like H is known as Toeplitz matrix, and its singular values were investigated three decades ago [15], [16]. According to them, the singular values of H are asymptotically equally distributed as absolute value of discrete time Fourier transform of $\{h_t\}$. More concretely, let $f(\omega)$ be the discrete time Fourier transform of $\{h_t\}$, $f(\omega) = \sum_{k=0}^{n-1} h_k e^{-i\omega k}$, and F be an arbitrary analytic function. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} F(\sigma_k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(|f(\omega)|) d\omega \tag{8}$$

where $\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_{n-1} \geq 0$ represents the singular values of H. Simplly assuming that n remains finite, the two sequences $\{\sigma_k\}$ and $\{|f(2\pi k/n - \pi)|\}$ must have absolutely the same elements each other (but the order of elements may be different) in order to keep relation (8) satisfied for an arbitrary function F. Therefore this property called asymptotically equal distribution.

Now, for the Fourier coefficients, noticing that $e^{i\omega n} = 1$ for $\omega = 2\pi k/n$ (where k is arbitrary integer), we can see

$$f(\omega) = \sum_{k=0}^{n-1} h_k e^{i\omega(n-k)} = h_t * e^{i\omega t}, \tag{9}$$

which indicates that $f(\omega)$ is deemed to the output signal for tone signal input, so that frequency response function. So we can estimate singular values of H from frequency response function of system.

Let us check this relation with a simple example. Assume that subsystem G is the LPF and H is the matrix which represents the linear transformation mapping original signal $\mathbf{x}$ to output $\mathbf{y}$. When we measure the step response function with time bins of $1\ \mu$s, then tone signal for $\omega = \pi$ represents the 500 kHz tone input. Equation (8) and above discussion claims that the distribution of singular values are same as frequency response function at the range of $f \in [0\text{ kHz}, 500\text{ kHz}]$. If sampling number $n = 10000$, there is n singular values of H and it must have the absolutely same value as frequency response function sampled at $500k/n$ kHz $= 50k$ Hz for $k = 0, 1, \cdots, 10000-1$. By assuming the frequency response function of LPF is ordered descending order, we can obtain the simplest relation $\sigma_k \simeq f(50k)$.

We measured this by using real LPF (Thorlabs EF120, f(1 dB)$=$ 10 kHz). Step response function is measured with time bins of $1\ \mu$s and $n = 10000$ by inputting 10 Hz square wave whose amplitude is 200 mV to LPF. Then we calculate matrix H and its singular values (Fig. 2 (a)). The frequency response function (Fig. 2 (b)) is measured for the same LPF, by inputting sinusoidal wave whose amplitude is 200 mV. Ignoring the long tail in (a) which is caused by small condition number of H, we can see these two graphs decays similarly.

To conclude this section, let us back to first question, whether $\mathbf{y} = H\mathbf{x}$ is solvable. This equation is known to unsolvable if the condition number σ_0/σ_{n-1} is large. Now, we see that singular values of H is asymptotically equally distributed as frequency response function of system. So if bandwidth limitation occurred, the smallest singular value σ_{n-1} is extremely small and it leads to large condition number. Therefore we can conclude that for the system with bandwidth limitation, we never be able to directly obtain original signal from output signal.

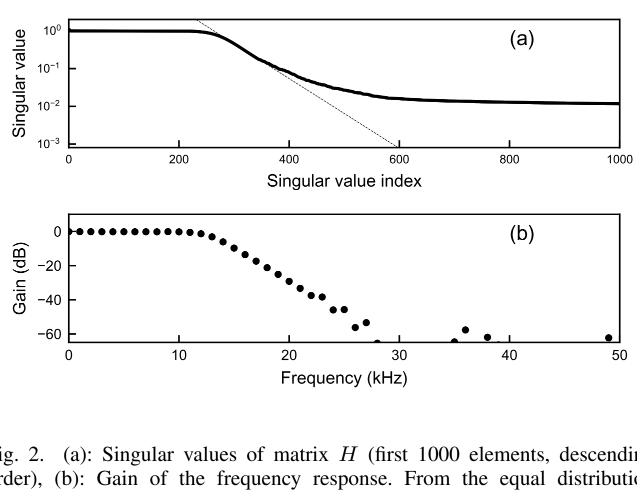

Fig. 2. (a): Singular values of matrix H (first 1000 elements, descending order), (b): Gain of the frequency response. From the equal distribution of singular values and Fourier coefficients, singular value indexed by n correspondes to gain (absolute value) of frequency response function at $f = 50n$ [Hz] (in this case). The dashed line in (a) is the linear fitting of the points which is calcurated using the point whose index is in the range of $[300, 350]$. Although small singular values of H is not properly calcurated, this line implies that singular values and gains are decreasing similarly.

C. Reconstruction Framework

Now let us describe the framework for reconstruction. The procedure of the framework is shown in Fig. 3.

To estimate the original signal from output signal, we must first convert the frequency of the signal outside the band and mix them with the signal in the passing band. In our framework, we multiply the random white noise signal to the original signal. As well known, the multiplication of two sinusoidal wave is given by summation of two waves whose frequency are the sum and the difference of frequencies of two waves, respectively. Then the multiplication of the original signal and the white noise signal, which contains finite components in each frequency, would spread one frequency component of the original signal on the entire range of frequency domain.

This process still linear on each time, although it is not time invariant because of time varying random signal, so we also be able to write this process in matrix multiplication. The original signal is multiplied with random signal $r(t)$ (and sampled random signal $\{r_t\}$) component-wisely. This can be represented as multiplication of the vector and the diagonal matrix, i.e.,

$$\text{(multiplied signal)} = R\mathbf{x}$$

where

$$R = \text{diag }(\{r_t\}) = \begin{pmatrix} r_0 & 0 & \cdots & 0 \\ 0 & r_1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & r_{n-1} \end{pmatrix}. \tag{10}$$

Using this, our problem changes its form a little, and is stated as,

$$\text{find the solution of } \mathbf{y} = HR\mathbf{x}.$$

This problem is, unfortunally, still unsolvable in direct way. It is intuitively described as below; This process mixes each frequency components randomly, i.e. let $\tilde{x}_1, \cdots \tilde{x}_{n-1}$ be discrete Fourier coefficients, and then the first discrete Fourier coefficient of $R\mathbf{x}$ would be

$$(\tilde{R\mathbf{x}})_0 = a_{00}\tilde{x}_0 + a_{01}\tilde{x}_1 + \cdots + a_{0(n-1)}\tilde{x}_{n-1},$$

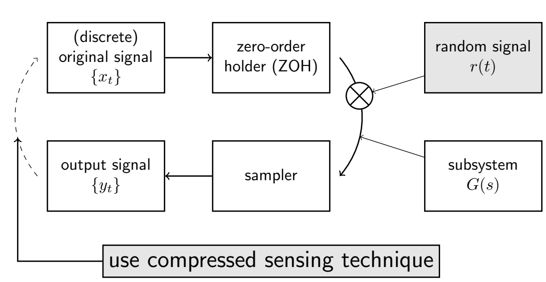

Fig. 3. System architecture to reconstruct the original signal. In this framework, random signal is multiplied to convert the frequency of the signal outside the band. To reconstruct the original signal, compressed sensing technique is used.

where a_{ij} is randomly distributed constant depending on random sequence r_t, and similarly,

$$
\begin{aligned}
(\tilde{R\mathbf{x}})_1 =& a_{10}\tilde{x}_0 + a_{11}\tilde{x}_1 + \cdots a_{1(n-1)}\tilde{x}_{n-1} \\
&\vdots \\
(\tilde{R\mathbf{x}})_{n-1} =& a_{(n-1)0}\tilde{x}_0 + a_{(n-1)1}\tilde{x}_1 + \cdots a_{(n-1)(n-1)}\tilde{x}_{n-1}.
\end{aligned}
$$

So the problem finding the solution of $\mathbf{y} = R\mathbf{x}$ is linear equation number of whose variables is n and number of whose constraints is also n. This is of course solvable since a_{ij} is randomly distributed. Then multiplying H with the right hand side of equation causes strong attenuation of some of the Fourier component of $R\mathbf{x}$, which can be assumed that some of constraints which appeares in previous problem does not appeare in the problem $\mathbf{y} = HR\mathbf{x}$. This indicates that these constraints are no longer obtained from the signal processed by bandwidth-limiting subsystem. Dropping off some constraints prevents from finding unique solution of linear equation, which means that we are still not be able to estimate original signal properly from output signal.

This type of problem, namely the linear equation whose some constraints are dropped off is known to be tackled with the compressed sensing technique. Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{y} \in \mathbb{R}^m$, and $m < n$. Compressed sensing technique then find the sparsest solution $\mathbf{x} \in \mathbb{R}^n$ of the linear equation $\mathbf{y} = A\mathbf{x}$, i.e.,

$$\mathbf{x}^{\text{est}} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} ||\mathbf{x}||_0 \text{ subject to } \mathbf{y} = A\mathbf{x} \tag{11}$$

where

$$||\mathbf{x}||_0 = \lim_{p\to 0}\left(\sum_i x_i^p\right)^{1/p} = (\#\text{ of nonzero elements in } \mathbf{x})$$

is the ℓ_0 norm (or sparsity) of $\mathbf{x}$.

Although $\mathbf{y} = A\mathbf{x}$ doesn't have the unique solution because number of constraints m is strictly smaller than the number of unknown variables n, it is known [9], [17] that if the problem (11) has the sufficiently sparse solution with respect to particular value depending on A, then that solution is unique. This indicates that if the vecter in our interest $\mathbf{x}_0$ satisfies the equation $\mathbf{y} = A\mathbf{x}_0$ and sufficiently sparse, then we can find $\mathbf{x}_0$ by finding the unique solution of (11).

While the signals in real world are not necessarily sparse, we can apply the compressed sensing technique by adding small modification on (11), like,

$$\mathbf{x}^{\text{est}} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} ||\Psi(\mathbf{x})||_0 \text{ subject to } \mathbf{y} = A\mathbf{x} \tag{12}$$

where Ψ, the dictionary, is the known transformation which makes $\mathbf{x}$ sparse.

This problem still has two issues. First, the condition $\mathbf{y} = A\mathbf{x}$ is too strict to satisfy. Second, the problem (11) and (12) are computationally difficult because these are optimization problems on all allowable combination of $\mathbf{x}$ indexes. To settle this matters, allowing linear constraint to a little difference ϵ in terms of ℓ_2 (Euclidean) norm, and changing the ℓ_0 norm to ℓ_1 (absolute-value) norm to make above problems simple linear programing problems are presented in previous works, which results in,

$$\mathbf{x}^{\text{est}} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} ||\Psi(\mathbf{x})||_1 \text{ subject to } ||\mathbf{y} - A\mathbf{x}||_2 < \epsilon, \tag{13}$$

or its Lagrange dual problem

$$\mathbf{x}^{\text{est}} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \left\{\lambda||\Psi(\mathbf{x})||_1 + \frac{1}{2}||\mathbf{y} - A\mathbf{x}||_2\right\}, \tag{14}$$

which is called as least absolute shrinkage and selection operator (LASSO) [18]. LASSO is also known [19] to give the unique solution if $\mathbf{x}$ is sufficiently sparse, but the condition is a bit stricter than the case (11).

Here, our framework is below; Our purpose is to estimate the discrete original signal $\mathbf{x}$. We can observe it only after processed by bandwidth-limiting subsystem $G(s)$. To achieve above purpose, we measure the step response of total system and calcurate the transfer matrix H in advance. And to avoid for frequency component outside the band diminishing, we mixes the random white noise signal $r(t)$ to original signal and observe output signal. Let this output signal be $\mathbf{y}$. The matrix R is calculated by this random signal. Now, we apply the compressed sensing technique, i.e. solve the optimization problem

$$\mathbf{x}^{\text{est}} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \left\{\lambda||\Psi(\mathbf{x})||_1 + \frac{1}{2}||\mathbf{y} - HR\mathbf{x}||_2\right\}. \tag{15}$$

Under the assumption that orginal signal is sufficiently sparse under the transformation by Ψ, it is guaranteed that $\mathbf{x}^{\text{est}}$ is sufficiently close to $\mathbf{x}$ from compressed sensing theory.

III. Experimental Results

We performed the experiment to demonstrate our new framework. The setup is shown in Fig. 4. The function generator (Hewlett-Packard 8116A) was used for generate original signal which is represented as $\mathbf{x}$. Besides, the field programmable gate array (FPGA, Xilinx Spartan-6) was configured to generate pseudorandom numbers with linear congruential generators. This sequence was transmitted to digital to analog converter (DAC), which consequently generated the random signal that contains wide range of frequencies. These two signals are multiplied at mixer (Analog Devices AD734), and the high frequency components in this signal was attenuated by passive LPF (Thorlabs EF120). This bandwidth limited signal was observed at oscilloscope (Tektronix MDO3104) and used for reconstruction as output signal $\mathbf{y}$, in above context. Likewise, random signal was branched into the oscilloscope and used for the reconstruction. The original signal was also branched (dashed line) and observed as reference, but it was not used for reconstruction.

We also used the information of step response function which is measured before the experiment. All of these signals were measured with time bins of 1 μs. Then, we estimated the output signal by solving the equation (15) with fast iterative shrinkage-thresholding algorithm [20]. We used discrete Fourier transform for the dictionary Ψ. Remainig parameter, the regularization constant λ, was determined by minimizing cross validation error.

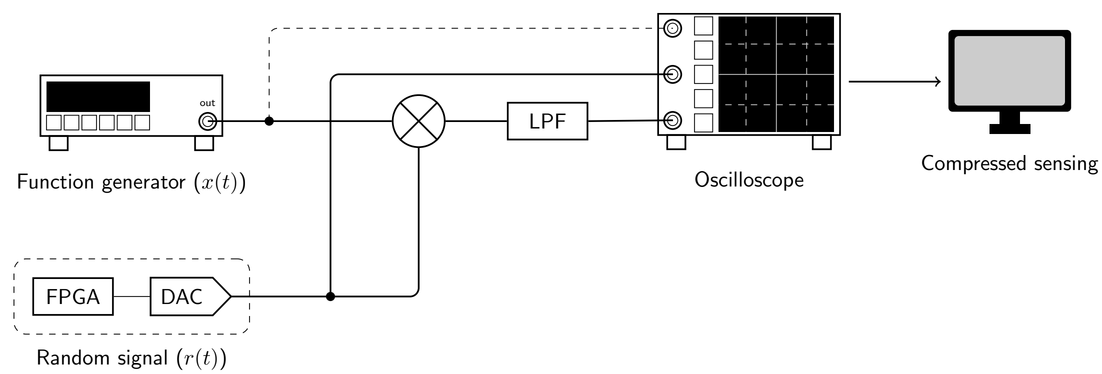

Fig. 4. The experimental apparatus used to demonstrate our framework. The original signal which is represented as $\mathbf{x}$ in main text is generated by function generator. The random signal are generated with FPGA and DAC as described in main text. The oscilloscope measure three signals; the input signal (for reference), the random signal and the output signal of LPF which is represented as $\mathbf{y}$. All data observed at oscilloscope are passed to compressed sensing process.

To improve the accuracy of estimated value, we added some modification on above framework. When we can genarate the same signal repeatedly, we can multiply several random signals to original signal and can observe several output signal. This circumstance can be represented by adding a simple change on our model. Suppose that m random signals $\{r_t^{(1)}\}, \cdots, \{r_t^{(m)}\}$ are multiplied to original signal. We can construct diagonal matrices correspond to each of random signal. These are represented as $R^{(1)}, \cdots, R^{(m)}$. The output signal varies on random signals, and let us represent them as $\{y_t^{(1)}\}, \cdots \{y_t^{(m)}\}$. Then, we denote another variables

$$\mathbf{y}' = \begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \\ \vdots \\ \mathbf{y}^{(m)} \end{bmatrix} \tag{16}$$

$$H' = \begin{bmatrix} H & & & \\ & H & & \\ & & \ddots & \\ & & & H \end{bmatrix} \tag{17}$$

$$R' = \begin{bmatrix} R^{(1)} \\ R^{(2)} \\ \vdots \\ R^{(m)} \end{bmatrix}, \tag{18}$$

and we can modify the equation (15) like

$$\mathbf{x}^{\text{est}} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \left\{ \lambda||\Psi(\mathbf{x})||_1 + \frac{1}{2}||\mathbf{y}' - H'R'\mathbf{x}||_2 \right\}. \tag{19}$$

We are also able to solve this new problem using typical compressed sensing technique.

Fig. 5 shows the result for this demonstration. While the output signal processed by LPF (dashed line) has no high frequency components, We were able to estimate them by compressed sensing technique. To evaluate our results, let us define fidelity as $F = \exp(-||\mathbf{x} - \mathbf{x}^{\text{est}}||_2/||\mathbf{x}||_2)$ which takes value in the range of $[0, 1]$ and which is 1 only if $\mathbf{x} = \mathbf{x}^{\text{est}}$. The sinusoidal wave, which is extremely sparse on the basis of discrete Fourier transform, was estimated with high fidelity of $F = 0.894$. We also conducted experiment for triangular wave input, which is also sparse but contains small components. Although the estimated wave is a bit deformed when compared with the former result, but still has enough fidelity of $F = 0.858$.

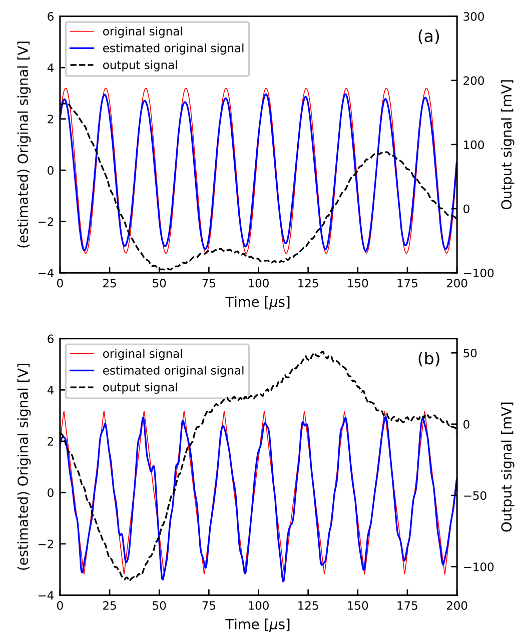

Fig. 5. Results for demonstration for (a): 50 kHz sinusoidal wave input. (b): 50 kHz triangular wave input. In both of graphs, dashed line is one of output signals, which is denoted as $\mathbf{y}$ in main text. Red (thin) line is original signal ($\mathbf{x}$), and blue (thick) line is estimated original signal ($\mathbf{x}^{\text{est}}$). All of them are showed only first 1000 data. Fidelity of estimated signal is calculated as $F = 0.894$ for sinusoidal wave input and $F = 0.858$ for triangular wave input.

Let us compare these results on the sparse basis, so that discrete Fourier coefficients. Low frequency components of spectrum are shown in Fig. 6. When input wave is 50 kHz sinusoidal wave, its Fourier coefficients takes non-zero value only at 50 kHz (and its aliasing frequencies). Fig. 6 (a) indicates this fact and (b) implies that we were successfully

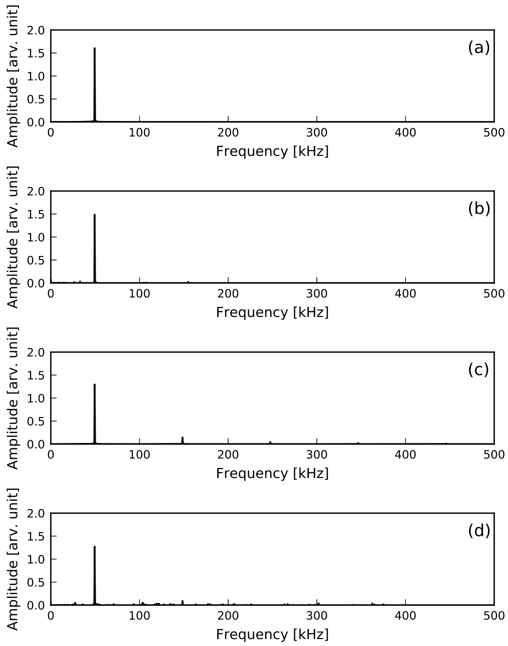

Fig. 6. Absolute value of discrete Fourier coefficients normalized with length of the vector for (a): sinusoidal wave input (Fig. 5 (a), red (thin) line), (b): estimated sinusoidal wave (Fig. 5 (a), blue (thick) line), (c): triangular wave input (Fig. 5 (b), red (thin) line), (d): estimated triangular wave (Fig. 5 (b), blue (thick) line)

reconstruct this signal with our framework. The triangular wave input case, Fourier coefficients of input wave takes non-zero value at 50 kHz, 150 kHz, …and so on. These coefficients are, however, decrease with the square of the frequency. From our experimental results Fig. 6 (c) and (d), we were able to extract the first (50 kHz) and second (150 kHz) dominant Fourier coefficients, and this led to enough fidelity.

IV. Conclusion

We showed the framework to reconstruct the signal which is processed by bandwidth limiting subsystem. To achive this, we first multiply the original signal and spread the frequency components on the spectrum domain. And then, we use the compressed sensing technique and estimate the original signal. We implement this framework with some electronic devices and showed that we can extract major coefficients of sparse basis with our framework. This highly general frameworks can be applied to various fields of natural science and help to overcome bandwidth limitation.

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Derivation of Equation (4)

From equation (1), the Laplace transform of $x(t)$ is calculated as

$$X(s) = \frac{(1-e^{-Ts})}{s}\sum_{k=0}^{n-1} x_k e^{-kTs}. \tag{20}$$

This signal is processed by subsystem and the output signal $y(t)$ is

$$y(t) = \mathcal{L}^{-1}\left[X(s)G(s)\right]. \tag{21}$$

Now we use the relation

$$\sum_{k=0}^{n-1} x_k e^{-kTs} = \int_0^\infty \sum_{k=0}^{n-1} x_k \delta(t-kT) e^{-ts} dt \tag{22}$$

$$= \mathcal{L}\left[\sum_{k=0}^{n-1} x_k \delta(t-kT)\right] \tag{23}$$

where $\delta(t)$ represents Dirac delta function, $y(t)$ is written as

$$y(t) = \mathcal{L}^{-1}\left[\frac{(1-e^{-Ts})G(s)}{s}\mathcal{L}\left[\sum_{k=0}^{n-1} x_k\delta(t-kT)\right]\right] \tag{24}$$

$$= \mathcal{L}^{-1}\left[\frac{(1-e^{-Ts})G(s)}{s}\right] * \left(\sum_{k=0}^{n-1} x_k\delta(t-kT)\right). \tag{25}$$

$*$ represents the convolution $f(t) * g(t) = \int f(t')g(t-t')dt'$. Sampling $y(t)$ by time bins of T, then $\{y_t\}$ becomes

$$y_k = \int y(t)\delta(t-kT)dt \tag{26}$$

$$= \iint \mathcal{L}^{-1}\left[\frac{(1-e^{Ts})G(s)}{s}\right](t-t') \sum_{k'=0}^{n-1} x'_k \delta(t'-k'T)\delta(t-kT)dtdt' \tag{27}$$

$$= \sum_{k'=0}^{n-1} x_{k'} \mathcal{L}^{-1}\left[\frac{(1-e^{Ts})G(s)}{s}\right](kT-k'T). \tag{28}$$

By calculating z transform of $\{y_t\}$,

$$Y(z) = \sum_{k=0}^{n-1} y_k z^{-k} \tag{29}$$

$$= \sum_{k=0}^{n-1} \mathcal{L}^{-1}\left[\frac{(1-e^{Ts})G(s)}{s}\right](kT) z^{-k} \sum_{k'=0}^{n-1} x_k z^{-k'} \tag{30}$$

$$= \left(\mathcal{Z}\left[\frac{G(s)}{s}\right] - z^{-1}\mathcal{Z}\left[\frac{G(s)}{s}\right]\right)X(z), \tag{31}$$

we can get the transfer function (4).

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