Inner product

If $V_1 = a_1 i + b_1 j + c_1 k$ and $V_2 = a_2 i + b_2 j + c_2 k$ are 2 vectors

then the inner product of

$$V_1 \cdot V_2 = a_1 a_2 + b_1 b_2 + c_1 c_2$$

eg: $U = 2i + 3j + k$

$V = 4i - 5j - 3k$

$$U \cdot V = (2 \times 4) + (3 \times {}^{-}5) + (1 \times {}^{-}3)$$

$$= 8 - 15 - 3$$

$$\boxed{= -10}$$

Orthogonal function

2 functions f_1 and f_2 are said to be orthogonal on an interval $[a, b]$

$$\boxed{f_1 f_2 = \int_a^b f(x_1) f(x_2)\, dx = 0}$$

eg: $f(x_1) = x^2$

$f(x_2) = x^3$

$[-1, 1]$

$$\int_{-1}^{1} x^2 x^3\, dx = \int_{-1}^{1} x^5\, dx$$

$$= \left.\frac{x^6}{6}\right]_{-1}^{1} = \frac{1}{6} - \frac{1}{6} = 0$$

orthonormal function

$$\boxed{f_1 f_2 = \int_a^b f(x_1)\, f(x_2)\, dx = 1}$$

? ST the set $\{1, \cos x, \cos 2x, \ldots\}$ is orthogonal on the interval $[-\pi, \pi]$

an: $f(x_1) = 1$

$f(x_2) = \cos x$

$$f_1 f_2 = \int_{-\pi}^{\pi} 1 \cdot \cos x \, dx$$

$$= \int_{-\pi}^{\pi} \cos x \, dx = \sin x \Big]_{-\pi}^{\pi}$$

$$= \sin \pi - \sin(-\pi) = 0$$

$$\boxed{\cos a \cos b = \frac{\cos(a+b) + \cos(a-b)}{2}}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \int_{-\pi}^{\pi} \frac{\cos(m+n)x + \cos(m-n)x}{2}$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi}$$

$$= \underline{\underline{0}}$$

? ST the set $\{1, \cos x, \cos 2x, \ldots\}$ is orthonormal on the interval $[-\pi, \pi]$

an: $$\|\phi_n(x)\|^2 = \int_a^b \phi_n(x)\, dx = \int_{-\pi}^{\pi} 1\, dx = [x]_{-\pi}^{\pi}$$

$$= \pi + \pi = 2\pi$$

$$\|\phi_n x\|^2 = 2\pi$$

$$\|\phi_n x\| = \sqrt{2\pi}$$

$$\|\phi_n x\|^2 = \int_a^b \phi_n x \, dx = \int_{-\pi}^{\pi} \cos^2 x \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 1 + \frac{\cos 2x}{2}\, dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\pi + 0 - -\pi + 0\right] = \underline{\underline{\cancel{\sqrt{\pi}}}} \ \pi$$

$$\|\phi_n x\| = \sqrt{\pi}$$

$$\|\phi_n x\|^2 = \int_a^b \phi_n x \, dx = \int_{-\pi}^{\pi} \cos^2 nx \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos 2nx \, dx = \frac{1}{2}\left[x + \frac{\sin 2nx}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\pi + 0 - -\pi + 0\right] = \underline{\underline{\sqrt{\pi}}}$$

? ST the given functions are orthogonal or not on the interval

1 $f_1(x) = x^3$

$f_2(x) = x^2 + 1$ $\quad [-1, 1]$

an: $$\int_{-1}^{1} x^3 \cdot (x^2 + 1)\, dx$$

$$= \int_{-1}^{1} x^5 + x^3 \, dx = \left[\frac{x^6}{6} + \frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{1}{6} + \frac{1}{4} - \left[\frac{1}{6} + \frac{1}{4}\right] = \underline{\underline{0}}$$

? ST the set of given function orthonormal or not. If not find its norm.

$\{\cos x, \cos 3x, \cos 5x, \ldots\}$ $[0, \frac{\pi}{2}]$

$\| \phi_n x \|^2 = \int_a^b \phi_n x \, dx$

$= \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$= \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$

$= \cancel{\frac{1}{2}\left[\frac{\pi}{2} + 0 - [0+\right.}$

$= \frac{1}{2}\left[\frac{\pi}{2} + 0 - [0+0]\right]$

$= \frac{\pi}{4}$

$\| \phi_n x \| = \frac{\sqrt{\pi}}{2}$

$\| \phi_n x \|^2 = \int \phi_n x \, dx = \int_0^{\frac{\pi}{2}} \cos^2 (2n-1) x \, dx$

$= \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2(2n-1)x}{2} dx$

$= \int_0^{\frac{\pi}{2}} \frac{1 + \cos 4nx - \cos 2x}{2} dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \cos 4nx - \cos 2x \, dx$

$= \frac{1}{2}\left[x + \frac{\sin 4nx}{4n} - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$

$\frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin 4n \times \frac{\pi}{2}}{4n} - 0\right]$

$= \frac{1}{2}\left[\frac{\pi}{2}\right]$

$= \frac{\pi}{4}$

$\| \phi_n x \| = \frac{\sqrt{\pi}}{2}$

$\left\{ \frac{\cos x}{\frac{\sqrt{\pi}}{2}}, \frac{\cos 3x}{\frac{\sqrt{\pi}}{2}}, \frac{\cos 5x}{\frac{\sqrt{\pi}}{2}} \ldots \right\}$

? $\{\sin nx\}$ $n = 1, 2, 3, \ldots$ on $[0, \frac{\pi}{2}]$

an: $\| \phi_n x \|^n = \int_a^b \phi_n x \, dx$ $\qquad \sin^2 x = \frac{1 - \cos 2x}{2}$

$= \int_0^{\frac{\pi}{2}} \sin^2 nx \, dx$

orthogonal series Expansion

suppose $\{\phi_n(x)\}$ is a infinite orthogonal set function $[a,b]$ the series

$$f(x) = c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_n\phi_n(x) = \sum_{n=0}^{\infty} (c_n\phi_n x)$$

where $$c_n = \frac{\int_a^b f(x)\,\phi_n(x)\,dx}{\int_a^b \phi_n(x)^2\,dx} \qquad n = 1, 2, 3, \dots$$

orthogonal set / weight function

A set of real valued function $\{\phi_0(x), \phi_1(x), \phi_2(x) \dots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on an interval $[a,b]$, $\int_a^b w(x)\,\phi_m(x)\,\phi_n(x)\,dx = 0$

complete sets

complete set is a set of functions such that the only continues function orthogonal to each member of the set is the zero function

Fourier series

The fourier series of a function f defined on a interval $(-p, p)$ is given by

$$\boxed{f(x) = \frac{a_0}{2} \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)}$$

$$\boxed{a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx} \qquad \cancel{a_n} = \frac{1}{p}$$

$$\boxed{a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx}$$

$$\boxed{b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx}$$

? Expand $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi - x & 0 \le x < \pi \end{cases}$ in fourier series.

an: $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi} \times \frac{\pi^2}{2}$$

$$= \frac{\pi}{2}$$

Integration by parts

$$\boxed{\int uv\,dx = u\int v\,dx - \int\left(u'\int v\,dx\right)dx}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos\frac{n\pi}{p}x\,dx + \int_0^{\pi}(\pi-x)\cos\frac{n\pi}{p}x\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi}(\pi-x)\cos\frac{n\pi}{\pi}x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\pi-x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi-x)\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left(\frac{d}{dx}(\pi-x)\cdot\int\cos nx\,dx\right)dx\right]$$

$$= \frac{1}{\pi}\left[(\pi-x)\left[\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}-1\left(\int\cos nx\,dx\right)dx\right]$$

$$= \frac{1}{\pi}\left[(\pi-x)(0-0) + \int_0^{\pi}\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{\sin nx}{n}\,dx = \frac{1}{\pi n}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= -\frac{1}{n^2\pi}\left[(-1)^n-1\right] = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin\frac{n\pi}{p}x\,dx + \int_0^{\pi}(\pi-x)\sin\frac{n\pi}{p}x\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi}(\pi-x)\sin\frac{n\pi}{\pi}x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\pi-x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi-x)\int_0^{\pi}\sin nx\,dx - \int_0^{\pi}\left(\frac{d}{dx}(\pi-x)\cdot\int\sin nx\,dx\right)dx\right]$$

$$= \frac{1}{\pi}\left[(\pi-x)\left[\frac{-\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi}-1\times\frac{-\cos nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[-(\pi-x)\left[\frac{\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-(\pi-x)}{n}\left[(-1)^n-1\right] - \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right]$$

$$= \cancel{\frac{1}{\pi}\left[\frac{-(\pi-x)}{n}\left[(-1)^n-1\right]\right] - 0}$$

$$= \frac{1}{\pi}\left[0-0-\pi\left(\frac{1}{n}\right)-0\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\right] = \frac{1}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x + b_n\sin\frac{n\pi}{p}x\right]$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right]$$

? Expand ~~f(x) = x²~~ the fourier series

$f(x) = x^2$ in $-1 < x < 1$

an: $a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$

$$= \frac{1}{1}\int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$= \frac{1}{1}\int_{-1}^{1} x^2\cos n\pi x\,dx$$

$$= x^2\int \cos n\pi x - \int \frac{d}{dx}x^2 \int \cos n\pi x\,dx$$

$$= x^2 \left.\frac{\sin n\pi x}{n\pi}\right] - \int 2x\left[\frac{\sin n\pi x}{n\pi}\right] dx$$

$$= x^2 \frac{\sin n\pi x}{n\pi} - \frac{2}{n\pi}\left[x \frac{-\cos n\pi x}{n\pi} - \int \frac{-\cos n\pi x}{n^2\pi^2}\right.$$

$$= \frac{x^2\sin n\pi x}{n\pi} - \frac{2}{n\pi}\left[\frac{\cos n\pi x}{n\pi} - \frac{x\cos n\pi x}{n^2\pi^2}\right]$$

$$2\left[\frac{x^2\sin n\pi x}{n\pi} - \frac{2}{n^2\pi^2}\left[\cos n\pi x - \frac{x \cancel{\cos}\overset{\sin}{} n\pi x}{n\pi}\right]\right]_0^1$$

$$\cancel{2\left[0 - \frac{2}{n^2\pi}\left((-1)^n - (-1)^n\right]\right] - \left[-\frac{2}{n^2\pi}(1)\right]}$$

$$\cancel{2\left[\frac{2}{n^2\pi} - \frac{2}{n^2\pi}(-1)^n - (-1)^n\right]}$$

$$2\left[\frac{x^2\sin n\pi x}{n\pi} - 2x\frac{\cos n\pi x}{n^2\pi^2} + 2\frac{\sin n\pi}{n^3\pi^3}x\right]_0^1$$

$$= 2\left[2\times 0 - \frac{2(-1)^n}{\pi^2 n^2} + 0 - 0\right]$$

$$= 2\left[-\frac{2(-1)^n}{n^2\pi^2}\right] = \frac{-4(-1)^n}{n^2\pi^2} = \frac{-4(-1)^n}{n^2\pi^2}$$