Laplace Transforms

Let 'f' be a function defined for $t \geq 0$ the integral

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ is said to be laplace transforms of 'f', provided the integral converg ence. It is denoted by $L\{f(t)\}$ / $L(f)$ / $F(s)$

$L\{f(t)\} = F(s)$

? Evaluate $L\{1\}$ or Find the laplace Transform of $f(t) = 1$

an: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$L\{1\} = \int_0^\infty e^{-st} \times 1\, dt = \int_0^\infty e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{-e^{-s\times\infty}}{s} + \frac{e^{-s\times 0}}{s} = \frac{-e^{-\infty}}{s} + \frac{e^0}{s} = \frac{0}{s} + \frac{1}{s}$$

$$= \underline{\underline{\frac{1}{s}}}$$

? $L\{t\}$.

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$L\{t\} = \int_0^\infty e^{-st} \times t\, dt$

$$= \int_0^\infty t \cdot e^{-st}\, dt$$

$$= t \cdot \left.\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt$$

$$= -\frac{1}{s} t \cdot e^{-st} \Big]_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt$$

$$= -\frac{1}{s} t e^{-st} \Big]_0^\infty + \frac{1}{s} \times \frac{e^{-st}}{-s} \Big]_0^\infty$$

$$= -\frac{1}{s} \left[t e^{-st} + \frac{1}{s} e^{-st} \right]_0^\infty$$

$$= -\frac{1}{s} \left[\infty \times e^{-s\times\infty} + \frac{1}{s} \times e^{-s\times\infty} \right] - \left(0 \cdot e^{-s\times 0} + \frac{1}{s} \times e^{-s0} \right.$$

$$= -\frac{1}{s} \left[e^{\infty} + \frac{1}{s} e^{-\infty} + 0 - \frac{1}{s} e^{0} \right]$$

$$= -\frac{1}{s} \left[0 + \frac{1}{s} \times 0 + 0 - \frac{1}{s} \times 1 \right]$$

$$= -\frac{1}{s} \left[-\frac{1}{s} \right]$$

$$= \frac{1}{s^2}$$

? ~~$L\{a$~~

? $L\{e^{at}\}$

an: $L\{f(t)\} = \int_0^\infty e^{-st} e^{at} dt$

$$= \int_0^\infty e^{-st+at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \frac{-e^{-(s-a)t}}{s-a} \Big]_0^\infty$$

$$= \frac{-e^{-(s-a)\times\infty}}{s-a} + \frac{e^{-(s-a)\times 0}}{s-a} \Big]$$

$$= \frac{-e^{-\infty}}{s-a} + \frac{e^{-0}}{s-a}$$

$$= \frac{0}{s-a} + \frac{1}{s-a} = \frac{1}{s-a}$$

This result is valid for $s-a>0$

$\Rightarrow s>a$

? Evaluate a) e^{-3t} b) e^{6t}

an: $L\{e^{-3t}\} = \int_0^\infty e^{-st} \times e^{-3t} dt$

$$= \int_0^\infty e^{-st-3t} dt = \int_0^\infty e^{-(s+3)t} dt$$

$$= \frac{-e^{-(s+3)t}}{s+3} \Big]_0^\infty$$

$$= \frac{-e^{-(s+3)\times\infty}}{s+3} + \frac{e^{-(s+3)\times 0}}{s+3}$$

$$= \frac{0}{s+3} + \frac{1}{s+3} = \frac{1}{s+3}$$

b) e^{6t}

$L\{e^{6t}\} = \int_0^\infty e^{-st} e^{6t} dt$

$$= \int_0^\infty e^{-(s-6)t} dt$$

$$= \frac{-e^{-(s-6)t}}{s-6} \Big]_0^\infty$$

$$= \frac{e^{-(s-6)\times\infty}}{s-6} + \frac{e^{-(s-6)\times 0}}{s-6}$$

$= \frac{1}{s-6}$

? Evaluate $L\{\sin at\}$

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$L\{\sin at\} = \int_0^\infty e^{-st} \sin at\,dt$

$= \int_0^\infty \sin at\, e^{-st}\,dt = \left[\frac{\sin at\, e^{-st}}{s}\right]_0^\infty -$

$\int_0^\infty a\cos at \times -\frac{e^{-st}}{s}\,dt$

$= 0 + 0 + \frac{a}{s}\int_0^\infty \cos at\, e^{-st}\,dt$

$= \frac{a}{s}\left[\left[\frac{(-\cos at\, e^{-st})}{s}\right]_0^\infty - \int_0^\infty -a\sin at\,\frac{e^{-st}}{-s}\,dt\right.$

$= \frac{a}{s}\left[(0 - (-\frac{1}{s}) - \frac{a}{s}\int_0^\infty \sin at\, e^{-st}\,dt\right]$

$= \frac{a}{s}\left[\frac{1}{s} - \frac{a}{s}\, L\{\sin at\}\right]$

$L\{\sin at\} = \frac{a}{s^2} - \frac{a^2}{s^2}\, L\{\sin at\}$

$L\{\sin at\} + \frac{a^2}{s^2}\, L\{\sin at\} = \frac{a}{s^2}$

$L\{\sin at\}\left[1 + \frac{a^2}{s^2}\right] = \frac{a}{s^2}$

$L\{\sin at\}\left[\frac{s^2+a^2}{s^2}\right] = \frac{a}{s^2}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$, provided $\frac{a}{s^2+a^2} > 0$

? Evaluate $L\{\sin 2t\}$

an: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$L\{\sin at\} = \cancel{\int_0^\infty e^{-st} f(t)\,dt}\ \frac{a}{s^2+a^2}$

$L\{\sin 2t\} = \frac{2}{s^2+a^2} = \frac{2}{s^2+4}$

Note :

* $\int_0^\infty e^{-st}[\alpha F(t) + \beta g(t)]\,dt$

$= \alpha\int_0^\infty e^{-st} f(t)\,dt + \beta\int_0^\infty e^{-st} g(t)\,dt$

Linearity property

? ST $L\{af(t) + bg(t)\} = aL\{f(t)\} + b\, L\{g(t)\}$.

an: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ —— ①

$L\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt$ —— ②

$L\{af(t) + bg(t)\} = \int_0^\infty e^{-st}[af(t) + bg(t)]\,dt$

$= \int_0^\infty [e^{-st} af(t) + e^{-st} bg(t)]\,dt$

$= \int_0^\infty e^{-st} af(t)\,dt + \int_0^\infty e^{-st} bg(t)\,dt$

$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$

$= aL\{f(t)\} + bL\{g(t)\}$. (by ① & ②)

? Evaluate $L\{1+5t\}$

an: $L\{1+5t\} = L\{1\} + 5\,L\{t\}$

$= \frac{1}{s} + 5 \times \frac{1}{s^2}$

$= \frac{1}{s} + \frac{5}{s^2}$

? Evaluate $L\{10e^{-3t} - 5t + 8\}$

$= 10L\{e^{-3t}\} - 5L\{t\} + 8L\{1\}$

$= 10 \times \frac{1}{s+3} - 5 \times \frac{1}{s^2} + 8 \times \frac{1}{s}$

$= \frac{10}{s+3} - \frac{5}{s^2} + \frac{8}{s}$

First shifting property

? ST IF $L\{f(t)\} = F(s)$ then prove that $L\{e^{at} f(t)\} = F(s-a)$ where $s-a > k$ for some k?

an: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = f(s)$

$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$

$= \int_0^\infty e^{-st+at} f(t)\,dt$

$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$= F(s-a)$

Transform of some basic functions

* $L\{1\} = \frac{1}{s}$
* $L\{t^n\} = \frac{n!}{s^{n+1}}$, $n = 1, 2, \ldots$
* $L\{e^{at}\} = \frac{1}{s-a}$
* $L\{\sin kt\} = \frac{k}{s^2+k^2}$
* $L\{\cos kt\} = \frac{s}{s^2+k^2}$
* $L\{\sinh kt\} = \frac{k}{s^2-k^2}$
* $L\{\cosh kt\} = \frac{s}{s^2-k^2}$
* $L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$

? Evaluate $L\{\sin^2 t\}$

an: $\sin^2 t = \frac{1-\cos 2t}{2}$

$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$

$= \frac{1}{2} L\{1-\cos 2t\}$

$= \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+2^2}\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

? Evaluate $L\{e^{a-bt}\}$

an: $L\{e^{a-bt}\} = L\{e^{a} \cdot e^{-bt}\}$

$= e^{a} L\{e^{-bt}\}$

$= e^{a} \times \frac{1}{s-(-b)}$

$= e^{a} \times \frac{1}{s+b}$

$= \frac{e^{a}}{s+b}$

? Find the Laplace transform of $(t+1)^2 e^t$

an: $L\{(t+1)^2 e^t\} = L\{(t^2+2t+1)e^t\}$

$= L\{t^2e^t + 2te^t + e^t\}$ —— ①

$= L\{t^2e^t\} + 2L\{te^t\} + L\{e^t\}$

$L\{t^2e^t\} = \frac{2!}{(s-1)^{2+1}}$

$= \frac{2}{(s-1)^3}$ —— ②

$L\{t \cdot e^t\} = \frac{1!}{(s-1)^{1+1}} = \frac{1}{(s-1)^2}$ —— ③

$L\{e^t\} = \frac{1}{s-1}$ —— ④

sub ②, ③ & ④ in ①

$L\{(t+1)^2 e^t\} = \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2} + \frac{1}{(s-1)}$

Exponential order

A function 'f' is said to be exponential order if there exist constants c, M > 0 and T > 0 such that

$\boxed{|f(t)| \leq Me^{ct}}$ for all $t > T$

Sufficient condition for existence of $L\{f(t)\}$

If $f(t)$ is piecewise ~~condition~~ continuous on the interval $[0, \infty]$ and of exponential order, then $L\{f(t)\}$ is exists for $s > c$

Proof

By the additive interval property of definite integrals,

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$

$= I_1 + I_2$

The integral I_1 exists because it can be written as the sum of integrals over intervals on which $e^{-st} f(t)$ is continuous.

Now f is of exponential order

so, there exists c, M > 0 and T > 0 so that

$|f(t)| \leq Me^{st}$ for all $t > T$

Then

$$|I_2| = \left| \int_T^\infty e^{-st} f(t)\,dt \right| \leq \int_T^\infty e^{-st} |f(t)|\,dt$$

$$\leq \int_T^\infty e^{-st} M e^{ct}\,dt$$

$$\leq M \int_T^\infty e^{-st+ct}\,dt$$

$$\leq M \int_T^\infty e^{-(s-c)t}\,dt.$$

$$\leq M \left[\frac{e^{-(s-c)t}}{(s-c)} \right]_T^\infty$$

$$\leq M \left[\frac{-e^{-\infty}}{(s-c)} + \frac{e^{-(s-c)T}}{(s-c)} \right]$$

$$\leq M \left[0 + \frac{e^{-(s-c)T}}{(s-c)} \right]$$

$$|I_2| \leq M \frac{e^{-(s-c)T}}{(s-c)}$$

for $s > c$

since $\int_T^\infty M e^{-(s-c)t}\,dt$ converges. The integral $\int_T^\infty |e^{-st} f(t)|\,dt$ converges by comparison test for improper integral.

This implies I_2 exists for $s > c$. The existence of I_1 and I_2 implies that $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ exists for $s > c$.