Enumerating classes of regular triangulations

ΕργΑ Laboratory of Algebraic and Geometric Algorithms

Vissarion Fisikopoulos

National and Kapodistrian UNIVERSITY OF ATHENS

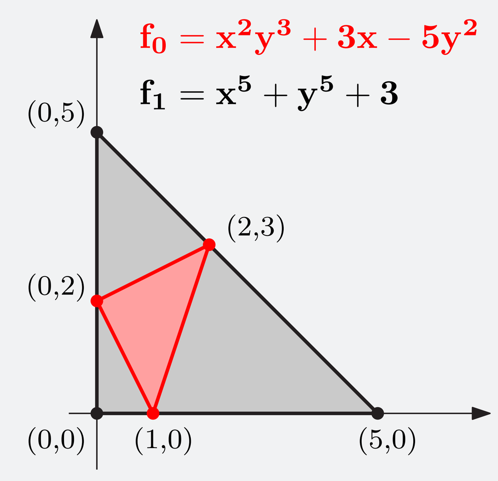
Fall School Shapes, Geometry, and Algebra, October 4-8 2010, Kolympari

Problem

Definitions

Let $f = f_0, f_1, \ldots, f_d$ polynomials and A_i the sets of its exponent vectors. The Newton polytope of f_i is the convex hull of A_i.

We want to compute the Newton polytope of the Resultant named the **Resultant polytope**



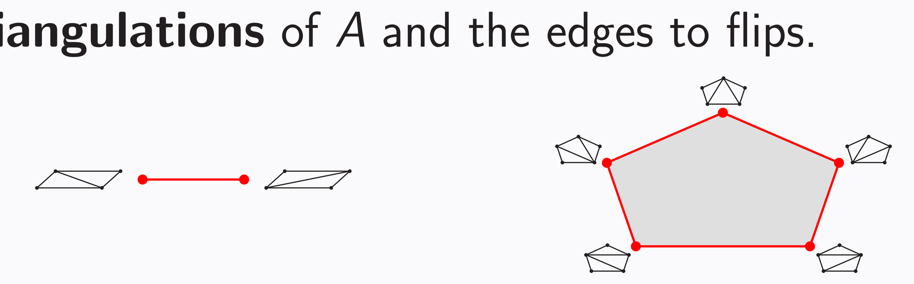
A Toolbox [Sturmfels94]

Given a regular fine mixed subdivision of the Minkowski sum $A = A_0 + A_1 + \cdots + A_d$ we get a unique vertex of the Resultant polytope.

regular fine mixed subdivisions $\xrightarrow[\text{to one}]{\text{many}}$ vertices of Resultant polytope

Secondary polytope

Let A be a set of n points in $\mathbb{R}^d$. To every A corresponds a Secondary polytope with dimension $n - d - 1$. The vertices correspond to the **regular triangulations** of A and the edges to flips.



Enumeration of regular triangulations:
TOPCOM [Rambau], Reverse search [Masada]

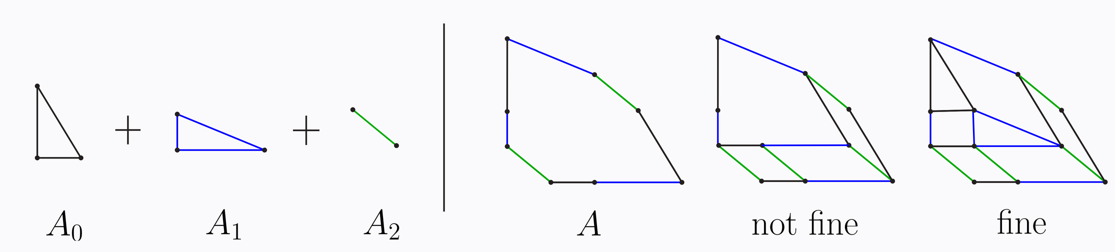
the Cayley trick

$$\left(\begin{array}{c}\textbf{regular}\\ \textbf{triangulations}\\ \textbf{of}\\ \mathcal{C}(\mathbf{A_0}, \ldots, \mathbf{A_d})\end{array}\right) \longleftrightarrow \left(\begin{array}{c}\textbf{regular}\\ \textbf{fine mixed}\\ \textbf{subdivisions}\\ \textbf{of}\\ \mathbf{A_0} + \ldots + \mathbf{A_d}\end{array}\right)$$

Mixed subdivisions

Let $A_0, A_1, \ldots, A_d$ point sets in $\mathbb{R}^d$. A **fine mixed subdivision** of $A = A_0 + A_1 + \ldots + A_d$ is a collection of subsets (cells) of A s.t.

- the cells cover $convex\_hull(A)$ and intersect properly
- every cell $\sigma = F_0 + \cdots + F_d$ for $F_0 \subseteq A_0, \ldots, F_d \subseteq A_d$
- all F_i are affinely independent and σ does not contain any other cell

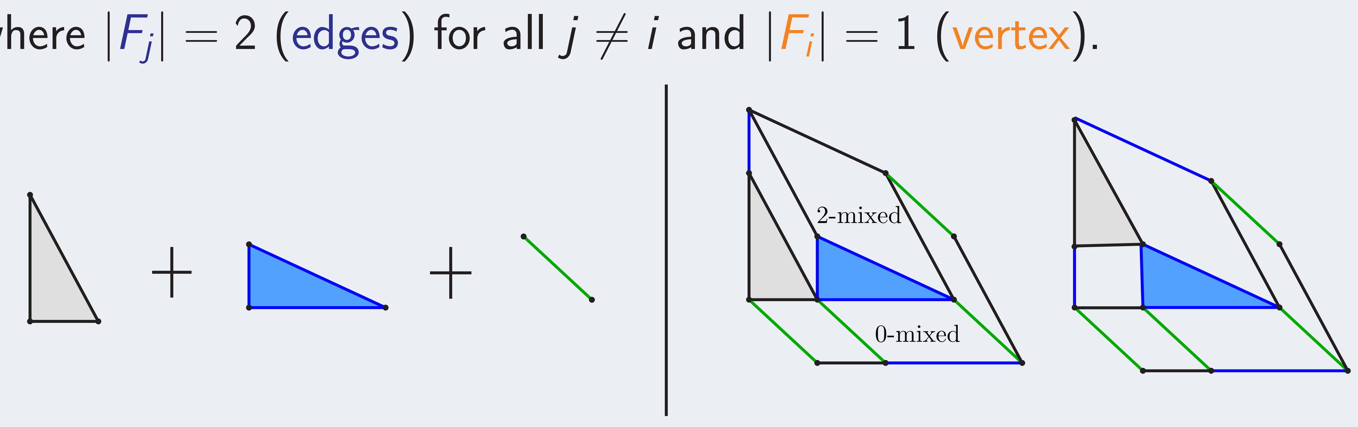


Equivalence classes and cubical flips

A cell σ of a mixed subdivision is called **i-mixed** if for all j exists $F_j \subseteq A_j$ s.t.

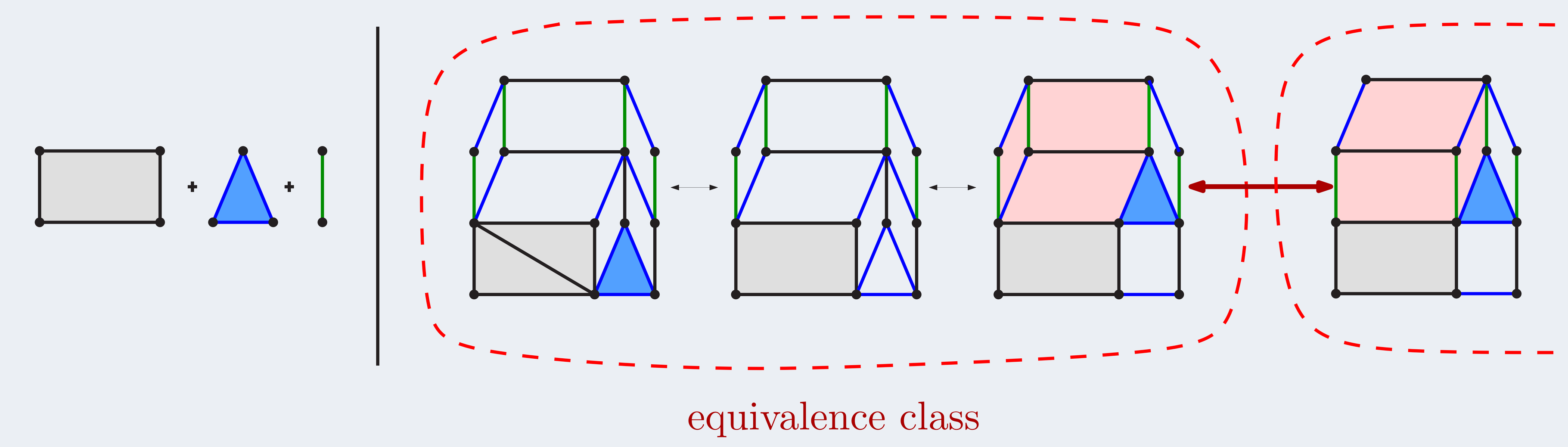
$$\sigma = F_0 + \cdots + F_{i-1} + F_i + F_{i+1} + \cdots + F_d$$

where $|F_j| = 2$ (edges) for all $j \neq i$ and $|F_i| = 1$ (vertex).



Resultant polytope's vertices define equivalence classes over the regular fine mixed subdivisions.

A flip is called **cubical** if and only if it takes us from one equivalence class to another.



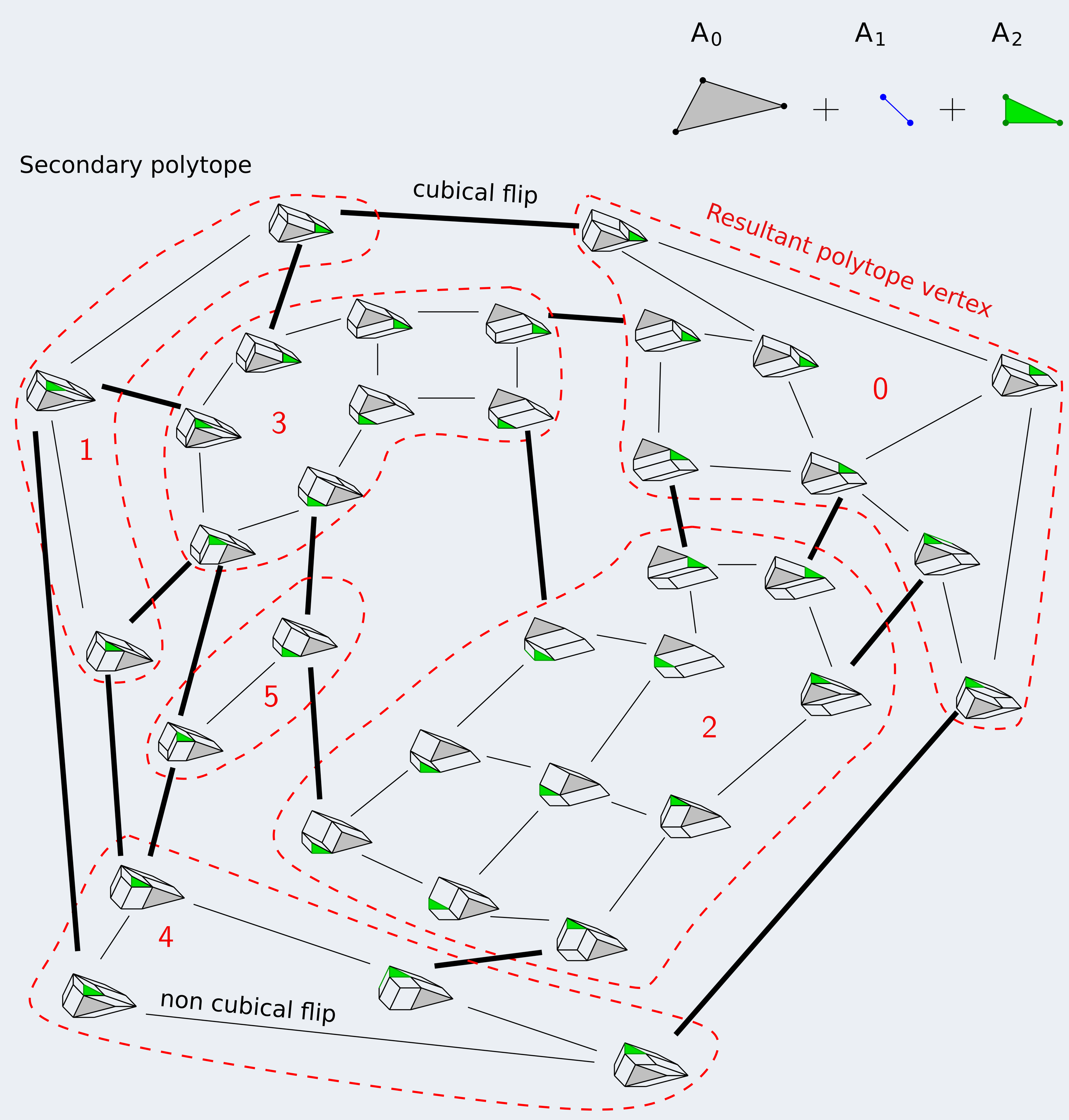
Algorithmic test: A flip is cubical if and only if it involves exactly 2 points from each A_i.

Enumerating the Resultant polytope

Complexity

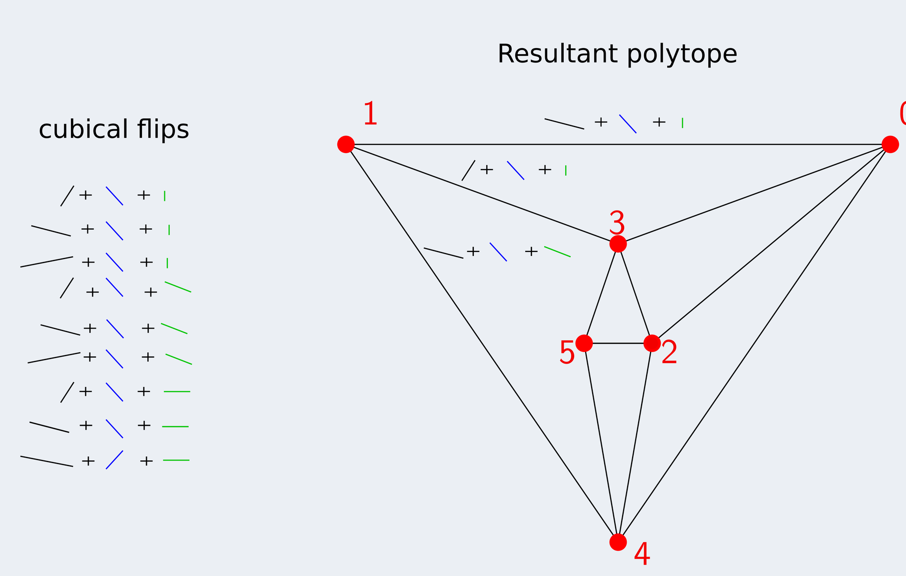
A_0	A_1	A_2	# Secondary polytope vertices	# Resultant polytope vertices
			108	6
			122	8
			3540	22
			76280	95
			17916	60
			104148	21

The computation of the Resultant polytope is infeasible if we enumerate the whole Secondary polytope first!



Our Approach

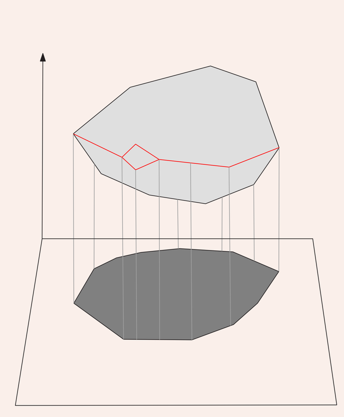
- Precompute all cubical flips. These correspond to the edges of the Resultant polytope.
- Compute a regular fine mixed subdivision. This will give a Resultant polytope vertex.
- Starting from this vertex we have to select the appropriate cubical flips from the precomputed to discover the neighbor Resultant vertices.
- We continue in a BFS manner to enumerate the whole Resultant polytope.



Open problem: Given a Resultant polytope vertex and the set of cubical flips, find an efficient way to select the cubical flips that produce another vertex of the Resultant polytope.

Application to Implicitization

- We need to compute only a silhouette w.r.t. a projection of Resultant polytope [EmirisKonaxisPalios07]
- Wiki page with experiments http://ergawiki.di.uoa.gr/index.php/Implicitization
- More information in the poster of Tatjana Kalinka.



Some References

[Sturmfels94] B. Sturmfels.
On the Newton polytope of the resultant.
J. Algebraic Comb., 3(2):207–236, 1994.

[EmirisKonaxisPalios07] Ioannis Z. Emiris, Christos Konaxis, and Leonidas Palios.
Computing the newton polytope of specialized resultants.
In *Proceeding of the MEGA 2007 conference.*

vissarion@di.uoa.gr