12. Solutions for Selected Problems

1. Mathematical Functions

2. To prove that $\exp(x+y) = \exp(x)\,\exp(y)$, we employ the infinite series formula

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{12.1}$$

Here, for notational convenience, we let the sum start from $n = 0$, so that the leading term 1 in the definition of the exponential is grouped with the rest of the sum as its first term. This relies on the understanding that $0! \equiv 1$, and that $x^0 = 1$ (the latter is consistent with the generalized definition of the power operation; but to avoid circular logic, treat this as the *definition* of x^0 just for the sake of this proof). We begin by substituting the series formula into the right-hand side of our target equation:

$$\exp(x)\exp(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{y^m}{m!}\right). \tag{12.2}$$

Note that we use the symbol n for the first sum, and the symbol m for the second sum; n and m are bound variables, whose terms run over the values specified by the summation signs. The actual choice of symbol used in either sum is unimportant, except that *we must not use the same symbol for both sums*, because the two variables belong to distinct sums. In other words:

$$\exp(x)\exp(x) \neq \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right). \quad \text{(Nonsense expression!)}$$

Next, we make use of the fact that the product of two series can be written as a double sum:

$$\exp(x)\exp(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^n}{n!}\frac{y^m}{m!}. \tag{12.3}$$

Here, we are summing over all possible pair-wise combinations of n and m, which is precisely what happens when one expands the product of two series according to the usual rules of algebra. The next step is to perform a *change of variables* on m and n. In the above expression, we are summing over all non-negative integer m and n; however, the bound variable n can be re-expressed in terms of a newly-defined variable,

$$N = m + n. \tag{12.4}$$

In the original double sum, n and m both run from 0 to $+\infty$, so it follows that their sum N runs from 0 to $+\infty$. For each given value of N, we can write $n = N - m$, and moreover the allowed values of m would only go from 0 to N (it can't exceed N, otherwise n would be negative). In this way, the double sum is converted to

$$\exp(x)\exp(x) = \sum_{N=0}^{\infty}\sum_{m=0}^{N} \frac{x^{N-m}}{(N-m)!}\frac{y^m}{m!} \tag{12.5}$$

Note that after this change of variables, the two summation signs are no longer interchangeable. In the $\sum_{m=0}^{N}$ sign, the variable N appears in the upper limit, so this needs to be written to the right of $\sum_{N=0}^{\infty}$. One sum is thus "encapsulated" inside the other; we could write the algebraic expression more rigorously like this:

$$\exp(x)\exp(x) = \sum_{N=0}^{\infty}\left(\sum_{m=0}^{N} \frac{x^{N-m}}{(N-m)!}\frac{y^m}{m!}\right). \tag{12.6}$$

Finally, we use the binomial theorem to simplify the inner sum:

$$\exp(x)\exp(x) = \sum_{N=0}^{\infty} \frac{(x+y)^N}{N!}, \quad \text{since } (x+y)^N = \sum_{m=0}^{N} \frac{N!}{m!(N-m)!} x^{N-m} y^m. \tag{12.7}$$

Referring again to the series definition of the exponential, we obtain the desired result:

$$\exp(x)\exp(x) = \exp(x+y) \tag{12.8}$$

4. The definition of non-natural powers is

$$a^b = \exp[b\ln(a)]. \tag{12.9}$$

Let $a = \exp(1) = e$ and $b = x$. Then

$$[\exp(1)]^x = \exp\Big[x \ln\Big(\exp(1)\Big)\Big]. \tag{12.10}$$

Since the logarithm is the inverse of the exponential function, $\ln(\exp(1)) = 1$. Hence,

$$e^x = \exp(x). \tag{12.11}$$

2. Derivatives

2. If $y = \ln(x)$, it follows from the definition of the logarithm that

$$\exp(y) = x. \tag{12.12}$$

Taking d/dx on both sides, and using the product rule, gives

$$\frac{dy}{dx}\exp(y) = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\exp(y)} = \frac{1}{x}. \tag{12.13}$$

8. For an ordinary differential equation for a scalar (one-component) function of order n, the general solution must contain n independent variables. In this case, $\vec{v}$ is a two-component function, so it requires $2n$ indpendent variables. The differential equation

$$\frac{d\vec{v}}{dx} = \mathbf{A}\vec{v} \tag{12.14}$$

has order $n = 1$, so a total of 2 independent variables is required for the general solution.

Let u be an eigenvector of $\mathbf{A}$ with eigenvalue λ, and suppose that $\vec{v}(x) = \vec{u}\, e^{\lambda x}$ (note that $\vec{u}$ itself does not depend on x). Then

$$\begin{aligned}\frac{d\vec{v}}{dx} &= \vec{u}\frac{d}{dx}\left(e^{\lambda x}\right) && (12.15)\\ &= \lambda\,\vec{u}\,e^{\lambda x} && (12.16)\\ &= (\mathbf{A}\vec{u})\,e^{\lambda x} && (12.17)\\ &= \mathbf{A}\left(\vec{u}e^{\lambda x}\right) && (12.18)\\ &= \mathbf{A}\vec{v}(x). && (12.19)\end{aligned}$$

Hence, $\vec{v}(x)$ satisfies the desired differential equation.

Let $\vec{u}_1$ and $\vec{u}_2$ be the eigenvectors of $\mathbf{A}$, with eigenvalues λ_1 and λ_2. The general solutions will be

$$\vec{v}(x) = c_1\vec{u}_1 e^{\lambda_1 x} + c_2\vec{u}_2 e^{\lambda_2 x}, \tag{12.20}$$

where c_1 and c_2 are independent variables.

3. Integrals

4. Let us define
$$I(\gamma) = \int_0^1 \frac{x^\gamma - 1}{\ln(x)}, \tag{12.21}$$
so that $I(2)$ is our desired integral. To take the derivative, first note that
$$\frac{d}{d\gamma}(x^\gamma) = \ln(x)\, x^\gamma, \tag{12.22}$$
which can be proven using the generalized definition of the power operation. Thus,
$$\frac{d}{d\gamma} I(\gamma) = \int_0^1 \frac{\ln(x) x^\gamma}{\ln(x)} \tag{12.23}$$
$$= \int_0^1 x^\gamma \tag{12.24}$$
$$= \frac{1}{1+\gamma}. \tag{12.25}$$
This can be integrated straightforwardly:
$$I(\gamma) = \int \frac{d\gamma}{1+\gamma} = \ln(1+\gamma) + c, \tag{12.26}$$
where c is a constant of integration, which we now have to determine. Referring to the original definition of $I(\gamma)$, observe that $I(0) = \int_0^1 (1-1)/\ln(x) = 0$. This implies that $c = 0$. Therefore, the answer is
$$I(2) = \ln(3). \tag{12.27}$$

6. We are provided with the following ansatz for the solution to the differential equation:
$$y(t) = y(0) + \int_0^t dt' e^{-\gamma(t-t')} g(t'). \tag{12.28}$$
First, note that when $t = 0$, the integral's range shrinks to zero, so the result is $y(0)$, as expected. In order to determine the appropriate function g, we perform a derivative in t. The tricky part is that t appears in two places: in the upper range of the integral, as well as in the integrand. So when we take the derivative, there should be two distinct terms (see problem 5):
$$\frac{dy}{dt} = \Big[e^{-\gamma(t-t')} g(t')\Big]_{t'=t} + \int_0^t dt' (-\gamma)\, e^{-\gamma(t-t')}\, g(t') \tag{12.29}$$
$$= g(t) - \gamma[y(t) - y(0)]. \tag{12.30}$$
In the last step, we again made use of the ansatz for $y(t)$. Finally, comparing this with the original differential equation for $y(t)$, we find that
$$g(t) - \gamma[y(t) - y(0)] = -\gamma y(t) + f(t) \quad \Rightarrow \quad g(t) = f(t) - \gamma y(0). \tag{12.31}$$
Hence, the solution to the differential equation is
$$y(t) = y(0) + \int_0^t dt'\, e^{-\gamma(t-t')}\, [f(t') - \gamma y(0)] \tag{12.32}$$
$$= y(0)\, e^{-\gamma t} + \int_0^t dt'\, e^{-\gamma(t-t')} f(t'). \tag{12.33}$$

4. Complex Numbers

2. Using the polar representation: let $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$. Then

$$\begin{aligned}
\big|z_1\, z_2\big| &= \big|r_1 e^{i\theta_1}\, r_2 e^{i\theta_2}\big| && (12.34)\\
&= \Big|(r_1 r_2)\, e^{i(\theta_1+\theta_2)}\Big| && (12.35)\\
&= r_1 r_2 && (12.36)\\
&= |z_1|\,|z_2|. && (12.37)
\end{aligned}$$

Using the Cartesian representation: let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. For convenience, we evaluate the squared magnitude:

$$\begin{aligned}
\big|z_1\, z_2\big|^2 &= |(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)|^2 && (12.38)\\
&= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 && (12.39)\\
&= x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2 && (12.40)\\
&= \left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right) && (12.41)\\
&= |z_1|^2\,|z_2|^2. && (12.42)
\end{aligned}$$

3. Using the polar representation: let $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$. Then

$$\begin{aligned}
\big(z_1\, z_2\big)^* &= \Big((r_1r_2)\, e^{i(\theta_1+\theta_2)}\Big)^* && (12.43)\\
&= (r_1r_2)\, e^{-i(\theta_1+\theta_2)} && (12.44)\\
&= \left(r_1 e^{-i\theta_1}\right)\left(r_2 e^{-i\theta_2}\right) && (12.45)\\
&= z_1^*\, z_2^*. && (12.46)
\end{aligned}$$

Using the Cartesian representation: let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

$$\begin{aligned}
\big(z_1\, z_2\big)^* &= [(x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1)]^* && (12.47)\\
&= (x_1x_2 - y_1y_2) - i\,(x_1y_2 + x_2y_1) && (12.48)\\
&= (x_1 - iy_1)\,(x_2 - iy_y) && (12.49)\\
&= z_1^*\, z_2^*. && (12.50)
\end{aligned}$$

4. The problem arises in this part of the chain: $i \cdot i = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)(-1)}$. The square root is a non-integer power, and non-integer powers are not allowed to take part in standard complex algebra equations in the same way as addition, subtraction, multiplication, division, and integer powers.

   As discussed in Chapter 7, square roots and other non-integer powers have multiple values. The definition of the imaginary unit is often written as $i = \sqrt{-1}$, but this is misleading. Actually, $\sqrt{-1}$ has two legitimate values; one of these values is (by definition) i, while the other value is $-i$.

5. Complex Oscillations

4. The general solution is

$$z(t) = A \exp\left[-i(\omega_1 - i\gamma)t\right]. \qquad (12.51)$$

   It can be verified by direct substitution that this is a solution to the differential equation. It contains one free parameter, and the differential equation is first-order, so it must be a general solution. Next,

$$\frac{d^2z}{dt^2} + 2\gamma\frac{dz}{dt} = (-i)^2(\omega_1 - i\gamma)^2 z(t) - 2i\gamma(\omega_1 - i\gamma)z(t) \tag{12.52}$$
$$= \left[-\omega_1^2 + \gamma^2 + 2i\gamma\omega_1 - 2i\gamma\omega_1 - 2\gamma^2)\right] z(t) \tag{12.53}$$
$$= -\left(\omega_1^2 + \gamma^2\right) z(t). \tag{12.54}$$

Hence, $z(t)$ obeys a damped harmonic oscillator equation with $\omega_0^2 = \omega_1^2 + \gamma^2$. This expression for the natural frequency ensures that $\omega_0^2 > \gamma^2$ (assuming the parameters γ and ω_1 are both real); hence, the harmonic oscillator is always under-damped.

6. Complex Waves

2. Writing $n = n' + in''$, where n' and n'' are real, the travelling wave solutions are

$$\psi(x) = A\exp\left[\pm i(n' + in'')\frac{\omega}{c}x\right]. \tag{12.55}$$

The magnitude and argument are:

$$\big|\psi(x)\big| = |A|\exp\left[\mp n''\frac{\omega}{c}x\right] \tag{12.56}$$
$$\arg\big[\psi(x)\big] = \arg(A) \pm n'\frac{\omega}{c}x. \tag{12.57}$$

The wave's propagation direction is determined by the argument: if the argument increases with x then it is right-moving, and if the argument decreases with x it is left-moving. Moreover, the wave is said to experience amplification if its amplitude grows along the propagation direction, and damping if its amplitude decreases along the propagation direction.

Consider the upper choice of sign (i.e., $+$ for the $\pm$ symbol and $-$ for the $\mp$ symbol). From the magnitude, we see that the wave's amplitude decreases with x if $n'' > 0$, and increases with x if $n'' < 0$. From the argument, the wave is right-moving if $n' > 0$, and left-moving if $n' < 0$. Hence, the wave is damped if $n'n'' > 0$ and amplified if $n'n'' < 0$.

(For example, consider the case $n' < 0$ and $n'' < 0$. The amplitude increases with x but the wave is moving in the $-x$ direction; this means the amplitude grows in the direction opposite to the propagation direction, so the wave is damped.)

For the lower choice of sign, we see from the magnitdue that the amplitude increases with x if $n'' > 0$, and decreases with x if $n'' < 0$. From the argument, we see that the wave is left-moving if $n' > 0$ and right-moving if $n' < 0$. Hence, the wave is damped if $n'n'' > 0$ and amplified if $n'n'' < 0$, exactly the same as in the previous case.

Hence, whether the wave is amplified or damped only depends on the relative signs of n' and n'', and is independent of the direction of propagation.

7. Complex Derivatives

3. We will use the Cauchy-Riemann equations. Decompose z, f, and g into real and imaginary parts as follows: $z = x + iy$, $f = u + iv$, and $g = p + iq$. Since $f(z)$ and $g(z)$ are analytic in D, they satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{12.58}$$
$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}, \quad -\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}. \tag{12.59}$$

This holds for all $z \in D$. Next, expand the product $f(z)\,g(z)$ into real and imaginary parts:

$$f(z)\,g(z) = A(x,y) + iB(x,y), \quad \text{where} \quad \begin{cases} A = up - vq \\ B = uq + vp. \end{cases} \tag{12.60}$$

Our goal is to prove that A and B satisfy the Cauchy-Riemann equations for $x+iy \in D$, which would then imply that fg is analytic in D. Using the product rule for derivatives:

$$\begin{aligned} \frac{\partial A}{\partial x} &= \frac{\partial u}{\partial x}p + u\frac{\partial p}{\partial x} - \frac{\partial v}{\partial x}q - v\frac{\partial q}{\partial x} && (12.61)\\ &= \frac{\partial v}{\partial y}p + u\frac{\partial q}{\partial y} + \frac{\partial u}{\partial y}q + v\frac{\partial p}{\partial y} && (12.62)\\ \frac{\partial B}{\partial y} &= \frac{\partial u}{\partial y}q + u\frac{\partial q}{\partial y} + \frac{\partial v}{\partial y}p + v\frac{\partial p}{\partial y}. && (12.63) \end{aligned}$$

By direct comparison, we see that the two expressions are equal. Similarly,

$$\begin{aligned} \frac{\partial A}{\partial y} &= \frac{\partial u}{\partial y}p + u\frac{\partial p}{\partial y} - \frac{\partial v}{\partial y}q - v\frac{\partial q}{\partial y} && (12.64)\\ &= -\frac{\partial v}{\partial x}p - u\frac{\partial q}{\partial x} - \frac{\partial u}{\partial x}q - v\frac{\partial p}{\partial x} && (12.65)\\ \frac{\partial B}{\partial x} &= \frac{\partial u}{\partial x}q + u\frac{\partial q}{\partial x} + \frac{\partial v}{\partial x}p + v\frac{\partial p}{\partial x}. && (12.66) \end{aligned}$$

These two are the negatives of each other. Q.E.D.

8. Branch Points and Branch Cuts

1. We can write i in polar coordinates as $\exp(i\pi/2)$. Hence,

$$\begin{aligned} (i)^i &= \exp\Big\{ i \ln \big[\exp(i\pi/2)\big] \Big\} && (12.67)\\ &= \exp\left\{ i \left[\frac{i\pi}{2} + 2\pi i n \right] \right\}, \quad n \in \mathbb{Z} && (12.68)\\ &= \exp\left[-2\pi \left(n + \frac{1}{4} \right) \right], \quad n \in \mathbb{Z}. && (12.69) \end{aligned}$$

2. Let $z = r\exp(i\theta)$, where $r > 0$. The values of the logarithm are

$$\ln(z) = \ln(r) + i(\theta + 2\pi n), \quad n \in \mathrm{Z}. \tag{12.70}$$

For each n, note that the first term is the real part and the second term is the imaginary part of a complex number w_n. The logarithm in the first term can be taken to be the real logarithm.

For $z \to 0$, we have $r \to 0$ and hence $\ln(r) \to -\infty$. This implies that w_n lies infinitely far to the left of the origin on the complex plane. Therefore, $w_n \to \infty$ (referring to the complex infinity) regardless of the value of n. Likewise, for $z \to \infty$, we have $r \to \infty$ and hence $\ln(r) \to +\infty$. This implies that w_n lies infinitely far to the right of the origin on the complex plane, so $w_n \to \infty$ regardless of the value of n. Therefore, 0 and ∞ are both branch points of the complex logarithm.

9. Contour Integration

2. By analytic continuation, consider the integral

$$I = \oint \frac{dz}{z^4+1}, \tag{12.71}$$

where the contour is closed in the upper half-plane (we could also choose to close below without changing the results). The contour integral over the large arc scales with the arc radius R as R^{-3}, so it vanishes as $R \to \infty$. Hence, I is exactly equal to the definite integral we are after.

To evaluate the loop integral, we need the poles of the integrand, which are the solutions to $z^4 = -1$. Writing $-1 = \exp(i\pi)$, we find that the roots are $\exp(i\pi/4) \times$ {4-roots of unity}. These can be written in the Cartesian representation as

$$z_1 = \frac{1+i}{\sqrt{2}} \tag{12.72}$$

$$z_2 = \frac{-1+i}{\sqrt{2}} \tag{12.73}$$

$$z_3 = \frac{-1-i}{\sqrt{2}} \tag{12.74}$$

$$z_4 = \frac{1-i}{\sqrt{2}}. \tag{12.75}$$

By closing the contour above, we enclose z_1 and z_2. Thus, by the residue theorem,

$$I = 2\pi i \left\{ \left[\mathrm{Res}\left(\frac{1}{z^4+1}\right)\right]_{z=z_1} + \left[\mathrm{Res}\left(\frac{1}{z^4+1}\right)\right]_{z=z_2} \right\} \tag{12.76}$$

$$= 2\pi i \left[\frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} + \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}\right] \tag{12.77}$$

$$= 2\pi i \left[\frac{\sqrt{8}}{(2)(2+2i)(2i)} + \frac{\sqrt{8}}{(-2)(2i)(-2+2i)}\right] \tag{12.78}$$

$$= \frac{\sqrt{2}\pi i}{2}\left[\frac{1}{-1+i} + \frac{1}{1+i}\right] \tag{12.79}$$

$$= \frac{\pi}{\sqrt{2}}. \tag{12.80}$$

6. A unit circle centered at the origin can be parameterized by $z = \exp(i\phi)$. Hence, along this circle,

$$\cos\phi = \frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right) \tag{12.81}$$

$$= \frac{1}{2}\left(e^{i\phi} + \frac{1}{e^{i\phi}}\right) \tag{12.82}$$

$$= \frac{1}{2}\left(z + \frac{1}{z}\right). \tag{12.83}$$

Also,

$$\frac{dz}{d\phi} = iz. \tag{12.84}$$

Let us denote this circular contour by Γ. We want to find a function $f(z)$ such that

$$\oint_\Gamma f(z)\, dz = \int_0^{2\pi} \frac{d\phi}{\cos\phi + 3}. \tag{12.85}$$

The contour integral on the left side can be parameterized as

$$\int_0^{2\pi} d\phi\, f\big(z(\phi)\big)\, \frac{dz}{d\phi}. \tag{12.86}$$

Therefore, we want $f(z)$ such that

$$f\big(z(\phi)\big)\frac{dz}{d\phi} = \frac{1}{\cos\phi + 3} \tag{12.87}$$

$$= f(z)\ (iz) = \frac{1}{\frac{1}{2}\left(z+\frac{1}{z}\right)+3}. \tag{12.88}$$

After some algebra, we obtain

$$f(z) = \frac{-2i}{z^2+6z+1}. \tag{12.89}$$

The denominator in $f(z)$ has two roots, which are both real:

$$z_+ = -3+2\sqrt{2} = -0.17157\dots \tag{12.90}$$

$$z_- = -3-2\sqrt{2} = -5.8284\dots \tag{12.91}$$

Only the z_+ pole is enclosed by the unit circle. Thus, we can use the residue theorem to evaluate the integral:

$$I = \oint_\Gamma \frac{-2i}{z^2+6z+1}\, dz = 2\pi i \;\mathrm{Res}\left[\frac{-2i}{(z-z_+)(z-z_-)}\right]_{z=z_+} \tag{12.92}$$

$$= 2\pi i\left(\frac{-2i}{z_+ - z_-}\right) \tag{12.93}$$

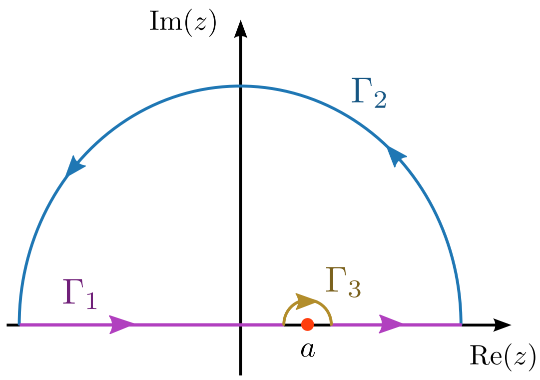
$$= \frac{4\pi}{\big(-3+2\sqrt{2}\big)-\big(-3-2\sqrt{2}\big)} \tag{12.94}$$

$$= \frac{\pi}{\sqrt{2}}. \tag{12.95}$$

7. To evaluate the principal-value integral

$$I = \mathcal{P}\left[\int_{-\infty}^{\infty} \frac{f(x)}{x-a} dx\right], \tag{12.96}$$

we introduce the following loop contour:



The solution procedure is very similar to the example worked out in Section 8.4.3. From the properties of $f(z)$ given in the problem statement, we can conclude that (i) the integrand is analytic on and within the loop contour, so the residue theorem can be used; and (ii) the integrand vanishes quickly enough far from the origin so that, by Jordan's lemma, the integral over Γ_2 vanishes. Hence,

$$I = i\pi f(a). \tag{12.97}$$

By relabelling the dummy variables $x \rightarrow y$ and $a \rightarrow x$, we can write

$$f(x) = -\frac{i}{\pi}\mathcal{P}\left[\int_{-\infty}^{\infty} \frac{f(y)}{y-x}dy\right]. \tag{12.98}$$

Let us now break up f into its real and imaginary parts:

$$\begin{aligned}\mathrm{Re}[f(x)] + i\mathrm{Im}[f(x)] &= -\frac{i}{\pi}\mathcal{P}\left[\int_{-\infty}^{\infty} \frac{\mathrm{Re}[f(y)] + i\mathrm{Im}[f(y)]}{y-x}\,dy\right] &(12.99)\\ &= \frac{1}{\pi}\mathcal{P}\left[\int_{-\infty}^{\infty} \frac{\mathrm{Im}[f(y)] - i\mathrm{Re}[f(y)]}{y-x}\,dy\right]. &(12.100)\end{aligned}$$

Equating the real and imaginary parts of the two sides, we obtain the following two real equations, which are the Kramers-Kronig relations:

$$\begin{aligned}\mathrm{Re}[f(x)] &= \;\;\frac{1}{\pi}\mathcal{P}\left[\int_{-\infty}^{\infty} \frac{\mathrm{Im}[f(y)]}{y-x}\,dy\right] &(12.101)\\ \mathrm{Im}[f(x)] &= -\frac{1}{\pi}\mathcal{P}\left[\int_{-\infty}^{\infty} \frac{\mathrm{Re}[f(y)]}{y-x}\,dy\right]. &(12.102)\end{aligned}$$

10. Fourier Series and Fourier Transforms

3. The Fourier coefficients are given by

$$f_n = \frac{1}{a}\int_{-a/2}^{a/2} dx\; e^{-ik_n x}\, f(x), \quad \text{where } k_n = \frac{2\pi n}{a}. \tag{12.103}$$

First, consider the case where $f(x)$ is real. Take the complex conjugate of both sides:

$$\begin{aligned}f_n^* &= \frac{1}{a}\int_{-a/2}^{a/2} dx\;\left(e^{-ik_n x}\, f(x)\right)^* &(12.104)\\ &= \frac{1}{a}\int_{-a/2}^{a/2} dx\; e^{ik_n x}\, f(x)^* &(12.105)\\ &= \frac{1}{a}\int_{-a/2}^{a/2} dx\; e^{ik_n x}\, f(x) &(12.106)\\ &= f_{-n}. &(12.107)\end{aligned}$$

Hence,

$$f_n = f_{-n}^*. \tag{12.108}$$

For the second case, $f(x) = f(-x)$, perform a change of variables $x = -u$ in the Fourier integral:

$$\begin{aligned}f_n &= \frac{1}{a}\int_{-a/2}^{a/2} du\; e^{ik_n u}\, f(u) &(12.109)\\ &= f_{-n}. &(12.110)\end{aligned}$$

For $f(x) = f(-x)^*$, the same change of variables gives

$$f_n = f_n^*. \tag{12.111}$$

7. From the definition of the delta function as the narrow-peak limit of a Gaussian wavepacket:
$$\delta(ax) = \lim_{\gamma\rightarrow 0} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikax}\, e^{-\gamma k^2}. \tag{12.112}$$
Perform a change of variables $k = q/a$ and $\gamma = \gamma'\, a^2$:
$$\begin{aligned}\delta(ax) &= \lim_{\gamma'\rightarrow 0} \int_{-\infty}^{\infty} \frac{1}{a}\frac{dq}{2\pi}\, e^{iqx}\, e^{-\gamma' q^2} && (12.113)\\ &= \frac{1}{a}\delta(x). && (12.114)\end{aligned}$$

8. Perform a change of variables from Cartesian coordinates (x, y) to polar coordinates (r, ϕ):
$$\begin{aligned}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; x^2\, \delta\left(\sqrt{x^2+y^2} - a\right) &= \int_0^{\infty} dr \int_0^{2\pi} r d\phi \,\cdot\, r^2 \cos^2\phi \; \delta(r-a) && (12.115)\\ &= \left(\int_0^{\infty} dr\, r^3\, \delta(r-a)\right)\left(\int_0^{2\pi} d\phi\, \cos^2\phi\right) && (12.116)\\ &= \begin{cases}\pi a^3, & a \ge 0\\ 0, & a < 0.\end{cases} && (12.117)\end{aligned}$$

11. Green's Functions

2. For the over-damped oscillator, the Green's function is
$$G(t,t') = \Theta(t-t')\, \frac{e^{-\gamma(t-t')}}{\Gamma} \sinh\big[\Gamma(t-t')\big], \quad \text{where } \Gamma = \sqrt{\gamma^2 - \omega_0^2}. \tag{12.118}$$
Hence, the response to the force f is
$$x(t) = \frac{1}{m\Gamma} \int_{-\infty}^{t} dt'\; e^{-\gamma(t-t')} \sinh\big[\Gamma(t-t')\big] f(t'). \tag{12.119}$$
From this, we get the following expression for the desired correlation function:
$$\begin{aligned}\langle x(t_1)\, x(t_2)\rangle = \frac{1}{m^2\Gamma^2} &\int_{-\infty}^{t_1} dt' \int_{-\infty}^{t_2} dt''\; e^{-\gamma(t_1 - t')}\, e^{-\gamma(t_2-t'')}\\ &\times \sinh\big[\Gamma(t_1-t')\big] \sinh\big[\Gamma(t_2 - t'')\big]\; \langle f(t') f(t'')\rangle. \qquad (12.120)\end{aligned}$$
Note that the $\langle\cdots\rangle$ can be shifted inside the integrals, because it represents taking the mean over independent sample trajectories. Now, without loss of generality, let us take
$$t_1 \ge t_2. \tag{12.121}$$
Since $\langle f(t')f(t'')\rangle = A\delta(t'-t'')$ which vanishes for $t' \ne t''$, the double integral only receives contributions from values of t' not exceeding t_2 (which is the upper limit of the range for t''). Thus, we revise $\int^{t_1} dt'$ into $\int^{t_2} dt'$. The delta function then reduces the double integral into a single integral, which can be solved and simplified with a

bit of tedious algebra:

$$\langle x(t_1)\, x(t_2)\rangle = \frac{A}{m^2\Gamma^2} e^{-\gamma(t_1+t_2)} \int_{-\infty}^{t_2} dt' e^{2\gamma t'} \sinh\big[\Gamma(t'-t_1)\big] \sinh\big[\Gamma(t'-t_2)\big] \tag{12.122}$$

$$= \frac{A}{8m^2\Gamma^2} e^{-\gamma(t_1+t_2)} \Bigg[\frac{e^{-\Gamma t_1} e^{(2\gamma+\Gamma)t_2}}{\gamma+\Gamma} + \frac{e^{\Gamma t_1} e^{(2\gamma-\Gamma)t_2}}{\gamma-\Gamma} - \frac{e^{-\Gamma t_1} e^{(\Gamma+2\gamma)t_2} + e^{\Gamma t_1} e^{(-\Gamma+2\gamma)t_2}}{\gamma}\Bigg] \tag{12.123}$$

$$= \frac{A}{8m^2\Gamma\gamma} \left[\frac{e^{-(\gamma-\Gamma)(t_1-t_2)}}{\gamma-\Gamma} - \frac{e^{-(\gamma+\Gamma)(t_1-t_2)}}{\gamma+\Gamma}\right]. \tag{12.124}$$

Hence,

$$\Big\langle [x(t+\Delta t) - x(t)]^2 \Big\rangle = 2\Big[\,\Big\langle x(t)^2 \Big\rangle - \langle x(t+\Delta t) x(t)\rangle\,\Big] \tag{12.125}$$

$$= \frac{A}{4m^2\Gamma\gamma} \left[\frac{1-e^{-(\gamma-\Gamma)\Delta t}}{\gamma-\Gamma} - \frac{1-e^{-(\gamma+\Gamma)\Delta t}}{\gamma+\Gamma}\right]. \tag{12.126}$$