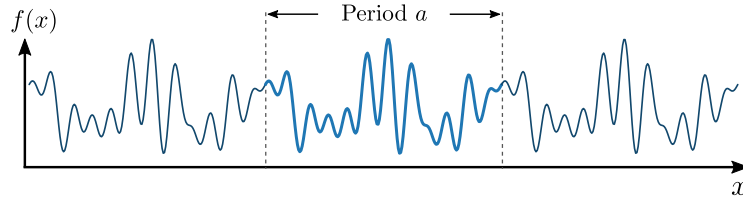


## 10. Fourier Series and Fourier Transforms

The **Fourier transform** is one of the most important mathematical tools used for analyzing functions. Given an arbitrary function  $f(x)$ , with a real domain ( $x \in \mathbb{R}$ ), we can express it as a linear combination of complex waves. The coefficients of the linear combination form a counterpart to  $f$ , which is a complex function  $F(k)$  defined in a wave-number domain ( $k \in \mathbb{R}$ ). Often,  $F$  turns out to be easier to deal with than  $f$ . In particular, differential equations for  $f$  can often be reduced to algebraic equations for  $F$ , which are much easier to solve.

### 10.1 Fourier series

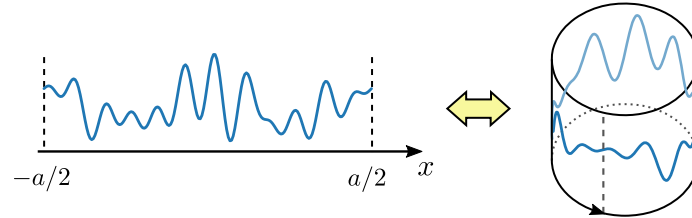
We begin by discussing the **Fourier series**, which is used to analyze functions that are periodic in their inputs. A **periodic function**  $f(x)$  is a function of a real variable  $x$  that repeats itself every time  $x$  changes by  $a$ , as shown in the figure below:



The constant  $a$  is called the **period**. We can write the periodicity condition as

$$f(x + a) = f(x), \quad \forall x \in \mathbb{R}. \quad (10.1)$$

The value of  $f(x)$  can be real or complex, but  $x$  should be real. You can think of  $x$  as representing a spatial coordinate. We can also think of the periodic function as being defined over a finite segment  $-a/2 \leq x < a/2$ , with periodic boundary conditions  $f(-a/2) = f(a/2)$ . In spatial terms, this is like wrapping the segment into a loop:



Let's consider what it means to specify a periodic function  $f(x)$ . One way to specify the function is to give an explicit mathematical formula for it. Another approach might be to specify the function values in  $-a/2 \leq x < a/2$ . Since there's an uncountably infinite number of points in this domain, we can generally only achieve an approximate specification of  $f$  this way, by giving the values of  $f$  at a large but finite set  $x$  points.

There is another interesting approach to specifying  $f$ . We can express it as a linear combination of simpler periodic functions, consisting of sines and cosines:

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{2\pi n x}{a}\right) + \sum_{m=0}^{\infty} \beta_m \cos\left(\frac{2\pi m x}{a}\right). \quad (10.2)$$

(Note that the  $n$  index does not include 0; since the sine term with  $n = 0$  vanishes for all  $x$ , it's redundant.) This is called a **Fourier series**. Given the numbers  $\{\alpha_n, \beta_m\}$ , which are called the **Fourier coefficients**,  $f(x)$  can be calculated for any  $x$ . The Fourier coefficients are real if  $f(x)$  is a real function, or complex if  $f(x)$  is complex.

The justification for the Fourier series formula is that the sine and cosine functions in the series are, themselves, periodic with period  $a$ :

$$\sin\left(\frac{2\pi n(x+a)}{a}\right) = \sin\left(\frac{2\pi nx}{a} + 2\pi n\right) = \sin\left(\frac{2\pi nx}{a}\right) \quad (10.3)$$

$$\cos\left(\frac{2\pi m(x+a)}{a}\right) = \cos\left(\frac{2\pi mx}{a} + 2\pi m\right) = \cos\left(\frac{2\pi mx}{a}\right). \quad (10.4)$$

Any linear combination of them thus automatically satisfies the periodicity condition for  $f$ .

### 10.1.1 Square-integrable functions

Can arbitrary periodic functions always be expressed as a Fourier series? This question turns out to be surprisingly intricate, and its resolution preoccupied mathematicians for much of the 19th century. The full discussion is beyond the scope of this course.

Luckily, it turns out that a certain class of periodic functions, commonly encountered in physical contexts, are guaranteed to always be expressible as Fourier series. These are **square-integrable functions**, for which

$$\int_{-a/2}^{a/2} dx |f(x)|^2 \text{ exists and is finite.} \quad (10.5)$$

Unless otherwise stated, we will always assume that the functions we're dealing with are square-integrable.

### 10.1.2 Complex Fourier series and inverse relations

We have written the Fourier series as sums over sine and cosine functions. But as we know, sines and cosines can be re-written in terms of exponentials by using Euler's formula. If we do this for the Fourier series, it takes the form

$$f(x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x/a} f_n. \quad (10.6)$$

Now it is just a single sum, which is neater than having two separate sums for sines and cosines. The sum includes negative integers  $n$ , and involves a new set of complex Fourier coefficients,  $\{f_n\}$ . (As an exercise, try working out how the old coefficients  $\{\alpha_n, \beta_n\}$  are related to the new coefficients  $\{f_n\}$ .)

If the Fourier coefficients  $\{f_n\}$  are known, then  $f(x)$  can be calculated using the above formula. The converse is also true: given  $f(x)$ , we can determine the Fourier coefficients. To see how, observe that

$$\int_{-a/2}^{a/2} dx e^{-2\pi i m x/a} e^{2\pi i n x/a} = a \delta_{mn} \quad \text{for } m, n \in \mathbb{Z}, \quad (10.7)$$

where  $\delta_{mn}$  is the **Kronecker delta**, defined as:

$$\delta_{mn} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n. \end{cases} \quad (10.8)$$

Due to this property, the set of functions  $\exp(2\pi i n x/a)$ , with integer values of  $n$ , are said to be **orthogonal** functions. (We won't go into the details now, but the term "orthogonality" is used here with the same meaning as in vector algebra, where a set of vectors  $\vec{v}_1, \vec{v}_2, \dots$  is

said to be orthogonal if  $\vec{v}_m \cdot \vec{v}_n = 0$  for  $m \neq n$ .) Hence,

$$\int_{-a/2}^{a/2} dx e^{-2\pi i m x/a} f(x) = \int_{-a/2}^{a/2} dx e^{-2\pi i m x/a} \left[ \sum_{n=-\infty}^{\infty} e^{2\pi i n x/a} f_n \right] \quad (10.9)$$

$$= \sum_{n=-\infty}^{\infty} \int_{-a/2}^{a/2} dx e^{-2\pi i m x/a} e^{2\pi i n x/a} f_n \quad (10.10)$$

$$= \sum_{n=-\infty}^{\infty} a \delta_{mn} f_n \quad (10.11)$$

$$= a f_m. \quad (10.12)$$

The procedure of multiplying by  $\exp(-2\pi i m x/a)$  and integrating over  $x$  acts like a sieve, filtering out all other Fourier components of  $f(x)$  and keeping only the one with the matching index  $m$ . Thus, we arrive at a pair of relations expressing  $f(x)$  in terms of its Fourier components, and vice versa:

$$f(x) = \sum_{n=-\infty}^{\infty} e^{i k_n x} f_n, \quad \text{where } k_n \equiv \frac{2\pi n}{a} \quad (10.13)$$

$$f_n = \frac{1}{a} \int_{-a/2}^{a/2} dx e^{-i k_n x} f(x). \quad (10.14)$$

The real numbers  $k_n$  are called **wave-numbers**. They form a discrete set, with one for each Fourier component. In physics jargon, we say that the wave-numbers are quantized to integer multiples of  $\Delta k \equiv 2\pi/a$ .

### 10.1.3 Example: Fourier series of a square wave

To get a feel for how the Fourier series behaves, let's look at a square wave: a function that takes only two values  $+1$  or  $-1$ , jumping between the two values at periodic intervals. Within one period, the function is

$$f(x) = \begin{cases} -1, & -a/2 \leq x < 0 \\ +1, & 0 \leq x < a/2. \end{cases} \quad (10.15)$$

Plugging this into the Fourier relation, and doing the straightforward integrals, gives the Fourier coefficients

$$f_n = -i \frac{[\sin(n\pi/2)]^2}{n\pi/2} \quad (10.16)$$

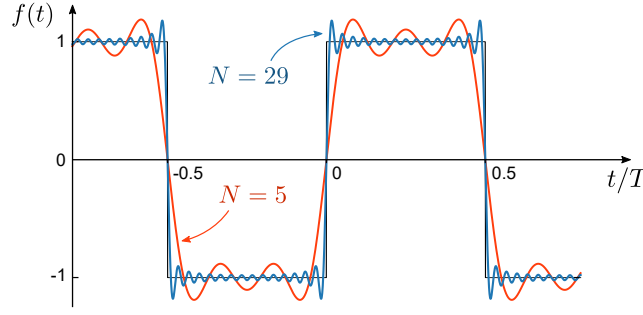
$$= \begin{cases} -2i/n\pi, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases} \quad (10.17)$$

As can be seen, the Fourier coefficients become small for large  $n$ . We can write the Fourier series as

$$f(x) \leftrightarrow \sum_{n=1,3,5,\dots} \frac{4 \sin(2\pi n x/a)}{n\pi}. \quad (10.18)$$

If this infinite series is truncated to a finite number of terms, we get an approximation to  $f(x)$ .

The plot below shows the graph of the square wave  $f(x)$  alongside the truncated Fourier series. As more terms are included in the truncated Fourier series, the approximation gets better and better.



One amusing consequence of the above result is that we can use it to derive a series expansion for  $\pi$ . If we set  $x = a/4$ ,

$$f(a/4) = 1 = \frac{4}{\pi} \left[ \sin(\pi/2) + \frac{1}{3} \sin(3\pi/2) + \frac{1}{5} \sin(5\pi/2) + \cdots \right], \quad (10.19)$$

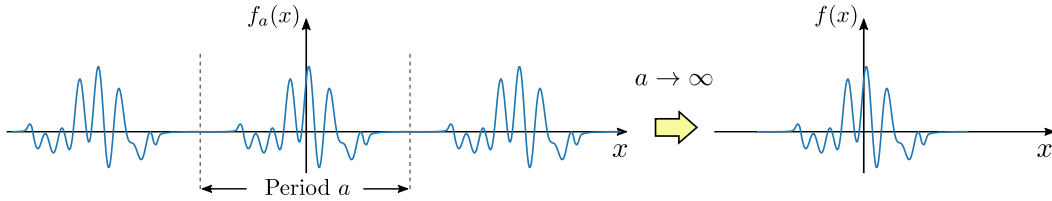
and hence

$$\pi = 4 \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right). \quad (10.20)$$

## 10.2 Fourier transforms

The Fourier series applies to periodic functions defined over the interval  $-a/2 \leq x < a/2$ . But the concept can be generalized to functions defined over the entire real line,  $x \in \mathbb{R}$ , if we take the limit  $a \rightarrow \infty$  carefully.

Suppose we have a function  $f$  defined over the entire real line,  $x \in \mathbb{R}$ , such that  $f(x) \rightarrow 0$  for  $x \rightarrow \pm\infty$ . Imagine there is a family of periodic functions  $\{f_a(x) \mid a \in \mathbb{R}^+\}$ , such that  $f_a(x)$  has periodicity  $a$ , and approaches  $f(x)$  in the limit  $a \rightarrow \infty$ . This is illustrated in the figure below:



In mathematical terms,

$$f(x) = \lim_{a \rightarrow \infty} f_a(x), \quad \text{where } f_a(x+a) = f_a(x). \quad (10.21)$$

Since  $f_a$  is periodic, it can be expanded as a Fourier series:

$$f_a(x) = \sum_{n=-\infty}^{\infty} e^{ik_n x} f_{an}, \quad \text{where } k_n = n\Delta k, \quad \Delta k = \frac{2\pi}{a}. \quad (10.22)$$

Here,  $f_{an}$  denotes the  $n$ -th complex Fourier coefficient of the function  $f_a(x)$ . Note that each Fourier coefficient depends implicitly on the periodicity  $a$ .

As  $a \rightarrow \infty$ , the wave-number quantum  $\Delta k$  goes to zero, and the set of discrete  $k_n$  turns into a continuum. During this process, each Fourier coefficient  $f_{an}$  goes to zero, because there are more and more Fourier components in the vicinity of each  $k$  value, and each individual component contributes less. This implies that we can replace the discrete sum with an integral. To accomplish this, we multiply the summand by a factor of  $(\Delta k/2\pi)/(\Delta k/2\pi) = 1$ :

$$f(x) = \lim_{a \rightarrow \infty} \left[ \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} e^{ik_n x} \left( \frac{2\pi f_{an}}{\Delta k} \right) \right]. \quad (10.23)$$

(In case you're wondering, the choice of  $2\pi$  factors is essentially arbitrary; we are following the usual convention.) Next, define

$$F(k) \equiv \lim_{a \rightarrow \infty} \left[ \frac{2\pi f_{an}}{\Delta k} \right]_{k=k_n}. \quad (10.24)$$

In the  $a \rightarrow \infty$  limit, the  $f_{an}$  in the numerator and the  $\Delta k$  in the denominator both go zero, but if their ratio remains finite, the Fourier sum turns into an integral:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(k). \quad (10.25)$$

### 10.2.1 The Fourier relations

The function  $F(k)$  is called the **Fourier transform** of  $f(x)$ . Just as we have expressed  $f(x)$  in terms of  $F(k)$ , we can also express  $F(k)$  in terms of  $f(x)$ . To do this, we apply the  $a \rightarrow \infty$  limit to Eq. (10.14):

$$F(k_n) = \lim_{a \rightarrow \infty} \frac{2\pi f_{an}}{\Delta k} \quad (10.26)$$

$$= \lim_{a \rightarrow \infty} \frac{2\pi}{2\pi/a} \left( \frac{1}{a} \int_{-a/2}^{a/2} dx e^{-ik_n x} \right) \quad (10.27)$$

$$= \int_{-\infty}^{\infty} dx e^{-ikx} f(x). \quad (10.28)$$

Hence, we arrive at a pair of equations called the **Fourier relations**:

$$F(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \quad (10.29)$$

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(k) \quad (10.30)$$

Eq. (10.29) is the Fourier transform, and Eq. (10.30) is called the **inverse Fourier transform**.

There are some differences between the two formulas. First,  $dk$  is accompanied by a factor of  $1/2\pi$ , but there is no such factor for  $dx$ ; this is a matter of convention, tied to our earlier definition of  $F(k)$ . Second, the integral over  $x$  contains a factor of  $e^{-ikx}$  but the integral over  $k$  contains a factor of  $e^{ikx}$ . One way to remember which equation has the positive sign in the exponent is to interpret the inverse Fourier transform equation (which has the form of an integral over  $k$ ) as the continuum limit of a sum over complex waves. In this sum,  $F(k)$  plays the role of the series coefficients, and the complex waves have the form  $\exp(ikx)$  in accordance with our usual convention (see Chapter 6).

As noted in Section 10.1.1, all the functions we deal with are assumed to be square integrable. This includes the  $f_a$  functions used to define the Fourier transform. In the  $a \rightarrow \infty$  limit, this implies that we are dealing with functions such that

$$\int_{-\infty}^{\infty} dx |f(x)|^2$$

exists and is finite.

### 10.2.2 A simple example

Consider the function

$$f(x) = \begin{cases} e^{-\eta x}, & x \geq 0 \\ 0, & x < 0, \end{cases} \quad \eta \in \mathbb{R}^+. \quad (10.31)$$

For  $x < 0$ , this is an exponentially-decaying function, and for  $x < 0$  it is identically zero. The real parameter  $\eta$  is called the decay constant; for  $\eta > 0$ , the function  $f(x)$  vanishes as  $x \rightarrow +\infty$  and can thus be shown to be square-integrable. Larger values of  $\eta$  correspond to faster exponential decay.

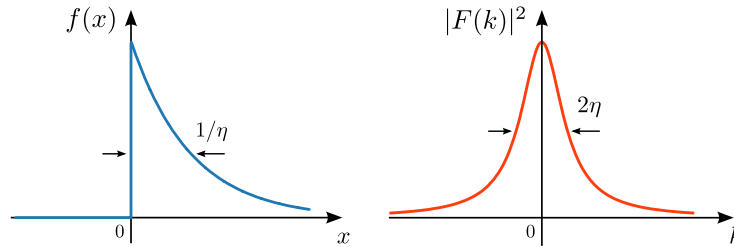
The Fourier transform can be found by directly calculating the Fourier integral:

$$F(k) = \int_0^{\infty} dx e^{-ikx} e^{-\kappa x} = \frac{-i}{k - i\eta}. \quad (10.32)$$

It is useful to plot the squared magnitude of the Fourier transform,  $|F(k)|^2$ , against  $k$ . This is called the **Fourier spectrum** of  $f(x)$ . In this case,

$$|F(k)|^2 = \frac{1}{k^2 + \eta^2}. \quad (10.33)$$

The Fourier spectrum is plotted below. It consists of a peak centered at  $k = 0$ , forming a curve called a **Lorentzian**. The width of the Lorentzian is dependent on the original function's decay constant  $\eta$ . For small  $\eta$ , i.e. weakly-decaying  $f(x)$ , the peak is narrow; for large  $\eta$ , i.e. rapidly-decaying  $f(x)$ , the peak is broad.



We can quantify the width of the Lorentzian by defining the **full-width at half-maximum** (FWHM)—the width of the curve at half the value of its maximum. In this case, the maximum of the Lorentzian curve occurs at  $k = 0$  and has the value of  $1/\eta^2$ . The half-maximum,  $1/2\eta^2$ , occurs when  $\delta k = \pm\eta$ . Hence, the original function's decay constant,  $\eta$ , is directly proportional to the FWHM of the Fourier spectrum, which is  $2\eta$ .

To wrap up this example, let's evaluate the inverse Fourier transform:

$$f(x) = -i \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{k - i\eta}. \quad (10.34)$$

This can be solved by contour integration. The analytic continuation of the integrand has a simple pole at  $k = i\eta$ . For  $x < 0$ , the numerator  $\exp(ikx)$  vanishes far from the origin in the lower half-plane, so we close the contour below. This encloses no pole, so the integral is zero. For  $x > 0$ , the numerator vanishes far from the origin in the upper half-plane, so we close the contour above, with a counter-clockwise arc, and the residue theorem gives

$$f(x) = \left(\frac{-i}{2\pi}\right) (2\pi i) \operatorname{Res} \left[ \frac{e^{ikx}}{k - i\eta}, k = i\eta \right] = e^{-\eta x} \quad (\text{for } x > 0), \quad (10.35)$$

in agreement with the form of  $f(x)$  we started out with.

### 10.3 Fourier transforms for time-domain functions

So far, we have been dealing with functions of a spatial coordinate  $x$ . Of course, mathematical relations don't care about what kind of physical variable we are dealing with, so the same equations could be applied to functions of time  $t$ . However, there is an important difference in *convention*. When dealing with functions of the time coordinate  $t$ , it is customary to use a different sign convention in the Fourier relations!

The Fourier relations for a function of time,  $f(t)$ , are:

$$\begin{cases} F(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} f(t) \\ f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} F(\omega). \end{cases} \quad (10.36)$$

Compared to the previously-derived Fourier relations, Eqs. (10.29) and (10.30), the signs of the  $\pm i\omega t$  exponents are flipped.

There's a good reason for this difference in sign convention: it arises from the need to describe propagating waves, which vary with both space *and* time. As discussed in Chapter 6, a propagating plane wave can be described by a wavefunction of the form

$$f(x, t) = Ae^{i(kx - \omega t)}, \quad (10.37)$$

where  $k$  is the wave-number and  $\omega$  is the angular frequency. We write the plane wave function this way so that positive  $k$  indicates forward propagation in space (i.e., in the  $+x$  direction), and positive  $\omega$  indicates forward propagation in time (i.e., in the  $+t$  direction). This requires the  $kx$  and  $\omega t$  terms in the exponent to have opposite signs. Thus, when  $t$  increases by some amount, a corresponding *increase* in  $x$  leaves the exponent unchanged.

As we have seen, the inverse Fourier transform relation describes how a wave-form is broken up into a superposition of elementary waves. For a wavefunction  $f(x, t)$ , the superposition is given in terms of plane waves:

$$f(x, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i(kx - \omega t)} F(k, \omega). \quad (10.38)$$

To be consistent with this, we need to treat space and time variables with oppositely-signed exponents:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(k) \quad (10.39)$$

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} F(\omega). \quad (10.40)$$

The other equations follow similarly.

## 10.4 Basic properties of the Fourier transform

The Fourier transform has several important properties. These can all be derived from the definition of the Fourier transform; the proofs are left as exercises.

1. The Fourier transform is linear: if we have two functions  $f(x)$  and  $g(x)$ , whose Fourier transforms are  $F(k)$  and  $G(k)$  respectively, then for any constants  $a, b \in \mathbb{C}$ ,

$$af(x) + bg(x) \xrightarrow{\text{FT}} aF(k) + bG(k). \quad (10.41)$$

2. Performing a coordinate translation on a function causes its Fourier transform to be multiplied by a phase factor:

$$f(x + b) \xrightarrow{\text{FT}} e^{ikb} F(k). \quad (10.42)$$

As a consequence, translations leave the Fourier spectrum  $|F(k)|^2$  unchanged.

3. If the Fourier transform of  $f(x)$  is  $F(k)$ , then

$$f^*(x) \xrightarrow{\text{FT}} F^*(-k). \quad (10.43)$$

As a consequence, the Fourier transform of a real function must satisfy the symmetry relation  $F(k) = F^*(-k)$ , meaning that the Fourier spectrum is symmetric about the origin in  $k$ -space:  $|F(k)|^2 = |F(-k)|^2$ .

4. When you take the derivative of a function, that is equivalent to multiplying its Fourier transform by a factor of  $ik$ :

$$\frac{d}{dx}f(x) \xrightarrow{\text{FT}} ikF(k). \quad (10.44)$$

For functions of time, because of the difference in sign convention discussed in Section 10.3, there is an extra minus sign:

$$\frac{d}{dt}f(t) \xrightarrow{\text{FT}} -i\omega F(\omega). \quad (10.45)$$

## 10.5 Fourier transforms of differential equations

The Fourier transform is a useful tool for solving many differential equations. As an example, consider a damped harmonic oscillator (previously studied in Chapter 5) that is subjected to an additional driving force  $f(t)$ . This force has an arbitrary time dependence, and is not necessarily harmonic. The equation of motion is

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2x(t) = \frac{f(t)}{m}. \quad (10.46)$$

To solve for  $x(t)$ , we first take the Fourier transform of both sides of the above equation. The result is:

$$-\omega^2X(\omega) - 2i\gamma\omega X(\omega) + \omega_0^2X(\omega) = \frac{F(\omega)}{m}, \quad (10.47)$$

where  $X(\omega)$  and  $F(\omega)$  are the Fourier transforms of  $x(t)$  and  $f(t)$  respectively. To obtain the left-hand side of this equation, we used the properties of the Fourier transform described in Section 10.4, specifically linearity and the Fourier transforms of derivatives. Note also that we are using the convention for time-domain functions (Section 10.3).

The Fourier transform has turned our ordinary differential equation into an algebraic equation which can be easily solved:

$$X(\omega) = \frac{F(\omega)/m}{-\omega^2 - 2i\gamma\omega + \omega_0^2} \quad (10.48)$$

Knowing  $X(\omega)$ , we can use the inverse Fourier transform to obtain  $x(t)$ :

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t} F(\omega)/m}{-\omega^2 - 2i\gamma\omega + \omega_0^2}, \quad \text{where } F(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} f(t). \quad (10.49)$$

To summarize, the solution procedure for the driven harmonic oscillator equation consists of (i) using the Fourier transform on  $f(t)$  to obtain  $F(\omega)$ , (ii) using the above equation to find  $X(\omega)$  algebraically, and (iii) performing an inverse Fourier transform to obtain  $x(t)$ . This is the basis for the Green's function method, which provides a way to systematically solve differential equations. We will explore this further in the next chapter.



## 10.6 Common Fourier transforms

To accumulate more intuition about Fourier transforms, let us examine the Fourier transforms of some interesting functions. We will just state the results; the calculations are left as exercises.

### 10.6.1 Damped waves

In Section 10.2.2, we saw that an exponentially decay function with decay constant  $\eta \in \mathbb{R}^+$  has the following Fourier transform:

$$f(x) = \begin{cases} e^{-\eta x}, & x \geq 0 \\ 0, & x < 0, \end{cases} \xrightarrow{\text{FT}} F(k) = \frac{-i}{k - i\eta}. \quad (10.50)$$

Observe that  $F(k)$  is given by a simple algebraic formula. If we “extend” the domain of  $k$  to complex values,  $F(k)$  corresponds to an analytic function with a simple in the upper half of the complex plane, at  $k = i\eta$ .

Now consider a decaying wave with wave-number  $q \in \mathbb{R}$  and decay constant  $\eta \in \mathbb{R}^+$ . The Fourier transform is a function with a simple pole at  $q + i\eta$ :

$$f(x) = \begin{cases} e^{i(q+i\eta)x}, & x \geq 0 \\ 0, & x < 0. \end{cases} \xrightarrow{\text{FT}} F(k) = \frac{-i}{k - (q + i\eta)}. \quad (10.51)$$

Next, consider a wave that grows exponentially with  $x$  for  $x < 0$ , and vanishes for  $x > 0$ . Using a similar calculation, we can show that the Fourier transform is a function with a simple pole in the lower half-plane:

$$f(x) = \begin{cases} 0, & x \geq 0 \\ e^{i(q-i\eta)x}, & x < 0. \end{cases} \xrightarrow{\text{FT}} F(k) = \frac{i}{k - (q - i\eta)}. \quad (10.52)$$

From these examples, we see that oscillations and amplification/decay in  $f(x)$  are related to the existence of poles in the algebraic expression for  $F(k)$ . The real part of the pole position gives the wave-number of the oscillation, and the distance from the pole to the real axis gives the amplification or decay constant. An exponentially decaying signal produces a pole in the upper half-plane, while an exponentially amplifying signal produces a pole in the lower half-plane. In both cases, the Fourier spectrum (i.e., the graph of  $|F(k)|^2$  versus  $k$ ) is a Lorentzian centered at  $k = q$ , with width  $2\eta$ .

### 10.6.2 Gaussian wave-packets

Consider a function with a decay envelope given by a Gaussian function (see Section 2.5):

$$f(x) = e^{iqx} e^{-\gamma x^2}, \quad \text{where } q \in \mathbb{C}, \gamma \in \mathbb{R}. \quad (10.53)$$

This is called a **Gaussian wave-packet**. The width of the envelope is usually characterized by the Gaussian function’s **standard deviation**, which is  $\Delta x = 1/\sqrt{2\gamma}$ .

We will show that  $f(x)$  has the following Fourier transform:

$$F(k) = \sqrt{\frac{\pi}{\gamma}} e^{-\frac{(k-q)^2}{4\gamma}}. \quad (10.54)$$

To derive this result, we perform the Fourier integral as follows:

$$F(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \quad (10.55)$$

$$= \int_{-\infty}^{\infty} dx \exp[-i(k-q)x - \gamma x^2]. \quad (10.56)$$

In the integrand, the expression inside the exponential is quadratic in  $x$ . We complete the square:

$$F(k) = \int_{-\infty}^{\infty} dx \exp \left[ -\gamma \left( x + \frac{i(k-q)}{2\gamma} \right)^2 + \gamma \left( \frac{i(k-q)}{2\gamma} \right)^2 \right] \quad (10.57)$$

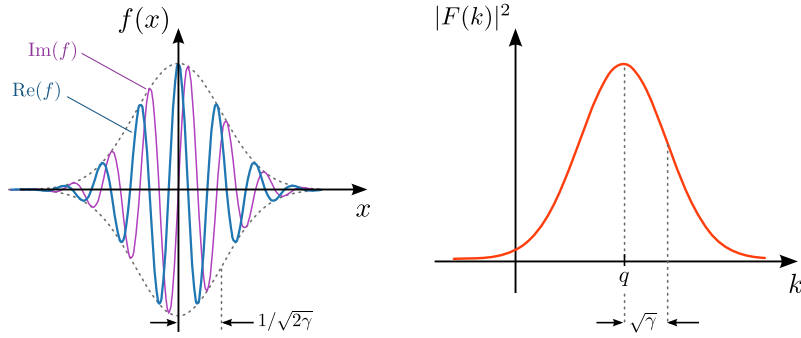
$$= \exp \left[ -\frac{(k-q)^2}{4\gamma} \right] \int_{-\infty}^{\infty} dx \exp \left[ -\gamma \left( x + \frac{i(k-q)}{2\gamma} \right)^2 \right]. \quad (10.58)$$

The remaining integral is the Gaussian integral (see Section 2.5) with a constant imaginary shift in  $x$ . By shifting the integration variable, one can show that this is equal the standard Gaussian integral,  $\sqrt{\pi/\gamma}$  (the details are left as an exercise for the reader). We thus arrive at the result stated above.

The Fourier spectrum,  $|F(k)|^2$ , is a Gaussian function. Its standard deviation is

$$\Delta k = \frac{1}{\sqrt{2(1/2\gamma)}} = \sqrt{\gamma}. \quad (10.59)$$

Once again, the Fourier spectrum is peaked at a value of  $k$  corresponding to the wave-number of the underlying sinusoidal wave in  $f(x)$ , and a stronger (weaker) decay in  $f(x)$  leads to a broader (narrower) Fourier spectrum.



## 10.7 The delta function

What happens when we feed the Fourier relations into one another? Plugging the Fourier transform into the inverse Fourier transform,

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(k) \quad (10.60)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \int_{-\infty}^{\infty} dx' e^{-ikx'} f(x') \quad (10.61)$$

$$= \int_{-\infty}^{\infty} dx' \left[ \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} e^{-ikx'} \right] f(x'). \quad (10.62)$$

Let us denote the term in brackets in the last line as

$$\delta(x - x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')}. \quad (10.63)$$

This is called the **delta function**. The delta function acts as a kind of filter. When we multiply it by any function  $f(x')$  and integrate over  $x'$ , the result is the value of that function at a particular point  $x$ :

$$\int_{-\infty}^{\infty} dx' \delta(x - x') f(x') = f(x). \quad (10.64)$$

But here's a problem: the integral that we used to define the delta function is non-convergent. In particular, the integrand does not vanish at  $\pm\infty$ . We can get around this by defining the delta function as a limiting case of a convergent integral:

$$\delta(x - x') \equiv \lim_{\gamma \rightarrow 0} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} e^{-\gamma k^2}. \quad (10.65)$$

The extra factor of  $\exp(-\gamma k^2)$ , which we have inserted into the integrand, ensures that the integrand vanishes at  $\pm\infty$ , making the integral convergent. As  $\gamma \rightarrow 0$ , this factor goes to one, and the integrand approaches what we had before. But note that the expression on the right is the Fourier transform for a Gaussian wave-packet (see Section 10.6.2), so

$$\delta(x - x') \equiv \lim_{\gamma \rightarrow 0} \frac{1}{\sqrt{4\pi\gamma}} e^{-\frac{(x-x')^2}{4\gamma}}. \quad (10.66)$$

This is a Gaussian function of width  $\sqrt{2\gamma}$  and area 1. Hence, the delta function can be regarded as the limit of a Gaussian function as its width goes to zero while keeping the area under the curve fixed at unity (which means the height of the peak goes to infinity).

The “filtering” behavior of the delta function  $\delta(x - x')$  can be understood from its nature as an infinitesimal-width Gaussian. When we multiply a function  $f(x')$  by  $\delta(x - x')$ , the product is non-zero only in the vicinity of  $x' = x$ . Since the area under the delta function is unity, integrating that product over all  $x'$  yields the value of  $f(x')$  at  $x' = x$ .

*Note*—In physics, the delta function is commonly used to represent the density distributions of point particles. For instance, the distribution of mass within an object can be represented by a mass density function. Assuming one-dimensional space for simplicity, we define the mass density  $\rho(x)$  as the mass per unit length at position  $x$ . By this definition,

$$M = \int_{-\infty}^{\infty} \rho(x) dx \quad (10.67)$$

is the total mass of the object. Now suppose the mass is distributed among  $N$  point particles, which are located at distinct positions  $x_1, x_2, \dots, x_N$ , and have masses  $m_1, m_2, \dots, m_N$ . To describe this situation, we can write the mass density function as

$$\rho(x) = \sum_{j=1}^N m_j \delta(x - x_j). \quad (10.68)$$

The reason for this is that if we integrate  $\rho(x)$  around the vicinity of the  $j$ -th particle, the result is just the mass of that single particle, thanks to the features of the delta function:

$$\lim_{\varepsilon \rightarrow 0^+} \int_{x_j - \varepsilon}^{x_j + \varepsilon} \rho(x) dx = \sum_{i=1}^N m_i \left[ \lim_{\varepsilon \rightarrow 0^+} \int_{x_j - \varepsilon}^{x_j + \varepsilon} \delta(x - x_i) dx \right] \quad (10.69)$$

$$= \sum_{i=1}^N m_i \delta_{ij} \quad (10.70)$$

$$= m_j. \quad (10.71)$$

Likewise, integrating  $\rho(x)$  over all  $x$  gives the total mass  $m_1 + m_2 + \dots + m_N$ .

## 10.8 Multi-dimensional Fourier transforms (optional topic)

When studying problems such as wave propagation, we often deal with Fourier transforms of several variables. This is conceptually straightforward. For a function  $f(x_1, x_2, \dots, x_d)$  which depends on  $d$  independent spatial coordinates  $x_1, x_2, \dots, x_d$ , we can Fourier transform each coordinate individually:

$$F(k_1, k_2, \dots, k_d) = \int_{-\infty}^{\infty} dx_1 e^{-ik_1 x_1} \int_{-\infty}^{\infty} dx_2 e^{-ik_2 x_2} \dots \int_{-\infty}^{\infty} dx_d e^{-ik_d x_d} f(x_1, x_2, \dots, x_d). \quad (10.72)$$

Each coordinate gets Fourier-transformed into its own independent  $k$  variable, so the result is also a function of  $d$  independent variables.

We can express the multi-dimensional Fourier transform more compactly using vector notation. If  $\vec{x}$  is a  $d$ -dimensional coordinate vector, the Fourier-transformed coordinates can be written as  $\vec{k}$ , and the Fourier transform is

$$F(\vec{k}) = \int d^d x \exp(-i \vec{k} \cdot \vec{x}) f(\vec{x}), \quad (10.73)$$

where  $\int d^d x$  denotes an integral over the entire  $d$ -dimensional space, and  $\vec{k} \cdot \vec{x}$  is the usual dot product of two vectors. The inverse Fourier transform is

$$f(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} \exp(i \vec{k} \cdot \vec{x}) F(\vec{k}). \quad (10.74)$$

The delta function can also be defined in  $d$ -dimensional space, as the Fourier transform of a plane wave:

$$\delta^d(\vec{x} - \vec{x}') = \int \frac{d^d k}{(2\pi)^d} \exp[i \vec{k} \cdot (\vec{x} - \vec{x}')]. \quad (10.75)$$

Note that  $\delta^d$  has the dimensions of  $[x]^{-d}$ . The multi-dimensional delta function has a filtering property similar to the one-dimensional delta function. For any  $f(x_1, \dots, x_d)$ ,

$$\int d^d x \delta^d(\vec{x} - \vec{x}') f(\vec{x}) = f(\vec{x}'). \quad (10.76)$$

## 10.9 Exercises

1. Find the relationship between the coefficients  $\{\alpha_n, \beta_m\}$  in the sine/cosine Fourier series and the coefficients  $f_n$  in the complex exponential Fourier series:

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{2\pi n x}{a}\right) + \sum_{m=0}^{\infty} \beta_m \cos\left(\frac{2\pi m x}{a}\right) \quad (10.77)$$

$$= \sum_{n=-\infty}^{\infty} f_n \exp\left(\frac{2\pi i n x}{a}\right). \quad (10.78)$$

2. Consider the triangular wave

$$f(x) = \begin{cases} -x, & -a/2 \leq x < 0, \\ x, & 0 \leq x < a/2 \end{cases} \quad (10.79)$$

- (a) Derive the Fourier series expansion.
- (b) Plot the Fourier series numerically, and show that it converges to the triangular wave as the number of terms increases.

3. A periodic function  $f(x)$  (with period  $a$ ) is written as a complex Fourier series with coefficients  $\{f_0, f_{\pm 1}, f_{\pm 2}, \dots\}$ . Determine the relationship(s) between the Fourier coefficients under each of the following scenarios:

- (a)  $f(x)$  is real for all  $x$ .  
 (b)  $f(x) = f(-x)$  for all  $x$   
 (c)  $f(x) = f(-x)^*$  for all  $x$ .

[solution available]

4. Prove the properties of the Fourier transform listed in Section 10.4.

5. Find the Fourier transform of  $f(x) = \sin(\kappa x)/x$ .

6. Prove that if  $f(x)$  is a real function, then its Fourier transform satisfies  $F(k) = F(-k)^*$ .

7. Prove that

$$\delta(ax) = \frac{1}{a} \delta(x), \quad (10.80)$$

where  $a$  is any nonzero real number.

[solution available]

8. Calculate

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy x^2 \delta(\sqrt{x^2 + y^2} - a), \quad (10.81)$$

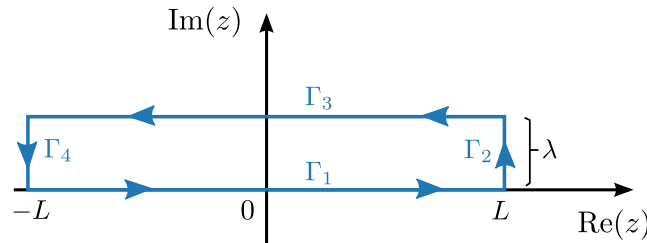
where  $a$  is a real number.

[solution available]

9. Consider the integral

$$I = \int_{-\infty}^{\infty} e^{-\gamma(x+i\lambda)^2} dx,$$

where  $\gamma \in \mathbb{R}^+$  and  $\lambda \in \mathbb{R}$ . (This is the Gaussian integral with an imaginary displacement in the integration variable, which we encountered while discussing Gaussian wave-packets (Section 10.6.2). To solve this integral, consult the contour shown in the figure below (for the case  $\lambda > 0$ ):



Show, by parameterization, that

(i)  $I = - \lim_{L \rightarrow \infty} \int_{\Gamma_3} e^{-\gamma z^2} dz$ , and

(ii)  $\lim_{L \rightarrow \infty} \int_{\Gamma_2 \text{ or } \Gamma_4} e^{-\gamma z^2} dz = 0$ .

Hence, explain why  $I = \sqrt{\pi/\gamma}$ .