# 5. Complex Oscillations

In physics and the other quantitative sciences, complex numbers are widely used for analyzing oscillations and waves. We begin our study of this topic with an elementary model called the **damped harmonic oscillator**.

5.1   The damped harmonic oscillator

A particle of mass m moves along one dimension, with $x(t)$ denoting its displacement at time t. It is subject to two forces: a spring force and a damping force. The spring constant is $k = m\omega_0^2$, and the damping coefficient is $2m\gamma$. The parameters m, γ, and ω_0 are all positive real numbers.
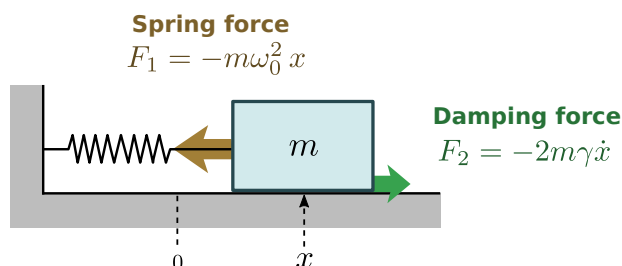
According to Newton's second law,

$$m\frac{d^2x}{dt^2} = F(x,t) = -2m\gamma\frac{dx}{dt} - m\omega_0^2 x(t). \tag{5.1}$$

Dividing by the common factor of m, and bringing everything to one side, gives

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x(t) = 0. \tag{5.2}$$

This is called the **damped harmonic oscillator equation**. It is a second-order ordinary differential equation (ODE), so its general solution must contain two free parameters. These parameters are usually (but not necessarily) specified by the initial displacement $x(0)$ and initial velocity $\dot{x}(0)$.

Note—Sometimes, we write the damped harmonic oscillator equation as:

$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right] x(t) = 0. \tag{5.3}$$

The quantity in square brackets is a linear differential operator acting on $x(t)$. The three terms in the operator correspond to the three ingredients of the damped harmonic oscillator model: (i) a second derivative term stemming from Newton's second law, (ii) a first derivative term representing damping, and (iii) a constant term representing the spring force.

Writing the differential equation this way emphasizes its linearity, a property that is important for finding the solutions, as discussed below.

5.1.1   Simple harmonic oscillator limit

For $\gamma = 0$ (zero damping), the system reduces to the **simple harmonic oscillator**. From previous physics courses, we know the general solution is

$$x(t) = A\cos(\omega_0 t + \phi), \tag{5.4}$$

where A and ϕ are free parameters. This is a sinusoidal oscillation with amplitude A, phase ϕ, and frequency ω_0.

The parameter ω_0 comes from the spring constant $k = m\omega_0^2$. (In fact, the spring constant was parameterized this way so that the solution ends up with this nice form.) We call ω_0 the **natural frequency**, meaning the frequency of the oscillator in the absence of damping or other disturbances.

> *Note*—Some authors call ω_0 the "angular frequency", reserving the term "frequency" for the quantity $f_0 = \omega_0/2\pi$. But since we will always deal with ω_0 rather than f_0, we will refer to ω_0 as simply "frequency".

5.1.2 Damped oscillations

For $\gamma > 0$, there is now a damping force opposing the motion of the oscillator. What form will the solutions take? Before launching into the mathematics, let's use our physical intuition to make a guess.

The damping force does work against the particle (since its sign is always opposite to the particle's velocity). If the damping force is very weak, the solution should not be too different from the simple harmonic oscillator solution—the particle should oscillate around the equilibrium point $x = 0$ with a frequency of around ω_0. But the damping force will cause it to lose a bit of energy every oscillatory cycle, resulting in an oscillation whose amplitude diminishes slowly over time. In the $t \to \infty$ limit, all the energy is lost, and x (as well as $\dot{x}$) should go asymptotically to zero.

So we would guess something like this:
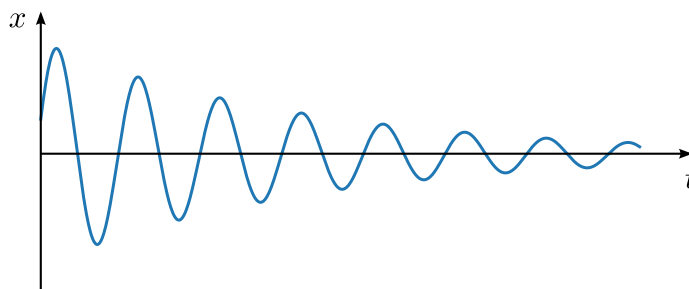
Let us see if the mathematical analysis agrees with this guess.

5.2 The complex damped harmonic oscillator equation

The variable $x(t)$ is the displacement of the particle, so it ought to be real. However, a good way to solve the damped harmonic oscillator equation is to generalize $x(t)$ to complex values. In other words, we convert the harmonic oscillator equation into a complex ODE:

$$\frac{d^2 z}{dt^2} + 2\gamma \frac{dz}{dt} + \omega_0^2 z(t) = 0, \quad z(t) \in \mathbb{C}. \tag{5.5}$$

The parameter-counting rule for real ODEs (see Section 1.3) generalizes to complex ODEs, except that the free parameters should be complex numbers. In this case, the complex damped harmonic oscillator equation is a second-order ODE, so its general solution must have two complex free parameters.

If we can find a solution $z(t)$ for the complex damped harmonic oscillator equation, then its real part $x(t) = \text{Re}[z(t)]$ would be a solution to the real damped harmonic oscillator

equation, since

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x(t) = \frac{d^2\mathrm{Re}[z]}{dt^2} + 2\gamma\frac{d\mathrm{Re}[z]}{dt} + \omega_0^2\,\mathrm{Re}[z(t)] \tag{5.6}$$

$$= \mathrm{Re}\left[\frac{d^2z}{dt^2} + 2\gamma\frac{dz}{dt} + \omega_0^2 z(t)\right] \tag{5.7}$$

$$= 0. \tag{5.8}$$

Here, we have used the fact that the $\mathrm{Re}[\cdots]$ operation can be freely shuffled in or out of derivatives and sums with real coefficients (see Section 3.1).

5.2.1   Complex ansatz

We now aim to derive the general solution for the complex damped harmonic oscillator equation.

First, note that the equation is linear. This means that for any two solutions $z_1(t)$ and $z_2(t)$, a linear superposition

$$z(t) = a_1\,z_1(t) \,+\, a_2\,z_2(t), \quad \text{where } a_1, a_2 \in \mathbb{C} \tag{5.9}$$

is also a solution. This can be verified by direct substitution into the ODE.

Due to linearity, a good strategy for finding the general solution is to identify two different specific solutions, $z_1(t)$ and $z_2(t)$. Then we can construct the above linear superposition, with the two complex coefficients a_1 and a_2 serving as free parameters. Any solution containing two free parameters is automatically the general solution.

So now we have to find some specific solutions. Let us make a guess, or **ansatz**:

$$z(t) = e^{-i\omega t}. \tag{5.10}$$

Here, ω is a constant to be determined. The first and second derivatives are:

$$\frac{dz}{dt} = -i\omega\,e^{-i\omega t}, \tag{5.11}$$

$$\frac{d^2z}{dt^2} = -\omega^2\,e^{-i\omega t}. \tag{5.12}$$

Substituting these into the damped harmonic oscillator equation gives

$$\left(-\omega^2 - 2i\gamma\omega + \omega_0^2\right)e^{-i\omega t} = 0. \tag{5.13}$$

This equation holds for all t if and only if the complex second-order polynomial on the left-hand side is zero:

$$-\omega^2 - 2i\gamma\omega + \omega_0^2 = 0. \tag{5.14}$$

The solutions to the polynomial can be obtained from the quadratic formula:

$$\omega = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \tag{5.15}$$

Hence, we have found the specific solutions

$$z_\pm(t) = \exp\left(-i\omega_\pm t\right), \quad \text{where } \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \tag{5.16}$$

Either ω_+ or ω_- gives a valid specific solution to the damped harmonic oscillator equation. Note that the solution is *specific* and contains no free parameters, since $\omega_\pm$ is entirely determined by γ and ω_0 (which are fixed parameters appearing in the ODE, not free parameters).

5.2.2   Complex frequencies

The specific solutions we found have the form $e^{-i\omega t}$, where ω is a complex "frequency". There are two choices of complex frequency,

$$\omega_{\pm} = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \tag{5.17}$$

The values depend on the oscillator parameters γ and ω_0. The plot below shows how $\omega_{\pm}$ move in the complex plane as γ and ω_0 are varied:
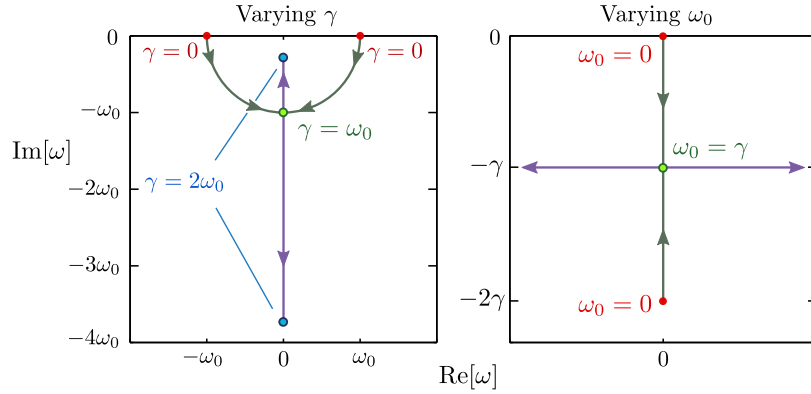
Note the following features:

1. For $\gamma = 0$, the frequencies are both real, with values $\pm \omega_0$.

2. If we increase γ from zero with ω_0 fixed, both ω_+ and ω_- move downwards in the complex plane, along a circular arc.

3. At $\gamma = \omega_0$, the frequencies meet along the imaginary axis.

4. For $\gamma > \omega_0$, the two frequencies move apart along the imaginary axis.

To understand the implications of these complex frequencies, let uswrite the real and imaginary parts of ω as $\omega_R + i\omega_I$. Then

$$z(t) = e^{-i\omega t} = e^{\omega_I t}\, e^{-i\omega_R t}. \tag{5.18}$$

If both ω_R and ω_I are non-zero, this describes a spiral trajectory in the complex plane (see Section 3.6) whose magnitude either increases or decreases with time, depending on the sign of ω_I. To see this explicitly, we can write

$$z(t) = e^{\omega_I t}\, e^{-i\omega_R t}. \tag{5.19}$$

Taking the real part,

$$\mathrm{Re}\,[z(t)] = e^{\omega_I t}\cos\left[\omega_R t\right]. \tag{5.20}$$

Assuming that $\gamma > 0$, the complex frequencies given by Eq. (5.17) always have $\omega_I < 0$. Referring to Eq. (5.20), this means the solutions are always damped. Moreover, if $\omega_R \neq 0$, then $z(t)$ executes an anticlockwise (for ω_+) or clockwise (for ω_-) inward spiral; in either case, $\mathrm{Re}[z(t)]$ describes an oscillation with diminishing amplitude, consistent with our guess from Section 5.1.2. On the other hand, if $\omega_R = 0$, then the solution is a pure exponential decay with no oscillation. In the next few sections, we will undertake a systematic examination of these two distinct behaviors.

5.3   General solution for the damped harmonic oscillator

For now, let us suppose that $\omega_0 \neq \gamma$. Then we have two distinct specific solutions,

$$z_\pm(t) = e^{-i\omega_\pm t}, \quad \text{where} \quad \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \tag{5.21}$$

By taking a linear superposition of these specific solutions, we obtain the general solution for the complex damped harmonic oscillator equation:

$$z(t) = a_+ e^{-i\omega_+ t} + a_- e^{-i\omega_- t}, \tag{5.22}$$

where a_+ and a_- are independent complex free parameters.

   (Note that if $\omega_0 = \gamma$, then $z_+(t) = z_-(t)$, so a_+ and a_- would be coefficients multiplying the same function, and we would not be allowed to treat them as two independent free parameters. We will discuss how to handle this case in Section 5.3.3.)

   To find solutions to the real damped harmonic oscillator equation, we take $x(t) = \text{Re}[z(t)]$. The resulting expression will depend on whether $\omega_0 > \gamma$ or $\omega_0 < \gamma$. These two cases lead to **under-damped solutions** and **over-damped solutions**, respectively, as discussed in the next two sections.

5.3.1   Under-damped motion

First, consider $\omega_0 > \gamma$. Let us define

$$\Omega = \sqrt{\omega_0^2 - \gamma^2} \in \mathbb{R}, \tag{5.23}$$

so that $\omega_\pm = -i\gamma \pm \Omega$. Plugging this into the complex general solution gives

$$z(t) = a_+ e^{-\gamma t} e^{-i\Omega t} + a_- e^{-\gamma t} e^{i\Omega t} \tag{5.24}$$

$$= e^{-\gamma t}\left[a_+ e^{-i\Omega t} + a_- e^{i\Omega t}\right]. \tag{5.25}$$

We can use Euler's formula to simplify the terms in the brackets:

$$a_+ e^{-i\Omega t} + a_- e^{i\Omega t} = a_+\left[\cos(\Omega t) - i\sin(\Omega t)\right] + a_-\left[\cos(\Omega t) + i\sin(\Omega t)\right] \tag{5.26}$$

$$= \left(a_+ + a_-\right)\cos(\Omega t) \; - \; i\left(a_+ - a_-\right)\sin(\Omega t). \tag{5.27}$$

Hence,

$$x(t) = \text{Re}\left[z(t)\right] \tag{5.28}$$

$$= e^{-\gamma t}\left[A\cos\left(\Omega t\right) + B\sin\left(\Omega t\right)\right], \quad \text{where} \quad \begin{cases} A = \text{Re}\left[a_+ + a_-\right] \\ B = \text{Im}\left[a_+ - a_-\right]. \end{cases} \tag{5.29}$$

This is called an **under-damped solution**. The coefficients A and B are two independent *real* parameters, so this serves as a general solution for the real damped harmonic oscillator equation. Using the trigonometric formulas, the solution can be equivalently written as

$$x(t) = C\, e^{-\gamma t}\cos\left(\Omega t + \Phi\right), \quad \text{where} \quad \begin{cases} C = \sqrt{A^2 + B^2}, \\ \Phi = -\tan^{-1}\left[B/A\right]. \end{cases} \tag{5.30}$$

This shows explicitly that it consists of a sinusoidal oscillation (with frequency Ω) overlaid on an "envelope" given by the exponentially decreasing function $\exp(-\gamma t)$. The graph of $x(t)$ versus t is plotted below:

$\gamma = 0.05, \omega_0 = 1.0$
$x_0 = 1, v_0 = 3$
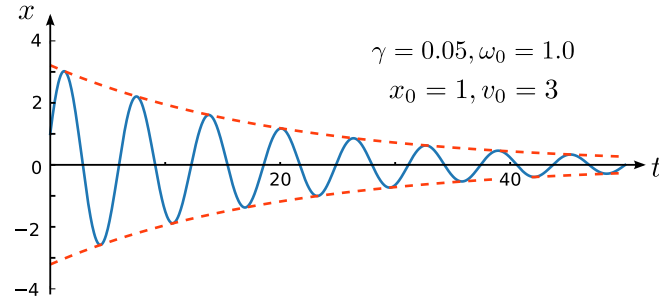
5.3.2   Over-damped motion

For $\omega_0 < \gamma$, the square root term in $\omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$ is imaginary. Let us define

$$\Gamma = \sqrt{\gamma^2 - \omega_0^2} \quad \Rightarrow \quad \omega_\pm = i\left(-\gamma \pm \Gamma\right). \tag{5.31}$$

Then the complex general solution can be simplified to

$$z(t) = a_+ e^{-(\gamma - \Gamma)\,t} + a_- e^{-(\gamma + \Gamma)\,t}, \tag{5.32}$$

and the real solution is

$$x(t) = \mathrm{Re}\left[z(t)\right] \tag{5.33}$$

$$= C_+ e^{-(\gamma - \Gamma)\,t} + C_- e^{-(\gamma + \Gamma)\,t}, \quad \text{where } C_\pm = \mathrm{Re}[a_\pm]. \tag{5.34}$$

This is called an **over-damped solution**. It consists of two exponentially decaying terms, with decay rates $(\gamma - \Gamma)$ and $(\gamma + \Gamma)$ respectively. Since $\Gamma < \gamma$, both decay rates are positive real numbers, but note that $(\gamma - \Gamma)$ *decreases* with γ, whereas $(\gamma + \Gamma)$ *increases* with γ, as shown below:
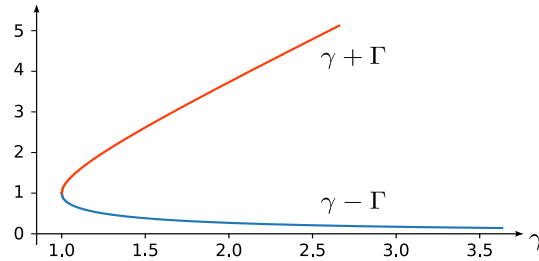
Since $\gamma + \Gamma$ is associated with a faster-decaying exponential, for large t the second term becomes negligible compared to the first term, and the solution has the limiting form

$$x(t) \approx C_+ e^{-(\gamma - \Gamma)t} \qquad \text{(for large } t\text{)}. \tag{5.35}$$

A plot of $x(t)$ versus t is shown below, with the limiting form plotted as dashes.

$\gamma = 0.05, \omega_0 = 1.0$
$x_0 = 1, v_0 = 3$
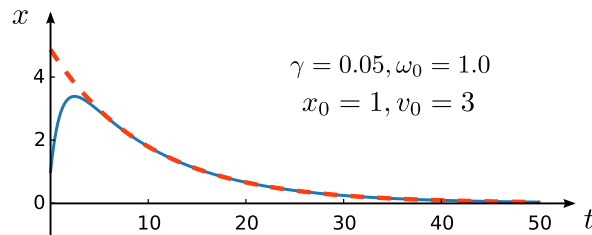
The over-damped solution has an interesting feature: *for stronger damping, the decay rate at long times is slower.* This is the opposite of the under-damped oscillator's behavior! Mathematically, it happens because the decay rate $(\gamma - \Gamma)$, appearing in the limiting form of $x(t)$ for large t, is a decreasing function of γ.

In the over-damped regime, the motion of the oscillator is dominated by the damping force rather than the spring force. Therefore, as the oscillator tries to return to its equilibrium position $x = 0$, the damping acts against this motion, and the stronger the damping, the slower the decay to equilibrium. By contrast, in the under-damped regime, the spring force is dominant, so stronger damping leads to faster decay via faster dissipation of the oscillator's kinetic energy.

5.3.3   Critical damping

Critical damping occurs when $\omega_0 = \gamma$. Under this special condition, the solution given in Eq. (5.22) reduces to

$$z(t) = (a_+ + a_-)\, e^{-\gamma t}. \tag{5.36}$$

This has only *one* independent complex parameter, i.e. the parameter $(a_+ + a_-)$. Therefore, it cannot be a general solution for the complex damped harmonic oscillator equation.

We will not go into detail here regarding the procedure for finding the general solution for the critically-damped oscillator, leaving it as an exercise for the interested reader. Basically, it involves Taylor expanding the solution near the critical point, and then showing that there is a solution of the form

$$z(t) = (A + Bt)\, e^{-\gamma t}, \tag{5.37}$$

which contains the desired two independent parameters.

The critically-damped solution contains an exponential decay constant of γ, which is the same as the decay constant for the envelope function in the under-damped regime (Section 5.3.1), and *smaller* than the (long-time) decay constants in the over-damped regime (Section 5.3.2). Hence, we can regard the critically-damped solution as the *fastest-decaying non-oscillatory solution.*

This feature of critical damping is employed in many engineering contexts, the most familiar being automatic door closers. If the damping is too weak or the spring force is too strong (under-damped), the door slams shut, whereas if the damping is too strong or the spring force is too weak (under-damping), the door takes unnecessarily long to swing shut. For best performance, an automatic door closer should be tuned to a "sweet spot" that corresponds to the critical point of a damped harmonic oscillator.

5.4   Stating the solution in terms of initial conditions

The general solution for the complex damped harmonic oscillator equation contains two undetermined parameters which are the complex amplitudes of the "clockwise" and "counterclockwise" complex oscillations:

$$z(t) = a_+ e^{-i\omega_+ t} + a_- e^{-i\omega_- t}, \quad \text{where } \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \tag{5.38}$$

However, mechanics problems are often expressed in terms of an **initial value problem**, specifying the state of the system at some initial time $t = 0$. In other words, given $z(0) \equiv x_0$ and $\dot{z}(0) \equiv v_0$, what is $z(t)$ in terms of x_0 and v_0?

We can solve the initial-value problem by finding $z(0)$ and $\dot{z}(0)$ in terms of the above general solution for $z(t)$:

$$z(0) = \quad a_+ + a_- \qquad\quad = x_0 \tag{5.39}$$

$$\dot{z}(0) = -i\omega_+ a_+ - i\omega_- a_- = v_0. \tag{5.40}$$

These two equations can be combined into a 2×2 matrix equation:

$$\begin{bmatrix} 1 & 1 \\ -i\omega_+ & -i\omega_- \end{bmatrix} \begin{bmatrix} a_+ \\ a_- \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}. \tag{5.41}$$

So long as $\omega_+ \neq \omega_-$, the matrix is non-singular, and we can invert it to obtain $a_\pm$:

$$\begin{bmatrix} a_+ \\ a_- \end{bmatrix} = \frac{1}{i(\omega_+ - \omega_-)} \begin{bmatrix} -i\omega_- x_0 - v_0 \\ i\omega_+ x_0 + v_0 \end{bmatrix}. \tag{5.42}$$

We can plug these coefficients back into the general solution. After some algebra, the result simplifies to

$$z(t) = e^{-\gamma t} \left[x_0 \cos(\Omega t) + \frac{\gamma x_0 + v_0}{\Omega} \sin(\Omega t) \right], \quad \text{where } \Omega \equiv \sqrt{\omega_0^2 - \gamma^2}. \tag{5.43}$$

For the under-damped case, Ω is real, and this solution is consistent with the one we derived om Section 5.3.1, except that it is now explicitly expressed in terms our initial conditions x_0 and v_0. As for the over-damped case, we can perform the replacement

$$\Omega \to i\Gamma = i\sqrt{\gamma^2 - \omega_0^2}. \tag{5.44}$$

Then, using the relationships between trigonometric and hyperbolic functions from Section 3.5.3, the solution can be re-written as

$$z(t) = e^{-\gamma t} \left[x_0 \cosh(\Gamma t) + \frac{\gamma x_0 + v_0}{i\Gamma} i \sinh(\Gamma t) \right] \tag{5.45}$$

$$= \left(\frac{x_0}{2} + \frac{\gamma x_0 + v_0}{2\Gamma} \right) e^{-(\gamma - \Gamma)t} + \left(\frac{x_0}{2} - \frac{\gamma x_0 + v_0}{2\Gamma} \right) e^{-(\gamma + \Gamma)t}, \tag{5.46}$$

which is consistent with the result found in Section 5.3.2.

In either case, so long as we plug in real values for x_0 and v_0, the solution is guaranteed to be real for all t. That's to be expected, since the real solution is also one of the specific solutions for the complex harmonic oscillator equation.

5.5 Exercises

1. In Section 5.2.2, we encountered the complex frequencies

$$\omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \tag{5.47}$$

For fixed ω_0 and $\omega_0 > \gamma$ (under-damping), prove that $\omega_\pm$ lie along a circular arc in the complex plane.

2. Derive the general solution for the critically damped harmonic oscillator by following these steps:

(a) Consider the complex ODE, in the under-damped regime $\omega_0 > \gamma$. We saw in Section 5.3 that the general solution has the form

$$z(t) = \psi_+ \exp\left[\left(-\gamma - i\sqrt{\omega_0^2 - \gamma^2} \right) t \right] + \psi_- \exp\left[\left(-\gamma + i\sqrt{\omega_0^2 - \gamma^2} \right) t \right] \tag{5.48}$$

for some complex parameters ψ_+ and ψ_-. Define the positive parameter $\varepsilon = \sqrt{\omega_0^2 - \gamma^2}$. Re-write $z(t)$ in terms of γ and ε (i.e., eliminating ω_0).

(b) The expression for $z(t)$ is presently parameterized by the independent parameters ψ_+, ψ_-, ε, and γ. We are free to re-define the parameters, by taking

$$\alpha = \psi_+ + \psi_- \tag{5.49}$$

$$\beta = -i\varepsilon(\psi_+ - \psi_-). \tag{5.50}$$

Using these equations, express $z(t)$ using a new set of independent complex parameters, one of which is ε. Explicitly identify the other independent parameters, and state whether they are real or complex.

(c) Expand the exponentials in $z(t)$ in terms of the parameter ε. Then show that in the limit $\varepsilon \to 0$, $z(t)$ reduces to the critically-damped general solution (5.37).

3. Repeat the above derivation for the critically-damped solution, but starting from the over-damped regime $\gamma > \omega_0$.

4. Let $z(t)$ be a complex function of a real input t, which obeys the differential equation

$$\frac{dz}{dt} = -i\left(\omega_1 - i\gamma\right) z(t), \tag{5.51}$$

where ω_1 and γ are real. Find the general solution for $z(t)$, and hence show that $z(t)$ satisfies the damped oscillator equation

$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right] z(t) = 0 \tag{5.52}$$

for some ω_0^2. Finally, show that this harmonic oscillator is always under-damped.

[solution available]