11. Green's Functions

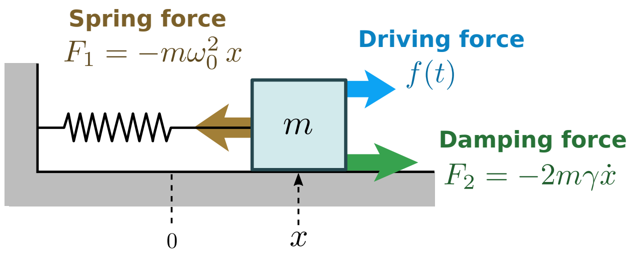
A **Green's function** is a solution to an inhomogenous differential equation with a delta function "driving term". It provides a convenient method for solving more complicated inhomogenous differential equations. In physics, Green's functions methods are used to describe a wide range of physical phenomena, such as the response of mechanical systems to impacts or the emission of sound waves from acoustic sources.

11.1 The driven harmonic oscillator

As an introduction to the Green's function technique, we will study the **driven harmonic oscillator**, which is a damped harmonic oscillator subjected to an additional arbitrary driving force. The equation of motion is

$$\left[\frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega_0^2\right] x(t) = \frac{f(t)}{m}. \tag{11.1}$$

Here, m is the mass of the particle, γ is the damping coefficient, and ω_0 is the natural frequency of the oscillator. The left side of the qquation is the same as in the damped harmonic oscillator equation (see Chapter 5). On the right side, we introduce a time-dependent driving force $f(t)$, which acts alongside the pre-existing spring and damping forces. Given an arbitrarily complicated $f(t)$, our goal is to determine $x(t)$.



11.1.1 Green's function for the driven harmonic oscillator

Prior to solving the driven harmonic oscillator problem for a general driving force $f(t)$, let us first consider the following equation:

$$\left[\frac{\partial^2}{\partial t^2} + 2\gamma \frac{\partial}{\partial t} + \omega_0^2\right] G(t,t') = \delta(t-t'). \tag{11.2}$$

The function $G(t,t')$, which depends on the two variables t and t', is called the **Green's function**. Note that the differential operator on the left side involves only derivatives in t.

The Green's function describes the motion of a damped harmonic oscillator subjected to a particular driving force that is a delta function (see Section 9.7), describing an infinitesimally sharp pulse centered at $t = t'$:

$$\frac{f(t)}{m} = \delta(t-t'). \tag{11.3}$$

Here's the neat thing about $G(t,t')$: once we know it, we can find a specific solution to the driven harmonic oscillator equation for *any* $f(t)$. The solution has the form

$$x(t) = \int_{-\infty}^{\infty} dt'\; G(t,t')\, \frac{f(t')}{m}. \tag{11.4}$$

To show that this is indeed a solution, plug it into the equation of motion:

$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right] x(t) = \int_{-\infty}^{\infty} dt' \, \left[\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \omega_0^2\right] G(t,t')\frac{f(t')}{m} \tag{11.5}$$

$$= \int_{-\infty}^{\infty} dt' \, \delta(t-t') \, \frac{f(t')}{m} \tag{11.6}$$

$$= \frac{f(t)}{m}. \tag{11.7}$$

Note that we can move the differential operator inside the integral because t and t' are independent variables.

The Green's function concept is based on the principle of superposition. The motion of the oscillator is induced by the driving force, but the value of $x(t)$ at time t does not depend only on the instantaneous value of $f(t)$ at time t; instead, it depends on the values of $f(t')$ over all past times $t' < t$. We can thus decompose f into a superposition of pulses described by delta functions at different times. Then $x(t)$ is a superposition of the oscillations produced by the individual pulses.

11.1.2 Finding the Green's function

To find the Green's function, we can use the Fourier transform (Chapter 10). Let us assume that the Fourier transform of $G(t,t')$ with respect to t is convergent, and that the oscillator is not critically damped (i.e., $\omega_0 \neq \gamma$; see Section 4.3.3). The Fourier transformation of the Green's function (also called the **frequency-domain Green's function**) is

$$G(\omega,t') = \int_{-\infty}^{\infty} dt \, e^{i\omega t} \, G(t,t'). \tag{11.8}$$

Here, we have used the sign convention for time-domain Fourier transforms (see Section 9.3). Applying the Fourier transform to both sides of the Green's function equation, we get

$$\left[-\omega^2 - 2i\gamma\omega + \omega_0^2\right] G(\omega,t') = \int_{-\infty}^{\infty} dt \, e^{i\omega t} \, \delta(t-t') = e^{i\omega t'}. \tag{11.9}$$

The differential equation for $G(t,t')$ has thus been converted into an *algebraic* equation for $G(\omega,t')$, whose solution is

$$G(\omega,t') = -\frac{e^{i\omega t'}}{\omega^2 + 2i\gamma\omega - \omega_0^2}. \tag{11.10}$$

Finally, we retrieve the time-domain solution by using the inverse Fourier transform:

$$G(t,t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega t} G(\omega,t') \tag{11.11}$$

$$= -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\gamma\omega - \omega_0^2}. \tag{11.12}$$

The denominator of the integral is a quadratic expression, so this can be re-written as:

$$G(t,t') = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \frac{e^{-i\omega(t-t')}}{(\omega-\omega_+)(\omega-\omega_-)} \quad \text{where } \; \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \tag{11.13}$$

This can be evaluated by contour integration. The integrand has two poles, which are precisely the complex frequencies of the damped harmonic oscillator; both lie in the negative complex plane. For $t < t'$, Jordan's lemma requires us to close the contour in the upper half-plane, enclosing neither pole, so the integral is zero. For $t > t'$, we must close the contour in the lower half-plane, enclosing both poles, so the result is

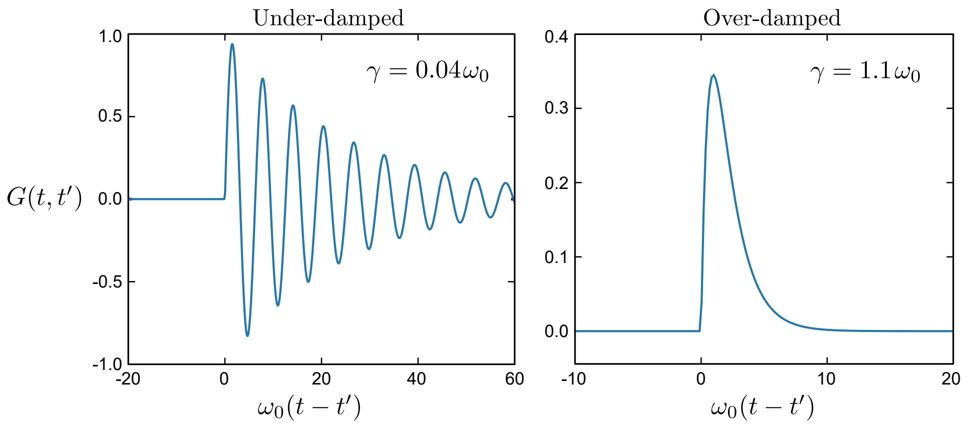
$$G(t,t') = i\Theta(t-t') \left[\frac{e^{-i\omega_+(t-t')}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_-(t-t')}}{\omega_- - \omega_+}\right] \tag{11.14}$$

$$= \Theta(t-t')\, e^{-\gamma(t-t')} \times \begin{cases} \frac{1}{\sqrt{\omega_0^2-\gamma^2}} \sin\left[\sqrt{\omega_0^2-\gamma^2}(t-t')\right], & \gamma < \omega_0, \\ \frac{1}{\sqrt{\gamma^2-\omega_0^2}} \sinh\left[\sqrt{\gamma^2-\omega_0^2}(t-t')\right], & \gamma > \omega_0. \end{cases} \tag{11.15}$$

Here, $\Theta(t-t')$ refers to the step function

$$\Theta(\tau) = \begin{cases} 1, & \text{for } \tau \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{11.16}$$

This result is plotted below. The solution for the critically damped case, $\gamma = \omega_0$, is left as an exercise.



11.1.3 Features of the Green's function

The time-domain Green's function represents the motion of the oscillator in response to a pulse of force, $f(t) = m\,\delta(t-t')$. Let us examine its features in greater detail.

The first thing to notice is that the Green's function depends on t and t' only in the combination $t - t'$. This makes sense: the response of the oscillator to the force pulse should only depend on the time elapsed since the pulse. We can exploit this property by re-defining the frequency-domain Green's function as

$$G(\omega) = \int_{-\infty}^{\infty} dt\; e^{i\omega(t-t')}\, G(t-t'), \tag{11.17}$$

which then obeys

$$\left[-\omega^2 - 2i\gamma\omega + \omega_0^2\right] G(\omega) = 1. \tag{11.18}$$

This is nicer to work with, as there is no extraneous t' variable present.

Next, note how the Green's function behaves just before and after the pulse. Its value is zero for all $t - t' < 0$ (i.e., prior to the pulse). This feature will be discussed in greater detail in the next section. Moreover, there is no discontinuity in $x(t)$ at $t - t' = 0$; the force pulse does not cause the oscillator to "teleport" instantaneously to a different position. Instead, it produces a discontinuity in the oscillator's velocity.

We can calculate the velocity discontinuity by integrating the Green's function equation over an infinitesimal interval of time surrounding t':

$$\lim_{\epsilon\to 0}\int_{t'-\epsilon}^{t'+\epsilon} dt \left[\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \omega_0^2\right] G(t,t') = \lim_{\epsilon\to 0}\int_{t'-\epsilon}^{t'+\epsilon} dt\;\delta(t-t') \tag{11.19}$$

$$= \lim_{\epsilon\to 0}\left\{ \frac{\partial G(t,t')}{\partial t}\Bigg|_{t=t'+\epsilon} - \frac{\partial G(t,t')}{\partial t}\Bigg|_{t=t'-\epsilon}\right\} = 1. \tag{11.20}$$

On the last line, the expression on the left-hand side represents the difference between the velocities just after and before the pulse. Evidently, the pulse imparts one unit of velocity at $t = t'$. Looking at the solutions obtained in Section 11.1.2, we can verify that indeed $\partial G/\partial t = 0$ right before the pulse, and $\partial G/\partial t = 1$ right after it.

For $t - t' > 0$, the applied force goes back to zero, and the system behaves like the undriven harmonic oscillator. If the oscillator is under-damped ($\gamma < \omega_0$), it undergoes a decaying oscillation around the origin. If the oscillator is over-damped ($\gamma > \omega_0$), it moves ahead for some distance, then settles exponentially back to the origin.

11.1.4 Causality

We have seen that the motion $x(t)$ ought to depend on the driving force $f(t')$ at all past times $t' < t$, but should *not* depend on the force at future times. Because of the relation

$$x(t) = \int_{-\infty}^{\infty} dt'\; G(t,t')\,\frac{f(t')}{m}, \tag{11.21}$$

this means that the Green's function ought to satisfy

$$G(t,t') = 0 \;\;\text{for}\;\; t - t' < 0. \tag{11.22}$$

This condition is referred to as **causality**, because it is equivalent to saying that *cause* must precede *effect*. A Green's function with this feature is called a **causal Green's function**.

For the driven harmonic oscillator, the time-domain Green's function satisfies a second-order differential equation, so its general solution must contain two free parameters. The specific solution we derived in Section 11.1.2 turns out to be the *only* causal solution. There are a couple of ways to see why.

The first way is to observe that for $t > t'$, the Green's function satisfies the differential equation for the *undriven* harmonic oscillator. But based on the discussion in Section 11.1.3, the causal Green's function needs to obey two conditions at $t = t' + 0^+$: (i) $G = 0$, and (ii) $\partial G/\partial t = 1$. These act as two boundary conditions for the undriven harmonic oscillator equation, giving rise to the specific solution that we found.

The other way to see that the causal Green's function is unique is to imagine adding to our specific solution any solution $x_1(t)$ for the undriven harmonic oscillator. It is easily verified that the resulting $G(t,t')$ is also a solution to the Green's function equation (11.2). Since the general solution for $x_1(t)$ contains two free parameters, we have thus found the general solution for $G(t,t')$. But the solutions for $x_1(t)$ are all infinite in the $t \to -\infty$ limit, *except* for the trivial solution $x_1(t) = 0$. That choice corresponds to the causal Green's function we found.

11.2 Space-time Green's functions (optional topic)

The Green's function method can also be used for studying waves. For simplicity, we restrict the following discussion to waves propagating through a uniform medium. Also, we will just consider 1D space; the generalization to higher spatial dimensions is straightforward.

As discussed in Chapter 6, wave propagation can be modelled by the wave equation

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(x,t) = 0, \tag{11.23}$$

where $\psi(x,t)$ is a complex wavefunction and c is the wave speed. Henceforth, to simplify the equations, we will set $c = 1$. (You can reverse this simplification by replacing all instances of t with ct, and ω with ω/c, in the subsequent formulas.)

The wave equation describes how waves propagate *after* they have already been created. To describe how the waves are generated in the first place, we must modify the wave equation by introducing a term on the right-hand side, called a **source**:

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right]\psi(x,t) = f(x,t). \tag{11.24}$$

The source term turns the wave equation into an inhomogenous partial differential equation, similar to the driving force for the driven harmonic oscillator.

11.2.1 Time-domain Green's function (optional topic)

The wave equation's **time-domain Green's function** is defined by setting the source term to delta functions in both space and time:

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right] G(x,x';t-t') = \delta(x-x')\,\delta(t-t'). \tag{11.25}$$

Here G is a function of two spatial variables, x and x', as well as two temporal variables t and t'. It corresponds to the wave generated by a pulse

$$f(x,t) = \delta(x-x')\,\delta(t-t'). \tag{11.26}$$

The differential operator in the Green's function equation only involves x and t, so we can regard x' and t' as parameters specifying where the pulse is localized in space and time. This Green's function ought to depend on the time variables only in the combination $t-t'$, as we saw in Section 11.1.3. To emphasize this, we have written it as $G(x,x';t-t')$.

The Green's function describes how a source localized at a space-time point influences the wavefunction at other positions and times. Once we have found the Green's function, it can be used to construct solutions for arbitrary sources:

$$\psi(x,t) = \int dx' \int_{-\infty}^{\infty} dt'\; G(x,x';t-t')\, f(x',t'). \tag{11.27}$$

11.2.2 Frequency-domain Green's function (optional topic)

The **frequency-domain Green's function** is obtained by Fourier transforming the time-domain Green's function in the $t-t'$ coordinate:

$$G(x,x';\omega) = \int_{-\infty}^{\infty} d\tau\; e^{i\omega\tau}\, G(x,x';\tau). \tag{11.28}$$

It obeys the differential equation

$$\left[\frac{\partial^2}{\partial x^2} + \omega^2\right] G(x,x';\omega) = \delta(x-x'). \tag{11.29}$$

Just as we can write the time-domain solution to the wave equation in terms of the time-domain Green's function, we can do the same for the frequency-domain solution:

$$\Psi(x,\omega) = \int dx'\; G(x,x';\omega)\, F(x',\omega), \tag{11.30}$$

where

$$\Psi(x,\omega) = \int_{-\infty}^{\infty} dt\; e^{i\omega t}\, \psi(x,t), \quad F(x,\omega) = \int_{-\infty}^{\infty} dt\; e^{i\omega t}\, f(x,t). \tag{11.31}$$

11.2.3 Outgoing boundary conditions (optional topic)

So far, we have not specified the boundary conditions along x. There are several possible choices of boundary conditions, corresponding to different physical scenarios. For example, if the waves are trapped within a finite domain $x \in (x_a, x_b)$, with reflecting walls, we would impose Dirichlet boundary conditions: $G(x,x';\omega) = 0$ for $x, x' \in \{x_a, x_b\}$.

We will focus on the interesting case of an unbounded spatial domain: $x \in (-\infty, \infty)$. This describes, for example, the case of a loudspeaker emitting sound waves into an infinite empty space. The relevant boundary conditions for this case are called **outgoing boundary conditions**. The Green's function should correspond to a left-moving wave for x to the left of the source, and to a right-moving wave for x to the right of the source.

We can guess the form of the Green's function obeying these boundary conditions:

$$G(x,x';\omega) = \left\{ \begin{array}{ll} A\, e^{-i\omega(x-x')}, & x \le x', \\ B\, e^{i\omega(x-x')}, & x \ge x' \end{array} \right. \quad \text{for some } \; A, B \in \mathbb{C}. \tag{11.32}$$

It is straightforward to verify that this formula for $G(x,x',\omega)$ satisfies the wave equation in both the regions $x < x'$ and $x > x'$, as well as satisfying outgoing boundary conditions. To determine the A and B coefficients, note that $G(x,x')$ should be continuous at $x = x'$, so $A = B$. Then, integrating the Green's function equation across x' gives

$$\lim_{\epsilon\rightarrow 0} \int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\partial^2}{\partial x^2} + \omega^2\right] G(x-x') = \lim_{\epsilon\rightarrow 0} \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x') \tag{11.33}$$

$$= \lim_{\epsilon\rightarrow 0} \left\{ \frac{\partial G}{dx}(x,x')\Big|_{x=x'+\epsilon} - \frac{\partial G}{\partial x}(x,x')\Big|_{x=x'-\epsilon} \right\} = i\omega(B+A) = 1. \tag{11.34}$$

Combining these two equations gives $A = B = 1/2i\omega$. Hence,

$$G(x,x';\omega) = \frac{e^{i\omega|x-x'|}}{2i\omega}. \tag{11.35}$$

11.3 Causality and the time-domain Green's function (optional topic)

Let us try converting the above result into a time-domain Green's function by using the inverse Fourier transform:

$$G(x,x';t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, G(x,x';\omega) \tag{11.36}$$

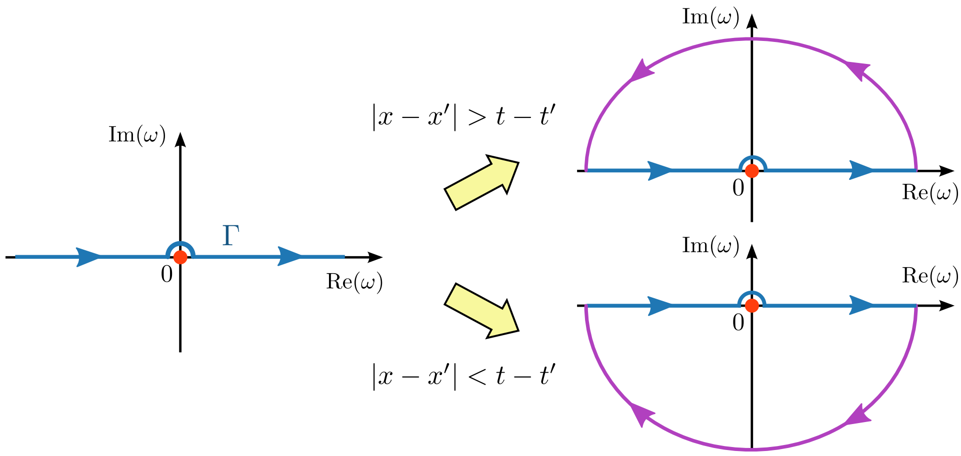
$$= \int_{-\infty}^{\infty} d\omega\, \frac{e^{i\omega\big[|x-x'|-(t-t')\big]}}{4\pi i\omega} \qquad (?!?) \tag{11.37}$$

There is a problem: the integral runs over the real-ω line, yet the integrand has a pole at $\omega = 0$, on the real axis, making the integral ill-defined.

To resolve this, we redefine $G(x,x';\omega)$ as an integral over a *deformed* contour Γ:

$$G(x,x';t-t') \equiv \int_{\Gamma} d\omega\, \frac{e^{i\omega\big[|x-x'|-(t-t')\big]}}{4\pi i\omega}. \tag{11.38}$$

We will choose to deform the contour in a very specific way, which turns out to be the choice that satisfies causality (Section 11.1.4). As shown in the left subplot of the figure below, it runs along the real axis, but skips *above* the pole at the origin.



The integral can be solved by either closing the contour in the upper half-plane, or in the lower half-plane. If we close the contour above, then the loop contour does not enclose the pole, and hence $G(x, x'; t-t') = 0$. According to Jordan's lemma, we must do this if the exponent in the integrand obeys

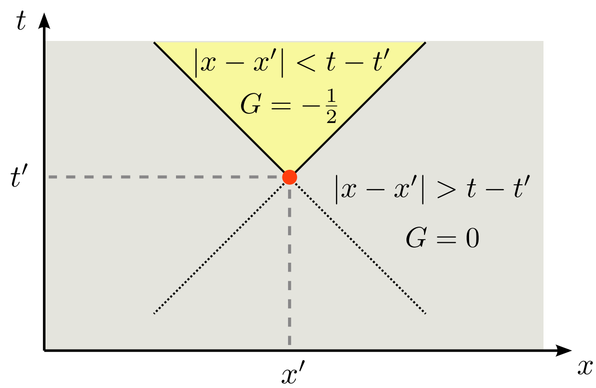
$$|x-x'| - (t-t') > 0 \quad \Rightarrow \quad |x-x'| > t-t'. \tag{11.39}$$

This inequality is satisfied in two cases: either (i) $t < t'$ (in which case the inequality is satisfied for all x, x' because $|x-x'|$ is strictly non-negative), or (ii) $t > t'$ but the value of $t-t'$ is smaller than $|x-x'|$. To understand the physical meaning of these two cases, recall that $G(x, x'; t-t')$ represents the field at position x and time t resulting from a pulse at the space-time point (x', t'). Thus, case (i) corresponds to times occurring before the pulse, and case (ii) corresponds to times occurring after the pulse but too far away from the pulse location for a wave to reach in time.

For the other case, $|x-x'| - (t-t') < 0$, the residue theorem gives

$$G(x, x'; t-t') = -1/2. \tag{11.40}$$

The space-time diagram below summarizes the results:



The resulting time-domain wavefunctions can be written as

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' \left[-\frac{1}{2}\,\Theta(t-t'-|x-x'|)\right] f(x', t'), \tag{11.41}$$

where Θ denotes the unit step function. In other words, the wavefunction at each space-time point (x,t) receives equal contribution from the sources $f(x', t')$ at space-time points (x', t') lying within the "past light cone".

11.4 Looking ahead (optional topic)

Green's functions are widely used in the study of acoustic and electromagnetic waves, which is a vast topic covered in advanced courses in theoretical physics, electrical engineering, and mechanical engineering. Here, we give a brief sketch of some future directions of study.

So far, we have focused our attentions on the simplest case of an infinite one-dimensional uniform medium. Most practical applications are concerned with three spatial dimensions and non-uniform media. For such cases, the wave equation's frequency-domain Green's function can be generalized to

$$\left[\nabla^2 + n^2(\vec{r}) \left(\frac{\omega}{c}\right)^2\right] G(\vec{r}, \vec{r}\,'; \omega) = \delta^3(\vec{r} - \vec{r}\,'), \tag{11.42}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the three-dimensional Laplacian operator, and $n(\vec{r})$ is a space-dependent refractive index (see Section 5.5.1). On the right-hand side of this equation is the three-dimensional delta function (see Section 9.8), which describes a point source located at position $\vec{r}\,'$ in the three-dimensional space.

When $n = 1$, the above equation is similar to the frequency-domain Green's function equation studied in Section 11.2.3, except that the problem is three-dimensional rather than one-dimensional. Again assuming outgoing boundary conditions, the Green's function in three dimensions can be found using contour integrals similar to those we have previously covered; the result is

$$G(\vec{r}, \vec{r}\,'; \omega) = -\frac{e^{i(\omega/c)|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r} - \vec{r}\,'|}. \tag{11.43}$$

Like the Green's function in one dimension, this depends on $|\vec{r} - \vec{r}\,'|$, and thence describes waves that are emitted isotropically from the source at $\vec{r}\,'$. However, the magnitude of G now decreases to zero with distance, due to the $|\vec{r} - \vec{r}\,'|$ in the denominator. This matches our everyday experience that the sound emitted from a point source grows fainter with distance, which is because the energy carried by the outgoing wave is spread out over a larger area with increasing distance from the source. This is unlike waves in one-dimensional space, which do not become weaker with distance.

When $n(\vec{r})$ is not a constant but varies with position $\vec{r}$, then the waves emitted by the source do not radiate outwards in a simple way. The variations in the refractive index cause the waves to scatter in complicated ways. In most situations, the exact solution for the Green's function cannot be obtained analytically, but must be computed using specialized numerical methods.

For electromagnetic waves, there is another important complication coming from the fact that electromagnetic fields are described by vectors (i.e., the electric field vector and the magnetic field vector), not scalars. The propagation of electromagnetic waves is therefore described by a vectorial wave equation, not the scalar wave equation that we have looked at so far. Moreover, electromagnetic waves are not generated by scalar sources, but by vector sources (typically, electrical currents). The corresponding Green's function is not a scalar function, but a multi-component entity called a **dyadic Green's function**, which describes the vector waves emitted by a vector source.

Finally, even though we have dealt so far with classical (non-quantum) waves, the Green's function concept extends to the theory of quantum mechanics. In quantum field theory, which is the principal theoretical framework used in fundamental physics, calculations typically involve quantum mechanical generalizations of the Green's functions we have studied above, whose values are no longer simple numbers but rather quantum mechanical operators.

11.5 Exercises

1. Find the time-domain Green's function of the critically-damped harmonic oscillator ($\gamma = \omega_0$).

2. Consider an overdamped harmonic oscillator ($\gamma > \omega_0$) subjected to a *random* driving force $f(t)$, which fluctuates between random values, which can be either positive or negative, at each time t. The random force satisfies

$$\langle f(t) \rangle = 0 \quad \text{and} \quad \langle f(t) f(t') \rangle = A\, \delta(t - t'), \tag{11.44}$$

where $\langle \cdots \rangle$ denotes an average taken over many realizations of the random force and A is some constant. Using the causal Green's function, find the correlation function $\langle x(t_1)\, x(t_2) \rangle$ and the mean squared deviation $\left\langle [x(t + \Delta t) - x(t)]^2 \right\rangle$.

[solution available]