ML for number theory

Team NT

December 4, 2025

Goal

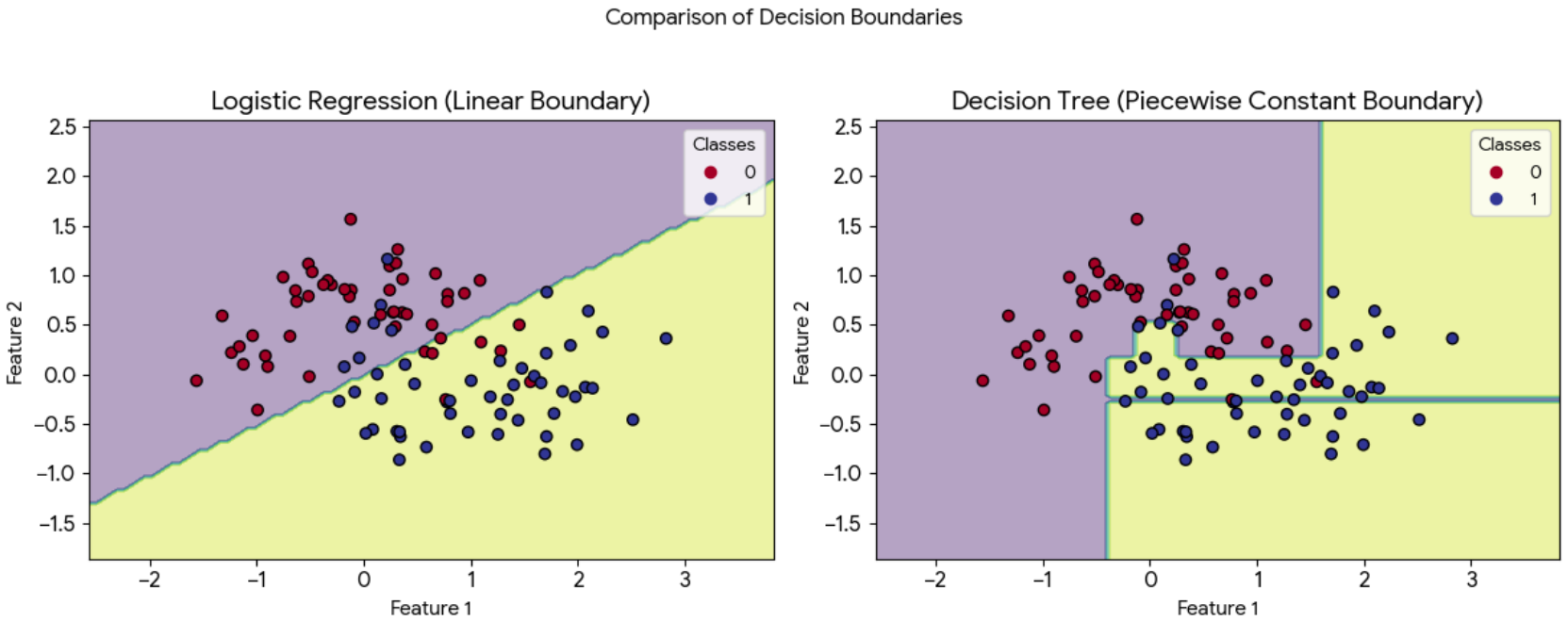
Use ML to solve classification problems in number theory.

Machine Learning

Classical ML models:

- Logistic regression: learn the best spearating hyperplane
- Decision tree: learn the best sequence of `if-else` rules, based on the size of features.

Logistic Regression vs Decision Tree



Data - LMFDB

LMFDB

⌂ → Artin representations → 3.1729.9t20.c.a

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Artin representation 3.1729.9t20.c.a

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Varieties
Elliptic curves over $\mathbb{Q}$
Elliptic curves over $\mathbb{Q}(\alpha)$
Genus 2 curves over $\mathbb{Q}$
Higher genus families
Abelian varieties over $\mathbb{F}_q$
Belyi maps

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Artin representations

Groups
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Abstract groups

Database

Basic invariants

Dimension:	3
Group:	$C_3 \wr S_3$
Conductor:	$1729 = 7 \cdot 13 \cdot 19$
Artin stem field:	Galois closure of 9.3.738391927.1
Galois orbit size:	2
Smallest permutation container:	$C_3 \wr S_3$
Parity:	odd
Determinant:	1.1729.6t1.a.b
Projective image:	$C_3^2 : C_6$
Projective stem field:	Galois closure of 9.1.8936757492481.1

Defining polynomial

$$f(x) = x^9 - 2x^7 + 2x^5 - 5x^4 + 2x^3 + 3x^2 - 3x + 1 \quad .$$

The roots of f are computed in an extension of $\mathbb{Q}_{71}$ to precision 10.

Minimal polynomial of a generator a of K over $\mathbb{Q}_{71}$: $x^3 + 4x + 64$

Roots:

$$r_1 = 22a^2 + 52a + 43 + (a^2 + 39a + 43) \cdot 71 + (56a^2 + a + 13) \cdot 71^2 + (a^2 + 68a + 24) \cdot 71^3 + (38a^2 + 59a + 41) \cdot 71^4 + (33a^2 + 54a) \cdot 71^5 + (24a^2 + 50a + 68) \cdot 71^6 + (66a^2 + 43a + 53) \cdot 71^7 + (23a^2 + 12a + 13) \cdot 71^8 + (4a^2 + 17a + 2) \cdot 71^9 + O(71^{10})$$

$$r_2 = 70a^2 + 55a + 29 + (20a^2 + 36a + 1) \cdot 71 + (54a^2 + 34a + 9) \cdot 71^2 + (20a^2 + 49a + 51) \cdot 71^3 + (62a^2 + 25a + 58) \cdot 71^4 + (31a^2 + 32a + 19) \cdot 71^5 + (10a^2 + 48a + 54) \cdot 71^6 + (31a^2 + 15a + 54) \cdot 71^7 + (25a^2 + 30a + 17) \cdot 71^8 + (32a^2 + 43a + 53) \cdot 71^9 + O(71^{10})$$

$$r_3 = 31a^2 + 53a + 5 + (27a^2 + 48a + 47) \cdot 71 + (31a^2 + 41a + 55) \cdot 71^2 + (14a^2 + 45a + 60) \cdot 71^3 + (12a^2 + 40a + 66) \cdot 71^4 + (32a^2 + 26a + 2) \cdot 71^5 + (15a^2 + 47a + 33) \cdot 71^6 + (44a^2 + 36a + 41) \cdot 71^7 + (26a^2 + 32a + 41) \cdot 71^8 + (37a^2 + 66a + 62) \cdot 71^9 + O(71^{10})$$

$$r_4 = 50a^2 + 35a + 23 + (48a^2 + 65a + 4) \cdot 71 + (31a^2 + 34a + 67) \cdot 71^2 + (48a^2 + 24a + 53) \cdot 71^3 + (41a^2 + 56a + 3) \cdot 71^4 + (5a^2 + 54a + 68) \cdot 71^5 + (36a^2 + 42a + 27) \cdot 71^6 + (44a^2 + 11a + 19) \cdot 71^7 + (21a^2 + 28a + 31) \cdot 71^8 + (34a^2 + 10a + 58) \cdot 71^9 + O(71^{10})$$

$$r_5 = 21 + 4 \cdot 71 + 30 \cdot 71^2 + 30 \cdot 71^3 + 20 \cdot 71^4 + 48 \cdot 71^5 + 44 \cdot 71^6 + 52 \cdot 71^7 + 67 \cdot 71^8 + 69 \cdot 71^9 + O(71^{10})$$

$$r_6 = 54a^2 + 9a + 19 + (42a^2 + 64a + 64) \cdot 71 + (33a^2 + 15a + 37) \cdot 71^2 + (21a^2 + 47a + 8) \cdot 71^3 + (29a^2 + 40a + 65) \cdot 71^4 + (58a^2 + 70a + 1) \cdot 71^5 + (18a^2 + 5a + 42) \cdot 71^6 + (37a^2 + 31a + 46) \cdot 71^7 + (57a^2 + 22a + 5) \cdot 71^8 + (a^2 + 48a + 15) \cdot 71^9 + O(71^{10})$$

$$r_7 = 6 + 69 \cdot 71 + 29 \cdot 71^3 + 7 \cdot 71^4 + 7 \cdot 71^5 + 14 \cdot 71^6 + 45 \cdot 71^7 + 20 \cdot 71^8 + 20 \cdot 71^9 + O(71^{10})$$

$$r_8 = 40 + 26 \cdot 71 + 34 \cdot 71^2 + 28 \cdot 71^3 + 49 \cdot 71^4 + 33 \cdot 71^5 + 28 \cdot 71^6 + 3 \cdot 71^7 + 8 \cdot 71^8 + 49 \cdot 71^9 + O(71^{10})$$

$$r_9 = 57a^2 + 9a + 27 + (29a + 23) \cdot 71 + (6a^2 + 13a + 35) \cdot 71^2 + (35a^2 + 49a + 68) \cdot 71^3 + (29a^2 + 60a + 41) \cdot 71^4 + (51a^2 + 44a + 30) \cdot 71^5 + (36a^2 + 17a + 42) \cdot 71^6 + (60a^2 + 3a + 37) \cdot 71^7 + (57a^2 + 16a + 6) \cdot 71^8 + (31a^2 + 27a + 24) \cdot 71^9 + O(71^{10})$$

Properties

Label	3.1729.9t20.c.a
Dimension	3
Group	$C_3 \wr S_3$
Conductor	1729
Root number	not computed
Indicator	0

Related objects
Field 9.3.738391927.1
Field 9.1.8936757492481.1
Determinant 1.1729.6t1.a.b
Galois orbit 3.1729.9t20.c

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Data - LMFDB

LMFDB

⌂ → Abelian varieties → Fq → 3 → 2 → ad_d_ac

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Abelian variety isogeny class 3.2.ad_d_ac over $\mathbb{F}_2$

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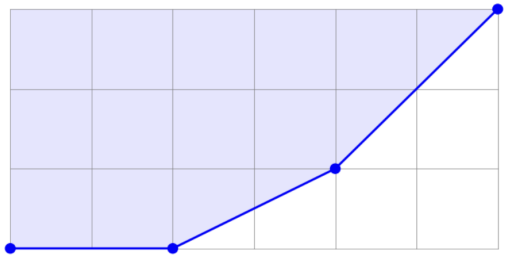
Invariants

Base field:	$\mathbb{F}_2$
Dimension:	3
L-polynomial:	$(1 - 2x + 2x^2)(1 - x - x^2 - 2x^3 + 4x^4)$
	$1 - 3x + 3x^2 - 2x^3 + 6x^4 - 12x^5 + 8x^6$
Frobenius angles:	± 0.0516399385854, ± 0.250000000000, ± 0.718306605252
Angle rank:	1 (numerical)
Jacobians:	1
Cyclic group of points:	yes

This isogeny class is not simple, primitive, not ordinary, and not supersingular. It is principally polarizable and contains a Jacobian.

Newton polygon



p-rank: 2
Slopes: $[0, 0, 1/2, 1/2, 1, 1]$

Point counts

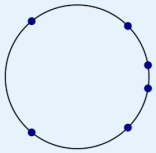
Point counts of the abelian variety

r	1	2	3	4	5
$A(\mathbb{F}_{q^r})$	1	35	208	6475	30791

Point counts of the curve

Properties

Label 3.2.ad_d_ac



Base field	$\mathbb{F}_2$
Dimension	3
p-rank	2
Ordinary	no
Supersingular	no
Simple	no
Geometrically simple	no
Primitive	yes
Principally polarizable	yes
Contains a Jacobian	yes

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Artin Representation

Artin Representation

Definition

Artin representation is a finite dimensional complex representation of a Galois group, i.e. a group homomorphism

$$\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$$

For each (unramified) prime p, there is a conjugacy class $\mathrm{Frob}_p \subset \mathrm{Gal}(K/\mathbb{Q})$ called the **Frobenius conjugacy class** at p. We denote by a_p for the trace of $\rho(\mathrm{Frob}_p)$ ($a_p = 0$ if p is ramified).

Task

a_p are important, e.g. they define the Artin L-function $L(\rho, s)$.

Question

Can we predict other invariants of ρ with a_p's?

Candidates:

- Parity
- Frobenius–Schur indicator
- **Projective image** (image in $\mathrm{PGL}_n(\mathbb{C})$)

3-dim representations

- Total $\approx 30,000$
- S_4 vs A_4.
- Logistic regression
- Decision tree

3-dim representation, further data analysis

Conjecture

Let $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_3(\mathbb{C})$ *be an irreducible Artin representation with projective image isomorphic to* A_4*. If* $a_p = -3$ *for some* p*, then* $a_{p'} = 1$ *for some* p'*.*

True for all data in LMFDB.

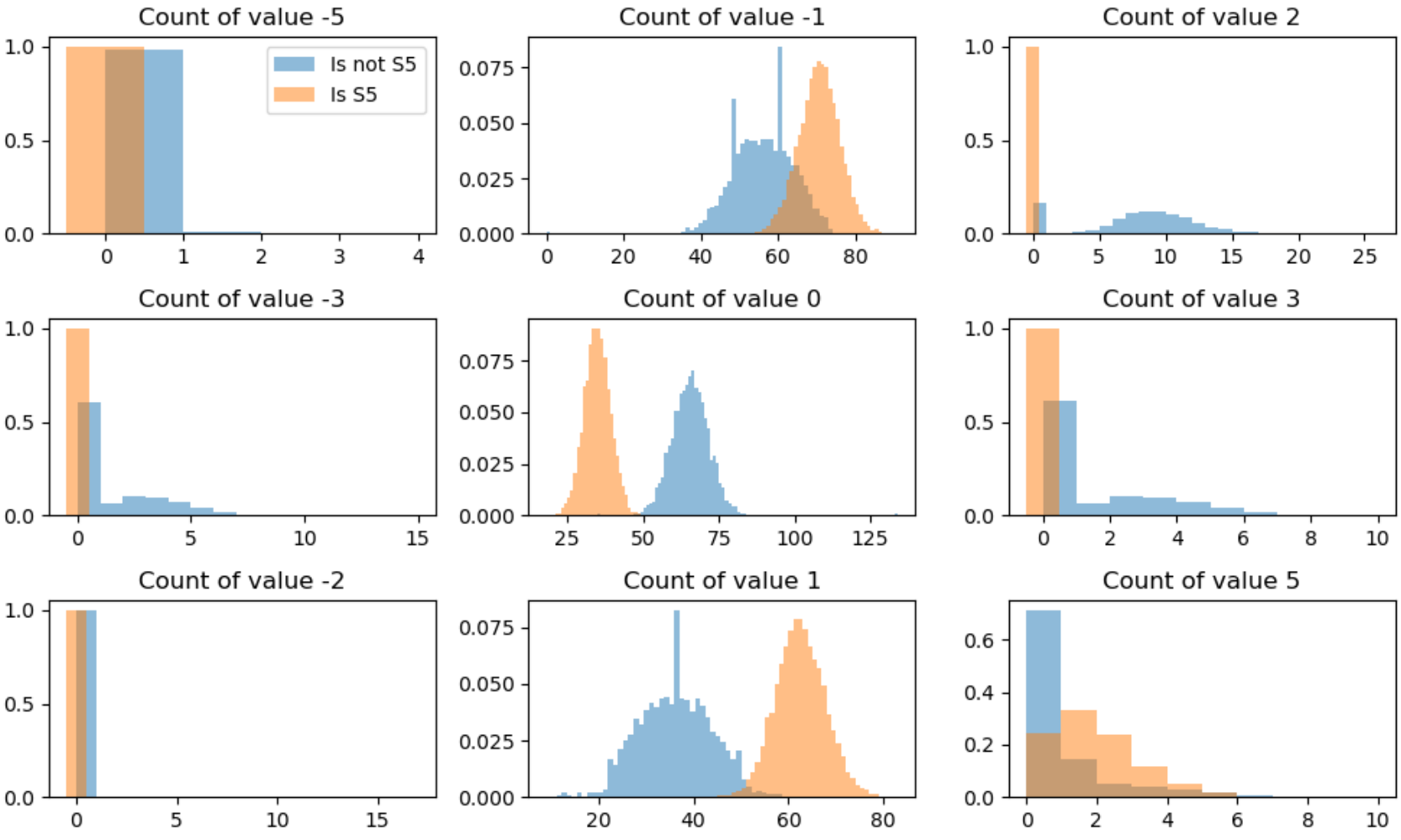
5-dim representations

- Total $\approx 20,000$
- Majority of them are S_5, A_6, S_6
- Logistic regression, one-hot encoding: 99.5% accuracy.
- Decision tree: $> 92\%$ accuracy.

5-dim representations, Logistic regression

- S_5 vs rest
- One-hot encoding: for each p and $a \in \{-5, -3, -2, -1, 0, 1, 2, 3, 5\}$, we create a feature which is 1 if $a_p = a$ and 0 otherwise.
- Number of p's with $a_p = 0$ is important, which can be explained by Chebotarev density theorem.

5-dim representations, a_p statistics



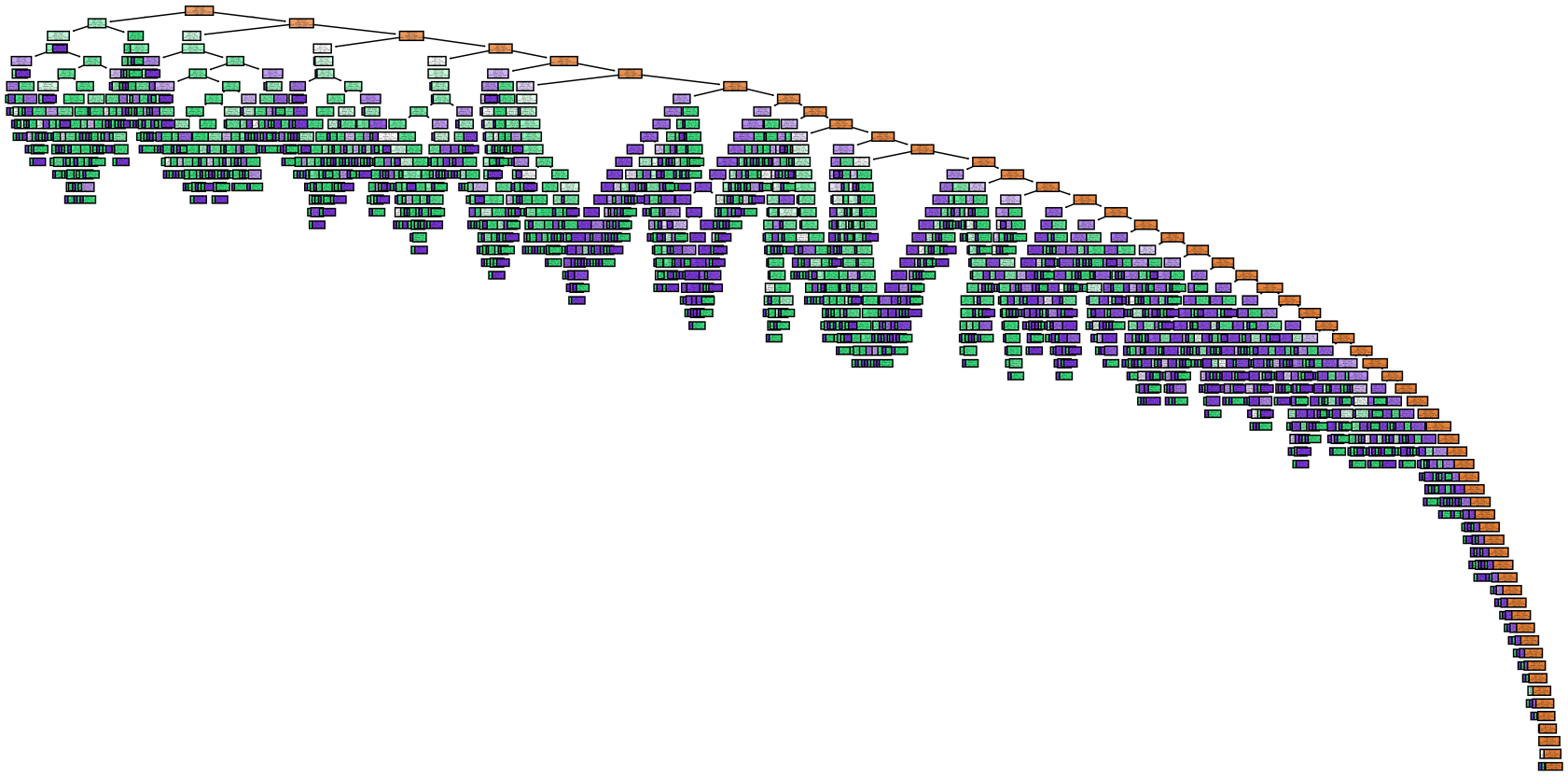
5-dim representations, Decision tree

- S_6 vs rest
- Existence of p with $a_p = 2$ is important, which can be explained by checking the character table.

2-dim odd representations, image in $\mathrm{GL}_2(\mathbb{C})$

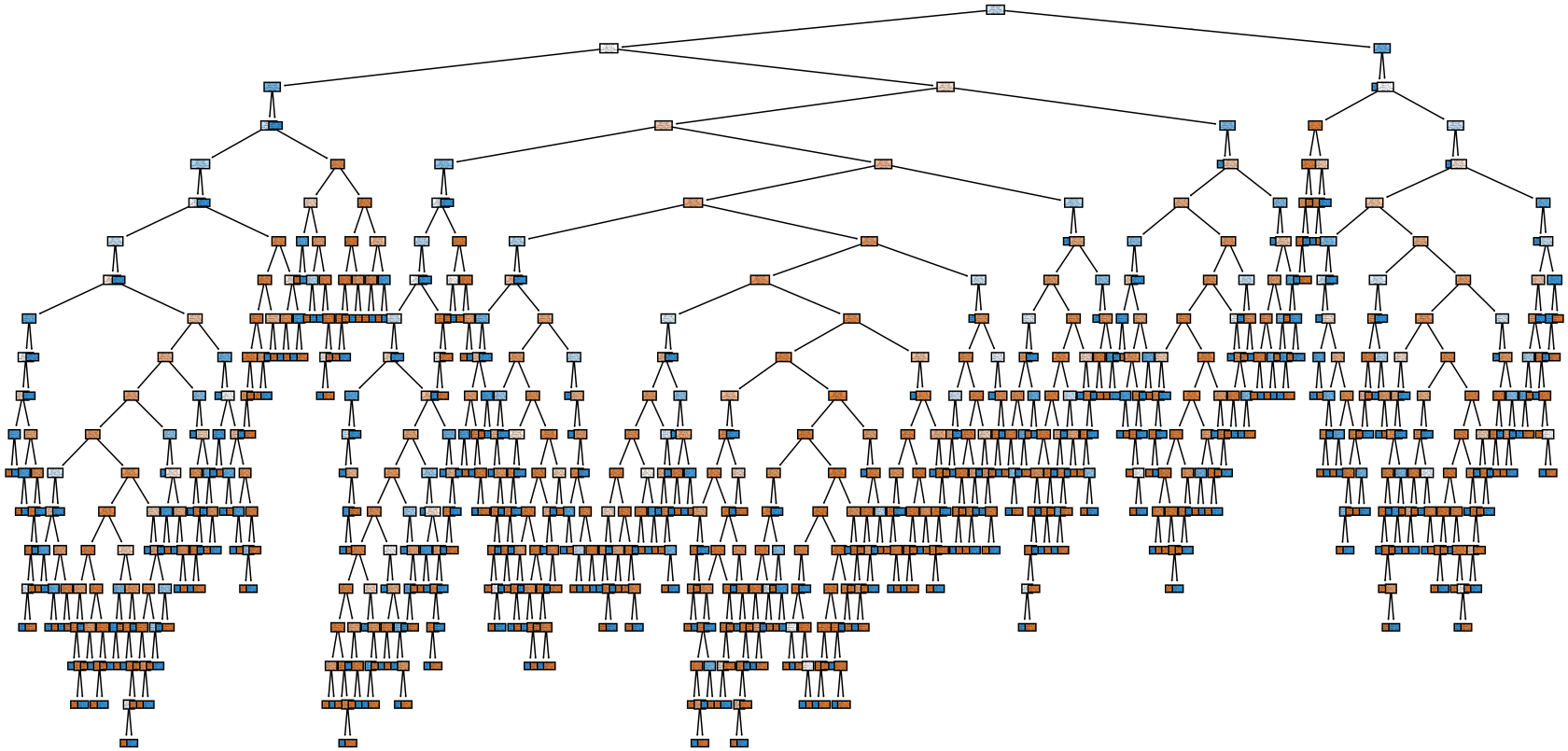
- Total $\approx 220,000$
- S_3 vs D_4 vs D_6
- Decision tree: $> 99\%$ accuracy.

Tree (S_3 vs D_4 vs D_6)



If $a_p = -2$ for some p, then the image is *not* S_3. Can be explained using character table.

Tree (D_4 vs D_6)



Future plans

- More interpretation of decision trees for 2-dim and 5-dim cases.
- Prove conjecture for 3-dim case.

Abelian Varieties over finite fields

Abelian Varieties

Definition

Abelian variety is a **variety** with an **abelian group** structure.

e.g. Elliptic curves (1-dim abelian varieties). $y^2 = x^3 + ax + b$.

L-polynomial

Definition

L-polynomial of an abelian variety $A_{/\mathbb{F}_q}$ is defined as

$$L_A(x) = \det(1 - x\mathrm{Frob}_q^{-1} | \mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell))$$

which is an integer polynomial of degree $2g$ where $g = \dim A$.

It has a form of

$$L_A(x) = 1 + a_1 x + \cdots + a_{2g-1} x^{2g-1} + q^g t^{2g}$$

and $a_{g+1}, a_{g+2}, \ldots, a_{2g}$ are multiples of p.

(Deligne) Zeros of $L_A(t)$ have an absolute value $q^{-1/2}$.

(Honda–Tate) Isogeny classes $\leftrightarrow$ L-polynomials.

Task

Question

Can we predict other invariants of A with a_i's?

Candidates:

- **p-rank**
- Principally polarizable
- **Jacobian**

p-rank

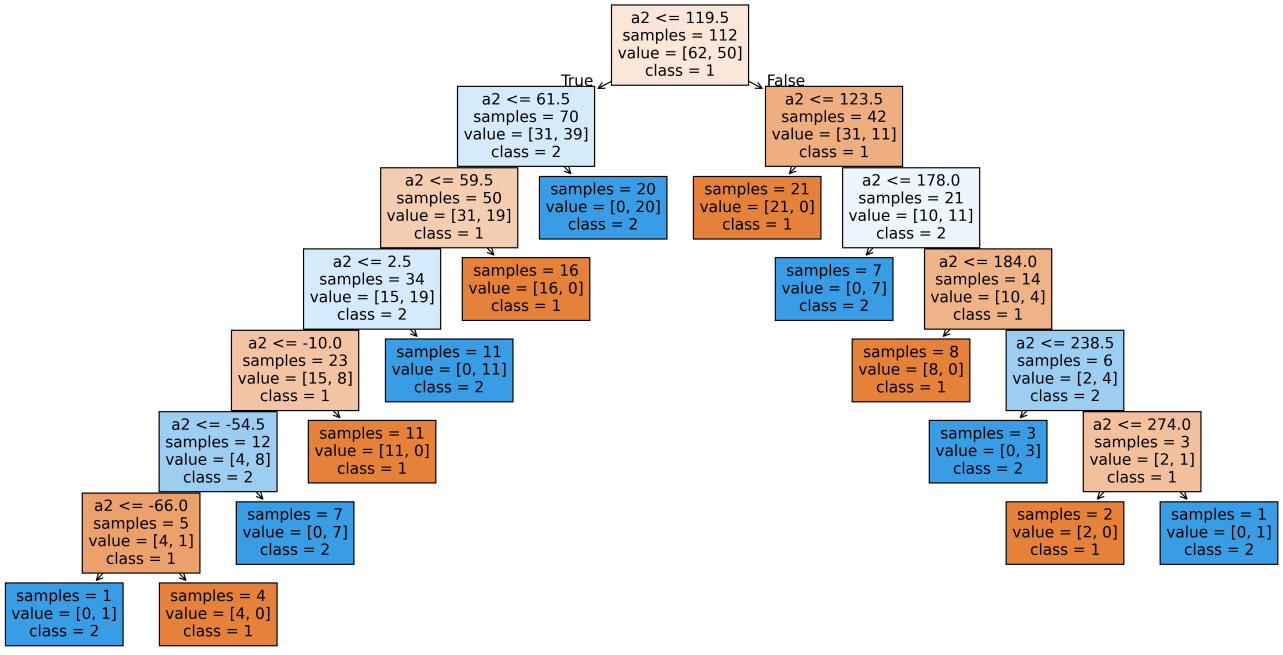
Definition

$$\mathrm{rank}_p(A) = \dim_{\mathbb{F}_p}(A(\overline{\mathbb{F}}_p)[p])$$

Possible values: $0, 1, \ldots, g = \dim A$.

When $g = 2$, there are very few rank 0 cases, so we focus on rank 1 vs rank 2 classification.

Tree



- Most of the nodes sees a_2. (a_1 and a_3 are not important.)
- When $a_2 = 61k + r$ with *small* $|r|$, then the p-rank is 1.

Why?

In fact, p-rank is determined by the *Newton polygon* of $L_A(t)$: the lower convex hull of the points

$$(0,0),\ (1, v_p(a_1)),\ \ldots,\ (2g-1, v_p(a_{2g-1})),\ (2g, g)$$

The rank is equal to the number of segments of slope 0.

When $g = 2$, $v_p(a_2) = 0 \Leftrightarrow a_2 \not\equiv 0 \pmod{p}$ implies p-rank 2.

Jacobian

For any genus g curve $C_{/\mathbb{F}_q}$, its Jacobian variety J_C is a g-dim abelian variety.

Question

For a given isogeny class of abelian variety, does it contain a Jacobian variety? Can we predict it from $L_A(t)$?

(Howe–Nart–Ritzenthaler) A complete answer for $g = 2$.

For $g \geq 3$, some entries in the LMFDB marked as *unknown*.

$g = 2$, split

p-rank of $\mathcal{A}$	Condition on p and q	Conditions on s and t
—	—	$\lvert s-t\rvert = 1$
2	—	$s = t$ and $t^2 - 4q \in \{-3, -4, -7\}$
	$q = 2$	$\lvert s\rvert = \lvert t\rvert = 1$ and $s \neq t$
1	q square	$s^2 = 4q$ and $s - t$ squarefree
0	$p > 3$	$s^2 \neq t^2$
	$p = 3$ and q nonsquare	$s^2 = t^2 = 3q$
	$p = 3$ and q square	$s - t$ is not divisible by $3\sqrt{q}$
	$p = 2$	$s^2 - t^2$ is not divisible by $2q$
	$q = 2$ or $q = 3$	$s = t$
	$q = 4$ or $q = 9$	$s^2 = t^2 = 4q$

Table 1.1. Conditions that ensure that the split isogeny class with Weil polynomial $(x^2 - sx + q)(x^2 - tx + q)$ does not contain a Jacobian. Here we assume that $\lvert s\rvert \geqslant \lvert t\rvert$.

Weil polynomial $= x^{2g} L_A(1/x)$

$g = 2$, split

Proposition (Howe–Nart–Ritzenhaler)

For non-square q and $p \neq 2, 3$, if L_A factors as $L_A(x) = (1 + sx + qx^2)(1 + tx + qx^2)$ and $|s - t| > 1$, then the isogeny class contains a Jacobian.

$s - t = \pm 1$ would work as a decision boundary. Decision tree cannot easily learn such "diagonal" boundary.

$g = 2$, split

Proposition (Howe–Nart–Ritzenhaler)

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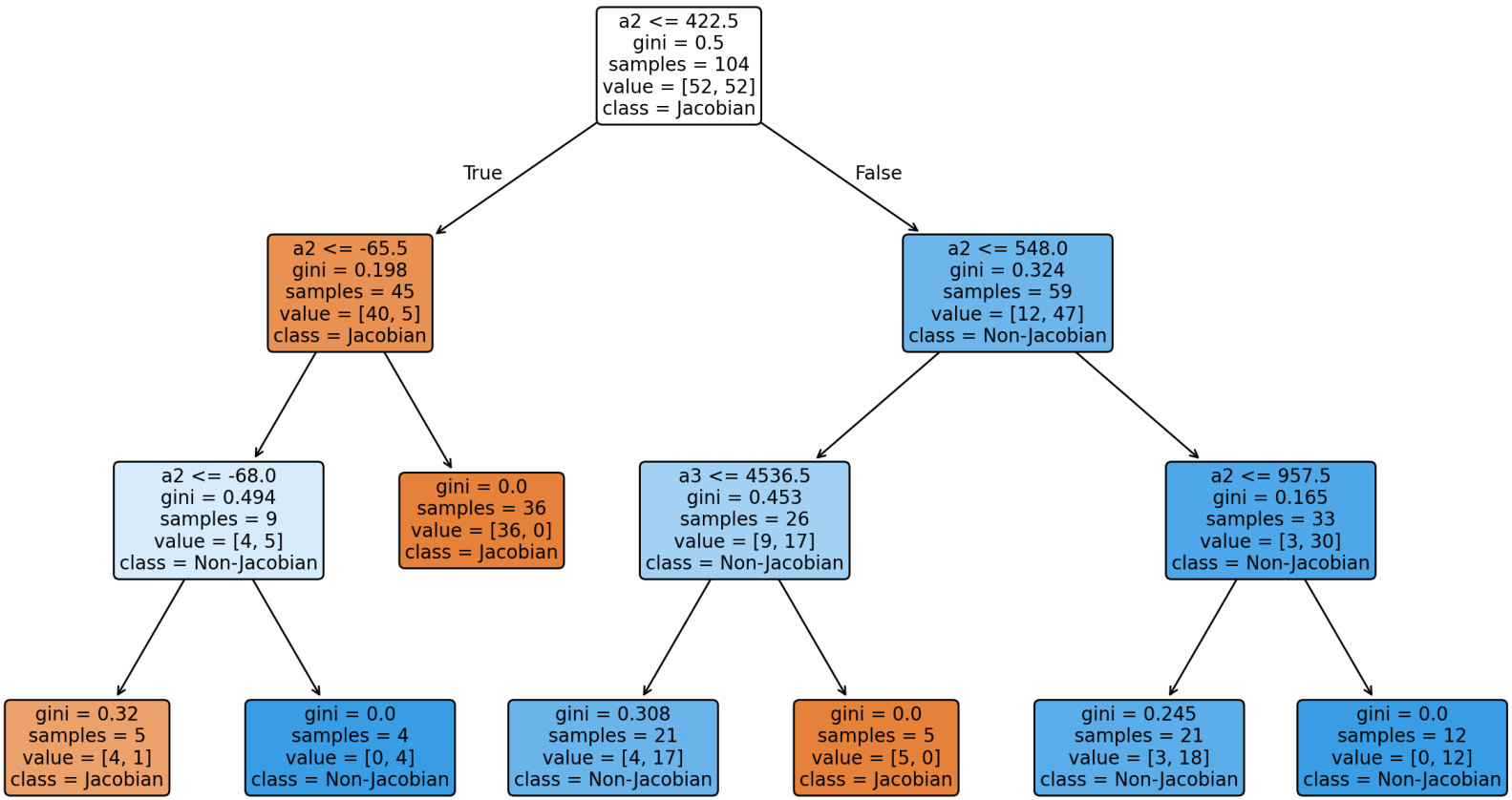
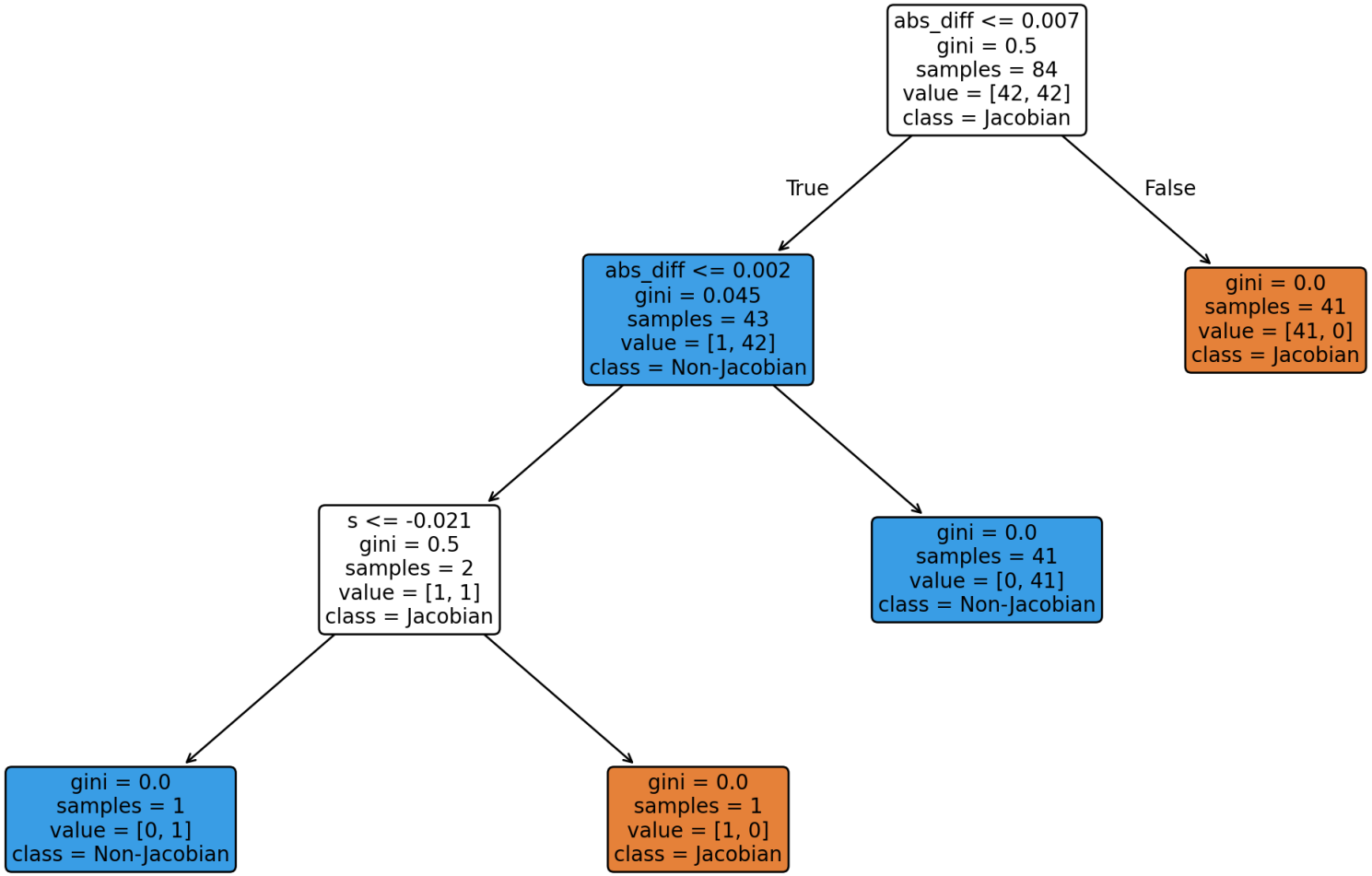
$s - t = \pm 1$ would work as a decision boundary. Decision tree cannot easily learn such "diagonal" boundary.

To mitigate this, we can simply add $|s - t|$ as a new feature.

Tree, without $|s - t|$ ($q = 211$, acc = 80.7%)



Tree, with $|s - t|$ ($q = 211$, acc = 100%)



$g = 3$, split

Question

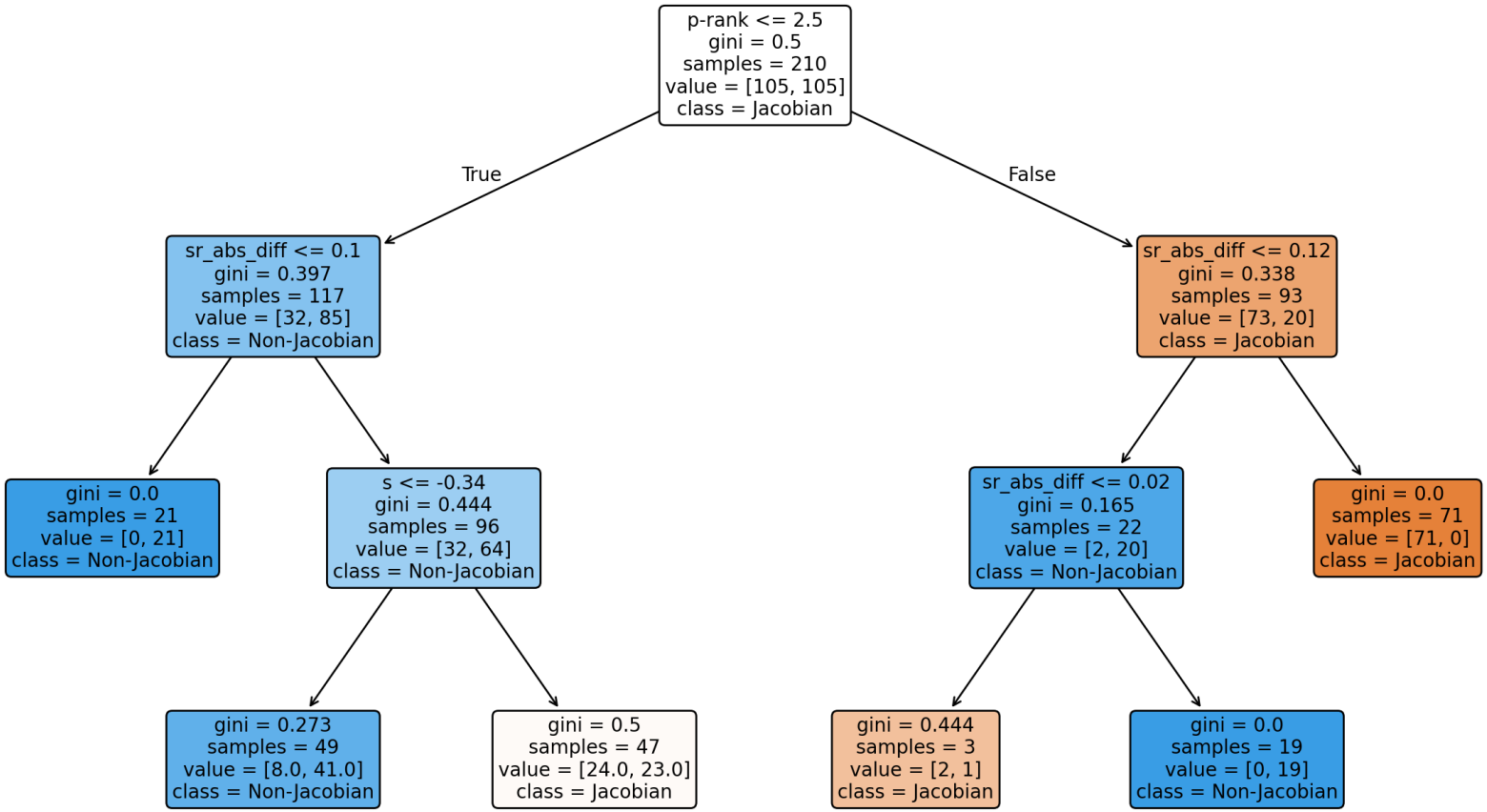
How about $g = 3$ split case? If

$$L_A(x) = (1 + sx + qx^2)(1 + tx + qx^2)(1 + rx + qx^2)$$

can we predict whether the isogeny class contains a Jacobian, using s, t, u?

We assume $s \leq t \leq r$. We will add $|s - t|, |t - r|, |r - s|$ as new features.

Tree ($q = 25$)



If $\text{rank}_p(A) = 3$ and $|r - s|$ is large, then the tree predicts that the isogeny class contains a Jacobian.

Future plans

- *Prove* the heuristics suggested from ML experiments and data for $g = 3$.
- Non-split cases?
- Data visualization for better understanding.
- $g \geq 4$?

Summary

- ML models tell us *rough* strategy to predict invariants. In particular, decision trees are good for
  - interpretability
  - when output depends on smallness/bigness of inputs
- Further investigation of data may lead to *precise* conjectures.
- If we are lucky, we can
  - prove the conjecture
  - come up with more efficient algorithms
  - filling missing entries in databases

Use ML for your own problem!