SOLUTION FOR "AN INTRODUCTION TO AUTOMORPHIC REPRESENTATION"

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Abstract. This is a solution for the exercises in J. Getz and H. Hahn's book *An Introduction to Automorphic Representation with a view toward Trace Formulae*.

1. Chapter 1

***Problem* 1.1** *** By Yoneda lemma, the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes corresponds to the k-algebra morphism $\phi : A \to B$. This induces a map on the underlying topological spaces by sending a prime ideal $\mathfrak{p} \subset B$ to $\phi^{-1}(\mathfrak{p}) \subset A$, which is also prime.

***Problem* 1.2** ***

***Problem* 1.3** By Yoneda lemma, we have

$$\mathrm{Mor}(\mathrm{Spec}(B), \mathrm{Spec}(A)) \simeq \mathrm{Nat}(h^B, h^A) \simeq h^B(A) = \mathrm{Hom}_k(A, B)$$

which gives an equivalence between $\mathbf{AffSch}_k^{\mathrm{op}}$ and $\mathbf{Alg}_k$.

***Problem* 1.4**
- Nonreduced: $\mathrm{Spec}(\mathbb{C}[x]/(x^2))$
- Reducible: $\mathrm{Spec}(\mathbb{C}[x, y]/(x, y))$
- Reduced and irreducible (i.e. integral): $\mathrm{Spec}(\mathbb{C}[x])$

***Problem* 1.5** We can assume that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A/I)$ for some k-algebra A and an ideal I of A. Then it is enough to show that the map $\mathrm{Hom}(A/I, R) \to \mathrm{Hom}(A, R)$, given by composing with the natural map $\pi : A \to A/I$, is injective. This follows from the surjectivity of π.

***Problem* 1.6** Let $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B), Z = \mathrm{Spec}(C)$. Then the statement is equivalent to

$$\mathrm{Hom}(A \otimes_B C, R) \simeq \mathrm{Hom}(A, R) \times_{\mathrm{Hom}(B,R)} \mathrm{Hom}(C, R).$$

We can define a bijection as follows. First, consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & A \otimes_B C \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\iota_C} \\ B & \xrightarrow{\gamma} & C \end{array}$$

Using the maps above, we define a map from LHS to RHS as $\phi \mapsto (\phi\iota_A, \phi\iota_C)$. Since $\iota_A\alpha = \iota_C\gamma$, we have $\phi\iota_A\alpha = \phi\iota_C\gamma$ and the map is well-defined. For the other direction, for given $(f, g) : A \times C \to R$ with $f\alpha = g\gamma$, universal property of the tensor product gives a unique map $\phi : A \otimes_B C \to R$ with $f = \phi\iota_A$ and $g = \phi\iota_C$. We can check that these maps are inverses for each other.

***Problem* 1.7** ***

***Problem* 1.8** *** We define an $\mathbb{R}$-algebra A as

$$A = \mathbb{R}[(x_{ij}, y_{ij})_{1 \le i,j \le n}]/I$$

where I is an ideal generated by elements of the form

$$\left(\sum_{k=1}^{n}(x_{ik}^2 + y_{ik}^2)\right) - 1,$$
$$\sum_{k=1}^{n}(x_{ik}x_{jk} - y_{ik}y_{jk}), \quad i \neq j$$
$$\sum_{k=1}^{n}(x_{ij}y_{jk} + y_{ik}x_{jk}), \quad i \neq j$$

for $1 \le i, j \le n$. Then we can identify $\mathrm{U}_n(R)$ with $\mathrm{Hom}(A, R)$ as follows: for given $\phi : A \to R$, let $\alpha_{ij} = \phi(x_{ij})$ and $\beta_{ij} = \phi(y_{ij})$. Then a matrix $g = (g_{ij})_{1\le i,j\le n}$ with $g_{ij} = 1 \otimes \alpha_{ij} + \sqrt{-1} \otimes \beta_{ij}$ becomes an element of $U_n(R)$ by the relations of x_{ij} and y_{ij}s defined by the ideal I. Similarly, for given $g = (g_{ij}) \in \mathrm{U}_n(R)$, we can write $g_{ij} = (a_{ij} + \sqrt{-1}b_{ij}) \otimes r_{ij} = 1 \otimes a_{ij}r_{ij} + \sqrt{-1} \otimes b_{ij}r_{ij}$ and we have a corresponding map $\phi : A \to R$ sending x_{ij} to $a_{ij}r_{ij}$ and y_{ij} to $b_{ij}r_{ij}$.

The group $\mathrm{U}_n(\mathbb{R})$ is a compact group (as a topological subgroup of $\mathrm{GL}_n(\mathbb{C})$) since it is closed (it is an inverse image of point I of a continuous map $g \to g\overline{g}^t$) and bounded (each row and column vectors have norm 1).

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***Problem* 1.9** Consider the following short exact sequence:

$$0 \to \ker(\epsilon)/\ker(\epsilon)^2 \to \mathcal{O}(G)/\ker(\epsilon)^2 \to k \to 0.$$

The map $O(G)/\ker(\epsilon)^2 \to k$ is defined as a composition of the natural map $\mathcal{O}(G)/\ker(\epsilon)^2 \to \mathcal{O}(G)/\ker(\epsilon)$ followed by ϵ. Then we have a section $k \to \mathcal{O}(G)/\ker(\epsilon)$ which is the composition $k \to \mathcal{O}(G) \to \mathcal{O}(G)/\ker(\epsilon)^2$ and the above sequence splits.

***Problem* 1.10** Let $g = (g_{ij}) \in GL_n(R)$ and $J = (\alpha_{ij}) \in \mathrm{GL}_n(k)$. Then $g^tJg = J$ is equivalent to

$$\sum_{k,l=1}^{n} \alpha_{kl} g_{ki} g_{lj} = \alpha_{ij}$$

for all $1 \le i, j \le n$. Hence G is an affine algebraic group with a coordinate ring

$$A = k[(X_{ij})_{1\le i,j\le n}]/\left(\sum_{k,l=1}^{n} \alpha_{kl} X_{ki} X_{lj} - \alpha_{ij}, 1 \le i, j \le n\right).$$

Since $\mathrm{Lie}\, G = \ker(G(k[t]/t^2) \to G(k))$, the elements of $\mathrm{Lie}\, G$ have a form of $I + tX$ for some $X \in \mathrm{M}_n(k)$. Then the defining equation $g^tJg = J$ is equivalent to

$$(I + tX)^t J(I + tX) = J \Leftrightarrow J + tX^tJ + tJX + t^2X^tJX = J + t(X^tJ + JX) = J,$$

(here every elements are in $\mathrm{GL}_n(k[t]/t^2)$) so we should have $X^tJ + JX = 0$. In other words, we have a map

$$\mathrm{Lie}\, G \xrightarrow{\sim} \{X \in \mathfrak{gl}_n(k) : X^tJ + JX\}, \quad I + tX \to X.$$

***Problem* 1.11** ***

***Problem* 1.12** ***

***Problem* 1.13** Using the equivalence of $\mathbf{Spl}_k$ and **RRD**, it is enough to check that the dual of the root datum of GL_n is isomorphic to itself in **RRD**. Recall that the root datum of GL_n with torus T of diagonal elements is given as follows: (Example 1.12)

- $X^*(T) = \{\alpha_{k_1,\dots,k_n} : \mathrm{diag}(t_1, \dots, t_n) \mapsto \prod_{1\le j\le n} t_j^{k_j},\ k_1, \dots, k_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $X_*(T) = \{\beta_{k_1,\dots,k_n} : t \mapsto \mathrm{diag}(t^{k_1}, \dots, t^{k_n}),\ t_1, \dots, t_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $\Phi(\mathrm{GL}_n, T) = \{e_{ij},\ 1 \le i \ne j \le n\},\ e_{ij}(\mathrm{diag}(t_1, \dots, t_n)) = t_i t_j^{-1}$
- $\Phi^\vee(\mathrm{GL}_n, T) = \{e_{ij},\ 1 \le i \ne j \le n\},\ e_{ij}^\vee(t) = \mathrm{diag}(1, \dots, t, \dots, t^{-1}, \dots, 1)$ (t in the i-th entry, t^{-1} in the j-th entry, 1 for other entries)

Then we define a map $f : X_*(T) \to X^*(T)$ and $\iota : \Phi(\mathrm{GL}_n, T) \to \Phi^\vee(\mathrm{GL}_n, T)$ as

$$f(\beta_{k_1,\dots,k_n}) = \alpha_{k_1,\dots,k_n}, \quad \iota(e_{ij}) = e_{ij}^\vee.$$

and define $f^\vee : X^*(T) \to X_*(T)$ and $\iota^\vee : \Phi^\vee(\mathrm{GL}_n, T) \to \Phi(\mathrm{GL}_n, T)$ similarly. Then these maps are inverse to each other and gives an isomorphism between two root data

$$(X^*(T), X_*(T), \Phi(\mathrm{GL}_n, T), \Phi^\vee(\mathrm{GL}_n, T)) \simeq (X_*(T), X^*(T), \Phi^\vee(\mathrm{GL}_n, T), \Phi(\mathrm{GL}_n, T))$$

(they are central isogenies) so we get $\widehat{\mathrm{GL}_n} = \mathrm{GL}_{n\mathbb{C}}$.

***Problem* 1.14** ***

***Problem* 1.15** ***

2. Chapter 2

***Problem* 2.1** ***

***Problem* 2.2** ***

***Problem* 2.3** It is compact since it is an intersection of closed subset $G(F)$ of $\mathrm{GL}_n(F)$ ($G \hookrightarrow \mathrm{GL}_n$ is closed immersion) and intersection of closed set with compact set is again compact. Openness follows from continuity of $G(F) \hookrightarrow \mathrm{GL}_n(F)$: $\rho(G(F)) \cap K$ is an inverse image of K under $G(F) \hookrightarrow \mathrm{GL}_n(F)$.

***Problem* 2.4** ***

***Problem* 2.5** Using the anti-equivalence of category $\mathbf{AffSch}_k$ and $\mathbf{Alg}_k$, we can reformulate the situation in terms of algebra as follows. Let $A = \mathcal{O}(Y)$ be $\mathfrak{o}$-algebra and $A_F := A \otimes_{\mathfrak{o}} F$. Let $X = \mathrm{Spec}(A_F/I)$ and $\mathcal{X}$ be schematic closure of X in Y, so that $\mathcal{O}(\mathcal{X}) = \mathrm{Im}(\pi^I \iota)$ where $\iota : A \hookrightarrow A_F$ and $\pi^I : A_F \twoheadrightarrow A_F/I$. Let $\mathcal{Z} = \mathrm{Spec}\, A/J$ (we have closed immersion $\mathcal{Z} \hookrightarrow \mathcal{Y}$), and we assume that the map on generic fibre, which corresponds to $A_F \twoheadrightarrow (A/J)_F$, induces an isomorphism $A_F/I = \mathcal{O}(X) \simeq \mathcal{O}(\mathcal{Z}) = (A/J)_F$. This means that there exists an isomorphism $\phi : A_F/I \to (A/J)_F$ such that the following diagram commutes:

$$\begin{tikzcd} (A/J)_F & \\ A_F/I \arrow[u, "\phi"] & A_F \arrow[l, two heads, "\pi^I"] \arrow[ul, two heads, "\pi^J_F"'] \end{tikzcd}$$

Now our goal is to show that there exists a unique map

$$f : \mathcal{O}(\mathcal{X}) = \mathrm{Im}(\pi^I \iota) \to \mathcal{O}(\mathcal{Z}) = A/J$$

such that the following diagram commutes:

$$\begin{tikzcd} A/J & \\ \mathrm{Im}(\pi^I \iota) \arrow[u, "f"] & A \arrow[l, two heads, "\pi^I \iota"] \arrow[ul, two heads, "\pi^J"'] \end{tikzcd}$$

The only way to define f that the above diagram commutes is following: for $x \in \mathrm{Im}(\pi^I \iota)$, choose $a \in A$ with $x = \pi^I \iota(a)$ and define $f(x) := \pi^J(a)$. Then we only need to show that the map is well-defined regardless of the choice of a. Let $a_1, a_2 \in A$ such that $\pi^I \iota(a_1) = \pi^I \iota(a_2) = x$. Since $\iota^J : A/J \hookrightarrow (A/J)_F$ is an injection, it is enough to show that $\iota^J \pi^J(a_1) = \iota^J \pi^J(a_2)$. By the commutativity of the following diagram

$$\begin{tikzcd} A/J \arrow[d, hook, "\iota^J"'] & A \arrow[l, two heads, "\pi^J"'] \arrow[d, hook, "\iota"] \\ (A/J)_F & A_F \arrow[l, two heads, "\pi^J_F"'] \end{tikzcd}$$

we have $\iota^J \pi^J = \pi^J_F \iota = \phi \pi^I \iota$, and this proves

$$\iota^J \pi^J(a_1) = \phi \pi^I \iota(a_1) = \phi(x) = \phi \pi^I \iota(a_2) = \iota^J \pi^J(a_2),$$

i.e. the map is well-defined.

***Problem* 2.6** ***

***Problem* 2.7** ***

***Problem* 2.8** Note that the coordinate ring of $\mathrm{GL}_{n,\mathbb{Q}}$ is

$$B = \mathcal{O}(\mathrm{GL}_{n,\mathbb{Q}}) = \mathbb{Q}[x_{ij}, y]_{1\le i,j\le n}/(\det(x_{ij})y - 1).$$

To show that $\mathcal{G}$ is a model of $\mathrm{GL}_{n,\mathbb{Q}}$ over $\mathbb{Z}$, we need to show that $A \hookrightarrow B$ and $A \otimes_{\mathbb{Z}} \mathbb{Q} \simeq B$. Latter isomorphism easily follows from

$$A \otimes \mathbb{Q} = \mathbb{Q}[x_{ij}, t_{ij}, y]/(\det(x_{ij})y - 1, \{x_{ij} - \delta_{ij} - mt_{ij}\}) \simeq B$$

since we can invert $m > 1$ in $\mathbb{Q}$ and get an isomorphism $A \otimes \mathbb{Q} \to B$ via $t_{ij} \mapsto (1 - x_{ij})/m$. Shoing $A \hookrightarrow B \simeq A \otimes_{\mathbb{Z}} \mathbb{Q}$ is equivalent to showing that A is a torsion-free $\mathbb{Z}$-module. Assume that we have $z \in \mathbb{Z}[x_{ij}, t_{ij}, y]$ and $0 \neq a \in \mathbb{Z}$ such that $az = 0$ in A. Then there exists $\alpha, \beta_{ij} \in \mathbb{Z}$ for $1 \le i, j \le n$ s.t.

$$\begin{aligned} az &= \alpha(\det(x_{ij})y - 1) + \sum_{ij} \beta_{ij}(x_{ij} - \delta_{ij} - mt_{ij}) \\ \Leftrightarrow z &= \frac{\alpha}{a}\det(x_{ij})y + \sum_{i,j} \frac{\beta_{ij}}{a} x_{ij} - \sum_{i,j} \frac{m\beta_{ij}}{a} t_{ij} - \frac{\alpha + \sum_i \beta_{ii}}{a} \end{aligned}$$

which implies $a|\alpha$ and $a|\beta_{ij}$, i.e. $z = 0$ in A. Hence $\mathcal{G}$ is a model of $\mathrm{GL}_{n,\mathbb{Q}}$ over $\mathbb{Z}$.

The set of $\mathbb{Z}$-points $\mathcal{G}(\mathbb{Z}) = \mathrm{Hom}(A, \mathbb{Z})$ can be identified with the set via map

$$\begin{aligned} \mathrm{Hom}(A, \mathbb{Z}) &\to \{g \in \mathrm{GL}_n(\mathbb{Z}) \,:\, g \equiv I_n \pmod{m\mathrm{M}_n(\mathbb{Z})}\} \\ \phi &\mapsto (g_{ij} = \phi(x_{ij})) \end{aligned}$$

since $\phi(x_{ij}) = \delta_{ij} + m\phi(t_{ij}) \Rightarrow g - I_n \in m\mathrm{M}_n(\mathbb{Z})$.

***Problem* 2.9** *** It is not hard to prove that if Z_1, Z_2 are dense subsets of a topological space Y_1, Y_2 respectively, then $Z_1 \times Z_2$ is dense in $Y_1 \times Y_2$. Combining with Exercise 1.6 and Theorem 2.2.1 (b), we get the desired results for both weak and strong approximation.

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***Problem* 2.10** By Exercise 2.7 and 2.9, $\mathrm{M}_n \simeq \mathbb{G}_a^{n^2}$ admits weak approximation over F. With embedding $\mathrm{GL}_n \hookrightarrow \mathrm{M}_n$ with $\mathrm{GL}_n(F) = \mathrm{M}_n(F) \cap \mathrm{GL}_n(F_S) \subset \mathrm{M}_n(F_S)$, we also have $\mathrm{GL}_n(F)$ dense in $\mathrm{GL}_n(F_S)$.

***Problem* 2.11** ***

***Problem* 2.12** ***

***Problem* 2.13** ***

***Problem* 2.14** ***

***Problem* 2.15** ***

***Problem* 2.16** ***

***Problem* 2.17** ***

***Problem* 2.18** ***

***Problem* 2.19** Let $N = p_1^{e_1} \cdots p_r^{e_r}$ be a prime factorization of N. Define $K_N \le \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ as

$$K_N = \prod_{i=1}^{r} (I_n + p_i^{e_i}\mathrm{M}_n(\mathbb{Z}_{p_i})) \times \prod_{p \neq p_i} \mathrm{GL}_n(\mathbb{Z}_p).$$

Then K_N is an open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ such that $K_N \cap \mathrm{GL}_n(\mathbb{Q}) = \Gamma(N)$.

($\Rightarrow$) Let H be a congruence subgroup of $\mathrm{GL}_n(\mathbb{Q})$, which means that there exists an open compact subgroup $K_H \leq \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ such that $H = K_H \cap \mathrm{GL}_n(\mathbb{Q})$. Then we can find an open compact neighborhood $U \leq K_H$ of I_n which has a form of

$$U = \prod_{p \in S}(I_n + p^{e_p}\mathrm{M}_n(\mathbb{Z}_p)) \times \prod_{p \notin S} \mathrm{GL}_n(\mathbb{Z}_p)$$

for some finite set of primes S (Note that $\{I_n + p^k \mathrm{M}_n(\mathbb{Z}_p)\}_{k \geq 1}$ is a decreasing sequence of open compact neighborhoods of I_n, which is also a subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$). Then $U = K_N$ for $N = \prod_{p \in S} p^{e_p}$, i.e. U is also an open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$, and it is a finite index subgroup of K_H since K_H is open and compact (consider all the cosets of K_N in K_H, which are all homeomorphic to K_N). Then $[H : \Gamma(N)] = [K_H : K_N]$ implies that H contains $\Gamma(N)$ as a finite index subgroup.

($\Leftarrow$) Let H be a subgroup of $\mathrm{GL}_n(\mathbb{Q})$ contains $\Gamma(N)$ with $[H : \Gamma(N)] < \infty$. Let K_H be an image of H in $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ under the diagonal embedding $\mathrm{GL}_n(\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ so that $K_H \cap \mathrm{GL}_n(\mathbb{Q}) = H$. Then K_H contains K_N and $[K_H : K_N] = [H : \Gamma(N)]$, so K_N is a finite index subgroup of K_H. for coset representatives $g_1, g_2, \ldots, g_t$ of K_H/K_N, $K_H = \cup_{j=1}^{t} g_j K_N$ and by openness (resp. compactness) of K_N, K_H is also open (resp. compact) subgroup.