ANALYTIC LANGLANDS PROGRAM

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ABSTRACT. This is a LaTeX-ed note for the special lecture on *Analytic Langlands Program* by Edward Frankel at UC Berkeley in 2022 Fall.

CONTENTS

1. Introduction to Langlands correspondence (August 25)

This course is on a new aspect of Langlands program, so-called *Analytic Langlands Program*. The classical Langlands program is originated from Langlands' letter to André Weil in 1967, and also from André Weil's letter to his sister (Simone Weil) on his conjecture (Weil's conjecture on zeta functions of curves over finite fields, which was resolved by Dwork, Grothendieck, and Deligne) in 1940. Weil's *Rosetta stone* relates two different topics in mathematics: number theory and complex curves (Riemann surfaces). A goal is to find something happens in parallel between two, and we need another bridge - curves over finite fields. The difference between complex curves and curves over finite fields is the fact that those are defined over different fields. A similarity between number theory sied and the curves over finite fields side is that the number fields (finite extensions of $\mathbb{Q}$) are similar to the function fields of curves (over finite fields - we denote it as $\mathbb{F}_q(X)$ for a curve $X/\mathbb{F}_q$). The most simplest example is a comparison between $\mathbb{Q}$ and $\mathbb{F}_q(\mathbb{P}^1) \simeq \mathbb{F}_q(t)$:

$$\mathbb{Q} = \left\{ \frac{p}{q} \,:\, p, q \,\text{rel. prime} \in \mathbb{Z} \right\} \leftrightarrow \mathbb{F}_q(t) = \left\{ \frac{P(t)}{Q(t)} \,:\, P, Q \,\text{rel. prime} \in \mathbb{F}_q[t] \right\}$$

$$\text{ring of integers: } \mathbb{Z} \leftrightarrow \mathbb{F}_q[t]$$

$$\text{completions: } \mathbb{Q}_p \leftrightarrow \mathbb{F}_q((t))$$

Sometimes we include one more topic in this Rosetta stone, which originates from Physics - Quantum Field Theory, Electro-Magnetic Duality, and Gauge Theory (developed by Edward Witten and other physicists).

The (classical) Langlands correspondence is about interplays between the Galois representations and Automorphic representations.[1] It dealts with two different (but similar) types of fields - number fields and function fields of curves over finite fields. Fix such a field F. Then we can describe a Langlands correspondence for GL_n over F as follows:

(1) **Galois side:** Let $\overline{F}$ be a (separable) algebraic closure of F, and $\mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group of F (i.e. the group of automorphisms of $\overline{F}$ that fix F pointwisely). This is one of the most important groups in number theory, and its structure is higly complicated. Hence, instead of studying the group $\mathrm{Gal}(\overline{F}/F)$ directly, we study the representations of it. Especially, we are going to consider the (equivalence) classes of n-dimensional representations of $\mathrm{Gal}(\overline{F}/F)$ over some field that would be determined later. This is just an equivalence class of homomorphisms

$$\sigma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(?).$$

(2) **Automorphic side:** We first need to define the notion of *Adele*. Let $\mathscr{V} = \mathscr{V}_F$ be the set of equivalent classes of the norms (places) on F. For each $v \in \mathscr{V}$, we can define a completion F_v with respect to v. For example, when $F = \mathbb{Q}$, Ostrowski's theorem states that the places of $\mathbb{Q}$ corresponds to the set of primes (each prime p gives p-adic norms, which is non-archimedean) along with the "infinite" prime (corresponds to the usual archimedean norm). In this case, we have two types of completions, either p-adic numbers $\mathbb{Q}_p$ or

[1]The *geometric* Langlands correspondence is about curves over $\mathbb{C}$, which is mainly developed by Drinfeld, Laumon, Beilinson, Gaitsgory, ...

real numbers $\mathbb{R}$. And we have the ring of p-adic integers $\mathbb{Z}_p$ as a subring of $\mathbb{Q}_p$.

In case of function field $F = \mathbb{F}_q(\mathbb{P}^1) \simeq \mathbb{F}_q(t)$ of $X = \mathbb{P}^1$, the places of F corresponds to the closed points of X, which again corresponds to the maximal ideals of $\mathbb{F}_q[t]$ (and the point at infinity). For example, any $a \in \mathbb{F}_q$ actually gives a closed point that corresponds to the maximal ideal $(x-a)$. Any other irreducible polynomials over $\mathbb{F}_q$ of higher degree also give closed points in X. Completion of F at $x \in X$ is isomorphic to the field of formal Laurent series $(\mathbb{F}_q)_x((t_x))$, where $(\mathbb{F}_q)_x$ is the residue field at x and t_x is some parameter. We also have a ring of integers in these completions, which is $(\mathbb{F}_q)_x[[t_x]]$.

The ring of adeles is defined as a restricted product of all completions of F, which is

$$\mathbb{A}_F = \prod_{v \in \mathscr{V}_F} F_v = \{(f_v) : f_v \in F_v, f_v \in \mathcal{O}_v \text{ for all but finitely many } v.\}$$

where $\mathcal{O}_v \subset F_v$ is the ring of integers of F_v, which is the set of elements with norm at most 1. Then we have a diagonal embedding $F \hookrightarrow \mathbb{A}_F$ that sends $a \in F$ to $(a, a, \dots) \in \mathbb{A}_F$, and this induces an embedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_F)$. Now we can think of a Hilbert space $\mathscr{H}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F))$ of L^2-functions on the quotient space $\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)$, with the Haar measure on the quotient space. Then we have a right regular representation of $\mathrm{GL}_n(\mathbb{A}_F)$ on $\mathscr{H}$, and it is known that this representation decomposes into continuous part and discrete part:

$$\mathscr{H} = \mathscr{H}_{\text{cont}} \oplus \mathscr{H}_{\text{disc}}$$

and the discrete part decomposes into irreducible representations as

$$\mathscr{H}_{\text{disc}} = \oplus_\pi \pi$$

without multiplicity (multiplicity one theorem).[2] The irreducible constituents of $\mathscr{H}_{\text{disc}}$ is called *cuspidal automorphic rerpesentations of* $\mathrm{GL}_n(\mathbb{A}_F)$, up to technical conditions on the center of the group and the archimedean places.

(3) **Correspondence:** The Langlands correspondence states that there is a one-to-one correspondence between these two different objects that preserves some special invariants. More precisely, the equivalence classes of n-dimensional Galois representation σ of $\mathrm{Gal}(\overline{F}/F)$ (over some field) corresponds to an irreducible automorphic representation $\pi = \pi(\sigma)$ of $\mathrm{GL}_n(\mathbb{A}_F)$, and vice versa. The irreducibility of σ corresponds to cuspidality of π.

One of the invariant of Galois representation σ is the conjugacy classes of $\sigma(\mathrm{Fr}_x)$ in GL_n, where Fr_x is the *Frobenius* element corresponds to a closed point $x \in X$ (for almost all x). On the automorphic side, there is so-called *Hecke operators* h_x for each $x \in X$. Then the conjectural Langlands correspondence should give correspondences between these two invariants.

[2]It is not always the case that any representation decomposes into irredubiles - consider the regular representation of $\mathbb{R}$ on $\mathrm{L}^2(\mathbb{R})$ that acts as a translation. The irreducible sub-representations of it correspons to the exponential function $\exp(i\lambda x)$ for $\lambda \in \mathbb{R}$, but we can't write a function $f(x)$ as a discrete sum of these in general. We can only write it as an integral of these, which is the Fourier transform. Note that the regular representation on $\mathrm{L}^2(\mathbb{Z}\backslash\mathbb{R})$ decomposes into irreducibles (which gives Fourier series), and the reason behind is that the circle group $\mathbb{S}^1 = \mathbb{Z}\backslash\mathbb{R}$ is compact.

A celebrated example of Langland's correspondence is the Shimura-Taniyama-Weil conjecture, which is now a theorem by Andrew Wiles and Richard Taylor (modularity theorem). It is a special case of Langland's correspondence for $n = 2$ that relates an elliptic curve and a modular form.

Let E be an elliptic curve over $\mathbb{Q}$, i.e. a smooth projective curve defined over $\mathbb{Q}$ by equation

$$y^2 = x^3 + ax + b$$

for $a, b \in \mathbb{Q}$ and $\Delta = 4a^3 + 27b^2 \neq 0$ (the discriminant of E). Then for each prime ℓ not dividing Δ (or any $\ell \neq p = \mathrm{char}(F)$ when F is a function field), the first étale cohomology $\mathrm{H}^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is isomorphic to $\mathbb{Q}_\ell^2$, 2-dimensional vector space over $\mathbb{Q}_\ell$. Since E is defined over $\mathbb{Q}$, we have a natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $E_{\mathbb{Q}} = E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ which induces an action on $\mathrm{H}^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$i, i.e. a 2-dimensional representation $\sigma_{E,\ell}$ over $\mathbb{Q}_\ell$ (so the mysterious field where the representation is defined that we didn't defined before is $\mathbb{Q}_\ell$ in this case). For $p \nmid \Delta$, Frobenius conjugacy class $\sigma_{E,\ell}(\mathrm{Fr}_p)$ has a trace[3]

$$\mathrm{Tr}(\sigma_{E,\ell}(\mathrm{Fr}_p)) = p + 1 - \#E(\mathbb{F}_p)$$

and determinant p, which completely determines the conjugacy class of $\sigma_{E,\ell}(\mathrm{Fr}_p)$ for GL_2. Here $\#E(\mathbb{F}_p)$ is the number of $\mathbb{F}_p$-points on E.

Assuming Langlands' correspondence, such $\sigma_{E,\ell}$ (or family of $\sigma_{E,\ell}$) for varying ℓ's) should corresponds to an irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. In case of GL_2, an automorphic representation π corresponds to certain holomorphic function f_π called *modular form* on the complex upper half plane $\mathfrak{H}$ satisfying a functional equation

$$f_\pi\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 f_\pi(\tau)$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), c \equiv 0 \pmod{N}\}$ where $N = N_E$ is the conductor of E. Also, $f_\pi(\tau) \to 0$ as $\tau \to \infty$ (in other words, f_π is a cusp form). It admits a Fourier expansion

$$f_\pi(\tau) = \sum_{n \geq 1} a_n q^n, \quad q = e^{2\pi i \tau}$$

and the Langlands correspondence for this case becomes the equality

$$a_p = \mathrm{Tr}(\sigma_{E,\ell}(\mathrm{Fr}_p)) = p + 1 - \#E(\mathbb{F}_p).$$

for all $p \nmid \ell N$.

Langlands correspondence for GL_n is now a theorem when F is a function field of some curve, and this is proven by Drinfeld ($n = 2$) and Laurent Lafforgue ($n > 2$).[4]

[3] This follows from the Lefschetz formula for étale cohomology.

[4] Lafforgue got a Fields medal for this work.

2. More on classical Langlands correspondence (August 30)

We are going to give more detailed explanations on the classical Langlands correspondence and give an explicit example of a correspondence between elliptic curves and modular forms (Taniyama-Shimura-Weil conjecture, now a theorem by Wiles-Taylor and Breuil-Conrad-Diamond-Taylor).

First, irreducible cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_F)$ always decomposes into *local* representations as[5]

$$\pi = \bigotimes_{v\in\mathscr{V}} \pi_v$$

(this is also a kind of restricted product). When $F = \mathbb{F}_q(X)$ is a function field, then there is a 1-1 correspondence between the set of places (completions) $\mathscr{V} = \mathscr{V}_F$ and the set of closed points $|X|$ of a curve X. (There are only non-archimedean places.) If a place $v \in \mathscr{V}$ corresponds to a point $x \in |X|$, and the completion of F by v is isomorphic to $(\mathbb{F}_q)_x((t_x))$, where t_x is a local coordinate at x. In this case, each π_v becomes a representation of $\mathrm{GL}_n(F_v)$. When F is a number field, there exist archimedean places, which has a different nature from nonarchimedean places. For example, when $F = \mathbb{Q}$, we have $\mathscr{V}_{\mathbb{Q}} = \{p : p \text{ prime}\} \cup \{\infty\}$, and π decomposes as

$$\pi = \left(\bigotimes_{p<\infty} \pi_p\right) \otimes \pi_\infty.$$

Although π_p's are representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, π_∞ is *not* an irreducible representation of $\mathrm{GL}_2(\mathbb{R})$. It is acually a representation of $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{O}_2(\mathbb{R}))$ - in other words, it is a representation of Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ and a (maximal compact subgroup) $\mathrm{O}_2(\mathbb{R})$ with compatibility condition on their actions.

Recall that the classical Langlands correspondence for GL_n is a correspondence between (equivalence classes of) n-dimensional irreducible (ℓ-adic) Galois representations σ of $\mathrm{Gal}(\overline{F}/F)$ and (equivalence classes of) cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. It is not an arbitrary 1-1 correspondence - certain *invariants* should match. The Galois-side invariant is semisimple Frobenius conjugacy classes in $\mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$: it is

$$\{\sigma(\mathrm{Fr}_v), v \in \mathscr{V}\backslash S_\sigma\}$$

where S_σ is a finite subset of $\mathscr{V}$. Note that the topology matters for Galois side - we have Krull topology (profinite topology) on $\mathrm{Gal}(\overline{F}/F)$ and we only consider continuous representations. On the automorphic side, there are certain semisimple conjugacy classes in $\mathrm{GL}_n(\mathbb{C})$, which we call Hecke conjugacy classes. These record eigenvalues of the (spherical) Hecke algebra associated to each $v \in \mathscr{V}$. We denote it as

$$\{\pi(h_v), v \in \mathscr{V}\backslash S_\pi\}$$

where S_π is a finite subset of $\mathscr{V}$. Note that we can identify $\overline{\mathbb{Q}_\ell}$ and $\mathbb{C}$ since they have the same transcendence degree over $\overline{\mathbb{Q}}$, and the correspondence is independent of the choice of identification. Also, the invariants uniquely determine representation themselves.

Now, we will introduce an explicit correspondence between a certain elliptic curve and a modular form. Let E be an elliptic curve over $\mathbb{Q}$ defined by

$$y^2 + y = x^3 - x^2.$$

[5]this is called Flath's theorem.

Then the only bad prime of reduction is 11, and the conductor of the elliptic curve is also 11. We can count the number of $\mathbb{F}_p$-points on the curve. For example, when $p = 5$, there are exactly 5 points: $\{(0,0),(1,0),(0,4),(1,4),\infty\}$. Now, consider the following function defined as an infinite product:

$$f(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2, \quad q = e^{2\pi i\tau}.$$

It turns out that this is a modular frm of weight 2 and level 11. Its expansion is

$$f(\tau) = q - 2q^2 - q^3 + 2q^r + q^5 + 2q^6 - 2q^7 + \cdots$$

and the 5th coefficient of f is $a_5(f) = 1$, which equals to $a_5(E) = 5 + 1 - 5 = 1$. In fact, this is the modular form corresponds to E, and $a_p(E) = a_p(f)$ holds for all $p \neq 11$.

As an aside, Langlands correspondence for GL_1 has long been known as *abelian class field theory*. Since GL_1 is an abelian group, 1-dimensional Galois represention should factor through $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ and the structure of the latter group is well known for some cases. For example, we have a Kronecker-Weber theorem when $\mathbb{Q}$, which states that $\mathbb{Q}^{\mathrm{ab}} = \cup_{n\geq 1}\mathbb{Q}(\zeta_n)$.

Now we will explain Frobenius automorphisms and conjugacy classes in detail. The Galois group of finite extension of finite fields has a simple structure. For the extension $\mathbb{F}_{q^n}/\mathbb{F}_q$, its Galois group is just a cyclic group of order n generated by the Frobenius automorphism $x \mapsto x^q$. Now let K/F be a finite extension of number fields, and $\mathcal{O}_F \subset \mathcal{O}_K$ be the ring of integers. These are Dedekind domain: any ideal admits a prime ideal factorization. For a prime ideal $v \subset \mathcal{O}_F$, regarding it as an ideal $\mathcal{O}_K$, it splits as a product of prime ideals in $\mathcal{O}_K$ as $v = w_1 \cdots w_g$. Then $\mathcal{O}_F/v \subset \mathcal{O}_K/w_j$ is a finite extension of finite fields, so is cyclic. Although we can't directly link $\mathrm{Gal}(K/F)$ with $\mathrm{Gal}((\mathcal{O}_K/w_j)/(\mathcal{O}_F/v))$, there exists a *decomposition group* $D_{w_j} \subset \mathrm{Gal}(K/F)$ defined as

$$D_{w_j} := \{g \in \mathrm{Gal}(K/F) : gw_j = w_j\} \xrightarrow{\alpha_{w_j}} \mathrm{Gal}((\mathcal{O}_K/w_j)/(\mathcal{O}_F/v))$$

where α_{w_j} is surjective. We also define *inertia subgroup* I_{w_j} as $\ker \alpha_{w_j}$, so that $D_{w_1}/I_{w_1} \simeq \mathbb{Z}/n\mathbb{Z}$ for some n. Now, when $I_{w_j} = 1$, we have $D_{w_j} \simeq \mathbb{Z}/n\mathbb{Z}$ and we can define a Frobenius conjugacy class in $\mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ by composing the isomorphism with $\sigma|_{D_{w_j}}$. It is known that $I_{w_j} = 1$ for all but finitely many v (we call such v *unramified*), and since different choices of w_j gives conjugated decomposition groups, the Frobenius conjugacy class $\sigma(\mathrm{Fr}_{w_j})$ does not depend on the choice of w_j and only on v.

3. Classical Langlands correspondence over function fields (September 1)

We are going to explain classical Langlands correspondence over function fields in (more) detail. Let X be a smooth, geometrically irreducible, projective curve over $\mathbb{F}_q$ and $F = \mathbb{F}_q(X)$ be a function field. Let $|X|$ be a set of closed points of X, which has a 1-1 correspondence with $\mathscr{V}$ - the set of places (completions) of F. Recall that the completion F_x at $x \in |X|$ is isomorphic to $(\mathbb{F}_q)_x((t_x))$, where $(\mathbb{F}_q)_x$ residue field at x and t_x is a rational functino on X with order 1 zero at x (In other words, it is a generator of maximal ideal $\mathfrak{m}_x$ corresponds to x). Then we have a ring of integer $\mathcal{O}_x \subset F_x$ isomorphic to the ring of formal power series $(\mathbb{F}_q)_x[[t_x]]$. We also defined the adéle ring $\mathbb{A}_F$ for F.

Now we define the *Weil group* $W(\overline{F}/F)$ as follows. Let $\overline{F}$ be a (separable) algebraic closure of F, then we have the action of $\mathrm{Gal}(\overline{F}/F)$ on the subfield $\overline{\mathbb{F}_q}$ (the field of constants) that fixes $\mathbb{F}_q$. Then we have a surjective map

$$\mathrm{Gal}(\overline{F}/F) \xrightarrow{\mathrm{res}} \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$$

and the latter group is an inverse limit of Galois groups of finite extensions of $\mathbb{F}_q$, so

$$\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \simeq \varprojlim \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \simeq \varprojlim \mathbb{Z}/n\mathbb{Z} =: \widehat{\mathbb{Z}},$$

which is the profinite completion of $\mathbb{Z}$. It is topologically generated by Frobenius automorphism Fr, and it has a subgroup isomorphic to $\mathbb{Z}$ generated (not topologically, but just algebraically) by Fr. Then we define the *Weil group* $W(\overline{F}/F)$ as an inverse image of $\mathbb{Z} \simeq \langle \mathrm{Fr} \rangle \subset \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ of restriction map, which is a subgroup of $\mathrm{Gal}(\overline{F}/F)$. For Galois side of Langlands correspondence over function field, we are going to consider irreducible representations of $W(\overline{F}/F)$ instead of $\mathrm{Gal}(\overline{F}/F)$. More precisely, we consider the (equivalence classes of) irreducible n-dimensional ℓ-adic representations of $W(\overline{F}/F)$,

$$\sigma : W(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$$

such that

(1) Image of σ in $\mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ is in $\mathrm{GL}_n(E)$ for some finite extension $E/\mathbb{Q}_\ell$.
(2) σ is continuous where $W(\overline{F}/F)$ is given Krull topology (profinite toppology) and $\mathrm{GL}_n(E)$ is given subspace topology of $M_n(E)$.[6]
(3) σ is unramified for all but finitely many $x \in |X|$. Note that the unramifiedness is defined using decomposition group and inertia group as before.

On the automorphic side, we wiil explain cuspidality and unramifiedness in more detail. The space of cusp forms $\mathrm{L}^2_{\mathrm{cusp}}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F), \chi)$[7] are functions satisfying the following vanishing condition: for $0 < n_1, n_2 < n$ with $n = n_1 + n_2$, we

[6]This explains somehow why we are considering ℓ-adic representations instead of complex representations. As a toy example, consider continuous 1-dimensional complex representations of $(\mathbb{Z}_\ell, +)$, i.e. an additive character $\sigma : \mathbb{Z}_\ell \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. Then it should factor through $\mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell$ for some n, so that the image is always finite. However, if we consider ℓ-adic characters $\sigma : \mathbb{Z}_\ell \to \mathbb{Q}_\ell^\times$, then there are non-trivial characters with infinite image, e.g. $x \mapsto \exp_\ell(\ell x)$ where $\exp_{ell}$ is an ℓ-adic exponential function.

[7]Here χ is a continuous unitary character on center $Z(\mathbb{A}_F)$ trivial on $Z(F)$, and $\mathrm{L}^2(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F), \chi)$ is a space of functions where the center acts as the character χ.

have
$$\int_{N_{n_1,n_2}(F)\backslash N_{n_1,n_2}(\mathbb{A}_F)} f(ng)dn = 0$$
for all $g \in \mathrm{GL}_n(\mathbb{A}_F)$, where $N_{n_1,n_2} < \mathrm{GL}_n$ is the unipotent group of matrices of the form
$$\begin{pmatrix} I_{n_1} & * \\ \mathbf{0} & I_{n_2} \end{pmatrix}$$
Note that non-example of cuspidal represenetation is Eisenstein series representation, which is obtained from two representations π_1, π_2 of $\mathrm{GL}_{n_1}(\mathbb{A}_F)$ and $\mathrm{GL}_{n_2}(\mathbb{A}_F)$ respectively, by inflation and (parabolic induction). Then it is a theorem (from Flath) that any irreducible cuspidal represenetations of $\mathrm{GL}_n(\mathbb{A}_F)$ decomposes as restricted product of local representations,
$$\pi \simeq \bigotimes_{x\in|X|} \pi_x$$
where each π_x are irreducible representation of $\mathrm{GL}_n(F_x)$. In this case, for all but finitely many x, $\mathrm{GL}_n(\mathcal{O}_x)$-fixed subspace $\pi^{\mathrm{GL}_n(\mathcal{O}_x)}$ is non-trivial and one-dimensional. We call that π is *unramified at* x for such x. For x where *pi* is unramified, we have a represenetation of *spherical Hecke algebra* $\mathcal{H}_x$, which is a sub-algebra of compactly supported functions on $\mathrm{GL}_n(F_x)$ that are $\mathrm{GL}_n(\mathcal{O}_x)$-biinvariant. Then $\mathcal{H}_x$ is a convolution algebra which is commutative and isomorphic to $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]^{S_n}$ and corresponds to semisimple conjugacy classes in $\mathrm{GL}_n(\mathbb{C})$, which we will denote $\pi(h_x)$.

Also, as in the case of Galois side, we impose some conditions on the automorphic side. We will only consider automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ with some finiteness conditions, i.e. for any compact subgroup K of $\mathrm{GL}_n(\mathbb{A}_F)$, the translates of any $f \in \pi$ span a finite dimensional vector space.

Then the Langlands correspondence becomes as follows. It is a 1-1 correspondence between the irreducible ℓ-adic n-dimensional representations of Weil group $W(\overline{F}/F)$ (with some conditions) and irreducible cupsidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ (with some conditions). The invariants, Frobenius conjugacy classes $\{\sigma(\mathrm{Fr}_x)\}$ on the Galois side, matches with the Hecke conjugacy classes $\{\pi(h_x)\}$, for all $x \notin S_\sigma \cup S_\pi$. Here S_σ (resp. S_π) is the set of unramified places for σ (resp. π), and we actually have $S_\sigma = \S_\pi$ for corresponding $\sigma - \pi$ pairs.