# SOLUTION FOR “AN INTRODUCTION TO AUTOMORPHIC REPRESENTATION”

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Abstract. This is a solution for the exercises in J. Getz and H. Hahn’s book *An Introduction to Automorphic Representation with a view toward Trace Formulae*.

## 1. Chapter 1

***Problem* 1.1** *** By Yoneda lemma, the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes corresponds to the $k$-algebra morphism $\phi : A \to B$. This induces a map on the underlying topological spaces by sending a prime ideal $\mathfrak{p} \subset B$ to $\phi^{-1}(\mathfrak{p}) \subset A$, which is also prime.

***Problem* 1.2** ***

***Problem* 1.3** By Yoneda lemma, we have

$$\mathrm{Mor}(\mathrm{Spec}(B), \mathrm{Spec}(A)) \simeq \mathrm{Nat}(h^B, h^A) \simeq h^B(A) = \mathrm{Hom}_k(A, B)$$

which gives an equivalence between $\mathbf{AffSch}_k^{\mathrm{op}}$ and $\mathbf{Alg}_k$.

***Problem* 1.4**

- Nonreduced: $\mathrm{Spec}(\mathbb{C}[x]/(x^2))$
- Reducible: $\mathrm{Spec}(\mathbb{C}[x, y]/(x, y))$
- Reduced and irreducible (i.e. integral): $\mathrm{Spec}(\mathbb{C}[x])$

***Problem* 1.5** We can assume that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A/I)$ for some $k$-algebra $A$ and an ideal $I$ of $A$. Then it is enough to show that the map $\mathrm{Hom}(A/I, R) \to \mathrm{Hom}(A, R)$, given by composing with the natural map $\pi : A \to A/I$, is injective. This follows from the surjectivity of $\pi$.

***Problem* 1.6** Let $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B), Z = \mathrm{Spec}(C)$. Then the statement is equivalent to

$$\mathrm{Hom}(A \otimes_B C, R) \simeq \mathrm{Hom}(A, R) \times_{\mathrm{Hom}(B,R)} \mathrm{Hom}(C, R).$$

We can define a bijection as follows. First, consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\iota_A"] & A \otimes_B C \\ B \arrow[u, "\alpha"] \arrow[r, "\gamma"] & C \arrow[u, "\iota_C"'] \end{tikzcd}$$

Using the maps above, we define a map from LHS to RHS as $\phi \mapsto (\phi\iota_A, \phi\iota_C)$. Since $\iota_A\alpha = \iota_C\gamma$, we have $\phi\iota_A\alpha = \phi\iota_C\gamma$ and the map is well-defined. For the other direction, for given $(f, g) : A \times C \to R$ with $f\alpha = g\gamma$, universal property of the tensor product gives a unique map $\phi : A \otimes_B C \to R$ with $f = \phi\iota_A$ and $g = \phi\iota_C$. We can check that these maps are inverses for each other.

***Problem* 1.7** ***

***Problem* 1.8** *** We define an $\mathbb{R}$-algebra $A$ as

$$A = \mathbb{R}[(x_{ij}, y_{ij})_{1\le i,j\le n}]/I$$

where $I$ is an ideal generated by elements of the form

$$\left(\sum_{k=1}^n (x_{ik}^2 + y_{ik}^2)\right) - 1,$$

$$\sum_{k=1}^n (x_{ik}x_{jk} - y_{ik}y_{jk}), \quad i \neq j$$

$$\sum_{k=1}^n (x_{ij}y_{jk} + y_{ik}x_{jk}), \quad i \neq j$$

for $1 \le i, j \le n$. Then we can identify $\mathrm{U}_n(R)$ with $\mathrm{Hom}(A, R)$ as follows: for given $\phi : A \to R$, let $\alpha_{ij} = \phi(x_{ij})$ and $\beta_{ij} = \phi(y_{ij})$. Then a matrix $g = (g_{ij})_{1\le i,j\le n}$ with $g_{ij} = 1 \otimes \alpha_{ij} + \sqrt{-1} \otimes \beta_{ij}$ becomes an element of $U_n(R)$ by the relations of $x_{ij}$ and $y_{ij}$s defined by the ideal $I$. Similarly, for given $g = (g_{ij}) \in \mathrm{U}_n(R)$, we can write $g_{ij} = (a_{ij} + \sqrt{-1}b_{ij}) \otimes r_{ij} = 1 \otimes a_{ij}r_{ij} + \sqrt{-1} \otimes b_{ij}r_{ij}$ and we have a corresponding map $\phi : A \to R$ sending $x_{ij}$ to $a_{ij}r_{ij}$ and $y_{ij}$ to $b_{ij}r_{ij}$.

The group $\mathrm{U}_n(\mathbb{R})$ is a compact group (as a topological subgroup of $\mathrm{GL}_n(\mathbb{C})$) since it is closed (it is an inverse image of point $I$ of a continuous map $g \to g\overline{g}^t$) and bounded (each row and column vectors have norm 1).

At last, NOT FINISHED

***Problem* 1.9** Consider the following short exact sequence:

$$0 \to \ker(\epsilon)/\ker(\epsilon)^2 \to \mathcal{O}(G)/\ker(\epsilon)^2 \to k \to 0.$$

The map $O(G)/\ker(\epsilon)^2 \to k$ is defined as a composition of the natural map $\mathcal{O}(G)/\ker(\epsilon)^2 \to \mathcal{O}(G)/\ker(\epsilon)$ followed by $\epsilon$. Then we have a section $k \to \mathcal{O}(G)/\ker(\epsilon)$ which is the composition $k \to \mathcal{O}(G) \to \mathcal{O}(G)/\ker(\epsilon)^2$ and the above sequence splits.

***Problem* 1.10** Let $g = (g_{ij}) \in GL_n(R)$ and $J = (\alpha_{ij}) \in \mathrm{GL}_n(k)$. Then $g^tJg = J$ is equivalent to

$$\sum_{k,l=1}^{n} \alpha_{kl} g_{ki} g_{lj} = \alpha_{ij}$$

for all $1 \le i, j \le n$. Hence $G$ is an affine algebraic group with a coordinate ring

$$A = k[(X_{ij})_{1\le i,j\le n}]/\left(\sum_{k,l=1}^{n} \alpha_{kl} X_{ki} X_{lj} - \alpha_{ij}, 1 \le i, j \le n\right).$$

Since $\mathrm{Lie}\, G = \ker(G(k[t]/t^2) \to G(k))$, the elements of $\mathrm{Lie}\, G$ have a form of $I + tX$ for some $X \in \mathrm{M}_n(k)$. Then the defining equation $g^tJg = J$ is equivalent to

$$(I + tX)^t J(I + tX) = J \Leftrightarrow J + tX^tJ + tJX + t^2X^tJX = J + t(X^tJ + JX) = J,$$

(here every elements are in $\mathrm{GL}_n(k[t]/t^2)$) so we should have $X^tJ + JX = 0$. In other words, we have a map

$$\mathrm{Lie}\, G \xrightarrow{\sim} \{X \in \mathfrak{gl}_n(k) : X^tJ + JX\}, \quad I + tX \to X.$$

***Problem* 1.11** ***

***Problem* 1.12** ***

***Problem* 1.13** Using the equivalence of $\mathbf{Spl}_k$ and **RRD**, it is enough to check that the dual of the root datum of $\mathrm{GL}_n$ is isomorphic to itself in **RRD**. Recall that the root datum of $\mathrm{GL}_n$ with torus $T$ of diagonal elements is given as follows: (Example 1.12)

- $X^*(T) = \{\alpha_{k_1,\dots,k_n} : \mathrm{diag}(t_1, \dots, t_n) \mapsto \prod_{1\le j\le n} t_j^{k_j},\, k_1, \dots, k_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $X_*(T) = \{\beta_{k_1,\dots,k_n} : t \mapsto \mathrm{diag}(t^{k_1}, \dots, t^{k_n}),\, t_1, \dots, t_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $\Phi(\mathrm{GL}_n, T) = \{e_{ij},\, 1 \le i \ne j \le n\}$, $e_{ij}(\mathrm{diag}(t_1, \dots, t_n)) = t_i t_j^{-1}$
- $\Phi^\vee(\mathrm{GL}_n, T) = \{e_{ij},\, 1 \le i \ne j \le n\}$, $e_{ij}^\vee(t) = \mathrm{diag}(1, \dots, t, \dots, t^{-1}, \dots, 1)$ ($t$ in the $i$-th entry, $t^{-1}$ in the $j$-th entry, 1 for other entries)

Then we define a map $f : X_*(T) \to X^*(T)$ and $\iota : \Phi(\mathrm{GL}_n, T) \to \Phi^\vee(\mathrm{GL}_n, T)$ as

$$f(\beta_{k_1,\dots,k_n}) = \alpha_{k_1,\dots,k_n}, \quad \iota(e_{ij}) = e_{ij}^\vee.$$

and define $f^\vee : X^*(T) \to X_*(T)$ and $\iota^\vee : \Phi^\vee(\mathrm{GL}_n, T) \to \Phi(\mathrm{GL}_n, T)$ similarly. Then these maps are inverse to each other and gives an isomorphism between two root data

$$(X^*(T), X_*(T), \Phi(\mathrm{GL}_n, T), \Phi^\vee(\mathrm{GL}_n, T)) \simeq (X_*(T), X^*(T), \Phi^\vee(\mathrm{GL}_n, T), \Phi(\mathrm{GL}_n, T))$$

(they are central isogenies) so we get $\widehat{\mathrm{GL}_n} = \mathrm{GL}_{n\mathbb{C}}$.

***Problem* 1.14** ***

***Problem* 1.15** ***

## 2. Chapter 2

***Problem* 2.1** ***

***Problem* 2.2** ***

***Problem* 2.3** ***

***Problem* 2.4** ***

***Problem* 2.5** ***

***Problem* 2.6** ***

***Problem* 2.7** ***

***Problem* 2.8** ***

***Problem* 2.9** ***

***Problem* 2.10** ***

***Problem* 2.11** ***

***Problem* 2.12** ***

***Problem* 2.13** ***

***Problem* 2.14** ***

***Problem* 2.15** ***

***Problem* 2.16** ***

***Problem* 2.17** ***

***Problem* 2.18** ***

***Problem* 2.19** ***