Work of Waldspurger

Re-typed by Seewoo Lee*

Ilya Piatetski-Shapiro
Yale University
and
University of Tel Aviv

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※ Introduction

The work of Waldspurger [10, 11, 12, 13, 14] is devoted to a very deep study of the automorphic forms on $\widetilde{\mathrm{SL}}_2$. The main tool for such a study is the correspondence between automorphic forms on SL_2 and automorphic forms on PGL_2. This correspondence was first discovered by Shintani and Niwa using the Weil representation. An earlier approach to this correspondence, based on L-functions, was suggested by Shimura [8]. Indeed, Shimura's work seemed to stimulate Shintani's and Niwa's work on the subject.

*seewoo5@berkeley.edu

R. Howe has outlined a general theory of duality correspondence based on the use of the Weil representation. He has introduced the general notion of a dual reductive pair, and has defined both a local and global duality correspondence. R. Howe has obtained many deep results in the general situation: but many important problems remain [4].

A systematic study of the duality correspondence for the simplest dual reductive pair $(\widetilde{\mathrm{SL}}_2, \mathrm{PGL}_2)$ from the point of view of representation theory has been carried out by Rallis and Schiffmann [7]. In his work, Waldspurger refers in many places to Rallis and Schiffmann, and, in a way, Waldspurger's work is a continuation of that of Rallis and Schiffmann. However, I would like to emphasize that Waldspurger's work contains many fundamental new ideas especially in the global case.

Flicker has studied a correspondence between the automorphic forms of GL_2 and those of GL_2 using the trace formula [1]. He has in fact obtained a complete description of this correspondence. Since $\widetilde{\mathrm{SL}}_2$ is a subgroup of $\widetilde{\mathrm{GL}}_2$ there is a close connection between the automorphic forms of these two groups. Waldspurger has used Flicker's results in a substantial way to obtain his own results. However, let me say that Waldspurger's results for $\widetilde{\mathrm{SL}}_2$ are quite surprising and were not predicted from the results for $\widetilde{\mathrm{GL}}_2$. It remains a mystery to me why the automorphic forms on $\widetilde{\mathrm{SL}}_2$ and $\widetilde{\mathrm{GL}}_2$ behave so differently. For example, strong multiplicity one is true for $\widetilde{\mathrm{GL}}_2$ but not for $\widetilde{\mathrm{SL}}_2$, Also, the descent (correspondence) of automorphic forms from GL_2 to $\widetilde{\mathrm{GL}}_2$ has only a local obstruction, while the correspondence from PGL_2 to $\widetilde{\mathrm{SL}}_2$ has a global obstruction, but no local obstruction.

Let me also mention work [2, 3] which deals with L-functions for GL_2. This work can be considered as an adélization of Shimura's work. It establishes an injection of the automorphic representations of $\widetilde{\mathrm{GL}}_2$ into those of GL_2.

In this talk, I would like to explain Waldspurger's work in the framework of representation theory. I will explain all of Waldspurger's work except [12], which deals with the Fourier coefficients of automorphic forms of half-integral weight. This latter work, which is based on the material explained here, is very important for number theory, but lies outside the framework of this talk. Despite the fact that I have omitted many local proofs, I hope this talk will be useful to the mathematical community. A beautiful exposition of Waldspurger's work from the classical point of view has been given in a talk by Marie-France Vigneras [9].

❊ Automorphic Forms on $\widetilde{\mathrm{SL}}_2$

Let k be a global field. The adéle group $\mathrm{SL}_2(\mathbb{A})$ has a unique non-trivial two-fold covering $\widetilde{\mathrm{SL}}_2(\mathbb{A})$:

$$1 \to \{\pm 1\} \to \widetilde{\mathrm{SL}}_2(\mathbb{A}) \to \mathrm{SL}_2(\mathbb{A}) \to 1.$$

There is a unique embedding of $\mathrm{SL}_2(k)$ into $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ such that the following diagram commutes.

$$\begin{tikzcd} & \widetilde{\mathrm{SL}}_2(\mathbb{A}) \arrow[d] \\ \mathrm{SL}_2(k) \arrow[ur] \arrow[r] & \mathrm{SL}_2(\mathbb{A}) \end{tikzcd}$$

This means covering splits over $\mathrm{SL}_2(k)$. Similarly, there is an embedding of $N(\mathbb{A})$ into $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, where N is the upper unipotent subgroup of SL_2.

Let A_0 denote the space of genuine cuspidal functions on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. In particular, if $f \in A_0$, then

1. $f(\xi\gamma g) = \xi f(g) \qquad (\xi \in \{\pm 1\}, \gamma \in \mathrm{SL}_2(k), g \in \widetilde{\mathrm{SL}}_2(\mathbb{A}))$

Under right translation, A_0 decomposes discretely into a countable number of irreducible subspaces. An irreducible representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ which occurs in A_0 is called a genuine automorphic cuspidal representation. Let A_{00} denote the subspace of forms in A_0 orthogonal to the Weil representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.

Theorem 1.1 (Multiplicity One [11])**.** The multiplicity of an irreducible genuine automorphic cuspidal representation in A_{00} is one.

Remark. If a is a genuine irreducible automorphic cuspidal representation lying in a Weil representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, then multiplicity one is obvious.

If ψ is a character of $k\backslash\mathbb{A}$, and $f \in A_{00}$, the ψ-Fourier coefficient of f is defined to be

$$f_\psi(g) = \int_{k\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} g\right)\psi(n)\mathrm{d}n \qquad (g \in \widetilde{\mathrm{SL}}_2(\mathbb{A}))$$

The multiplicity result follows from the uniqueness of Whittaker models for $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, and the following result of Waldspurger.

Theorem 1.2 ([12, 13]). Let (σ, V) be a genuine irreducible automorphic cuspidal representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. If $v \mapsto \varphi(v)$ $(v \in V, \varphi(v) \in A_{00})$ is an embedding of (σ, V) into A_{00}, then the vanishing of the ψ-Fourier coefficient $\varphi(v)_\psi$ depends only on (σ, V) as an abstract representation, and not on the embedding φ.

Proof of the multiplicity one. Suppose $v \mapsto \varphi'(v)$ and $v \mapsto \varphi''(v)$ $(v \in v)$ are two distinct embeddings of an irreducible genuine automorphic cuspidal representation (σ, V) into A_0. We may select a character ψ of $k\backslash\mathbb{A}$ so that the ψ-Fourier coefficient $\varphi'(v)_\psi$ does not vanish for some $v \in V$. Let us consider the ψ-Fourier coefficient $\varphi''(v)_\psi$. If $\varphi''(v)_\psi$ vanishes, then Theorem 1.2 says $\varphi'(v)_\psi$ must also vanish, a contradiction. If $\varphi''(v)$ does not vanish, then the uniqueness of Whittaker models for $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ tells us that $\varphi''(v) = c\varphi'(v)$ for some constant c. Since φ' and φ'' are assumed to be distinct embeddings of (σ, V) into A_{00}, the map $w \mapsto \varphi''(w) - c\varphi'(w)$ is a non-trivial embedding of (σ, V) into A_{00}. The ψ-Fourier coefficient of $\varphi''(v) - c\varphi'(v)$ vanishes. This again contradicts Theorem 1.2; therefore (σ, V) must occur in A_{00} with multiplicity one. □

Two irreducible genuine automorphic cuspidal representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, $\sigma = \otimes_v \sigma_v$ and $\sigma' = \otimes_v \sigma'_v$, are said to be nearly equivalent if $\sigma_v \simeq \sigma'_v$ for almost all places v. Let $\ell(\sigma)$ denote the set of irreducible genuine automorphic cuspidal representations nearly equivalent to σ. $\ell(\sigma)$, of course, just measures departure from strong multiplicity one. In order to determine the set $\ell(\sigma)$, Waldspurger has defined an involution $\sigma \mapsto \sigma^W$ whenever σ is a discrete series representation of $\widetilde{\mathrm{SL}}_2(k_v)$. If $\sigma = \otimes_v \sigma_v \subset A_{00}$, define

$$\Sigma = \{v : \sigma_v \text{ is a discrete series representation}\}.$$

If $M \subseteq \Sigma$, and $|M|$ is even, put

$$\sigma^M = \otimes_v \sigma_v^M \text{ where } \sigma_v^M = \begin{cases} \sigma_v & \text{if } v \in M \\ \sigma_V^W & \text{if } v \notin M. \end{cases}$$

The relationship of the σ^M's and $\ell(\sigma)$ is given in the following theorem.

Theorem 1.3 ([14]). Any representation in $\ell(\sigma)$ is of the form σ^M for some $M \subseteq \Sigma$.

Corollary 1.4. $|\ell(\sigma)| = 2^{|\Sigma|-1}$.

Remark. Recall that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ lies in the center of $\widetilde{\mathrm{SL}}_2(k_v)$. Waldspurger has shown that $\sigma_v^M \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\sigma_v \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(k)$, it follows that if $M \subseteq \Sigma$ has an odd number of elements, σ^M cannot be an automorphic representation.

※ The Oscillator Representation Over a Local Field

Let k be a local field, and let X be a $2n$-dimensional vector space over k with a symplectic form $\langle\,,\,\rangle$. If $X = X_1 \oplus X_2$ is a polarization of X, let P be the subgroup of $\mathrm{Sp}(X)$ which preserves X_2. If ψ is a non-trivial character of k, let ω_ψ be the oscillator representation of $\mathrm{Mp}_{2n}(k) = \mathrm{Mp}(X)$, the double cover of $\mathrm{Sp}_{2n}(k) = \mathrm{Sp}(X)$, acts on the Schwartz-Bruhat space $\mathcal{S}(X_1)$.

Let us now consider the 3-dimensional vector space $M = \{m \in \mathrm{M}_2(k) : \mathrm{Tr}(m) = 0\}$. PGL_2 acts on M by conjugation:

$$m \mapsto g^{-1}mg \qquad (g \in \mathrm{PGL}_2, m \in M).$$

This conjugation action preserves the symmetric form $q(x) = -\det(x)$. Let Y be a 2-dimensional vector space over k with a symplectic form $\langle\,,\,\rangle$. Define a symplectic vector space X by $X = M \otimes_k Y$, $\langle m_1 \otimes y_1, m_2 \otimes y_2\rangle = (m_1, m_2)\langle y_1, y_2\rangle$. Since PGL_2 and SL_2 preserve the forms $(\,,\,)$ and $\langle\,,\,\rangle$ respectively, there is a natural embedding of $\mathrm{PGL}_2 \times \mathrm{SL}_2$ into $\mathrm{Sp}(X) = \mathrm{Sp}_6$. Our aim is to use the oscillator representation of Mp_6 to define a correspondence between certain irreducible representations of PGL_2 and certain irreducible representations of $\widetilde{\mathrm{SL}}_2$. Waldspurger has given a different definition of the correspondence based on explicit integral formulas. These integral formulas, though complicated and defined only for the case PGL_2, $\widetilde{\mathrm{SL}}_2$, yield much more information about the correspondence.

Let T be a subgroup of $G = \mathrm{PGL}_2$ and let N a subgroup of $H = \mathrm{SL}_2$. Let α and β be characters of T and N respectively. Let $X = X_1 \oplus X_2$ be a polarization of X such that $T \times N \subset P$. Let us suppose that $x_1 \in X_1$ is a vector such that the linear functional

$$\phi \mapsto \phi(x_1) \qquad (\phi \in \mathcal{S}(X_1))$$

transforms under $T \times N$ by $\alpha \times \beta$, i.e.,

$$\omega_\psi(t, n)\phi(x_1) = \alpha(t)\beta(n)\phi(x_1).$$

Let (π, V) be an irreducible admissible representation of PGL_2 and let us assume that ℓ is a linear functional on V such that $\ell(\pi(t)v) = \alpha(t)^{-1}\ell(v)$ $(t \in T)$. If the integral

$$F(h) = \int_{T\backslash G} \omega_\psi(g, h)\phi(x_1)\ell(\pi(g)v)\mathrm{d}g \qquad (h \in H)$$

converges, then $F(nh) = \beta(n)F(h)$ $(n \in N)$. Let W be the space of all the functions F obtained in this fashion by varying ϕ and v. $\widetilde{\mathrm{SL}}_2$ acts on W by right translation. We shall denote this representation by $\theta(\pi,\psi)$. Conversely, given an irreducible admissible genuine representation σ of $\widetilde{\mathrm{SL}}_2$ it is possible to define a representation $\theta(\sigma,\psi)$ of PGL_2, which may be a zero representation.

In order to explain Waldspurger's integral formulas for the correspondence, we have to consider two polarizations of X. For the first polarization, let $y_1, y_2 \in Y$ be a symplectic basis, i. e., $\langle y_1, y_2\rangle = 1$, and put $X_1 = M \otimes y_1, X_2 = M \otimes y_2$. Let m_1 be an element of M such that $\det m_1 \neq 0$ and let $T = \mathrm{Stab}(m_1)$. T is a torus in G. Let N be the unipotent subgroup of SL_2 which preserves Y_2. Let α be the trivial character, and β the character $\beta\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) = \psi(q(m_1)n)$. We now describe the second polarization which has the property that the unipotent subgroups of PGL_2 and $\widetilde{\mathrm{SL}}_2$ both lie in P. Let e_1, e_2, e_3 be a basis of M such that the matrix of the symmetric form is

$$\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}.$$

Define $X_1 = e_1 \otimes Y + e_2 \otimes ky_1$ and $X_2 = e_3 \otimes Y + e_2 \otimes ky_2$. It is clear that the unipotent subgroup of $G = \mathrm{PGL}_2$ which preserves e_3 also preserves X_2. We shall denote this subgroup by T. Similarly, the unipotent subgroup N of SL_2 which preserves y_2 preserves X_2. Let $x_1 = e_1 \otimes y_2 + \lambda e_2 \otimes y_1$ and define α and β by

$$\alpha\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} = \psi(-\lambda t)$$
$$\beta\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} = \psi(\lambda^2 n).$$

Waldspurger has proved the following theorems:

Theorem 2.1 ([11]). Let T and N be as above. If (π, V) (respectively (σ, V)) is an irreducible admissible representation of PGL_2 (respectively $\widetilde{\mathrm{SL}}_2$), then the representation of $\widetilde{\mathrm{SL}}_2$ (respectively PGL_2) obtained from the above integral formulas is irreducible admissible and depends only on the additive character ψ. It is independent of the choice of the subgroups T and N and the characters α and β.

Theorem 2.2 ([14]). Let $\xi \in k^\times$, and let χ_ξ be the quadratic character of $k^\times$ associated to $k(\sqrt{\xi})$. If and $\theta(\sigma,\psi)$ and $\theta(\sigma,\psi^\xi)$ are both nonzero representations of PGL_2, then $\theta(\sigma,\psi^\xi) = \theta(\sigma,\psi)\otimes\chi_\xi$.

Remark. $\theta(\pi,\psi)$ is non-zero for any irreducible admissible representation π of PGL_2 and any ψ. It follows from this that any irreducible admissible representation of PGL_2 admits a linear functional which is invariant with respect to the split torus. $\theta(\sigma, \psi)$ is nonzero if and only if σ admits a linear functional which transforms under N by ψ^{-1}.

Let us now make a few remarks about a similar construction for the quaternion algebra D over k. Let M' be the elements of trace zero in D, and let q be the symmetric form on M' given by $q(m) = -\mathrm{N}_D(x)$. $\mathrm{P}D^\times$ acts on M' by conjugation, and this action preserves the form q. We can introduce a symplectic space $X' = M' \otimes_k Y$ and as above, we have an embedding $\mathrm{P}D^\times \times \mathrm{SL}_2 \hookrightarrow \mathrm{Sp}_6$. In an analogous fashion, we can also introduce integral formulas to describe a correspondence between some of the irreducible admissible representations of $\mathrm{P}D^\times$ and some of the irreducible genuine admissible representations of $\widetilde{\mathrm{SL}}_2$ The analogues of Theorems 2.1 and 2.2 are also true for the quaternion algebra. If σ (respectively π) is an irreducible admissible representation of $\widetilde{\mathrm{SL}}_2$ (respectively $\mathrm{P}D^\times$), we shall denote the corresponding representation of $\mathrm{P}D^\times$ (respectively $\widetilde{\mathrm{SL}}_2$) by $\theta'(\sigma, \psi)$ (respectively $\theta'(\pi, \psi)$).

From the explicit integral formulas, it is easy to show that $\theta'(\pi, \psi)$ does not admit a linear functional which transforms under N by ψ^{-1}. This together with the remark after Theorem 2.2 implies that the representations $\theta(\sigma, \psi)$ and $\theta'(\sigma, \psi)$ cannot both be non-zero representations. However, Waldspurger has the following result.

Theorem 2.3 ([14])**.** One of the representations $\theta(\sigma, \psi)$ and $\theta'(\sigma, \psi)$ is always non-zero.

Claim. $\theta'(\pi', \psi')$ is non-zero if and only if π' is a spherical representation, i.e., π' possesses a T-invariant vector for some $T \subset \mathrm{P}D^\times$.

Proof. (Waldspurger) Consider for the moment, an irreducible admissible representation π of PGL_2. If χ is a quadratic character of $k^\times$, we define the Waldspurger symbol as follows. Let $\varepsilon(\pi, s, \psi)$ be the ε-factor introduced in [6]. It is easy to check that $\varepsilon(\pi, \frac{1}{2}, \psi) = \pm 1$ does not depend on ψ. Let $\varepsilon(\pi, \frac{1}{2})$ denote $\varepsilon(\pi, \frac{1}{2}, \psi)$. We then define $\left(\frac{\chi}{\pi}\right)$ by

$$\varepsilon\left(\pi \otimes \chi, \frac{1}{2}\right) = \left(\frac{\chi}{\pi}\right) \chi(-1) \varepsilon\left(\pi, \frac{1}{2}\right).$$

$\left(\frac{\chi}{\pi}\right) = 1$, and if χ is the trivial character, then $\left(\frac{\chi}{\pi}\right) = 1$. It is easy to see that if π is an irreducible principal series representation, then $\left(\frac{\chi}{\pi}\right) = 1$ for all χ. On the other hand, if π is a discrete series representation, then there exists a χ such that $\left(\frac{\chi}{\pi}\right) = -1$ [14]. Let us now return to the proof of the claim. Let π be the discrete series representation of PGL_2 associated to π' under the Jacquet-Langlands map. Let χ be a quadratic character of $k^\times$ such that $\left(\frac{\chi}{\pi}\right) = -1$, and denote by $K = k(\sqrt{\xi})$ the field corresponding to χ. Put

$$\sigma_1 = \theta(\pi \otimes \chi, \psi^\xi), \qquad \sigma = \theta(\pi, \psi).$$

Waldspurger has proved that

$$\sigma_1 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \left(\frac{\chi}{\pi}\right) \sigma \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $\left(\frac{\chi}{\pi}\right) = -1$, $\sigma_1 \neq \sigma$. This means that σ does not admit a ψ^ξ-linear functional, for if it did, $\theta(\sigma, (\psi^\xi)^{-1}) \neq 0$, so $\theta(\sigma_1, (\psi^\xi)^{-1}) = \theta(\sigma, \psi^{-1}) \otimes \chi_\xi$ which would imply $\sigma_1 = \sigma$, a contradiction. Now, Theorem 2.3 tells us that $\theta'(\sigma, (\psi^\xi)^{-1}) \neq 0$ and so $\theta(\pi', \psi^\xi) \neq 0$ which means π' is spherical. □

The next theorem defines Waldspurger's involution.

Theorem 2.4 ([14]). Let σ be an irreducible representation of $\widetilde{\mathrm{SL}}_2$, and let ψ be a character of k such that $\theta(\sigma, \psi) \neq 0$. The composition of 3 maps

$$\sigma \mapsto \theta(\sigma, \psi) = \pi \overset{\mathrm{JL}}{\longmapsto} \pi' \mapsto \theta(\pi', \psi^{-1})$$

(where JL means the Jacquet-Langlands map) is independent of ψ and defines an involution.

Finally, it is not difficult to prove

Theorem 2.5. If $\pi = \theta(\sigma, \psi) \neq 0$, then

$$\theta(\pi \otimes \chi_\xi, \psi^\xi) = \begin{cases} \sigma & \text{if } \left(\frac{\chi_\xi}{\pi}\right) = 1 \\ \sigma^W & \text{if } \left(\frac{\chi_\xi}{\pi}\right) = -1 \end{cases}$$

where χ_ξ is the character associated to $k(\sqrt{\xi})$.

※ The θ-correspondence

Let k be a global field. We shall use the same notion globally as was previously introduced locally. The global Weil (oscillator) representation ω_ψ acts on

$\mathcal{S}(X_1(\mathbb{A}))$. It is easy to see that it is the tensor product of the local Weil representations. Let $X = X_1 \oplus X_2$ be the standard polarization of X, and identify X_1 wiht M. For $\phi \in \mathcal{S}(X_1(\mathbb{A}))$,

$$\vartheta_\psi^\phi(g,h) = \sum_{x \in X_1(k)} \omega_\psi(g,h)\phi(x) \qquad (g \in G(\mathbb{A}), h \in \widetilde{\mathrm{SL}}_2(\mathbb{A})).$$

Here, G is either PGL_2 or $\mathrm{P}D^\times$. It is well known that ϑ_ψ^ϕ is an automorphic function on $G(\mathbb{A}) \times \widetilde{\mathrm{SL}}_2(\mathbb{A})$ of moderate growth.

The theta function ϑ_ψ^ϕ's can be used to define a correspondence between the automorphic representation of $G(\mathbb{A})$ and those of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. To describe this correspondence, let π be an irreducible automorphic cuspidal representation of $G(\mathbb{A})$. If $f \in \pi \subset A_0$, put

$$\varphi(h) := \int_{G(k)\backslash G(\mathbb{A})} \vartheta_\psi^\phi(g,h) f(g) \mathrm{d}g.$$

In the case $G = \mathrm{P}D^\times$, we assume that $\int_{G(k)\backslash G(\mathbb{A})} f(g)\mathrm{d}g = 0$. The fact that ϑ_ψ^ϕ is a function of moderate grwoth on $(G(k) \times \mathrm{SL}_2(k))\backslash(G(\mathbb{A}) \times \widetilde{SL}_2(\mathbb{A}))$ means that the integral is well-defined, and that φ is a function on $\mathrm{SL}_2(k)\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A})$.

Claim. φ is a cusp form.

Proof. It is enough to show that $\int_{k\backslash\mathbb{A}} \varphi\left(\begin{smallmatrix} 1 & z \\ 0 & 1\end{smallmatrix}\right) \mathrm{d}z = 0$.

$$\begin{aligned}\int_{k\backslash\mathbb{A}} \varphi\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix} \mathrm{d}z &= \int_{k\backslash\mathbb{A}} \int_{G(k)\backslash G(\mathbb{A})} \sum_{x \in X(k)} \omega_\psi\left(g\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}\right)\phi(x) f(g)\mathrm{d}g\mathrm{d}z \\ &= \int_{G(k)\backslash G(\mathbb{A})} \sum_{x \in X(k)} \omega_\psi(g)\phi(x) f(g) \int_{k\backslash\mathbb{A}} \psi(zq(x))\mathrm{d}z\mathrm{d}g.\end{aligned}$$

The inner integral $\int_{k\backslash\mathbb{A}} \psi(zq(x))\mathrm{d}z$ is zero unless $q(x) = 0$. If $G = \mathrm{P}D^\times$, then $q(x) = 0$ if and only if $x = 0$, and the integral becomes

$$\int_{k\backslash\mathbb{A}} \varphi\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix} \mathrm{d}z = \int_{G(k)\backslash G(\mathbb{A})} \phi(0) f(g)\mathrm{d}g = 0.$$

If $G = \mathrm{PGL}_2$, then $q(x) = 0$ means either $x = 0$ or x is a non-zero nilpotent element of $M_2(k)$. The integral in this situation is

$$\int_{k\backslash\mathbb{A}} \varphi\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix} \mathrm{d}z = \int_{G(k)\backslash G(\mathbb{A})} \phi(0) f(g)\mathrm{d}g$$

$$
\begin{aligned}
&+ \int_{G(k)\backslash G(\mathbb{A})} \sum_{N(k)\backslash G(k)} \phi\left(g^{-1}\gamma_{-1}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\gamma g\right) f(g)\mathrm{d}g \\
&= 0 + \int_{N(\mathbb{A})\backslash G(\mathbb{A})} \phi(g^{-1}xg)f(\gamma^{-1}g)\mathrm{d}g \\
&= \int_{N(\mathbb{A})\backslash G(\mathbb{A})} \omega_\psi(g)\phi(x)\int_{N(k)\backslash N(\mathbb{A})} f(ng)\mathrm{d}n\mathrm{d}g = 0.
\end{aligned}
$$

Here N is centralizer in G of $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, and $\int_{N(k)\backslash N(\mathbb{A})} f(ng)\mathrm{d}n = 0$, $\int_{N(\mathbb{A})\backslash G(\mathbb{A})} f(g)\mathrm{d}g = 0$ since f is a cusp form. $\square$

Let $\theta(\pi,\psi)$ denote the representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ spanned by the φ's ($\phi \in \mathcal{S}(X_1(\mathbb{A})), f \in \pi$). $\theta(\pi,\psi)$ is a genuine automorphic cuspidal representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.

Theorem 3.1 ([11])**.** The θ-correspondence $\pi \mapsto \theta(\pi,\psi)$ is compatible with the local correspondences introduced in §2.

Proof. Let π be an irreducible automorphic cuspidal representation of $G(\mathbb{A})$. For $f \in \pi$, and $\phi \in \mathcal{S}(X_1(\mathbb{A}))$, let φ again be the cusp form

$$\varphi(h) = \int_{G(k)\backslash G(\mathbb{A})} \vartheta_\psi^\phi(g,h)f(g)\mathrm{d}g.$$

If $a \in k^\times$, then a calculation similar to the one used to show φ is a cusp form shows

$$
\begin{aligned}
\varphi_a(1) &:= \int_{k\backslash\mathbb{A}} \varphi\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\overline{\psi(az)}\mathrm{d}z \\
&= \int_{T^a(\mathbb{A})\backslash G(\mathbb{A})} \omega_\psi(g)\phi(x_a)\int_{T^a(k)\backslash T^a(\mathbb{A})} f(tg)\mathrm{d}t\mathrm{d}g. \qquad (1)
\end{aligned}
$$

Here, x_a is any element in X, such that $q(x_a) = a$ (if $G = PD^\times$, we assume a is representable by q), and T^a is the stabilizer of x_a. T^a is a torus in G. Put

$$U(f,g) := \int_{T^a(k)\backslash T^a(\mathbb{A})} f(tg)\mathrm{d}t \qquad (g \in G(\mathbb{A}), f \in \pi).$$

The function $U(f,-)$ satisfies the property $U(f,tg = U(f,g)$ for $t \in T^a(\mathbb{A})$, and the linear function $\ell : f \mapsto U(f,1)$ is a linear functional on ($f \in \pi$) for which $\ell(\pi(t)f) = \ell(f)$ ($t \in T^a(\mathbb{A})$). Locally, such a linear functional is unique, hence ℓ is globally unique and

$$U(f,-) = \otimes_v U_v(-),$$

where U_v is a function on G_v such that $U_v(t_v g_v) = U_v(g_v)$ $(t_v \in T^a(k_v), g_v \in G(k_v))$. Under right translations by G_v on $T_v^a \backslash G_v$, U_v generates a representation equivalent to π_v. In analogy with the global formula

$$\varphi_a(h) = \int_{T^a(\mathbb{A})\backslash G(\mathbb{A})} \omega_\psi(g,h)\phi(x_a)U(f,g)\mathrm{d}g,$$

if U is an element in the space generated by U_v, and if

$$W_{\psi^a}(h) := \int_{T^a\backslash G_v} \omega_{\phi,v}(g,h)\phi(x_a)U(g)\mathrm{d}g$$

then $W_{\psi^a}\left(\begin{smallmatrix}1 & z\\ 0 & 1\end{smallmatrix} h\right) = \psi_v(za)W_{\psi_a}(h)$. □

Theorem 3.2 ([11]). The θ-correspondence is a 1-1 correspondence between certain automorphic cuspidal irreducible representations of $G(\mathbb{A})$ and certain genuine automorphic cuspidal irreducible representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.

Theorem 3.3 ([11, 5]). Let $G = \mathrm{PGL}_2$. Suppose $\sigma \in A_\infty$, and π is an automorphic cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$. Then

1. $\theta(\sigma,\psi^{-1}) \neq 0$ if and only if σ possesses a nonvanishing ψ-Fourier coefficient.

2. $\theta(\pi,\psi) \neq 0$ if and only if $L(1/2,\pi) \neq 0$.

Proof. In order to prove this theroem, we must use a polarization for which the usual subgroups of $\mathrm{PGL}_2(\mathbb{A})$ and $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ lie inside P. As before, let M be the elements of $M_2(k)$ of trace zero, and let $q(m) = -\det(m)$. Let Y be a 2-dimensional symplectic vector spcaes over k with form $\langle\,,\,\rangle$ and symplectic basis y_1, y_2. Let e_1, e_2, e_3 be a basis of M such that q has the matrix $\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}\right)$. Put $X_1 = e_1 \otimes Y + e_2 \otimes ky_1, X_2 = e_3 \otimes Y + e_2 \otimes ky_2$. Suppose σ is an irreducible genuine automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ lying in A_{00}. If $\varphi \in \sigma$, let $f(g) = \int_{\mathrm{SL}_2(k)\backslash \mathrm{SL}_2(\mathbb{A})} \vartheta_\psi^\phi(g,h)\varphi(h)\mathrm{d}h$. We can identify X_1 with $Y \oplus k$, and we can choose ϕ in the form $\phi = \phi_1\phi_2$, where $\phi_1 \in \mathcal{S}(Y(\mathbb{A})), \phi_2 \in \mathcal{S}(\mathbb{A})$. In this situation,

$$\vartheta_\psi^\phi(1,h) = F_1(h)F_2(h)$$

where

$$F_1(h) = \sum_{Y(k)} \phi_1(yh) = \phi_1(0) + \sum_{\gamma\in B(k)\backslash \mathrm{SL}_2(k)} \phi_1(y_2\gamma)$$
$$F_2(h) = \sum_{t\in k} \omega'_\psi(h)\phi_2(t)$$

In the formula for F_2, ω'_ψ is the 1-dimensional Weil representation.

$$f(1) = \int_{\mathrm{SL}_(k)\backslash \mathrm{SL}_2(\mathbb{A})} \phi_1(0)F_2(h)\varphi(h)\mathrm{d}h + \int_{N(k)\backslash \mathrm{SL}_2(\mathbb{A})} \phi_1(y_2h)F_2(h)\varphi(h)\mathrm{d}h.$$

Since $\sigma \in A_{00}$, and F_2 lies in the space of the Weil representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, the first integral is zero. It follows that $\theta(\sigma, \psi^{-1}) \neq 0$ if and only if the second integral does not vanish identically.

$$\begin{aligned} f(1) &= \int_{N(k)\backslash \mathrm{SL}_2(\mathbb{A})} \phi_1(y_2h)F_2(h)\varphi(h)\mathrm{d}h \\ &= \sum_{t\in k} \int_{N(k)\backslash \mathrm{SL}_2(\mathbb{A})} \phi_1(y_2h)\omega'_\psi(h)\phi_2(t)\varphi(h)\mathrm{d}h. \end{aligned}$$

Since $\phi_1(y_2nh) = \phi_1(y_2h)$ and $\omega'_\psi(nh)\phi_2(t) = \psi(t^2n)\phi_2(t)$ ($n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in N(\mathbb{A})$) it follows that

$$f(1) = \sum_{t\in k} \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \phi_1(y_2h)\omega'_\psi(h)\phi_2(t)\varphi_{\psi^{t^2}}(h)\mathrm{d}h.$$

Thus, if $\theta(\sigma, \psi^{-1}) \neq 0$, then there exists a t for which φ_{t^2} is non-zero. This means σ possesses a non-zero ψ-Fourier coefficient. Conversely, now suppose σ possesses a non-vanishing ψ-Fourier coefficient. Let

$$f_t(1) = \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \phi_1(y_2h)\omega'_\psi(h)\phi_2(t)\varphi_{\psi^{t^2}}(h)\mathrm{d}h.$$

The latter formula allows us to define $f_t(1)$ for arbitrary ϕ. In this situation, we still have $f(1) = \sum_{t\in k^\times} f_t(1)$. ($f_0(1) = 0$, since f is a cusp form). Let N be the unipotent subgroup of PGL_2. For $n \in N$

$$\omega_\psi(n)\phi(y_2, t) = \psi(tn)\phi(y_2, t).$$

It follows form this formula that $f_t(1)$ is a Fourier coefficient of f. Therefore, if $\varphi_\psi \neq 0$, then $f_t(1) \neq 0$ and so $\theta(\sigma, \psi^{-1}) \neq 0$. To prove the second part of the theorem, we use the standard polarization. If $\sigma = \theta(\pi, \psi) \neq 0$, then $\theta(\sigma, \psi^{-1})$ equals π. This means by part 1 that σ possesses a non-zero ψ-Fourier coefficient. If T is the split torus in PGL_2, then formula (1) in the proof of Theorem 3.1 tells us that

$$\int_{T(k)\backslash T(\mathbb{A})} f(t)\mathrm{d}t \neq 0.$$

From the Jacquet-Langlands theory of L-functions, it is known that for an appropriate choice of f,

$$L(s, \pi) = \int_{T(k)\backslash T(\mathbb{A})} f(t)|t|^{s-\frac{1}{2}}\mathrm{d}t.$$

In particular,

$$L\left(\frac{1}{2},\pi\right)=\int_{T(k)\backslash T(\mathbb{A})} f(t)\mathrm{d}t\neq 0.$$

Conversely, if $L(\frac{1}{2},\pi)\neq 0$, then it is clear that $\int_{T(k)\backslash T(\mathbb{A})} f(t)\mathrm{d}t\neq 0$, and hence that $\theta(\pi,\psi)\neq 0$. □

※ Non-vanishing of a Fourier Coefficient

In this section we will prove Theorem 1.2 that the non-vanishing of the ψ-Fourier coefficient of σ depends on σ only as an abstract representation. Let $\sigma\in A_{00}$ and let ψ be a non-trivial character of $k\backslash\mathbb{A}$. There exists a $\xi\in k^\times$ such that $\theta(\sigma,\psi^\xi)\neq 0$. Define $W(\sigma,\psi)$ to be $\theta(\sigma,\psi^\xi)\otimes\chi_\xi$. By Theorem 2.3, $W(\sigma,\psi)$ depends only on ψ. Define $L_\psi(s,\sigma)$ to be $L(s,W(\sigma,\psi))$.

Theorem 4.1. Let $\sigma=\otimes_v\sigma_v\subseteq A_{00}$. σ admits a non-zero ψ-Fourier coefficient if and only if

(i) at each place v, there is a linear functional ℓ_v on the space W of σ_v such that

$$\ell_v\left(\sigma\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}w\right)=\psi_v(t)\sigma_v(w)\qquad (w\in W),$$

(ii) $L(\frac{1}{2},\sigma)\neq 0$.

Proof. If σ admits a non-zero ψ-Fourier coefficient, it is clear that (i) is satisfied. (ii) follows from Theorem 3.3. In order to prove the converse state, Waldspurger developed a remarkable method, based on the generalization of the Siegel-Weil formula. We shall now describe Waldspurger's generalization of the Siegel-Weil formula. The Siegel-Weil formula for the simplest dual reductive pair Sp_{2n} and O_m expresses the integral $\int_{\mathrm{O}_m(k)\backslash\mathrm{O}_m(\mathbb{A})}\vartheta_\psi^\phi(g,h)\mathrm{d}g$ in terms of an Eisenstein series on Sp_{2n}, when m is sufficiently large compared to m. Waldspurger' s generalization of the Siegel-Weil formula considers the case when m is small. Let T be an anisotropic form of SO_2; thus, T is isomorphic to the norm one elements of some quadratic extension K of k. Let χ be the idéle class character associated to K. Let X be the 2-dimensional space on which T acts, and let Y be 2-dimensional with a symplectic form $\langle\ ,\ \rangle$. Put $Z=X\otimes_k Y$, $\langle x_1\otimes y_1,x_2\otimes y_2\rangle=(x_1,x_2)\langle y_1,y_2\rangle$. As usual, $\mathrm{SO}_2\times\mathrm{SL}_2\hookrightarrow\mathrm{Sp}_4$. For $h_v\in\mathrm{SL}_2(k_v)$, we have an Iwasawa decomposition $h_v=\left(\begin{smallmatrix}\alpha & *\\ 0 & \alpha^{-1}\end{smallmatrix}\right)u$ with $u\in\mathrm{SL}_2(\mathcal{O}_v)$ if k_v is non-archimedean, and $u\in\mathrm{SO}_2(\mathbb{R})$ or

$u \in \mathrm{SU}_2(\mathbb{C})$ in the archimedean case. Define $A_v(h_v)$ to be $|\alpha|$, and if $h \in \mathrm{SL}_2(\mathbb{A})$ put

$$A(h) = \prod_v A_v(h_v).$$

We define Eisenstein series by

$$E^\phi(h,s) = L\left(s+\frac{1}{2},\chi\right) \sum_{\gamma \in B(k)\backslash \mathrm{SL}_2(k)} A(\gamma h)^{s-\frac{1}{2}} \omega_\psi(\gamma h)\phi(0)$$

where ϕ is a Schwartz-Bruhat function on $X(\mathbb{A})$. Using the standard theory of Eisenstein series, it is easy to show that this Eisenstein series converges absolutely in some half-plane and admits a meromorphic continuation to the entire plane. We have the following Sieqel-Weil-Waldspurger identity

$$E^\phi\left(h,\frac{1}{2}\right) = c\int_{T(k)\backslash T(\mathbb{A})} \vartheta_\psi^\phi(g,h)\mathrm{d}g$$

where c is a constant depending only on K/k. This identity can be proved by Poisson summation. We now return to the proof of Theorem 4.1. According to Theorem 3.3, it is sufficient to prove for the dual reductive pair $\mathrm{PGL}_2, \widetilde{\mathrm{SL}}_2$ such that

$$\zeta = \int_{\mathrm{SL}_2(k)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(h)\vartheta_\psi^\phi(g,h)\mathrm{d}h$$

is non-zero for some choice of ϕ, φ, and g. Suppose $\zeta \equiv 0$. Since $\varphi \neq 0$, there is $a \in k^\times$ such that $\varphi_{\psi^a} \neq 0$. If $a \in (k^\times)^2$, then since φ_ψ and $\varphi_{\psi^{\lambda^2}}$ is related in an elementary fashion, our statement is true. Thus, we may assume that $k \notin (k^\times)^2$. Let x_a be an element of X so that $q(x_a) = a$, and decompose X into the line $(x_a) = kx_a$ generated by x_a and the orthogonal complement X'_a. We may take a ϕ of the form $\phi(\lambda x_a + x') = \phi_1(\lambda x_a)\phi_2(x')$ $(x' \in X'_a)$. For $g \in T = \mathrm{Stab} x_a$, we have

$$0 \equiv \zeta = \int_{\mathrm{SL}_2(k)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(h)\vartheta_\psi^{\phi_1}(h)\vartheta_\psi^{\phi_2}(g,h)\mathrm{d}h.$$

Let $K = k(\sqrt{a})$, and let T be the anisotropic torus of the norm-one elements in $K^\times$. We can integrate with respect to $g \in T(k)\backslash T(\mathbb{A})$. Since $T(k)\backslash T(\mathbb{A})$ is compact, we can change the order of integration to obtain

$$0 \equiv \int_{\mathrm{SL}_2(k)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(h)\vartheta_\psi^{\phi_1}(h)E^{\phi_2}\left(h,\frac{1}{2}\right)\mathrm{d}h.$$

Let

$$\zeta(s) = \int_{\mathrm{SL}_2(k)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(h)\vartheta_\psi^{\phi_1}(h)E^{\phi_2}(h,s)\mathrm{d}h.$$

For $\Re(s)$ sufficiently large, the Eisenstein series converges absolutely, and hence we can write

$$\begin{aligned}
\zeta(s) &= \int_{B(k)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(h)\vartheta_\psi^{\phi_1}(j) L\left(s+\frac{1}{2},\chi\right) A(h)^{s-\frac{1}{2}}\omega_\phi(h)\phi_2(0)\mathrm{d}h \\
&= \int_{N(k)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(h)\omega'_\psi(h)\phi_1(x_a) L\left(s+\frac{1}{2},\chi\right) A(h)^{s-\frac{1}{2}}\omega_\psi(h)\phi_2(0)\mathrm{d}h \\
&= L\left(s+\frac{1}{2},\chi\right)\int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \varphi_{\psi^0}(h)\omega'_\psi(h)\phi_1(x_a)\omega_\psi(h)\phi_2(0)A(h)^{s-\frac{1}{2}}\mathrm{d}h \\
&= L\left(s+\frac{1}{2},\chi\right)\int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \varphi_{\psi^0}(h)\omega_\psi(h)\phi(x_a)A(h)^{s-\frac{1}{2}}\mathrm{d}h.
\end{aligned}$$

Each function in the integral factorizes as a product of local factors so

$$\zeta(s) = L\left(s+\frac{1}{2},\chi\right)\prod_v \int_{N(k_v)\backslash \mathrm{SL}_2(k_v)} \ell_v(\sigma(h_v)w)\omega_\psi(h_v)\phi_v(x_a)A(h_v)^{s-\frac{1}{2}}\mathrm{d}h_v$$

By §2, we know that the local integral $\int_{N(k_v)\backslash \mathrm{SL}_2(k_v)} \ell_v(\sigma(h_v)w)\omega_\psi(h_v)\phi(x_a)A(h_v)^{s-\frac{1}{2}}\mathrm{d}h_v$ does not vanish identically if and only if $\theta(\sigma_v,\psi_v^{-1}) \neq 0$, which in turn is equivalent to the existence of linear functional ℓ_v, which transforms under $N(k_v)$ by ψ_v^a. We have

$$\frac{\zeta(s)}{L_\psi(s,\sigma)} = \prod_v R_v(s),$$

where $R_v(s) \equiv 1$ for almost all v, and $R_v(\frac{1}{2}) \neq 0$ for all v. Since $L_\psi(\frac{1}{2},\sigma) \neq 0$, we obtain $0 \neq \zeta(\frac{1}{2}) = \zeta$, a contradiction. Thus $\zeta \neq 0$. Thus, there exists a $\phi = \prod_v \phi_v$ and a $w = \otimes_v w_v$ such that for all v, we have

$$\int_{N(k_v)\backslash \mathrm{SL}_2(k_v)} \ell_v(\sigma(h_v)w_v)\omega_{\psi_v}(h_v)\phi_v(x_a)\mathrm{d}h_v \neq 0.$$

□

Proof of Theorem 1.3. Let $\sigma_1 = \otimes_v\sigma_{1,v}$, $\sigma_2 = \otimes_v\sigma_{2,v} \subset A_{00}$, and assume that they are nearly equivalent. In this situation, $\pi_1 = W(\sigma,\psi)$ and $\pi_2 = W(\sigma_2,\psi)$ will have the same local components at almost all places. By the strong multiplicity theorems for PGL_2, it follows that $\pi_1 \simeq \pi_2$. This means that $\sigma_{1,v} \simeq \sigma_{2,v}$ at all places v for which $\sigma_{1,v}$ is not a discrete series representation. Furthermore, at the places v for which $\sigma_{1,v}$ is in the discrete series, it follows from Theorem 2.5 and the local Wa1dspurger involution, that either $\sigma_{2,v} = \sigma_{1,v}$ or $\sigma_{2,v} = \sigma_{1,v}^W$. Since

$$\sigma_{1,v}^W\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = -\sigma_{1,v}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},$$

and $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right) \in \mathrm{SL}_2(k_v)$, the number of places for which $\sigma_{2,v} = \sigma_{1,v}^W$ is even. From this, we conclude that the representations in A_{00} nearly equivalent to σ_1 must be of the form of σ_1^M (see §1 for the definition of σ_1^M). To complete the proof of Theorem 1.3, we must show that every σ^M (with $|M|$ even) lies in A_{00}. To do this, Waldspurger used a result of Flicker [1] which we shall now describe. Flicker established a correspondence between the representations of $\widetilde{\mathrm{GL}}_2$ and GL_2 ($\rho \mapsto \pi$). A representation $\pi = \otimes_v \pi_v$ of GL_2 lies in the image of the Flicker correspondence if and only if at each place v for which π_v is a principal series representation $\pi_v = \pi_v(\mu_v^1, \mu_v^2)$ with $\mu_v^1(-1) = \mu_v^2(-1) = 1$. It is known [14, 2] that π is in the Wa1dspurger correspondence if and only if there is an idéle character ω such that $\pi \otimes \omega$ is in the Flicker correspondence. Waldspurger used this fact to prove that σ^M is automorphic. □

※ Appendix: A Conjecture of Howe

R. Howe introduced in his Corvallis talk "θ-series and invariant theory" (1977) the notion of a dual reductive pair and defined a duality correspondence between the irreducible admissible representations of the members of a dual reductive pair. Howe also conjectured the following: Let (G, H) be a dual reductive pair over a global field k. Suppose that $\pi = \otimes_v \pi_v$ is an automorphic representation of $G(\mathbb{A})$, and suppose that locally at each place v, σ_v is the associated representation of $H(k_v)$ under the local duality correspondence. Then, $\sigma = \otimes_v \sigma_v$ is an automorphic representation of $H(\mathbb{A})$.

The pair $G = \mathrm{PGL}_2, H = \widetilde{\mathrm{SL}}_2$ is one of the simplest examples of a dual reductive pair. If D is a quaternion algebra over k, then the pair $(\mathrm{P}D^\times, \widetilde{\mathrm{SL}}_2)$ is also a dual reductive pair. We shall see that if σ is an automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ then the associated representation π of $\mathrm{PGL}_2(\mathbb{A})$ is automorphic. However, we will give an example which the correspondence in the opposite direction does not send an automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ to an automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. Finally, we shall show that Howe's conjecture in weak form is true for $(\mathrm{P}D^\times, \widetilde{\mathrm{SL}}_2)$, i.e. there exists a nearly equivalent automorphic representation.

By the definition of the Waldspurger map, we have that $W(\sigma, \psi) = \otimes_v \theta(\sigma_v, \psi_v)$ where $\sigma_v \mapsto \theta(\sigma_v, \psi_v)$ is the local Waldspurger map. The Waldspurger map is always defined; thus, the Howe conjecture is true in the direction from $\widetilde{\mathrm{SL}}_2$ to PGL_2 or $\mathrm{P}D^\times$.

Let us now consider the other direction. If π is an automorphic cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$, and denote by $H(\pi,\psi)$ the corresponding representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ under the Howe correspondence. The following theorems are a consequence of Waldspurger's work.

Theorem A.1. $\sigma = H(\pi,\psi)$ is an automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ if and only if there is a quadratic character χ_ξ such that 1) $L(\frac{1}{2},\pi\otimes\chi_\xi) \neq 0$, and 2) $\left(\frac{\chi_{\xi,v}}{\pi_v}\right)$ for all local place v. If 2) is not satisfied then there exists σ' which is an automorphic and nearly equivalent to σ.

Proof. Assume that σ is automorphic. The L-function $L_\psi(s,\sigma,\omega)$ (ω any idéle class character) is entire since π is automorphic cuspidal for GL_2. This means, since σ is automorphic, that it must be cuspidal and in fact $\sigma \subset A_{00}$. π is equal to $W(\sigma,\psi^{-1})$. If ψ^ξ is a character for which σ possesses a non-zero ψ^ξ-Fourier coefficient, then $\pi\otimes\chi_\xi = \theta(\sigma,\psi^\xi)$. By Theorem 3.3, we have $L(\frac{1}{2},\pi\otimes\chi_\xi)\neq 0$. Conversely, if $L(\frac{1}{2},\pi\otimes\chi_\xi)\neq 0$, then $\sigma' = \theta(\pi\otimes\chi_\xi,(\psi^\xi)^{-1})$ and so $\sigma'\subset A_{00}$. It is easy to see that σ' is nearly equivalent to σ and $\sigma'\simeq\sigma$ iff $\left(\frac{\chi_{\xi,v}}{\pi_v}\right) = 1$ for all v. □

Theorem A.2. Let $\pi = \otimes_v \pi_v$ be an automorphic cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$. If either of the following conditions satisfied,

(i) there is a v for which π_v lies in a discrete series

(ii) $\varepsilon(\frac{1}{2},\pi) = 1$ (see §2),

then there is χ_ξ such that $L(\frac{1}{2},\chi_\xi)\neq 0$. Also, if $L(\frac{1}{2},\chi_\xi)\neq 0$, then π satisfies one of the above two conditions.

Proof. We shall show that $L(\frac{1}{2},\pi\otimes\chi_\xi) = 0$ for all χ_ξ equivalent to all the π_v's being in the principal series, and $\varepsilon(\frac{1}{2},\pi) = -1$. If $\varepsilon(\frac{1}{2},\pi) = -1$ and all the π_v's are principal series, then $\varphi(\pi\otimes\chi_\xi,\frac{1}{2}) = -1$ for any χ_ξ. This means $L(\frac{1}{2},\pi\otimes\chi_\xi) = 0$. The converse result was proved by Waldspurger using the result of Flicker [1] formulated in §4. □

We shall now construct a counterexample to Howe's conjecture. Let $\pi = \otimes_v\pi_v$ be an automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$ for which π_∞ lies in the holomorphic discrete series, and π_v for v finite is unramified. In classical language, such a representation corresponds to a holomorphic modular form with respect to the full modular group $\mathrm{PSL}_2(\mathbb{Z})$. Let K be any imaginary quadratic extension of $\mathbb{Q}$ and denote by Π, the base change lift of π to $\mathrm{PGL}_2(\mathbb{A}_K)$. Π_v lies in the principal series for all v, and $\varepsilon(\frac{1}{2},\Pi) = -1$. Thus, $\varepsilon(\frac{1}{2},\Pi\otimes\chi_\xi) = -1$ for all χ_ξ. By Theorem A.1, $\sigma = H(\pi,\psi)$ is not an automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_K)$.

Let us now consider the dual-reductive pair $(\mathrm{P}D^\times, \widetilde{\mathrm{SL}}_2)$. Let π' be an infinite dimensional automorphic representation of $\mathrm{P}D^\times$. Denote by $\pi = \otimes_v \pi_v$ the automorphic cuspidal representation PGL_2 associated to π' by the Jacquet-Langlands correspondence. For some place v, π_v will lie in the discrete series. It follows from Theorem A.2 that there is χ_ξ for which $L(\frac{1}{2}, \pi \otimes \chi_\xi) \neq 0$. This means that there exists $\sigma \subset A_{00}$, which is nearly equivalent to $H(\pi', \psi)$.

Additional note on Theorem A.1 Waldspurger's results [14] imply that assumptions (1) and (2) of A.1 are equivalent to saying that $\varepsilon(\frac{1}{2}, \pi) = 1$, where $\varepsilon(\frac{1}{2}, \pi)$ was defined after Theorem 2.3. For all v such that π_v is unramified, we have $\varepsilon(\frac{1}{2}, \pi_v) = 1$. Hence, in order to verify this assumption, we have to check that

$$\prod_{v \in S} \varepsilon\left(\frac{1}{2}, \pi_v\right) = 1$$

where S is a certain finite set.

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