Automorphic forms and L-functions for the unitary group*

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Introduction

Our purpose is to define and analyze L-functions attached to automorphic cusp forms on the unitary group $G = \mathrm{U}_{2,1}$ and a six-dimensional representation

$$\rho : {}^L G \to \mathrm{GL}_6(\mathbb{C})$$

of its L-group.

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The motivation for this work is three fold.

Firstly, we use these L-functions to analyze the lifting of cusp forms from $U_{1,1}$ to $U_{2,1}$; here the model for our work is Waldspurger's L-function theoretic characterization of the image of Shimura's map for modular forms of half-integral weight (cf. [Wald]).

A second motivation comes from the need to relate the poles of the L-functions for G, to integrals of cusp forms over cycles coming from $U_{1,1}$. The prototvpe here is the recent proof of Tate's conjecture for Hilbert modular surfaces due to Harder, Langlands, and Rapaport.

Thirdly, we view this work as a special contribution to the general program of constructing local L and ε factors of Langlands type for representations of arbitrary reductive groups. In [PS1], such a program was sketched generalizing classical methods of Heeke, Rankin–Selberg, and Shimura. Related developments are discussed in [Jacquet], [Novod], [PS2], and [PS3]. For the unitary group $U_{2,1}$ the present paper extends the developments initiated in [PS3].

Contents

Notation

(i) F is a field (sometimes local, somtimes a global field), E is a quadratic extension of F with Galois involution $z \mapsto \bar{z}$.

(ii) V is a 3-dimensional vector space over E, with basis $\{\ell_{-1}, \ell_0, \ell_1\}$. $(-,-)_V$ is a Hermitian form on V, with matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

with respect to $\{\ell_{-1}, \ell_0, \ell_1\}$.

(iii) $G = U_{2,1} = U(V)$ is the unitary group for the form $(-,-)_V$. P=parabolic subgroup stabilizing the isotropic line through $\ell_{-1} = MN$ with

$$M = \left\{ \begin{bmatrix} \delta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \bar{\delta}^{-1} \end{bmatrix} : \delta \in E^\times, \beta \in E^1 = \{z : z\bar{z} = 1\} \right\}$$

and unipotent radical

$$N = \left\{ \begin{bmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{bmatrix} : z, b \in E, z + \bar{z} = -b\bar{b} \right\}.$$

The center of N is

$$Z = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \bar{z} = -z \right\}$$

※ Whittaker Models (Ordinary and Generalized)

Some kind of Whittaker model is needed in order to introduce L-functions on G.

Fix F local (not of characteristic two), and suppose (π, H_π) is an irreducible admissible representation of G. Naively, we should look for functionals on H_π which·transform under N according to a one-dimensional representation. However, since such functionals need not exist in general, and since there are irreducible representations of N which are not 1-dimensional, it is natural to pursue a more general approach.

1.1

Recall N is the maximal unipotent subgroup of G and E is a quadratic extension of F. We fix, once and for all, an element i in E such that $\bar{i} = -i$, so $\Im(z) = (z - \bar{z})/2i$. Regarding E as a 2-dimensional symplectic space over F with skew-form $\langle z_1, z_2 \rangle = \Im(z_1 \bar{z_2})$ we have

$$N = \left\{ \begin{bmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{bmatrix} : z, b \in E, z + \bar{z} = -b\bar{b} \right\} \simeq H(E),$$

the Heisenberg group attached to E over F. In particular, N is non-abelian, with commutator subgroup

$$[N, N] = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = Z,$$

the center of N. The maximal abelian subgroup of N is

$$N' = \left\{ \begin{bmatrix} 1 & b & z \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \in N : b \in F \right\}.$$

1.2

The irreducible representations of the Heisenberg group, and hence those of N, are well known:

(i) σ is 1-dimensional.

In this case, σ must be trivial on

$$Z = [N, N]$$

and define a character of N/Z. So

$$N/Z \simeq \left\{ \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{bmatrix} \right\} \simeq E$$

implies σ corresponds to a character of E, i.e.

$$\sigma = \psi_N \left(\begin{bmatrix} 1 & a & z \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{bmatrix} \right) = \psi(\Im a)$$

with ψ a character of F.

(ii) σ is infinite-dimensional.

In this case (by the Stone-von Neumann uniqueness theorem), σ is completely determined by its "central" character. In particular, if

$$\sigma \left(\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \psi(\Im z) I$$

for some (additive) character ψ of F, then

$$\sigma = \rho_\psi = \mathrm{Ind}_{N'}^{N} \psi_{N'},$$

with $\psi_{N'}$ the character of (the maximal abelian subgroup) N' obtained by trivially extending ψ from Z to N'.

1.3

Definition. By a (generalized) Whittaker functional for (π, H_π) we understand N-map from N_π to some irreducible representation of (σ, L_σ) of N (possibly infinite dimensional).

1.4

Remark. The torus

$$T = \left\{ \begin{bmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar{\delta}^{-1} \end{bmatrix} : \delta \in E^\times \right\}$$

acts by conjugation on N, taking

$$\begin{bmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & \delta b & \delta\bar{\delta} z \\ 0 & 1 & -\overline{\delta b} \\ 0 & 0 & 1 \end{bmatrix}.$$

So if ψ_N denotes the 1-dimensional representation of N corresponding to the fixed character of F as in 1.2 (i), Pontrygin duality for $E \simeq N/Z$ implies that any other 1-dimensional representation is trivial or of the form

$$\psi_N^\delta(n) = \psi_N\left(\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} n \begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix}^{-1} \right)$$

for some $\delta \in E^\times$.

1.5

If σ is a one-dimensional representation of N of the form ψ_N, a given irreducible admissible representation (π, H_π) need not possess a nontrivial ψ_N-Whittaker functional $\mathcal{L}$. However, if it does, then by 1.4 it possesses a σ-Whittaker functional for any one-dimensional representation ψ_N^δ, given by the formula

$$\mathcal{L}^\delta(v) = \mathcal{L}\left(\pi\left(\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} \right) v \right), \quad v \in H_\pi.$$

In this case, we call (π, H_π) non-degenerate. By a well-known theorem of Shalika and Gelfand-Kazhdan (cf. [Sha1]), the space of such σ-Whittaker functionals is one-dimensional. In particular, the corresponding Whittaker models

$$\mathcal{W}(\pi, \psi) = \{W(g) = \mathcal{L}(\pi(g)v) : v \in H_\pi\}$$

are unique.

1.6

In general, (π, H_π) is not non-degenerate, examples being provided by the Weil representations discussed in §**??**. Thus it is necessary to consider σ-Whittaker models for infinite dimensional σ as well. Such σ, however, are completely determined by their central character ψ_Z, so it is convenient to work with a slight thickening of N. More precisely, consider the stabilizer R in P of the central character ψ_Z of Z. Because $\begin{bmatrix} \delta & & \\ & \beta & \\ & & \bar{\delta}^{-1} \end{bmatrix}$ conjugates $\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 & \delta\bar{\delta}z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$R = \left\{ \begin{bmatrix} \delta & * & * \\ 0 & \beta & * \\ 0 & 0 & \delta \end{bmatrix} \in P : \delta, \beta \in E^1 \right\} \simeq (E^1 \times E^1) \ltimes N.$$

In particular, each irreducible infinite dimensional representation ρ_ψ of N extends to a like representation ρ_ψ^α of R with α a character of $E^1 \times E^1$.

Theorem (Existence and Uniqueness of Generalized Whittaker Models: [PS3)**.**] Any (π, H_π) possesses a ρ_ψ^α-Whittaker functional for some choice of ρ_ψ^α; moreover, the space of such functionals is at most one dimensional.

We shall discuss this result in more detail in the global context of §**??**.