

Automorphic forms and L-functions for the unitary group^{*}

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Stephen Gelbart
Department of Mathematics
Cornell University
Ithaca, New York 14853/USA

and

Ilya Piatetski-Shapiro
Department of Mathematics
Yale University, New Haven, CT. 06520/USA
Tel Aviv University, Ramat-Aviv, Israel

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Introduction

Our purpose is to define and analyze L-functions attached to automorphic cusp forms on the unitary group $G = \mathrm{U}_{2,1}$ and a six-dimensional representation

$$\rho : {}^L G \rightarrow \mathrm{GL}_6(\mathbb{C})$$

of its L -group.

^{*}Notes based on the lectures by S. G. at the University of Maryland Special Year on Lie Group Representations, 1982-83.

[†]seewoo5@berkeley.edu

The motivation for this work is three fold.

Firstly, we use these L -functions to analyze the lifting of cusp forms from $U_{1,1}$ to $U_{2,1}$; here the model for our work is Waldspurger's L -function theoretic characterization of the image of Shimura's map for modular forms of half-integral weight (cf. [Wald]).

A second motivation comes from the need to relate the poles of the L -functions for G , to integrals of cusp forms over cycles coming from $U_{1,1}$. The prototype here is the recent proof of Tate's conjecture for Hilbert modular surfaces due to Harder, Langlands, and Rapaport.

Thirdly, we view this work as a special contribution to the general program of constructing local L and ε factors of Langlands type for representations of arbitrary reductive groups. In [PS1], such a program was sketched generalizing classical methods of Hecke, Rankin–Selberg, and Shimura. Related developments are discussed in [Jacquet], [Novod], [PS2], and [PS3]. For the unitary group $U_{2,1}$ the present paper extends the developments initiated in [PS3].

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Notation

- (i) F is a field (sometimes local, sometimes a global field), E is a quadratic extension of F with Galois involution $z \mapsto \bar{z}$.
- (ii) V is a 3-dimensional vector space over E , with basis $\{\ell_{-1}, \ell_0, \ell_1\}$. $(-, -)_V$ is a Hermitian form on V , with matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

with respect to $\{\ell_{-1}, \ell_0, \ell_1\}$.

- (iii) $G = U_{2,1} = U(V)$ is the unitary group for the form $(-, -)_V$. P =parabolic

subgroup stabilizing the isotropic line through $\ell_{-1} = MN$ with

$$M = \left\{ \begin{bmatrix} \delta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \bar{\delta}^{-1} \end{bmatrix} : \delta \in E^\times, \beta \in E^1 = \{z : z\bar{z} = 1\} \right\}$$

and unipotent radical

$$N = \left\{ \begin{bmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{bmatrix} : z, b \in E, z + \bar{z} = -b\bar{b} \right\}.$$

The center of N is

$$Z = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \bar{z} = -z \right\}$$

❖ Whittaker Models (Ordinary and Generalized)

Some kind of Whittaker model is needed in order to introduce L -functions on G .

Fix F local (not of characteristic two), and suppose (π, H_π) is an irreducible admissible representation of G . Naively, we should look for functionals on H_π which transform under N according to a one-dimensional representation. However, since such functionals need not exist in general, and since there are irreducible representations of N which are not 1-dimensional, it is natural to pursue a more general approach.

1.1

Recall N is the maximal unipotent subgroup of G and E is a quadratic extension of F . We fix, once and for all, an element i in E such that $\bar{i} = -i$, so $\Im(z) = (z - \bar{z})/2i$. Regarding E as a 2-dimensional symplectic space over F with skew-form $\langle z_1, z_2 \rangle = \Im(z_1\bar{z}_2)$ we have

$$N = \left\{ \begin{bmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{bmatrix} : z, b \in E, z + \bar{z} = -b\bar{b} \right\} \simeq H(E),$$

the Heisenberg group attached to E over F . In particular, N is non-abelian, with commutator subgroup

$$[N, N] = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = Z,$$

the center of N . The maximal abelian subgroup of N is

$$N' = \left\{ \begin{bmatrix} 1 & b & z \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \in N : b \in F \right\}.$$

1.2

The irreducible representations of the Heisenberg group, and hence those of N , are well known:

(i) σ is 1-dimensional.

In this case, σ must be trivial on

$$Z = [N, N]$$

and define a character of N/Z . So

$$N/Z \simeq \left\{ \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{bmatrix} \right\} \simeq E$$

implies σ corresponds to a character of E , i.e.

$$\sigma = \psi_N \left(\begin{bmatrix} 1 & a & z \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{bmatrix} \right) = \psi(\mathfrak{I}a)$$

with ψ a character of F .

(ii) σ is infinite-dimensional.

In this case (by the Stone-von Neumann uniqueness theorem), σ is completely determined by its “central” character. In particular, if

$$\sigma \left(\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \psi(\mathfrak{I}z)I$$

for some (additive) character ψ of F , then

$$\sigma = \rho_\psi = \text{Ind}_{N'}^N \psi_{N'},$$

with $\psi_{N'}$ the character of (the maximal abelian subgroup) N' obtained by trivially extending ψ from Z to N' .

1.3

Definition. By a (generalized) Whittaker functional for (π, H_π) we understand N -map from N_π to some irreducible representation of (σ, L_σ) of N (possibly infinite dimensional).

1.4

Remark. The torus

$$T = \left\{ \begin{bmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar{\delta}^{-1} \end{bmatrix} : \delta \in E^\times \right\}$$

acts by conjugation on N , taking

$$\begin{bmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & \delta b & \delta \bar{\delta} z \\ 0 & 1 & -\bar{\delta} \bar{b} \\ 0 & 0 & 1 \end{bmatrix}.$$

So if ψ_N denotes the 1-dimensional representation of N corresponding to the fixed character of F as in 1.2 (i), Pontrygin duality for $E \simeq N/Z$ implies that any other 1-dimensional representation is trivial or of the form

$$\psi_N^\delta(n) = \psi_N \left(\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} n \begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix}^{-1} \right)$$

for some $\delta \in E^\times$.

1.5

If σ is a one-dimensional representation of N of the form ψ_N , a given irreducible admissible representation (π, H_π) need not possess a nontrivial ψ_N -Whittaker

functional \mathcal{L} . However, if it does, then by 1.4 it possesses a σ -Whittaker functional for any one-dimensional representation ψ_N^δ , given by the formula

$$\mathcal{L}^\delta(v) = \mathcal{L}\left(\pi\left(\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix}v\right)\right), \quad v \in H_\pi.$$

In this case, we call (π, H_π) non-degenerate. By a well-known theorem of Shalika and Gelfand-Kazhdan (cf. [Sha1]), the space of such σ -Whittaker functionals is one-dimensional. In particular, the corresponding Whittaker models

$$\mathcal{W}(\pi, \psi) = \{W(g) = \mathcal{L}(\pi(g)v) : v \in H_\pi\}$$

are unique.

1.6

In general, (π, H_π) is not non-degenerate, examples being provided by the Weil representations discussed in §??. Thus it is necessary to consider σ -Whittaker models for infinite dimensional σ as well. Such σ , however, are completely determined by their central character ψ_Z , so it is convenient to work with a slight thickening of N . More precisely, consider the stabilizer R in P of the central character ψ_Z of Z . Because $\begin{bmatrix} \delta & & \\ & \beta & \\ & & \bar{\delta}^{-1} \end{bmatrix}$ conjugates $\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 & \delta\bar{\delta}z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$R = \left\{ \begin{bmatrix} \delta & * & * \\ 0 & \beta & * \\ 0 & 0 & \delta \end{bmatrix} \in P : \delta, \beta \in E^1 \right\} \simeq (E^1 \times E^1) \ltimes N.$$

In particular, each irreducible infinite dimensional representation ρ_ψ of N extends to a like representation ρ_ψ^α of R with α a character of $E^1 \times E^1$.

Theorem (Existence and Uniqueness of Generalized Whittaker Models: [PS3].) Any (π, H_π) possesses a ρ_ψ^α -Whittaker functional for some choice of ρ_ψ^α ; moreover, the space of such functionals is at most one dimensional.

We shall discuss this result in more detail in the global context of §??.

❖ Some Fourier Expansions and Hypercuspidality

Now F is a global field not of characteristic 2, and π is an automorphic cuspidal representation of $G(\mathbb{A})$ which we suppose realized in some subspace of cusp

forms H_π in $L_0^2(G(F)\backslash G(\mathbb{A}))$. To attach an L -function to π , it is useful to take forms f in H_π and examine their Fourier coefficients along the maximal unipotent subgroup N . When such coefficients are non-zero, π is non-degenerate, and we are led back to the local Whittaker models of 1.5; in this case, we can (and eventually do) introduce L -functions using Jacquet's generalization of the "Rankin-Selberg method".

On the other hand, if these Fourier coefficients represent zero, then π is hypercuspidal; in this case, looking at Fourier expansions along Z will bring us back to the generalized Whittaker models of 1.6, and ultimately allow us to introduce an L -function for π using the so-called "Shimura method".

Henceforth, let us fix a non-trivial character ψ of $F\backslash\mathbb{A}$, and define characters ψ_N and ψ_Z of $N = N(\mathbb{A})$ and $Z = Z(\mathbb{A})$ by

$$\psi_N \left(\begin{bmatrix} 1 & a & z \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{bmatrix} \right) = \psi(\Im a)$$

and

$$\psi_Z \left(\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \psi(\Im z).$$

2.1

Fix f in H_π . To obtain a Fourier expansion of f "along N ", we introduce the familiar ψ -th coefficient

$$W_f^\psi(g) = \int_{N(F)\backslash N(\mathbb{A})} f(n g) \overline{\psi_N(n)} dn.$$

The transitivity of $T(\mathbb{A}) = \left\{ \begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} \right\}$ acting on $N(\mathbb{A})\backslash Z(\mathbb{A})$ implies - as in the local theory - that

$$\begin{aligned} W_f^{\psi^\delta}(g) &= \int_{N(F)\backslash N(\mathbb{A})} f(n g) \overline{\psi_N^\delta(n)} dn \\ &= \int_{N(F)\backslash N(\mathbb{A})} f(n g) \psi_N \left(\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} n \begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix}^{-1} \right) dn \\ &= W_f^\psi \left(\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} g \right). \end{aligned}$$

In other words, knowing W_f^ψ determines $W_f^{\psi^\delta}$ for all $\psi^\delta, \delta \in E^\times$.

However, though $N(F)\backslash N(\mathbb{A})$ is compact, it is not abelian; to obtain a nice Fourier expansion, we must bring into play the compact abelian group $N(F)Z(\mathbb{A})\backslash N(\mathbb{A})$.

2.2