How automorphic forms and elliptic curves fly?

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Abstract

This is an expository note on *murmurations*, which was initially discovered by He, Lee, Oliver, and Pozdnyakov for elliptic curves. We focus on the cases where the mumuration density is computed (under GRH), including the work of Zubrilina, Lee–Oliver–Pozdnyakov, and Sawin–Sutherland.

1 Introduction

2 Murmuration of Elliptic Curves

2.1 He–Lee–Oliver–Pozdnyakov's Murmuration

Murmuration of elliptic curves refers to the following average of Frobenius traces. Fix a nonnegative integer r and $N_1 < N_2$. Let $\mathcal{E}_r[N_1, N_2]$ be the set of isomorphism classes of elliptic curves $E_{/\mathbb{Q}}$ with conductor $N(E) \in [N_1, N_2]$ and rank r. For a fixed prime p, we consider the following average

$$\mathbb{E}_{E\in\mathcal{E}_r[N_1,N_2]}[a_p(E)] = \frac{\sum_{E\in\mathcal{E}_r[N_1,N_2]} a_p(E)}{\sum_{E\in\mathcal{E}_r[N_1,N_2]} 1} \tag{1}$$

as a function of p. What He, Lee, Oliver, and Pozdnyakov [10] observed is that this yields a surprising oscillation pattern as in Figure 1. Especially, it appears to have the same oscillation pattern for different conductor ranges, where the pattern seems to only depend on the rank r.

2.2 Sutherland's observation

Later, Sutherland [24] (and further works by several people) showed that one really needs to view the murmuration density as a function of p/N rather than p for a fixed N. He found that, for different dyadic intervals of the form $(2^k, 2^{k+1}]$, the murmuration patterns look the same (and become clearer as k increases), even if the averages consider completely different sets of elliptic curves (Figure 2). Also, instead of considering each rank separately, it seems better to consider all ranks together, where we weight $a_p(E)$ by the root number $\epsilon(E)$ of E. One can separate into two groups depending on the parity of the rank. So the major open question is to *compute* the density function, i.e. to find a function $M : (0, \infty) \to \mathbb{R}$ such that

$$\mathbb{E}_{E\in\mathcal{E}_r[N,2N]}[a_p(E)] = M\left(\frac{p}{N}\right) + \text{error} \tag{2}$$

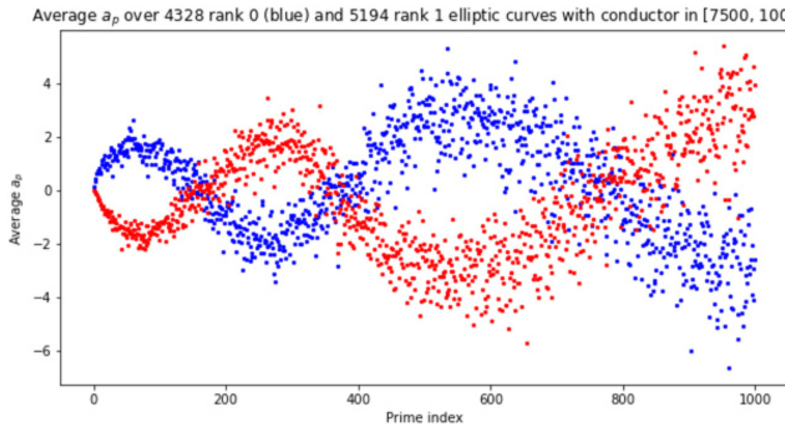

Figure 1: Murmuration of elliptic curves with conductor in $[7500, 1000]$ *and rank* $r = 0$ *(blue) and* $r = 1$ *(red) [10].*

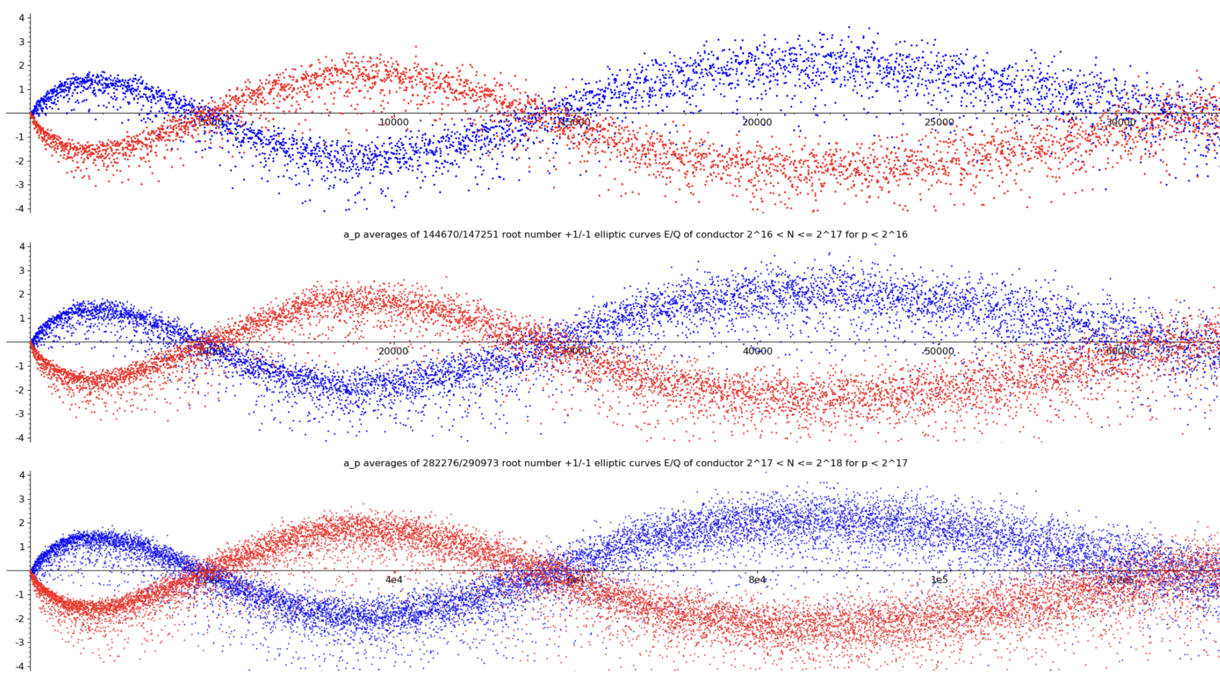

Figure 2: Murmuration of elliptic curves with conductor in $[2^k, 2^{k+1})$ *and primes* $p < 2^{k-1}$ *for* $k = 15, 16, 17$ *[24]. Blue (resp. red) curves correspond to* $\epsilon(E) = +1$ *(resp.* -1*) elliptic curves.*

where the error term goes to zero as $N \to \infty$. More generally, one can fix $0 < C_1 < C_2$ and consider the interval $[C_1 N, C_2 N]$.

Sutherland also observed that the murmuration disappears when elliptic curves are ordered by other measures, such as naive height, discriminant, or j-invariants, although further local averaging gives murmuration for naive heights (see Section 5). This shows that the murmuration is a phenomenon that is sensitive to the ordering of elliptic curves.

2.3 What is the role of Machine Learning?

Although there seems to be no machine learning involved in the previous discussions, I will make a brief comment on the relation between machine learning and murmuration, as I found that existing literature is often misleading in distinguishing the machine learning part from the murmuration part. I have read a few articles on the internet which basically say that "AI found new mathematics," which is false.

One of the main motivations of the paper [10] is to study elliptic curves via machine learning. Especially, they were interested in predicting the rank of elliptic curves (which is widely known to be hard to compute in general) by means of machine learning, where the coefficients $a_p(E)$ of Hasse–Weil L-functions are used as features. Surprisingly, they found that a simple logistic regression model can already distinguish between rank 0 and 1 elliptic curves with high accuracy of $> 90\%$ (see also [9]). Along these lines, they (more precisely, He, Lee, and Oliver) were curious about what was actually going on, and Pozdnyakov (who was an undergraduate student of Lee at that time) figured out the murmuration pattern. This somehow gives an explanation for the high accuracy of the model, since the murmuration patterns for rank 0 and 1 elliptic curves are noticeably different. But the correct way to say it is that the machine learning experiments *motivated* them to study what the models were doing, which is essentially the work of humans, not the ML models. You can find more of the story in the Quanta Magazine article [5].

2.4 Sato–Tate conjecture and Murmuration

One should not confuse murmuration with the (vertical) Sato–Tate conjecture, which we will explain here. The original (i.e. *horizontal*) Sato–Tate conjecture is about the distribution of $a_p(E)$ for a *fixed* $E_{/\mathbb{Q}}$ *and varying* p. The Hasse–Weil bound says that $|a_p(E)| \leq 2\sqrt{p}$, and the conjecture predicts that for a non-CM elliptic curve E, the distribution of $a_p(E)$ is semicircular with radius $2\sqrt{p}$, i.e., the density function is $\frac{1}{2\pi}\sqrt{4 - x^2}\mathrm{d}x$ for the normalized traces $a_p(E)/\sqrt{p}$. Equivalently, if we write $a_p(E) = 2\sqrt{p}\cos\theta_p$ for $\theta_p \in [0, \pi]$, then θ_p follows the distribution $\frac{2}{\pi}\sin^2\theta \mathrm{d}\theta$. The distributions for CM elliptic curves are different, and we also expect that the Frobenius traces for abelian varieties of higher dimension will follow certain distributions, which are conjecturally the pushforward of the Haar measure of a certain compact Lie group, called the *Sato–Tate group*. See [26] for more about the Sato–Tate conjecture and recent progress on it.

The *vertical* Sato–Tate conjecture fixes p and varies E over $\mathbb{F}_p$ instead, where there are only finitely many isomorphism classes of E over $\mathbb{F}_p$. Birch [1] proved that the distribution converges to the above semicircular distribution as $p \to \infty$. This is different from the murmuration for two reasons: vertical Sato–Tate considers the elliptic curves over $\mathbb{F}_p$, and there's no conductor involved in vertical Sato–Tate.

3 Murmuration of Dirichlet Series

Although the original murmuration density for elliptic curves is still unknown, there are a few works where murmuration exists and is even computed (under GRH). Historically, the first such example is the work of Zubrilina on modular forms [28], but we will start with the simplest case of Dirichlet characters. Lee, Oliver, and Pozdnyakov computed the murmuration density for Dirichlet characters [14][1]. For complex characters, the corresponding murmuration densities are given by the following theorem.

Theorem 3.1 (Lee–Oliver–Pozdnyakov [14, Theorem 1.1])**.** Let $\mathcal{D}_+(N)$ (resp. $\mathcal{D}_-(N)$) denote the set of primitive even (resp. odd) Dirichlet characters modulo N. For $x \in \mathbb{R}_{>0}$, let $\lceil x \rceil^{\mathfrak{p}}$ be the smallest prime $\geq x$. For $c > 1$, $\delta > 0$, and $y > 0$, define

$$P_\pm(y, X, c) := \frac{\log X}{X} \sum_{\substack{N \in [X, cX] \\ N \text{ prime}}} \sum_{\chi \in \mathcal{D}_\pm(N)} \frac{\chi(\lceil yX \rceil^{\mathfrak{p}})}{\tau(\chi)}, \tag{3}$$

$$P_\pm(y, X, \delta) := \frac{\log X}{X^\delta} \sum_{\substack{N \in [X, X+X^\delta] \\ N \text{ prime}}} \sum_{\chi \in \mathcal{D}_\pm(N)} \frac{\chi(\lceil yX \rceil^{\mathfrak{p}})}{\tau(\chi)}. \tag{4}$$

Then

$$\lim_{X\to\infty} P_\pm(y, X, c) = \begin{cases} \int_1^c \cos\left(\frac{2\pi y}{x}\right) \mathrm{d}x & \text{if } +, \\ -i \int_1^c \sin\left(\frac{2\pi y}{x}\right) \mathrm{d}x & \text{if } -, \end{cases} \tag{5}$$

and assuming RH, if $\frac{1}{2} < \delta < 1$, we have

$$\lim_{X\to\infty} P_\pm(y, X, \delta) = \begin{cases} \cos(2\pi y) & \text{if } +, \\ -i \sin(2\pi y) & \text{if } -. \end{cases} \tag{6}$$

See Figure 3 for the plot of the above murmuration densities. As you can see, there are two versions of murmurations: the *long interval* $[X, cX]$ and the *short interval* $[X, X + X^\delta]$. Note that one needs to assume RH to get the short interval version, to guarantee the existence of primes in short intervals. The summand $\chi(p)/\tau(\chi)$ is the p-th Fourier coefficient of $\overline{\chi}$ when expanded in terms of additive characters, which justifies the normalization. Also, the above averages only consider prime moduli, though the authors also studied the case of composite moduli in [14, Section 6.1].

The proof of Theorem 3.1 is much simpler than the case of modular forms (Section 4). The main ingredient of the proof is the following formulas [14, Lemma 2.6]: for two distinct primes p and N,

$$\sum_{\chi \in \mathcal{D}_+(N)} \frac{\chi(p)}{\tau(\chi)} = \left(\frac{N-1}{N}\right) \cos\left(\frac{2\pi p}{N}\right) + \frac{1}{N}, \tag{7}$$

$$\sum_{\chi \in \mathcal{D}_-(N)} \frac{\chi(p)}{\tau(\chi)} = -i \left(\frac{N-1}{N}\right) \sin\left(\frac{2\pi p}{N}\right), \tag{8}$$

which can be proved by using the orthogonality of Dirichlet characters. Combined with the prime number theorem (which gives equidistribution results of primes in $[X, cX]$ normalized by X), we get (5), and assuming RH gives (6).

[1] These can be thought of as automorphic forms on GL_1 over $\mathbb{Q}$.

They also proved similar results for real Dirichlet characters, but the proof is more complicated. Let $\mathcal{G}$ be the set of odd square-free integers and let $\chi_d = \left(\frac{d}{\cdot}\right)$. For a compactly supported smooth function $\Phi \geq 0$ on $\mathbb{R}$, define

$$M_\Phi(y, X, \delta) = \frac{\log X}{X^{1+\delta}} \sum_{\substack{p \in [yX, yX+X^\delta] \\ p \text{ prime}}} \sum_{d \in \mathcal{G}} \Phi\left(\frac{d}{X}\right) \chi_{8d}(p)\sqrt{p}. \tag{9}$$

(Here we only consider characters of conductor $8d$ due to technical reasons, see [22].)

Theorem 3.2 (Lee–Oliver–Pozdnyakov [14, Theorem 1.2])**.** Fix $y > 0$ and assume $\frac{3}{4} < \delta < 1$. Assuming GRH, we have

$$M_\Phi(y, \delta) := \lim_{X \to \infty} M_\Phi(y, X, \delta) = \frac{1}{2} \sum_{\substack{a \geq 1 \\ a \text{ odd}}} \frac{\mu(a)}{a^2} \sum_{m \geq 1} (-1)^m \widetilde{\Phi}\left(\frac{m^2}{2a^2 y}\right), \tag{10}$$

where

$$\widetilde{\Phi}(\xi) = \int_{-\infty}^{\infty} (\cos(2\pi\xi x) + \sin(2\pi\xi x))\Phi(x)\mathrm{d}x. \tag{11}$$

The proof is more involved and is based on the Polya–Vinogradov inequality

$$\left| \sum_{\substack{p \in [yX, yX+X^\delta] \\ p \text{ prime}}} \chi_d(p) \right| \ll (yX)^{\frac{1}{2}+\epsilon}$$

(for non-principal χ_d with $\frac{1}{2} < \delta < 1$, which uses GRH [8]), and a summation formula

$$\frac{1}{X} \sum_{\substack{d \in \mathbb{Z} \\ d \text{ odd}}} \left(\sum_{\substack{a^2 || d| \\ a \leq A}} \mu(a) \right) \Phi\left(\frac{d}{X}\right) \left(\frac{d}{p}\right) \sqrt{p} = \frac{1}{2} \left(\frac{2}{p}\right) \sum_{\substack{0 < a \leq A \\ (a, 2p) = 1}} \frac{\mu(a)}{a^2} \sum_{k \in \mathbb{Z}} (-1)^k \left(\frac{k}{p}\right) \widetilde{\Phi}\left(\frac{kX}{2a^2 p}\right),$$

which can be proved using the Poisson summation formula.

4 Murmuration of Modular Forms

The first murmuration density that is computed ever is for modular forms by Zubrilina [28].

4.1 Statement

Before we state the result, we define some notations first.

- For $n \in \mathbb{Z}_{\geq 0}$, *Chebyshev polynomial of the second kind* is defined as

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

- For $r \in \mathbb{Z}_{\geq 1}$, define

$$\nu(r) := \prod_{p | r} \left(1 + \frac{p^2}{p^4 - 2p^2 - p + 1}\right)$$

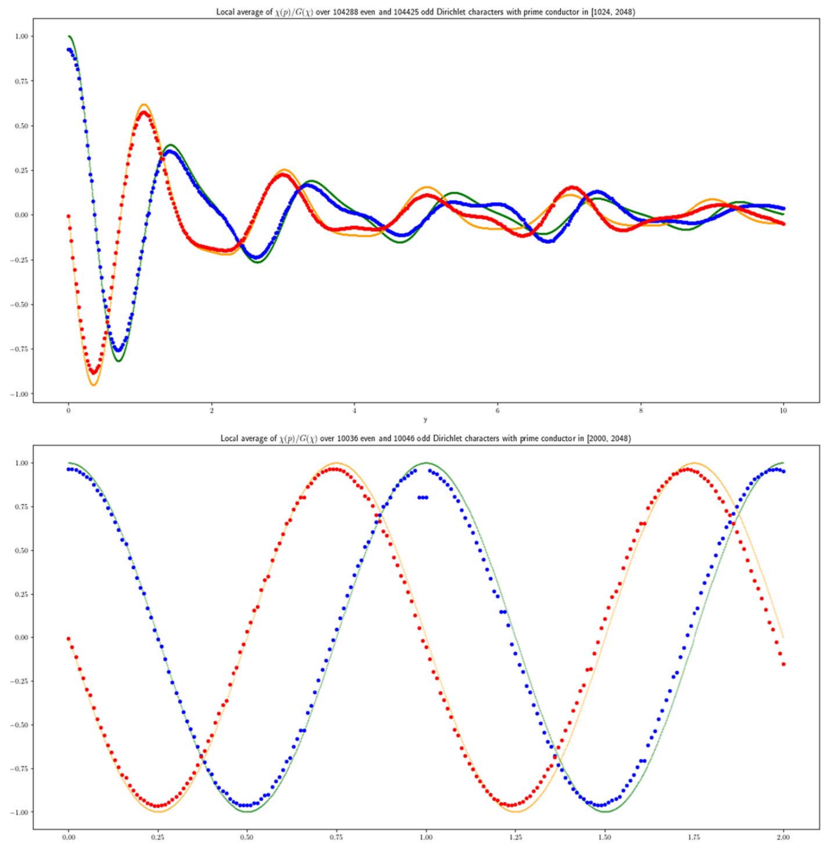

Figure 3: Murmuration of Dirichlet characters. The top figure presents $P_{\pm}(y, 2^{10}, 2)$ *for* $y \in [0, 10]$ *with* $+$ *in blue and (the imaginary part of)* $-$ *in red. The bottom figure presents* $\widetilde{P}_{\pm}(y, 2002, 0.51)$ *for* $y \in [0, 2]$ *with* $+$ *in blue and (the imaginary part of)* $-$ *in red. The discontinuity of* $\widetilde{P}_{+}(y, 2002, 0.51)$ *at* $y = 1$ *corresponds to the term* $p = N$ *in* (4).

- Define constants α, β, γ as

$$\alpha := 2\pi \prod_p \frac{p^4 - 2p^2 - p + 1}{p^4 - 2p^2 + p},$$
$$\beta := 2\pi \prod_p \frac{p^3 + p^2 - 1}{p(p^2 + p - 1)},$$
$$\gamma := 12 \prod_p \frac{p(p+1)}{p^2 + p - 1}.$$

Theorem 4.1 (Zubrilina [28])**.** Let X, Y, P be parameters going infinite with $X, Y > 0$ and P prime; assume further that $Y = (1 + o(1))X^{1-\delta_2}$ and $P \ll X^{1+\delta_1}$ for some δ_1, δ_2 with $2\delta_1 < \delta_2 < 1$. Let $y = P/X$. Then

$$\frac{\sum^{\square}_{N \in [X, X+Y]} \sum_{f \in H^{\mathrm{new}}(N,k)} \sqrt{P} \lambda_f(P) \varepsilon(f)}{\sum^{\square}_{N \in [X, X+Y]} \sum_{f \in H^{\mathrm{new}}(N,k)} 1} = \mathcal{M}_k(y) + O_\varepsilon \left(X^{-\delta' + \varepsilon} + \frac{1}{P} \right) \tag{12}$$

where

$$\mathcal{M}_k(y) = \frac{\alpha (-1)^{k/2-1}}{k-1} \sum_{1 \le r \le 2\sqrt{y}} \nu(r) \sqrt{4y - r^2} U_{k-2} \left(\frac{r}{2\sqrt{y}} \right) + \frac{\beta}{k-1} \sqrt{y} - \gamma \delta_{k=2} y. \tag{13}$$

4.2 Eichler–Selberg trace formula

To prove Theorem 4.1, one need to understand how to estimate the numerator on the LHS. Recall that $a_f(P) = P^{(k-1)/2} \lambda_f(P)$ is the P-th Fourier coefficient of f, which is also the eigenvalue of the Hecke operator T_P acting on f. Also, $(-1)^{k/2} \varepsilon(f)$ is equal to the eigenvalue of the Atkin–Lehner involution $W_N = T_N$ acting on f. Thus the sum appears in the numerator of LHS of (12) can be interpreted as the trace of the operator $(-1)^{k/2} T_P \circ W_N$ acting on the space of cusp forms of weight k and level N (multiplied by $P^{1-k/2}$). Eichler [7] studied such a sum of traces and proved that it can be expressed in terms of (Hurwitz) class numbers, which is generalized by Selberg [19]. To account the root number $\varepsilon(f)$, i.e. eigenvalue of W_N, we used the following version of Eichler–Selberg trace formula by Skoruppa and Zagier [21].

Theorem 4.2 (Skoruppa–Zagier [21])**.** For square-free N and prime $P \nmid N$,

$$\sum_{f \in H^{\mathrm{new}}(N,k)} \sqrt{P} \lambda_f(P) \varepsilon(f)$$
$$= \frac{H_1(-4PN)}{2} + (-1)^{k/2-1} U_{k-2} \left(\frac{r\sqrt{N}}{2\sqrt{P}} \right) \sum_{0 < r \le 2\sqrt{P/N}} H_1(r^2 N^2 - 4PN) - \delta_{k=2}(P+1)$$

Here $H_1(-d)$ $(d > 0)$ is the Hurwitz class number, the number of equivalence classes of positive definite binary quadratic forms of discriminant $-d$ weighted by the number of automorphisms, i.e. with forms correspond to $x^2 + y^2$ or $x^2 + xy + y^2$ counted with multiplicity $1/2$ and $1/3$ respectively.

Hurwitz class number can be expressed as a sum of usual class numbers as

$$H_1(-d) = \sum_{f^2 | d} h(-d/f^2) + O(1)$$

where the "error term" $O(1)$ disappears if $d \neq 3 \cdot \square$ or $4 \cdot \square$. Using this, we can rewrite the Skoruppa–Zagier trace formula as

$$\sum_{f \in H^{\mathrm{new}}(k,N)} \sqrt{P} \lambda_f(P) \varepsilon(f)$$
$$= \frac{h(-4PN)}{2} + \frac{h(-PN)}{2} - \delta_{k=2} P + O(1) + (-1)^{k/2-1} U_{k-2}\left(\frac{r\sqrt{N}}{2\sqrt{P}}\right) \sum_{1 \le r \le 2\sqrt{P/N}} \sum_{d^2 \mid r^2N-4P} h\left(\frac{N(r^2N-4P)}{d^2}\right)$$

From this, our new goal is to estimate the average of class numbers over short intervals, i.e. when $N \in [X, X+Y]$ with $Y = o(X)$. The main idea is to use class number formula to write class numbers as special L-values at $s = 1$, e.g.

$$h(-d) = \frac{\sqrt{d}}{\pi} L(1, \chi_d)$$

when $d > 4$ and $-d \not\equiv 2, 3 \pmod 4$, and $\chi_d = \left(\frac{d}{\cdot}\right)$ is the Kronecker symbol. Then the sum (average) of the corresponding L-values can be estimated via truncation and Polya–Vinogradov inequality. For example, we have an estimate

$$L(1, \chi_d) = \sum_{n \ge 1} \frac{\chi_d(n)}{n} = \sum_{1 \le n \le T} \frac{\chi_d(n)}{n} + O\left(\frac{\sqrt{d}\log d}{T}\right).$$

With some hard analysis, one get the following estimations on the sum of $h(-PN)$ and $h(-4PN)$.

Proposition 4.3 (Zubrilina [28, Poposition 3.1])**.** Let P be an odd prime and let $[X, X+Y]$ be an interval with $Y = o(X)$. Then as $X \to \infty$,

$$\frac{\zeta(2)\pi}{XY} \sum_{\substack{N \in [X, X+Y] \\ P \nmid N}}^{\square} \left(\frac{h(-PN)}{2} + \frac{h(-4PN)}{2}\right) = A\sqrt{y} + O_\varepsilon\left(\frac{1}{P^{3/2}X^{1/2}} + \frac{P^{7/12}}{Y^{5/6}X^{5/12}} + \frac{YP^{1/2}}{X^{3/2}}\right)(XYP)^\varepsilon$$

where

$$A = \prod_p \left(1 + \frac{p}{(p+1)^2(p-1)}\right).$$

The summation of $H_1(r^2N^2 - 4PN)$ terms can be bounded in a similar way, although the computation is much more complicated.

Proposition 4.4 (Zubrilina [28, Proposition 3.2])**.** Let P be an odd prime, $r \in \mathbb{N}$, and $X > Y > 0$ be such that $r^2(X+Y) < 4P$ for each $r > 2\sqrt{P/X}$. Let $y = P/X$. Then

$$\frac{\zeta(2)\pi}{XY} \sum_{r \le 2\sqrt{P/X}} \sum_{\substack{N \in [X, X+Y] \\ P \nmid N}}^{\square} H_1\left(r^2N^2 - 4PN\right)$$
$$= \sum_{r \le 2\sqrt{P/X}} B\nu(r)\sqrt{4y - r^2} + O\left(\frac{P^{11/10}}{Y^{2/5}X^{9/10}} + \frac{YP}{X^2} + \frac{PY^{1/2}}{X^{3/2}} + \frac{P}{X^{1/2}Y^{13/18}} + \frac{P}{XY^{1/9}}\right)(XYP)^\varepsilon$$

where

$$B = \prod_p \frac{p^4 - 2p^2 - p + 1}{(p^2-1)^2}.$$

4.3 Geometric intervals

Theorem 4.1 considers the average over "short intervals" $[X, X+Y]$ with $Y = o(X)$. By integrating it in a suitable sence (and assuming GRH), one can also get the average over "geometric intervals" $[X, cX]$ for some constant $c > 1$.

Theorem 4.5 ([28, Theorem 2]). Let $P \ll X^{6/5}$, $c > 1$ be a constant, and $y = P/X$. As $X \to \infty$,

$$\mathbb{E}_{\substack{N \in [X, cX] \\ N \text{ squarefree} \\ f \in H^{\text{new}}(N,k)}} [\sqrt{P}\lambda_f(P)\epsilon(f)] = \frac{2}{c^2 - 1} \int_1^c u\mathcal{M}_k\left(\frac{y}{u}\right) \mathrm{d}u + o_y(1) \tag{14}$$

where $\mathcal{M}_k(y)$ is as in Theorem 4.1. In particular, for $k = c = 2$, the dyadic average

$$\frac{\sum_{N \in [X,2X]}^{\square} \sum_{f \in H^{\text{new}}(N,2)} a_f(P)\epsilon(f)}{\sum_{N \in [X,2X]}^{\square} \sum_{f \in H^{\text{new}}(N,2)} 1} \tag{15}$$

converges to

$$\begin{cases} a\sqrt{y} - by & 0 \le y \le \frac{1}{4} \\ a\sqrt{y} - by + c\pi y^2 - c(1-2y)\sqrt{y - \frac{1}{4}} - 2cy^2 \arcsin\left(\frac{1}{2y} - 1\right) & \frac{1}{4} \le y \le \frac{1}{2} \\ a\sqrt{y} - by + 2cy^2\left(\arcsin\left(\frac{1}{y} - 1\right) - \arcsin\left(\frac{1}{2y} - 1\right)\right) & \\ \qquad -c(1-2y)\sqrt{y - \frac{1}{4}} + 2c(1-y)\sqrt{2y-1} & \frac{1}{2} \le y \le 1 \end{cases} \tag{16}$$

for explicit constants a, b, c.

Proof. The main idea of the proof is to divide the interval $[X, cX]$ into short intervals $[X_g, X_{g+1}]$ for $X_g = X + (g-1)Y$ where $Y \sim X^{1-\delta_2}$. Then use Theorem 4.1 to approximate the sum over short intervals as an integral of $u\mathcal{M}_k(y/u)$. The case of $k = c = 2$ can be done by elementary computations. $\square$

The murmuration density function $\mathcal{M}_k(y)$ have many interesting properties. Especially, their properties can be related to Katz and Sarnak's 1-level density conjecture [12]; see Section 6 for more details.

4.4 Doesn't Zubrilina's result prove murmuration for elliptic curves, because of the modularity?

No! The reason is because elliptic curves over $\mathbb{Q}$ corresponds to modular forms of weight 2 *with coefficient field (Hecke field)* $\mathbb{Q}$ (recall that $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$ are integers). The family (see also Section 6) that Zubrilina considered is much larger than the family of elliptic curves in [10], and it seems hard to isolate such family from whole family of Hecke eigenforms (of weight 2). It is conjectured that the *conductor dimension* (See 6 for the definition) of elliptic curves is $\frac{5}{6}$ [20], while that of the weight 2 modular forms is 2.

5 Murmuration of Elliptic Curves, Revisited

Recently, Will Sawin and Andrew Sutherland announced a murmuration theorem for elliptic curves, which is slightly different from the formulation in [10]. Especially, they proved a version of the murmuration theorem *ordered by height*:

Theorem 5.1 (Sawin–Sutherland [18]). Let $\mathcal{E}(X) := \{y^2 = x^3+ax+b : a, b \in \mathbb{Z}, p^4 \mid a \Rightarrow p^6 \nmid b, \max\{4|a|^3, 27b^2\} \le X\}$ be the set of naive isomorphism classes of elliptic curves over $\mathbb{Q}$ ordered by height. Let $0 < C_1 < C_2$ be real numbers. For any smooth function $W : \mathbb{R}_{>0} \to \mathbb{R}$ with compact support, the limit

$$\lim_{P\to\infty} \lim_{X\to\infty} \mathbb{E}_{E:H(E)\le X}\left[\frac{\prod_{p\le P}(1-p^{-1})^{-1}}{N(E)} \sum_{\substack{n\ge1 \\ p\nmid n\, \forall p\le P}} W\left(\frac{n}{N(E)}\right) a_n(E)\epsilon(E)\right] \tag{17}$$

exists and is equal to

$$\int_0^\infty W(u)\sqrt{u}\left(2\pi \sum_{q\ge1}{}^{\square} \sum_{m\ge1} \frac{\mu(\gcd(m,q))}{qm\phi\left(\frac{q}{\gcd(m,q)}\right)} J_1\left(\frac{4\pi\sqrt{u}m}{q}\right) \prod_{p\mid q} \hat{\ell}_{p,2v_p(m)} \prod_{p\mid m, p\nmid q} \ell_{p,2v_p(m)}\right) \mathrm{d}u \tag{18}$$

where $\ell_{p,\nu}$ and $\hat{\ell}_{p,\nu}$ are certain local factors that can be written in terms of traces of the Hecke operator T_p (see [18, Lemma 3, 4]).

You should have a question at this point. I said that Sutherland observed no murmuration pattern in [24] when elliptic curves are ordered by height, but Theorem 5.1 seems to suggest that there is a murmuration pattern. In fact, the difference comes from *local averaging*, which I'm going to explain now.

The difference between the original murmuration observed in HLOP [10] and the one in Sawin–Sutherland is well-explained in [18, Section 1.1]. The original murmuration considered the averages of the form

$$\mathbb{E}_{\substack{N(E)\in[N_1,N_2] \\ \mathrm{rank}(E)=r}}[a_p(E)]$$

as a function in p for fixed r, N_1, N_2 (initially $[N_1, N_2] = [7500, 10000]$ in [10]). As mentioned earlier, subsequent works (especially [24]) found that we need to view the murmuration density as a function in p/N, not p. Also, it seems better to consider all elliptic curves with same root numbers at once, or weight a_p by root numbers. Hence the reformulated HLOP's murmuration would be

$$\mathbb{E}_{N(E)\in[X,2X]}[\epsilon(E)a_p(E)]$$

In [18], the authors mentioned that Bober suggested that one may need *local averaging* in p before we average over different elliptic curves.

$$\mathbb{E}_{N(E)\in[X,2X]}\left[\mathbb{E}_{\substack{p\in(C_1N(E),C_2N(E)) \\ p \text{ prime}}}[\epsilon(E)a_p(E)]\right]$$

The main idea of the proof of Theorem 5.1 is the Voronoi summation formula.

Theorem 5.2 ([18, Lemma 11]). Let $E_{/\mathbb{Q}}$ be an elliptic curves, q be a positive integer, a a positive integer coprime to q, and $W : (0,\infty) \to \mathbb{R}$ a smooth function with compact support. Then

$$\epsilon(E)\sum_{n\ge1} \frac{a_n(E)}{\sqrt{n}} W\left(\frac{n}{N(E)}\right) e\left(\frac{an}{q}\right) = \frac{\sqrt{N(E)}}{q} \sum_{n\ge1} \frac{a_n(E)}{\sqrt{n}} e\left(\frac{\overline{aN(E)}n}{q}\right) \int_0^\infty 2\pi W(u) J_1\left(\frac{4\pi\sqrt{un}}{q}\right) \mathrm{d}u \tag{19}$$

where $e(x) = e^{2\pi i x}$ and $\overline{aN(E)}$ is the multiplicative inverse of $aN(E)$ modulo q.

Note that summation of n instead over primes is built-in inside the formula. Based on the theorem, they also conjectured that:

Conjecture 5.3 ([18, Conjecture 1]).

$$\lim_{X\to\infty} \mathbb{E}_{H(E)\le X}\left[\frac{\log\left(N(E)^{\frac{C_1+C_2}{2}}\right)}{N(E)} \sum_{p\in(C_1N(E),C_2N(E))} \epsilon(E)a_p(E)\right]$$

$$= \int_{C_1}^{C_2} 2\pi\sqrt{u} \sum_q \sum_{m\in\mathbb{N}} \frac{\mu(\gcd(m,q))}{qm\phi\left(\frac{q}{\gcd(m,q)}\right)} J_1\left(\frac{4\pi\sqrt{u}m}{q}\right) \prod_{p|q} \hat{\ell}_{p,2v_p(m)} \prod_{p|m,p\nmid q} \ell_{p,2v_p(m)}\mathrm{d}u$$

The main two differences between the conjecture and Theorem 5.1 are that (1) the summation is over primes and (2) the (smooth, compactly supported) weight function W is replaced by the characteristic function of the interval (C_1, C_2). Heuristics like Crámer's random model suggests that these changes do not affect the density function.

You can find more on the Sutherland's lecture [25] at Tate conference (*The legacy of John Tate, and beyond* at Harvard university). He considered it as *a* murmuration theorem, and might not be *the* murmuration theorem since the density formula is too complicated.

6 General formulation of Murmuration

6.1 Family

Sarnak suggested a general framework of murmuration in his letter to Sutherland and Zubrilina [16]. Let $\mathcal{F}$ be a family of L-functions in a suitable sense (e.g. See [17]). For a smooth nonnegative function $\Phi : (0,\infty) \to \mathbb{R}$ with compact support and $f : \mathcal{F} \to \mathbb{C}$, consider the Φ-weighted average of f:

$$\mathbb{E}_{\pi\in\mathcal{F}}[f;\Phi,N] := \frac{\sum_{\pi\in\mathcal{F}} \Phi\left(\frac{N_\pi}{N}\right) f(\pi)}{\sum_{\pi\in\mathcal{F}} \Phi\left(\frac{N_\pi}{N}\right)} = \frac{A_{\mathcal{F}}(f;\Phi,N)}{A_{\mathcal{F}}(1;\Phi,N)} \tag{20}$$

where

$$A_{\mathcal{F}}(f;\Phi,N) := \sum_{\pi\in\mathcal{F}} \Phi\left(\frac{N_\pi}{N}\right) f(\pi). \tag{21}$$

Here N_π is the "conductor" of π (e.g. conductor of an elliptic curve or analytic conductor of an automorphic form). When we order the family by the conductor, we say that $\mathcal{F}$ has *conductor dimension* δ if

$$\#\{\pi \in \mathcal{F} : N_\pi \le N\} \sim \alpha N^\delta \tag{22}$$

as $N \to \infty$ for some $\alpha > 0$ and $\delta = \delta(\mathcal{F}) > 0$. For such family, we have

$$A_{\mathcal{F}}(1;\Phi,N) \sim \alpha\delta N^\delta \int_0^\infty \Phi(x)x^\delta \frac{\mathrm{d}x}{x}.$$

Most of the known murmuration results consider the function

$$f(\pi) = a_\pi(p) := \sqrt{p}\lambda_\pi(p) \tag{23}$$

for a given prime p, where $\lambda_\pi(p)$ is the normalized trace of Frobenius at p so that the Ramanujan–Petersson conjecture says $|\lambda_\pi(p)| \le n$ for GL_n automorphic forms π. Furthermore, if $\mathcal{F}$ is self-dual, then $a_\pi(p)$ are real and the global root number w_π is either 1 or -1. Then we can separate by root number and consider the averages

$$\mathbb{E}_{\pi\in\mathcal{F}^w}[a_\pi(p);\Phi,N] \tag{24}$$

for $w \in \{\pm 1\}$ and $\mathcal{F}^w = \{\pi \in \mathcal{F} : w_\pi = w\}$.

When π is self-dual, functional equation relates $L(s,\pi)$ and $L(1-s,\pi)$ and the completed L-function $\Lambda(s,\pi\times\pi)$ of $\pi\times\pi$ factors as

$$\Lambda(s,\pi\times\pi) = \Lambda(s,\pi,\mathrm{Sym}^2)\Lambda(s,\pi,\wedge^2).$$

π is said to be *orthogonal* if the first factor $\Lambda(s,\pi,\mathrm{Sym}^2)$ has a pole at $s=1$, and *symplectic* if the second factor $\Lambda(s,\pi,\wedge^2)$ has a pole at $s=1$. The symplectic case occur only if n is even, and root number of orthogonal π is always 1.

Katz and Sarnak [12, 13] studied statistics of zeros of L-functions via random matrix models. In particular, they considered *one-level density* of low-lying zeros: for an even function ϕ with rapid decay as $|x|\to\infty$, the one-level density of a family $\mathcal{F}$ is

$$\mathrm{OLD}(\mathcal{F};\phi) = \lim_{N\to\infty} \mathbb{E}_{\pi\in\mathcal{F}(N)}\left[\sum_{\gamma_\pi}\phi\left(\frac{\gamma_\pi \log N}{2\pi}\right)\right] \tag{25}$$

where $\mathcal{F}(N) := \{\pi\in\mathcal{F} : N_\pi = N\}$ and γ_π runs through the ordinates of nontrivial zeros of $L(s,\pi)$ on the critical line, i.e. $L\left(\frac{1}{2}+i\gamma_\pi,\pi\right)=0$. The factor $\frac{\log N}{2\pi}$ guarantees that the nontrivial zeros have unit spacing on average. Katz–Sarnak philosophy claims that there is a measure $W_{\mathcal{F}}$ coming from matrices related to the "type" of $\mathcal{F}$ such that

$$\mathrm{OLD}(\mathcal{F};\phi) = \int_{\mathbb{R}} \widehat{\phi}(x)\widehat{W_{\mathcal{F}}}(x)\mathrm{d}x \tag{26}$$

for any nice text function ϕ.

One such example is the following theorem on the family of Hecke eigenforms by Iwaniec, Luo, and Sarnak [11].

Theorem 6.1 (Iwaniec–Luo–Sarnak [11])**.** Assume GRH. Let ϕ be an even Schwartz function with $\mathrm{supp}(\widehat{\phi}) \subset (-2,2)$. Let $H_k^\pm$ be a set of Hecke eigenforms of weight k and root number $\epsilon = \pm 1$. Then

$$\mathrm{OLD}(H_k^\pm;\phi) = \int_{\mathbb{R}} \widehat{\phi}(x)\widehat{W_{\mathrm{SO}(\pm)}}(x)\mathrm{d}x \tag{27}$$

where

$$W_{\mathrm{SO}(+)}(x) = 1 + \frac{\sin(2\pi x)}{2\pi x}, \quad W_{\mathrm{SO}(-)}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta_0(x). \tag{28}$$

There is an explicit formula relates the summation over zeros of L-functions and over primes, which is given by [11, Section 4]

$$\sum_{\gamma_\pi}\phi\left(\frac{\gamma_\pi\log N}{2\pi}\right) = C - 2\sum_p\sum_{\nu\ge 1}\left(\sum_j \alpha_j(p)^\nu\right)\widehat{\phi}\left(\frac{\nu\log p}{\log N}\right)\frac{\log p}{p^{\nu/2}\log N}$$

and since $\widehat{\phi}$ is compactly supported, the main contribution comes from $\nu = 1$ summand, so we are mostly interested in

$$\sum_p \frac{\lambda_\pi(p)}{p^{1/2}} \widehat{\phi}\left(\frac{\log p}{\log N}\right) \frac{\log p}{\log N} \tag{29}$$

The Fourier transforms of (28) are

$$\widehat{W_{\mathrm{SO}(+)}}(y) = \delta_0(y) + \frac{2 - \mathbb{1}_{[-1,1]}(y)}{2}, \quad \widehat{W_{\mathrm{SO}(-)}}(y) = \delta_0(y) + \frac{\mathbb{1}_{[-1,1]}(y)}{2} \tag{30}$$

and there are obvious discontinuities at $y = \pm 1$.

[15]

6.2 Katz–Sarnak philosophy

6.3 Revisiting the murmuration theorems

Let's see how the previous works [28, 14, 18] fit into the above framework.

6.3.1 Dirichlet characters

6.3.2 Modular forms

6.3.3 Elliptic curves (ordered by heights)

The conductor dimension of a family of elliptic curves ordered by heights is $\frac{5}{6}$, so we may need further local averaging over $X^{1/6+\epsilon}$ many primes. Sawin and Sutherland introduced local averaging in Theorem 5.1, but it is slightly different from Sarnak's suggestion, since they take local average over $\Theta(N_E)$-many primes, not $O(H_E^{1/6+\epsilon})$. Cowan [6]

7 Other known cases

After the success of Zubrilina, a lot of people are interested in murmuration density for different objects in number theory. We list the known works here.

7.1 Flying Hecke characters of imaginary quadratic fields

Wang [27] computed murmuration density for Hecke characters of imaginary quadratic fields.

Theorem 7.1 (Wang [27])**.** Let $\mathcal{F}$ be the family of nontrivial Hecke characters of $\mathbb{Q}(\sqrt{-D})$ for square-free $D > 3$, $D \equiv 3 \pmod 4$. Then the average of normalized trace $\lambda_f(p) = a_f(p)\sqrt{p}$ over $f \in \mathcal{F}$ with $N_f \in [X, X+Y]$ is

$$\frac{\sum_{\substack{f\in\mathcal{F} \\ N_f\in[X,X+Y]}} \lambda_f(p)}{\sum_{\substack{f\in\mathcal{F} \\ N_f\in[X,X+Y]}} 1} = c(p) \sum_{1\le m\le 2\sqrt{y}} \delta_m(p) M_m(y) + M_-(y) + \text{error} \tag{31}$$

where

$$\begin{aligned}
M_m(y) &= \frac{11\zeta(2)}{4A}\sqrt{\frac{y}{4y-m^2}}\vartheta(m) \\
M_-(y) &= -\frac{11\pi}{A}\sqrt{y} \\
c(p) &= \frac{p+1}{3p}\prod_{\ell>2,\left(\frac{p}{\ell}\right)=1}\left(1-2\ell^{-2}-\frac{2\ell^{-3}}{1-\ell^{-2}}\right) \\
\delta_m(p) &= \begin{cases} \mathbb{1}_{\left(\frac{p}{q}\right)=1} & m=q^k,\ q \text{ is odd prime} \\ \mathbb{1}_{p\equiv 3 \pmod 4} & m=2 \\ \mathbb{1}_{p\equiv 5 \pmod 4} & m=4 \\ \mathbb{1}_{p\equiv 1 \pmod 8} & m=2^\nu,\ \nu\ge 3 \end{cases}
\end{aligned}$$

See [27, Theorem 1] for the missing definitions. Note that the main term of (31) depends on the arithmetic of p, so it is not a murmuration in the sense of [16]. However, the dependence on p is explicit and $c(p)\delta_m(p)$ is almost periodic in m, where such an almost periodicity does not appear in other families. He also proved that the value of the murmuration function at 0 and ∞ agrees with the prediction from 1-level density conjecture (Theorem 3). The main ingredients of the proof are orthogonality of characters and summation of class numbers in short intervals with class number formula, similar as Zubrilina's approach.

7.2 Flying modular forms (in weight direction)

Recall that Zubrilina computed murmuration density for a *fixed weight k and varying level N*. In [2], Bober, Booker, M. Lee, and Lowry-Duda considered the opposite case, where they fix the level $N=1$ and vary the weight k. In this case, the considered family of Hecke newforms whose *analytic conductor*

$$\mathcal{N}(k) := \left(\frac{\exp\psi(k/2)}{2\pi}\right)^2 \approx \left(\frac{k-1}{4\pi}\right)^2 + O(1)$$

are in certain range, where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the digamma function.

Theorem 7.2 (Bober–Booker–Lee–Lowry-Duda [2, Theorem 1.1])**.** Fix $\epsilon\in(0,\frac{1}{12})$, $\delta\in\{0,1\}$, and a compact interval $E\subset\mathbb{R}_{>0}$ with $|E|>0$. Let $K,H>0$ with $K^{\frac{5}{6}+\epsilon}<H<K^{1-\epsilon}$, and let $N=\mathcal{N}(K)$. Then as $K\to\infty$, we have

$$\frac{\sum_{\substack{p \text{ prime} \\ p/N\in E}}\log p\sum_{\substack{k\equiv 2\delta \pmod 4 \\ |k-K|\le H}}\sum_{f\in H_k(1)}\lambda_f(p)}{\sum_{\substack{p \text{ prime} \\ p/N\in E}}\log p\sum_{\substack{k\equiv 2\delta \pmod 4 \\ |k-K|\le H}}\sum_{f\in H_k(1)}1} = \frac{(-1)^\delta}{\sqrt{N}}\left(\frac{\nu(E)}{|E|}+o_{E,\epsilon}(1)\right) \tag{32}$$

where

$$\nu(E) = \frac{1}{\zeta(2)}\sum_{\substack{a,q\in\mathbb{Z}_{>0} \\ (a,q)=1 \\ q^2/a^2\in E}}\frac{\mu(q)^2}{\varphi(q)^2\sigma(q)}\left(\frac{a}{q}\right)^{-3} \tag{33}$$

$$= \frac{1}{2} \sum_{t \in \mathbb{Z}} \prod_{p \nmid t} \frac{p^2 - p - 1}{p^2 - p} \int_E \cos\left(\frac{2\pi t}{\sqrt{y}}\right) \mathrm{d}y \tag{34}$$

where the summation $\sum^*$ indicates that the terms occuring at the endpoints of E are halved.

The main tool for the proof is the (original) Eichler–Selberg trace formula that does not include Atkin–Lehner operators (e.g. [4, Theorem 2.1]). Then apply class number formula to replace class numbers with the special values of Dirichlet L-functions at $s = 1$, which can be estimated under GRH.

7.3 Flying Maass forms

Booker, Lee, Lowry-Duda, Seymour-Howell, and Zubrilina computed murmuration densities for weight 0 and level 1 Maass forms [3]. They considered a family of Maass forms where the spectral parameter (R with $\lambda = \frac{1}{4} + R^2$) goes ot ∞, which is equivalent to the *analytic conductor* $\mathcal{N}(R)$ going to ∞.

Theorem 7.3 (Booker–Lee–Lowry-Duda–Seymour-Howell–Zubrilina [3, Theorem 1.1])**.** Let $E \subset \mathbb{R}_{>0}$ be a fixed compact interval with $|E| > 0$. Let $R, H > 0$ with $R^{\frac{5}{6}+\delta} < H < R^{1-\delta}$ for some $\delta > 0$ and $N = \mathcal{N}(R)$. Assumming GRH for L-functions of Dirichlet characters and Maass forms, as $R \to \infty$ we have

$$\frac{\sum_{\substack{p \text{ prime} \\ p/N \in E}} \log p \sum_{|r(f)-R| \le H} \epsilon(f) a_f(p)}{\sum_{\substack{p \text{ prime} \\ p/N \in E}} \log p \sum_{|r(f)-R| \le H} 1} \to \frac{1}{\sqrt{N}|E|} \sideset{}{^*}\sum_{\frac{q^2}{a^2} \in E} \frac{\mu(q)^2}{\varphi(q)^2 \sigma(q)} \left(\frac{a}{q}\right)^{-3} \tag{35}$$

where the summation $\sum^*$ indicates that the terms occuring at the endpoints of E are halved.

Proof uses an explicit Selberg trace formula due to Strömbergsson in his unpublished work [23], which requires an analytic test function and cannot be compactly supported, where GRH is needed to control the cutoff error term. The remaining proof is similar to the weight aspect case of the modular forms [2].

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