How automorphic forms and elliptic curves fly?

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Last updated: August 23, 2025

Abstract

This is an expository note on *murmurations*, which was initially discovered by He, Lee, Oliver, and Pozdnyakov for elliptic curves. We focus on the cases where the mumuration density is computed (under GRH), including the work of Zubrilina, Lee–Oliver–Pozdnyakov, and Sawin–Sutherland.

1 Introduction

2 Murmuration of Elliptic Curves

2.1 He–Lee–Oliver–Pozdnyakov's Murmuration

Murmuration of elliptic curves refers to the following average of Frobenius traces. Fix a nonnegative integer r and $N_1 < N_2$. Let $\mathcal{E}_r[N_1, N_2]$ be the set of isomorphism classes of elliptic curves $E_{/\mathbb{Q}}$ with conductor $N(E) \in [N_1, N_2]$ and rank r. For a fixed prime p, we consider the following average

$$\mathbb{E}_{E\in\mathcal{E}_r[N_1,N_2]}[a_p(E)] = \frac{\sum_{E\in\mathcal{E}_r[N_1,N_2]} a_p(E)}{\sum_{E\in\mathcal{E}_r[N_1,N_2]} 1} \tag{1}$$

as a function of p. What He, Lee, Oliver, and Pozdnyakov [9] observed is that this yields a surprising oscillation pattern as in Figure 1. Especially, it appears to have the same oscillation pattern for different conductor ranges, where the pattern seems to only depend on the rank r.

2.2 Sutherland's observation

Later, Sutherland [17] (and further works by several people) showed that one really needs to view the murmuration density as a function of p/N rather than p for a fixed N. He found that, for different dyadic intervals of the form $(2^k, 2^{k+1}]$, the murmuration patterns look the same (and become clearer as k increases), even if the averages consider completely different sets of elliptic curves (Figure 2). Also, instead of considering each rank separately, it seems better to consider all ranks together, where we weight $a_p(E)$ by the root number $\epsilon(E)$ of E. One can separate into two groups depending on the parity of the rank. So the major open question is to *compute* the density function, i.e. to find a function $M : (0, \infty) \to \mathbb{R}$ such that

$$\mathbb{E}_{E\in\mathcal{E}_r[N,2N]}[a_p(E)] = M\left(\frac{p}{N}\right) + \text{error} \tag{2}$$

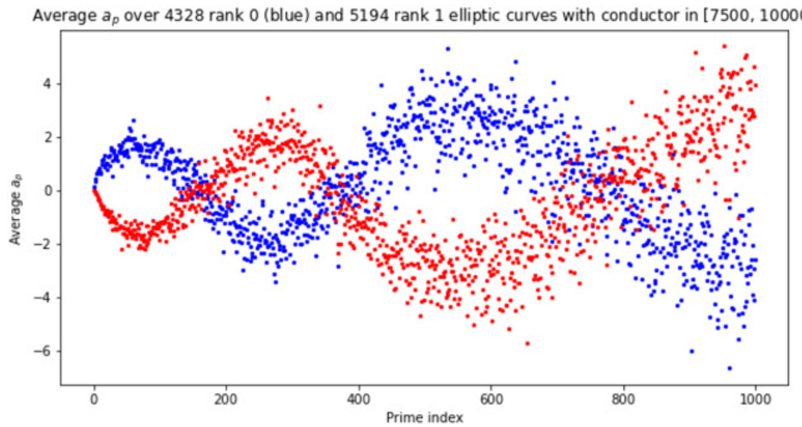

Figure 1: Murmuration of elliptic curves with conductor in $[7500, 1000]$ *and rank* $r = 0$ *(blue) and* $r = 1$ *(red) [9].*

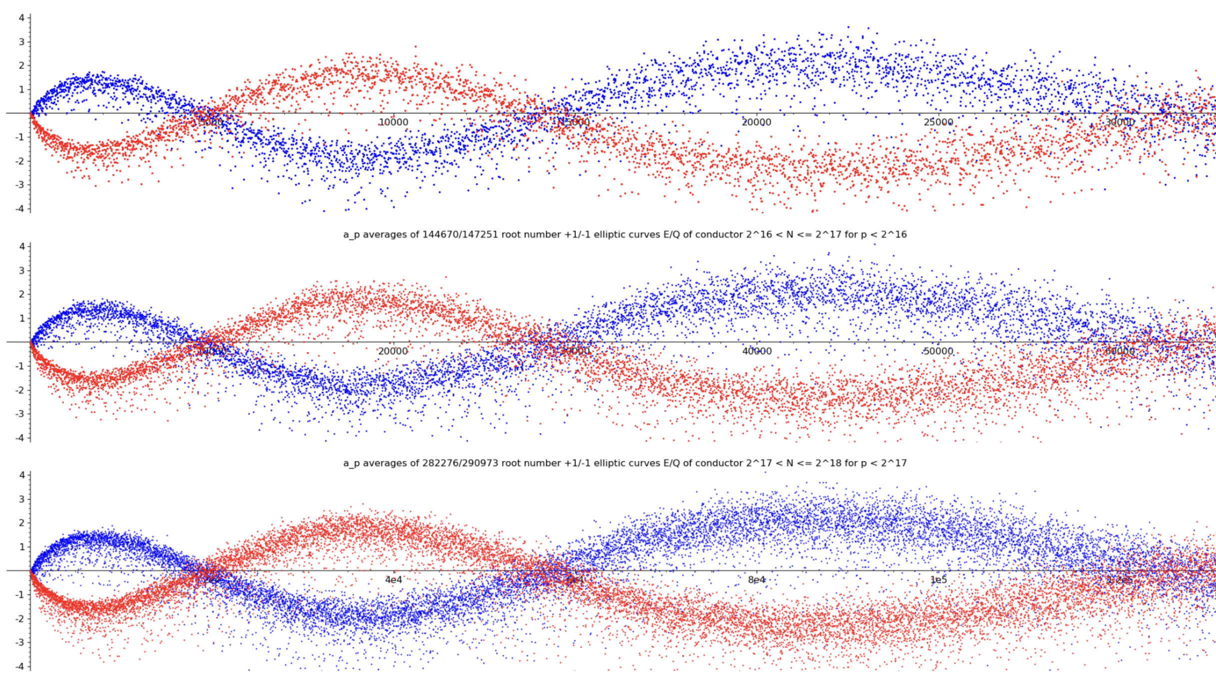

Figure 2: Murmuration of elliptic curves with conductor in $[2^k, 2^{k+1})$ *and primes* $p < 2^{k-1}$ *for* $k = 15, 16, 17$ *[17]. Blue (resp. red) curves correspond to* $\epsilon(E) = +1$ *(resp.* -1*) elliptic curves.*

where the error term goes to zero as $N \to \infty$. More generally, one can fix $0 < C_1 < C_2$ and consider the interval $[C_1 N, C_2 N]$.

Sutherland also observed that the murmuration disappears when elliptic curves are ordered by other measures, such as naive height, discriminant, or j-invariants, although further local averaging gives murmuration for naive heights (see Section 5). This shows that the murmuration is a phenomenon that is sensitive to the ordering of elliptic curves.

2.3 What is the role of Machine Learning?

Although there seems to be no machine learning involved in the previous discussions, I will make a brief comment on the relation between machine learning and murmuration, as I found that existing literature is often misleading in distinguishing the machine learning part from the murmuration part. I have read a few articles on the internet which basically say that "AI found new mathematics," which is false.

One of the main motivations of the paper [9] is to study elliptic curves via machine learning. Especially, they were interested in predicting the rank of elliptic curves (which is widely known to be hard to compute in general) by means of machine learning, where the coefficients $a_p(E)$ of Hasse–Weil L-functions are used as features. Surprisingly, they found that a simple logistic regression model can already distinguish between rank 0 and 1 elliptic curves with high accuracy of $> 90\%$ (see also [8]). Along these lines, they (more precisely, He, Lee, and Oliver) were curious about what was actually going on, and Pozdnyakov (who was an undergraduate student of Lee at that time) figured out the murmuration pattern. This somehow gives an explanation for the high accuracy of the model, since the murmuration patterns for rank 0 and 1 elliptic curves are noticeably different. But the correct way to say it is that the machine learning experiments *motivated* them to study what the models were doing, which is essentially the work of humans, not the ML models. You can find more of the story in the Quanta Magazine article [5].

2.4 Sato–Tate conjecture and Murmuration

One should not confuse murmuration with the (vertical) Sato–Tate conjecture, which we will explain here. The original (i.e. *horizontal*) Sato–Tate conjecture is about the distribution of $a_p(E)$ for a *fixed* $E_{/\mathbb{Q}}$ *and varying* p. The Hasse–Weil bound says that $|a_p(E)| \leq 2\sqrt{p}$, and the conjecture predicts that for a non-CM elliptic curve E, the distribution of $a_p(E)$ is semicircular with radius $2\sqrt{p}$, i.e., the density function is $\frac{1}{2\pi}\sqrt{4 - x^2}\mathrm{d}x$ for the normalized traces $a_p(E)/\sqrt{p}$. Equivalently, if we write $a_p(E) = 2\sqrt{p}\cos\theta_p$ for $\theta_p \in [0, \pi]$, then θ_p follows the distribution $\frac{2}{\pi}\sin^2\theta \mathrm{d}\theta$. The distributions for CM elliptic curves are different, and we also expect that the Frobenius traces for abelian varieties of higher dimension will follow certain distributions, which are conjecturally the pushforward of the Haar measure of a certain compact Lie group, called the *Sato–Tate group*. See [19] for more about the Sato–Tate conjecture and recent progress on it.

The *vertical* Sato–Tate conjecture fixes p and varies E over $\mathbb{F}_p$ instead, where there are only finitely many isomorphism classes of E over $\mathbb{F}_p$. Birch [1] proved that the distribution converges to the above semicircular distribution as $p \to \infty$. This is different from the murmuration for two reasons: vertical Sato–Tate considers the elliptic curves over $\mathbb{F}_p$, and there's no conductor involved in vertical Sato–Tate.

3 Murmuration of Dirichlet Series

Although the original murmuration density for elliptic curves is still unknown, there are few works where murmuration still exists and even computed (under GRH). Historically, the first such example is the work of Zubrilina for the modular forms [21], but we will start with the simplest case of Dirichlet characters. Lee, Oliver, and Pozdnyakov computed murmuration density for Dirichlet characters [10][1]. For complex characters, the corresponding murmuration densities are given by the following theorem.

Theorem 3.1 (Lee–Oliver–Pozdnyakov [10, Theorem 1.1])**.** Let $\mathcal{D}_+(N)$ (resp. $\mathcal{D}_-(N)$) denote the set of primitive even (resp. odd) Dirichlet characters modulo N. For $x \in \mathbb{R}_{>0}$, let $\lceil x \rceil^{\mathfrak{p}}$ be the smallest prime $\geq x$. For $c > 1$, $\delta > 0$, and $y > 0$, define

$$P_\pm(y, X, c) := \frac{\log X}{X} \sum_{\substack{N \in [X, cX] \\ N \text{ prime}}} \sum_{\chi \in \mathcal{D}_\pm(N)} \frac{\chi(\lceil yX \rceil^{\mathfrak{p}})}{\tau(\chi)}$$

$$P_\pm(y, X, \delta) := \frac{\log X}{X^\delta} \sum_{\substack{N \in [X, X+X^\delta] \\ N \text{ prime}}} \sum_{\chi \in \mathcal{D}_\pm(N)} \frac{\chi(\lceil yX \rceil^{\mathfrak{p}})}{\tau(\chi)}.$$

Then

$$\lim_{X \to \infty} P_\pm(y, X, c) = \begin{cases} \int_1^c \cos\left(\frac{2\pi y}{x}\right) \mathrm{d}x & \text{if } + \\ -i \int_1^c \sin\left(\frac{2\pi y}{x}\right) \mathrm{d}x & \text{if } - \end{cases} \tag{3}$$

and assumming RH, if $\frac{1}{2} < \delta < 1$ we have

$$\lim_{X \to \infty} P_\pm(y, X, \delta) = \begin{cases} \cos(2\pi y) & \text{if } + \\ -i \sin(2\pi y) & \text{if } - \end{cases} \tag{4}$$

See Figure 3 for the plot of the above murmuration densities. As you can see, there are two version of murmurations: *long interval* $[X, cX]$ and *short interval* $[X, X + X^\delta]$. Note that one needs to assume RH to get the short interval version, to guarantee the existence of primes in short intervals. The summand $\chi(p)/\tau(\chi)$ is the p-th Fourier coefficient of $\overline{\chi}$ when expanded in terms of additive characters, which justifies the normalization. Also, the above averages only consider prime moduli, where the authors also studied the case of composite moduli in [10, Section 6.1].

The proof of Theorem 3.1 is much simpler than the case of modular forms (Section 4). The main ingredient of the proof is the following formulas [10, Lemma 2.6]: for two distinct primes p and N,

$$\sum_{\chi \in \mathcal{D}_+(N)} \frac{\chi(p)}{\tau(\chi)} = \left(\frac{N-1}{N}\right) \cos\left(\frac{2\pi p}{N}\right) + \frac{1}{N} \tag{5}$$

$$\sum_{\chi \in \mathcal{D}_-(N)} \frac{\chi(p)}{\tau(\chi)} = -i \left(\frac{N-1}{N}\right) \sin\left(\frac{2\pi p}{N}\right) \tag{6}$$

Combined with the prime number theorem (which gives equidistribution results of primes in $[X, cX]$ normalized by X), we get (3), and assuming RH gives (4).

[1] which can be thought as automorphic forms on GL_1 over $\mathbb{Q}$.

They also proved similar results for real Dirichlet characters, but the proof is more complicated. Let $\mathcal{G}$ be the set of odd square-free integers and let $\chi_d = \left(\frac{d}{\cdot}\right)$. For a compactly supported smooth function $\Phi \geq 0$ on $\mathbb{R}$, define

$$M_\Phi(y, X, \delta) = \frac{\log X}{X^{1+\delta}} \sum_{\substack{p \in [yX, yX+X^\delta] \\ p \text{ prime}}} \sum_{d \in \mathcal{G}} \Phi\left(\frac{d}{X}\right) \chi_{8d}(p)\sqrt{p}. \tag{7}$$

Theorem 3.2 (Lee–Oliver–Pozdnyakov [10, Theorem 1.2])**.** Fix $y > 0$ and assume $\frac{3}{4} < \delta < 1$. Assumming GRH, we have

$$M_\Phi(y, \delta) := \lim_{X \to \infty} M_\Phi(y, X, \delta) = \frac{1}{2} \sum_{\substack{a \geq 1 \\ a \text{ odd}}} \frac{\mu(a)}{a^2} \sum_{m \geq 1} (-1)^m \widetilde{\Phi}\left(\frac{m^2}{2a^2 y}\right), \tag{8}$$

where

$$\widetilde{\Phi}(\xi) = \int_{-\infty}^{\infty} (\cos(2\pi\xi x) + \sin(2\pi\xi x))\Phi(x)\mathrm{d}x. \tag{9}$$

The proof is more involved, which is based on the Polya–Vinogradov inequality

$$\left| \sum_{\substack{p \in [yX, yX+X^\delta] \\ p \text{ prime}}} \chi_d(p) \right| \ll (yX)^{\frac{1}{2}+\epsilon}$$

(for non-principal χ_d with $\frac{1}{2} < \delta < 1$, which uses GRH [7]) and a summation formula

$$\frac{1}{X} \sum_{\substack{d \in \mathbb{Z} \\ d \text{ odd}}} \left(\sum_{\substack{a^2 || d| \\ a \leq A}} \mu(a) \right) \Phi\left(\frac{d}{X}\right) \left(\frac{d}{p}\right) \sqrt{p} = \frac{1}{2} \left(\frac{2}{p}\right) \sum_{\substack{0 < a \leq A \\ (a, 2p) = 1}} \frac{\mu(a)}{a^2} \sum_{k \in \mathbb{Z}} (-1)^k \left(\frac{k}{p}\right) \widetilde{\Phi}\left(\frac{kX}{2a^2 p}\right)$$

which can be proved by using Poisson summation formula.

4 Murmuration of Modular Forms

In this section, we sketch Zubrilina's computation of murmuration density for modular forms.

4.1 Statement

Before we state the result, we define some notations first.

- For $n \in \mathbb{Z}_{\geq 0}$, *Chebyshev polynomial of the second kind* is defined as

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

- For $r \in \mathbb{Z}_{\geq 1}$, define

$$\nu(r) := \prod_{p | r} \left(1 + \frac{p^2}{p^4 - 2p^2 - p + 1}\right)$$

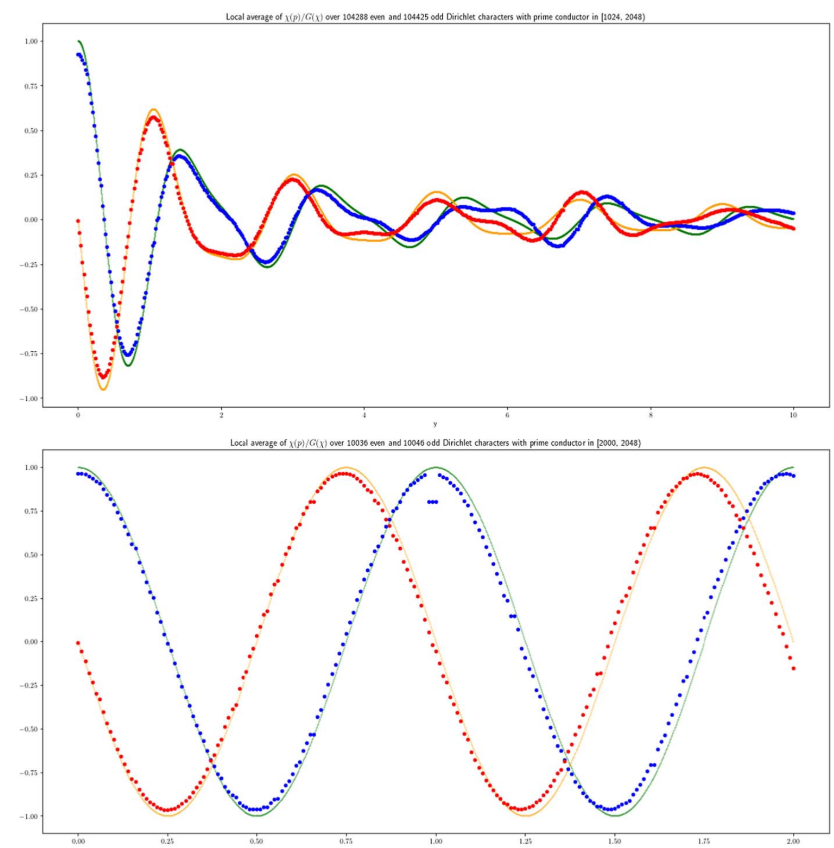

Figure 3: Murmuration of Dirichlet characters. The top figure presents $P_{\pm}(y, 2^{10}, 2)$ for $y \in [0, 10]$ with + in blue and (imaginary part of) − in red. The bottom figure presents $\widetilde{P}_{\pm}(y, 2002, 0.51)$ for $y \in [0, 2]$ with + in blue and (imaginary part of) − in red.

- Define constants α, β, γ as

$$\alpha := 2\pi \prod_p \frac{p^4 - 2p^2 - p + 1}{p^4 - 2p^2 + p},$$
$$\beta := 2\pi \prod_p \frac{p^3 + p^2 - 1}{p(p^2 + p - 1)},$$
$$\gamma := 12 \prod_p \frac{p(p+1)}{p^2 + p - 1}.$$

Theorem 4.1 (Zubrilina [21])**.** Let X, Y, P be parameters going infinite with $X, Y > 0$ and P prime; assume further that $Y = (1 + o(1))X^{1-\delta_2}$ and $P \ll X^{1+\delta_1}$ for some δ_1, δ_2 with $2\delta_1 < \delta_2 < 1$. Let $y = P/X$. Then

$$\frac{\sum^{\square}_{N\in[X,X+Y]} \sum_{f\in H^{\mathrm{new}}(N,k)} \sqrt{P}\lambda_f(P)\varepsilon(f)}{\sum^{\square}_{N\in[X,X+Y]} \sum_{f\in H^{\mathrm{new}}(N,k)} 1} = \mathcal{M}_k(y) + O_\varepsilon\left(X^{-\delta'+\varepsilon} + \frac{1}{P}\right) \tag{10}$$

where

$$\mathcal{M}_k(y) = \frac{\alpha(-1)^{k/2-1}}{k-1} \sum_{1\le r\le 2\sqrt{y}} \nu(r)\sqrt{4y - r^2}\,U_{k-2}\left(\frac{r}{2\sqrt{y}}\right) + \frac{\beta}{k-1}\sqrt{y} - \gamma\delta_{k=2}\,y. \tag{11}$$

4.2 Eichler–Selberg trace formula

To prove Theorem 4.1, one need to understand how to estimate the numerator on the LHS. Recall that $a_f(P) = P^{(k-1)/2}\lambda_f(P)$ is the P-th Fourier coefficient of f, which is also the eigenvalue of the Hecke operator T_P acting on f. Also, $(-1)^{k/2}\varepsilon(f)$ is equal to the eigenvalue of the Atkin–Lehner involution $W_N = T_N$ acting on f. Thus the sum appears in the numerator of LHS of (10) can be interpreted as the trace of the operator $(-1)^{k/2}T_P \circ W_N$ acting on the space of cusp forms of weight k and level N (multiplied by $P^{1-k/2}$). Eichler [6] studied such a sum of traces and proved that it can be expressed in terms of (Hurwitz) class numbers, which is generalized by Selberg [14]. To account the root number $\varepsilon(f)$, i.e. eigenvalue of W_N, we used the following version of Eichler–Selberg trace formula by Skoruppa and Zagier [15].

Theorem 4.2 (Skoruppa–Zagier [15])**.** For square-free N and prime $P \nmid N$,

$$\begin{aligned}
&\sum_{f\in H^{\mathrm{new}}(N,k)} \sqrt{P}\lambda_f(P)\varepsilon(f) \\
&= \frac{H_1(-4PN)}{2} + (-1)^{k/2-1} U_{k-2}\left(\frac{r\sqrt{N}}{2\sqrt{P}}\right) \sum_{0<r\le 2\sqrt{P/N}} H_1(r^2N^2 - 4PN) \\
&\quad - \delta_{k-2}(P+1)
\end{aligned}$$

Here $H_1(-d)$ ($d > 0$) is the Hurwitz class number, the number of equivalence classes of positive definite binary quadratic forms of discriminant $-d$ weighted by the number of automorphisms, i.e. with forms correspond to $x^2 + y^2$ or $x^2 + xy + y^2$ counted with multiplicity $1/2$ and $1/3$ respectively.

Hurwitz class number can be expressed as a sum of usual class numbers as

$$H_1(-d) = \sum_{f^2|d} h(-d/f^2) + O(1)$$

where the "error term" $O(1)$ disappears if $d \neq 3 \cdot \square$ or $4 \cdot \square$. Using this, we can rewrite the Skoruppa–Zagier trace formula as

$$\begin{aligned}
&\sum_{f \in H^{\mathrm{new}}(k,N)} \sqrt{P}\lambda_f(P)\varepsilon(f) \\
&= \frac{h(-4PN)}{2} + \frac{h(-PN)}{2} - \delta_{k=2}P + O(1) \\
&+ (-1)^{k/2-1}U_{k-2}\left(\frac{r\sqrt{N}}{2\sqrt{P}}\right) \sum_{1 \le r \le 2\sqrt{P/N}} \sum_{d^2 | r^2N-4P} h\left(\frac{N(r^2N-4P)}{d^2}\right)
\end{aligned}$$

From this, our new goal is to estimage the average of class numbers over short intervals, i.e. when $N \in [X, X+Y]$ with $Y = o(X)$. The main idea is to use class number formula to write class numbers as special L-values at $s = 1$, e.g.

$$h(-d) = \frac{\sqrt{d}}{\pi} L(1, \chi_d)$$

when $d > 4$ and $-d \equiv 1 \pmod 4$. Then the sum (average) of the corresponding L-values can be estimated via truncation and Polya–Vinogradov inequality. For example, we have an estimate

$$L(1, \chi_d) = \sum_{n \ge 1} \frac{\chi_d(n)}{n} = \sum_{1 \le n \le T} \frac{\chi_d(n)}{n} + O\left(\frac{\sqrt{d}\log d}{T}\right).$$

With some hard analysis, one get the following estimations.

Proposition 4.3 (Zubrilina [21, Proposition 3.1])**.** Let P be an odd prime and let $[X, X+Y]$ be an interval with $Y = o(X)$. Then as $X \to \infty$,

$$\begin{aligned}
&\frac{\zeta(2)\pi}{XY} \sum_{\substack{N \in [X,X+Y] \\ P \nmid N}}^{\square} \left(\frac{h(-PN)}{2} + \frac{h(-4PN)}{2}\right) \\
&= A\sqrt{y} + O_\varepsilon\left(\frac{1}{P^{3/2}X^{1/2}} + \frac{P^{7/12}}{Y^{5/6}X^{5/12}} + \frac{YP^{1/2}}{X^{3/2}}\right)(XYP)^\varepsilon
\end{aligned}$$

Proposition 4.4 (Zubrilina [21, Proposition 3.2])**.** Let P be an odd prime, $r \in \mathbb{N}$, and $X > Y > 0$ be such that $r^2(X+Y) < 4P$ for each $r > 2\sqrt{P/X}$. Let $y = P/X$. Then

$$\begin{aligned}
&\frac{\zeta(2)\pi}{XY} \sum_{r \le 2\sqrt{P/X}} \sum_{\substack{N \in [X,X+Y] \\ P \nmid N}}^{\square} H_1\left(r^2N^2 - 4PN\right) \\
&= \sum_{r \le 2\sqrt{P/X}} \nu(r)\sqrt{4y - r^2} \\
&+ O\left(\frac{P^{11/10}}{Y^{2/5}X^{9/10}} + \frac{YP}{X^2} + \frac{PY^{1/2}}{X^{3/2}} + \frac{P}{X^{1/2}Y^{13/18}} + \frac{P}{XY^{1/9}}\right)(XYP)^\varepsilon
\end{aligned}$$

5 Murmuration of Elliptic Curves, Revisited

Recently, Will Sawin and Andrew Sutherland announced a murmuration theorem for elliptic curves, which is slightly different from the formulation in [9].

Theorem 5.1 (Sawin–Sutherland [13]). Let $\mathcal{E}(X) := \{y^2 = x^3+ax+b : a, b \in \mathbb{Z}, p^4 \mid a \Rightarrow p^6 \nmid b, \max\{4|a|^3, 27b^2\} \leq X\}$ be the set of naive isomorphism classes of elliptic curves over $\mathbb{Q}$ ordered by height. Let $0 < C_1 < C_2$ be real numbers. For any smooth function $W : \mathbb{R}_{>0} \to \mathbb{R}$ with compact support, the limit

$$\lim_{P\to\infty} \lim_{X\to\infty} \mathbb{E}_{E:H(E)\leq X} \left[\frac{\prod_{p\leq P}(1-p^{-1})^{-1}}{N(E)} \sum_{\substack{n\geq 1 \\ p\nmid n\, \forall p\leq P}} W\left(\frac{n}{N(E)}\right) a_n(E)\epsilon(E) \right] \tag{12}$$

exists and is equal to

$$\int_0^\infty W(u)\sqrt{u} \left(2\pi \sum_{q\geq 1}{}^{\square} \sum_{m\geq 1} \frac{\mu(\gcd(m,q))}{qm\phi\left(\frac{q}{\gcd(m,q)}\right)} J_1\left(\frac{4\pi\sqrt{u}m}{q}\right) \prod_{p\mid q} \hat{\ell}_{p,2v_p(m)} \prod_{p\mid m, p\nmid q} \ell_{p,2v_p(m)} \right) \mathrm{d}u \tag{13}$$

where $\ell_{p,v}$ and $\hat{\ell}_{p,v}$ are certain local factors that can be written in terms of traces of the Hecke operator T_p (see [13, Lemma 3, 4]).

The difference between the original murmuration observed in HLOP [9] and the one in Sawin–Sutherland is well-explained in [13, Section 1.1]. The original murmuration considered the averages of the form

$$\mathbb{E}_{\substack{N(E)\in[X,X+1000] \\ \mathrm{rank}(E)=r}}[a_p(E)]$$

as a function in p for fixed r. However, subsequent works found that the dyadic intervals like $[X, 2X]$ or slightly smaller intervals like $[X, X + X^{1-\delta}]$ for $\delta > 0$ are more appropriate, since it make analysis more tractable and plots smoother. Hence the reformulated HLOP's murmuration would be

$$\mathbb{E}_{\substack{N(E)\in[X,2X] \\ \mathrm{rank}(E)=r}}[a_p(E)] \tag{14}$$

Also, later study found that the oscillations would converge to a continuous function in p/X, so we can understand (14) as (1) the limit of $X \to \infty$ with fixed p/X value, or (2) the limit of the average over p with p/X lies in a fixed interval.

Another subsequent observation is that considering all elliptic curves with different ranks would be better to study, where we weight $a_p(E)$ by the ϵ factor of E. Also, rather than p/X, the crucial ratio might be $p/N(E)$. In other words, we can consider further averaging over p where p/N lies in a certain interval, such as

$$\mathbb{E}_{N(E)\in[X,2X]} \left[\mathbb{E}_{p\in(C_1N(E),C_2N(E))}[\epsilon(E)a_p(E)]\right]$$

for $0 < C_1 < C_2$. Note that it is slightly easier to work with

$$\mathbb{E}_{N(E)\in[X,2X]} \left[\frac{\log\left(N(E)\frac{C_2-C_1}{2}\right)}{N(E)} \sum_{p\in(C_1N(E),C_2N(E))} \epsilon(E)a_p(E) \right]$$

instead of the previous double expectation, where the term $N/\log(N\frac{C_2-C_1}{2})$ roughly counts the number of primes in the interval (C_1N, C_2N). What Sawin and Sutherland proved is a naive height variation of the above average.

The main idea of the proof of Theorem 5.1 is the Voronoi summation formula.

Theorem 5.2 ([13, Lemma 11]). Let $E_{/\mathbb{Q}}$ be an elliptic curves, q be a positive integer, a a positive integer coprime to q, and $W : (0, \infty) \to \mathbb{R}$ a smooth function with compact support. Then

$$\epsilon(E) \sum_{n \geq 1} \frac{a_n(E)}{\sqrt{n}} W\left(\frac{n}{N(E)}\right) e\left(\frac{an}{q}\right) = \frac{\sqrt{N(E)}}{q} \sum_{n \geq 1} \frac{a_n(E)}{\sqrt{n}} e\left(\frac{\overline{aN(E)}n}{q}\right) \int_0^\infty 2\pi W(u) J_1\left(\frac{4\pi\sqrt{u}n}{q}\right) \mathrm{d}u \tag{15}$$

where $e(x) = e^{2\pi i x}$ and $\overline{aN(E)}$ is the multiplicative inverse of $aN(E)$ modulo q.

Note that summation of n instead over primes is built-in inside the formula. Based on the theorem, they also conjectured that:

Conjecture 5.3 ([13, Conjecture 1]).

$$\begin{aligned} &\lim_{X \to \infty} \mathbb{E}_{H(E) \leq X}\left[\frac{\log\left(N(E)^{\frac{C_1+C_2}{2}}\right)}{N(E)} \sum_{p \in (C_1 N(E), C_2 N(E))} \epsilon(E) a_p(E)\right] \\ &= \int_{C_1}^{C_2} 2\pi\sqrt{u} \sum_q \sum_{m \in \mathbb{N}} \frac{\mu(\gcd(m,q))}{qm\phi\left(\frac{q}{\gcd(m,q)}\right)} J_1\left(\frac{4\pi\sqrt{u}m}{q}\right) \prod_{p \mid q} \hat{\ell}_{p,2v_p(m)} \prod_{p \mid m, p \nmid q} \ell_{p,2v_p(m)} \mathrm{d}u \end{aligned}$$

The main two differences between the conjecture and Theorem 5.1 are that (1) the summation is over primes and (2) the (smooth, compactly supported) weight function W is replaced by the characteristic function of the interval (C_1, C_2). Heuristics like Crámer's random model suggests that these changes do not affect the density function.

You can find more on the Sutherland's lecture [18] at Tate conference (*The legacy of John Tate, and beyond* at Harvard university). He considered it as *a* murmuration theorem, and might not be *the* murmuration theorem since the density formula is too complicated.

6 Other known cases

After the success of Zubrilina, a lot of people are interested in murmuration density for different objects in number theory. We list the known works here.

6.1 General formulation

In fact, all the above works fit into the general framework suggested by Sarnak, in his letter to Sutherland and Zubrilina [12].

[11]

6.2 Flying Hecke characters of imaginary quadratic fields

Wang [20] computed murmuration density for Hecke characters of imaginary quadratic fields.

Theorem 6.1 (Wang [20]). Let $\mathcal{F}$ be the family of nontrivial Hecke characters of $\mathbb{Q}(\sqrt{-D})$ for square-free $D > 3$, $D \equiv 3 \pmod 4$. Then the average of normalized trace $\lambda_f(p) = a_f(p)\sqrt{p}$ over $f \in \mathcal{F}$ with $N_f \in [X, X+Y]$ is

$$\frac{\sum_{\substack{f\in\mathcal{F}\\ N_f\in[X,X+Y]}} \lambda_f(p)}{\sum_{\substack{f\in\mathcal{F}\\ N_f\in[X,X+Y]}} 1} = c(p) \sum_{1\le m\le 2\sqrt{y}} \delta_m(p) M_m(y) + M_-(y) + \text{error} \tag{16}$$

where

$$\begin{aligned}
M_m(y) &= \frac{11\zeta(2)}{4A}\sqrt{\frac{y}{4y-m^2}}\vartheta(m)\\
M_-(y) &= -\frac{11\pi}{A}\sqrt{y}\\
c(p) &= \frac{p+1}{3p}\prod_{\ell>2,\left(\frac{p}{\ell}\right)=1}\left(1-2\ell^{-2}-\frac{2\ell^{-3}}{1-\ell^{-2}}\right)\\
\delta_m(p) &= \begin{cases} \mathbb{1}_{\left(\frac{p}{q}\right)=1} & m = q^k,\ q \text{ is odd prime}\\ \mathbb{1}_{p\equiv 3 \pmod 4} & m = 2\\ \mathbb{1}_{p\equiv 5 \pmod 4} & m = 4\\ \mathbb{1}_{p\equiv 1 \pmod 8} & m = 2^\nu,\ \nu \ge 3 \end{cases}
\end{aligned}$$

See [20, Theorem 1] for the missing definitions. Note that the main term of (16) depends on the arithmetic of p, so it is not a murmuration in the sense of [12]. However, the dependence on p is explicit and $c(p)\delta_m(p)$ is almost periodic in m, where such an almost periodicity does not appear in other families. He also proved that the value of the murmuration function at 0 and ∞ agrees with the prediction from 1-level density conjecture (Theorem 3). The main ingredients of the proof are orthogonality of characters and summation of class numbers in short intervals with class number formula, similar as Zubrilina's approach.

6.3 Flying modular forms (in weight direction)

Recall that Zubrilina computed murmuration density for a *fixed weight k and varying level N*. In [2], Bober, Booker, M. Lee, and Lowry-Duda considered the opposite case, where they fix the level $N = 1$ and vary the weight k. In this case, the considered family of Hecke newforms whose *analytic conductor*

$$\mathcal{N}(k) := \left(\frac{\exp\psi(k/2)}{2\pi}\right)^2 \approx \left(\frac{k-1}{4\pi}\right)^2 + O(1)$$

are in certain range, where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

Theorem 6.2 (Bober–Booker–Lee–Lowry-Duda [2, Theorem 1.1]). Fix $\epsilon \in (0, \frac{1}{12})$, $\delta \in \{0, 1\}$, and a compact interval $E \subset \mathbb{R}_{>0}$ with $|E| > 0$. Let $K, H > 0$ with $K^{\frac{5}{6}+\epsilon} < H < K^{1-\epsilon}$, and let $N = \mathcal{N}(K)$. Then as $K \to \infty$, we have

$$\frac{\sum_{\substack{p \text{ prime}\\ p/N\in E}} \log p \sum_{\substack{k\equiv 2\delta \pmod 4\\ |k-K|\le H}} \sum_{f\in H_k(1)} \lambda_f(p)}{\sum_{\substack{p \text{ prime}\\ p/N\in E}} \log p \sum_{\substack{k\equiv 2\delta \pmod 4\\ |k-K|\le H}} \sum_{f\in H_k(1)} 1} = \frac{(-1)^\delta}{\sqrt{N}}\left(\frac{\nu(E)}{|E|} + o_{E,\epsilon}(1)\right) \tag{17}$$

where

$$\nu(E) = \frac{1}{\zeta(2)} \sum_{\substack{a,q\in\mathbb{Z}_{>0} \\ (a,q)=1 \\ q^2/a^2\in E}} \frac{\mu(q)^2}{\varphi(q)^2\sigma(q)} \left(\frac{a}{q}\right)^{-3} \tag{18}$$

$$= \frac{1}{2} \sum_{t\in\mathbb{Z}} \prod_{p\nmid t} \frac{p^2-p-1}{p^2-p} \int_E \cos\left(\frac{2\pi t}{\sqrt{y}}\right) \mathrm{d}y \tag{19}$$

where the summation $\sum^*$ indicates that the terms occuring at the endpoints of E are halved.

The main tool for the proof is the (original) Eichler–Selberg trace formula that does not include Atkin–Lehner operators (e.g. [4, Theorem 2.1]). Then apply class number formula to replace class numbers with the special values of Dirichlet L-functions at $s = 1$, which can be estimated under GRH.

6.4 Flying Maass forms

Booker, Lee, Lowry-Duda, Seymour-Howell, and Zubrilina computed murmuration densities for weight 0 and level 1 Maass forms [3]. They considered a family of Maass forms where the spectral parameter (R with $\lambda = \frac{1}{4} + R^2$) goes ot ∞, which is equivalent to the *analytic conductor* $\mathcal{N}(R)$ going to ∞.

Theorem 6.3 (Booker–Lee–Lowry-Duda–Seymour-Howell–Zubrilina [3, Theorem 1.1])**.** Let $E \subset \mathbb{R}_{>0}$ be a fixed compact interval with $|E| > 0$. Let $R, H > 0$ with $R^{\frac{5}{6}+\delta} < H < R^{1-\delta}$ for some $\delta > 0$ and $N = \mathcal{N}(R)$. Assumming GRH for L-functions of Dirichlet characters and Maass forms, as $R \to \infty$ we have

$$\frac{\sum_{\substack{p \text{ prime} \\ p/N\in E}} \log p \sum_{|r(f)-R|\le H} \epsilon(f) a_f(p)}{\sum_{\substack{p \text{ prime} \\ p/N\in E}} \log p \sum_{|r(f)-R|\le H} 1} \to \frac{1}{\sqrt{N}|E|} \sum_{\frac{q^2}{a^2}\in E}^{*} \frac{\mu(q)^2}{\varphi(q)^2\sigma(q)} \left(\frac{a}{q}\right)^{-3} \tag{20}$$

where the summation $\sum^*$ indicates that the terms occuring at the endpoints of E are halved.

Proof uses an explicit Selberg trace formula due to Strömbergsson in his unpublished work [16], which requires an analytic test function and cannot be compactly supported, where GRH is needed to control the cutoff error term. The remaining proof is similar to the weight aspect case of the modular forms [2].

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