Modular forms on G_2

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Abstract

This is a survey note on the smallest exceptional group G_2 and modular forms on it. After reviewing the theory of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$, we introduce a parallel theory for the exceptional group G_2.

1 Introduction

The main goal of this note is to give evidences (not proof) of the following claim:

Claim. Modular forms on the exceptional group G_2 are as much interesting as classical modular forms.

While reading the note, you will find that many of the features of the classical modular forms also possessed by the modular forms on G_2, including:

- Fourier coefficients and expansions via cubic rings (Section 4.2 and 6.2)
- Local representation theory (Section 5)
- Eisenstein series and theta series (Section 6.4)
- Hecke operators (Section 6.3)

Gan–Gross–Savin [13] initiated the theory of G_2 modular forms, based on the structure theory [1], *quaternionic* discrete series [17], Hecke algebra [15], multiplicity one [32], etc. We start with covering the basic theory of automorphic forms on $\mathrm{GL}_{2,\mathbb{Q}}$ in Section 2. Section 3 covers the definition of (split) G_2 and its root system. We define the Heisenberg parabolic subgroup of G_2 in Section 4, along with its relation with the space of cubic rings. Archimedean and non-archimedean representation theory of G_2 is coverd in Section 5, which will be used to define *weights* of modular forms on G_2 in Section 6. We end the note by introducing related works that are not covered here (Section 7) for the readers who are interested in the topic.

2 Holomorphic modular forms and automorphic representations of GL_2

In this section, we recall the theory of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, and how to associate such an automorphic representation with a holomorphic modular form. The main references are Bump [6], Getz–Hahn [14], and Booher's note [4]. We assume that the readers are familiar with the theory of holomorphic modular forms - if not, the standard references for the classical theory are Serre [30], Diamond–Shurman [11], Zagier [34], and Bump [6, Chapter 1] again.

2.1 Adelizing holomorphic modular forms

Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic modular form of (even) weight k and level 1, defined on the complex upper-half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. We can upgrade f to a function φ_f on $\mathrm{GL}_2(\mathbb{A})$ as follows. By the strong approximation theorem, any $g \in \mathrm{GL}_2(\mathbb{A})$ can be written as $g = \gamma g_\infty g_{\mathrm{fin}}$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ (diagonally embedded), $g_\infty \in \mathrm{GL}_2^+(\mathbb{R})$, $g_{\mathrm{fin}} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) = \prod_{p<\infty} \mathrm{GL}_2(\mathbb{Z}_p)$. Then we define

$$\varphi_f(\gamma g_\infty g_{\mathrm{fin}}) = (f|_k g_\infty)(i) = (ad-bc)^{k/2}(ci+d)^{-k} f\left(\frac{ai+b}{ci+d}\right)$$

where $g_\infty = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$. Using the transformation law of f, one can show that φ_f is well-defined and becomes an *automorphic form on* $\mathrm{GL}_2(\mathbb{A})$: it is

- left $\mathrm{GL}_2(\mathbb{Q})$-invariant,
- right $K = K_\infty K_{\mathrm{fin}} = \mathrm{O}(2)\mathrm{GL}_2(\widehat{\mathbb{Z}})$-finite,
- has moderate growth,
- $Z = Z(\mathrm{GL}_2(\mathbb{A}))$-invariant,
- $\mathcal{Z} = \mathcal{Z}(\mathfrak{gl}_2(\mathbb{R}))$-finite,
- if f is a cusp form, then

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}x = 0$$

  for all $g \in \mathrm{GL}_2(\mathbb{A})$.

We denote the space of functions on $\mathrm{GL}_2(\mathbb{A})$ satisfying the above conditions (resp. including the last condition) as $\mathcal{A}(\mathrm{GL}_2)$ (resp. $\mathcal{A}_0(\mathrm{GL}_2)$). It admits an action of $(\mathfrak{gl}_2, O(2)) \times \mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$ via *differentiation and right translation*

$$((X, k, g_{\mathrm{fin}}), \varphi) \mapsto (x \mapsto X\varphi(xkg_{\mathrm{fin}})).$$

Now, we can associate a representation of $(\mathfrak{gl}_2, \mathrm{O}(2)) \times \mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$ with f by considering the space generated by the right translations of φ_f, denoted as $\pi = \pi_f$. The representation is irreducible if and only if f is a *Hecke eigenform*, and the corresponding π becomes an *automorphic representation*, i.e. irreducible and admissible subquotient of the space $\mathcal{A}(\mathrm{GL}_2)$ (with right translation). Especially, any irreducible admissible automorphic representation factors as a (restricted) tensor product of local representations $\pi \simeq \otimes'_{p\leq\infty}\pi_p$, proven by Flath. Theorem 2.1 tells you how these local components are related to the original modular form f.

Theorem 2.1. Let $f = \sum_{n\geq 0} a_n(f)q^n$ be an eigenform of weight k and level 1. Then the associated automorphic representation factors as $\pi = \otimes'_{p\leq\infty}\pi_p$ where

1. π_∞ is a discrete series of weight k.

2. π_p is an unramified principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ induced from (unramified) characters χ_1, χ_2 with $\chi_i(p) = \alpha_i / p^{\frac{k-1}{2}}$ satisfying

$$\alpha_1 + \alpha_2 = a_p(f), \quad \alpha_1\alpha_2 = p^{k-1}.$$

The definitions of *discrete series* and *unramified principal series* will be explained in the following sections 2.2 and 2.3, along with the classification of local representations.

If we consider modular forms of higher level or Maass wave forms, then we can get more interesting local representations (e.g. Steinberg or supercuspidal at finite places or other principal series at the archimedean place). See [22] for the computation of local representations at nonarchimedean places when they are supercuspidal.

2.2 Archimedean representation theory of GL_2

To study a representation of $\mathrm{GL}_2(\mathbb{R})$ or a general Lie group $G(\mathbb{R})$, we study the associated $(\mathfrak{g}, K)$-modules instead, which are more "algebraic" in nature. Here $\mathfrak{g}$ is (the complexification of) the Lie algebra of G, and K is a maximal compact subgroup of G. Then $(\mathfrak{g}, K)$-module is a vector space equipped with actions of $\mathfrak{g}$ and K which are compatible in a certain sense. A $(\mathfrak{g}, K)$-module V is *admissible* if $V(\sigma)$ (σ-isotypic part of V) is finite dimensional for any unitary representation σ of K. Now, for any Hilbert space representation π of $G(\mathbb{R})$, we can always associate a $(\mathfrak{g}, K)$-module (by differentiating and restricting the original representation), and it determines the original representation when π is unitary.

In case of $G = \mathrm{GL}_2$, we have a complete classification of irreducible admissible $(\mathfrak{gl}_2, \mathrm{O}(2))$-modules. Representations of $\mathfrak{gl}_2(\mathbb{C})$ is well-understood; especially, the center of the universal enveloping algebra $\mathcal{Z}(\mathfrak{gl}_2(\mathbb{C})) = \mathcal{Z}(\boldsymbol{\mathcal{U}}(\mathfrak{gl}_2(\mathbb{C})))$ is generated by the two elements $Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and Δ (Casimir operator), and it is enough to understand how these elements act (which are just constants since they live in the center). All irreducible representations of $\mathrm{O}(2)$ are either trivial, determinant, or $\tau_n = \mathrm{Ind}_{\mathrm{SO}(2)}^{\mathrm{O}(2)}(\epsilon_n)$ induced from 1-dimensional characters $\epsilon_n : \mathrm{SO}(2) \to \mathbb{S}^1, \kappa_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto e^{in\theta}$ for $n \geq 1$. Using this, we can obtain the following classification result.

Theorem 2.2. Irreducible admissible $(\mathfrak{gl}_2, O(2))$-modules is one of the following:

1. Principal series $\pi_{s,\mu,\varepsilon} = \mathrm{pInd}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi_1 \boxtimes \chi_2)$, with

$$\chi_i : \mathbb{R}^\times \to \mathbb{R}^\times, \quad y \mapsto \mathrm{sgn}(y)^{\varepsilon_i}|y|^{s_i}$$

   where $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ and $s_1, s_2 \in \mathbb{C}$ satisfy $\varepsilon \equiv \varepsilon_1 + \varepsilon_2 \pmod 2$, $\mu = s_1 + s_2, s = \frac{1}{2}(s_1 - s_2 + 1)$. Z (resp. Δ) acts as μ (resp. $\lambda = s(1-s)$).

2. Discrete series $\pi_{k,\mu}$ for $k \geq 2$.

Here pInd stands for *parabolic induction*. Discrete series appear discretely in the decomposition of the regular representation of $\mathrm{GL}_2(\mathbb{R})$ on $L^2(\mathrm{GL}_2(\mathbb{R}))$. These can be realized as a representation on a space of holomorphic functions (and they are often called *holomorphic* discrete series) of $\mathbb{H}$ with Petersson norm. Especially, they are unitarizable. As we see in Theorem 2.1, discrete series appear as archimedean component of the automorphic representation associated with a holomorphic modular form of weight k. Note that there is a *limit* of discrete series $\pi_{1,\mu}$, which is actually the principal series $\pi_{\frac{1}{2},\mu,\varepsilon}$.

For general G, we can still use "parabolic induction" technique by inducing (tempered) representations of Levi subgroups, and this gives all irreducible admissible representations by Langlands [14, Theorem 4.9.2].

2.3 Nonarchimedean representation theory of GL_2

As in the archimedean case, we are only interested in the special class of representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, which are *admissible* representations. A representation $\pi : G \to \mathrm{GL}(V)$ of G on a complex vector space V is *smooth* if the corresponding map $G \times V \to V$ is continuous where we endow V with the discrete topology. A representation is *admissible* if it is smooth and $\dim V^K < \infty$ for any compact open subgroup $K \leq G$. We have a classification of *admissible* representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$, which is somewhat similar to the classification of $(\mathfrak{g}, K)$-modules for $\mathrm{GL}_2(\mathbb{R})$:

Theorem 2.3. Irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is one of the following:

1. Principal series $\pi(\chi_1, \chi_2) = \mathrm{pInd}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1 \boxtimes \chi_2)$, for quasicharacters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ with $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$,
2. (Twisted) Steinberg representations,
3. 1-dimensional representations of the form $g \mapsto \chi(\det(g))$ for $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$,
4. Supercuspidal representations.

Among the above representations, we will focus on the principal series. Especially, when $p \nmid N$ and π_p is a local component of an automorphic representation associated to a level N modular form, then π_p is an *unramified* principal series, i.e. it has a nonzero $K = \mathrm{GL}_2(\mathbb{Z}_p)$-fixed vector (*spherical vector*). For such representations, one can "linearize" it as a representation of the *spherical Hecke algebra*

$$\mathcal{H}_p = \mathcal{H}_p(\mathrm{GL}_2) := C_c^\infty(K \backslash \mathrm{GL}_2(\mathbb{Q}_p)/K),$$

where the multiplication on $\mathcal{H}_p$ is given by the convolution, and $e_p := \frac{1}{|K|}\mathbb{1}_K$ is an identity. $\mathcal{H}_p$ acts on the space of K-fixed vectors π_p^K via

$$\pi_p(f)v = \int_{\mathrm{GL}_2(\mathbb{Q}_p)} f(g)\pi_p(g)v\,\mathrm{d}g.$$

One can show that $\mathcal{H}_p$ is commutative: in fact, Cartan decomposition allows us to understand the structure of $\mathcal{H}_p$ fairly well.

Theorem 2.4. $\mathcal{H}_p$ is generated by three characteristic functions:

$$T_p = \mathbb{1}_{K\begin{pmatrix} p & \\ & 1\end{pmatrix}K}$$

$$R_p = \mathbb{1}_{K\begin{pmatrix} p & \\ & p \end{pmatrix}K}$$
$$R_p^{-1} = \mathbb{1}_{K\begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}K}.$$

By the theorem, we have $\dim \pi_p^K = 1$ and the representation of $\mathcal{H}_p$ is just a character. In fact, unramified representations are completely determined by the associated characters of $\mathcal{H}_p$ and it is enough to understand the characters. Again, using Cartan decomposition, we can explicitly write down the structure of $\mathcal{H}_p$ in terms of characteristic functions of certain double cosets, and write down the action of these explicitly.

Theorem 2.5. Let π be an irreducible admissible unramified representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Then π is either 1-dimensional representation of the form $g \mapsto \chi(\det(g))$ for an unramified quasicharacter $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, or a principal series associated to unramified quasicharacters χ_1, χ_2. For the latter case, the associated Hecke character acts on a spherical vector $\phi^\circ \in \pi^K$ via

$$T_p\phi^\circ = p^{1/2}(\chi_1(p) + \chi_2(p))\phi^\circ, \quad R_p\phi^\circ = \chi_1(p)\chi_2(p)\phi^\circ.$$

Under the Satake isomorphism, spherical Hecke algebra is isomorphic to the representation ring of the Langlands dual group $\widehat{\mathrm{GL}_2} = \mathrm{GL}_2(\mathbb{C})$ [15]. The isomorphism $\mathcal{S} : \mathcal{H}_p \simeq R(\widehat{\mathrm{GL}_2})$ is given by

$$\mathcal{S}(T_p) = p^{1/2} \cdot \chi,$$
$$\mathcal{S}(R_p) = \det,$$
$$\mathcal{S}(R_p^{-1}) = \det{}^{-1}$$

where $\chi = \chi_{\mathrm{std}}$ is the character of the standard representation.

2.4 Fourier expansion and Whittaker model

Recall that holomorphic modular forms of level 1 are 1-periodic and admit a Fourier expansion of the form

$$f(z) = \sum_{n\geq 0} a_n(f)q^n = \sum_{n\geq 0} a_n(f)e^{2\pi i n z}$$

for $z \in \mathbb{H}$ and $q = e^{2\pi i z}$. One can upgrade f as a function on $\mathrm{GL}_2(\mathbb{R})^+ = \{g \in \mathrm{GL}_2(\mathbb{R}) : \det(g) > 0\}$ via

$$\phi_f(g) = (ad - bc)^{k/2}(ci + d)^{-k} f\left(\frac{ai + b}{ci + d}\right), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$$

and consider a "Fourier expansion" of this function as follows.

Let $\phi : \mathrm{GL}_2(\mathbb{R})^+ \to \mathbb{C}$ be an automorphic form on $\mathrm{GL}_2(\mathbb{R})$[1]. For a fixed $g \in \mathrm{GL}_2(\mathbb{R})^+$, consider the following function

$$\phi_g : \mathbb{R} \to \mathbb{C}, \quad x \mapsto \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right).$$

By the automorphy of ϕ, ϕ_g becomes a function on $\mathbb{Z}\backslash\mathbb{R} \simeq \mathbb{S}^1$. Hence ϕ_g admits a Fourier expansion

$$\phi_g(x) = \sum_{\psi \in \widehat{\mathbb{Z}\backslash\mathbb{R}}} W_{\phi,\psi}(g)\psi(x)$$

where

$$W_{\phi,\psi}(g) = \int_0^1 \phi_g(x)\overline{\psi(x)}\mathrm{d}x.$$

Here the sum is over all (unitary) characters of $\mathbb{Z}\backslash\mathbb{R}$. Any such characters are of the form $\psi_n(x) = e^{2\pi i n x}$, and by the moderate growth condition on ϕ, it gives

$$\phi_g(x) = \sum_{n \geq 0} a_n(\phi_g) e^{2\pi i n x}$$

where

$$a_n(\phi_g) = \int_0^1 \phi_g(x)\overline{\psi_n(x)}\mathrm{d}x = \int_0^1 \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) e^{-2\pi i n x}\mathrm{d}x$$

which recovers the usual Fourier expansion when $\phi = \phi_f$ is associated with a holomorphic modular form $f(z)$. Also, there is an adelic version of this (replace $\mathbb{Z}, \mathbb{R}, \mathrm{GL}_2(\mathbb{R})^+$ with $\mathbb{Q}, \mathbb{A}, \mathrm{GL}_2(\mathbb{A})$), but we will concentrate on the *archimedean* version for the latter purpose; Fourier expansions of the modular forms on $G_2(\mathbb{R})$ will be introduced in Section 6.2.

One can easily check that the function $W_{\phi,\psi} : \mathrm{GL}_2(\mathbb{R})^+ \to \mathbb{C}$ satisfy the equation

$$W_{\phi,\psi}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x) W_{\phi,\psi}(g)$$

for all $x \in \mathbb{R}$. Let $\mathcal{W}(\psi)$ be the space of functions in $\mathrm{GL}_2(\mathbb{R})^+$ satisfying the above equation and having moderate growth, where $\mathrm{GL}_2(\mathbb{R})^+$ acts as a right translation. One can understood a representation V of $\mathrm{GL}_2(\mathbb{R})^+$ or the associated $(\mathfrak{gl}_2, \mathrm{O}(2))$-modules through an embedding $V \hookrightarrow \mathcal{W}(\psi)$, i.e. via *Whittaker model*, when such ψ exists. Equivalently, we can consider the space of *Whittaker functionals*

$$\mathrm{Wh}(\pi, \psi) = \mathrm{Hom}_{N(\mathbb{R})}(\pi, \psi) = \{\ell : \pi \to \mathbb{C}, \ell(\pi(n)v) = \psi(n)\ell(v)\,\forall n \in N(\mathbb{R})\}$$

[1]Although I only explained the definition of $\mathrm{GL}_2(\mathbb{A})$ automorphic forms, we have a similar definition of automorphic forms on Lie groups. See Booher's note [4] for the case of $\mathrm{GL}_2(\mathbb{R})$.

for a representation π of $\mathrm{GL}_2(\mathbb{R})^+$, where $N(\mathbb{R}) = \{(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}) : x \in \mathbb{R}\}$ and $\psi((\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix})) = \psi(x)$. Then we have a *uniqueness* of a Whittaker model.

Theorem 2.6. Let π be an irreducible admissible $(\mathfrak{g}, K)$-module for $\mathrm{GL}_2(\mathbb{R})$. Then $\dim_{\mathbb{C}} \mathrm{Wh}(\pi, \psi) \le 1$. When the dimension is one, the corresponding Whittaker model can be given as a solution of a certain differential equation, which can be expressed as a Bessel function.

Note that local multiplicity one is true for $\mathrm{GL}_n(F)$ when $n \ge 2$ is arbitrary and F is a local field (both archimedean and non-archimedean). In fact, we also have a *global* uniqueness result, which implies (strong) multiplicity one result for GL_n (see [8, Chapter 4]).

3 G_2: definitions and properties

Let's move on to the modular forms on G_2. First, we need to understand about the group G_2. In short:

G_2 is an *automorphism group of octonions.*

You may have heard about the word octonion, but not the definition itself (I didn't know the actual definition until I read about G_2). First of all, over $\mathbb{R}$, we have *two* octonions: split one and non-split one. These two give two different G_2: split G_2 and non-split/compact G_2. You can consider them as an analogue of $\mathrm{M}_2(\mathbb{R})$ vs Hamilton's quaternion. In this note, we will only consider *split* octonions and *split* G_2, and modular forms on it. The main reference is Baez's article on octonions [1].

3.1 Split octonion

The split octonion, which we denote as $\mathbb{O}$, was first constructed by Cayley and Dickson. As a set, it is just two copies of Hamilton's $\mathbb{H}$. We define a multiplication of two pairs as:

$$(a, b) \cdot (c, d) = (ac + \bar{d}b, da + b\bar{c}).$$

$(1, 0)$ becomes an identity. This multiplication is not even associative; but still form a *composition algebra*: it admits a multiplicative quadratic norm. We first define a conjugate of an element as

$$(a, b)^* = (\bar{a}, -b)$$

and the norm is defined by $\mathrm{N}(x) = x^*x$. If we write the elements of $\mathbb{H}$ in a usual way, $a = a_0 + a_1 i + a_2 j + a_3 k$ and $b = b_0 + b_1 i + b_2 j + b_3 k$, then the norm $\mathrm{N}((a, b))$ becomes

$$\mathrm{N}((a, b)) = (a_0^2 + a_1^2 + a_2^2 + a_3^2) - (b_0^2 + b_1^2 + b_2^2 + b_3^2).$$

Especially, it gives a quadratic form of a signature $(4, 4)$. We can decompose an arbitrary element as a sum of the "real part" and the "imaginary part": $(a, b) = a_0(1, 0) + (a - a_0, b)$. Then the trace of an element becomes $\mathrm{Tr}((a, b)) = (a, b) + (a, b)^* = 2a_0(1, 0)$. We will denote the space of purely imaginary octonions (i.e. $a_0 = 0$) by $\mathfrak{I}(\mathbb{O})$, which is orthogonal to $\mathfrak{R}(\mathbb{O}) \simeq \mathbb{R}$. Note that we have a trilinear form on $\mathbb{O}$, given by

$$\mathbb{O} \times \mathbb{O} \times \mathbb{O} \to \mathbb{R}, \ (x, y, z) \mapsto \mathrm{Tr}((xy)z)$$

(we have $\mathrm{Tr}((xy)z) = \mathrm{Tr}(x(yz))$, even if the multiplication is not associative).

3.2 Definition and dimension

Now, let us get back to the definition. G_2 is an automorphism group of $\mathbb{O}$:

$$G_2 = \mathrm{Aut}(\mathbb{O}) = \{g \in \mathrm{GL}(\mathbb{O}) : g(x \cdot y) = (gx) \cdot (gy) \ \forall x, y \in \mathbb{O}\}.$$

Then any $g \in G_2$ also preserves conjugations, norms and inner products. Especially, we have an embedding $G_2 \hookrightarrow \mathrm{SO}(4, 4)$. We can do slightly better: $\mathfrak{I}(\mathbb{O})$ is stable under G_2, and we get $G_2 \hookrightarrow \mathrm{SO}(3, 4)$. Thus, we get a 7-dimensional faithful representation of G_2, which is in fact the smallest irreducible representation of G_2.

What is the dimension of G_2? Since it sits inside $\mathrm{SO}(3, 4)$, we have an upper bound $\dim \mathrm{SO}(3, 4) = \binom{8}{2} = 28$. In fact, the actual dimension is exactly half of it:

Proposition 3.1. $\dim G_2 = 14$.

Proof. Each element $g \in G_2$ is completely determined by its image of a *basic triple* $\{e_1, e_2, e_3\}$, that is, the set of orthonormal generators $\mathbb{O}$: $\mathrm{N}(e_1) = \mathrm{N}(e_2) = \mathrm{N}(e_3) = 1$, and $\langle e_i, e_j \rangle = 0$ for all $i \neq j$. They generate $\mathbb{O}$ in the sense that $\{1, e_1, e_2, e_3, e_1e_2, e_2e_3, e_3e_1, e_1e_2e_3\}$ becomes a $\mathbb{R}$-basis of $\mathbb{O}$. Now, consider how many "choices" we have for $e_1' = ge_1, e_2' = ge_2, e_3' = ge_3$:

1. e_1' is a purely imaginary element of norm 1, so it is a codimenion 1 space of $\mathfrak{I}(\mathbb{O}) \simeq \mathbb{R}^7$, hence dimension is 6.

2. e_2' is a norm 1 element in $\mathrm{Span}(1, e_1)^\perp \simeq \mathbb{R}^6$, hence the dimension is $6-1=5$.

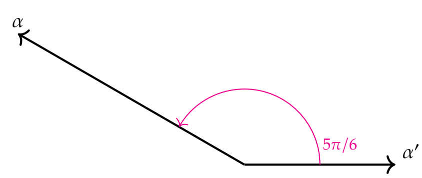
3. e_3' is a norm 1 element in $\mathrm{Span}(1, e_1, e_2, e_1e_2)^\perp \simeq \mathbb{R}^4$, hence the dimension is $4-1=3$.

Thus, we get $\dim G_2 = 6+5+3 = 14$. □

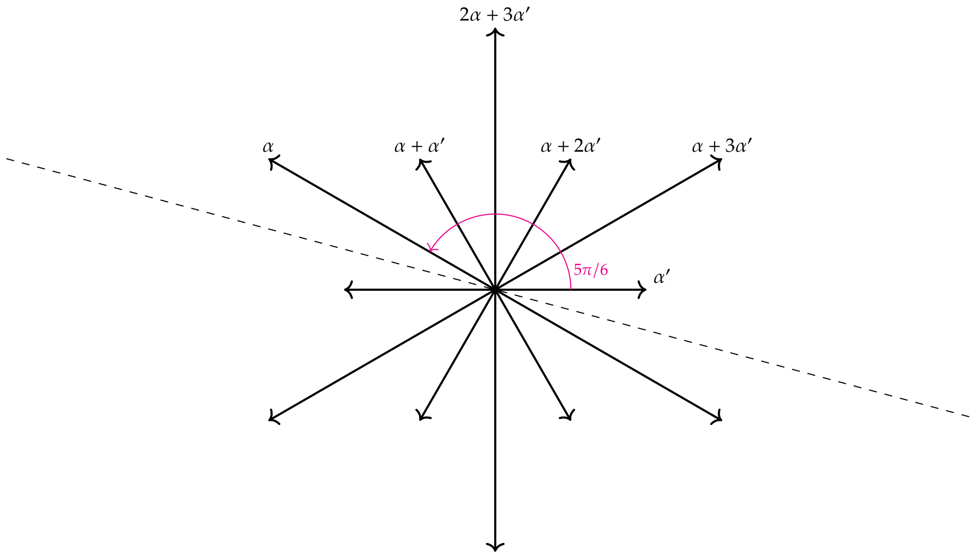
3.3 Lie algebra $\mathfrak{g}_2$

Let's move on to the semisimple Lie algebra $\mathfrak{g}_2 = \mathrm{Lie}(G_2)$. Instead of giving detailed proofs of the known facts on $\mathfrak{g}_2$, we will start with the Dynkin diagram of $\mathfrak{g}_2$ and see what we can say about $\mathfrak{g}_2$ from it. Here is the tiny little cute Dynkin diagram of $\mathfrak{g}_2$:

Now, let us see what this diagram tells us about $\mathfrak{g}_2$. First of all, we have two vertices, so the rank of $\mathfrak{g}_2$ is 2 and there are two simple roots. In addition, there are three lines between two vertices, which tell us the angle between the simple roots and the length ratio: the angle is $5\pi/6$, where the ratio between the lengths of the simple roots is $\sqrt{3}$. Let's call the longer one (resp. the shorter one) as α (resp. α').



By taking suitable $\mathbb{Z}$-linear combinations, we can build more roots out of these simple roots:



Recall that the only possible degrees between two root vectors are multiples of $\pi/6$, and no scaled vectors of given roots other than the vector and its opposite can appear as roots (reduced). The ones above the dashed line (i.e. ones with names) are the chosen positive roots. This gives the root space decomposition of $\mathfrak{g}_2$, where we can find the dimension of $\mathfrak{g}_2$ (hence Proposition 3.1) from this:

$$\dim \mathfrak{g}_2 = (\text{rank}) + (\text{number of roots}) = 2 + 12 = 14.$$

The highest root is $\beta_0 = 2\alpha+3\alpha'$ (literally the "highest" one in the above diagram), and the Weyl group of $\mathfrak{g}_2$ is the symmetric group of a regular hexagon, which is the dihedral group of order 12.

3.4 G_2 as a symmetry group of rolling balls

Here we introduce an another definition of G_2, as a symmetry group of *rolling balls*, discovered by Baez and Huerta [2]. This section is not necessary for the upcoming discussions on modular forms[2], but we include this because of its own interest.

We have a large ball of radius $R > 1$, and we are going to roll a unit ball around it, without slipping or twisting. Then the corresponding configuration space is $\mathbb{S}^2 \times \mathrm{SO}(3)$: $\mathbb{S}^2$ for the point of contact and $\mathrm{SO}(3)$ for the rotation of the small ball. Now, we define "lines" on the configuration space as paths along a great circle on the large ball, i.e. two points are connected if one can move the small ball from one state to the other state by rolling over a great circle (without slipping or twisting, of course).

[2]Maybe related? Who knows!

Now, we consider another (incidence) geometry, comes from our one and only[3] octonion $\mathbb{O}$. The "points" are 1-dimensional *null subalgebra* of $\mathfrak{I}(\mathbb{O})$: $p_x := \langle x\rangle \subset \mathfrak{I}(\mathbb{O})$ with $\mathrm{N}(x) = 0$. The "lines" are 2-dimensional null subalgebras spanned: i.e. two points $p_x = \langle x\rangle$ and $p_y = \langle y\rangle$ are connected if and only if $xy = 0 = yx$. Then the group G_2 acts naturally on this space, which we denote as

$$\mathbb{P}C = \{\langle x\rangle : 0 \neq x \in \mathfrak{I}(\mathbb{O}), \mathrm{N}(x) = 0\}.$$

One can check that this space is isomorphic to

$$\frac{\mathbb{S}^2 \times \mathbb{S}^3}{(a,b) \sim (-a,-b)} \simeq \mathbb{RP}^2 \times \mathbb{S}^3$$

which is very close to the configuration space of the rolling ball, but not exactly. To make them coincide, we can replace

- small unit ball with *spinor*: it needs to roll *twice* to get back to the original position
- large ball with $\mathbb{RP}^2$: in other words, rolling a pair of spinor together in sync around the large ball.

Now we have the following amazing theorem.

Theorem 3.2 (Baez–Huerta [2]). The symmetry group of a rolling spinor over $\mathbb{RP}^2$ is G_2 if and only if $R = 3$.

4 Heisenberg parabolic subgroup and cubic rings

For G_2, we have a very special parabolic subgroup called *Heisenberg parabolic subgroup*, which encode all information of Fourier coefficients of modular forms on G_2 (Section 6.2). Especially, the orbit of the character group under the adjoint action (of Levi component) is in bijection with the isomorphism classes of the *cubic rings*, and these will parametrize Fourier coefficients of modualar forms on G_2. Most of the discussions in the following can be found in Gan–Gross–Savin [13].

4.1 Heisenberg parabolic subgroup of G_2

Here G_2 is considered as an algebraic group over $\mathbb{Z}$. Recall that we have a bijection between (conjugacy classes of) parabolic subgroups as follows. Let G

[3]Of course, there are two, but let's focus on the split one. Sorry for the non-split octonion...

be a split group with a maximal (split) torus T and a Borel subgroup B. Let $J \subseteq \Delta$ be a subset and define $\Phi(J) := \mathbb{Z}J \cap \Phi(G,T)$, where $\Phi(G,T)$ is the set of roots of $T \subset G$. Then there exists a unique parabolic subgroup $P_J \supseteq B$ with a unipotent radical U_J such that

$$\mathrm{Lie} U_J = \bigoplus_{\alpha \in \Phi^+ - (\Phi(J) \cap \Phi^+)} \mathfrak{g}_\alpha$$

Theorem 4.1. We have a bijection

$$\begin{aligned}\{J \subseteq \Delta\} &\leftrightarrow \{\text{parabolic subgroups of } G \text{ containing } B\}\\ J &\mapsto P_J.\end{aligned}$$

This is an order-preserving bijection: we have $P_\emptyset = B$ and $P_\Delta = G$. The Levi subgroup L_J of P_J is also equal to the subgroup of G generated by the centralizer $C_G(T)$ and G_α for $\alpha \in J$. See [14, Theorem 1.9.2] or Chapter 1.9 of loc. cit. for a general theory that covers quasi-split G.

Let's specialize it to the *maximal* parabolic subgroups. These subgroups correspond to the subset of Δ of the form $\theta = \Delta - \{\alpha\}$ for some $\alpha \in \Delta$. The parabolic subgroup $P = P_\theta$ admits a Levi decomposition $P = LU$, where

$$\mathrm{Lie} L = \mathrm{Lie} T \oplus \left(\bigoplus_{\beta : m_\alpha(\beta) = 0} \mathfrak{g}_\beta \right)$$

where $m_\alpha(\beta)$ is the multiplicity of α in the root β. The unipotent radical U has a Lie algebra

$$V = \mathrm{Lie} U = \bigoplus_{\beta : m_\alpha(\beta) > 0} \mathfrak{g}_\beta.$$

It admits an action of the center $Z(L) \simeq \mathbb{G}_m$, which gives a grading of V:

$$\begin{aligned} V &= \bigoplus_{n \geq 1} V_n, \\ V_n &= \bigoplus_{\beta : m_\alpha(\beta) = n} \mathfrak{g}_\beta. \end{aligned}$$

We have a canonical L-stable filtration of U, corresponds to the filtration of V: we have

$$U = U_1 \supset U_2 \supset \cdots \supset U_d \supset \{1\}$$

with $\mathrm{Lie}(U_i/U_{i+1}) = V_i$. The commutator on U gives a map $U_i \times U_j \longrightarrow U_{i+j}$, and this corresponds to the Lie bracket $V_i \times V_j \longrightarrow V_{i+j}$ by passing to the quotients.

In our case, we first fix our set of simple roots $\Delta = \{\alpha, \alpha'\}$ above and have a corresponding Borel subgroup $B \subset G_2$. We consider the maximal parabolic subgroup $P = LU$ associated with the subset $J = \Delta - \{\alpha\} = \{\alpha'\} \subset \Delta$. Then

$$\Phi^+ - (\Phi(J) \cap \Phi^+) = \{\alpha, \alpha + \alpha', \alpha + 2\alpha', \alpha + 3\alpha', \beta_0\}$$

where the first four of them has $m_\alpha(-) = 1$ (contribute to V_1) and $m_\alpha(\beta_0) = 2$ (contribute to V_2). The filtration of U is

$$U = U_1 \supset U_2 = U_{\beta_0} \supset \{1\}$$

where $U_{\beta_0} = Z(U) = [U, U]$, and $U^{\mathrm{ab}} = U/[U, U] \simeq V_1$ has dimension 4.

4.2 Cubic rings

The Levi subgroup L acts on U by conjugation, and this induces an action of L on $\mathrm{Hom}(U, \mathbb{G}_a) = \mathrm{Hom}(U^{\mathrm{ab}}, \mathbb{G}_a)$. We have the following description of the representation:

Proposition 4.2. The representation L on the space $\mathrm{Hom}(U, \mathbb{G}_a) = \mathrm{Hom}(U^{\mathrm{ab}}, \mathbb{G}_a)$ is isomorphic to the twisted representation of GL_2 on the space of binary cubic forms

$$p(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

defined as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot p(x, y) = \frac{1}{\det(\gamma)} \cdot p(Ax + Cy, Bx + Dy), \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2.$$

Later, this space will serve as a parametrizing space of the Fourier coefficients of modular forms on G_2. Proposition 4.2 is even true over $\mathbb{Z}$, and in fact, each orbit is in bijection with an isomorphism class of *cubic rings*: a ring which is a free $\mathbb{Z}$-module of rank 3.

Proposition 4.3 (Delone–Fadeev [10], Gan–Gross–Savin [13])**.** There is a bijection between the $\mathrm{GL}_2(\mathbb{Z})$-orbits of the space of binary cubic forms with integer coefficients and the set of isomorphism class of cubic rings. This bijection preserves discriminants.

Proof. Here we only introduce the bijection without proof. For a cubic ring A, we can always find a *good basis* $(1, \alpha, \beta)$ so that $A = \mathbb{Z} + \mathbb{Z} \cdot \alpha + \mathbb{Z} \cdot \beta$ and

$$\begin{cases} \alpha\beta = -ad \\ \alpha^2 = -ac + b\alpha - a\beta \\ \beta^2 = -bd - d\alpha + c\beta \end{cases}$$

for $a, b, c, d \in \mathbb{Z}$, and the corresponding binary cubic form is $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$. Both A and f has the same discriminant

$$\Delta = b^2c^2 + 18abcd - 4ac^3 - 4db^3 - 27a^2d^2.$$

□

For a binary cubic form $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$, we define it's *content* simply as $e = \gcd(a, b, c, d)$. Then the associated cubic ring A can be written as $A = \mathbb{Z} + eA_0$ for some other cubic ring A_0. Especially, f is primitive (i.e. the content is 1) if and only if the corresponding cubic ring is *Gorenstein* (i.e. $\mathrm{Hom}(A, \mathbb{Z})$ is projective). We also have a local variant of the content, namely p-depth for a prime p, which is the exponent of p in the prime factorization of e.

5 Local representation theory of G_2

5.1 Archimedean

For a holomorphic modular form, the archimedean component of the associated automorphic representation is a *holomorphic* discrete series. For a general Lie group G with a maximal compact subgroup K, *when* G/K *possesses a* G*-invariant holomorphic structure*, one can construct holomorphic discrete series using holomorphic line bundles on the homogeneous space G/K (e.g. see [29]). One can try a similar construction of discrete series for $G_2(\mathbb{R})$, where it has a maximal compact subgroup $K = \mathrm{SU}_4 = (\mathrm{SU}_2 \times \mathrm{SU}_2)/\{\pm 1\}$. Unfortunately, this does not work: we do not have a $G_2(\mathbb{R})$-invariant holomorphic structure on $G_2(\mathbb{R})/K$. Instead, Gross and Wallach [17] considered the "next-best" discrete series representation for $G_2(\mathbb{R})$: *quaternionic* discrete series representation. These representations are constructed via H^1 of certain holomorphic line bundles on the "twistor space covering" $\mathcal{D} = G_2(\mathbb{R})/(L \cap K) \twoheadrightarrow G_2(\mathbb{R})/K$, which is a $\mathbb{P}^1(\mathbb{C})$-bundle over $G_2(\mathbb{R})/K$. These are parametrized by integers $k \geq 2$,[4] which have infinitesimal character $\rho + (k-2)\beta_0$ where $\rho = \frac{1}{2}\sum_{\beta \in \Phi^+} \beta = 3\alpha + 5\alpha'$ is the Weyl root. Its restriction to K decomposes as

$$(\pi_k)|_K \simeq \bigoplus_{n \geq 0} \mathrm{Sym}^{2k+n}(\mathbb{C}^2) \boxtimes \mathrm{Sym}^n(\mathrm{Sym}^3\mathbb{C}^2),$$

[4] There are also *limits* of discrete series representations when $k = 0$ and 1, but we'll ignore these representations.

and the minimal K-type is the representation

$$\mathbb{V}_k := \mathrm{Sym}^{2k}(\mathbb{C}^2) \boxtimes \mathbf{1} \quad \text{of } (\mathrm{SU}_2 \times \mathrm{SU}_2)/\{\pm 1\}$$

of dimension $2k+1$. π_k is a submodule of $\mathrm{Ind}_{P(\mathbb{R})}^{G_2(\mathbb{R})} \lambda_k$, where

$$\lambda_k = (\mathrm{sgn})^k \cdot |\det|^{-k-1}.$$

Recall that the adjoint representation of $L(\mathbb{R})$ on $\mathrm{Hom}(U(\mathbb{R}), \mathbb{R})$ is isomorphic to the twisted representation of $\mathrm{GL}_2(\mathbb{R})$ on the space of binary cubic forms with coefficients in $\mathbb{R}$ (Proposition 4.2). We have $\mathrm{Hom}(U(\mathbb{R}), \mathbb{R}) \simeq \mathrm{Hom}(U(\mathbb{R}), \mathbb{S}^1)$ via $f \mapsto \chi = e^{2\pi i f}$ (non-algebraic isomorphism), which takes the lattice $\mathrm{Hom}(U(\mathbb{Z}), \mathbb{Z})$ to $\mathrm{Hom}(U(\mathbb{Z})\backslash U(\mathbb{R}), \mathbb{S}^1) = \{\chi \in \mathrm{Hom}(U(\mathbb{R}), \mathbb{S}^1) : \chi|_{U(\mathbb{Z})} = 1\}$. The representation of $L(\mathbb{Z})$ on the later subgroup is isomorphic to twisted action of $\mathrm{GL}_2(\mathbb{Z})$ on the space of integral binary cubic forms. Then a character χ is called *generic* if the corresponding binary cubic form has nonzero discriminant (which we will denote as $\Delta(\chi)$).[5] Then the $L(\mathbb{R})$-action preserves sign of the discriminant, hence the set of generic characters break up into two orbits: those with $\Delta > 0$ (correponds to the real cubic algebra $\mathbb{R}^3$) and those with $\Delta < 0$ (corresponds to the cubic algebra $\mathbb{R} \times \mathbb{C}$). Wallach proved the following uniqueness result of Whittaker models of π_k [32]:

Proposition 5.1. Let χ be a generic character of $U(\mathbb{R})$, and $k \geq 0$. Let

$$\mathrm{Wh}_{k,\chi} = \mathrm{Hom}_{U(\mathbb{R})}(\pi_k, \chi) = \{\ell : \pi_k \to \mathbb{C}, \ell(\pi_k(u)v) = \chi(u)\ell(v)\, \forall u \in U(\mathbb{R})\}$$

be the space of χ-Whittaker functionals on π_k. Then

- If $\Delta(\chi) < 0$, $\dim_{\mathbb{C}} \mathrm{Wh}_{k,\chi} = 0$.
- If $\Delta(\chi) > 0$, $\dim_{\mathbb{C}} \mathrm{Wh}_{k,\chi} = 1$, and it affords the representation $(\mathrm{sgn})^k$ of $S_3 \simeq \mathrm{Stab}(\chi) \subset \mathrm{GL}_2(\mathbb{R})$.

The above result will be used to define *Fourier coefficients* of modular forms on G_2 (Section 6.2).

5.2 Nonarchimedean

In Section 2.3, we studied (unramified) representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and Hecke algebras. A similar theory for G_2 is developed in [13], which we are going to

[5] Usually, a character $\psi : N(F) \to \mathbb{C}^\times$ is called generic if it is nontrivial on each root (sub)group $N_\beta \leq N$, and I think our definition also fits into this definition, but I haven't checked myself.

introduce here. Using this, we can describe the action of Hecke operators on G_2 modular forms on their Fourier coefficients (Section 6.3).

We have two fundamental representations of G_2: the 7-dimensional standard representation (corresponds to the embedding $G_2 \hookrightarrow \mathrm{SO}_7$ explained in 3.2), and the 14-dimensional adjoint representation. Let χ_1 and χ_2 be the characters of these representations, respectively. Then the representation ring $R(\widehat{G_2})$ of the dual group $\widehat{G_2} = G_2(\mathbb{C})$ (dual of G_2 is again G_2!) is a polynomial ring in χ_1 and χ_2, with highest weights λ_1 and λ_2 identified with coroots

$$\lambda_1 = \beta_0^\vee = (2\alpha + 3\alpha')^\vee$$
$$\lambda_2 = (\alpha + 2\alpha')^\vee.$$

We have the following identities between χ_1 and χ_2:

$$\begin{cases} \wedge^2\chi_1 = \chi_1 + \chi_2 \\ \wedge^3\chi_1 = \chi_1^2 - \chi_2 \\ \wedge^{7-n}\chi_1 = \wedge^n\chi_1. \end{cases}$$

For the spherical Hecke algebra $\mathcal{H}_p(G_2) := C_c^\infty(G_2(\mathbb{Z}_p)\backslash G_2(\mathbb{Q}_p)/G_2(\mathbb{Z}_p))$, Gross [15] computed Satake transform $\mathcal{S}_{G_2} : \mathcal{H}_p(G_2) \simeq R(\widehat{G_2})$ as

$$\begin{cases} \varphi_1 = p^3\chi_1 = \mathcal{S}_{G_2}(K\lambda_1(p)K) + 1, \\ \varphi_2 = p^5\chi_2 = \mathcal{S}_{G_2}(K\lambda_2(p)K) + p^4 + \varphi_1. \end{cases}$$

Using the equation above, we can find a decomposition of the double cosets $K\lambda_i(p)K$ into single K-cosets of the form ulK, where $u \in U(\mathbb{Q}_p)$ and $l \in L(\mathbb{Q}_p) \simeq \mathrm{GL}_2(\mathbb{Q}_p)$, which will be used to compute the action of the corresponding Hecke operators on Fourier coefficients (Section 6.3). We can use *relative* Satake transform corresponds to the restriction map $R(\widehat{G_2}) \to R(\widehat{L})$, and this gives the number of distinct cosets in the decomposition of $K\lambda_i(p)K$ for each l. More precisely, the relative Satake transform $\mathcal{S}_{G_2/L} : \mathcal{H}_p(G_2) \to \mathcal{H}_p(L)$ is defined as

$$\mathcal{S}_{G_2/L}(c[t])(l) = |\delta_P(l)|^{1/2} \cdot \int_U c[t](lu)\mathrm{d}u$$

and it fits into the following commutative diagram

$$\begin{tikzcd} \mathcal{H}_p(G_2) \arrow[r, "\mathcal{S}_{G_2}"] \arrow[d, "\mathcal{S}_{G_2/L}"'] & R(\widehat{G_2}(\mathbb{C})) \arrow[d, "\mathrm{Res}"] \\ \mathcal{H}_p(L) \arrow[r, "\mathcal{S}_L"'] & R(\widehat{L}(\mathbb{C})) \end{tikzcd}$$

Proposition 5.2. Fix $t \in G_2$ and $l \in L$. Let $c[t] = \mathbb{1}_{KtK} \in \mathcal{H}_p(G_2, K)$ be the characteristic function. Then

$$\mathcal{S}_{G_2/L}(c[t])(l) = |\delta_P(l)|^{1/2} \cdot \#\{ulK \subset KtK, u \in U\}$$

Combining Proposition 5.2 with the restriction formula

$$\begin{aligned}
\mathrm{Res}(\chi_1) &= \det + \chi + 1 + \chi^* + \det{}^{-1} \\
\mathrm{Res}(\chi_2) &= \mathrm{Res}(\chi_1) + \det \cdot \chi + \chi \cdot \chi^* + \det{}^{-1} \cdot \chi^* - 1,
\end{aligned}$$

we can compute the number of distinct cosets of the form ulK in $K\lambda_1(p)K$ and $K\lambda_2(p)K$ for given l (Corollary 13.3 and 13.4 of [13]). One can find single cosets for each l with carefully chosen u match with the number: for example, we have the following result [13, Proposition 14.2].

Proposition 5.3. Let l lies in the double coset of either

$$\begin{pmatrix} p & \\ & p \end{pmatrix}, \quad \begin{pmatrix} p & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}, \quad \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}$$

in $L \simeq \mathrm{GL}_2$, and u lies in $U(\mathbb{Z}_p)$, then ulK is contained in the K-double coset of $\lambda_1(p)$ in G. For such l, the representatives u of the distinct right cosets of $U(\mathbb{Z}_p) \cap lU(\mathbb{Z}_p)l^{-1}$ in $U(\mathbb{Z}_p)$ give the distinct right cosets of the form ulK in $K\lambda_1(p)K$.

For other $l \in L$ and the double coset $K\lambda_2(p)K$, u can be chosen as elements in certain root groups (See [13, Section 14] for details).

6 Modular forms on G_2

6.1 Definition

Fix the *weight* $k \geq 2$ and a quaternionic discrete series representation π_k of $G_2(\mathbb{R})$ introduced in Section 5.1. Let $\mathcal{A} = \mathcal{A}(G_2)$ be the space of automorphic forms on G_2: the are the functions on $G_2(\mathbb{A})$ which are

- left $G_2(\mathbb{Q})$-invariant,
- right-invariant under some open compact group $K_f \subseteq G_2(\mathbb{A}_{\mathrm{fin}})$,
- annihilated by an ideal $J \subseteq \mathcal{Z}(\mathfrak{g}_2)$ of finite codimension in the center $\mathcal{Z}(\mathfrak{g}_2)$ of the universal enveloping algebra of $\mathfrak{g}_2 = \mathrm{Lie}G_2(\mathbb{R})$,

- has uniform moderate growth.

Note that the definition is slightly different from the literature, e.g. we are not assuming K-finiteness (compare this with the definition in Section 2.1). More explanation can be found in [13, Section 7]. Also, we are mainly interested in the modular forms of "level 1", i.e. when $K_f = G_2(\widehat{\mathbb{Z}})$.

Definition 6.1. The space of modular forms of weight k and level 1 on G_2 is

$$M_k(G_2) = \mathrm{Hom}_{G_2(\mathbb{R})\times G_2(\widehat{\mathbb{Z}})}(\pi_k \otimes \mathbb{C}, \mathcal{A}),$$

and the subspace of cusp forms is

$$S_k(G_2) = \mathrm{Hom}_{G_2(\mathbb{R})\times G_2(\widehat{\mathbb{Z}})}(\pi_k \otimes \mathbb{C}, \mathcal{A}_0).$$

By definition, $f \in M_k(G_2)$ is neither a function on $G_2(\mathbb{R})$ nor $G_2(\mathbb{A})$, but a $G_2(\mathbb{R}) \times G_2(\widehat{\mathbb{Z}})$-equivariant linear map from $\pi_k \otimes \mathbb{C}$ to $\mathcal{A}$. Once you choose a vector $v \in \pi_k$, then $f(v)$ is indeed an automorphic form on G_2. By the theorem of Harish-Chandra [5, Theorem 1.7], these spaces are finite dimensional. Also, it admits an action of the spherical Hecke algebra

$$\mathcal{H}(G_2(\mathbb{A}_{\mathrm{fin}}), G_2(\widehat{\mathbb{Z}})) \simeq \widehat{\bigotimes_p} \mathcal{H}_p(G_2)$$

and the action on Fourier coefficients will be explained in Section 6.3.

6.2 Fourier coefficients and expansion

Let $f \in M_k(G_2)$ and $v \in \pi_k$. Then $f(v)$ can be viewed as a function on the double coset space

$$G_2(\mathbb{Q})\backslash G_2(\mathbb{A})/G_2(\widehat{\mathbb{Z}}) \simeq G_2(\mathbb{Z})\backslash G_2(\mathbb{R})$$

where the homeomorphism comes from the strong approximation theorem. For $\chi \in \mathrm{Hom}(U(\mathbb{Z})\backslash U(\mathbb{R}), \mathbb{C}^\times)$ define a linear functional ℓ_χ on π_k as

$$\ell_\chi(v) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(v)(u)\overline{\chi(u)}\mathrm{d}u.$$

Then $\ell_\chi \in \mathrm{Wh}_{k,\chi}$, and for $\gamma \in L(\mathbb{Z})$, $\ell_{\gamma\cdot\chi} = \gamma \cdot \ell_\chi$. By Proposition 5.1, $\ell_\chi = 0$ for $\Delta(\chi) < 0$, and ℓ_χ lies in a 1-dimensional space if $\Delta(\chi) > 0$. For the latter case, fix χ_0 with $\Delta(\chi_0) > 0$ and a basis l_0 of Wh_{k,χ_0}. There exists $g \in L(\mathbb{R})$ with $\chi = g \cdot \chi_0$, well defined up to the right multiplication by $\mathrm{Stab}(\chi_0) \simeq S_3$. The linear functional $\lambda_k(g) \cdot (g \cdot \ell_0)$ is a well-defined basis element of $\mathrm{Wh}_{k,\chi}$ (independent of the choice of g), hence

$$\ell_\chi = c_\chi(f) \cdot \lambda_k(g) \cdot (g \cdot \ell_0)$$

for some constant $c_\chi(f)$. For *even* k, $c_\chi(f)$ depends only on the $L(\mathbb{Z})$-orbit of χ, and these orbits are indexed by (isomorphism classes of) cubic rings A with $\mathrm{disc}(A) > 0$, so $A \otimes \mathbb{R} \simeq \mathbb{R}^3$. We write $c_A(f)$ for the constants $c_\chi(f)$ and call it as A-th Fourier coefficient of f. For *odd* k, the situation is more subtle, since $c_\chi(f)$ depends on the $L(\mathbb{Z})$-orbit *and* the orientation of A, i.e. the choice of a basis element e of $\bigwedge^3 A \simeq \mathbb{Z}$. In this case, coefficients are only determined up to sign.

How much do Fourier coefficients $c_A(f)$ know about f itself? First of all, cusp forms are determined by the Fourier coefficients:

Proposition 6.2 (Gan–Gross–Savin [13, Proposition 8.4])**.** If $f \in S_k(G_2)$ satisfies $c_A(f) = 0$ for all cubic rings A, then $f = 0$.

The proof in [13] utilizes another maximal parabolic subgroup $Q = P_{\Delta - \{\alpha'\}} = P_{\{\alpha\}}$. Also, we have the following analogue of *Hecke bound* of Fourier coefficients for cusp forms:[6]

Proposition 6.3 (Gan–Gross–Savin [13, Proposition 8.6])**.** For $f \in S_k(G_2)$, there exists a constant $C_f > 0$ such that

$$|c_A(f)| \le C_f \cdot |\mathrm{disc}(A)|^{(k+1)/2}$$

for any totally real cubic ring A.

Recall that we have Fourier *expansions* of holomorphic modular forms, where the basis elements are exponential functions. Similarly, we have a Fourier expansion for Maass wave forms (i.e. non-holomorphic analogue of holomorphic modular forms), with the basis elements given by Bessel functions (see [6, Section 1.9]). Pollack developed a similar theory for all exceptional groups, including G_2, F_4, E_6, E_7, and E_8 [24]. To do this, he considered the modular forms $f \in M_k(G_2)$ as the associated vector-valued functions $F = F_f : G_2(\mathbb{A}) \to \mathbb{V}_k^\vee$ via $F_f(g)(v) := f(v)(g)$. Especially, he consider the restriction of F onto the real

[6] I believe that this bound is not optimal - we may expect a smaller exponent possibly from Ramanujan conjecture (as in the case of modular forms). Unfortunately, I have no idea what the optimal exponent would be. Note that we expect generalized Ramanujan conjecture (temperedness of local factors π_p of $\pi = \pi_f$) for G_2 [28], but it is not clear how this could be related to the Fourier coefficients.

points $G_2(\mathbb{R})$. He proved that, for each character $\chi : U(\mathbb{Z})\backslash U(\mathbb{R}) \to \mathbb{S}^1$, there exist (vector-valued) basis functions W_χ on $G_2(\mathbb{R})$ are vector-valued functions which are given by solutions of *Schmid operators*. He solved the equations explicitly and expressed the solutions in terms of Bessel functions. Before we state the result, note that we have a GL_2-invariant symplectic form on $V_1(\mathbb{R}) \simeq \det^{-1} \otimes \mathrm{Sym}^3(\mathbb{R}^2)$, given by $\langle f, f' \rangle = ad' - \frac{1}{3}bc' + \frac{1}{3}cb' - da'$ for $f = ax^3 + bx^2y + cxy^2 + dy^3$ and $f' = a'x^3 + b'x^2y + c'xy^2 + d'y^3$. Now, we have the following theorem.

Theorem 6.4 (Pollack [24, 25])**.** Let F be a modular form on G_2 of weight k and level 1, considered as a vector-valued function $F : G_2(\mathbb{R}) \to \mathbb{V}_k^\vee$. Let

$$F_0(g) = \int_{U_{\beta_0}(\mathbb{Z})\backslash U_{\beta_0}(\mathbb{R})} F(ng)\mathrm{d}n$$

be the constant term of F along the center $U_{\beta_0} = Z(U)$ of the unipotent part of the Heisenberg parabolic subgroup. For $x \in V_1(\mathbb{R}) \simeq \mathrm{Lie}(U(\mathbb{R})^{\mathrm{ab}})$ and $g \in L(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R})$, F_0 has a Fourier expansion of the form

$$F_0(\exp(x)g) = F_{00}(g) + \sum_A c_A(F)e^{-2\pi i\langle f_A, x\rangle} W_A(g)$$

where

- The sum is over all cubic rings with $A \otimes \mathbb{R} \simeq \mathbb{R}^3$.
- $c_A(F)$ is the A-th Fourier coefficient of F.
- f_A is the binary cubic form corresponds to A.
- $W_A : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{V}_k^\vee$ is the basis function given by

  $$W_A(g) = \sum_{-k\le v\le k} W_{A,v}(g)\frac{x_\ell^{k+v}y_\ell^{k-v}}{(k+v)!(k-v)!}$$

  with $W_{A,v} : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$,

  $$W_{A,v}(g) = \det(g)^k|\det(g)|\left(\frac{|j(g,i)p_A(g\cdot i)|}{j(g,i)p_A(g\cdot i)}\right)^v K_v(|j(g,i)p_A(g\cdot i)|).$$

  Here x_ℓ, y_ℓ are standard basis of weight vectors of the standard representation of SU_2 (so that $\{x_\ell^{k+v}y_\ell^{k-v}\}_{-k\le v\le k}$ are a basis of $\mathbb{V}_k$)[7], $p_A(z) = 2\pi f_A(z,1) = 2\pi(az^3 + bz^2 + cz + d)$ is the 2π-multiple of the cubic polynomial corresponds to A, $j(g,i) = \det(g)^{-1}(ci+d)^3$ is the automorphy factor of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and $K_v(y) = \frac{1}{2}\int_0^\infty t^v e^{-y(\frac{t+t^{-1}}{2})}\frac{\mathrm{d}t}{t}$ is the v-th Bessel function.

[7] ℓ for *long* root SU_2.

- F^{00} is the constant term of F^0, which has a form of

$$F_{00}(g) = \Phi(g)\frac{x_\ell^{2n}}{(2n)!} + \beta\frac{x_\ell^n y_\ell^n}{n!n!} + \Phi'(g)\frac{y_\ell^{2n}}{(2n)!}$$

  where $\beta \in \mathbb{C}$ is a constant and $\Phi : L(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$ is associated with a holomorphic modular form of weight $3k$, and $\Phi'(g) = \Phi(gw_0)$ with $w_0 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. When $F \in S_k(G_2)$ is a cusp form, then this term vanishes.

6.3 Hecke operators and Fourier coefficients

For a holomorphic modular form $f = \sum_{n\geq 1} a_n(f)q^n \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, the Hecke operator T_p acts on the coefficients via

$$a_n(T_p f) = a_{np}(f) + p^{k-1}a_{n/p}(f),$$

where T_p is the Hecke operator corresponds to the characteristic function of $\mathrm{GL}_2(\mathbb{Z}_p)\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)\mathrm{GL}_2(\mathbb{Z}_p)$ in the spherical Hecke algebra $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p))$. Gan–Gross–Savin [13] gave a similar description for the modular forms on G_2, using the Hecke algebra structure and explicit coset decompositions in Section 5.2.

Proposition 6.5 (Gan–Gross–Savin [13, Proposition 15.6-15.8])**.** *Let A be a cubic ring of p-depth zero, and for $i \geq 0$ define $A_i := \mathbb{Z} + p^i A$, which has p-depth i. For k even and $f \in M_k(G_2)$,*

$$\begin{aligned}
c_{A_i}(\chi_1|f) &= p^{2k-1}c_{A_{i-1}}(f) + p^{k-1}\sum_{A_i\subset B\subset A_{i-1}} c_B(f) + c_{A_i}(f) \\
&\quad + p^{-k}\sum_{A_{i+1}\subset B\subset A_i} c_B(f) + p^{1-2k}c_{A_{i+1}}(f) \qquad (i\geq 1), \\
c_A(\chi_1|f) &= p^{k-1}\sum_{A\subset_p B} c_B(f) + p^{-1}(n_A - 1)c_A(f) \\
&\quad + p^{-k}\sum_{A_1\subset B\subset A} c_B(f) + p^{1-2k}c_1(f), \\
c_{A_i}(\chi_2|f) &= c_{A_i}(\chi_1|f) + p^{3k-2}\sum_{A_{i-1}\subset B\subset A_{i-2}} c_B(f) + p^{-1}\sum_{A_{i+1}\subset C\subset A_{i-1}} c_C(f) \\
&\quad + p^{-1}c_{A_i}(f) + p^{1-3k}\sum_{A_{i+2}\subset B\subset A_{i+1}} c_B(f) \qquad (i\geq 2)
\end{aligned}$$

where $n_A = \#\{B : A_1 \subset B \subset A\}$. For the last equation, each C in the second sum is a ring with $C/A_{i+1} \simeq \mathbb{Z}/p^2\mathbb{Z}$.

For example, when A/pA is a field, the first equation simplifies as

$$c_A(\chi_1|f) = -\frac{1}{p}c_A(f) + p^{1-k}c_{A_1}(f).$$

We[8] have a similar description of $c_A(\chi_2|f)$ and $c_{A_1}(\chi_2|f)$, but more complicated. When A/pA is a field, then [13, Corollary 15.9]

$$\begin{aligned} c_A(\chi_2|f) &= \left(\frac{1}{p} + \frac{1}{p^2}\right)c_A(f) - p^{-2k}c_{A_1}(f) + p^{1-3k}\sum_{A_2\subset B\subset A_1} c_B(f) \\ c_{A_1}(\chi_2|f) &= -p^{2k-2}c_A(f) + \left(1 + \frac{1}{p}\right)c_{A_1}(f) + p^{1-2k}c_{A_2}(f) \\ &\quad + \sum_{A_3\subset B\subset A_2} p^{1-3k}c_B(f). \end{aligned}$$

When f is a Hecke eigenform, we have a stronger result than Proposition 6.2.

Theorem 6.6 (Gan–Gross–Savin [13, Theorem 16.2])**.** Let $f \in M_k(G_2)$ be a Hecke eigenform. If $c_A(f) = 0$ for all *Gorenstein* rings, then all the Fourier coefficients of f vanish. In particular, if f is a nonzero cuspidal Hecke eigenform, then $c_A(f) \neq 0$ for some Gorenstein ring A.

The analogous result is true for the holomorphic modular forms: if $f \in S_k(\Gamma_1)$ is a Hecke eigenform, then $a_1(f) \neq 0$. This is because $a_n(f)$ is completely determined by $a_1(f)$ and the Hecke eigenvalues of f, and the above theorem is also proved with a similar argument.

6.4 Examples

If we cannot find any single example of a nonzero modular form, then there's no reason to develop such a theory. Here we introduce examples from [13]: Eisenstein series and theta series.

Let $k \geq 2$ be an even integer. Recall that we have an embedding (Section 5.1)

$$i : \pi_k \hookrightarrow \mathrm{Ind}_{P(\mathbb{R})}^{G_2(\mathbb{R})}\lambda_k.$$

The character λ_k is the archimedean component of the global character

$$\chi_k = |\det|^{-k-1} : P(\mathbb{A}) \to \mathbb{C}^\times$$

[8] To be precise, they [13] have, not me.

which is unramified at all finite places. Consider the induced representation

$$I(k) = \mathrm{Ind}_{P(\mathbb{A})}^{G_2(\mathbb{A})} \chi_k = \bigotimes_v I_v(k).$$

For each finite $p < \infty$, choose the unique normalized vector $\varphi_p^\circ \in I_p(k)$ fixed by $G_2(\mathbb{Z}_p)$ and $\varphi_p^\circ(1) = 1$. For $\varphi_\infty \in \pi_k$, let

$$\varphi = i(\varphi_\infty) \otimes \left(\bigotimes_p \varphi_p^\circ \right) \in I(k)$$

and form the Eisenstein series

$$E(\varphi, g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G_2(\mathbb{Q})} \varphi(\gamma g).$$

This converges absolutely when $k > 2$, and defines an element of $\mathcal{A}$ right-invariant under $G_2(\widehat{\mathbb{Z}})$. Thus, we get a nonzero element

$$E_k : \varphi_\infty \mapsto E(\varphi, g)$$

in M_k. Now, for each character $\chi : U(\mathbb{R}) \to \mathbb{S}^1$ trivial on $U(\mathbb{Z})$, we can consider it as a character on $U(\mathbb{A})$ trivial on $U(\mathbb{Q})$ and $U(\widehat{\mathbb{Z}})$ (by strong approximation). To compute the corresponding Fourier coefficients, one needs to observe

$$\begin{aligned} \ell_\chi(\varphi) &= \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} E(u) \overline{\chi(u)} \mathrm{d}u \\ &= \int_{U(\mathbb{Q}) \backslash U(\mathbb{Q})} \left(\sum_{P_2(\mathbb{Q}) \backslash G_2(\mathbb{Q})} \varphi(\gamma u) \right) \overline{\chi(u)} \mathrm{d}u. \end{aligned}$$

The double coset space $P(\mathbb{Q}) \backslash G_2(\mathbb{Q}) / P(\mathbb{Q})$ has four representatives, say w_0, w_1, w_2, w_3, with

$$P(\mathbb{Q}) w_0 P(\mathbb{Q}) = P(\mathbb{Q}) w_0 U(\mathbb{Q})$$

an open orbit P, and only this double coset contributes to the integral above. Hence we get a factorizable integral

$$\begin{aligned} \ell_\chi(\varphi) &= \int_{U(\mathbb{A})} \varphi(w_0 u) \overline{\chi(u)} \mathrm{d}u \\ &= \left(\int_{U(\mathbb{R})} \varphi_\infty(w_0 u_\infty) \mathrm{d}u_\infty \right) \left(\prod_{p<\infty} \int_{U(\mathbb{Q}_p)} \varphi_p^\circ(w_0 u_p) \overline{\chi(u_p)} \mathrm{d}u_p \right). \end{aligned}$$

Jiang and Rallis [19] computed the non-archimedean factors above (under certain assumptions):

Proposition 6.7 ([19, Theorem 2]). Assume χ corresponds to a maximal cubic ring A. If $A \otimes \mathbb{Q}_p$ is one of the following:

$$\begin{cases} \mathbb{Q}_p \times \mathbb{Q}_p \times \mathbb{Q}_p & \\ \mathbb{Q}_p \times \mathbb{Q}_{p^2} & p \neq 2 \\ \mathbb{Q}_{p^3} & \mathbb{Q}_p \text{ containing all cube roots of unity,} \end{cases}$$

($\mathbb{Q}_{p^m}$ is the unique unramified extension of $\mathbb{Q}_p$ of degree m), then

$$\int_{U(\mathbb{Q}_p)} \varphi_p^\circ(w_0 u_p)\overline{\chi(u_p)}\mathrm{d}u_p = c_p \cdot \zeta_{A\otimes\mathbb{Z}_p}(k),$$

where c_p is an explicit universal constant independent of A.

As a result, up to a constant, we have

$$\ell_\chi(\varphi) = \zeta_A(k) \cdot \left(\int_{U(\mathbb{R})} \varphi_\infty(w_0 u_\infty)\overline{\chi(u_\infty)}\mathrm{d}u_\infty \right)$$

and it remains to compute the archimedean factor. It defines a nonzero linear form

$$\varphi_\infty \mapsto \int_{U(\mathbb{R})} \varphi_\infty(w_0 u_\infty)\overline{\chi(u_\infty)}\mathrm{d}u_\infty$$

in $\mathrm{Hom}_{U(\mathbb{R})}(I_\infty(k), \chi)$, but unfortunately, we do not know whether its restriction to π_k is also nonzero or not. If we assume that the restriction is also nonzero for some χ with $\Delta(\chi) > 0$ (recall Proposition 5.1 that the space of the Whittaker functional is zero if $\Delta(\chi) < 0$), then the restriction is nonzero for *all* χ with $\Delta(\chi) > 0$, so is $c_A(E_k)$. Now, fix χ_0 that corresponds to the cubic form $f(x,y) = x^2y + xy^2$, and let $\ell_0 = \ell_{\chi_0}$. Choose any $g \in L(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R})$ with $\chi = g \cdot \chi_0$. Then we can show that

$$(g \cdot \ell_0)(\varphi) = \delta_P(g)^{(k-2)/3} \cdot \left(\int_{U(\mathbb{R})} \varphi_\infty(w_0 u_\infty)\overline{\chi(u_\infty)}\mathrm{d}u_\infty \right)$$

and using (...), $\delta_P = |\det|^{-3}$ and $|\det(g)|^2 = \Delta(g \cdot \chi_0) = \mathrm{disc}(A)$, we can conclude

$$c_A(E_k) = \zeta_A(k) \cdot \mathrm{disc}(A)^{k-1/2} = c \cdot \zeta_A(1-k)$$

for some constant c, where the last equality comes from the functional equation of ζ_A. Considering the Proposition 6.3 and $\zeta_A(k) = O(1)$ (as $\mathrm{disc}(A) \to \infty$), one can conclude that E_k is not a cusp form.

Gan, Gross, and Savin gave another example in [13], which are theta series of weight 4. For a Gorenstein cubic ring A, A-th Fourier coefficients $N(A, J_E)$,

$N(A, J_I)$ of these theta series θ_E and θ_I count the number of embeddings of A into certain Jordan structures J_I and J_E on the cubic Jordan algebra $J = H_3(\mathbb{O})$ of 3 by 3 Hermitian matrices over octonions. Especially, the linear combination

$$N(A) = 91N(A, J_I) + 600N(A, J_E)$$

is studied in [16], and Gan proved that the corresponding theta series

$$\theta = 91\theta_I + 600\theta_E$$

is a constant multiple of E_4 [12]. Gan and Gross proved the corresponding formula [16, Theorem 3]

$$N(A) = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13 \cdot \zeta_A(-3),$$

but without using the theory of G_2 modular forms.

Unfortunately, all the examples above are not cusp forms, and it is not clear whether nonzero cusp forms exist at all. In fact, Dalal [9] computed the dimension of the space of G_2 modular forms of weight $k \geq 3$, using Arthur's trace formula. Especially, there exists a unique normalized nonzero cusp form of weights 9 and 11 respectively, and it is natural to ask if one can compute Fourier coefficients of the form.[9] Using the exceptional theta correspondences, Pollack [26, 27] proved that the coefficients of these forms are all integers and that all the coefficients $c_A(f)$ for cubic rings of the form $A \simeq \mathbb{Z} \times B$ vanish. More generally, he also proved that there exists a basis of $S_k(G_2)$ whose Fourier coefficients lie in $\mathbb{Q}^{\mathrm{cyc}} = \mathbb{Q}(\mu_\infty)$, for $k \geq 6$[10]. Note that a similar algebraicity result is known for holomorphic modular forms (if f is a normalized Hecke eigenform, then its coefficient lies in a certain number field $K = K_f$).

7 Further remarks

I end this note by introducing other works relevant to modular forms on G_2, which I don't have enough space (and knowledge) to write down the details.

[9]The minimal weight ≥ 3 with a nonzero cusp form is $k = 6$, but Pollack didn't prove/conjecture that the normalized form in $S_6(G_2)$ has integer Fourier coefficients. At least, we know algebraicity of the Fourier coefficients.

[10]During the seminar talk, I explained this as an "interesting" fact, especially because the "coefficient field" is always abelian. Note that the coefficient fields of classical modular forms can be non-abelian (examples can be found in LMFDB), but only when *we increase the levels*. If we keep the level as 1, then the coefficient field is just $\mathbb{Q}$ (generated by Eisenstein series E_4 and E_6). Hence, we might even expect that the coefficient field of any G_2-modular form of level 1 is in a (abelian) number field, or even $\mathbb{Q}$.

Further remarks

- We have a theory of (standard) L-functions for G_2 modular forms, especially their Rankin–Selberg type integral representations, developed by Gurevich–Segal [18] and Çiçek et. al. [7]. Especially, we have functional equations and Dirichlet series representations.

- There's an analogue of half-integral weight modular forms for G_2 by Leslie and Pollack [21], as automorphic forms on the double cover of G_2. Especially, they construct a modular form of weight $\frac{1}{2}$ on G_2, whose Fourier coefficients measure the size of 2-torsions of the narrow class groups.

- There are four other exceptional groups F_4, E_6, E_7, E_8, and we can define the notion of modular forms on these groups, too. In fact, there is a more uniform way to treat all five exceptional groups at once, via Jordan algebra and Freudenthal construction. This is well explained in an aribitrary paper of Pollack, e.g. [24].

- Somehow, reductive groups of type D_4 also behave like exceptional groups. Weissman developed a similar theory of D_4 modular forms (to be precise, modular forms on $\mathrm{Spin}(4,4)$ and $\mathrm{Spin}(8)$), including Fourier coefficients, local representation theory, and exceptional theta correspondences [33]. Interestingly, the Fourier coefficients of the D_4 modular forms are parameterized via Bhargava's cube [3].

- Gross and Lucianovic [23, 17] proved that there is a one-to-one correspondence between the space of *ternary quadratic forms* and the *quaternion algebras*, hence Fourier coefficients of genus 3 Siegel modular forms are parametrized by quaternion algebras. One of the main example comes from Kim's exceptional Eisenstein series on E_7 [20], whose restriction on the Siegel upper half plane gives a weight 4 Siegel modular form with Fourier coefficients counding the number of embeddings of quaternion algebras into the Coxeter's order inside the non-split octonion. The *anisotropic/compact* G_2^a form a reductive dual pair with GSp_6 in E_8, and Volpato [31] proved that the lift of the constant function on G_2^a coincides with the Siegel modular form mentioned above.

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