On Waldspurger's result

Sur un résultat de Waldspurger

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Translated and Re-TeXed by Seewoo Lee

Hervé Jacquet

Last updated: October 2, 2024

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Note:

Most of the translations are done by Google Translator and ChatGPT, where I made the translation more readable. Here are some notations that I used in this translation, which are different from the original paper or do not occur.

- For a number field F, we denote the ring of adéles over F as $\mathbb{A}_F$, instead of F_A as in the original paper.
- G would be the group of invertible 2 by 2 matrices (view as an algebraic group) and T be a maximal torus of G, chosen as a subgroup of diagonal matrices in G (the original paper uses A), and Z be the center of G. We will denote inner forms of G as G' and the corresponding maximal tori and centers as T' and Z' (the original paper uses T without $'$). Also, elements, functions or anything that are related to the inner form G' would be denoted with $'$ ($g' \in G'$, $f' : G' \to \mathbb{C}$, ...).
- For the above G and T, we will use notations $[G] = G(\mathbb{A}_F)/G(F)Z(\mathbb{A}_F)$ and $[T] = T(\mathbb{A}_F)/T(F)Z(\mathbb{A}_F)$, and similarly for G' and T' as $[G']$ and $[T']$.

Also, all the footnotes are added by myself. If you find any typos or errors, please contact seewoo5@berkeley.edu.

0 Introduction

0.1

We present a new proof of a remarkable result by Waldspurger [5, Theorem 2]. While Waldspurger's original proof relies on the properties of Weil's representation, our approach is based on a variant of the trace formula. We believe that this alternative perspective offers some interest.

We begin by recalling the statement of the result. Let F be a number field, and E a quadratic extension of F. Denote by η the character of the idéle class group of F associated with E. Consider the group GL(2) as an algebraic group G defined over F, and let Z be its center. Let T be a maximal torus in G, namely the group of diagonal matrices. Let π be an automorphic representation of $G(\mathbb{A}_F)$ that is trivial on the center $Z(\mathbb{A}_F)$.

We say that π satisfies Waldspurger's first condition (denoted **W1**) if there exist automorphic forms ϕ_1 and ϕ_2 in the space of π such that the following integrals are nonzero:

$$\int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} \phi_1(a)\mathrm{d}a, \qquad \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} \phi_2(a)\eta(\det a)\mathrm{d}a. \tag{0.1.1}$$

Next, define the set $X(E:F)$, or simply X, as the set of isomorphism classes of pairs (G',T'), where G' is an inner form of G and T' is a maximal torus in G' isomorphic to $E^\times$ over F. Such a pair arises from another pair (H,L), where H is a quaternion algebra over F and L is a subfield of H isomorphic to E over F, by taking G' as the multiplicative group of H and T' as that of L. We denote the center of G' by Z'.

We identify the set X with a set of representatives for each isomorphism class in X. Let $X(\pi)$ be the set of triples (G',T',π'), where the pair (G',T') lies in X and π' is a cuspidal automorphic representation of $G'(\mathbb{A}_F)$ associated with π through the condition outlined in [1]. This condition can be stated as follows: there exists a finite set of places S of F such that, for $v \notin S$, the groups G_v and G'_v are isomorphic, and the representations π_v and π'_v are equivalent under this isomorphism.[1] We say that π satisfies Waldspurger's second condition (denoted **W2**) if there exists a triple (G',T',π') in $X(\pi)$ and an automorphic form ϕ' in the space of π' such that the following integral is nonzero:

$$\int_{T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)} \phi'(t)\mathrm{d}t. \tag{0.1.2}$$

[1] Jacquet–Langlands correspondence.

We now state the result:

Theorem 0.1.1 (Waldspurger)**.** Conditions **W1** and **W2** are equivalent.

0.2

We now outline the main ideas behind our proof. First, we identify the set of double cosets $T\backslash G/T$ with the disjoint union of the double cosets $T'\backslash G/T'$ (§1). For this identification, we restrict to "regular" double cosets. This motivates the introduction of a compactly supported smooth function f on $G(\mathbb{A}_F)/Z(\mathbb{A}_F)$, and for each (G',T'), a corresponding compactly supported smooth function f' on $G'(\mathbb{A}_F)/Z'(\mathbb{A}_F)$. In fact, f' will be zero for almost all (G',T'). We associate a cuspidal kernel K_c to f, and similarly, a cuspidal kernel K'_c to each f'. The relevant conditions imposed on these functions are (§7 and §10):

$$\int_{[T]}\int_{[T]} K_c(a,b)\eta(\det b)\mathrm{d}b\mathrm{d}a = \sum_{(G',T')}\int_{[T']}\int_{[T']} K'_c(s,t)\mathrm{d}t\mathrm{d}s. \tag{0.2.1}$$

The relation between f and f' is as follows. These functions are products of local functions. If v is a place of F that splits in E, then for all (G',T'), the groups G'_v and G_v are isomorphic, and we define f_v and f'_v to be identical. In this case, the set $X(E_v : F_v)$ consists of two elements (G'_{v_i}, T'_{v_i}), $i = 1,2$, with G'_{v_1} being split. We can still identify the regular double cosets of T_v with the disjoint union of the regular double cosets of T'_1 and T'_2. We show that for a given function f, there are functions f'_i on G'_i such that:

$$\int_{T_v/Z_v}\int_{T_v/Z_v} f(agb)\eta_v(\det b)\mathrm{d}b\mathrm{d}a = \int_{T'_{v_i}/Z'_{v_i}}\int_{T'_{v_i}/Z'_{v_i}} f'_i(sg't)\mathrm{d}t\mathrm{d}s,$$

whenever g corresponds to g' (§2 and §4). If v is unramified and f_v is a Hecke function[2], then we can set $f'_1 = f_v$ and $f'_2 = 0$ (§5). The condition simplifies to $f'_v = f'_i$ if $G'_v = G'_i$. Waldspurger's result then follows easily from identity (0.2.1). Auxiliary results can be found in Section 6.

The proof of formula (0.2.1) relies on a generalization of the trace formula, which can be stated as follows. Let G be a semisimple group defined over F, and let A and B be subgroups of G defined over F. Let λ and μ be characters of $[A]$ and $[B]$, respectively.[3] Let f be a compactly supported smooth function on $[G]$,

[2]The characteristic function of the spherical subgroup.

[3]i.e. the characters of $A(\mathbb{A}_F)$ and $B(\mathbb{A}_F)$ that are trivial on $A(F)$ and $B(F)$, respectively.

and consider the integral:

$$\int_{[A]} \int_{[B]} K_c(a,b)\lambda(a)\mu(b)\mathrm{d}b\mathrm{d}a$$

where K_c is the cuspidal kernel attached to f. This kernel has a complex expression, which involves the sum:

$$\sum_{\zeta \in G(F)} f(x^{-1}\zeta y).$$

Choose a system of representatives for the double cosets of the groups $A(F)$ and $B(F)$. For an element η of $G(F)$ let H_η the subgroup of $H = A \times B$ of tuples (α, β) such that $\alpha^{-1}\eta\beta = \eta$. Then any element of $G(F)$ can be uniquely written in the form:

$$\zeta = \alpha^{-1}\eta\beta, \qquad \eta \in A(F)\backslash G(F)/B(F), \quad (\alpha, \beta) \in H_\eta(F)\backslash(A(F) \times B(F)).$$

By formal computation, we arrive at the following expression for the integral:

$$\sum_{\eta} \mathrm{vol}(H_\eta(F)\backslash H(\mathbb{A}_F)) \int_{[A]} \int_{[B]} f(a^{-1}\eta b)\lambda(a)\mu(b)\mathrm{d}b\mathrm{d}a$$

where sum on the left hand side is over all η such that $\lambda(a)\mu(b) = 1$ if $a^{-1}\eta b = 1$. Note that we have ignored convergence issues and other terms in the expression of K_c.

0.3

I extend my sincere thanks to the Institute for Advanced Study and its permanent members for their hospitality, as the majority of this work was completed during my stay there during the special year 1983-1984 on L-functions. I am particularly grateful to Langlands for his interest in this work. Lastly, I owe a great debt of gratitude to Piatetski-Shapiro, whose deep understanding of Waldspurger's work proved invaluable. A conversation with him was, in fact, the starting point of this work.

1 Double cosets

1.1

In this section, let F be any field of characteristic zero, and let E be a quadratic extension of F. Denote by $\mathrm{N}(E : F)$, or simply N, the subgroup of norms from

E to the multiplicative group of F. The set $X(E:F)$, or simply X, is defined as in the previous section. Consider an element (G',T') from this set. There exists a quaternion algebra H over F and a subfield L of H isomorphic to E, such that G' is the multiplicative group of H, and T' is the group of units of L. We now aim to parametrize the double cosets $T'\backslash G'/T'$. To this end, choose an element ε from the normalizer $N(T')$ of T' that is not in T'. Then, every element $h \in H$ can be uniquely written as:

$$h = h_1 + \varepsilon h_2, \qquad h_i \in L. \tag{1.1.1}$$

Moreover, if $\bar{z}$ denotes the non-trivial F-automorphism of L, then:

$$\varepsilon z \varepsilon^{-1} = \bar{z}. \tag{1.1.2}$$

The square $c = \varepsilon^2$ lies in Z', i.e., in F. Furthermore, the class of c modulo N is determined by the isomorphism class of the pair (G',T'), and conversely, it determines this class.

Next, define two involutions j^+ and j^- on H by:

$$j^{\pm}(h) = \bar{h}_1 \pm \varepsilon h_2, \quad h = h_1 + \varepsilon h_2. \tag{1.1.3}$$

It is easy to verify that these are the only involutions of H that induce the non-trivial F-automorphism of L. For any $h \in G'$, we define[4]

$$X'(h) = \frac{\frac{1}{2}\mathrm{tr}(hj^+(h))}{\frac{1}{2}\mathrm{tr}(hj^-(h))}. \tag{1.1.4}$$

Since the denominator in this expression is the reduced norm of h, $X'(h)$ is a well-defined element of F, depending only on the double coset of h modulo T'. We also introduce the function $P'(h:T)$, or simply $P'(h)$, by:

$$X'(h) = \frac{1+P'(h)}{1-P'(h)}, \tag{1.1.5}$$

or equivalently,

$$P'(h) = ch_2\bar{h}_2(h_1\bar{h}_1)^{-1}, \qquad c = \varepsilon^2. \tag{1.1.6}$$

Thus, P' is a function valued in the projective line[5], constant on the double cosets of T' in G'. According to the above formula, if $P'(h)$ is neither zero nor infinity, then it belongs to the class cN$'$ determined by the tuple (G',T'). Additionally, $P'(h)$ cannot be equal to 1, otherwise $X'(h)$ would be infinite. We say that h (or its double coset) is T'-singular if $P'(h)$ is either zero or infinity, and T'-regular otherwise.

[4]Here, tr is the trace map from H to F, given by $\mathrm{tr}(h_1 + \varepsilon h_2) = h_1 + \bar{h}_1$.

[5]$F \cup \{\infty\}$

Proposition 1.1. Two elements h and h' in G' are in the same double coset of T' if and only if $P'(h) = P'(h')$. Moreover, if x is in $c\mathrm{N}$ and is not equal to 1, then there exists an element $h \in G'$ such that $P'(h) = x$.

The proof is left to the reader.

1.2

The following proposition justifies the use of the adjective T'-regular.

Proposition 1.2. Suppose h is T'-regular. The relations

$$sht = hz, \quad s \in T', \quad t \in T', \quad z \in Z'$$

imply

$$s \in Z', \quad t \in Z', \quad st = z.$$

The proof is left to the reader.

1.3

The above results apply *mutatis mutandis* to a tuple of the form (G, T) where G is the group GL(2) and T is a maximal split torus, say the group diagonal matrices in G. In this case, H is the algebra of 2×2 matrices, and L is the subalgebra of diagonal matrices. We may take[6]

$$\epsilon = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad c = 1.$$

The functions X and $P(\cdot : T)$ (or simply P) are defined analogously. In particular:

$$P(h) = bc(ad)^{-1}, \quad h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

These functions are constant on the double cosets of T in G. As before, $P(h)$ cannot be equal to 1. We will also say that an element $h \in G$ is T-singular if $P(h)$ is either zero or infinity, and T-regular otherwise. There are six T-singular double cosets, corresponding to the cosets where $P(h)$ takes the value zero:

$$T, \quad Tn_+T, \quad Tn_-T, \quad \text{where } n_+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, n_- = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{1.3.1}$$

[6]Here, the non-trivial automorphism of L swaps the two diagonal elements, i.e., $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} d & 0 \\ 0 & a \end{smallmatrix}\right]$.

and the cosets where $P(h)$ takes the value infinity:

$$\varepsilon T, \quad T\varepsilon n_+T, \quad T\varepsilon n_-T. \tag{1.3.2}$$

Thus, these singular cosets cannot be distinguished using the function P.

1.4

However, P distinguishes between T-regular cosets:

Proposition 1.3. Let h and h' be T-regular elements of G. Then h and h' are in the same double coset if and only if $P(h) = P(h')$. Moreover, if $x \in F$ is neither 1 nor 0, there exists a T-regular element h such that $P(h) = x$.

The proof is left to the reader.

2 Orbital integrals: compact torus

2.1

Let's keep the notations from Section 1, but now assume that F is a local field. In this case, the quotient T'/Z' is compact. We choose a non-trivial additive character Ψ of F. The additive group F is endowed with the self-dual measure $\mathrm{d}x$ relative to Ψ, while the multiplicative group $F^\times$ is equipped with the measure $L(1, 1_F)|x|^{-1}\mathrm{d}x$ (the Tamagawa measure relative to Ψ). Similarly, we endow the multiplicative group $E^\times$ with a Tamagawa measure corresponding to the character $\Psi \circ \mathrm{tr}$. These measures induce corresponding measures on T' and Z', and we equip T'/Z' with the quotient measure. Let f' be a compactly supported smooth function on G'/Z'. Define

$$H'(g' : f' : T') = \int_{T'/Z'} \int_{T'/Z'} f(sg't)\mathrm{d}s\mathrm{d}t. \tag{2.1.1}$$

It is clear that $H'(g' : f' : T')$ depends only on the double coset of g' modulo T'. Now, let x be an element of $F^\times$. Define $H'(x : f' : T')$ as $H'(g' : f' : T')$ if there's g' in G' such that $P'(g' : T') = x$, and 0 otherwise. Thus, we obtain a function $H'(f' : T')$ on $F^\times$, and we will now characterize the functions H' on $F^\times$ that arise in the form $H' = H'(f' : T')$ for some appropriate function f'.

2.2

Consider a function $H' = H'(f' : T')$. By construction, H' vanishes on the complement of cN, and therefore is smooth there. Let x be a point of the form $P'(h' : T')$. Since the norm is a submersive map from $E^\times$ to $F^\times$, the map $g' \mapsto P'(g' : T')$ is also submersive at the point h'. Consequently, H' is smooth at x. Next, assume that 1 belongs to cN (i.e., the group G' splits), and without loss of generality, we take $c = 1$. We will show that H' vanishes near 1.

Since f' is compactly supported modulo Z', there exists a compact subset C of G' such that $H'(g' : f' : T') \neq 0$ implies $g' \in T'CT'$. Thus, it suffices to show the existence of a number K such that $|P'(g' : T') - 1| > K$ for $g' \in T'CT'$. Suppose no such number exists. Then, there would exist a sequence g'_i of elements in $T'CT'$ such that $P'(g'_i : T') \to 1$. By enlarging C and multiplying the elements of the sequence by elements of T', we can assume that

$$g'_i = 1 + \varepsilon t'_i = c_i z'_i$$

where $t'_i \in T'$, $c_i \in C$, and $z'_i \in Z'$. Thus,

$$P'(g'_i : T') = t'_i \bar{t}'_i = 1 + a_i$$

where $a_i \to 0$. On the other hand, we have:

$$\det g'_i = -a_i = (z'_i)^2 \det c_i.$$

which implies that $z'_i \to 0$, and consequently, $g'_i \to 0$. Since the projection of g'_i onto L is 1, this leads to a contradiction. Hence, H' must vanish near 1.

2.3

Now, let us examine the behavior of H' near 0 and near infinity. We will show that there exists a neighborhood U of 0 in F and a smooth function A' on U such that:

$$H'(x) = A'(x)(1 + \eta(cx)), \quad x \in U \tag{2.3.1}$$

$$2A'(0) = \mathrm{vol}(T'/Z') \int_{T'/Z'} f'(t)dt. \tag{2.3.2}$$

Similarly, there exists a neighborhood U of 0 in F and a smooth function B' on U such that:

$$H'(x) = B'(x^{-1})(1 + \eta(cx)), \quad x^{-1} \in U \tag{2.3.3}$$

$$2B'(0) = \mathrm{vol}(T'/Z') \int_{T'/Z'} f'(\varepsilon t)\mathrm{d}t. \tag{2.3.4}$$

Since $P'(\varepsilon g' : T') = P'(g' : T')^{-1}$, we have:

$$\int_{T'/Z'} \int_{T'/Z'} f'(s\varepsilon g' t)\mathrm{d}s\mathrm{d}t = H'(x^{-1} : f' : T')$$

or

$$H'(x^{-1} : f' : T') = H'(x : f'_0 : T'), \quad f'_0(g') = f'(\varepsilon g').$$

Thus, it suffices to prove the assertions near 0. Let us consider the non-archimedean case first. Suppose $x \in c\mathrm{N}$. Then $x = cl\bar{l}$ for some $l \in L$, and thus $x = P(h)$ for $h = 1 + \varepsilon l$. We can express

$$H'(x : f' : T') = \int_{T'/Z'} \int_{T'/Z'} f(t_1(1 + \varepsilon l)t_2)\mathrm{d}t_1\mathrm{d}t_2$$

or, after a change of variables:

$$H'(x : f' : T') = \int_{T'/Z'} \int_{T'/Z'} f\left(\left(1 + \varepsilon l \frac{\bar{t}_1}{t_1}\right) t_2\right) \mathrm{d}t_1\mathrm{d}t_2. \tag{2.3.5}$$

Since f' is smooth, there exists an ideal V of E such that for all $l \in V$, we have:

$$f'(g') = f((1 + \varepsilon l)g')$$

for all g'. Therefore, there exists an ideal U of F such that $l\bar{l} \in U$ is equivalent to $l \in V$. For $x \in cU$, we have $H'(x) = 0$ if $x \notin c\mathrm{N}$; if $x \in c\mathrm{N}$, then $x = cl\bar{l}$ for $l \in V$, and from (2.3.5):

$$H'(x) = \mathrm{vol}(T'/Z') \int_{T'/Z'} f'(t)\mathrm{d}t.$$

Our assertion follows immediately from this.

In the archimedean case, i.e., when $F = \mathbb{R}$ and $L = \mathbb{C}$, let $K(x) = H'(cx)$. Consider a disk $V = \{z : z\bar{z} < a\}$ in L such that $1 + \varepsilon V \subset G$. Then the right-hand side of (**??**) defines a smooth function on V, say $C(l)$, which depends only on the norm of l. Thus,

$$K(x) = \begin{cases} 0 & x < 0 \\ C(l) & x > 0 \text{ and } x = l\bar{l} \text{ for } x \in V. \end{cases}$$

In particular, the restriction of C to the real axis is even and smooth, and we obtain:

$$K(x) = \begin{cases} 0 & x < 0 \\ C(y) & 0 < x < a \text{ and } x = y^2, y \in \mathbb{R}. \end{cases}$$

Thus, the existence of a smooth function D on F such that $D(x) = K(x)$ for $0 < x < a$ follows from Whitney's approximation theorem.[7]

2.4

The following properties characterizes the functions $H'(f' : T')$:

Proposition 2.1. Let H' be a function on $F^\times$. There exists a compactly supported smooth function f' on G'/Z' such that $H' = H'(f' : T')$ if and only if the following conditions hold:

(1) H' vanishes on the complement of cN,

(2) H' vanishes on a neighborhood of 1,

(3) There exists a smooth function A' on a neighborhood of 0 in F such that, for x near 0, we have:

$$H'(x) = A'(x)(1 + \eta(cx)),$$

(4) There exists a smooth function B' on a neighborhood of 0 in F such that, for sufficiently large $|x|$,

$$H'(x) = B'(x^{-1})(1 + \eta(cx)).$$

If f', A', and B' satisfy these conditions, then:

$$2A'(0) = \mathrm{vol}(T'/Z') \int_{T'/Z'} f'(t)\mathrm{d}t, \quad 2B'(0) = \mathrm{vol}(T'/Z') \int_{T'/Z'} f'(\varepsilon t)\mathrm{d}t.$$

We have demonstrated the necessity of conditions (1) to (4). The proof of their sufficiency is left to the reader. The last assertion of the proposition has already been established.

3 Orbital integrals: split torus

3.1

In this section, let F be a local field, E a quadratic extension, η the quadratic character of $F^\times$ associated with E, G the group GL(2), and T the subgroup of

[7]Whitney's approximation theorem: a smooth function on $\mathbb{R}$ can be approximated by analytic functions.

diagonal matrices. We define the Tamagawa measure on $F^\times$ and its product measure on $F^\times \times F^\times$. This induces a measure on T, and we give T/Z the quotient measure. For a compactly supported smooth function f on G/Z and a T-regular element $g \in G$, we define the orbital integrals as:

$$H(g : f : A) = H(g : f : 1) = \int_{T/Z} \int_{T/Z} f(agb) \mathrm{d}a \mathrm{d}b \tag{3.1.1}$$

$$H(g : f : \eta) = \int_{T/Z} \int_{T/Z} f(agb) \eta(\det b) \mathrm{d}a \mathrm{d}b. \tag{3.1.2}$$

The first integral depends only on the projection $P(g : T)$, and we denote it as $H(x : f : T)$ or $H(x : f : 1)$, where x is such that $P(g : T) = x$. Additionally, we set $H(1 : f : T) = H(1 : f : 1) = 0$. For x in F different from 0 or 1, we define a matrix $g(x)$ as:

$$g(x) = \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix}. \tag{3.1.3}$$

Since $P(g(x)) = x$, $g(x)$ defines a section of the space of double cosets of A in G. We set $H(x : f : \eta) = H(g(x) : f : \eta)$ for $x \neq 0, 1$, and define $H(x : f : \eta) = 0$ for $x = 1$. Let w be the matrix

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.1.4}$$

Let N_+ denote the group of strictly upper triangular matrices and N_- the group of strictly lower triangular matrices. Then G can be covered by the two open sets:

$$G = TN_+N_- \cup TN_+wN_+. \tag{3.1.5}$$

Then we can write f as a sum $f_1 + f_2$, where f_1 is supported on the first open set, and f_2 on the second. We define the function $\phi(g)$ by:

$$\phi(g) = \int_{T/Z} f(ag) \mathrm{d}a, \tag{3.1.6}$$

and similarly define ϕ_1 and ϕ_2 using f_1 and f_2, respectively. These functions are left-invariant under T and compactly supported modulo T. Furthermore, the functions Φ_1 and Φ_2 defined by:

$$\Phi_1(u, v) = \phi_1 \left(\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \right), \tag{3.1.7}$$

$$\Phi_1(u,v) = \phi_1\left(\begin{bmatrix}1 & u\\ 0 & 1\end{bmatrix} w \begin{bmatrix}1 & 0\\ v & 1\end{bmatrix}\right) \tag{3.1.8}$$

which are compactly supported on $F \times F$. Since

$$\begin{aligned} g(x)\begin{bmatrix}a & 0\\ 0 & 1\end{bmatrix} &= \begin{bmatrix}a(1-x) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & a^{-1}(1-x)^{-1}x\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ a & 1\end{bmatrix} \\ &= \begin{bmatrix}1-x & 0\\ 0 & a\end{bmatrix}\begin{bmatrix}1 & a(1-x)^{-1}\\ 0 & 1\end{bmatrix} w \begin{bmatrix}1 & a^{-1}\\ 0 & 1\end{bmatrix}, \end{aligned} \tag{3.1.9}$$

for $x \neq 0, 1$, we have

$$\begin{aligned} H(x:f:T) = &\int_{T/Z} \Phi_1(a^{-1}(1-x)^{-1}x, a)\mathrm{d}^x a \\ &+ \int_{T/Z} \Phi_2(a(1-x)^{-1}, a^{-1})\mathrm{d}^\times a. \end{aligned} \tag{3.1.10}$$

To prove the convergence of these integrals, we can assume that f is positive. The right-hand side of (3.1.10) is compactly supported, ensuring the integral converges. Similarly, for x different from 0 and 1,

$$\begin{aligned} H(x:f:\eta) = &\int_{T/Z} \Phi_1(a^{-1}(1-x)^{-1}x, a)\eta(a)\mathrm{d}^\times a \\ &+ \int_{T/Z} \Phi_2(a(1-x)^{-1}, a^{-1})\eta(a)\mathrm{d}^\times a. \end{aligned} \tag{3.1.11}$$

3.2

We will now study the properties of the functions $H(f:\eta)$. The equation (3.1.11) already indicates that $H(x:f:\eta)$ is smooth at any $x \neq 0, 1$. On the other hand, if Φ_1 and Φ_2 are supported within the region where teh both absolute values $|x|$ and $|y|$ are less than some constant C, then, in the second integral, the condition $|a(1-x)^{-1}| < C$ and $|a^{-1}| < C$ holds on the support of Φ_2, which implies $C^{-2} < |1-x|$ if the second integral is not zero.

Similarly, if the first integral is non-zero, we deduce $|(1-x)^{-1}x| < C^2$, which also implies $D < |1-x|$ for a suitable constant $D > 0$. It follows that $H(x:f:\eta)$ vanishes in the neighborhood around $x = 1$. This shows that equation (3.1.11) holds for all nonzero x.

Next, let's examine $H(f:\eta)$ near $x = 0$. In (3.1.11) the second integral is obviously a smooth at $x = 0$. To analyze the first integral, we will apply the following lemma, leaving the proof as an exercise for the reader:

Lemma 3.1. Let Φ be a Schwartz–Bruhat function of two variables defined on $F \times F$. There exist two Schwartz–Bruhat functions $A_1(x)$ and $A_2(x)$ on F, such that for all $x \neq 0$ in F, we have

$$\int_{F^\times} \Phi(a^{-1}x, a)\eta(a)\mathrm{d}^\times a = A_1(x) + A_2(x)\eta(x).$$

If Φ is real valued and compactly supported, then we can take A_1 and A_2 to be compactly supported.

Returning to the first integral in equation (3.1.11), and using the notations from the lemma, we find that the integral is equal to:

$$A_1(x(1-x)^{-1}) + A_2(x(1-x)^{-1})\eta(x(1-x)^{-1}). \tag{3.2.1}$$

When x is sufficiently close to 0, then $1-x$ is a norm and $\eta(x(1-x)^{-1}) = \eta(x)$. Moreover, the functions $A_i(x(1-x)^{-1})$ are smooth functions of x in a neighborhood of 0 for $i = 1, 2$. As the second integral of (3.1.11) is clearly smooth at point $x = 0$, we conclude that, in a neighborhood of 0, $H(x : f : \eta)$ can be expressed as

$$H(x : f : \eta) = A_1(x) + A_2(x)\eta(x) \tag{3.2.2}$$

where both A_1 and A_2 are smooth functions.

Next, we study $H(x : f : \eta)$ for large $|x|$. We have

$$\varepsilon g(x) = g(x^{-1}) \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \quad \text{if } \varepsilon = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This leads to

$$H(x^{-1} : f : \eta) = H(x : f_0 : \eta)\eta(x), \quad f_0(g) = f(\varepsilon g). \tag{3.2.3}$$

Thus, there exist functions B_i, $i = 1, 2$, smooth at $x = 0$, such that

$$H(x : f : \eta) = B_1(x^{-1}) + B_2(x^{-1})\eta(x) \tag{3.2.4}$$

for x with sufficiently large $|x|$.

3.3

In summary,

Proposition 3.1. Let H be a function on $F^\times$, and suppose there exists a smooth, compactly supported function f on G/Z with $H(x : f : \eta) = H(x)$. Then the following hold:

1. H is smooth on $F^\times$,
2. H vanishes on a neighborhood of 1,
3. there exists a neighborhood U of 0 and smooth functions A_i on U for $i = 1, 2$ such that, for $x \in U$, we have
$$H(x) = A_1(x) + A_2(x)\eta(x),$$
4. there exists a neighborhood U of 0 and smooth functions B_i on U for $i = 1, 2$ such that, for x with sufficiently large $|x|$, we have
$$H(x) = B_1(x^{-1}) + B_2(x^{-1})\eta(x).$$

3.4

We will now discuss the significance of the zero sets of the functions A_i and B_i from Proposition 3.1. To do so, let us first recall a few key facts. If ϕ is a Schwartz–Bruhat function on F, the integral
$$\int_{F^\times} \phi(x)|x|^s \mathrm{d}^\times x,$$
(or rather its analytic continuation) has a pole at $s = 0$. The residue at this point takes the form $C\phi(0)$, where C is a constant depending on the choice of the Haar measure on $F^\times$. On the other hand, the integral
$$\int_{F^\times} \phi(x)|x|^s \eta(x) \mathrm{d}^\times x$$
admits analytic continuation at $s = 0$, and its value at this point is written as:
$$\int_{F^\times} \phi(x)\eta(x) \mathrm{d}^\times x.$$
Next, we define the following quantities:
$$H(n_+ : f : \eta) = \int_{F^\times}\int_{F^\times} f\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right)\eta(b)\mathrm{d}^\times a \mathrm{d}^\times b, \tag{3.4.1}$$
$$H(n_- : f : \eta) = \int_{F^\times}\int_{F^\times} f\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\right)\eta(b)\mathrm{d}^\times a \mathrm{d}^\times b, \tag{3.4.2}$$
$$H(\varepsilon n_+ : f : \eta) = \int_{F^\times}\int_{F^\times} f\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\varepsilon\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right)\eta(b)\mathrm{d}^\times a \mathrm{d}^\times b, \tag{3.4.3}$$

$$H(\varepsilon n_- : f : \eta) = \int_{F^\times}\int_{F^\times} f\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \varepsilon \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\right) \eta(b) \mathrm{d}^\times a \mathrm{d}^\times b. \tag{3.4.4}$$

In general, these integrals are divergent but can be interpreted as meromorphic continuations. For example, the first integral is the value at $s = 0$ of a meromorphic function which, for $\Re(s) > 0$, is given by the convergent integral

$$\int_{F^\times}\int_{F^\times} f\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right) \eta(b)|b|^s \mathrm{d}^\times a \mathrm{d}^\times b. \tag{3.4.5}$$

However, if f is supported in the open set TN_+wN_+, the integrals (3.4.1) and (3.4.2) are convergent. Specifically, the integral (3.4.1) is zero, as the sets TN_+ and TN_+wN_+ are disjoint. Furthermore, the intersection of TN_- with a compact set contained in TN_+wN_+ is compact and disjoint from T. This implies that in (3.4.2), the integrand is compactly supported in $F^\times \times F^\times$, ensuring the integral converges. Similarly, the integrals (3.4.3) and (3.4.4) converge if f is supported in TN_+N_+.

Proposition 3.2. With the notation from Proposition 3.1, we have:

$$H(n_+ : f : \eta) = A_2(0) \tag{3.4.6}$$
$$H(n_- : f : \eta) = A_1(0) \tag{3.4.7}$$
$$H(\varepsilon n_+ : f : \eta) = B_1(0) \tag{3.4.8}$$
$$H(\varepsilon n_- : f : \eta) = B_2(0). \tag{3.4.9}$$

Proof. We will prove equations (3.4.6) and (3.4.7). By the decomposition in (3.1.5), it is sufficient to prove the result when f is supported on either TN_+N_- or AN_+wN_+. Then we have

$$H(x : f : \eta) = \int_{F^\times} \Phi(a(1-x)^{-1}, a^{-1})\eta(a) \mathrm{d}^\times a \tag{3.4.10}$$

where

$$\Phi(u, v) = \phi\left(\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} w \begin{bmatrix} 1 & a^{-1} \\ 0 & 1 \end{bmatrix}\right), \tag{3.4.11}$$

$$\phi(g) = \int_{T/Z} f(ag)da. \tag{3.4.12}$$

Since H is smooth at $x = 0$, we deduce that $A_2 = 0$ and $A_1 = H(f : \eta)$. Hence

$$A_1(0) = \int_{F^\times} \phi\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} w \begin{bmatrix} 1 & a^{-1} \\ 0 & 1 \end{bmatrix}\right).$$

Using the invariance of f under the center, we can rewrite the integral as:

$$A_1(0) = \int_{F^\times}\int_F f\left(b\begin{bmatrix} a^2 & 0 \\ a & 1\end{bmatrix}\right)\eta(a)\mathrm{d}b\mathrm{d}^\times a.$$

A change of variable shows that this is equivalent to $H(n_- : f : \eta)$. (3.4.6) also holds as both A_2 and $H(n_+ : f : \eta)$ vanish.

Now assume that f is supported in TN_+N_-. Then we have

$$H(x : f : \eta) = \int_{F^\times}\Phi(a^{-1}(1-x)^{-1}x, a)\eta(a)\mathrm{d}^\times a \tag{3.4.13}$$

where

$$\Phi(u, v) = \phi\left(\begin{bmatrix} 1 & u \\ 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 \\ v & 1\end{bmatrix}\right), \tag{3.4.14}$$

$$\phi(g) = \int_{T/Z} f(ag)\mathrm{d}a. \tag{3.4.15}$$

Recall the definitions of A_1 and A_2:

$$H(x) = A_1(x) + A_2(x)\eta(x), \tag{3.4.16}$$

on the other hand, lemma (3.1) implies

$$\int_{F^\times}\Phi(a^{-1}x, a)\eta(a)\mathrm{d}^\times a = C_1(x) + C_2(x)\eta(x). \tag{3.4.17}$$

Comparing with (3.4.4) we get $C_i((1-x)^{-1}x) = A_i(x)$. Hence C_i and A_i vanish on the same set. By taking the Mellin transform of the above equation we get

$$\int_{F^\times}\int_{F^\times}\Phi(x, a)|x|^s\eta(a)|a|^s\mathrm{d}^\times a\mathrm{d}^\times x = \int_{F^\times} C_1(x)|x|^s\mathrm{d}^\times x + \int_{F^\times} C_2(x)|x|^s\eta(x)\mathrm{d}^\times x.$$

Comparing the residue of both sides at $s = 0$ we get

$$\int_{F^\times}\Phi(0, a)\eta(a)\mathrm{d}^\times a = C_1(0).$$

By (3.4.14) and (3.4.15) the left hand side is nothing but $H(n_- : f : \eta)$. On the other hand, the right hand side equals to $A_1(0)$, hence (3.4.7) is proven, and (3.4.6) can be proved in a similar way. Equations (3.4.8) and (3.4.9) follow from the equations (3.4.6) and (3.4.7), applied to the function f_0 defined as $f_0(g) = f(\varepsilon g)$. $\square$

4 Matching functions

4.1

In this section, E is a local field that is a quadratic extension of F, and η is a quadratic character attached to E. We again consider the pair (G,T) of the group GL(2) and the subgroup of diagonal matrices, and the set $X = X(E:F)$ that contains two elements (G'_i, T'_i), $i = 1, 2$, where G'_1 split. Let f be a compactly supported smooth function on G/Z and f'_i, $i = 1, 2$ compactly supported smooth functions on G'_i/Z'_i. We say that f and the pair (f'_1, f'_2) are matched if the following condition holds: For all $x \in F\backslash\{0,1\}$, choose i and $g' \in G'_i$ such that $x = P'(g' : T'_i)$ ($i = 1$ if x is a norm of E, and $i = 2$ otherwise). Then

$$H(x : f : \eta) = H'(g' : f'_i : T'_i).$$

Proposition 4.1. For a given function f, there exists a pair of functions (f'_1, f'_2) that matches f. Moreover, we have

$$\mathrm{vol}(T'_i/Z'_i)\int_{T'_i} f'_i(t'_i)\mathrm{d}t'_i = H(n_+ : f : \eta) \pm H(n_- : f : \eta)$$
$$\mathrm{vol}(T'_i/Z'_i)\int_{T'_i} f'_i(\varepsilon t'_i)\mathrm{d}t'_i = H(\varepsilon n_+ : f : \eta) \pm H(\varepsilon n_- : f : \eta)$$

where the sign is + for $i = 1$ and − for $i = 2$.

Proof. This result follows from the propositions 2.1, 3.1, and 3.2. □

4.2

When $F = \mathbb{R}$, let K denote the orthogonal subgroup in G. We define U to be the set of pairs (f'_1, f'_2) that match a compactly supported smooth function f on G/Z, where f is K-finite if $F = \mathbb{R}$.

Let U_1 (resp. U_2) be a projection of U to the first and second components, respectively. Then the sets U_i is *dense* in the following sense.

Proposition 4.2. Let ϕ' be a continuous function on G'_i/Z'_i that is bi-invariant under T'_i. If $\int_{G'_i/Z'_i} \phi'(g')f'_i(g')\mathrm{d}g' = 0$ for all $f'_i \in U_i$, then ϕ' must be a zero function.

The proof of 4.2 will occupy the remainder of this section.

4.3

Let f be a compactly supported smooth function on G/Z. Then

$$\int_{G/Z} f(g)\mathrm{d}g = c\int_{x\in F^\times} H(x:f:A)|1-x|^{-2}\mathrm{d}x \tag{4.3.1}$$

where c is a constant that does not depend on f. We also need an estimate for the functions $H(x:f:T)$, wehre f is compactly supported continuous on G/Z.

Lemma 4.1. Let f be a compactly supported function on G/Z, and $H(x) = H(x : f : T)$. Then H vanishes in a neighborhood of 1 and behaves as $O(\log|x|)$ for both small or large $|x|$.

Proof. The proof is similar to that given in Section 3.2, except that Lemma 3.1 is replaced by the following assertion: if Φ is a Schwarts–Bruhat function of two variables, then there exist two Schwartz–Bruhat functions $A_1(x)$ and $A_2(x)$, such that

$$\int_{F^\times}\Phi(a^{-1}xa)\mathrm{d}^\times a = A_1(x) + A_2(x)\log|x|.$$

We need integration formulas for groups G_i' analogous to (4.3.1):

$$\int_{G_1'/Z_1'} f_1'(g)\mathrm{d}g = c_1\int_0^\infty H'(x:f_1':T_1')|1-x|^{-2}\mathrm{d}x, \tag{4.3.2}$$

$$\int_{G_2'/Z_2'} f_2'(g)\mathrm{d}g = c_2\int_{-\infty}^0 H'(x:f_2':T_2')|1-x|^{-2}\mathrm{d}x, \tag{4.3.3}$$

where c_i is a constant and f_i' is a compactly supported continuous function on G_i'/Z_i', for $i = 1, 2$. $\square$

4.4

Proof of the proposition 4.2. Suppose $i = 1$. Let $H'(x) = H'(x : \phi' : T_1')$. In particuler, $H'(x) = 0$ if x is not a norm. We will compute the integrals up to multiplicative constants. Suppose f and (f_1', f_2') match with f' being K-finite if $F = \mathbb{R}$. By (4.3.2),

$$\int_{G_1'/Z_1'}\phi'(g_1')f_1'(g_1')\mathrm{d}g_1' = \int_0^\infty H'(x)H'(x:f_1':T_1')|1-x|^{-2}\mathrm{d}x.$$

Define ϕ_0 as a function on G by

$$\phi_0(g) = H'(x)\eta(\det b) \quad \text{if } g = ag(x)b.$$

By the properties of H' and the integration formula (4.3.1) ϕ_0 is locally integrable and

$$\int_{G/Z} \phi_0(g)f(g) = \int_{F^\times} H'(x)H'(x : f : \eta)|1-x|^{-2}\mathrm{d}x.$$

From $H(x : f : \eta) = H'(x : f_1' : T_1')$, when x is a norm, we find that

$$\int_{G/Z} \phi_0(g)f(g)\mathrm{d}g = \int_{G_1'/Z_1'} \phi'(g_1')f_1'(g_1')\mathrm{d}g_1'.$$

By assumption, the second integral vanishes. Therefore ϕ_0 is orthogonal to any smooth function (resp. any K-finite smooth function if F is real), implying that ϕ_0 must be zero. Consequently, H' must also be zero: since ϕ' is bi-invariant under T_1', it is completely determined by H' and we get $\phi' = 0$.

5 Orbital integrals: unramified case

5.1

In this section, let F be a non-archimedean local field, and let E be an unramified quadratic extension of F. Assume that the residual characteristic of F is not 2, and that the character ψ has order 0.[8] Consider the pair (G, T), where $G = \mathrm{GL}(2)$ and T is the diagonal subgroup. Let R denote the ring of integers of F, $\mathfrak{p}$ its maximal ideal, and ϖ a uniformizer. Set $K = \mathrm{GL}(2, R)$. The set $X = X(E : F)$ consists of two elements: (G_1', T_1') and (G_2', T_2'). Now assume that $G_1' = G$, and that T_1' is contained in the subgroup ZK. For simplicity, we will refer to T_1' as T'. The measures of $T \cap K/Z \cap K$ and $T' \cap K/Z \cap K$ are equal to 1. The goal of this section is to prove the following proposition.

Proposition 5.1. Let f be a K-bi-invariant, compactly supported function on G/Z. Then f matches with the pair $(f, 0)$. Moreover, we have

$$H(n_+ : f : \eta) = H(n_- : f : \eta) = \frac{1}{2}\mathrm{vol}(T'/Z)\int_{T'/Z} f(t')\mathrm{d}t'.$$

For convenience, we will consider functions with compact support on G rather than on G/Z. Since the measures of the sets $T \cap K$, $T' \cap K$, and $Z \cap K$ are

[8] The order of a character ψ is the smallest integer $r \geq 0$ such that $\varpi^r R \subseteq \ker\psi$. Order 0 character means it is unramified.

all equal to 1, if f is a K-bi-invariant function with compact support on G, we define

$$H'(g : f : T') = \int_{T'/Z} \int_{T'} f(s'gt')\mathrm{d}s'\mathrm{d}t'. \tag{5.1.1}$$

Since T' is contained in ZK, this simplifies to

$$H'(g : f : T') = \int_{Z} f(zg)\mathrm{d}z. \tag{5.1.2}$$

Similarly, we define

$$H(g : f : \eta) = \int_{T/Z} \int_{T} f(agb)\eta(\det b)dadb. \tag{5.1.3}$$

We will write $H(x : f : \eta)$ to denote $H(g(x) : f : \eta)$. We now aim to prove the following identities:

$$H(x : f : \eta) = \int_{Z} f(zg)\mathrm{d}z \quad \text{if } v(x) \text{ is even and } P(g : T') = x \tag{5.1.4}$$

$$H(x : f : \eta) = 0 \quad \text{if } v(x) \text{ is odd.} \tag{5.1.5}$$

By linearity, we can assume that f is either the characteristic function f_0 of K or the characteristic function f_m of the set

$$K \begin{bmatrix} \varpi^m & 0 \\ 0 & 1 \end{bmatrix} K, \quad m > 0. \tag{5.1.6}$$

Note that $f_m(g) \neq 0$ if and only if the following conditions hold:

- the entries of g are integers,
- $v(\det g) = m$,
- at least one of the entries of g is a unit.

These conditions trivially hold when $m = 0$.

5.2

We will now compute $H(x : f_m : \eta)$, which we will denote for simplicity as $H(x : m)$. Let us first assume $m > 0$.

Proposition 5.2. Suppose $m > 0$. Then $H(x : m)$ is given by the following formulas:

1. If $v(x)$ is odd, then $H(x:m) = 0$.

2. If $v(x)$ is even, then $H(x:m) = 0$ unless $v(x) = 0$ and $v(1-x) = m$, in which case $H(x:m) = 1$.

Proof. We will use the following lemma:

Lemma 5.1. Let

$$S = \sum_{i,j} (-1)^{i+j}$$

where the summation is over all the pairs of integers (i, j) on the edge of the rectangle defined by the inequalities

$$0 \leq i \leq P, \quad 0 \leq j \leq Q.$$

Then S is given by the following formulas:

1. if $PQ > 0$ then $S = 0$,

2. if $P = 0$ and $Q > 0$, then $S = 1$ if Q is even and $S = 0$ if Q is odd,

3. if $Q = 0$ and $P > 0$, the $S = 1$ if P is even and $S = 0$ if P is odd,

4. if $P = Q = 0$ then $S = 1$.

We now proceed to prove the proposition. Using the notation $\mathrm{Mat}[a, b, c, d]$ for matrices with entries a, b, c, d, we can write

$$H(x:m) = \sum_{i,j,k} f_m(\mathrm{Mat}[\varpi^{i+k}, x\varpi^{j+k}, \varpi^i, \varpi^j])(-1)^{i+j}, \tag{5.2.1}$$

where the sum is taken over all triples of integers (i, j, k). Since the determinant of the matrices in (5.2.1) has valuation $i+j+k+v(1-x)$, the condition $v(\det g) = m$ restricts the sum to triples (i, j, k) satisfying

$$i + j + k + v(1-x) = m.$$

This allows us to eliminate k. By the previous conditions on the integrality of the entries of g, we find

$$H(x:m) = \sum_{i,j} (-1)^{i+j} \tag{5.2.2}$$

where the sum is taken over all pairs of integers (i, j) such that

$$0 \leq i \leq m - v(1-x) + v(x) \tag{5.2.3}$$

$$0 \le j \le m - v(1-x) \tag{5.2.4}$$
$$ij(m - v(1-x) + v(x) - i)(m - v(1-x) - j) = 0. \tag{5.2.5}$$

The sum is empty, and $H(x : m)$ is zero unless

$$m - v(1-x) \ge 0 \quad \text{and} \quad m - v(1-x) + v(x) \ge 0. \tag{5.2.6}$$

Suppose (5.2.6) holds. Then we can apply the lemma, and we have $H(x : m) = 0$ unless

$$(m - v(1-x))(m - v(1-x) + v(x)) = 0. \tag{5.2.7}$$

Then the proposition follows from elementary calculations. □

5.3

Let's compute $H(x : 0)$.

Proposition 5.3. *$H(x : 0)$ is given by the following formulas:*

1. *if $v(x)$ is odd then $H(x : 0) = 0$,*
2. *if $v(x)$ is even then $H(x : 0) = 1$, unless $v(x) = 0$ and $v(1-x) > 0$ in which case $H(x : 0) = 0$.*

Proof. We have

$$H(x : 0) = \sum_{i,j,k} f_0(\mathrm{Mat}[\varpi^{i+k}, x\varpi^{j+k}, \varpi^i, \varpi^j])(-1)^{i+j}, \tag{5.3.1}$$

where the sum is over all triples of integers (i, j, k). As above, based on the conditions on g to be $f_0(g) \neq 0$, we can eliminate k and write

$$H(x : 0) = \sum_{i,j} (-1)^{i+j} \tag{5.3.2}$$

where the sum is over all pairs of integers (i, j) such that

$$0 \le i \le v(x) - v(1-x) \tag{5.3.3}$$
$$0 \le j \le -v(1-x). \tag{5.3.4}$$

Then the proposition follows from elementary calculations. □

5.4

We now compute $\int_Z f_m(zg)\mathrm{d}z$. It depends only on $x = P(g : T')$, and we denote $H'(x : m : T')$ for its value. Recall that by definition, x is a norm, meaning that the valuation of x is even. We begin with the case $m > 0$.

Proposition 5.4. Suppose $m > 0$. Then $H'(x : m : T') = 0$, unless $v(x) = 0$ and $v(1 - x) = m$ in which case $H'(x : m : T') = 1$.

Proof. We can assume that E is an extension generated by the square root of τ, where τ is a unit. Then we can take T' to be the multiplicative group of the following algebra

$$L = \left\{ \begin{bmatrix} a & b \\ b\tau & a \end{bmatrix} \right\} \tag{5.4.1}$$

and ε is a matrix

$$\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5.4.2}$$

Now, let's compute $H'(x : m : T')$ for $x = P'(g : T')$. We can assume that

$$g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} u & v \\ v\tau & u \end{bmatrix}. \tag{5.4.3}$$

so $\det g = 1 - x$ and $x = y^2 - v^2\tau$. We have

$$H'(x : m : T') = \sum_k f_m(\varpi^k g) \tag{5.4.4}$$

where

$$\varpi^k g = \begin{bmatrix} \varpi^k(1+u) & \varpi^k v \\ -\varpi^k v\tau & \varpi^k(1-u) \end{bmatrix}. \tag{5.4.5}$$

By the integrality condition on the entries of g, this sum consists of a single term where k is determined by the equation

$$k = \frac{1}{2}(m - v(1 - x)). \tag{5.4.6}$$

Thus, $H'(x : m : T')$ is either 0 or 1. By the integrality conditions again and $\det g = m$, we find that $H(x : m : T') = 1$ if and only if the followings hold:

- $m = v(1-x) \mod 2$,
- the entries of the matrix in (5.4.5) with k given by (5.4.6) are integers,
- at least one of the entries of this matrix is a unit.

First assume $v(x) < 0$. We have $v(1-x) = v(x)$, and since $x = u^2 - v^2\tau$ and τ is not a square, $v(x)$ is even. Thus $H'(x : m : T') = 0$ unless m is even. When m is even, we write

$$u = u_0\varpi^{v(x)/2}, \quad v = v_0\varpi^{v(x)/2},$$

where u_0, v_0 are integers, with at least one being a unit. Then the entries of the matrix (5.4.5) are:

$$\begin{gathered} \varpi^{(m-v(x))/2}(1 + u_0\varpi^{v(x)/2}), \quad \varpi^{m/2}v_0 \\ -\varpi^{m/2}v_0\tau, \quad \varpi^{(m-v(x))/2}(1 - u_0\varpi^{v(x)/2}). \end{gathered}$$

All of these are in $\mathfrak{p}$, so $H'(x : m : T') = 0$.

Supose $v(x) > 0$. We have $v(1-x) = 0$. From $m \equiv v(1-x) \pmod 2$, it follows that $H'(x : m : T') = 0$ unless m is even. If m is even, since $k = m/2$, u and v are integral. The entries of the matrix in (5.4.5) are

$$\begin{gathered} \varpi^{m/2}(1+u), \quad \varpi^{m/2}v, \\ -\varpi^{m/2}v\tau, \quad \varpi^{m/2}(1-u). \end{gathered}$$

All of these are in $\mathfrak{p}$, so $H'(x : m : T') = 0$.

Lastly, assume $v(x) = 0$. We have $v(1-x) \geq 0$. If $m - v(1-x)$ is odd, then $H'(x : m : T') = 0$. Let's assume that $m - v(1-x)$ is even. Then the entries of the matrix in (5.4.5) are

$$\begin{gathered} \varpi^{(m-v(1-x))/2}(1+u), \quad \varpi^{(m-v(1-x))/2}v \\ -\varpi^{(m-v(1-x))/2}v\tau, \quad \varpi^{(m-v(1-x))/2}(1-u). \end{gathered}$$

Since x is a unit, u and v are both integral and at least one of them is a unit. If $1+u$ and $1-u$ are both in $\mathfrak{p}$ we would have $2 \in \mathfrak{p}$, which is a contradiction. So at least one of the numbers $1+u$ and $1-u$ is a unit. If $H(x : m : T')$ is not zero, integrality condition gives $m = v(1-x)$. The entries of the matrix (5.4.5) are therefore reduced to

$$1+u, \quad v, \quad -v\tau, \quad 1-u,$$

which are all integral, with at least one being a unit. Hence $H'(x : m : T') = 1$.

So we have computed H' completely and the proposition follows. $\square$

5.5

Now, let's compute $H'(x : 0 : T')$. Recall that $v(x)$ is even.

Proposition 5.5. *$H'(x : 0 : T') = 1$, unless $v(x) = 0$ and $v(1-x) > 0$ in which case $H'(x : 0 : T') = 0$.*

Proof. As before, we have

$$H'(x : m : T') = \sum_k f_0(\varpi^k g), \tag{5.5.1}$$

where the sum has at most one term, with the index k is given by

$$k = -\frac{v(1-x)}{2}. \tag{5.5.2}$$

In particular, $H'(x : 0 : T') = 0$ or 1. Moreover $H'(x : 0 : T') = 1$ if and only if $v(1-x)$ is even, and the matrix

$$\varpi^k g = \begin{bmatrix} \varpi^k(1+u) & \varpi^k v \\ -\varpi^k v\tau & \varpi^k(1-u) \end{bmatrix} \tag{5.5.3}$$

with k given by (5.5.2), is in $\mathrm{GL}(2, R)$.

Assume that $v(x) < 0$ and $v(1-x)$ is even. Then $v(1-x) = v(x)$, $v(x)$ is even and

$$u = u_0\varpi^{v(x)/2}, \quad v = v_0\varpi^{v(x)/2}$$

where u_0 and v_0 are integral, with at least one being unit. Then the entries of the matrix (5.5.3) are

$$\varpi^{-v(x)/2} + u_0, \quad v_0, \quad -v_0\tau, \quad \varpi^{-v(x)/2} - u_0,$$

whih are all integral. As the determinant of the matrix (5.5.3) is a unit according to the choice of k the matrix (5.5.3) is in $\mathrm{GL}(2, R)$ and $H'(x : 0 : T') = 1$.

Suppose $v(x) \geq 0$ and $v(1-x) = 0$ (note that $v(x) > 0$ implies $v(1-x) = 0$). Then $k = 0$ and the entries of the matrices (5.5.3) reduce to the numbers

$$\begin{gathered} \varpi^{-v(1-x)/2}(1+u), \quad \varpi^{-v(1-x)/2}v \\ \varpi^{-v(1-x)/2}v\tau, \quad \varpi^{-v(1-x)/2}(1-u). \end{gathered}$$

Since either $1+u$ or $1-u$ is a unit, at least one of these entries is not integral. Therefore, (5.5.3) is not in $\mathrm{GL}(2, R)$ and $H'(x : 0 : T') = 0$. □

5.6

By comparing the propositions 5.2, 5.3, 5.4 and 5.5 we see that we have proved the identities (5.1.4) and (5.1.5). This completes the proof of the first assertion of proposition 5.1. The second then follows from proposition 4.1.

5.7

To establish the convergence of the global orbital integrals, we will require the following additional results. The proof is left to the reader.

Lemma 5.2. Let h in KZ, and define $x = P(h : T)$. Suppose $v(x) = 0$ and $v(1-x) = 0$. Then the relation

$$ahb \in KZ, \quad a \in T, \quad b \in T$$

implies

$$a \in Z(K \cap T), \quad b \in Z(K \cap T).$$

The lemma implies the following proposition.

Proposition 5.6. Let f be the characteristic function of KZ. Suppose E is an unramified quadratic extension, and T' contained in KZ. Let h be an element of KZ and $x = P(h : T)$. If x and $1-x$ are units, then

$$H(h : f : T) = 1, \quad H(h : f : \eta) = 1.$$

6 Review on local representations of GL(2)

6.1

Let F be a local field and E a quadratic extension of E. We will consider again the set X which is reduced to two elements (G'_1, T'_1) and (G'_2, T'_2), with say G'_1 splits. It will be convenient to use the following result:

Proposition 6.1. Let π' be an irreducible unitary representation of G'_i/Z'_i. Then the dimension of the space of continuous and T'_i-invariant linear functionals on the space of smooth vectors of π' is at most one. Moreover such a functional is given by the inner product with a smooth T'_i-invariant vector.

If F is non-archimedean then the assertion on the dimension is proven in [6], Proposition 9. It is well-known for $F = \mathbb{R}$. The rest of the proposition is obvious.

6.2

Likewise:

Proposition 6.2. Let π' be an infinite-dimensional irreducible unitary representation of G'_1/Z'_1. Then the dimension of the space of T-invariant linear functionals (resp. invariant under the character $\eta \circ \det$) on the space of smooth functions of π' is one.

These are the propositions 9 and 10 of [4].

6.3

For $i = 1, 2$, consider irreducible unitary representations π'_i of G'_i/Z'_i. Assume that the tuple (π'_1, π'_2) satisfies the conditions of the theorem (15.1) of [1]; in particular π'_1 is a discrete series representation.

Proposition 6.3. The representations π'_i cannot both have a nonzero invariant vector under the group T'_i.

If F is non-archimedean then our assertion is found in Theorem 2 of [6]. It is well-known for $F = \mathbb{R}$.

7 Global orbital integrals: split torus

7.1

In the rest of this work F will be a number field and E a quadratic extension of F, η the quadratic character of the ideles of F attached to E. In this and the next section we will consider the pair (G, T) and a compactly supported smooth function f on $G(\mathbb{A}_F)/Z(\mathbb{A}_F)$. Denote K_c for the cuspidal kernel attached to f. Let ϕ_j be an orthonormal basis of the space of cusp forms of the group G/Z.

Then, by definition

$$K_c(x, y) = \sum_j \rho(f)\phi_j(x)\overline{\phi}_j(y) \tag{7.1.1}$$

and

$$\rho(f)\phi(x) = \int f(g)\phi(xg)\mathrm{d}g. \tag{7.1.2}$$

In this and the following section we will give a nice expression of the integral

$$\int_{[T]}\int_{[T]} K_c(a,b)\eta(\det b)da\,db \tag{7.1.3}$$

We have chosen a nontrivial character ψ of $\mathbb{A}_F/F$. Then for each place v we have the Tamagawa measure attached to ψ_v induced on A_v and Z_v. We therefore have the product measure on $A(\mathbb{A}_F/F)$ and the quotient measure on $T(\mathbb{A}_F)/Z(\mathbb{A}_F)$. We will denote by S a finite set of places containing the infinite places, the ramified places in E and the places of residual characteristic 2. For each place v of F we will denote K_v for the usual maximal compact subgroup. In particular $K_v = \mathrm{GL}(2, R_v)$ if v is finite. We will take the function f product of local functions f_v which are K_v-finite at all places. We will assume that f_v is bi-K_v-invariant for all v not in S. We have a decomposition of K_c as a following sum:

$$K_c(x,y) = \sum_{\gamma\in G(F)/Z(F)} f(x^{-1}\gamma y) - K_{\mathrm{sp}}(x,y) - K_{\mathrm{ei}}(x,y), \tag{7.1.4}$$

where K_{sp} denotes the special kernel and K_{ei} the Eisenstein kernel (the definition will be recalled later). We can write the first term of this sum as the sum of two other terms K_r and K_s where

$$K_r(x,y) = \sum_{\gamma\in G(F)/Z(F)} f(x^{-1}\gamma y), \quad \gamma \text{ is } T\text{-regular} \tag{7.1.5}$$

$$K_s(x,y) = \sum_{\gamma\in G(F)/Z(F)} f(x^{-1}\gamma y), \quad \gamma \text{ is } T\text{-singular} \tag{7.1.6}$$

Then K_c can be written as

$$K_c = K_r + K_s - K_{\mathrm{sp}} - K_{\mathrm{et}}. \tag{7.1.7}$$

7.2

We first consider the integral of K_r. Any T-regular element γ of $G(F)/Z(F)$ can be uniquely written in the form

$$\gamma = \alpha g(\xi)\beta, \quad \alpha,\beta \in T(F)/Z(F) \text{ and } \xi \neq 0,1 \tag{7.2.1}$$

(cf. (3.1.3) for the notation and §1) This implies

$$\int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)}\int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} K_r(a,b)\eta(\det b)da\,db = \sum_{\xi\in F^\times-\{1\}} H(\xi : f : \eta), \tag{7.2.2}$$

where

$$H(\xi : f : \eta) = \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} f(ag(\xi)b)\eta(\det b)dadb. \tag{7.2.3}$$

Let's justify our formal computations. Assme that the support of f has only a finite number of regular classes. The function X introduced in the section 1 (equation (1.1.4)) defines a continuous function of the group $G(\mathbb{A}_F)/Z(\mathbb{A}_F)$ over $\mathbb{A}_F$. Hence it only takes a finite number of values on the intersection of the support of f with the set of rational points: the same is therefore true for the function $P(\cdot : A)$, which gives us our assertion. On the other hand, each of the integrals (7.2.3) converges absolutely: it suffices to prove it for the integral

$$H(\xi : f : T) = \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} f(ag(\xi)b)dadb. \tag{7.2.4}$$

Each of the local integrals $H(\xi : f_v : T_v)$ converges; almost all are equal to 1 (cf. (5.7)). Hence (7.2.4) converges. It is true for (3) and (3) is the product of the corresponding local integrals

$$H(\xi : f : \eta) = \prod_v H(\xi : f_v : \eta_v). \tag{7.2.5}$$

All but finitely many local integrals in the product are equal to 1 (cf. (5.7)).

7.3

Consider the integral of K_s. It is not absolutely convergent, but it is weakly convergent in the following sense. Let c be a number greater than 1. Define

$$\int_{c^{-1}}^{c} \int_{c^{-1}}^{c} K_s(a,b)\eta(\det b)dadb, \tag{7.3.1}$$

the integral of $K_s(a,b)\eta(\det b)$ over the set of pairs (a,b) satisfying $c^{-1} < |a_1/a_2| < c$, $c^{-1} < |b_1/b_2| < c$; where a_1 and a_2 are the diagonal entries a (and similar for b_1 and b_2). The integral exists since it is over a compact set. We will see that the integral (7.3.1) converges as c goes to infinity. Then we define the weak integral of $K_s(a,b)\eta(\det b)$ as the limit. We have seen in (1.3) that there are 6 singular double cosets in T, namely the double cosets of the following elements: $e, n_+, n_-, \varepsilon, \varepsilon n_+, \varepsilon n_-$. Let's number them from 1 to 6. Then we have a decomposition of K_s into 6 terms K_i, $1 \le i \le 6$, where K_i is the sum of the $f(x^{-1}\gamma y)$ for all γ in the i-th double coset. Let's study the integral of K_1 for example. We have

$$K_1(x,y) = \sum_{\alpha \in T(F)/Z(F)} f(x^{-1}\alpha y).$$

We have

$$\int_{c^{-1}}^{c}\int_{c^{-1}}^{c} K_1(a,b)\eta(\det b)da\,db = \int_{c^{-1}}^{c}\int_{c^{-1}}^{c} f(ab)\eta(\det b)da\,db,$$

in the left integral a and b vary in the compact subset of $[T]$ defined above; in the right integral b still varies in the compact subset of $[T]$ defined by $c^{-1} < |b_1/b_2| < c$, but a varies in the subset of $T(\mathbb{A}_F)/Z(\mathbb{A}_F)$ defined by $c^{-1} < |a_1/a_2| < c$.[9] Apply change of variable from a to ab^{-1} in the left integral. We get a double integral, with the inner integral only depending on $|b_1/b_2|$. This inner integral is written as[10]

$$\int_{c^{-1}}^{c} \eta(\det b)db.$$

It is 0 because the restriction of η to the group of idéles of absolute value 1 is nontrivial. The integral of K_1 is therefore weakly convergent and its value is 0. The same holds for the integral of K_4.

Let's examie the integrals of the other terms, K_2 for example. We have

$$K_2(x,y) = \sum_{\alpha,\beta\in T(F)/Z(F)} f(x^{-1}\alpha n_+\beta y). \tag{7.3.2}$$

It follows that

$$\int_{c^{-1}}^{c}\int_{c^{-1}}^{c} K_2(a,b)\eta(a,b)da\,db = \int_{c^{-1}}^{c}\int_{c^{-1}}^{c} \sum_{\beta\in T(F)/Z(F)} f(an_+\beta b)\eta(\det b)da\,db;$$

in the right integral b still varies in the compact subset of $[T]$ defined by $c^{-1} < |b_1/b_2| < c$, but a varies the subset of $T(\mathbb{A}_F)/Z(\mathbb{A}_F)$ defined by $c^{-1} < |a_1/a_2| < c$. Let's introduce the function ϕ on $\mathbb{A}_F^\times \times \mathbb{A}_F$ defined by

$$\phi(x,y) = f\left(\begin{bmatrix} x & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} 1 & y \\ 0 & 1\end{bmatrix}\right). \tag{7.3.3}$$

It has a compact support. Write our integral as

$$\int_{\mathbb{A}_F^\times/F^\times}\sum_{\zeta\in F^\times}\int_{\mathbb{A}_F^\times} \phi(ab^{-1}, b\zeta)\eta(b)da\,db, \quad c^{-1} < |a| < c,\ c^{-1} < |b| < c.$$

[9]The integral with respect to a over $[T] = T(\mathbb{A}_F)/T(F)Z(\mathbb{A}_F)$ and the summation over $T(F)$ are combined as an integral over $T(\mathbb{A}_F)/Z(\mathbb{A}_F)$.

[10]Integration over the elements $b = \left[\begin{smallmatrix} b_1 & \\ & b_2\end{smallmatrix}\right] \in [T]$ satisfying $c^{-1} < |b_1/b_2| < c^2$.

Using the Poisson summation formula with respect to the second variable and taking the Fourier transform with respect to the second variable we obtain for this integral the expression

$$\int_{\mathbb{A}_F^\times/F^\times}\sum_{\zeta\in F^\times}\int_{\mathbb{A}_F^\times}\phi(ab^{-1},b\zeta)\eta(b)da\,db+\int_{\mathbb{A}_F^\times/F^\times}\sum_{\zeta\in F^\times}\int_{\mathbb{A}_F^\times}\hat{\phi}(ab,b\zeta)|b|\eta(b)da\,db$$

with $c^{-1}<|a|<c$ and $1<|b|<c$. It is obvious that the integrals extend to the domain

$$a\in\mathbb{A}_F^\times,\quad b\in\mathbb{A}_F^\times/F^\times,\quad 1<|b|,$$

that converges absolutely. Moreover in the integrals with extended domains we can change a variable a into $ab^{\pm1}$. We conclude that the integral of K_2 is weakly convergent and that its value is the following sum:

$$\int_{\mathbb{A}_F^\times/F^\times}\int_{\mathbb{A}_F^\times}\sum_{\zeta\in F^\times}\phi(a,b\zeta)\eta(b)da\,db+\int_{\mathbb{A}_F^\times/F^\times}\int_{\mathbb{A}_F^\times}\sum_{\zeta\in F^\times}\hat{\phi}(a,b\zeta)|b|\eta(b)da\,db,$$

with $1<|b|$. The integral is nothing but the value of the analytical continuation of the following integral at $s=0$:

$$\int_{\mathbb{A}_F^\times}\int_{\mathbb{A}_F^\times}\phi(a,b)|b|^s\eta(b)da\,db. \tag{7.3.4}$$

The value will be denoted as an integral

$$\int_{\mathbb{A}_F^\times}\int_{\mathbb{A}_F^\times}\phi(a,b)\eta(b)da\,db. \tag{7.3.5}$$

With this convention we can write that the weak integral of K_2 as

$$H(n_+:f:\eta)=\int_{\mathbb{A}_F^\times}\int_{\mathbb{A}_F^\times}f\left(\begin{bmatrix}a&0\\0&1\end{bmatrix}\begin{bmatrix}1&b\\0&1\end{bmatrix}\right)\eta(b)da\,db. \tag{7.3.6}$$

An analogous result is valid for the integrals of the other K_i. Finally we see that the weak integral of K_s exists and is equal to the sum

$$H(n_+:f:\eta)+H(n_-:f:\eta)+H(n\varepsilon_+:f:\eta)+H(\varepsilon n_-:f:\eta), \tag{7.3.7}$$

where the first term is defined by (7.3.6) and the others are defined similarly:

$$H(n_-:f:\eta)=\iint f\left(\begin{bmatrix}a&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\b&1\end{bmatrix}\right)\eta(b)da\,db, \tag{7.3.8}$$

$$H(\varepsilon n_+:f:\eta)=\iint f\left(\begin{bmatrix}a&0\\0&1\end{bmatrix}\varepsilon\begin{bmatrix}1&b\\0&1\end{bmatrix}\right)\eta(b)da\,db, \tag{7.3.9}$$

$$H(\varepsilon n_-:f:\eta)=\iint f\left(\begin{bmatrix}a&0\\0&1\end{bmatrix}\varepsilon\begin{bmatrix}1&0\\b&1\end{bmatrix}\right)\eta(b)da\,db. \tag{7.3.10}$$

7.4

Let's move on to the integral of K_{sp}. Recall the definition of K_{sp}:

$$K_{\mathrm{sp}}(x,y) = \mathrm{vol}([\mathrm{G}])^{-1} \sum_{\chi} \int f(g)\chi(\det g)\mathrm{d}g \cdot \chi(\det x)\chi(\det y^{-1})$$

where the sum over all the quadratic characters χ of the group of idéle classes of F and $\mathrm{vol}([G])$ is the volume of the quotient $[G]$. If χ is such a character then either χ or $\chi\eta$ has a non-trival restriction to the groups of idéle classes of norm 1. Reasoning as for K_1 we immediately see that K_{sp} is weakly integrable and the integral vanishes.

8 Eisenstein kernel

8.1

We continue with the same notations in section 7. We will see that the integral

$$\int_{[T]} \int_{[T]} K_{\mathrm{ei}}(a,b)\eta(\det b)dadb \tag{8.1.1}$$

weakly converges. For the value of the integral, that is a classic application of the trace formula, we will only need a fairly small result. Choose a place u not in S that splits in E. Fix the components of f at the other places and view the integral (8.1.1) as a function of f_u. Denote $\hat{f}_u$ for the Satake transform of f_u. We prove the following result:

Proposition 8.1. There exists an integrable function ϕ on $\mathbb{R}$ and a constant c such that

$$\int_{[T]} \int_{[T]} K_{\mathrm{ei}}(a,b)\eta(\det b)dadb = \int_{-\infty}^{\infty} \phi(t)\hat{f}_u(q_u^{-it})dt + c\hat{f}_u(q^{-1}). \tag{8.1.2}$$

8.2

We need standard results on the Mellin transform of an Eisenstein series. We fix a subgroup C of $\mathbb{A}_F^\times$ isomorphic to the group of real numbers > 0 such that $\mathbb{A}_F^\times$ is the product of C and $\mathbb{A}_F^1$, the group of idéles of norm 1. The group C is equipped with the pullback measure of the measure $t^{-1}dt$ by the map $c \to |c|$ and $\mathbb{A}_F^1$ admits the quotient measure. We assume that all characters on the idéle class group are trivial on C. Let χ be such a character and $V(\chi)$ the space of functions ϕ on K (the product of K_v) such that

$$\phi\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} k\right) = \chi(ab^{-1})\phi(k) \tag{8.2.1}$$

if

$$\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in K.$$

Now consider a function ϕ on $K \times \mathbb{C}$ such that for each $u \in \mathbb{C}$ the function $\phi(\cdot, u)$ is in $V(\chi)$. The function is assumed to be holomorphic, or at least meromorphic with respect to u; for example it can be independent of u. We will extend ϕ into a function $\phi(g, u, \chi)$ on $G(\mathbb{A}_F)$ such that

$$\phi\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} g, u, \chi\right) = \chi(ab^{-1})|ab^{-1}|^{u+1/2}\phi(g, u, \chi). \tag{8.2.2}$$

Then the Eisenstein series is the analytic continuation of the series

$$E(g, \phi, u, \chi) = \sum_{\gamma \in G(F)/T(F)N_+(F)} \phi(\gamma g, u, \chi). \tag{8.2.3}$$

The series converges absolutely if $\Re u > 1/2$. The constant term of E along N_+, the group of strictly upper triangular matrices, is by definition the integral

$$E_{N_+}(g, \phi, u, \chi) = \int_{N_+(\mathbb{A}_F)/N_+(F)} E(ng, \phi, u, \chi)dn. \tag{8.2.4}$$

It has a form of

$$E_{N_+}(g, \phi, u, \chi) = \phi(g, u, \chi) + M(u, \chi)\phi(g, -u, \chi^{-1}) \tag{8.2.5}$$

where $M(u, \chi)$ is the intertwining operator from $V(\chi)$ to $V(\chi^{-1})$. We also need another Fourier coefficient of E, namely

$$W(g, \phi, u, \chi) = \int_{\mathbb{A}_F/F} E\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g, \phi, u, \chi\right)\psi(-x)\mathrm{d}x, \tag{8.2.6}$$

where ψ is a fixed character on the group $\mathbb{A}_F/F$. Then the Fourier series of E is written as

$$E(g, \phi, u, \chi) = \phi(g, u, \chi) + M(u, \chi)\phi(g, -u, \chi^{-1}) + \sum_{\alpha \in T(F)/Z(F)} W(\alpha g, \phi, u, \chi). \tag{8.2.7}$$

We can also consider a Fourier series for the group N_- of strictly lower triangular matrices. Since

$$N_- = wN_+w^{-1}, \quad w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{8.2.8}$$

the Fourier series is written as

$$E(g,\phi,u,\chi) = \phi(wg,u,\chi) + M(u,\chi)\phi(wg,-u,\chi^{-1}) + \sum_{\alpha\in A(F)/Z(F)} W(\alpha wg,\phi,u,\chi). \tag{8.2.9}$$

The Mellin transform $L(s,\lambda : \phi : u,\chi)$ of E is defined by the following integral (or its analytic continuation)

$$L(s,\lambda : \phi : u,\chi) = \int_{\mathbb{A}_F^\times/F^\times} \left(E\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) - E_{N_+}\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) \right) |a|^{s-1/2}\lambda(a)da. \tag{8.2.10}$$

Let's ignore the variables of E for simplification. By replacing E with its Fourier series, we immediately obtain the following in the Mellin transform

$$\int_{\mathbb{A}_F^\times/F^\times} W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) |a|^{s-1/2}\lambda(a)da. \tag{8.2.11}$$

We can also write the Mellin transform of E as follows:

$$L(s,\dots) = \int_1^\infty (E - E_{N_+}) + \int_0^1 (E - E_{N_-}) + \int_1^\infty E_{N_+} + \int_0^1 E_{N_-} \tag{8.2.12}$$

In each of these integrals, the function is evaluated at the point $\mathrm{diag}(a,1)$ and integrated against $|a|^{s-1/2}\lambda(a)$ on a subset of the idéle class group. For the first integral, for example, we integrate over the subset of a with $1 < |a|$. Using the Fourier series of E we easily obtain another expression for the Mellin transform:

$$\begin{aligned} &\int_1^\alpha W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) |a|^{s-1/2}\lambda(a)da \\ &+ \int_1^a W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} w \right) |a|^{s-1/2}\lambda(a)da \\ &+ \int_1^\alpha (|a|^{s+u}(\lambda\chi)(a)\phi(e) + |a|^{s-u}(\lambda\chi^{-1})(a)M(u,\chi)\phi(e))da \\ &+ \int_0^1 (|a|^{s-u-1}(\lambda\chi^{-1})(a)\phi(w) + |a|^{s+u-1}(\lambda\chi)(a)M(u,\chi)\phi(w))da. \end{aligned} \tag{8.2.13}$$

The first two integrals converge for all s and the last two for $\Re s > 1/2$. The last two integrals can be easily computed. In particular, for $s = 1/2$ and $u \in i\mathbb{R}$, we obtain the following expression for the Melline transform of E at the point $s = 1/2$:

$$\begin{aligned} L(1/2, \lambda : \phi : u, \chi) = &\int_1^\infty W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right)\lambda(a)da \\ &+ \int_1^\infty W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} w\right)\lambda(a)da \\ &- \frac{1}{u+1/2}(\phi(w)\delta(\lambda\chi^{-1}) + \phi(e)\delta(\lambda\chi)) \\ &+ \frac{1}{u-1/2}(M(u,\chi)\phi(w)\delta(\lambda\chi) + M(u,\chi)\phi(e)\delta(\lambda\chi^{-1})), \end{aligned} \tag{8.2.14}$$

where

$$\delta(\chi) = \int_{\mathbb{A}_F^1/F^\times} \chi(a)da$$

for a character χ on the idéle class group. We have to compute the difference between the Mellin transform and the following integral

$$\int_{c^{-1}}^{c} E\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right)\lambda(a)da. \tag{8.2.15}$$

Recall that this notation means that the integral is taken over the compact subset of idéle classes a such that $c^{-1} < |a| < c$. Instead of (8.2.12) we have for the integral (8.2.15) the expression:

$$\int_1^c (E - E_{N_+}) + \int_{c^{-1}}^1 (E - E_{N_-}) + \int_1^c E_{N_+} + \int_{c^{-1}}^1 E_{N_-}. \tag{8.2.16}$$

Replacing E again by its Fourier series we obtain for (8.2.15) the expression:

$$\begin{aligned} &\int_1^c W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right)\lambda(a)da \\ &+ \int_1^c W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} w\right)\lambda(a)da \\ &+ \int_1^c (|a|^{1/2+u}(\lambda\chi)(a)\phi(e) + |a|^{1/2-u}(\lambda\chi^{-1})(a)M(u,\chi)\phi(e))da \\ &+ \int_{c^{-1}}^1 (|a|^{-u-1/2}(\lambda\chi^{-1})(a)\phi(w) + |a|^{u-1/2}(\lambda\chi)(a)M(u,\chi)\phi(w))da \end{aligned} \tag{8.2.17}$$

Calculating the last two integrals and comparing to (8.2.14) we finally get the expression we had in mind:

$$\begin{aligned}
&\int_{c^{-1}}^{c} E\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right)\lambda(a)da \\
&= L(1/2, \lambda : \phi : u, \chi) \\
&+ \frac{c^{u+1/2}}{u+1/2}\delta(\chi\lambda)\phi(e) + \frac{c^{-u+1/2}}{-u+1/2}\delta(\chi^{-1}\lambda)M(u,\chi)\phi(e) \\
&+ \frac{c^{u+1/2}}{u+1/2}\delta(\chi^{-1}\lambda)\phi(w) + \frac{c^{-u+1/2}}{-u+1/2}\delta(\chi\lambda)M(u,\chi)\phi(w) + R(c)
\end{aligned} \tag{8.2.18}$$

where $R(c)$ is defined as

$$-R(c) = \int_{c}^{\infty} W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right)\lambda(a)da + \int_{c}^{\infty} W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} w\right)\lambda(a)da. \tag{8.2.19}$$

It is clear that $R(c)$ tends to zero as c goes to infinity.

8.3

We need precise estimates for $R(c)$. Recall that R depends not only on c, but also on u, λ and ϕ. Our estimates will be a consequence of the following lemma:

Lemma 8.1. Assume that ϕ is independent of u. There exists a Schwartz-Bruhat function Φ such that, for $u \in i\mathbb{R}$, we have

$$\left| W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \phi, u, \chi\right)\right| \leq \Phi(a)|a|^{-1/2}|L(2u+1, \chi^{2S})|^{-1}$$

Here $L(s, \chi^S)$ is defined as a product of local factors $L(s, \chi_v)$ for $v \notin S$ We also assume that ϕ is invariant under K_v for $v \notin S$.

Proof. There is a two-variable Schwartz-Bruhat function Φ such that

$$\phi(g, u, \chi) = \int \Phi((0,t)g)\chi^2(t)|t|^{2u+1}dt \times \chi(\det g)|\det g|^{u+1/2} \times L(2u+1, \chi^{2S})^{-1}.$$

A formal computation (see [1], chapter 3 for details) gives

$$W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \dots\right) = L(2u+1, \chi^{2S})\chi(a)|a|^{u+1/2}\int \hat{\Phi}(at, t^{-1})\chi^2(t)|t|^{2u+1}dt,$$

where $\hat{\Phi}$ is a Fourier transform with respect to the second variable. Hence it is suffice to prove the following assertion: given a Schwartz-Bruhat function $\Phi \geq 0$ with two variables, there exists a Schwartz-Bruhat function with one variable $\phi \geq 0$ such that for idéle a we have

$$\int \Phi(at, t^{-1})dt \leq \phi(a)|a|^{-1}.$$

Consider the analogous local problem. To be precise, let us first consider the case where the local field F is non-archimedean and the function Φ is the characteristic function of the integers. Then the integral is 0 except for the set defined by the inequalities $|a| \leq |t| \leq 1$. The integral is therefore 0 unless a is integral. In this case the integral is $1 + v(a)$. Since $q \geq a$ this is smaller than $q^{v(a)}$. So our integral is at most $\phi(a)|a|^{-1}$, where ϕ is the characteristic function of the integers. Then the integral, considered as a function of a, has the form

$$\int \Phi(at, t^{-1})dt = \phi_1(a) + \phi_2(a) \log |a|$$

for some Schwartz-Bruhat functions ϕ_i (cf. (4.3)). It is clear that the right hand side is bounded by $\phi(a)|a|^{-1}$, where ϕ is a suitable Schwartz-Bruhat function. By multiplying these local inequalites we easily obtain the required global inequality. □

It is well known that the function $L(2u+1, \chi^{2S})^{-1}$ has polynomial growth on the line $\Re(u) = 0$. On the other hand, if ϕ is a Schwartz-Bruhat function, there exists for all $N > 0$ a constant $C(N)$ such that

$$\int_c^\infty \phi(a)|a|^{-1}da \leq C(N)c^{-N}.$$

By comparing with definition (8.2.19) of R we immediately obtain the following:

Lemma 8.2. For all N there exist constants $C(N)$ and M such that for all imaginary u we have

$$|R(c, u)| \leq C(N)|c|^{-N}|u|^M.$$

In the same way using the expression for the Mellin transform and the fact that the operator $M(u, \chi)$ is unitary on the imaginary axis we obtain the following estimate:

Lemma 8.3. On the imaginary axis $M(u, \chi)\phi(k)$ and $L(1/2, \lambda : \phi : u, \chi)$ have polynomial growth.

8.4

Let's study the integral of the kernel K_{et}. Let's recall its definition. For any character χ choose an orthonormal basis ϕ_i of the Hilbert space $V(\chi)$; denote by $\rho(u,\chi)$ the representation of $G(\mathbb{A}_F)$ by right translations in the space of functions ϕ such that

$$\phi\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} g\right) = \chi(ab^{-1})|ab^{-1}|^{u+1/2}\phi(g). \tag{8.4.1}$$

We can identify the space of $\rho(u,\chi)$ with $V(\chi)$ and set:

$$F(u,\chi : i,j) = (\rho(u,\chi)\phi_i,\phi_j). \tag{8.4.2}$$

We will write $E_{\mathrm{ei}}(x,i,\ldots)$ for $E_{\mathrm{ei}}(x,\phi_i,\ldots)$. With these notations

$$K_{\mathrm{ei}}(x,y) = \sum_\chi K_\chi(x,y) \tag{8.4.3}$$

where, for each character of the idéle class group,

$$K_\chi(x,y) = \frac{1}{2i\pi}\sum_{i,j}\int_{-i\infty}^{i\infty} F(u,\chi : i,j)E(x,j,u,\chi)\overline{E(y,i,u,\chi)}du. \tag{8.4.4}$$

For a given f the sum (8.4.3) and (8.4.4) are finite. Define

$$I(c,\chi) = \int_{c^{-1}}^{c}\int_{c^{-1}}^{c} K_\chi(a,b)\eta(\det b)da db. \tag{8.4.5}$$

We can obviously change the order of integrations for u and the tuple (a,b). Using (8.2.12) we obtain the following expression on $I(c,\chi)$:

$$\begin{aligned}
&\frac{1}{i\pi}\sum_{i,j}\int_{-i\infty}^{i\infty} F(u,\chi : i,j) \\
&\times\Bigg(L(1/2,1 : j,u,\chi) + R(c,u) + \frac{c^{u+1/2}}{u+1/2}\delta(\chi)\phi_j(e) + \frac{c^{-u+1/2}}{-u+1/2}\delta(\chi^{-1})M(u,\chi)\phi_j(c) \\
&+ \frac{c^{u+1/2}}{u+1/2}\delta(\chi^{-1})\phi_j(w) + \frac{c^{-u+1/2}}{-u+1/2}\delta(\chi)M(u,\chi)\phi_j(w)\Bigg) \\
&\times\Bigg(\overline{L(1/2,\eta,i,u,\chi)} + R'(c,u) + \frac{c^{-u+1/2}}{-u+1/2}\delta(\eta\chi)\overline{\phi_i}(e) + \frac{c^{u+1/2}}{u+1/2}\delta(\chi^{-1}\eta)M(u,\chi)\overline{\phi_i}(e) \\
&+ \frac{c^{-u+1/2}}{-u+1/2}\delta(\chi^{-1}\eta)\overline{\phi_i}(w) + \frac{c^{u+1/2}}{u+1/2}\delta(\chi\eta)M(u,\chi)\overline{\phi_i}(w)\Bigg)du
\end{aligned} \tag{8.4.6}$$

For each (i, j), the terms $R(c, u)$ and $R'(c, u)$ satisfy the conclutions of the Lemma 8.3. For each f, $F(u, \chi : i, j)$ is zero for all but finitely many (i, j). In particular, $F(u, \chi : i, j)$ is zero unless ϕ_i and ϕ_j are both invariant under all K_v with v not in S. Moreover, on the imaginary axis, $F(u, \chi : i, j)$ rapidly decreases (faster than the inverse of a polynomial in u). On the other hand, according to (8.3), the terms $L(\dots)$ and the terms containing the powers of c moderately grows (at most polynomial in u). It follows that when we expand expression (8.4.6) we find a number of terms which tend to zero as c tends to infinity, and we can ignore these terms. The remaining terms become an integral independent of c:

$$\sum_{i,j} \int_{-i\infty}^{i\infty} F(u, \chi : i, j) L(1/2, 1 : j : u, \chi) \overline{L(1/2, \eta, i, u, \chi)} du. \tag{8.4.7}$$

The other terms does not vanish only if $\chi = 1$ or $\chi = \eta$. Each of these terms is of one of the following types:

$$\int F(u, 1 : i, j) \overline{L(1/2, \eta : i : u, 1)} \frac{c^{1/2+u}}{1/2 + u} (\phi_j(e) + \phi_j(w)) du, \tag{8.4.8}$$

$$\int F(u, \eta : i, j) L(1/2, 1 : j : u, \eta) \frac{c^{1/2-u}}{1/2 - u} (\overline{\phi_i}(e) + \overline{\phi_i}(w)) du, \tag{8.4.9}$$

$$\int F(u, 1 : i, j) \overline{L(1/2, \eta : i : u, 1)} \frac{c^{1/2-u}}{1/2 - u} M(u, 1)(\phi_j(e) + \phi_j(w)) du, \tag{8.4.10}$$

$$\int F(u, \eta : i, j) L(1/2, 1 : j : u, \eta) \frac{c^{1/2+u}}{1/2 + u} M(u, \eta)(\overline{\phi_i}(e) + \overline{\phi_i}(w)) du. \tag{8.4.11}$$

Integral (8.4.7) obviously has the properties required by Proposition 8.1. To prove the proposition, it suffices to show that each of the expressions (8.4.8) to (8.4.11) converges when c tends to infinity and that, moreover, the limit is zero if the Satake transform of the function f_u is zero at q^{-1}. This last condition implies that the integral of f_u over G_u/Z_u is zero and $F(u, 1 : u, j)$ and $F(u, \eta : i, j)$ cancel out at points $u = 1/2$ and $u = -1/2$.

8.5

Let's study (8.4.8). We will move the contour of integration from line $\Re u = 0$ to line $\Re u = -1/2$; but for the latter one, we will replace the segment joining the point $-1/2 - i\varepsilon$ and the point $-1/2 + i\varepsilon$ by the semi-circle centered at $-1/2$ and of radius ε which passes through the points $-1/2 - \varepsilon i$, $\varepsilon - 1/2$ and $-1/2 + i\varepsilon$. Let's prove that such a transformation of cantour is valid. The factor

$$F(u) = F(u, 1 : i, j)(\phi_j(e) + \phi_j(w))$$

and its derivatives, are holomorphic and rapidly decreasing on the vertical strip $-1/2 \leq \Re u \leq 0$. The exponential function remains bounded. The factor $(1/2+u)^{-1}$ also remains bounded at infinity on this vertical strip. Now we study the Mellin transform. Recall that we have an integral representation of $\phi_i(g,u,1)$:

$$\phi_i(g,u,1) = \int \Phi((0,t)g)|t|^{2u+1}dt \times |\det g|^{u+1/2}L(2u+1,1^S)^{-1}.$$

A formal computation gives the following expression of the Mellin transform (denoted as $L(u)$ in short):

$$L(u) = L(2u+1,1^S)^{-1}\iint \hat{\Phi}(a,b)|a|^{1/2+u}\eta(a)|b|^{1/2-u}\eta(b)dadb, \tag{8.5.1}$$

where $\hat{\Phi}$ is the Fourier transform of Φ with respec to the second variable. By taking the complex conjugation of the second variable we obtain

$$\overline{L(-\bar{u})} = L(-2u+1,1^S)^{-1}T(u), \tag{8.5.2}$$

where

$$T(u) = \iint \Phi_1(a,b)|a|^{1/2-u}\eta(a)|b|^{1/2+u}\eta(b)dadb. \tag{8.5.3}$$

Here Φ_1 is a Schwartz-Bruhat function; the *Tate's double integral* $T(u)$, as well as all its derivatives, is bounded on the vertical strip $-1/2 \leq \Re u \leq 0$. At last, on the vertical strip we have $1 \leq \Re(-2u+1) \leq 2$ and the function $L(-2u+1,1^S)^{-1}$ is holomorphic and bounded by a polynomial in $\Im u$. We can write the integral as

$$\int F(u)L(1-2u,1^S)^{-1}T(u)c^{1/2+u}(1/2+u)^{-1}du.$$

Hence our transformation of the contour of the integration is valid. By replacing u with $u-1/2$, we obtain the following expression for (8.4.8):

$$\int F(u-1/2)L(2-2u,1^S)^{-1}T(u-1/2)c^u u^{-1}du \tag{8.5.4}$$

In (8.5.4) the contour of the integration is the line $\Re u = 0$, except that the segment joining the point $-i\varepsilon$ to the point $i\varepsilon$ is replaced by the semicircle centered at 0 which goes through the points $-i\varepsilon$, ε, $i\varepsilon$. Now let $\varepsilon \to 0$. Then the integral over the semicircle tends to

$$i\pi F(-1/2)L(2,1^S)^{-1}T(-1/2)$$

while the integral on the linnear part of the contour tends to the Cauchy's principal value. In terms of the real variable t the integral (8.5.4) is also equal to

$$\int_{-\infty}^{\infty} F(it-1/2)L(2-2it,1^S)^{-1}T(it-1/2)c^{it}t^{-1}dt + i\pi F(-1/2)L(2,1^S)^{-1}T(-1/2). \tag{8.5.5}$$

For real r the function $L(-2it+2,1^S)$ is given by an absolutely and uniformly convergent infinite product (or a Dirichlet series). Its derivatives are therefore bounded and its inverse is also bounded. The derivatives of the factor $L(-2it+2,1^S)^{-1}$ are therefore bounded. In (8.5.5) the product of the first three terms is therefore a Schwartz function of t. When c tends to infinity the Cauchy integral tends to $i\pi$ times the value of the Schwartz function at point 0. In total we see that (8.5.5), i.e. the term (8.4.8), converges as $c \to \infty$, to

$$2i\pi F(-1/2)L(2,1^S)^{-1}T(-1/2),$$

where the limit vanishes if and only if $F(-1/2,1:i,j)$ does. This is what we had to prove. Similar argument holds for (8.4.9).

8.6

Let's move on to (8.4.10). We'll simplify the notation as

$$F(u) = F(u,1:i,j).$$

We are going to use a slightly different expression from the one we have used so far for the Mellin transform.

Write ϕ for ϕ_j and suppose that ϕ is a product of local functions ϕ_v. We can also assume that, for each place v, ϕ_v is either K_v invariant, or has a vanishing integral over K_v. Let S_0 denote the set of places where this last condition is satisfied. Then S_0 is finite and contains S. We can find an integral representation for $\phi(g,u,1)$ of the form

$$\phi(g,u,1) = \int \Phi((0,t)g)|t|^{2u+1}dt \times |\det g|^{u+1/2}L(2u+1,1^{S_0})^{-1} \tag{8.6.1}$$

We can conclude that, as before, the Mellin transform in (8.4.10) can be written as

$$L(-2u+1,1^{S_0})T(u) \tag{8.6.2}$$

where $T(u)$ is defined as a Tate's double integral, holomorphic in u. On the other hand we can write the intertwining operator $M(u,1)$ as a product

$$M(u,1) = L(2u,1)L(2u+1,1)^{-1}N(u,1) \tag{8.6.3}$$

where N is the normalized intertwining operator. Now the quotient of $L(2u,1)$ by $L(-2u+1,1)$ is an exponential function ab^u. It follows that the product of factors (8.6.2) and (8.6.3) reduces to

$$ab^u L(-2u+1,1_{S_0})L(2u+1,1_{S_0})^{-1}L(2u+1,1^{S_0})^{-1}T(u)N(u,1). \tag{8.6.4}$$

Then (8.4.10) is given by the following integral

$$\int F(u)L(2u+1,1^{S_0})^{-1}T(u)c^{1/2-u}(1/2-u)^{-1}A(u)du, \tag{8.6.5}$$

with

$$A(u) = ab^u L(-2u+1,1_{S_0})L(2u+1,1_{S_0})^{-1}N(u,1)(\phi(e)+\phi(w)).$$

We are going to move the contour of integration. The present contour is the line $\Re u = 0$. The new contour will be the line $\Re u = 1/2$, except that the segment joining the points $1/2 - i\varepsilon$ and $1/2 + i\varepsilon$ will be replaced by the semicircle passing through the points $1/2 - i\varepsilon, 1/2 - \varepsilon, 1/2 + i\varepsilon$. Remaining part of the proof will then be the same as in the previous case, except that we have to show that the factor $A(u)$ is holomorphic and has a moderate growth on the strip $0 \leq \Re u \leq 1/2$. The ratio of the factors L which appears in A is the product of the ratios

$$L(-2u+1,1_v)L(2u+1,1_v)^{-1}$$

for all v in S_0. If v is finite, then the ratio is a rational function in q_v^{-u} and has a moderate growth. If v is infinite, then the Stirling's formula implies that the ratio has a moderate growth. Recall that ϕ_v equals to 1 on K_v for all v not in S_0. For such v, we have $N(u,1_v)\phi_v(k_v) = 1$ for all u. So $N(u,1)\phi(e)$ is in fact the product over all v in S_0 of

$$N(u,1_v)\phi_v(e).$$

If v is finite this is still has a moderate growth. If v is infinite, this is a polynomial in u, and so A has a moderate growth. At last, let's prove that A is holomorphic at the poles of the factor $L(-2u+1,1_{S_0})$ on the strip. Let's prove, for example, holomorphy at $1/2$ of

$$L(-2u+1,1_{S_0})L(2u+1,1_{S_0})^{-1}N(u,1)\phi(e).$$

The previous product can be written as

$$\prod_{v \in S_0} L(-2u+1, 1_v) L(2u+1, 1_v)^{-1} N(u, 1_v) \phi_v(e).$$

Take a v in S_0. As the integral of φ_v over K_v is zero, $N(u, 1_v)\phi_v(e)$ vanishes at the point $u = 1/2$ and this zero cancel out the pole of the factor $L(-2u+1, 1_v)$ at the same point. The product is therefore holomorphic at the point $1/2$ and this concludes our proof for the term (8.4.10). A similar argument applies to the term (8.4.11). Hence we complete the proof of the assertions in (8.1).

9 Global orbital integrals: compact torus

9.1

In this section F is again a number field and E a quadratic extension of F. We will fix an element (G', T') of the set $X(E : F)$ and an element ε of $N(T') - T'$. Then the square c of ε is an element of $F^\times$ and the class cN of the group of norms N of E determines the isomorphism class of (G', T'). Let f' be a compactly supported smooth function on the group $G'(\mathbb{A}_F)/Z'(\mathbb{A}_F)$. We have a cuspidal kernel K'_c attached to f'. Let ϕ'_i be an orthonormal basis of the space of automorphic forms which are cuspidal and orthogonal to the functions s$g \mapsto \chi(\det g)$, where χ is a character of idéle class group whose square is trivial. By definition:

$$K'_c(x, y) = \sum_j \rho(f')\phi'_j(x)\overline{\phi'_j}(y). \tag{9.1.1}$$

We will give a useful expression of the integral

$$\int_{[T']} \int_{[T']} K'_c(s, t) ds dt. \tag{9.1.2}$$

Of course $\psi \circ \mathrm{tr}$ is a character of $\mathbb{A}_E/E$ and we therefore have for each place v of E the Tamagawa measure on the group $E_v^\times$ a and an induced measure on T'_v. We also have the product measure on the group $T'(\mathbb{A}_E)$ and the quotient measure on $T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)$. We will denote by S a finite set of places of F containing the places at infinity, the ramified places in E, the places where G' does not split, the places where ψ'_v is not of order 0 and places of residual characteristic 2. We choose for all v a maximal compact subgroup K'_v of G'_v such that T'_v is contained in $K'_v Z'_v$ if v does not split in E and $G'(\mathbb{A}_F)$ be the restricted product of G'_v with respect to K'_v. We suppose that f' is the product of compactly supported smooth local functions f'_v on G'_v/Z'_v. We assume f'_v bi-K'_v-invariant for each v not in S. For v that does not split in E we replace f_v for (9.1.1) as f'_v defined by

$$f'_{v0}(g') = \frac{1}{\mathrm{vol}(T_v)} \int_{T'_v} \int_{T'_v} f'_v(s_v g' t_v) ds_v dt_v.$$

We can therefore assume that each v which does not split in E the function f'_v is bi-T'_v-invariant, in particular bi-K'_v-finite. At last we assume that f'_v is bi-K'_v-finite at v's split in E. Then we have:

$$K'_c(x,y) = \sum_{\gamma \in G'(F)/Z'(F)} f'(x^{-1}\gamma y) - K'_{\mathrm{sp}}(x,y) - K'_{\mathrm{ei}}(x,y). \tag{9.1.3}$$

where K'_{sp} denote the special kernel and K'_{ei} is the Eisenstein kernel. The Eisenstein kernel is zero if G' does not split. The kernel K'_{sp} is defined by the following sum

$$K'_{\mathrm{sp}}(x,y) = \sum_{\chi} \frac{1}{\mathrm{vol}([G'])} \int f'(\det g') \mathrm{d}g' \cdot \chi(\det x)\chi^{-1}(\det y) \tag{9.1.4}$$

where the sum is over all the quadratic characters χ of the idéle class group of F. We define two other kernels

$$K'_r(x,y) = \sum_{\gamma \text{ is } T'\text{-regular}} f(x^{-1}\gamma y) \tag{9.1.5}$$

$$K'_s(x,y) = \sum_{\gamma \text{ is } T'\text{-singular}} f(x^{-1}\gamma y), \tag{9.1.6}$$

then K'_c can be written as a sum

$$K'_c = K'_r + K'_s - K'_{\mathrm{sp}} - K'_{\mathrm{ei}}. \tag{9.1.7}$$

Since $[T']$ is compact, (9.1.2) is simply an integral of each term in (9.1.7).

9.2

Let's consider K'_r first. Each T'-regular element $\gamma \in G'(F)/Z'(F)$ can be uniquely written as a form

$$\gamma = \sigma^{-1}\mu\tau, \tag{9.2.1}$$

where σ and τ are in $T'(F)/Z'(F)$ and μ is a representative of a T'-regular double coset of $T'(F)$ in $G'(F)$ (Proposition (1.2)). Then we get[11]

$$\iint K'_r(s,t) ds dt = \sum_{\mu} \iint f'(s^{-1}\mu t) ds dt, \tag{9.2.2}$$

[11] Of course, the summation on the RHS is over $T'(F)$-regular double coset representatives of $T'(F)$ in $G'(F)$.

where the integral on the RHS is over $T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)$. The double integral of the right-hand side only depends on $\zeta = P'(\mu : T')$ and we will note $H'(\zeta : f' : T')$ its value. Then we can write as

$$\iint K'_r(s,t)dsdt = \sum_{\zeta \in cN-\{1\}} H(\zeta : f' : T'), \tag{9.2.3}$$

since the function P' parametrizes the regular double cosets and its values, on the regular elements, are all the points of the class cN associated with the pair (G', T') minus identity (Proposition (1.1)). Of course the orbital integral $H'(\zeta : f' : T')$ is the product of the local orbital integrals:

$$H'(\zeta : f' : T') = \prod_v H'(\zeta : f'_v : T'_v). \tag{9.2.4}$$

Almost all the factors are equal to 1. Indeed when v be a place of F which is not in S; suppose that f'_v is the characteristic function of $Z'_v K'_v$. If v does not split in E, then T'_v is contained in $Z'_v K'_v$ and the integral is 1. If v splits in E then the local integral is still 1 by Proposition (5.7).

9.3

Let's consider the integral of the term K'_s. There are only two singular double cosets, $T'(F)$ and $\varepsilon T'(F)$. Then we get

$$\iint K'_s(s,t) = \mathrm{vol}([T']) \int_{T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)} f'(t)dt + \mathrm{vol}([T']) \int_{T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)} f'(\varepsilon t)dt. \tag{9.3.1}$$

9.4

Consider K'_{sp}. By (9.1.4), we have

$$\iint K'_{\mathrm{sp}}(s,t)dsdt = \sum_\chi \frac{1}{\mathrm{vol}([G'])} \int f'(g)\chi(\det g)\mathrm{d}g \int \chi(\det s)ds \int \chi^{-1}(\det t)dt, \tag{9.4.1}$$

each of the integral of characters over $[T']$ is 0 unless $s \mapsto \chi(\det s)$ is trivial over $T'(\mathbb{A}_F)$; this is the case if and only if $\chi = 1$ or $\chi = \eta$. The integral of K_{sp} therefore reduces to two terms:

$$\iint K'_{\mathrm{sp}}(s,t)dsdt = \frac{\mathrm{vol}([T'])^2}{\mathrm{vol}([G'])} \left(\int f'(g)\chi(\det g)\mathrm{d}g + \int f'(g)\mathrm{d}g \right) \tag{9.4.2}$$

In particular, let's choose as in (8.1) a place z of F not in S, fix the components of f at places other than z and look at the integral as a function of $\hat{f}'_z$. Then the integral (5) is of the form $c\hat{f}'_z(q_z^{-1})$ for a constant c.

9.5

Consider K_{ei}. It vanishes if G' does not split over F. Suppose G' splits and recall the notations in (8.4). we have:

$$K'_{\mathrm{ei}}(x, y) = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} A(x, y, u) \tag{9.5.1}$$

where

$$A(x, y, u) = \sum_{\chi, j} (\rho(f')E)(x, j, u, \chi)\overline{E(y, j, u, \chi)}. \tag{9.5.2}$$

Note that the maximal compact subgroup implicit in the definition of Eisenstein series is now the product of the groups K'_v, with $K'_v Z'_v = T'_v$ if v is infinite. In particular the series (9.5.2) is finite. As we integrate over a compact set we get:

$$\int_{[T']} \int_{[T']} K'_{\mathrm{ei}}(s, t) ds dt = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} A(u : f') du, \tag{9.5.3}$$

where

$$A(u : f') = \sum_{\chi, j} \int_{[T']} (\rho(f')E)(s, j, u, \chi) ds \int_{[T']} \overline{E(t, j, u, \chi)} dt. \tag{9.5.4}$$

Now $(\rho(f')E)(x, j, u, \chi)$ is zero unless ϕ'_j is K'_v-invariant for all places v not in S. In particular, let's choose as above a place z of F which is not in S and splits in E. Then $f' = f^{z\prime} f'_z$ where $f^{z\prime}$ is the product of f'_v for $v \neq z$ and

$$(\rho(f')E)(x, j, u, \chi) = \hat{f}'_z(q_z^{-2iu})(\rho(f^{z\prime})E)(x, j, u, \chi). \tag{9.5.5}$$

Hence we get

$$\int_{[T']} \int_{[T']} K'_{\mathrm{ei}}(s, t) ds dt = \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \hat{f}'_z(q_z^{-2iu}) A(u : f^{z\prime}) du, \tag{9.5.6}$$

where $A(u : f^{z\prime})$ is integrable, a result which will be sufficient for our purpose.

10 Fundamental lemma

10.1

In this section, we again assume that F is a number field, E is a quadratic extension of G, and η is a quadratic character attached to E. We consider the pair (G, T) of a group GL(2) and a subgroup T of diagonal matrices; fix an element

ε in the normalizer of T which is not in T. We denote by K_v the usual maximal compact subgroup of G_v and we assume ε contained in K_v for all v. We define a finite set S of places of F, containing the infinite places, the places which ramify in E, the places where ψ has no order 0 and the places of residual characteristic 2. It will be convenient to assume that S has an even number of elements. Let $X(S)$ be the set of pairs (G', T') in $X(E : F)$ such that G' splits outside S. For each (G', T') in $X(S)$ and each place v, we choose a maximal compact subgroup K'_v of G'_v so that $G'(\mathbb{A}_F)$ is the restricted product of G'_v with respect to K'_v. We assume that if v does not split in E then T'_v is contained in $K'_v Z'_v$. For all v not in S the measures of $T'_v \cap K'_v / K'_v \cap Z'_v$ and $T_v \cap K_v / K_v \cap Z_v$ are 1. We fix an element in the normalizer of T' which is not in T', and assume that ε' is in K'_v for all v not in S. Let f be a compactly supported smooth function on $G(\mathbb{A}_F)/Z(\mathbb{A}_F)$ and, for each (G', T') in $X(S)$, f' be a compactly supported smooth function on $G'(\mathbb{A}_F)/Z'(\mathbb{A}_F)$. Of course, these functions are assumed to be products of local functions. We also make the following assumptions:

1. Let $v \in S$ be a place that does not split in E. Then f'_v is T'_v-bi-invariant. Moreover if x is an element of F_v that is not 1 or zero, $(G', T') \in X(S)$ and $g' \in G'_v$ with $x = P'(g' : T'_v)$ then
$$H(x : f_v : \eta_v) = H(g' : f'_v : T'_v).$$

2. Let $v \in S$ be a place that splits in E. Then f_v is K_v-finite and f'_v is K'_v-finite. Let g be a A_v-regular element in G_v. If $(G', T') \in X(S)$ and $g' \in G_v$ satisfy
$$P(g : T_v) = P(g' : T'_v)$$
then

   (a)
$$H(g : f_v : T_v) = H(g' : f'_v : T'_v)$$

   (b)
$$\int_{T_v} f_v(a_v) da_v = \int_{T'_v} f'_v(t'_v) dt'_v$$

   (c)
$$\int_{T_v} f_v(\varepsilon a_v) da_v = \int_{T'_v} f'_v(\varepsilon' t'_v) dt'_v.$$

3. If v is not in S then f_v is K_v-bi-invariant, f'_v is K'_v-bi-invariant and the isomorphism between (G_v, K_v) and (G'_v, K'_v) induces a map from f_v to f'_v.

4. *Remark*. In the situation of assumption (2) there is an isomorphism between (G_v, T_v) and (G'_v, T'_v). Condition (2) is satisfied if we take for f'_v the image of f_v under the isomorphism. Indeed this is clear for (2.a) and (2.b). For (2.c), the integral of the right hand side does not change if we replace ε' by the image of ε under the isomorphism in question and then our assertion is obvious.

For given function f, we have a cuspidal kernel K_c for the group G associated to it. Similarly, for each (G', T'), theres a cuspidal kernel K'_c for the group G' attached to the function g'. In this section we will prove the following result:

Theorem 10.1.1. With the previous assumptions, we have

$$\iint K_c(a,b)\eta(\det b)dadb = \sum_{(G',T')\in X(S)} \iint K'_c(s,t)dsdt. \tag{10.1.1}$$

10.2

To prove the identity, as in the section 7 and 9, we write

$$K_c = K_r + K_s - K_{\mathrm{sp}} - K_{\mathrm{ei}}, \tag{10.2.1}$$

$$K'_c = K'_r + K'_s - K'_{\mathrm{sp}} - K'_{\mathrm{ei}}. \tag{10.2.2}$$

We will first prove the following identities:

$$\iint K_r(a,b)\eta(\det b)dadb = \sum_{(G',T')\in X(S)} \iint K'_r(s,t)dsdt, \tag{10.2.3}$$

$$\iint K_s(a,b)\eta(\det b)dadb = \sum_{(G',T')\in X(S)} \iint K'_s(s,t)dsdt. \tag{10.2.4}$$

We first assume these identities and will show how the theorem follows from. Consider the difference

$$\iint K_c(a,b)\eta(\det b)dadb - \sum_{(G',T')\in X(S)} \iint K'_c(s,t)dsdt. \tag{10.2.5}$$

Considering (10.2.3) and (10.2.4), we write

$$\begin{aligned} &-\iint K_{\mathrm{sp}}(a,b)\eta(\det b)dadb + \sum_{(G',T')\in X(S)} \iint K'_{\mathrm{sp}}(s,t)dsdt \\ &-\iint K_{\mathrm{ei}}(a,b)\eta(\det b)dadb + \sum_{(G',T')\in X(S)} \iint K'_{\mathrm{ei}}(s,t)dsdt. \end{aligned}$$

Recall that these are weak integrals for the group G.

Now choose a place z of E which is not in S and splits in E. Fix local factors of f and f' at the other places. At the place z the Satake transforms of f_z and f'_z are the same. Hence we can regard our integrals as functions of $\hat{f}_z$. Then by (8.1), (9.3) and (9.4) the above sum has the form of

$$\int_{-\infty}^{\infty} \phi(t)\hat{f}_z(q_z^{-2it})dt + c\hat{f}_z(q_z^{-1}), \tag{10.2.6}$$

where ϕ is integrable. We finish the proof as in [2] by using the fact that the integrals of K_c and K'_c also have the form

$$\sum_t a_t \hat{f}_z(t), \tag{10.2.7}$$

where the complex numbers t are either on the unit circle or on the real axis between q_z^{-1} and q_z and the series $\sum_t a_t$ absolutely converges. The uniqueness of the decomposition of a measure into an atomic measure and a continuous measure implies that the difference (10.2.5) is zero.

10.3

Let's prove (10.2.3). LHS can be written as

$$\sum_\zeta H(\zeta : f : \eta), \quad \zeta \neq 0, 1,$$

where RHS can be written as a double sum

$$\sum_{(G',T')} \sum_\zeta H'(\zeta : f' : T')$$

the inner sum is over all $1 \neq \zeta \in cN$, determined by the pair (G', T'). We can combine the two sums and write RHS as a sum

$$\sum_\zeta H(\zeta : f' : T'), \quad \zeta \in N(S) - \{1\},$$

where $N(S)$ is the union of the classes cN corresponds to the elements of $X(S)$. According to the class field theory the elements of $F^\times - N(S)$ are exactly the ζ in $F^\times$ which satisfy the following condition: there exists a place v of F, which is not in S, inert in E and not a norm of the quadratic extension E_v of F_v. By Proposition (5.1) we have, for such a ζ, $H(\zeta : f_v : \eta_v) = 0$ if v is the place in question. This results in $H(\zeta : f : \eta) = 0$. Therefore it is sufficient to show the equality of the orbital integrals $H(\zeta : f : \eta)$ and $H(\zeta : f' : T')$ when ζ is in $N(S)$. Decompose these integrals into products of local integrals $H(\zeta : f_v : \eta_v)$ and $H(\zeta : f'_v : T'_v)$ respectively. For v in S the equality of these integrals results from hypotheses (1) and (2). For v not in S the equality follows from hypothesis (3) and proposition (5.1). This proves the equality of the global orbital integrals, and the formula (3).

10.4

Let's prove (10.2.4). We can use (7.3.7) to compute LHS and (9.3.1) to compute RHS. The equality (10.2.4) will then be a consequence of the following two identities

$$H(n_+ : f : \eta) + H(n_- : f : \eta) = \sum_{(G',T')\in X(S)} \mathrm{vol}([T']) \int_{[T']} f'(t')\mathrm{d}t', \tag{10.4.1}$$

$$H(\varepsilon n_+ : f : \eta) + H(\varepsilon n_- : f : \eta) = \sum_{(G',T')\in X(S)} \mathrm{vol}([T']) \int_{[T']} f'(\varepsilon' t')\mathrm{d}t'. \tag{10.4.2}$$

The second identity results from the first identity applied to the function f_1 defined by $f'_1(g) = f'(\varepsilon' g)$. It is indeed easy to verify that the conditions (10.1.1) to (10.1.3) are satisfied by f_1 and f'_1. Therefore let's prove the first identity.

Let's compute (10.4.1). Let a and b be idéles of E and F. The analytic continuation of the Tate integral is

$$\int \phi(t)|t|^s \eta(t)dt \tag{10.4.3}$$

where ϕ is a Schwartz-Bruhat function, whose value at $s = 0$ is

$$L(0,\eta) \prod_{v\in W} \int_{T_v} \phi_v(t_v)\eta(t_v)dt_v L(0,\eta_v)^{-1} \prod_{v\in V} \phi_v(0)|a_v|^{1/2},$$

where W is a set of places of F that do not split in E and V is a set of places split in E.

Apply this formula to the functions ϕ_+ and ϕ_- defined as

$$\phi_+(x) = \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} f\left(a \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) da,$$

$$\phi_-(x) = \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} f\left(a \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\right) da.$$

The local components of ϕ_+ and ϕ_- are defined analogously in terms of the local decomponents of f. Then the right hand side of (10.4.1) is nothing but the sum of the values of the Tate integrals (10.4.3) of ϕ_+ and ϕ_- at the point $s = 0$. Moreover we obviously have for each v in V:

$$\phi_{+v}(0) = \phi_{-v}(0) = \int_{T_v/Z_v} f_v(a_v)da_v.$$

On the other hand, for each v in W the values at point 0 of the Tate integrals of ϕ_{+v} and ϕ_{-v} are nothing but the singular orbital integrals of the points n_+ and n_-. To simplify the notations, define

$$\begin{aligned} M_v &= \int_{T_v/Z_v} f_v(a_v)da_v, \quad v \in V, \\ M_{v\pm} &= 2H(n_\pm : f_v : \eta_v), \quad v \in W. \end{aligned}$$

Then the LHS of (10.4.1) can be written as

$$L(0,\eta)\prod_{v\in W}\frac{1}{2L(0,\eta_v)}\prod_{v\in V}|a_v|^{1/2} \times \prod_{v\in V} M_v \left(\prod_{v\in W} M_{v+} + \prod_{v\in W} M_{v-}\right). \tag{10.4.4}$$

Note that all but finitely many factors of each product are equal to 1.

Let's move on to RHS of (10.4.1). The integral is obviously the product of similar local integrals:

$$\int_{T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)} f'(t')\mathrm{d}t' = \prod_v \int_{T'_v/Z'_v} f'_v(t'_v)dt'_v.$$

If $v \in V$, the local integral is M_v by the assumption (2.b). If $v \in W$ the integral equals to

$$\frac{1}{2\mathrm{vol}(T'_v/Z'_v)}(M_{v-} + \eta_v(c)M_{v+})$$

by Proposition (4.1) and Proposition (5.1). The volume that appears in this formula is $|b_w|^{1/2}|a_v|^{-1/2}$, where w is the only place of E above v. Hence RHS of (10.4.1) is equal to the product

$$2L(1,\eta)\sum_{c\in N(S)/N}\prod_{v\in W}\frac{1}{2L(0,\eta_v)}\prod_{v\in W}\left|\frac{a_v}{b_w}\right|^{1/2}\prod_{v\in V}M_v\prod_{v\in W}\frac{1}{2}(M_{v-} + \eta_v(c)M_{v-}). \tag{10.4.5}$$

Comparing with (10.4.4), we can find that it suffices to prove the following identities:

$$L(0,\eta)\prod_{v\in V}|a_v|^{1/2} = L(1,\eta)\prod_{v\in W}\left|\frac{a_v}{b_w}\right|^{1/2}, \tag{10.4.6}$$

$$\prod_{v \in W} M_{v+} + \prod_{v \in W} M_{v-} = 2 \sum_{c \in N(S)/N} \prod_{v \in W} \frac{1}{2}(M_{v-} + \eta_v(c) M_{v+}). \tag{10.4.7}$$

The equality (10.4.6) immediately follows from the functional equations of the terms $L(s, 1_E)$ and $L(s, 1_F)$ and their relation to $L(s, \eta)$.

Let's move on to (10.4.7). For $v \in W - S$ we have $\eta_v(c) = 1$ by the definition of $N(S)$ and $M_{v+} = M_{v-}$ (Proposition (5.1)); moreover for almost all $v \in W - V$, $M_{v+} = M_{v-} = 1$. For $U = W \cap S$, we see that the identity (10.4.7) reduces to

$$\prod_{v \in U} M_{v+} + \prod_{v \in U} M_{v-} = 2 \sum_{c \in N(S)/N} \prod_{v \in U} \frac{1}{2}(M_{v-} + \eta_v(c) M_{v+}).$$

Let $H = \{1, -1\}^U$. For a place $v \in U$ define a character χ_v of H by the formula $\chi_v(h) = h_v$. Let H' be a subgroup of H defined by the equation $\prod_{v \in U} \chi_v(h) = 1$. Then the map $c \mapsto (\eta_v(c))$ defines a bijection between $N(S)/N$ and H'. Then the formula that we want to prove can be written as

$$\prod_{v \in U} M_{v+} + \prod_{v \in U} M_{v-} = 2 \sum_{h \in H} \prod_{v \in U} \frac{1}{2}(M_{v-} + \chi_v(h) M_{v+}).$$

Since $|H| = 2|H'|$, RHS can be written as

$$\frac{1}{|H'|} \sum_{Y} \sum_{h \in H'} \prod_{v \in Y} \chi_v(h) \prod_{v \in Y} M_{v+} \prod_{v \in U - Y} M_{v-},$$

where the outer sum is over the subset Y of U. The character $\prod_{v \in Y} \chi_v$ is nontrivial on H', unless Y is empty or equal to U. Therefore only the terms corresponding to the empty set and U contributes to the sum; this gives us our equality.

11 Waldspurger's result

11.1

We can finally prove Waldspurger's result using the identites from section 10. We will denote by S a finite set of places of F satisfying the conditions of section 10. By taking S sufficiently large, we can only consider the cuspidal representations of G unramified outside of S, pairs (G', T') belongto $X(S)$ and, for such a pair, cuspidal representations of G' unramified outside S. We will denote by K (resp. K^S) for the product of the compact subgroups K_v for all v (resp. all v not in S) and G^S the restricted product of G_v for v not in S. For a pair (G', T') in $X(S)$, we denote as K', K'^S and G'^S for the analogous groups.

To describe Waldspurger's first condition, note that if

$$\int \phi_1(a)da \quad \text{and} \quad \int \phi_2(b)\eta(\det b)db$$

is nonzero for some pair (ϕ_1, ϕ_2) of smooth vectors, then it is also nonzero for some K-finite vectors (ϕ_1, ϕ_2); moreover, if S is large enough, then ϕ and ϕ' are K^S-invariant. Similarly if there exists a pair (G', T') in X, a cuspidal representation π' and a smooth vector ϕ' in the space of π' such that the integral $\int \phi'(t')\mathrm{d}t'$ is nonzero, then we can take ϕ' to be K'-finite; moreover, if S is large enough, (G', T') is in $X(S)$ and ϕ' is invariant under K'^S.

11.2

Consider a set S and functions f and f' satisfying the conditions in section 10. In particular f_v (resp. f'_v) is bi-invariant under K_v (resp. K'_v). Consider the kernel K_c. We can write as

$$K_c = \sum_\pi K_\pi \tag{11.2.1}$$

where, for each cuspidal automorphic representation π (unramified outside of S), we have

$$K_\pi(x, y) = \sum_j \rho(f)\phi_j(x)\overline{\phi_j}(y), \tag{11.2.2}$$

where $\{\phi_j\}$ is an orthonormal basis of the subspace of K^S-invariant vectors in the space of π. The series (11.2.1) not only converges in the Hilbert space of square integrable functions on the quotient $[G]$, but also in the space of rapidly decreasing functions on the quotient $[G]$. Moreover, since f_v is K_v-finite for all v infinite, the series (11.2.2) has only a finite number of nonzero terms for given f. Denote by $H(S)$ the Hecke algebra of the group G^S relative to the subgroup K^S. Write $f = f_S f^S$, where f_S (resp. f^S) is the product of f_v for v in S (resp. not in S). Let Λ_π be the character of $H(S)$ attached to a representation π. Then we have

$$\iint K_c(a, b)\eta(\det b)dadb = \sum_\pi a(\pi, f_S)\Lambda_\pi(f^S), \tag{11.2.3}$$

where $a(\pi, f_S)$ is

$$a(\pi, f_S) = \sum_j \int \rho(f_S)\phi_j(a)da \int \overline{\phi_j}(b)\eta(\det b)db. \tag{11.2.4}$$

11.3

Consider $(G', T') \in X(S)$. We have a similar decomposition

$$K'_c = \sum_{\pi'} K_{\pi'}, \tag{11.3.1}$$

$$K_{\pi'}(x, y) = \sum_j \rho(f')\phi'_j(x)\overline{\phi'_j}(y), \tag{11.3.2}$$

where $\{\phi'_j\}$ is an orthonormal basis of the subspace of K'^S-invariant vectors in the space of π'. The series (11.3.1) also converges in the space of rapidly decreasing functions and the series (11.3.2) has only finitely many nonzero terms. By integrating term by term, we get

$$\iint K'_\pi(s, t)dsdt = a(\pi', f'_S)\Lambda'_\pi(f'^S) \tag{11.3.3}$$

where $a(\pi', f'_S)$ is given by

$$a(\pi', f'_S) = \sum_j \int \rho(f)\phi'_j(s)ds \int \overline{\phi'_j}(t)dt. \tag{11.3.4}$$

Then the whole integral of K'_c becomes

$$\iint K'_c(s, t)dsdt = \sum_{\pi'} a(\pi', f'_S)\Lambda_{\pi'}(f'^S).$$

11.4

Now let's use our fundamental lemma. Note that if π' is a cuspidal representation of G' and π the corresponding cuspidal representation of G, then $\Lambda_\pi(f_S) = \Lambda_{\pi'}(f'_S)$. Since π' determines π we can write $a(\pi, f'_S)$ for $a(\pi', f'_S)$. On the other hand, for a representation π of the group G, it will be convenient to set $a(\pi, f'_S) = 0$ if there is no representation π' of G' corresponding to π. Then our fundamental lemma is written as

$$\sum_\pi a(\pi, f_S)\Lambda_\pi(f^S) = \sum_\pi \sum_{(G',T')} a(\pi, f'_S)\Lambda_\pi(f^S) \tag{11.4.1}$$

Here f^S is an arbitrary element of $H(S)$. Let v be a place in S. Then the function f_v is an arbitrary K_v-finite function. The function f'_v matches with f_v by the conditions (10.1.1) and (10.1.2). If v splits then f'_v is in fact can be chosen as an arbitrary K'_v-finite function. If v does not split, then f'_v is no longer arbitrary but satisfies a density condition: if a function h is bi-T_v-invariant on G_v/Z_v and orthogonal to all f'_v then h is zero (Proposition 4.2). Suppose that π satisfies Waldspurger's first condition; then there exist K-finite vectors ϕ_1 and ϕ_2 in the space of π such that:

$$\int \phi_1(a)da \neq 0 \quad \text{and} \quad \int \phi_2(b)\eta(\det b)db \neq 0,$$

and we can assume ϕ_1 and ϕ_2 to be K^S-invariant. Choose a basis $\{\widetilde{\phi_j}\}$ such that $\widetilde{\phi_1} = \phi_2/\|\phi_2\|$. There exists f_S such that $\rho(f_S)\widetilde{\phi_1} = \phi_1$ and $\rho(f_S)\widetilde{\phi_j} = 0$ if $j \neq 1$. Then we have

$$a(\pi, f_S) = \int \phi_1(a)da \overline{\int \widetilde{\phi_1}(b)\eta(\det b)db} \neq 0.$$

According to the principle of *infinite* linear independence of the characters of $H(S)$ [2], there exists a pair (G', T') such that $a(\pi, f'_S)$ is nonzero. It clearly follows that there exists ϕ' in the space of π' such that $\int \phi'(t')\mathrm{d}t'$ is nonzero. Thus π satisfies the second condition of Waldspurger.

Now assume that there exists a pair (G', T'), a representation π' and a K'-finite vector ϕ' in the space of π' such that the integral $\int \phi'(t')\mathrm{d}t'$ is nonzero; we can assume that (G', T') is in $X(S)$ and ϕ' is invariant under K'^S. We will see that we can choose f'_S so that $a(\pi', f'_S)$ is nonzero. The integral over T' defines a continuous linear functional on the space of smooth vectors of π' fixed by K'^S. Let's write it as the dot product with a *generalized* vector $e_{T'}$:

$$\int \phi'(t')\mathrm{d}t' = (\phi', e_{T'}).$$

If h is a compactly supported smooth function on G_S/Z_S, then $\pi'(h)(e_{T'})$ is defined: it is a smooth vector satisfies $(\phi, \pi'(h)e_{T'}) = (\pi'(h^*)\phi, e_{T'})$ for any vector ϕ, smooth or not. With this notation we have

$$a(\pi, f'_S) = (\pi'(f'_S)e_{T'}, e_{T'}).$$

The subspace of K'^S-invariant vectors of π' is isomorphic to the tensor product of the spaces of π'_v with v in S. For each v in S, there exists a continuous linear functional on the space of smooth vectors of π' nonzero at e'_v which is invariant under T'_v. This functional is unique up to scalar (cf. (6.1) and (6.2)), and we can therefore write:

$$a(\pi', f'_S) = (\pi'(f'_S)e_{T'}, e_{T'}) = C \prod_{v \in S} (\pi'_v(f'_v)e'_v, e'_v),$$

where C is a nonzero constant. We want to show that we can choose f'_v so that $(\pi_v(f'_v)e'_v, e'_v)$ is nonzero. It is obvious when v splits since f'_v is then arbitrary K'_v-finite. If v splits e'_v is in fact an ordinary vector since T'_v is compact. Then $(\pi_v(f'_v)e'_v, e'_v)$ is the scalar product of the function f'_v with the continuous matrix coefficient $(\pi'(g)e'_v, e'_v)$; therefore it cannot be zero for any f'_v according to the density property of f'_v. On the other hand if (G'', T'') is another element of $X(S)$, then $a(\pi, f''_S) = 0$; otherwise there would exist at least one place v in S where the groups G'_v and G''_v are not isomorphic and the representations π'_v and π''_v admit nonzero vectors that are invariant under T'_v and T''_v respectively. But this is impossible (Proposition (6.3)). The coefficient of Λ_π in the RHS of (11.4.1) is therefore nonzero, for a suitable choice of f'_S (i.e. f_S). This results that $a(\pi, f_S)$ is not zero. This obviously implies that π satisfies Waldspurger's first condition. Therefore this completes the proof of the equivalence of the two Waldspurger conditions.

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