LANGLANDS FUNCTORIALITY CONJECTURE - A SURVEY

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1. INTRODUCTION

Conjecture 1.1 (Langlands functoriality conjecture). *Let G and G' be reductive groups over a global field F.*

This is an introductory note on Langlands functoriality conjecture view towards classical examples. Here is a list of topics we are going to study:

(1) Automorphic induction
(2) Base change
(3) Rankin-Selberg product
(4) Symmetric power lifting and Selberg's 1/4 conjecture
(5) Jacquet-Langlands correspondence
(6) Theta correspondence and Howe duality

2. AUTOMORPHIC INDUCTION

2.1. **From GL_1/K to $\mathrm{GL}_2/\mathbb{Q}$.** Let K be a quadratic field (over $\mathbb{Q}$) and ξ be a Hecke character for K. By Hecke and Maass, it was proven that one can associate GL_2 automorphic forms. Hecke attached modular forms to Hecke characters for imaginary quadratic fields, and Maass attached Maass forms to those for real quadratic fields. More precisely, they proved the following:

Theorem 2.1 (Hecke). *Let ξ (mod $\mathfrak{m}$) be a primitive Hecke character of $K = \mathbb{Q}(\sqrt{D})$ of discriminant $D < 0$ such that*

$$\xi((a)) = \left(\frac{a}{|a|}\right)^u \quad \text{if } a \equiv 1 \, (\mathrm{mod}\, \mathfrak{m})$$

where u is a non-negative integer. Then

$$f(z) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \xi(\mathfrak{a})(N\mathfrak{a})^{\frac{u}{2}} e^{2\pi(N\mathfrak{a})z}$$

is a modular form of weight $k = u + 1$ and level $N = |D| \cdot N\mathfrak{m}$ with Nebentypus χ (mod N), which is a Dirichlet character defined as

$$\chi(n) = \chi_D(n)\xi((n)) \quad n \in \mathbb{Z}.$$

Theorem 2.2 (Maass). *Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of discriminant $D > 0$ and ξ (mod $\mathfrak{m}$) a Hecke character such that*

$$\xi((a)) = \frac{a}{|a|} \quad \text{if } a \equiv 1 \, (\mathrm{mod}\, \mathfrak{m})$$

or

$$\xi((a)) = \frac{a'}{|a'|} \quad \text{if } a \equiv 1 \, (\mathrm{mod}\, \mathfrak{m})$$

1

where a' is a conjugate of a over $\mathbb{Q}$. Then

$$u(z) = \sum_{\mathfrak{a} \in \mathcal{O}_K} \xi(\mathfrak{a}) y^{\frac{1}{2}} e^{2\pi i(N\mathfrak{a})z}, \quad z = x + yi$$

is a Maass form of level N, eigenvalue $1/4$, and a Nebentypus $\chi \pmod N$ for $N = D \cdot N\mathfrak{m}$.

Both theorem can be proved using converse theorems for L-functions. By showing that the L-function attached to Hecke character ξ satisfies suitable functional equations, converse theorem shows that the L function should coincides with one comes from modular forms or Maass forms.

2.2. From GL_n/K to GL_{rn}/F.
In view of Langlands functoriality conjecture, Hecke and Maass' results can be considered as a special case when $G = \mathrm{GL}_1/K$ and $G' = \mathrm{GL}_2/\mathbb{Q}$. Automorphic induction, which is a vast generalization of this, is a functoriality from GL_n/K to GL_{rn}/F, where K/F is a degree r extension. On Galois side, this actually corresponds to the *induction* of Galois representation of $G_K = \mathrm{Gal}(\overline{K}/K)$ to $G_F = \mathrm{Gal}(\overline{F}/F)$. In other words, if one has a GL_n/K automorphic representation π with corresponding Galois representation $\sigma = \sigma(\pi) : G_K \to \mathrm{GL}_n(\mathbb{C})$, then automorphic induction predicts the existence of GL_{rn}/F automorphic representation Π that corresponds to the Galois representation

$$\Sigma = \mathrm{Ind}_{G_K}^{G_F} \sigma : G_F \to \mathrm{GL}_{rn}(\mathbb{C})$$

via Langlands correspondence.

Conjecture 2.1 (Automorphic induction). *Let K/F be a degree r extension of number fields. Let π be a GL_n/K automorphic representation. Then there exists a GL_{rn}/F automorphic representation $\Pi = \mathrm{AI}_F^K(\pi)$ such that*

(1) *the Galois representations*

$$\sigma = \sigma(\pi) : G_K \to \mathrm{GL}_n(\mathbb{C}), \quad \Sigma = \Sigma(\Pi) : G_F \to \mathrm{GL}_{rn}(\mathbb{C})$$

corresponds to π and Π via Langlands correspondence satisfies

$$\Sigma \simeq \mathrm{Ind}_{G_K}^{G_F} \sigma.$$

(2) *Local L-functions of π and Π are related as*

$$L(s, \Pi_v) = \prod_{w|v} L(s, \pi_w)$$

for all but finitely many v.

This is open in general, but proven to be true for some cases. We give a sketch of proofs for known cases.

2.2.1. *Local automorphic induction (Henniart-Herb).*
In [5], Henniart and Herb proved automorphic induction over local fields of characteristic zero, i.e. for p-adic fields. To give a precise statement, we first define some terminologies. Let F be a characteristic zero local field and E be a finite extension of F of degree d.

2.2.2. *Cyclic Galois extension of prime degree (Arthur-Clozel).*

Theorem 2.3 (Arthur-Clozel, [1])**.**

3. Base change

3.1. Doi-Naganuma lifting.

In [3], Doi and Naganuma constructed a lifting from the space of elliptic modular forms to the space of Hilbert modular forms of the same (parallel) weight. The proved the following theorem:

Theorem 3.1 (Doi-Naganuma [3]). *Let p be a prime such that the real quadratic field $F = \mathbb{Q}(\sqrt{p})$ has class number 1, and let ϕ_p be the Dirichlet character associated to F. Let $f(z) = \sum_{n \geq 1} a_n q^n \in S_k(p, \phi_p)$ be a weight k Hecke eigenform of level p with Nebentypus ϕ_p. Then there exists a Hilbert modular form $h := \mathrm{DN}(f)$ with respect to $\mathrm{GL}_2(\mathcal{O}_F)$, the Doi-Naganuma lift of f, that satisfies*

(1) *h is also an Hecke eigenform of weight k,*
(2)

$$L(s, \mathrm{DN}(f)) = L(s, f)L(s, f^\rho)$$

where $f^\rho(z) := \sum_{n \geq 1} \overline{a_n} q^n$ is a complex conjugate of $f(z)$,
(3) *has a Fourier expansion*

$$h(z_1, z_2) = -\frac{B_k}{2k} \tilde{a}_0 + \sum_{\substack{\nu \in \mathfrak{d}_F^{-1} \\ \nu \gg 0}} \sum_{d \mid \nu} d^{k-1} \tilde{a}_{p\nu\nu'/d^2} q_1^\nu q_2^{\nu'}$$

where B_k denotes the k-th Bernoulli number, ν' is a conjugate of ν, and $q_j = e^{2\pi i z_j}$.

4. Rankin-Selberg product

4.1. Rankin-Selberg convolution of modular forms.

Let f, g be two holomorphic cusp forms of weight k and level 1. Assume that two forms have Fourier expansions

$$f(z) = \sum_{n \geq 1} a_n e^{2\pi i n}, \quad g(z) = \sum_{n \geq 1} b_n e^{2\pi i n}.$$

If f, g are Hecke eigenforms, then their L-functions admit euler products as

$$L(s, f) = \sum_{n \geq 1} \frac{a_n}{n^s} = \prod_p \frac{1}{(1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})}$$

$$L(s, g) = \sum_{n \geq 1} \frac{b_n}{n^s} = \prod_p \frac{1}{(1 - \alpha_p' p^{-s})(1 - \beta_p' p^{-s})}$$

where $\alpha_p + \beta_p = a_p$, $\alpha_p' + \beta_p' = b_p$, and $\alpha_p \beta_p = \alpha_p' \beta_p' = p^{k-1}$ for all p. Rankin (1939) and Selberg (1940) independently studied the *convolution* of two L-series attached to f and g, which is

$$L(s, f \times g) = \sum_{n \geq 1} \frac{a_n \overline{b_n}}{n^s} = \prod_p \frac{1}{(1 - \alpha_p \alpha_p' p^{-s})(1 - \alpha_p \beta_p' p^{-s})(1 - \beta_p \alpha_p' p^{-s})(1 - \beta_p \beta_p' p^{-s})}$$

and studied its analytic properties. They proved that the new L-function also satisfy similar properties as original L-functions $L(s, f)$ and $L(s, g)$: it admits a meromorphic continuation, bounded on vertical strips, and satisfy a functional equation.

Theorem 4.1 (Rankin-Selberg convolution). *Let f, g be two holomorphic cusp eigenforms of weight k on $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Let*

$$\Lambda(s, f \times g) = (2\pi)^{-2s}\Gamma(s)\Gamma(s - k + 1)\zeta(2s - 2k + 2)L(s, f \times g)$$

be a completed L-function. Then $\Lambda(s, f \times g)$, which is originally defined for large $\Re(s)$, admits a meromorphic continuation to all s except for at most simple poles at $s = k$ and $k - 1$. Also, it satisfies a functional equation

$$\Lambda(s, f \times g) = \Lambda(2k - 1 - s, f \times g).$$

Essence of the proof is using real-analytic Eisenstein series with *unfolding trick*. For $s \in \mathbb{C}$, define a real-analytic Eisenstein series $E_s(z)$ as

$$E_s(z) := \sum_{\gamma \in P \backslash \mathrm{SL}(2,\mathbb{Z})} \Im(\gamma z)^s$$

where $P = \left\{ \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix} \right) \right\} \subset \Gamma$ is a standard parabolic subgroup of Γ. Clearly, this is a Γ-invariant function, and it converges for $\Re(s) > 1$. Also, using $\Delta y^s = s(1 - s)y^s$, one can show that $E_s(z)$ is also a Maass form with eigenvalue $s(1 - s)$ (but not a cusp form). By computing its Fourier expansion, we can see that $E_s(z)$, as a function in s, satisfies the functional equation

$$\xi(2s)E_s(z) = \xi(2 - 2s)E_{1-s}(z)$$

where $\xi(s)$ is the completed zeta function

$$\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

that satisfies $\xi(s) = \xi(1 - s)$ for all s.

Proposition 4.1.

$$\langle f \cdot E_s, g \rangle = (4\pi)^{-(s+2k-1)}\Gamma(s + 2k - 1)\sum_{n \geq 1} L(s + 2k - 1, f \times g)$$

where $\langle -, - \rangle$ is the Petersson inner product.

Proof. The idea is to unfold the integral. If φ is a P-invariant function on $\mathfrak{H}$, then Fubini's theorem gives

$$\int_{\Gamma \backslash \mathfrak{H}} \sum_{\gamma \in P \backslash \Gamma} \varphi(\gamma z) \frac{dxdy}{y^2} = \int_{P \backslash \mathfrak{H}} \varphi(z) \frac{dxdy}{y^2} = \int_0^\infty \int_0^1 \varphi(z) \frac{dxdy}{y^2}$$

(the fundamental domain of $P \backslash \mathfrak{H}$ is $\{z = x + iy \in \mathfrak{H} : 0 \leq x < 1\}$.) Once we apply this for $\varphi(z) = y^s f(z)\overline{g(z)}y^{2k}$, we get

$$\langle f \cdot E_s, g \rangle = \int_0^\infty \int_0^1 y^s f(z)\overline{g(z)}y^{2k} \frac{dxdy}{y^2}$$

$$= \sum_{m,n \geq 1} a_m \overline{b_n} y^{s+2k-1} e^{-2\pi(m+n)y} \left(\int_0^1 e^{2\pi i(m-n)x}dx \right) \frac{dy}{y}$$

$$= \sum_{n \geq 1} a_n \overline{b_n} \int_0^\infty y^{s+2k-1} e^{-4\pi ny} \frac{dy}{y}$$

$$= (4\pi)^{-(s+2k-1)}\Gamma(s + 2k - 1)L(s + 2k - 1, f \times g).$$

$\square$

4.2. **Modularity of** $\mathrm{GL}(2) \times \mathrm{GL}(2)$**.** It is also possible to construct Rankin-Selberg L-function attached to two Maass cusp forms with similar properties. In general, for given automorphic representations π_1, π_2 on $\mathrm{GL}(2)$, one can define Rankin-Selberg L-function $L(s, \pi_1 \times \pi_2)$. According to the philosophy of Langlands, there should exists a $\mathrm{GL}(4)$ automorphic representation whose L-function is $L(s, \pi_1 \times \pi_2)$. This is proven by Ramakrishnan in 2000.

Theorem 4.2 (Ramakrishnan, [11]). *Let π_1, π_2 be automorphic forms on $\mathrm{GL}(2, \mathbb{A})$. Then there exists an automorphic representation $\pi_1 \boxtimes \pi_2$ on $\mathrm{GL}(4, \mathbb{A})$ whose L-function equals the Rankin-Selberg L-function, i.e*

$$L(s, \pi_1 \boxtimes \pi_2) = L(s, \pi_1 \times \pi_2).$$

As a corollary, he proved multiplicity one result for $\mathrm{SL}(2)$.

4.3. $\mathrm{GL}(n) \times \mathrm{GL}(n)$**.**

5. Symmetric power lifting

Automorphic forms on GL_2 are often *classified* into two kinds of objects: *modular forms* and *Maass forms*[1]. These functions oftenly considered as a starting point for studying automorphic forms and representations for GL_n and other groups. However, there are not many references for GL_3.

5.1. **Automorphic forms on** GL_3**.** We first introduce the theory of automorphic forms on GL_3 (We follow the Bump's book [2]). We only consider the level 1 automorphic forms. Before we start, let's revisit the GL_2. Modular forms and Maass forms are certain functions defined on the complex upper half plane $\mathfrak{H}$, and one can lift the functions as a function on $\mathrm{GL}_2(\mathbb{R})$ by viewing $\mathfrak{H}$ as a symmetric space

$$\mathfrak{H} \simeq \mathrm{GL}_2(\mathbb{R})/Z(\mathbb{R})\mathrm{O}(2).$$

Here $Z(\mathbb{R}) \simeq \mathbb{R}^\times$ is the center of $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{O}(2)$ is a group of orthogonal matrices. The above isomorphism holds since $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathbb{H}$ transitively and the stabilizer of i is $Z(\mathbb{R})\mathrm{O}(2)$. To develop a theory of automorphic forms on GL_3, it is natural to consider them as a function defined on the symmetric space

$$\mathfrak{H}_3 := \mathrm{GL}_3(\mathbb{R})/Z(\mathbb{R})\mathrm{O}(3).$$

Using Iwasawa decomposition, each coset has a unique representation of the form

$$\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}, \quad y_1, y_2 > 0.$$

Especially, the space is parametrized with 5 real variables and has a real dimension 5, so we can't expect any *holomorphic* automorphic form over GL_3, in constrast to the GL_2 case. Also, we have an involution ι on $\mathfrak{H}_3$ defined as

$$\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -x_1 & x_1 x_2 - x_3 \\ 0 & 1 & -x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}.$$

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[1]and constant functions.

What about the Fourier expansion of GL_3 automorphic forms? In case of GL_2, the algebra of $GL_2(\mathbb{R})$-invariant differential operators on $\mathfrak{h}_2$ is isomorphic to a polynomial ring of single variable, generated by the following hyperbolic Laplacian

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

A 1-periodic function $f(z) = \sum_{n \geq 0} a_n(y) e^{2\pi i n x}$ become an eigenfunction with respect to Δ if the coefficients $a_n(y)$ satisfy certain degree 2 linear differential equations. More precisely, when $\Delta f = \left(\frac{1}{4} - \nu^2 \right) f$, the n-th coefficient $a_n(y)$ satisfies

$$y^2 \frac{\partial^2}{\partial y^2} a_n(y) + \left(\frac{1}{4} - \nu^2 - 4\pi^2 n^2 y \right) a_n(y) = 0.$$

Among two linearly independent solutions, only one satisfies the required growth condition (the other one grows exponentially), which can be expressed with a Bessel function of second kind:

$$a_n(y) = c_n \sqrt{y} K_\nu(2\pi|n|y), \quad K_\nu(y) := \frac{1}{2} \int_0^\infty e^{\frac{y(t+t^{-1})}{2}} t^\nu \frac{dt}{t}.$$

For GL_3, the algebra of $GL_3(\mathbb{R})$-invariant differential operators on $\mathfrak{h}_3$ is a polynomial ring in *two* variables, with two specific generators Δ_1, Δ_2. Then the automorphic forms of $GL(3, \mathbb{R})$ would be defined as functions that are eigenforms with respect to these two operators. Then the coefficients of the Fourier expansion (which will be defined explicitly later) of the automorphic forms would satisfy specific differential equations. In fact, for given λ and μ, there are 6 linearly independent functions that are

(1) eigenfunctions with respect to Δ_1, Δ_2, i.e.

$$\Delta_1 F = \lambda F$$
$$\Delta_2 F = \mu F$$

(2) and satisfies the equation

$$F\left(\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \tau \right) = e(x_1 + x_2) F(\tau)$$

for all τ and $x_1, x_2, x_3 \in \mathbb{R}$, where $e(x) := \exp(2\pi i x)$.

Among these 6 solutions, only one of them *decays rapidly*, which is the appropriate substitute of $K_\nu(y)$ for GL_3 (This is multiplicity one theorem for GL_3). It can be written as an inverse Mellin transform of a certain 2-variable function $V(s_1, s_2)$,

$$W(y_1, y_2) = W_{\nu_1, \nu_2}(y_1, y_2)$$

$$= \frac{1}{4} \frac{1}{(2\pi i)^2} \int_{\sigma - i\infty}^{\sigma + i\infty} \int_{\sigma - i\infty}^{\sigma + i\infty} V(s_1, s_2)(\pi y_1)^{1 - s_1}(\pi y_2)^{1 - s_2} ds_1 ds_2$$

where

$$V(s_1, s_2) = \frac{\Gamma\left(\frac{s_1 + \alpha}{2} \right) \Gamma\left(\frac{s_1 + \beta}{2} \right) \Gamma\left(\frac{s_1 + \gamma}{2} \right) \Gamma\left(\frac{s_2 - \alpha}{2} \right) \Gamma\left(\frac{s_2 - \beta}{2} \right) \Gamma\left(\frac{s_2 - \gamma}{2} \right)}{\Gamma\left(\frac{s_1 + s_2}{2} \right)}.$$

Here α, β, γ are auxillary parameters satisfy

$$\alpha = -\nu_1 - 2\nu_2 + 1$$
$$\beta = -\nu_1 + \nu_2$$
$$\gamma = 2\nu_1 + \nu_2 - 1$$
$$\lambda = -1 - \alpha\beta - \beta\gamma - \gamma\alpha$$
$$\mu = -\alpha\beta\gamma.$$

The Whittaker function $W(y_1, y_2)$ also can be written as an integral of Bessel functions

$$W_{\nu_1,\nu_2}(y_1, y_2) = 4y_1^{1-\frac{\beta}{2}} y_2^{1+\frac{\beta}{2}} \int_0^\infty K_{\frac{\gamma-\alpha}{2}}(2\pi y_2 \sqrt{1+u^{-2}}) K_{\frac{\gamma-\alpha}{2}}(2\pi y_1 \sqrt{1+u^2}) u^{\frac{-3\beta}{2}} \frac{du}{u}.$$

Definition 5.1 (Automorphic form on $\mathrm{GL}_3(\mathbb{R})$). *Let $\nu_1, \nu_2 \in \mathbb{C}$. An automorphic form of type (ν_1, ν_2) on $\mathrm{GL}_3(\mathbb{R})$ is a function $\phi : \mathfrak{H}_3 \to \mathbb{C}$ such that*

(1) *$\phi(g\tau) = \phi(\tau)$ for all $g \in \mathrm{GL}_3(\mathbb{Z})$ and $\tau \in \mathrm{GL}_3(\mathbb{R})$.*
(2) *ϕ is an eigenfunction of Δ_1, Δ_2 with eigenvalues λ, μ defined above,*
(3) *there exists n_1, n_2 such that*

$$y_1^{n_1} y_2^{n_2} \phi \left(\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \right)$$

is bounded on the subset of $\mathfrak{H}_2$ determined by $y_1, y_2 > 1$.

In addition, for all $\tau \in \mathfrak{H}_3$, if

$$\int_0^1 \int_0^1 \phi \left(\begin{pmatrix} 1 & & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} \tau \right) d\xi_1 d\xi_3 = 0$$

and

$$\int_0^1 \int_0^1 \phi \left(\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \\ & & 1 \end{pmatrix} \tau \right) d\xi_1 d\xi_3 = 0$$

then ϕ is called cusp form.

Note that, for a given automorphic form ϕ of type (ν_1, ν_2), the *dual form* $\tilde{\phi}(\tau) := \phi({}^\iota\tau)$ is also an automorphic form, but of type (ν_2, ν_1).

Any ϕ has a Fourier expansion with double indices

$$\phi(\tau) = \sum_{g \in \Gamma_\infty^2 \backslash \Gamma^2} \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \hat{\phi}_{n_1, n_2} \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} z \right)$$

where $\Gamma_\infty^2, \Gamma^2$ are the subgroups of $\mathrm{GL}_3(\mathbb{Z})$ defined as

$$\Gamma^2 = \left\{ \begin{pmatrix} * & * & \\ * & * & \\ & & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \right\}, \Gamma_\infty^2 = \Gamma^2 \cap \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \right\}$$

and $\hat{\phi}_{n_1, n_2}(z)$ is

$$\hat{\phi}_{n_1, n_2}(z) = \int_0^1 \int_0^1 \int_0^1 \phi(xz) e^{-2\pi i(n_1 x_1 + n_2 x_2)} dx$$

where

$$x = \begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}$$

and $dx = dx_1 dx_2 dx_3$. One can check that $\hat{\phi}_{n_1, n2}(z)$ is a rapidly decreasing Whittaker function, and multiplicity one theorem gives us that it should be a multiple of (suitable modification of) $W_{\nu_1, \nu_2}(y_1, y_2)$, that is

$$\phi(\tau) = \sum_{g \in \Gamma_\infty^2 \backslash \Gamma^2} \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{a_{n_1, n_2}}{n_1 n_2} W_{1,1}^{(\nu_1, \nu_2)} \left(\begin{pmatrix} n_1 n_2 & & \\ & n_1 & \\ & & 1 \end{pmatrix} g\tau \right).$$

Here

$$W_{1,1}^{(\nu_1, \nu_2)}(\tau) = W_{\nu_1, \nu_2}(y_1, y_2) e(x_1 + x_2).$$

We call $\{a_{n_1, n_2}\}$ the matrix of Fourier coefficients of ϕ. From this, we define the corresponding L-function as

$$L(s, \phi) = \sum_{n \geq 1} \frac{a_{1,n}}{n^s}.$$

As we expect, this function admits an analytic continuation and satisfies certain functional equation.

Theorem 5.1 (L-function of an automorphic form on $\mathrm{GL}_3(\mathbb{R})$)**.** *The L-function $L(s, \phi)$ of an $\mathrm{GL}_3(\mathbb{R})$ automorphic form ϕ admits an analytic continuation for all $\mathbb{C}$ and satisfies the functional equation*

$$\Phi(s) L(s, \phi) = \tilde{\Phi}(1-s) L(1-s, \tilde{\phi})$$

where $\Phi(s), \tilde{\Phi}(s)$ are Gamma factors

$$\Phi(s) = \pi^{-\frac{3s}{2}} \Gamma\left(\frac{s-\alpha}{2}\right) \Gamma\left(\frac{s-\beta}{2}\right) \Gamma\left(\frac{s-\gamma}{2}\right)$$

$$\tilde{\Phi}(s) = \pi^{-\frac{3s}{2}} \Gamma\left(\frac{s+\alpha}{2}\right) \Gamma\left(\frac{s+\beta}{2}\right) \Gamma\left(\frac{s+\gamma}{2}\right)$$

and $L(s, \tilde{\phi})$ is the L-function of the dual automorphic form which equals

$$L(s, \tilde{\phi}) = \sum_{n \geq 1} \frac{a_{n,1}}{n^s}.$$

It is also possible to define Hecke operators on the space of $\mathrm{GL}_3(\mathbb{R})$ automorphic forms. We define them via double cosets, and the ring of Hecke operators became commutative. Note that, for each $n \geq 1$, there are *two* Hecke operators T_n, S_n, where

Definition 5.2 (Hecke operators)**.** *Let $\mathcal{H} = \mathbb{Z}[\Gamma \backslash G / \Gamma]$ be a $\mathbb{Z}$-algebra of double cosets where $G = \mathrm{GL}_3(\mathbb{R})$ and $\Gamma = \mathrm{GL}_3(\mathbb{Z})$, which is called* Hecke algebra. *It decomposes as a (internal) tensor product*

$$\mathcal{H} = \bigotimes_p \mathcal{H}_p$$

where $\mathcal{H}_p$ is a subalgebra of $\mathcal{H}$ corresponding to the double cosets whose elementary divisors are powers of a given prime p. For each prime p, $\mathcal{H}_p$ is generated by three elements

$$T_p := \Gamma \begin{pmatrix} p & & \\ & 1 & \\ & & 1 \end{pmatrix} \Gamma, \quad S_p := \Gamma \begin{pmatrix} p & & \\ & p & \\ & & 1 \end{pmatrix} \Gamma, \quad R_p := \Gamma \begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix} \Gamma.$$

The whole $\mathcal{H}$ is generated by the operators T_n, S_n, R_n where

$$T_n := \sum_{n_0^3 n_1^2 n_2 = n} \Gamma \begin{pmatrix} n_0 n_1 n_2 & & \\ & n_0 n_1 & \\ & & 1 \end{pmatrix} \Gamma$$

$$S_n := \sum_{n_0^3 n_1^2 n_2 = n} \Gamma \begin{pmatrix} n_0^2 n_1^2 n_2 & & \\ & n_0^2 n_1 n_2 & \\ & & n_0^2 n_1 \end{pmatrix} \Gamma$$

$$R_n := \Gamma \begin{pmatrix} n & & \\ & n & \\ & & n \end{pmatrix} \Gamma$$

which satisfies certain relations given as the formal power series

$$\sum_{n \geq 1} \frac{T_n}{n^s} = \prod_p \frac{1}{1 - T_p \cdot p^{-s} + S_p \cdot p^{1-2s} - R_p \cdot p^{3-3s}}.$$

If ϕ is an automorphic form on $\mathrm{GL}_3(\mathbb{R})$, then the action of Hecke algebra on the form is defined as

$$(\phi | \Gamma \alpha \Gamma)(\tau) := \sum_i \phi(\alpha_i \tau)$$

where α_i's are the representatives of the double coset $\Gamma \alpha \Gamma$, i.e. $\Gamma \alpha \Gamma = \cup_i \Gamma \alpha_i$.

Note that the Hecke operators also commutes with the differential operators Δ_1 and Δ_2, so the space of automorphic forms of type (ν_1, ν_2) has a basis consisting of simultaneous eigenforms for all Hecke operators. Also, the coefficients of $\phi | T_n$ and $\phi | S_n$ can be expressed as certain sums of coefficients of ϕ - see the equations (9.8) and (9.9) in [2].

In GL_2, L-function attached to an automorphic form admits an Euler product if and only if the form is Hecke eigenform, and the local factors has a form of $P_p(p^{-s})^{-1}$, where $P_p(x)$ is a polynomial of degree 2. The same thing also holds for $\mathrm{GL}_3(\mathbb{R})$, where the local factors are inverses of cubic polynomials in p^{-s}.

Theorem 5.2 (Euler product of L-function)**.** *If ϕ is a normalized Hecke eigenform on $\mathrm{GL}_3(\mathbb{R})$ with matrix coefficients $\{a_{n_1, n_2}\}$, then its L-function has an Euler product*

$$L(s, \phi) = \prod_p \frac{1}{1 - a_{1,p} p^{-s} + a_{p,1} p^{-2s} - p^{-3s}}.$$

5.2. Symmetric square lifting by Gelbart-Jacquet. Let f be a level 1 Maass cusp form (of weight 0) on $\mathrm{GL}_2(\mathbb{R})$ with eigenvalue $\lambda = \nu(1 - \nu)$ which is also a normalized eigenform. Let $\{a_n\}_{n \geq 1}$ be Fourier coefficients of f. Consider the

Rankin-Selberg L-function of $f \times f$, i.e.

$$L(s, f \times f) = \zeta(2s) \sum_{n \geq 1} \frac{|a_n|^2}{n^s} = \zeta(2s) \prod_p \frac{1}{(1 - \alpha_p^2 p^{-s})(1 - p^{-s})^2(1 - \beta_p^2 p^{-s})}$$

where $a_p = \alpha_p + \beta_p, \alpha_p \beta_p = 1$. By the theory of Rankin-Selberg convolution, the L-function admits an analytic continuation to $\mathbb{C}$ with functional equation

$$\Lambda(s, f \times f) = G(s)L(s, f \times f) = \Lambda(1 - s, f \times f)$$

where $G(s)$ is the Gamma factor

$$G(s) := \pi^{-2s} \Gamma\left(\frac{s + 1 - 2\nu}{2}\right) \Gamma\left(\frac{s}{2}\right)^2 \Gamma\left(\frac{s - 1 + 2\nu}{2}\right)$$

If we divide the functional equation of $\zeta(s)$ from both sides, we get

$$\pi^{-\frac{3s}{2}} \Gamma\left(\frac{s + 1 - 2\nu}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s - 1 + 2\nu}{2}\right) \frac{L(s, f \times f)}{\zeta(s)}$$
$$= \pi^{-\frac{3(1-s)}{2}} \Gamma\left(\frac{(1 - s) + 1 - 2\nu}{2}\right) \Gamma\left(\frac{1 - s}{2}\right) \Gamma\left(\frac{(1 - s) - 1 + 2\nu}{2}\right) \frac{L(1 - s, f \times f)}{\zeta(1 - s)}$$

One may expect that the degree 3 L-function $L(s, f \times f)/\zeta(s)$ is attached to certain self-dual GL_3 Maass eigenform of type $(2\nu/3, 2\nu/3)$. Indeed, the twisted L-functions by Dirichlet characters admits Euler product, satisfies EBV (entire and bounded in vertical strips) conditions and certain functional equation, so the GL_3 converse theorem gives the desired result. Let's write the corresponding GL_3 Maass form as $\phi = \phi(\tau)$. From $L(s, \phi)\zeta(s) = L(s, f \times f)$, the Fourier coefficients matrix $\{b_{n_1,n_2}\}$ of ϕ and the Fourier coefficients of $f(z)$ should be related as

$$a_n^2 = \sum_{d|n} b_{d,1} \iff b_{n,1} = \sum_{d|n} \mu(d) a_{n/d}^2.$$

Now we will interpret the situation in the context of representation theory. Let $\pi = \otimes_v \pi_v$ be an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$, and let φ_v be the 2-dimensional representations of the Weil-Deligne group $W_v := W_{F_v}$ of F_v attached to π_v via Local Langlands correspondence. The symmetric square representation of $\mathrm{GL}_2(\mathbb{C})$

$$\mathrm{Sym}^2 : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_3(\mathbb{C}), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

gives a 3-dimensional representation $\mathrm{Sym}^2(\varphi_v) := \mathrm{Sym}^2 \circ \varphi$ of W_v, which should corresponds to an irreducible admissible representation of $\mathrm{GL}_3(F_v)$ via Local Langlands correspondence again. Global Langlands correspondences predicts that the representation $\mathrm{Sym}^2(\pi) := \otimes_v \mathrm{Sym}^2(\pi_v)$ is an automorphic represenetation of $\mathrm{GL}_3(\mathbb{A})$, which is proven by Gelbart-Jacquet.

Theorem 5.3 (Gelbart-Jacquet, [4])**.** *Let F be a number field and π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with $\mathbb{A} = \mathbb{A}_F$. Then $\mathrm{Sym}^2(\pi)$ is an automorphic representation of $\mathrm{GL}_3(\mathbb{A})$.*

In [12], Ramakrishnan proved the following converse of the Gelbart-Jacquet, by using L-functions.

Theorem 5.4 (Ramakrishnan, [12]). *Let F be a number field and Π be a cuspidal automorphic representation of $GL_3(\mathbb{A}_F)$, which is self-dual. Then, up to quadratic twist, it can be realized as an adjoint of a $GL_2(\mathbb{A}_F)$ automorphic representation. More precisely, there exists an automorphic form π of $GL_2(\mathbb{A}_F)$ and a grössencharacter η of F with $\eta^2 = 1$ such that*

$$\Pi = \mathrm{Ad}(\pi) \otimes \eta$$

where $\mathrm{Ad}(\pi) = \mathrm{Sym}^2(\pi) \otimes \omega_\pi^{-1}$.

5.3. **Higher symmetric power.** Since symmetric power map $\mathrm{Sym}^r : GL_2 \to GL_{r+1}$ is defined for arbitrary power r, we expect the presence of lifting from GL_2 automorphic representations to GL_{r+1} automorphic representations. Until now, this is proved for $r = 3, 4$ cases.

Theorem 5.5 (Kim-Shahidi, [7]). *Let F be a number field and π be a cuspidal automorphic representation of $GL_2(\mathbb{A})$ with $\mathbb{A} = \mathbb{A}_F$. Then $\mathrm{Sym}^3(\pi)$ is an automorphic representation of $GL_4(\mathbb{A})$. $\mathrm{Sym}^3(\pi)$ is cuspidal unless π is either dihedral or tetrahedral type. In particular, if $F = \mathbb{Q}$ and π is the automorphic representation attached to nondihedral modular form of level ≥ 2, then $\mathrm{Sym}^3(f)$ is cuspidal.*

Theorem 5.6 (Kim, [6]). *Let F be a number field and π be a cuspidal automorphic representation of $GL_2(\mathbb{A})$ with $\mathbb{A} = \mathbb{A}_F$. Then $\mathrm{Sym}^4(\pi)$ is an automorphic representation of $GL_5(\mathbb{A})$. If $\mathrm{Sym}^3(\pi)$ is cuspidal, then $\mathrm{Sym}^4(\pi)$ is either cuspidal or induced from cuspidal representation of $GL_2(\mathbb{A})$ and $GL_3(\mathbb{A})$.*

To prove Theorem 5.5, Kim and Shaidi first proved the functoriality $GL_2 \times GL_3 \to GL_6$, i.e. existence of an automorphic representation $\pi_1 \boxtimes \pi_2$ for GL_2 automorphic representation π_1 and GL_3 automorphic representation π_2. Then they obtained the result by applying it for $\pi_1 = \pi$ and $\pi_2 = \mathrm{Ad}(\pi_1)$, where Ad is the automorphic representation of $GL_3(\mathbb{A})$ obtained with adjoint representation $\mathrm{Ad} : GL_2 \to PGL_2 \to GL_3$. Note that $\mathrm{Sym}^2(\pi) = \mathrm{Ad}(\pi) \otimes \omega_\pi$, where ω_π is the central character of π.

For Theorem 5.6, Kim first proved exterior square lifting for GL_4, which corresponds to the map $\wedge^2 : GL_4(\mathbb{C}) \to GL_6(\mathbb{C})$. Then he obtained the result on the fourth power by applying exterior square to $\mathrm{Sym}^3(\pi) \otimes \omega_\pi^{-1}$, showing that

$$\wedge^2(\mathrm{Sym}^3(\pi) \otimes \omega_\pi^{-1}) = (\mathrm{Sym}^4(\pi) \otimes \omega_\pi^{-1}) \boxplus \omega_\pi.$$

Recently, it is proved that the functoriality holds for arbitrary power when π is a *regular algebraic cuspidal* representation, which corresponds to twists of cuspidal modular forms.

Theorem 5.7 (Newton-Thorne [9, 10]). *Let π be a regular algebraic cuspidal representation of $GL_2(\mathbb{A}_\mathbb{Q})$ of level 1, or without complex multiplication. For any $n \geq 1$, $\mathrm{Sym}^n(f)$ is a regular algebraic cuspidal representation of $GL_{n+1}(\mathbb{A})$.*

5.4. **Ramanujan's conjecture, Selberg's 1/4 conjecture, and Sato-Tate conjecture.** The importance of symmetric power lifting is due to it's application on the Ramanujan conjecture, Selberg's eigenvalue conjecture, and the Sato-Tate conjecture.

Conjecture 5.1 (Selberg's 1/4 conjecture). *For any Maass form on a congruence subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$, its eigenvalue is at least 1/4.*

It is known that the conjecture is false for non-congruence subgroups (See [13] for Sarnak's argument). Also, it is widely believed that the Maass forms with eigenvalue $1/4$ are *algebraic*, in a sense that they comes from *even* 2-dimensional Galois representations. Selberg himself proved a weaker bound $3/16$ for $\Gamma = \Gamma(N)$ in [14],

Proposition 5.1. *Assume that symmetric power lifting holds for arbitrary power, i.e. for any cuspidal automorphic representation π on $\mathrm{GL}_2(\mathbb{A})$, $\mathrm{Sym}^r(\pi)$ is an automorphic representation of $\mathrm{GL}_{r+1}(\mathbb{A})$ for any r. Then the Ramanujan's conjecture and Selberg's conjecture are true.*

Proof. By Jacquet-Shalika [REFERENCE], it is proven that the Satake parameters of automorphic forms of $\mathrm{GL}_n(\mathbb{A})$ satisfy

$$q_v^{-1/2} < |\alpha_{i,v}| < q_v^{1/2}$$

for all $1 \le i \le n$ and unramified places v (including archimedean places). Now, assume that symmetric power lifting holds for arbitrary power. If $\Pi = \otimes_v \Pi_v = \mathrm{Sym}^r(\pi)$ is the corresponding representation, then the Satake parameters at place v are given as

$$\begin{pmatrix} \alpha_{1,v}^r & & & & \\ & \alpha_{1,v}^{r-1}\alpha_{2,v} & & & \\ & & \ddots & & \\ & & & \alpha_{1,v}\alpha_{2,v}^{r-1} & \\ & & & & \alpha_{2,v}^r \end{pmatrix}$$

and Jacquet-Shalika's bound gives

$$q_v^{-1/2} < |\alpha_{i,v}^r| < q_v^{1/2} \iff q_v^{-1/2r} < |\alpha_{i,v}| < q_v^{1/2r}$$

for all $r \ge 1$. Now taking the limit $r \to \infty$ proves both conjecture. $\square$

Combined with Theorem 5.6, Proposition 5.1 gives the current best bound for the Selberg's conjecture.

Corollary 5.1. *Eigenvalus of Maass forms on a congruence subgroup is at least*

$$\frac{1}{4} - \left(\frac{7}{64}\right)^2 = \frac{975}{4096} \approx 0.238037\ldots$$

6. Jacquet-Langlands correspondence

6.1. **Quaternionic modular forms and Basis problem.** [8]

6.2. **Jacquet-Langlands correspondence.**

7. Theta correspondence and Howe duality

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