Langlands functoriality

view towards classical examples

by

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Survey on the Langlands functoriality conjecture with many examples

Abstract

Survey on the Langlands functoriality conjecture.

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1 Introduction

Conjecture 1.0.1 (Langlands functoriality conjecture)**.** *Let G and G' be reductive groups over a global field F.*

This is an introductory note on Langlands functoriality conjecture view towards classical examples. Here is a list of topics we are going to study:

1. Automorphic induction
2. Base change
3. Rankin-Selberg product
4. Symmetric power lifting and Selberg's 1/4 conjecture
5. Jacquet-Langlands correspondence
6. Theta correspondence and Howe duality

2 Base change

2.1 Doi-Naganuma lifting

In [10], Doi and Naganuma constructed a lifting from the space of elliptic modular forms to the space of Hilbert modular forms of the same (parallel) weight. The proved the following theorem:

Theorem 2.1.1 (Doi-Naganuma [10])**.** *Let p be a prime such that the real quadratic field $F = \mathbb{Q}(\sqrt{p})$ has class number 1, and let ϕ_p be the Dirichlet character associated to F. Let $f(z) = \sum_{n\geq 1} a_n q^n \in S_k(p, \phi_p)$ be a weight k Hecke eigenform of level p with Nebentypus ϕ_p. Then there exists a Hilbert modular form $h := \mathrm{DN}(f)$ with respect to $\mathrm{GL}_2(\mathcal{O}_F)$, the* Doi-Naganuma lift of f, *that satisfies*

1. *h is also an Hecke eigenform of weight k,*
2. $$L(s, \mathrm{DN}(f)) = L(s, f)L(s, f^\rho)$$
   where $f^\rho(z) := \sum_{n\geq 1} \overline{a_n} q^n$ is a complex conjugate of $f(z)$,
3. *has a Fourier expansion*
   $$h(z_1, z_2) = -\frac{B_k}{2k}\tilde{a}_0 + \sum_{\substack{\nu \in \mathfrak{d}_F^{-1} \\ \nu >> 0}} \sum_{d|\nu} d^{k-1} \tilde{a}_{p\nu\nu'/d^2} q_1^{\nu} q_2^{\nu'}$$
   where B_k denotes the k-th Bernoulli number, ν' is a conjugate of ν, and $q_j = e^{2\pi i z_j}$.

3 Rankin-Selberg product

3.1 Rankin-Selberg convolution of modular forms

Let f, g be two holomorphic cusp forms of weight k and level 1. Assume that two forms have Fourier expansions

$$f(z) = \sum_{n \geq 1} a_n e^{2\pi i n}, \quad g(z) = \sum_{n \geq 1} b_n e^{2\pi i n}.$$

If f, g are Hecke eigenforms, then their L-functions admit euler products as

$$\begin{aligned} L(s, f) &= \sum_{n \geq 1} \frac{a_n}{n^s} = \prod_p \frac{1}{(1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})} \\ L(s, g) &= \sum_{n \geq 1} \frac{b_n}{n^s} = \prod_p \frac{1}{(1 - \alpha'_p p^{-s})(1 - \beta'_p p^{-s})} \end{aligned}$$

where $\alpha_p + \beta_p = a_p$, $\alpha'_p + \beta'_p = b_p$, and $\alpha_p \beta_p = \alpha'_p \beta'_p = p^{k-1}$ for all p. Rankin (1939) and Selberg (1940) independently studied the *convolution* of two L-series attached to f and g, which is

$$\begin{aligned} L(s, f \times g) &= \sum_{n \geq 1} \frac{a_n \overline{b_n}}{n^s} \\ &= \prod_p \frac{1}{(1 - \alpha_p \alpha'_p p^{-s})(1 - \alpha_p \beta'_p p^{-s})(1 - \beta_p \alpha'_p p^{-s})(1 - \beta_p \beta'_p p^{-s})} \end{aligned}$$

and studied its analytic properties. They proved that the new L-function also satisfy similar properties as original L-functions $L(s, f)$ and $L(s, g)$: it admits a meromorphic continuation, bounded on vertical strips, and satisfy a functional equation.

Theorem 3.1.1 (Rankin-Selberg convolution)**.** *Let f, g be two holomorphic cusp eigenforms of weight k on $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Let*

$$\Lambda(s, f \times g) = (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 1) \zeta(2s - 2k + 2) L(s, f \times g)$$

be a completed L-function. Then $\Lambda(s, f \times g)$, which is originally defined for large $\Re(s)$, admits a meromorphic continuation to all s except for at most simple poles at $s = k$ and $k - 1$. Also, it satisfies a functional equation

$$\Lambda(s, f \times g) = \Lambda(2k - 1 - s, f \times g).$$

Essence of the proof is using real-analytic Eisenstein series with *unfolding trick*. For $s \in \mathbb{C}$, define a real-analytic Eisenstein series $E_s(z)$ as

$$E_s(z) := \sum_{\gamma \in P \backslash \mathrm{SL}(2,\mathbb{Z})} \Im(\gamma z)^s$$

where $P = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subset \Gamma$ is a standard parabolic subgroup of Γ. Clearly, this is a Γ-invariant function, and it converges for $\Re(s) > 1$. Also, using $\Delta y^s = s(1-s)y^s$, one can show that $E_s(z)$ is also a Maass form with eigenvalue $s(1-s)$ (but not a cusp form). By computing its Fourier expansion, we can see that $E_s(z)$, as a function in s, satisfies the functional equation

$$\xi(2s)E_s(z) = \xi(2-2s)E_{1-s}(z)$$

where $\xi(s)$ is the completed zeta function

$$\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

that satisfies $\xi(s) = \xi(1-s)$ for all s.

Proposition 3.1.1.

$$\langle f \cdot E_s, g\rangle = (4\pi)^{-(s+2k-1)}\Gamma(s+2k-1)\sum_{n\geq 1} L(s+2k-1, f\times g)$$

where $\langle -, -\rangle$ is the Petersson inner product.

Proof. The idea is to unfold the integral. If φ is a P-invariant function on $\mathfrak{H}$, then Fubini's theorem gives

$$\int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\backslash\Gamma} \varphi(\gamma z)\frac{dxdy}{y^2} = \int_{P\backslash\mathfrak{H}} \varphi(z)\frac{dxdy}{y^2} = \int_0^\infty \int_0^1 \varphi(z)\frac{dxdy}{y^2}$$

(the fundamental domain of $P\backslash\mathfrak{H}$ is $\{z = x+iy \in \mathfrak{H} \;:\; 0 \leq x < 1\}$.) Once we apply this for $\varphi(z) = y^s f(z)\overline{g(z)}y^{2k}$, we get

$$\begin{aligned}
\langle f\cdot E_s, g\rangle &= \int_0^\infty \int_0^1 y^s f(z)\overline{g(z)}y^{2k}\frac{dxdy}{y^2} \\
&= \sum_{m,n\geq 1} a_m\overline{b_n}y^{s+2k-1}e^{-2\pi(m+n)y}\left(\int_0^1 e^{2\pi i(m-n)x}dx\right)\frac{dy}{y} \\
&= \sum_{n\geq 1} a_n\overline{b_n}\int_0^\infty y^{s+2k-1}e^{-4\pi ny}\frac{dy}{y} \\
&= (4\pi)^{-(s+2k-1)}\Gamma(s+2k-1)L(s+2k-1, f\times g).
\end{aligned}$$

□

3.2 Modularity of $\mathrm{GL}(2) \times \mathrm{GL}(2)$

It is also possible to construct Rankin-Selberg L-function attached to two Maass cusp forms with similar properties. In general, for given automorphic representations π_1, π_2 on $\mathrm{GL}(2)$, one can define Rankin-Selberg L-function $L(s, \pi_1 \times \pi_2)$. According to the philosophy of Langlands, there should exists a $\mathrm{GL}(4)$ automorphic representation whose L-function is $L(s, \pi_1 \times \pi_2)$. This is proven by Ramakrishnan in 2000.

Theorem 3.2.1 (Ramakrishnan, [30]). *Let π_1, π_2 be automorphic forms on $\mathrm{GL}(2, \mathbb{A})$. Then there exists an automorphic representation $\pi_1 \boxtimes \pi_2$ on $\mathrm{GL}(4, \mathbb{A})$ whose L-function equals the Rankin-Selberg L-function, i.e*

$$L(s, \pi_1 \boxtimes \pi_2) = L(s, \pi_1 \times \pi_2).$$

As a corollary of Theorem 3.2.1, he proved multiplicity one result for $\mathrm{SL}(2)$. This was conjectured by Labesse and Langlands before [25].

Theorem 3.2.2 (Ramakrishnan, [30]). *Multiplicity one theorem holds for $\mathrm{SL}(2)$. More precisely, any smooth irreducible admissible representation of $\mathrm{SL}(2, \mathbb{A})$ occurs with multiplicity at most one in the space of cusp forms $L_0^2(\mathrm{SL}(2, F) \backslash \mathrm{SL}(2, \mathbb{A}_F))$.*

In the context of modular forms, this implies the following. Let f, g be cusp forms of level N and M respectively. Assume that, for all but finitely many p, we have

$$a_p^2 = b_p^2$$

where a_p (resp. b_p) is the p-th Fourier coefficient of f (resp. g). Then there exists a quadratic Dirichlet character χ such that

$$a_p = \chi(p) b_p$$

for all but finitely many p. Also, if N, M are in addition square-free, then $\chi = 1$ and so $f = g$ by strong multiplicity one.

Proof of Theorem 3.2.2 goes as follows. In [25], the authors proved that the multiplicity one theorem for $\mathrm{SL}(2)$ holds if one can show the following.

Theorem 3.2.3 (Ramakrishnan, [30]). *If two automorphic representation π, π' on $\mathrm{GL}(2, \mathbb{A}_F)$ satisfy $\mathrm{Ad}(\pi) \simeq \mathrm{Ad}(\pi')$, then $\pi' \simeq \pi \otimes \chi$ for some idele class character χ of F. Here Ad is the adjoint lift from $\mathrm{GL}(2)$ to $\mathrm{GL}(3)$ [14].*

He first show this when at least one of π or π' is *dihedral* (i.e. has a form of $\mathrm{AI}_K^F(\mu)$ for some quadratic extension K of F and idele class character μ of K). If both π, π' are not dihedral, then he proved that $\pi \boxtimes \pi'$ is not cuspidal by analyzing poles of L-functions at $s = 1$. which shows that $\pi' \simeq \pi \otimes \chi$ for some χ by cuspidality criterion that is also proved by him in loc. cit.

3.3 $\mathrm{GL}(n) \times \mathrm{GL}(n)$

4 Symmetric power lifting

Automorphic forms on $\mathrm{GL}(2)$ are often *classified* into two kinds of objects: *modular forms* and *Maass forms*[1]. These functions oftenly considered as a starting point for studying automorphic forms and representations for $\mathrm{GL}(n)$ and other groups. However, there are not many references for $\mathrm{GL}(3)$.

4.1 Automorphic forms on $\mathrm{GL}(3, \mathbb{R})$

We first introduce the theory of automorphic forms on $\mathrm{GL}(3)$ (We follow the Bump's book [5]). We only consider the level 1 automorphic forms. Before we start, let's revisit the $\mathrm{GL}(2)$. Modular forms and Maass forms are certain functions defined on the complex upper half plane $\mathfrak{H}$, and one can lift the functions as a function on $\mathrm{GL}(2, \mathbb{R})$ by viewing $\mathfrak{H}$ as a symmetric space

$$\mathfrak{H} \simeq \mathrm{GL}(2, \mathbb{R})/Z(\mathbb{R})\mathrm{O}(2).$$

Here $Z(\mathbb{R}) \simeq \mathbb{R}^\times$ is the center of $\mathrm{GL}(2, \mathbb{R})$ and $\mathrm{O}(2)$ is a group of orthogonal matrices. The above isomorphism holds since $\mathrm{GL}(2, \mathbb{R})$ acts on $\mathbb{H}$ transitively and the stabilizer of i is $Z(\mathbb{R})\mathrm{O}(2)$. To develop a theory of automorphic forms on $\mathrm{GL}(3)$, it is natural to consider them as a function defined on the symmetric space

$$\mathfrak{H}_3 := \mathrm{GL}(3, \mathbb{R})/Z(\mathbb{R})\mathrm{O}(3).$$

Using Iwasawa decomposition, each coset has a unique representation of the form

$$\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}, \quad y_1, y_2 > 0.$$

Especially, the space is parametrized with 5 real variables and has a real dimension 5, so we can't expect any *holomorphic* automorphic form over $\mathrm{GL}(3)$, in constrast to the $\mathrm{GL}(2)$ case. Also, we have an involution ι on $\mathfrak{H}_3$ defined as

$$\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -x_1 & x_1 x_2 - x_3 \\ 0 & 1 & -x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}.$$

[1] and constant functions.

What about the Fourier expansion of GL(3) automorphic forms? In case of GL(2), the algebra of $\mathrm{GL}(2,\mathbb{R})$-invariant differential operators on $\mathfrak{h}_2$ is isomorphic to a polynomial ring of single variable, generated by the following hyperbolic Laplacian

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

A 1-periodic function $f(z) = \sum_{n\geq 0} a_n(y)e^{2\pi i n x}$ become an eigenfunction with respect to Δ if the coefficients $a_n(y)$ satisfy certain degree 2 linear differential equations. More precisely, when $\Delta f = \left(\frac{1}{4} - \nu^2\right)f$, the n-th coefficient $a_n(y)$ satisfies

$$y^2\frac{\partial^2}{\partial y^2}a_n(y) + \left(\frac{1}{4} - \nu^2 - 4\pi^2 n^2 y\right)a_n(y) = 0.$$

Among two linearly independent solutions, only one satisfies the required growth condition (the other one grows exponentially), which can be expressed with a Bessel function of second kind:

$$a_n(y) = c_n\sqrt{y}K_\nu(2\pi|n|y), \quad K_\nu(y) := \frac{1}{2}\int_0^\infty e^{\frac{y(t+t^{-1})}{2}}t^\nu\frac{dt}{t}.$$

For GL(3), the algebra of $\mathrm{GL}(3,\mathbb{R})$-invariant differential operators on $\mathfrak{h}_3$ is a polynomial ring in *two* variables, with two specific generators Δ_1, Δ_2. Then the automorphic forms of $\mathrm{GL}(3,\mathbb{R})$ would be defined as functions that are eigenforms with respect to these two operators. Then the coefficients of the Fourier expansion (which will be defined explicitly later) of the automorphic forms would satisfy specific differential equations. In fact, for given λ and μ, there are 6 linearly independent functions that are

1. eigenfunctions with respect to Δ_1, Δ_2, i.e.

$$\begin{aligned}\Delta_1 F &= \lambda F\\ \Delta_2 F &= \mu F\end{aligned}$$

2. and satisfies the equation

$$F\left(\begin{pmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{pmatrix}\tau\right) = e(x_1 + x_2)F(\tau)$$

for all τ and $x_1, x_2, x_3 \in \mathbb{R}$, where $e(x) := \exp(2\pi i x)$.

Among these 6 solutions, only one of them *decays rapidly*, which is the appropriate substitute of $K_\nu(y)$ for $\mathrm{GL}(3)$ (This is multiplicity one theorem for $\mathrm{GL}(3)$). It can be written as an inverse Mellin transform of a certain 2-variable function $V(s_1,s_2)$,

$$\begin{aligned} W(y_1,y_2) &= W_{\nu_1,\nu_2}(y_1,y_2) \\ &= \frac{1}{4}\frac{1}{(2\pi i)^2}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty} V(s_1,s_2)(\pi y_1)^{1-s_1}(\pi y_2)^{1-s_2}ds_1ds_2 \end{aligned}$$

where

$$V(s_1,s_2) = \frac{\Gamma\left(\frac{s_1+\alpha}{2}\right)\Gamma\left(\frac{s_1+\beta}{2}\right)\Gamma\left(\frac{s_1+\gamma}{2}\right)\Gamma\left(\frac{s_2-\alpha}{2}\right)\Gamma\left(\frac{s_2-\beta}{2}\right)\Gamma\left(\frac{s_2-\gamma}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)}.$$

Here α,β,γ are auxillary parameters satisfy

$$\begin{aligned} \alpha &= -\nu_1 - 2\nu_2 + 1 \\ \beta &= -\nu_1 + \nu_2 \\ \gamma &= 2\nu_1 + \nu_2 - 1 \\ \lambda &= -1 - \alpha\beta - \beta\gamma - \gamma\alpha \\ \mu &= -\alpha\beta\gamma. \end{aligned}$$

The Whittaker function $W(y_1,y_2)$ also can be written as an integral of Bessel functions

$$W_{\nu_1,\nu_2}(y_1,y_2) = 4y_1^{1-\frac{\beta}{2}}y_2^{1+\frac{\beta}{2}}\int_0^\infty K_{\frac{\gamma-\alpha}{2}}(2\pi y_2\sqrt{1+u^{-2}})K_{\frac{\gamma-\alpha}{2}}(2\pi y_1\sqrt{1+u^2})u^{\frac{-3\beta}{2}}\frac{du}{u}.$$

Definition 4.1.1 (Automorphic form on $\mathrm{GL}(3,\mathbb{R})$)**.** *Let* $\nu_1,\nu_2\in\mathbb{C}$*. An automorphic form of type* (ν_1,ν_2) *on* $\mathrm{GL}(3,\mathbb{R})$ *is a function* $\phi:\mathfrak{H}_3\to\mathbb{C}$ *such that*

1. $\phi(g\tau)=\phi(\tau)$ *for all* $g\in\mathrm{GL}(3,\mathbb{Z})$ *and* $\tau\in\mathrm{GL}_3(\mathbb{R})$.

2. ϕ *is an eigenfunction of* Δ_1,Δ_2 *with eigenvalues* λ,μ *defined above,*

3. *there exists* n_1,n_2 *such that*

$$y_1^{n_1}y_2^{n_2}\phi\left(\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right)$$

is bounded on the subset of $\mathfrak{H}_2$ *determined by* $y_1,y_2>1$.

In addition, for all $\tau\in\mathfrak{H}_3$*, if*

$$\int_0^1\int_0^1 \phi\left(\begin{pmatrix} 1 & & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\tau\right)d\xi_1 d\xi_3 = 0$$

and

$$\int_0^1\int_0^1 \phi\left(\begin{pmatrix}1 & \xi_2 & \xi_3\\ & 1 & \\ & & 1\end{pmatrix}\tau\right)d\xi_1 d\xi_3 = 0$$

then ϕ is called cusp form.

Note that, for a given automorphic form ϕ of type (ν_1,ν_2), the *dual* form $\tilde{\phi}(\tau) := \phi({}^{\iota}\tau)$ is also an automorphic form, but of type (ν_2,ν_1).

Any ϕ has a Fourier expansion with double indices

$$\phi(\tau) = \sum_{g\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{n_1\geq 1}\sum_{n_2\geq 1}\hat{\phi}_{n_1,n_2}\left(\begin{pmatrix}g & \\ & 1\end{pmatrix}z\right)$$

where $\Gamma^2_\infty, \Gamma^2$ are the subgroups of $\mathrm{GL}(3,\mathbb{Z})$ defined as

$$\Gamma^2 = \left\{\begin{pmatrix}* & * & \\ * & * & \\ & & 1\end{pmatrix}\in \mathrm{GL}(2,\mathbb{Z})\right\}, \Gamma^2_\infty = \Gamma^2 \cap \left\{\begin{pmatrix}1 & * & *\\ & 1 & *\\ & & 1\end{pmatrix}\in\mathrm{GL}(2,\mathbb{Z})\right\}$$

and $\hat{\phi}_{n_1,n_2}(z)$ is

$$\hat{\phi}_{n_1,n_2}(z) = \int_0^1\int_0^1\int_0^1 \phi(xz)e^{-2\pi i(n_1x_1+n_2x_2)}dx$$

where

$$x = \begin{pmatrix}1 & x_2 & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}$$

and $dx = dx_1dx_2dx_3$. One can check that $\hat{\phi}_{n_1,n2}(z)$ is a rapidly decreasing Whittaker function, and multiplicity one theorem gives us that it should be a multiple of (suitable modification of) $W_{\nu_1,\nu_2}(y_1,y_2)$, that is

$$\phi(\tau) = \sum_{g\in\Gamma^2_\infty\backslash\Gamma^2}\sum_{n_1\geq 1}\sum_{n_2\geq 1}\frac{a_{n_1,n_2}}{n_1n_2}W^{(\nu_1,\nu_2)}_{1,1}\left(\begin{pmatrix}n_1n_2 & & \\ & n_1 & \\ & & 1\end{pmatrix}g\tau\right).$$

Here

$$W^{(\nu_1,\nu_2)}_{1,1}(\tau) = W_{\nu_1,\nu_2}(y_1,y_2)e(x_1+x_2).$$

We call $\{a_{n_1,n_2}\}$ the matrix of Fourier coefficients of ϕ. From this, we define the corresponding L-function as

$$L(s,\phi) = \sum_{n\geq 1}\frac{a_{1,n}}{n^s}.$$

As we expect, this function admits an analytic continuation and satisfies certain functional equation.

Theorem 4.1.1 (L-function of an automorphic form on $\mathrm{GL}(3,\mathbb{R})$)**.** *The L-function $L(s,\phi)$ of an $\mathrm{GL}(3,\mathbb{R})$ automorphic form ϕ admits an analytic continuation for all $\mathbb{C}$ and satisfies the functional equation*

$$\Phi(s)L(s,\phi) = \tilde{\Phi}(1-s)L(1-s,\tilde{\phi})$$

where $\Phi(s), \tilde{\Phi}(s)$ are Gamma factors

$$\Phi(s) = \pi^{-\frac{3s}{2}}\Gamma\left(\frac{s-\alpha}{2}\right)\Gamma\left(\frac{s-\beta}{2}\right)\Gamma\left(\frac{s-\gamma}{2}\right)$$
$$\tilde{\Phi}(s) = \pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+\alpha}{2}\right)\Gamma\left(\frac{s+\beta}{2}\right)\Gamma\left(\frac{s+\gamma}{2}\right)$$

and $L(s,\tilde{\phi})$ is the L-function of the dual automorphic form which equals

$$L(s,\tilde{\phi}) = \sum_{n\geq 1}\frac{a_{n,1}}{n^s}.$$

It is also possible to define Hecke operators on the space of $\mathrm{GL}(3,\mathbb{R})$ automorphic forms. We define them via double cosets, and the ring of Hecke operators became commutative. Note that, for each $n \geq 1$, there are *two* Hecke operators T_n, S_n, where

Definition 4.1.2 (Hecke operators)**.** *Let $\mathcal{H} = \mathbb{Z}[\Gamma\backslash G/\Gamma]$ be a $\mathbb{Z}$-algebra of double cosets where $G = \mathrm{GL}(3,\mathbb{R})$ and $\Gamma = \mathrm{GL}(3,\mathbb{Z})$, which is called* Hecke algebra*. It decomposes as a (internal) tensor product*

$$\mathcal{H} = \bigotimes_p{}' \mathcal{H}_p$$

where $\mathcal{H}_p$ is a subalgebra of $\mathcal{H}$ corresponding to the double cosets whose elementary divisors are powers of a given prime p. For each prime p, $\mathcal{H}_p$ is generated by three elements

$$T_p := \Gamma\begin{pmatrix} p & & \\ & 1 & \\ & & 1\end{pmatrix}\Gamma, \quad S_p := \Gamma\begin{pmatrix} p & & \\ & p & \\ & & 1\end{pmatrix}\Gamma, \quad R_p := \Gamma\begin{pmatrix} p & & \\ & p & \\ & & p\end{pmatrix}\Gamma.$$

The whole $\mathcal{H}$ is generated by the operators T_n, S_n, R_n where

$$T_n := \sum_{n_0^3 n_1^2 n_2 = n}\Gamma\begin{pmatrix} n_0n_1n_2 & & \\ & n_0n_1 & \\ & & 1\end{pmatrix}\Gamma$$
$$S_n := \sum_{n_0^3 n_1^2 n_2 = n}\Gamma\begin{pmatrix} n_0^2n_1^2n_2 & & \\ & n_0^2n_1n_2 & \\ & & n_0^2n_1\end{pmatrix}\Gamma$$
$$R_n := \Gamma\begin{pmatrix} n & & \\ & n & \\ & & n\end{pmatrix}\Gamma$$

which satisfies certain relations given as the formal power series

$$\sum_{n\geq 1}\frac{T_n}{n^s}=\prod_p \frac{1}{1-T_p\cdot p^{-s}+S_p\cdot p^{1-2s}-R_p\cdot p^{3-3s}}.$$

If ϕ is an automorphic form on $\mathrm{GL}(3,\mathbb{R})$*, then the action of Hecke algebra on the form is defined as*

$$(\phi|\Gamma\alpha\Gamma)(\tau):=\sum_i \phi(\alpha_i\tau)$$

where α_i's are the representatives of the double coset $\Gamma\alpha\Gamma$, i.e. $\Gamma\alpha\Gamma=\cup_i\Gamma\alpha_i$.

Note that the Hecke operators also commutes with the differential operators Δ_1 and Δ_2, so the space of automorphic forms of type (ν_1,ν_2) has a basis consisting of simultaneous eigenforms for all Hecke operators. Also, the coefficients of $\phi|T_n$ and $\phi|S_n$ can be expressed as certain sums of coefficients of ϕ - see the equations (9.8) and (9.9) in [5].

In $\mathrm{GL}(2)$, L-function attached to an automorphic form admits an Euler product if and only if the form is Hecke eigenform, and the local factors has a form of $P_p(p^{-s})^{-1}$, where $P_p(x)$ is a polynomial of degree 2. The same thing also holds for $\mathrm{GL}(3,\mathbb{R})$, where the local factors are inverses of cubic polynomials in p^{-s}.

Theorem 4.1.2 (Euler product of L-function)**.** *If ϕ is a normalized Hecke eigenform on* $\mathrm{GL}(3,\mathbb{R})$ *with matrix coefficients $\{a_{n_1,n_2}\}$, then its L-function has an Euler product*

$$L(s,\phi)=\prod_p\frac{1}{1-a_{1,p}p^{-s}+a_{p,1}p^{-2s}-p^{-3s}}.$$

4.2 Symmetric square lifting by Gelbart-Jacquet

Let f be a level 1 Maass cusp form (of weight 0) on $\mathrm{GL}(2,\mathbb{R})$ with eigenvalue $\lambda=\nu(1-\nu)$ which is also a normalized eigenform. Let $\{a_n\}_{n\geq 1}$ be Fourier coefficients of f. Consider the Rankin-Selberg L-function of $f\times f$, i.e.

$$L(s,f\times f)=\zeta(2s)\sum_{n\geq 1}\frac{|a_n|^2}{n^s}=\zeta(2s)\prod_p\frac{1}{(1-\alpha_p^2p^{-s})(1-p^{-s})^2(1-\beta_p^2p^{-s})}$$

where $a_p=\alpha_p+\beta_p, \alpha_p\beta_p=1$. By the theory of Rankin-Selberg convolution, the L-function admits an analytic continuation to $\mathbb{C}$ with functional equation

$$\Lambda(s,f\times f)=G(s)L(s,f\times f)=\Lambda(1-s,f\times f)$$

where $G(s)$ is the Gamma factor

$$G(s):=\pi^{-2s}\Gamma\left(\frac{s+1-2\nu}{2}\right)\Gamma\left(\frac{s}{2}\right)^2\Gamma\left(\frac{s-1+2\nu}{2}\right)$$

If we divide the functional equation of $\zeta(s)$ from both sides, we get

$$\pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+1-2\nu}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-1+2\nu}{2}\right)\frac{L(s,f\times f)}{\zeta(s)}$$
$$=\pi^{-\frac{3(1-s)}{2}}\Gamma\left(\frac{(1-s)+1-2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{(1-s)-1+2\nu}{2}\right)\frac{L(1-s,f\times f)}{\zeta(1-s)}$$

One may expect that the degree 3 L-function $L(s,f\times f)/\zeta(s)$ is attached to certain self-dual $\mathrm{GL}(3)$ Maass eigenform of type $(2\nu/3, 2\nu/3)$. Indeed, the twisted L-functions by Dirichlet characters admits Euler product, satisfies EBV (entire and bounded in vertical strips) conditions and certain functional equation, so the $\mathrm{GL}(3)$ converse theorem gives the desired result. The detailed proof can be found in Chapter 7 of Goldfeld's book [15], where the proof of EBV condition is based on Shimura's brilliant idea that considers Rankin-Selberg product of f with a theta function (see also [39]). Let's write the corresponding $\mathrm{GL}(3)$ Maass form as $\phi=\phi(\tau)$. From $L(s,\phi)\zeta(s)=L(s,f\times f)$, the Fourier coefficients matrix $\{b_{n_1,n_2}\}$ of ϕ and the Fourier coefficients of $f(z)$ should be related as

$$a_n^2=\sum_{d|n}b_{d,1}\Longleftrightarrow b_{n,1}=\sum_{d|n}\mu(d)a_{n/d}^2.$$

Now we will interpret the situation in the context of representation theory. Let $\pi=\otimes'_v\pi_v$ be an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$, and let φ_v be the 2-dimensional representations of the Weil-Deligne group $W_v:=W_{F_v}$ of F_v attached to π_v via Local Langlands correspondence. The symmetric square representation of $\mathrm{GL}(2,\mathbb{C})$

$$\mathrm{Sym}^2:\mathrm{GL}(2,\mathbb{C})\to\mathrm{GL}(3,\mathbb{C}),\qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\mapsto\begin{pmatrix}a^2 & 2ab & b^2\\ ac & ad+bc & bd\\ c^2 & 2cd & d^2\end{pmatrix}$$

gives a 3-dimensional representation $\mathrm{Sym}^2(\varphi_v):=\mathrm{Sym}^2\circ\varphi$ of W_v, which should corresponds to an irreducible admissible representation of $\mathrm{GL}(3,F_v)$ via Local Langlands correspondence again. Global Langlands correspondences predicts that the representation $\mathrm{Sym}^2(\pi):=\otimes_v\mathrm{Sym}^2(\pi_v)$ is an automorphic represenetation of $\mathrm{GL}(3,\mathbb{A})$, which is proven by Gelbart-Jacquet.

Theorem 4.2.1 (Gelbart-Jacquet, [14]). *Let F be a number field and π be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ with $\mathbb{A}=\mathbb{A}_F$. Then $\mathrm{Sym}^2(\pi)$ is an automorphic representation of $\mathrm{GL}(2,\mathbb{A})$.*

The idea of the proof is based on Shimura's proof of analytic continutation and functional equation for symmetric square L-function $L(s,\mathrm{Sym}^2 f)$ of modular forms [38], where he used Rankin-Selberg convolution of f with theta series of half-integral weight. In the context of representation theory, theta functions correspond to the automorphic representations on *metaplectic groups*, which are double cover of symplectic groups. (See Chapter 7 for more details about half-integral modular forms.) This proves analytic properties of $L(s,\mathrm{Sym}^2(\pi))$ (and its twists), and $\mathrm{GL}(3)$ converse theorem by Jacquet, Piatetski-Shapiro and Shalika [18, 19] concludes the proof.

In [29], Ramakrishnan proved the following converse of the Gelbart-Jacquet, by using L-functions.

Theorem 4.2.2 (Ramakrishnan, [29]). *Let F be a number field and Π be a cuspidal automorphic representation of $GL(3, \mathbb{A}_F)$, which is self-dual. Then, up to quadratic twist, it can be realized as an adjoint of a $GL(2, \mathbb{A}_F)$ automorphic representation. More precisely, there exists an automorphic form π of $\mathrm{GL}(2, \mathbb{A}_F)$ and a grössencharacter η of F with $\eta^2 = 1$ such that*

$$\Pi = \mathrm{Ad}(\pi) \otimes \eta$$

where $\mathrm{Ad}(\pi) = \mathrm{Sym}^2(\pi) \otimes \omega_\pi^{-1}$.

4.3 Higher symmetric power

Since symmetric power map $\mathrm{Sym}^r : \mathrm{GL}(2) \to \mathrm{GL}(r+1)$ is defined for arbitrary power r, we expect the presence of lifting from $\mathrm{GL}(2)$ automorphic representations to $\mathrm{GL}(r+1)$ automorphic representations. Until now, this is proved for $r = 3, 4$ cases.

Theorem 4.3.1 (Kim-Shahidi, [24]). *Let F be a number field and π be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ with $\mathbb{A} = \mathbb{A}_F$. Then $\mathrm{Sym}^3(\pi)$ is an automorphic representation of $\mathrm{GL}(4, \mathbb{A})$. $\mathrm{Sym}^3(\pi)$ is cuspidal unless π is either dihedral or tetrahedral type. In particular, if $F = \mathbb{Q}$ and π is the automorphic representation attached to nondihedral modular form of level ≥ 2, then $\mathrm{Sym}^3(f)$ is cuspidal.*

Theorem 4.3.2 (Kim, [21]). *Let F be a number field and π be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ with $\mathbb{A} = \mathbb{A}_F$. Then $\mathrm{Sym}^4(\pi)$ is an automorphic representation of $\mathrm{GL}(5, \mathbb{A})$. If $\mathrm{Sym}^3(\pi)$ is cuspidal, then $\mathrm{Sym}^4(\pi)$ is either cuspidal or induced from cuspidal representation of $\mathrm{GL}(2, \mathbb{A})$ and $\mathrm{GL}(3, \mathbb{A})$.*

To prove Theorem 4.3.1, Kim and Shahidi first proved the functoriality $\mathrm{GL}(2) \times \mathrm{GL}(3) \to \mathrm{GL}(6)$, i.e. existence of an automorphic representation $\pi_1 \boxtimes \pi_2$ for $\mathrm{GL}(2)$ automorphic representation π_1 and $\mathrm{GL}(3)$ automorphic representation π_2. Then they obtained the result by applying it for $\pi_1 = \pi$ and $\pi_2 = \mathrm{Ad}(\pi_1)$, where Ad is the automorphic representation of $\mathrm{GL}(3, \mathbb{A})$ obtained with adjoint representation $\mathrm{Ad} : \mathrm{GL}(2) \to \mathrm{PGL}(2) \to \mathrm{GL}(3)$. Note that $\mathrm{Sym}^2(\pi) = \mathrm{Ad}(\pi) \otimes \omega_\pi$, where ω_π is the central character of π.

For Theorem 4.3.2, Kim first proved exterior square lifting for $\mathrm{GL}(4)$, which corresponds to the map $\wedge^2 : \mathrm{GL}(4, \mathbb{C}) \to \mathrm{GL}(6, \mathbb{C})$. Then he obtained the result on the fourth power by applying exterior square to $\mathrm{Sym}^3(\pi) \otimes \omega_\pi^{-1}$, showing that

$$\wedge^2(\mathrm{Sym}^3(\pi) \otimes \omega_\pi^{-1}) = (\mathrm{Sym}^4(\pi) \otimes \omega_\pi^{-1}) \boxplus \omega_\pi.$$

Recently, it was proven that the functoriality holds for arbitrary power when π is a *regular algebraic cuspidal* representation, which corresponds to twists of cuspidal modular forms.

Theorem 4.3.3 (Newton-Thorne [27, 28]). *Let π be a regular algebraic cuspidal representation of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ of level 1, or without complex multiplication. For any $n \geq 1$, $\mathrm{Sym}^n(f)$ is a regular algebraic cuspidal representation of $\mathrm{GL}(n+1, \mathbb{A})$.*

4.4 Ramanujan's conjecture and Selberg's 1/4 conjecture

The importance of symmetric power lifting is due to it's application on Ramanujan's conjecture and Selberg's eigenvalue conjecture.

In 1916, Ramanujan conjectured that the Fourier coefficients $\tau(n)$ of disciminant function

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n, \qquad q = e^{2\pi i z}$$

(which is a weight 12 and level 1 holomorphic cusp form) satisfy

1. $\tau(mn) = \tau(m)\tau(n)$ for $(m, n) = 1$ (i.e. τ is multiplicative),
2. $\tau(p^{k+1}) = \tau(p)\tau(p^k) - p^{11}\tau(p^{k-1})$ for prime p and $k \geq 1$,
3. $|\tau(p)| \leq p^{11/2}$.

The first two statements were first proved by Mordell in 1917, and Hecke showed that similar phenomena can be found in other modular forms, by defining so-called *Hecke operators*. Third statement remained as a conjecture until 1974 when it was proved by Deligne as a consequence of his proof of Weil's conjecture, an analogue of Riemann's hypothesis for varieties over finite fields which states that the *Zeta function* of a variety is a rational function which can be written as an alternating product of certain polynomials, and the zeros of each polynomial have the same norm. In fact, he proved that every normalized eigenforms have similar bound for their Fourier coefficients: $|a_p| \leq 2p^{\frac{k-1}{2}}$ for weight k modular forms. Deligne himself proved it for $k \geq 2$ [8], and weight 1 case was proven by Deligne and Serre [9].

For Maass forms, there is a well-known conjecture on eigenvalues by Selberg.

Conjecture 4.4.1 (Selberg's 1/4 conjecture)**.** *For any Maass form on a congruence subgroup* $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$*, its eigenvalue is at least* $1/4$*.*

Selberg himself proved a weaker bound $3/16$ for $\Gamma = \Gamma(N)$ in [34]. The conjecture is false for non-congruence subgroups (See [32]). Also, it is widely believed that the Maass forms with eigenvalue $1/4$ are *algebraic*, in a sense that they comes from *even* 2-dimensional Galois representations. For example, Cohen constructed an explicit Maass form of level 4 with eigenvalue $\frac{1}{4}$ that is originated from one of q-series in Ramanujan's lost notebook (this is actually the first explicitly known example of Maass cusp form), and proved that its L-function equals to certain *even* 2-dimensional Galois representation induced from a Hecke character of $\mathbb{Q}(\sqrt{6})$ [7]. Also, once can find additional computational examples from Buzzard's note [6].

In [33], Satake formulated both Ramanujan's conjecture and Selberg's conjecture in terms of automorphic representations. He conjectured that, for any automorphic representation $\pi = \otimes'_v \pi_v$ of $\mathrm{GL}(2)$, the local components π_v are *tempered*, i.e. the matrix coefficients of π_v are in $L^{2+\epsilon}(\mathrm{GL}(2, F_v))$ for all $\epsilon > 0$. For example, let's consider the automorphic representation

$\pi = \pi_\Delta$ associated to the disciminant function Δ. For $p \neq 2, 3$, π_p is an unramified principal series representation, which is

$$\begin{aligned}\pi(\chi_1, \chi_2) &= \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}(2,\mathbb{Q}_p)}(\chi_1 \boxtimes \chi_2) \\ &= \left\{ f : \mathrm{GL}(2, F_v) \to \mathbb{C} : f(bg) = \left|\frac{b_1}{b_2}\right|^{\frac{1}{2}} \chi_1(b_1)\chi_2(b_2)f(g), b = \begin{pmatrix} b_1 & * \\ & b_2 \end{pmatrix} \right\}\end{aligned}$$

where $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ are unramified characters ($\mathbb{Z}_p^\times \subseteq \ker \chi_i$) with $\chi_i(p) = \alpha_i$ where α_1, α_2 are zeros of the polynomial $x^2 - \tau(p)x + p^{11}$.

Several authors found counterexamples in some groups like $\mathrm{U}(2, 1)$ and $\mathrm{Sp}(4)$ by constructing corresponding automorphic forms that are non-tempered almost everywhere. In this case, the correct version of Ramanujan's conjecture is for *generic* representations, i.e. automorphic representations which has a Whittaker model. (Note that all the automorphic representations of $\mathrm{GL}(n)$ are generic.)

Proposition 4.4.1. *Assume that symmetric power lifting holds for arbitrary power, i.e. for any cuspidal automorphic representation π on $\mathrm{GL}(2, \mathbb{A})$, $\mathrm{Sym}^r(\pi)$ is an automorphic representation of $\mathrm{GL}(r + 1, \mathbb{A})$ for any r. Then the Ramanujan's conjecture and Selberg's conjecture are true.*

Proof. By Jacquet-Shalika [20], it was proven that the Satake parameters of automorphic forms of $\mathrm{GL}(n, \mathbb{A})$ satisfy

$$q_v^{-1/2} < |\alpha_{j,v}| < q_v^{1/2}$$

for all $1 \leq j \leq n$ and finite places v where π_v is unramified. Similarly, one has

$$|\Re(\mu_{j,v})| < \frac{1}{2}$$

for unramified archimedean places v. This follows from non-vanishing result of Rankin-Selberg L-function $L(s, \pi \times \pi^\vee)$ proved by the authors in [20]. Now, assume that symmetric power lifting holds for arbitrary power. If $\Pi = \otimes_v' \Pi_v = \mathrm{Sym}^r(\pi)$ is the corresponding representation, then the Satake parameters at place v are given as

$$\begin{pmatrix} \alpha_{1,v}^r & & & & \\ & \alpha_{1,v}^{r-1}\alpha_{2,v} & & & \\ & & \ddots & & \\ & & & \alpha_{1,v}\alpha_{2,v}^{r-1} & \\ & & & & \alpha_{2,v}^r \end{pmatrix}$$

and Jacquet-Shalika's bound gives

$$q_v^{-1/2} < |\alpha_{j,v}^r| < q_v^{1/2} \Longleftrightarrow q_v^{-1/2r} < |\alpha_{j,v}| < q_v^{1/2r}$$

for unramified finite places v and Similarly

$$|\Re(\mu_{j,v})| < \frac{1}{2r}$$

for unramified archimedean places v. Now taking the limit $r \to \infty$ proves both conjecture. □

Combined with Theorem 4.3.2, Proposition 4.4.1 gives the following bound.

Corollary 4.4.1. *Let f be a Maass form. Its' Fourier coefficient is bounded by*

$$|a_p| \leq p^{1/8} + p^{-1/8}.$$

Also, eigenvalue of f is at least

$$\frac{1}{4} - \left(\frac{1}{8}\right)^2 = \frac{15}{64} = 0.234375$$

The current best known bound for $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ is the following result from Kim and Sarnak [22].

Theorem 4.4.1. *Let $\pi = \otimes'_p \pi_p$ be an automorphic cusp form on $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$. If π is unramified at p, then its Satake parameter $\alpha_{1,p}, \alpha_{2,p}$ satisfies*

$$|\log_p |\alpha_{j,p}|| \leq \frac{7}{64}, \;\; j = 1, 2.$$

If π_∞ is unramified, then

$$|\Re(\mu_{j,\infty})| \leq \frac{7}{64}, \;\; j = 1, 2.$$

If f is a classical Maass form on a congruence subgroup $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$, then the above inequality gives the following bound on Fourier coefficients

$$|a_p| \leq p^{7/64} + p^{-7/64}$$

and its eigenvalue is at least

$$\frac{1}{4} - \left(\frac{7}{64}\right)^2 = \frac{975}{4096} \approx 0.238037\cdots.$$

Proof. When Π is a $\mathrm{GL}(n)$ automorphic form, it is known that the symmetric square L-function $L(s, \mathrm{Sym}^2(\Pi))$ satisfies desired analytic properties by Kim and Shahidi [23, 37]. Using this, the authors proved

$$|\log_p |\alpha_{j,p}|| \leq \frac{1}{2} - \frac{1}{\frac{n(n+1)}{2} + 1}, \;\; 1 \leq j \leq n.$$

for Satake parameters of unramified Π_p with $p < \infty$, and

$$|\Re(\mu_{j,\infty})| \le \frac{1}{2} - \frac{1}{\frac{n(n+1)}{2}+1}, \quad 1 \le j \le n.$$

for Satake parameters $\mu_{j,\infty}$ when Π_∞ is unramified. By Theorem 4.3.2, $\Pi = \mathrm{Sym}^4(\pi)$ is automorphic and applying the above bound for Π with $n = 5$ gives the desired results. □

The same $7/64$ bound also holds over general number fields, due to Blomer and Brumley [2]. We refer to [3, 31, 41] for further discussion about Ramanujan's conjecture.

4.5 Sato-Tate conjecture

Another consequence of symmetric power lifting is Sato-Tate conjecture. It is a conjecture about equidistribution of *Frobenius angle*, which we are going to explain briefly. Let E be an elliptic curve over $\mathbb{Q}$ without CM. For every prime p of good reduction for E, the number of points over $\mathbb{F}_p$ satisfies an inequality

$$|\#E(\mathbb{F}_p) - (p+1)| \le 2\sqrt{p}$$

which was proven by Hasse in 1933. The quantity $t_p := (p+1) - \#E(\mathbb{F}_p)$ is called *trace*, since it is actually the trace of $\rho_{E,\ell}(\mathrm{Frob}_p)$, where $\rho_{E,\ell}$ is a (family of) ℓ-adic Galois representation attached to E and $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a Frobenius automorphism. Because of Hasse's inequality, we can write t_p as $t_p = 2\sqrt{p}\cos\theta_p$ for some $\theta_p \in [0, \pi]$, where we call θ_p as *Frobenius angle*. Sato-Tate conjecture states that the Frobenius angle is equidistributed over $[0, \pi]$, and it was proven in 2011 by Barnet-Lamb, Geraghty, Harris, and Taylor [1].

Theorem 4.5.1 (Sato-Tate conjecture, [16])**.** *Let $E/\mathbb{Q}$ be an elliptic curve without CM. The sequence of Frobenius angles $\{\theta_p\}$ is uniformly distributed on the interval $[0, \pi]$. In terms of traces $\{t_p\}$, for every subinterval $[a, b]$ of $[-2, 2]$,*

$$\lim_{B\to\infty} \frac{\#\{p \le B \,:\, t_p \in [a, b]\}}{\#\{p \le B\}} = \int_a^b \frac{1}{2\pi}\sqrt{4 - t^2}\,dt.$$

A key idea for the proof of Theorem 4.5.1 is the following equivalence between equidistribution and holomorphicity & nonzeroness of a L-function. Let G be a compact group and $X = \mathrm{conj}(G)$ be a space of conjugacy classes of G. Let K be a number field, and $P = (\mathfrak{p}_1, \mathfrak{p}_2, \dots)$ be a sequence of all but finitely many primes of K ordered by norm. Let $(x_\mathfrak{p})_{\mathfrak{p}\in P}$ be a sequence in X indexed by P, and for each irreducible representation $\rho : G \to \mathrm{GL}(d, \mathbb{C})$, define the L-function

$$L(s, \rho) := \prod_{\mathfrak{p}\in P} \det(1 - \rho(x_\mathfrak{p})N(\mathfrak{p})^{-s})^{-1},$$

for $s \in \mathbb{C}$ with $\Re s > 1$.

Theorem 4.5.2. *Let G and $(x_\mathfrak{p})$ as above, and assume that $L(s, \rho)$ is meromorphic on $\Re s \ge 1$ with nozeros or poles except possibly at $s = 1$, for every irreducible representation ρ of G. The sequence*

$(x_\mathfrak{p})$ *is equidistributed if and only if for each* $\rho \neq 1$, *the* L*-function* $L(\rho, s)$ *extends analytically to a function that is holomorphic and nonvanishing on* $\Re s \geq 1$.

Proof. See Theorem 2.3 of [12]. □

Now let $G = \mathrm{SU}(2)$ (compact group of complex 2 by 2 unitary matrices of determinant 1) and $K = \mathbb{Q}$. The irreducible representations of $\mathrm{SU}(2)$ are of the form $\rho_m = \mathrm{Sym}^m \rho_1$ where $\rho_1 : \mathrm{SU}(2) \hookrightarrow \mathrm{GL}(2, \mathbb{C})$ is the representation given by inclusion. In this case, each element of $X = \mathrm{conj}(\mathrm{SU}(2))$ has a representative of the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

with $\theta \in [0, \pi]$, and the L-function of ρ_m can be written as

$$L(s, \rho_m) = \prod_{p \nmid N} \det(1 - \rho_m(x_p)p^{-s})^{-1} = \prod_{p \nmid N} \prod_{k=0}^{m} (1 - e^{i(m-2k)\theta_p} p^{-s})^{-1}$$

where θ_p is Frobenius angle and N is a conductor of elliptic curve E. If we set $\alpha_p := e^{i\theta_p} p^{1/2}$ and

$$L_m^1(s) := \prod_{p \nmid N} \prod_{k=0}^{m} (1 - \alpha_p^{m-k} \bar{\alpha}_p^k p^{-s})^{-1},$$

then we have $L(s, \rho_m) = L_m^1(s - m/2)$. By Theorem 4.5.2, Sato-Tate theorem (for non-CM elliptic curve) would follow from holomorphicity and nonzeroness of $L_m^1(s - m/2)$ for $\Re s \geq \frac{m}{2} + 1$. Now, the celebrated modularity theorem by several mathematicians (Wiles, Taylor, Brueil, Conrad, ...) states that one can find a modular form f whose L-function $L(s, f)$ coincides with the Hasse-Weil L-function $L(s, E)$ attached to E, and both coincides with $L_1^1(s)$ up to finitely many Euler factors at bad primes. It is easy to show holomorphicity and nonvanishing property of $L(s, f)$, and one essentially have $L_m^1(s) = L(s, \mathrm{Sym}^m f)$. Hence, Sato-Tate conjecture reduces to some analytic properties of $L(s, \mathrm{Sym}^m f)$.

Sato-Tate conjecture is also true for CM elliptic curves, but with different measures. Actually it is much easier to prove for such cases since the corresponding *Sato-Tate group* is $U(1)$ (embedded in $\mathrm{GL}(2, \mathbb{C})$ via $u \mapsto \left(\begin{smallmatrix} u & 0 \\ 0 & \bar{u} \end{smallmatrix}\right)$) or $N(U(1))$ (normalizer of $U(1)$ in $GL(2, \mathbb{C})$) and $U(1)$ is abelian, so all the irreducible representations are 1-dimensional. Also, it is possible to define *a* Sato-Tate group $\mathrm{ST}(A)$ and formulate the analogous conjecture for abelian varieties A: the normalized images of the Frobenius elements in a Sato-Tate group is equidistributed with respect to the pushforward of the Haar measure of $\mathrm{ST}(A)$ to its space of conjugacy classes. However, we don't even know whether *the* Sato-Tate group is well-defined or not for $g \geq 2$, and only few examples of abelian varieties are known to be that the conjecture holds (e.g. [13]). One can find more details in Sutherland's excellent survey paper [40].

5 Jacquet-Langlands correspondence

5.1 Basis problem and quaternionic modular forms

Here's a fundamental question that we can ask about modular forms.

Question 5.1.1. *Find a basis of a space $S_k(N)$ for given k, N.*

For $N = 1$, it is possible to construct a basis using the Eisenstein series E_4 and E_6. In particular, they are algebraically independent over $\mathbb{C}$ (i.e. there's no nonzero polynomial $p(x, y) \in \mathbb{C}[x, y]$) such that $p(E_4, E_6) = 0$ identically), and they generate the space of modular forms of level 1 (with trivial characters). We have $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$, where $\Delta(z) = e^{2\pi i z}\prod_{n\geq 1}(1 - e^{2\pi i n z})^{24}$ is a discriminant function, and multiplying Δ induces an isomorphism $M_k(1)$ to $S_{k+6}(1)$. With $S_k(1) = 0$ for $k = 2, 4, 6, 8, 10$, we can inductively construct the basis of $S_k(1)$. (See [35] or Zagier's article in [4] for details.)

For general N, Hecke [17] conjectured that one can construct a basis using *theta series associated to orders in certain quaternion algebra*. Before we give definition of such theta series, let's introduce some notations. For a prime p, let B_p be a quaternion algebra over $\mathbb{Q}$ which is ramified at p and infinity. It known that quaternion algebra over $\mathbb{Q}$ is uniquely determined by the set of primes that ramify.

[11]

5.2 Jacquet-Langlands correspondence

5.3 Basis problem by means of Jacquet-Langlands correspondence

In this section, we will give a solution to the basis problem by using Jacquet-Langlands correspondence. Our goal is to prove the following theorem.

Theorem 5.3.1 (Kimball, [26])**.** *Let F be a totally real number field of degree $d = [F : \mathbb{Q}]$ and B be a quaternion algebra over F with discriminant $\mathfrak{D}$. Let $\mathcal{O}$ be a special order of level $\mathfrak{N}$. This means that $\mathcal{O}_\mathfrak{p}$ is an Eichler order for $\mathfrak{p} \nmid \mathfrak{D}$ and $\mathcal{O}_\mathfrak{p}$ contains the ring of integers of a quadratic extension $E_\mathfrak{p}/F_\mathfrak{p}$ for $\mathfrak{p}|\mathfrak{D}$, and that the product of levels of the local orders is $\mathfrak{N}$. Let $\mathbf{k} = (k_1, \ldots, k_d) \in (\mathbb{Z}_{\geq 0})^d$ and $S_\mathbf{k}(\mathcal{O})$ be the space of quaternionic cusp forms of level $\mathbf{k}$.*

There is a Hecke-module homomorphism

$$\mathrm{JL} : S_\mathbf{k}(\mathcal{O}) \to S_{\mathbf{k}+\mathbf{2}}(\mathfrak{N})$$

such that

1. *any newform* $f \in S_{\mathbf{k+2}}(\mathfrak{N})$ *which is* $\mathfrak{p}$*-primitive for* $\mathfrak{p}|\mathfrak{D}$ *is contained in the image;*
2. *if* $v_{\mathfrak{p}}(\mathfrak{N})$ *is odd for all* $\mathfrak{p}|\mathfrak{D}$*, then* JL *is injective and yields an isomorphism*

$$S_{\mathbf{k}}(\mathcal{O}) \simeq \bigoplus_{\mathfrak{d}} S_{\mathbf{k+2}}^{\mathfrak{d}-new}(\mathfrak{d}\mathfrak{M})$$

where $\mathfrak{d}$ *runs over all divisors of* $\mathfrak{N}'$ *such that* $v_{\mathfrak{p}}(\mathfrak{d})$ *is odd for all* $\mathfrak{p}|\mathfrak{D}$*, and* $\mathfrak{M}$ *is* $\mathfrak{D}$*-prime part of* $\mathfrak{N}$*. Here* $S_{\mathbf{k+2}}^{\mathfrak{d}-new}$ *is the subspace of* $S_{\mathbf{k+2}}(\mathfrak{N})$*consisting of forms that are* $\mathfrak{p}$*-new for all* $\mathfrak{p}|\mathfrak{d}$*.*

This section closely follow Kimball's article [26].

6 Theta correspondence and Howe duality

6.1 Half-integral weight modular forms

The theta series

$$\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}, \qquad q = e^{2\pi i z}$$

is regarded as a powerful tool to study lattices and quadratic forms. For example, power of the theta series $\theta(z)^k$ is a generating function of $r_k(n)$, the number of ways to represent an integral n as a sum of k squares. For *even* k, $\theta(z)^k$ is a weight $k/2$ modular form, which makes us to analyze $\theta(z)^k$ more closely and even find the formula for $r_k(n)$. For example, $\theta(z)^2$ is a weight 1 modular form on $\Gamma_1(4)$ with character (Nebentypus) χ_4, the primitive Dirichlet character of level 4. The space $S_1(\Gamma_1(4), \chi_4)$ of such modular forms has dimension 1, so that $\theta(z)^2$ is actually a non-zero multiple of certain weight 1 Eisenstein series, and this gives a formula

$$r_2(n) = 4 \sum_{2\nmid d|n} (-1)^{(d-1)/2}$$

and this gives a one-line proof for the Fermat's theorem on sum of two squares. Similarly, $\theta(z)^4$ is also a modular form (of weight 2 on $\Gamma_0(4)$), and the similar argument gives a formula

$$r_4(n) = 8 \sum_{4\nmid d|n} d$$

and Lagrange's four square theorem is a direct consequence of this (see Zagier's article *Elliptic Modular Forms and Their Applications* in the book [4] for details).

How about the *odd* powers of $\theta(z)$? For example, what is $\theta(z)$ itself? Since $\theta(z)^2$ is a weight 1 modular form, we have $\theta(\gamma z)^2 = \chi_4(d)(cz+d)\theta(z)^2$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(4)$. So $\theta(z)$ itself satisfies a transformation law $\theta(\gamma z) = j(\gamma, z)\theta(z)$ where $j(\gamma, z)$ is, by definition, $j(\gamma, z) := \theta(\gamma z)/\theta(z)$. Indeed, it can be written as

$$j(\gamma, z) = \begin{cases} \epsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2} & c \neq 0 \\ 1 & c = 0 \end{cases}$$

where the branch of $(cz+d)^{1/2}$ is choosen so that its real part is positive. ϵ_d is 1 if $d \equiv 1 \pmod 4$, and i otherwise.

In general, half-integral weight modular forms are defined as follows. Define $\mathcal{G}$ to be the group with elements (γ, ϕ) where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}^+(2, \mathbb{R})$ (group of matrices with positive determinant) and $\phi : \mathfrak{H} \to \mathbb{C}$ is a holomorphic function satisfying $\phi(z)^2 = t\det(\gamma)^{-1/2}(cz+d)$ with $t = t(\gamma, \phi) \in \mathbb{C}$ independent of z satisfying $|t| = 1$. This has a group structure defined as

$$(\gamma_1, \phi_1)(\gamma_2, \phi_2) = (\gamma_1\gamma_2, z \mapsto \phi_1(\gamma_2 z)\phi_2(z))$$

and it is an extension of the group $\mathrm{GL}^+(2, \mathbb{R})$ with fiber $\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}$.

Some other examples of half-integral weight modular forms are the theta series with characters. For example, an even primitive character ψ of conductor r defines a theta series

$$\theta_\psi(z) = \sum_{n\in\mathbb{Z}} \psi(n) q^{n^2}$$

that belongs to $M_{1/2}(\Gamma_0(4r^2), \psi)$ (Proposition 2.2 of [38], proven by Poisson summation formula). From this, we also have $\theta_{\psi,t}(z) := \theta_\psi(tz) \in M_{1/2}(\Gamma_0(4r^2t), \psi)$. Note that $\theta_\psi(z)$ is different from $(\theta \otimes \psi)(z) = \sum_{n\in\mathbb{Z}} \psi(n)^2 q^{n^2}$, the series obtained by *twisting* θ with character ψ. Serre and Stark proved that every modular form of weight $1/2$ is a linear combination of theta series $\theta_{\psi,t}$. Indeed, they find a bases for $M_{1/2}(\Gamma_0(N), \chi)$.

Theorem 6.1.1 (Serre-Stark, [36])**.** *Let $\Omega(N, \chi)$ be the set of pairs (ψ, t) where $t \geq 1$ is an integer and ψ is an even primitive character of conductor $r = r(\psi)$ such that*

1. $4r^2 t | N$
2. $\chi(n) = \psi(n)\left(\frac{n}{t}\right)$ *for all n prime to N (i.e. ψ is the primitive character associated with $\chi\left(\frac{\cdot}{t}\right)$).*

Then $\{\theta_{\psi,t} \,:\, (\psi, t) \in \Omega(N, \chi)\}$ forms a basis of $M_{1/2}(\Gamma_0(N), \chi)$.

We have Hecke operators on the space of half-integral modular forms too, but it is quite different from that of integral weights. As in the integral weight cases, we define the Hecke operator as an action of double cosets.

6.2 Shimura correspondence and Shintani lift

Let $f(z) = \sum_{n\geq 1} a_n q^n$ be a Hecke eigenform of half-integral weight $k/2$ and level N with character χ, i.e. $T_{\chi,p^2} f = \lambda_p f$ for all p with $p \nmid N$. Then for every square-free integer t, we have

$$\sum_{n\geq 1} \frac{a_{n^2}}{n^s} \prod_p \left(1 - \chi(p)\left(\frac{-1}{p}\right)^{\frac{k-1}{2}} p^{\frac{k-1}{2}-1-s}\right)^{-1} = \prod_p (1 - \lambda_p p^{-s} + \chi(p)^2 p^{k-2-2s})^{-1}$$

In [38], Shimura established a lift from the space of half integral weight modular forms to the space of integral weight modular forms. More precisely, he proved that the RHS of above equation becomes an L-function attached to certain weight $k-1$ modular form.

Theorem 6.2.1 (Shimura, [38]). *Let $F(z) = \sum_{n\geq 1} A_n q^n$ where*

$$\sum_{n\geq 1} \frac{A_n}{n^s} = \prod_p (1 - \lambda_p p^{-s} + \chi(p)^2 p^{k-2-2s})^{-1}.$$

Then $F(z)$ is a weight $k-1$ modular form satisfying

$$F(\gamma z) = \chi(d)^2 (cz+d)^{k-1} F(z)$$

for all $\gamma \in \Gamma_0(N_0)$, where N_0 is an integer only depends on N and χ. This defines a map

$$S_{k/2}(N, \chi) \to M_{k-1}(N_0, \chi^2),$$

which is called as Shimura correspondence. $F(z)$ is a cusp form if $k \geq 5$.

The proof is based on Weil's converse theorem.

6.3 Waldspurger's work

6.4 Theta correspondence and Howe duality

6.5 Gan-Gross-Prasad conjecture

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