Automorphic forms and L-functions for the unitary group*

Re-TEXed by Seewoo Lee†

Stephen Gelbart
Department of Mathematics
Cornell University
Ithaca, New York 14853/USA

and

Ilya Piatetski-Shapiro
Department of Mathematics
Yale University, New Haven, CT. 06520/USA
Tel Aviv University, Ramat-Aviv, Israel

Last updated: October 18, 2023

Introduction

Our purpose is to define and analyze L-functions attached to automorphic cusp forms on the unitary group $G = \mathrm{U}_{2,1}$ and a six-dimensional representation

$$\rho : {}^L G \to \mathrm{GL}_6(\mathbb{C})$$

of its L-group.

*Notes based on the lectures by S. G. at the University of Maryland Special Year on Lie Group Representations, 1982-83.

†seewoo5@berkeley.edu

The motivation for this work is three fold.

Firstly, we use these L-functions to analyze the lifting of cusp forms from $U_{1,1}$ to $U_{2,1}$; here the model for our work is Waldspurger's L-function theoretic characterization of the image of Shimura's map for modular forms of half-integral weight (cf. [Wald]).

A second motivation comes from the need to relate the poles of the L-functions for G, to integrals of cusp forms over cycles coming from $U_{1,1}$. The prototvpe here is the recent proof of Tate's conjecture for Hilbert modular surfaces due to Harder, Langlands, and Rapaport.

Thirdly, we view this work as a special contribution to the general program of constructing local L and ε factors of Langlands type for representations of arbitrary reductive groups. In [PS1], such a program was sketched generalizing classical methods of Heeke, Rankin–Selberg, and Shimura. Related developments are discussed in [Jacquet], [Novod], [PS2], and [PS3]. For the unitary group $U_{2,1}$ the present paper extends the developments initiated in [PS3].

Contents

Whittaker Models (Ordinary and Generalized) **3**

Notation

(i) F is a field (sometimes local, somtimes a global field), E is a quadratic extension of F with Galois involution $z \mapsto \bar{z}$.

(ii) V is a 3-dimensional vector space over E, with basis $\{\ell_{-1}, \ell_0, \ell_1\}$. $(-,-)_V$ is a Hermitian form on V, with matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

with respect to $\{\ell_{-1}, \ell_0, \ell_1\}$.

(iii) $G = U_{2,1} = U(V)$ is the unitary group for the form $(-,-)_V$. P=parabolic subgroup stabilizing the isotropic line through $\ell_{-1} = MN$ with

$$M = \left\{ \begin{bmatrix} \delta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \bar{\delta}^{-1} \end{bmatrix} : \delta \in E^\times, \beta \in E^1 = \{z : z\bar{z} = 1\} \right\}$$

and unipotent radical

$$N = \left\{ \begin{bmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{bmatrix} : z, b \in E, z + \bar{z} = -b\bar{b} \right\}.$$

The center of N is

$$Z = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \bar{z} = -z \right\}$$

※ Whittaker Models (Ordinary and Generalized)