SOLUTION FOR "AN INTRODUCTION TO AUTOMORPHIC REPRESENTATION"

SEEWOO LEE

Abstract. This is a solution for the exercises in J. Getz and H. Hahn's book *An Introduction to Automorphic Representation with a view toward Trace Formulae*.

1. Chapter 1

***Problem* 1.1** NOT FINISHEDBy Yoneda lemma, the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes corresponds to the k-algebra morphism $\phi : A \to B$. This induces a map on the underlying topological spaces by sending a prime ideal $\mathfrak{p} \subset B$ to $\phi^{-1}(\mathfrak{p}) \subset A$, which is also prime.

***Problem* 1.2** NOT FINISHED

***Problem* 1.3** By Yoneda lemma, we have

$$\mathrm{Mor}(\mathrm{Spec}(B), \mathrm{Spec}(A)) \simeq \mathrm{Nat}(h^B, h^A) \simeq h^B(A) = \mathrm{Hom}_k(A, B)$$

which gives an equivalence between $\mathbf{AffSch}_k^{\mathrm{op}}$ and $\mathbf{Alg}_k$.

***Problem* 1.4**
- Nonreduced: $\mathrm{Spec}(\mathbb{C}[x]/(x^2))$
- Reducible: $\mathrm{Spec}(\mathbb{C}[x, y]/(x, y))$
- Reduced and irreducible (i.e. integral): $\mathrm{Spec}(\mathbb{C}[x])$

***Problem* 1.5** We can assume that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A/I)$ for some k-algebra A and an ideal I of A. Then it is enough to show that the map $\mathrm{Hom}(A/I, R) \to \mathrm{Hom}(A, R)$, given by composing with the natural map $\pi : A \to A/I$, is injective. This follows from the surjectivity of π.

***Problem* 1.6** Let $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B), Z = \mathrm{Spec}(C)$. Then the statement is equivalent to

$$\mathrm{Hom}(A \otimes_B C, R) \simeq \mathrm{Hom}(A, R) \times_{\mathrm{Hom}(B,R)} \mathrm{Hom}(C, R).$$

We can define a bijection as follows. First, consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\iota_A"] & A \otimes_B C \\ B \arrow[u, "\alpha"] \arrow[r, "\gamma"] & C \arrow[u, "\iota_C"'] \end{tikzcd}$$

Using the maps above, we define a map from LHS to RHS as $\phi \mapsto (\phi\iota_A, \phi\iota_C)$. Since $\iota_A\alpha = \iota_C\gamma$, we have $\phi\iota_A\alpha = \phi\iota_C\gamma$ and the map is well-defined. For the other direction, for given $(f, g) : A \times C \to R$ with $f\alpha = g\gamma$, universal property of the tensor product gives a unique map $\phi : A \otimes_B C \to R$ with $f = \phi\iota_A$ and $g = \phi\iota_C$. We can check that these maps are inverses for each other.

***Problem* 1.7** NOT FINISHED

***Problem* 1.8** NOT FINISHEDWe define an $\mathbb{R}$-algebra A as

$$A = \mathbb{R}[(x_{ij}, y_{ij})_{1\le i,j\le n}]/I$$

where I is an ideal generated by elements of the form

$$\left(\sum_{k=1}^{n}(x_{ik}^2 + y_{ik}^2)\right) - 1,$$

$$\sum_{k=1}^{n}(x_{ik}x_{jk} - y_{ik}y_{jk}), \quad i \neq j$$

$$\sum_{k=1}^{n}(x_{ij}y_{jk} + y_{ik}x_{jk}), \quad i \neq j$$

for $1 \le i,j \le n$. Then we can identify $\mathrm{U}_n(R)$ with $\mathrm{Hom}(A,R)$ as follows: for given $\phi : A \to R$, let $\alpha_{ij} = \phi(x_{ij})$ and $\beta_{ij} = \phi(y_{ij})$. Then a matrix $g = (g_{ij})_{1\le i,j\le n}$ with $g_{ij} = 1 \otimes \alpha_{ij} + \sqrt{-1} \otimes \beta_{ij}$ becomes an element of $U_n(R)$ by the relations of x_{ij} and y_{ij}s defined by the ideal I. Similarly, for given $g = (g_{ij}) \in \mathrm{U}_n(R)$, we can write $g_{ij} = (a_{ij} + \sqrt{-1}b_{ij}) \otimes r_{ij} = 1 \otimes a_{ij}r_{ij} + \sqrt{-1} \otimes b_{ij}r_{ij}$ and we have a corresponding map $\phi : A \to R$ sending x_{ij} to $a_{ij}r_{ij}$ and y_{ij} to $b_{ij}r_{ij}$.

The group $\mathrm{U}_n(\mathbb{R})$ is a compact group (as a topological subgroup of $\mathrm{GL}_n(\mathbb{C})$) since it is closed (it is an inverse image of point I of a continuous map $g \to g\overline{g}^t$) and bounded (each row and column vectors have norm 1).

At last, NOT FINISHED

***Problem* 1.9** Consider the following short exact sequence:

$$0 \to \ker(\epsilon)/\ker(\epsilon)^2 \to \mathcal{O}(G)/\ker(\epsilon)^2 \to k \to 0.$$

The map $O(G)/\ker(\epsilon)^2 \to k$ is defined as a composition of the natural map $\mathcal{O}(G)/\ker(\epsilon)^2 \to \mathcal{O}(G)/\ker(\epsilon)$ followed by ϵ. Then we have a section $k \to \mathcal{O}(G)/\ker(\epsilon)$ which is the composition $k \to \mathcal{O}(G) \to \mathcal{O}(G)/\ker(\epsilon)^2$ and the above sequence splits.

***Problem* 1.10** Let $g = (g_{ij}) \in GL_n(R)$ and $J = (\alpha_{ij}) \in \mathrm{GL}_n(k)$. Then $g^tJg = J$ is equivalent to

$$\sum_{k,l=1}^{n} \alpha_{kl} g_{ki} g_{lj} = \alpha_{ij}$$

for all $1 \le i,j \le n$. Hence G is an affine algebraic group with a coordinate ring

$$A = k[(X_{ij})_{1\le i,j\le n}]/\left(\sum_{k,l=1}^{n} \alpha_{kl} X_{ki} X_{lj} - \alpha_{ij}, 1 \le i,j \le n\right).$$

Since $\mathrm{Lie}\, G = \ker(G(k[t]/t^2) \to G(k))$, the elements of $\mathrm{Lie}\, G$ have a form of $I + tX$ for some $X \in \mathrm{M}_n(k)$. Then the defining equation $g^tJg = J$ is equivalent to

$$(I+tX)^t J(I+tX) = J \Leftrightarrow J + tX^tJ + tJX + t^2X^tJX = J + t(X^tJ + JX) = J,$$

(here every elements are in $\mathrm{GL}_n(k[t]/t^2)$) so we should have $X^tJ + JX = 0$. In other words, we have a map

$$\mathrm{Lie}\, G \xrightarrow{\sim} \{X \in \mathfrak{gl}_n(k) \,:\, X^tJ + JX\}, \quad I + tX \to X.$$

***Problem* 1.11** NOT FINISHED

***Problem* 1.12** NOT FINISHED

***Problem* 1.13** Using the equivalence of $\mathbf{Spl}_k$ and **RRD**, it is enough to check that the dual of the root datum of GL_n is isomorphic to itself in **RRD**. Recall that the root datum of GL_n with torus T of diagonal elements is given as follows: (Example 1.12)

- $X^*(T) = \{\alpha_{k_1,\dots,k_n} : \mathrm{diag}(t_1,\dots,t_n) \mapsto \prod_{1\le j\le n} t_j^{k_j},\, k_1,\dots,k_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $X_*(T) = \{\beta_{k_1,\dots,k_n} : t \mapsto \mathrm{diag}(t^{k_1},\dots,t^{k_n}),\, t_1,\dots,t_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $\Phi(\mathrm{GL}_n, T) = \{e_{ij},\, 1 \le i \ne j \le n\}$, $e_{ij}(\mathrm{diag}(t_1,\dots,t_n)) = t_it_j^{-1}$
- $\Phi^\vee(\mathrm{GL}_n, T) = \{e_{ij},\, 1 \le i \ne j \le n\}$, $e_{ij}^\vee(t) = \mathrm{diag}(1,\dots,t,\dots,t^{-1},\dots,1)$ (t in the i-th entry, t^{-1} in the j-th entry, 1 for other entries)

Then we define a map $f : X_*(T) \to X^*(T)$ and $\iota : \Phi(\mathrm{GL}_n, T) \to \Phi^\vee(\mathrm{GL}_n, T)$ as

$$f(\beta_{k_1,\dots,k_n}) = \alpha_{k_1,\dots,k_n}, \quad \iota(e_{ij}) = e_{ij}^\vee.$$

and define $f^\vee : X^*(T) \to X_*(T)$ and $\iota^\vee : \Phi^\vee(\mathrm{GL}_n, T) \to \Phi(\mathrm{GL}_n, T)$ similarly. Then these maps are inverse to each other and gives an isomorphism between two root data

$$(X^*(T), X_*(T), \Phi(\mathrm{GL}_n, T), \Phi^\vee(\mathrm{GL}_n, T)) \simeq (X_*(T), X^*(T), \Phi^\vee(\mathrm{GL}_n, T), \Phi(\mathrm{GL}_n, T))$$

(they are central isogenies) so we get $\widehat{\mathrm{GL}_n} = \mathrm{GL}_{n\mathbb{C}}$.

***Problem* 1.14** We will show that complex dual of SL_n is PGL_n, and vice versa. Let's compute root datum for SL_n. We choose a maximal torus $T = T_{\mathrm{SL}_n} \leq \mathrm{SL}_n$ of diagonal matrices, so that

$$T(R) = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} : t_k \in R, \prod_{1 \leq k \leq n} t_k = 1 \right\}.$$

Then the characters $X^*(T)$ is almost same as the GL_n case, but we get a quotient of it. For given $\lambda = (\lambda_1, \dots, \lambda_n), \lambda' = (\lambda_1', \dots, \lambda_n') \in \mathbb{Z}^n$, two characters $\alpha_\lambda, \alpha_{\lambda'}$ are the same when $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$. Hence we have

$$X^*(T) \simeq \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n\}/\sim$$

where $\lambda \sim \lambda'$ if $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$. Similarly, cocharacter $\beta_\lambda(t) = \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_1})$ is well-defined only when $\sum_{k=1}^n \lambda_k = 0$, so that

$$X_*(T) \simeq \left\{ \lambda = (\lambda_1, \dots, \lambda_n) : \sum_{k=1}^n \lambda_k = 0 \right\} =: H \subset \mathbb{Z}^n.$$

The set of roots and coroots for SL_n is the same as that of GL_n: only Cartan subalgebra $\mathfrak{t}$ is changed from diagonal matrices in $\mathfrak{gl}_n$ to traceless diagonal matrices.

For PGL_n, we choose the maximal torus $T' = T_{\mathrm{PGL}_n}$ of diagonal matrices, and characters of T' has a form of $\alpha'_\lambda : \mathrm{diag}(t_1, \dots, t_n) \mapsto \prod_{k=1}^n t_k^{\lambda_k}$ for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, and we should have $\sum_{k=1}^n \lambda_k = 0$ for the character to be well-defined on T'. Hence we have $X^*(T') \simeq H \subset \mathbb{Z}^n$. Similarly, any cocharacter on T' has a form of $\beta'_\lambda : t \mapsto \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_n})$, and two different $\lambda, \lambda \in \mathbb{Z}^n$ define same cocharacter as a map to T' if $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$, so $X_*(T)'$ is isomorphic to the quotient of $\mathbb{Z}^n$ by $\mathbb{Z} \cdot (1, \dots, 1)$. The set of roots and coroots are the same as GL_n and SL_n. Note that the Lie algebra $\mathfrak{pgl}_n$ of PGL_n can be thought as a quotient of $\mathfrak{gl}_n(R)$ by $R \cdot I_n$, where I_n is the $n \times n$ identity matrix.

Observe that we can natually identify $X^*(T) \simeq \mathbb{Z}^n/(\mathbb{Z} \cdot (1, \cdots, 1)) \simeq X_*(T')$ and $X_*(T) \simeq H \simeq X^*(T')$. We can define a map between two root data of SL_n and $\widehat{\mathrm{PGL}_n}$ as follows:

$$\begin{aligned} f : X^*(T) \to X_*(T'), &\quad \alpha_\lambda \mapsto \beta'_\lambda \\ f^\vee : X_*(T) \to X^*(T'), &\quad \beta_\lambda \mapsto \alpha'_\lambda \\ \iota : \Phi(\mathrm{SL}_n, T) \to \Phi^\vee(\mathrm{PGL}_n, T'), &\quad e_{ij} \mapsto e_{ij}^\vee \\ \iota^\vee : \Phi^\vee(\mathrm{SL}_n, T) \to \Phi(\mathrm{PGL}_n, T'), &\quad e_{ij}^\vee \mapsto e_{ij} \end{aligned}$$

and this gives $\mathrm{SL}_n \simeq \widehat{\mathrm{PGL}_n}$. Similarly, we have $\mathrm{PGL}_n \simeq \widehat{\mathrm{SL}_n}$.

***Problem* 1.15** NOT FINISHED

2. Chapter 2

***Problem* 2.1** NOT FINISHED

***Problem* 2.2** NOT FINISHED

***Problem* 2.3** It is compact since it is an intersection of closed subset $G(F)$ of $\mathrm{GL}_n(F)$ ($G \hookrightarrow \mathrm{GL}_n$ is closed immersion) and intersection of closed set with compact set is again compact. Openness follows from continuity of $G(F) \hookrightarrow \mathrm{GL}_n(F)$: $\rho(G(F)) \cap K$ is an inverse image of K under $G(F) \hookrightarrow \mathrm{GL}_n(F)$.

***Problem* 2.4** NOT FINISHED

***Problem* 2.5** Using the anti-equivalence of category $\mathbf{AffSch}_k$ and $\mathbf{Alg}_k$, we can reformulate the situation in terms of algebra as follows. Let $A = \mathcal{O}(Y)$ be $\mathfrak{o}$-algebra and $A_F := A \otimes_{\mathfrak{o}} F$. Let $X = \mathrm{Spec}(A_F/I)$ and $\mathcal{X}$ be schematic closure of X in Y, so that $\mathcal{O}(\mathcal{X}) = \mathrm{Im}(\pi^I \iota)$ where $\iota : A \hookrightarrow A_F$ and $\pi^I : A_F \twoheadrightarrow A_F/I$. Let $\mathcal{Z} = \mathrm{Spec}\, A/J$ (we have closed immersion $\mathcal{Z} \hookrightarrow \mathcal{Y}$), and we assume that the map on generic fibre, which corresponds to $A_F \twoheadrightarrow (A/J)_F$, induces an isomorphism $A_F/I = \mathcal{O}(X) \simeq \mathcal{O}(\mathcal{Z}) = (A/J)_F$. This means that there exists an isomorphism $\phi : A_F/I \to (A/J)_F$ such that the following diagram commutes:

$$\begin{tikzcd}
(A/J)_F & \\
A_F/I \arrow[u, "\phi"] & A_F \arrow[l, two heads, "\pi^I"] \arrow[ul, two heads, "\pi^J_F"']
\end{tikzcd}$$

Now our goal is to show that there exists a unique map

$$f : \mathcal{O}(\mathcal{X}) = \mathrm{Im}(\pi^I \iota) \to \mathcal{O}(\mathcal{Z}) = A/J$$

such that the following diagram commutes:

$$\begin{tikzcd}
A/J & \\
\mathrm{Im}(\pi^I \iota) \arrow[u, "f"] & A \arrow[l, two heads, "\pi^I \iota"] \arrow[ul, two heads, "\pi^J"']
\end{tikzcd}$$

The only way to define f that the above diagram commutes is following: for $x \in \mathrm{Im}(\pi^I \iota)$, choose $a \in A$ with $x = \pi^I \iota(a)$ and define $f(x) := \pi^J(a)$. Then we only need to show that the map is well-defined regardless of the choice of a. Let $a_1, a_2 \in A$ such that $\pi^I \iota(a_1) = \pi^I \iota(a_2) = x$. Since $\iota^J : A/J \hookrightarrow (A/J)_F$ is an injection, it is enough to show that $\iota^J \pi^J(a_1) = \iota^J \pi^J(a_2)$. By the commutativity of the following diagram

$$\begin{tikzcd}
A/J \arrow[d, hook, "\iota^J"'] & A \arrow[l, two heads, "\pi^J"'] \arrow[d, hook, "\iota"] \\
(A/J)_F & A_F \arrow[l, two heads, "\pi^J_F"']
\end{tikzcd}$$

we have $\iota^J \pi^J = \pi^J_F \iota = \phi \pi^I \iota$, and this proves

$$\iota^J \pi^J(a_1) = \phi \pi^I \iota(a_1) = \phi(x) = \phi \pi^I \iota(a_2) = \iota^J \pi^J(a_2),$$

i.e. the map is well-defined.

***Problem* 2.6** NOT FINISHED

***Problem* 2.7** NOT FINISHED

***Problem* 2.8** Note that the coordinate ring of $\mathrm{GL}_{n,\mathbb{Q}}$ is

$$B = \mathcal{O}(\mathrm{GL}_{n,\mathbb{Q}}) = \mathbb{Q}[x_{ij}, y]_{1\leq i,j\leq n}/(\det(x_{ij})y - 1).$$

To show that $\mathcal{G}$ is a model of $\mathrm{GL}_{n,\mathbb{Q}}$ over $\mathbb{Z}$, we need to show that $A \hookrightarrow B$ and $A \otimes_{\mathbb{Z}} \mathbb{Q} \simeq B$. Latter isomorphism easily follows from

$$A \otimes \mathbb{Q} = \mathbb{Q}[x_{ij}, t_{ij}, y]/(\det(x_{ij})y - 1, \{x_{ij} - \delta_{ij} - mt_{ij}\}) \simeq B$$

since we can invert $m > 1$ in $\mathbb{Q}$ and get an isomorphism $A \otimes \mathbb{Q} \to B$ via $t_{ij} \mapsto (1 - x_{ij})/m$. Shoing $A \hookrightarrow B \simeq A \otimes_{\mathbb{Z}} \mathbb{Q}$ is equivalent to showing that A is a torsion-free $\mathbb{Z}$-module. Assume that we have $z \in \mathbb{Z}[x_{ij}, t_{ij}, y]$ and $0 \neq a \in \mathbb{Z}$ such that $az = 0$ in A. Then there exists $\alpha, \beta_{ij} \in \mathbb{Z}$ for $1 \leq i, j \leq n$ s.t.

$$\begin{aligned} az &= \alpha(\det(x_{ij})y - 1) + \sum_{ij} \beta_{ij}(x_{ij} - \delta_{ij} - mt_{ij}) \\ \Leftrightarrow z &= \frac{\alpha}{a}\det(x_{ij})y + \sum_{i,j} \frac{\beta_{ij}}{a} x_{ij} - \sum_{i,j} \frac{m\beta_{ij}}{a} t_{ij} - \frac{\alpha + \sum_i \beta_{ii}}{a} \end{aligned}$$

which implies $a|\alpha$ and $a|\beta_{ij}$, i.e. $z = 0$ in A. Hence $\mathcal{G}$ is a model of $\mathrm{GL}_{n,\mathbb{Q}}$ over $\mathbb{Z}$.

The set of $\mathbb{Z}$-points $\mathcal{G}(\mathbb{Z}) = \mathrm{Hom}(A, \mathbb{Z})$ can be identified with the set via map

$$\begin{aligned} \mathrm{Hom}(A, \mathbb{Z}) &\to \{g \in \mathrm{GL}_n(\mathbb{Z}) : g \equiv I_n \pmod{m\mathrm{M}_n(\mathbb{Z})}\} \\ \phi &\mapsto (g_{ij} = \phi(x_{ij})) \end{aligned}$$

since $\phi(x_{ij}) = \delta_{ij} + m\phi(t_{ij}) \Rightarrow g - I_n \in m\mathrm{M}_n(\mathbb{Z})$.

***Problem* 2.9** NOT FINISHEDIt is not hard to prove that if Z_1, Z_2 are dense subsets of a topological space Y_1, Y_2 respectively, then $Z_1 \times Z_2$ is dense in $Y_1 \times Y_2$. Combining with Exercise 1.6 and Theorem 2.2.1 (b), we get the desired results for both weak and strong approximation.

***Problem* 2.10** By Exercise 2.7 and 2.9, $\mathrm{M}_n \simeq \mathbb{G}_a^{n^2}$ admits weak approximation over F. With embedding $\mathrm{GL}_n \hookrightarrow \mathrm{M}_n$ with $\mathrm{GL}_n(F) = \mathrm{M}_n(F) \cap \mathrm{GL}_n(F_S) \subset \mathrm{M}_n(F_S)$, we also have $\mathrm{GL}_n(F)$ dense in $\mathrm{GL}_n(F_S)$.

***Problem* 2.11** NOT FINISHED

***Problem* 2.12** NOT FINISHED

***Problem* 2.13** NOT FINISHED

***Problem* 2.14** NOT FINISHED

***Problem* 2.15** NOT FINISHED

***Problem* 2.16** NOT FINISHEDThe center Z_{GL_2} of GL_2 is $Z_{\mathrm{GL}_2}(R) = R^\times I_2$. Hence the largest $\mathbb{F}_p(t)$ split torus in $\mathrm{Res}_{F/\mathbb{F}_p(t)} Z_{\mathrm{GL}_2}$ is just $\mathrm{Res}_{F/\mathbb{F}_p(t)} Z_{\mathrm{GL}_2}$ itself which has degree $d = [F : \mathbb{F}_q(t)]$.

***Problem* 2.17** NOT FINISHED

***Problem* 2.18** NOT FINISHED

***Problem* 2.19** Let $N = p_1^{e_1} \cdots p_r^{e_r}$ be a prime factorization of N. Define $K_N \leq \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^\infty)$ as

$$K_N = \prod_{i=1}^r (I_n + p_i^{e_i}\mathrm{M}_n(\mathbb{Z}_{p_i})) \times \prod_{p \neq p_i} \mathrm{GL}_n(\mathbb{Z}_p).$$

Then K_N is an open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^\infty)$ such that $K_N \cap \mathrm{GL}_n(\mathbb{Q}) = \Gamma(N)$.

($\Rightarrow$) Let H be a congruence subgroup of $\mathrm{GL}_n(\mathbb{Q})$, which means that there exists an open compact subgroup $K_H \leq \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ such that $H = K_H \cap \mathrm{GL}_n(\mathbb{Q})$. Then we can find an open compact neighborhood $U \leq K_H$ of I_n which has a form of

$$U = \prod_{p \in S}(I_n + p^{e_p}\mathrm{M}_n(\mathbb{Z}_p)) \times \prod_{p \notin S} \mathrm{GL}_n(\mathbb{Z}_p)$$

for some finite set of primes S (Note that $\{I_n + p^k\mathrm{M}_n(\mathbb{Z}_p)\}_{k\geq 1}$ is a decreasing sequence of open compact neighborhoods of I_n, which is also a subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$). Then $U = K_N$ for $N = \prod_{p\in S} p^{e_p}$, i.e. U is also an open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$, and it is a finite index subgroup of K_H since K_H is open and compact (consider all the cosets of K_N in K_H, which are all homeomorphic to K_N). Then $[H : \Gamma(N)] = [K_H : K_N]$ implies that H contains $\Gamma(N)$ as a finite index subgroup.

($\Leftarrow$) Let H be a subgroup of $\mathrm{GL}_n(\mathbb{Q})$ contains $\Gamma(N)$ with $[H : \Gamma(N)] < \infty$. Let K_H be an image of H in $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ under the diagonal embedding $\mathrm{GL}_n(\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ so that $K_H \cap \mathrm{GL}_n(\mathbb{Q}) = H$. Then K_H contains K_N and $[K_H : K_N] = [H : \Gamma(N)]$, so K_N is a finite index subgroup of K_H. for coset representatives $g_1, g_2, \ldots, g_t$ of K_H/K_N, $K_H = \cup_{j=1}^t g_j K_N$ and by openness (resp. compactness) of K_N, K_H is also open (resp. compact) subgroup.

3. Chapter 3

***Problem* 3.1** NOT FINISHED

***Problem* 3.2** Since G is compact, the image of the modular quasi-character $\delta_G : G \to \mathbb{R}_{>0}^\times$ is a compact subgroup of $\mathbb{R}_{>0}^\times$. Then it should be trivial - otherwise, there exists $g \in G$ with $\delta_G(g) > 1$ (we can choose g or g^{-1}), and then $\delta_G(g^n) = \delta_G(g)^n \to \infty$ as $n \to \infty$, i.e. the image is not bounded. Hence G is unimodular.

***Problem* 3.3** NOT FINISHED

***Problem* 3.4** NOT FINISHED

***Problem* 3.5** NOT FINISHED

***Problem* 3.6** NOT FINISHED

***Problem* 3.7** Let k be a residue field and ϖ be a uniformizer of F. We have $\mathcal{O}_F^\times = \coprod_{a\in k^\times}(a + \varpi\mathcal{O}_F)$ and

$$\begin{aligned}
d^\times x(\mathcal{O}_F^\times) &= \int_{\mathcal{O}_F^\times} \frac{dx}{|x|} \\
&= \int_{\mathcal{O}_F^\times} dx \\
&= dx(\mathcal{O}_F^\times) \\
&= \sum_{a\in k^\times} dx(a + \varpi\mathcal{O}_F) \\
&= \sum_{a\in k^\times} q^{-1} dx(\mathcal{O}_F) \\
&= (q-1)q^{-1}dx(\mathcal{O}_F) = (1-q^{-1})dx(\mathcal{O}_F).
\end{aligned}$$

***Problem* 3.8** NOT FINISHED

***Problem* 3.9** NOT FINISHED

***Problem* 3.10** NOT FINISHED

***Problem* 3.11** Let $x, g, y \in \mathrm{GL}_n(F)$ with $y = xg$ (regard g as a constant matrix). Then we have $y_{ij} = \sum_{1\le k\le n} x_{ik}g_{kj}$ and $dy_{ij} = \sum_{1\le k\le n} g_{kj}dx_{ik}$. This gives

$$\begin{aligned}
&dy_{11} \wedge dy_{12} \wedge \cdots \wedge dy_{1n} \\
&= (g_{11}dx_{11} + g_{21}dx_{12} + \cdots + g_{n1}dx_{1n}) \wedge \cdots \wedge (g_{1n}dx_{11} + \cdots + g_{nn}dx_{1n}) \\
&= |\det(g^t)| dx_{11} \wedge \cdots \wedge dx_{1n} \\
&= |\det(g)| dx_{11} \wedge \cdots \wedge dx_{1n}
\end{aligned}$$

and along with $\det(xg) = \det(x)\det(g)$, we have

$$\frac{\wedge_{i,j} dy_{ij}}{|\det(y)|^n} = \frac{|\det(g)|^n \wedge_{i,j} dx_{ij}}{|\det(xg)|^n} = \frac{\wedge_{i,j} dx_{ij}}{|\det(x)|^n}$$

so $d(x_{ij})$ is right Haar measure. Since GL_n is reductive, it is unimodular and so $d(x_{ij})$ is also a left Haar measure.

***Problem* 3.12** NOT FINISHED

***Problem* 3.13** NOT FINISHED

***Problem* 3.14** NOT FINISHEDConsider a reduction map $\mathrm{GL}_n(\mathcal{O}_{F_v}) \twoheadrightarrow \mathrm{GL}_n(k_v)$ where k_v is a residue field of F_v with $\#k_v = q_v$, which is surjective. The kernel H

of the map is $1 + \varpi_v \mathrm{M}_n(\mathcal{O}_{F_v})$ where ϖ_v is a uniformizer of F_v. Then we have

$$|\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = |\omega|_v(H) \cdot \#\,\mathrm{GL}_n(k_v).$$

The order of $\mathrm{GL}_n(k_v)$ is $(q_v^2-1)(q_v^2-q_v)$: there are q_v^2-1 choices for the first column vector (all but zero vector), and $q_v^2-q_v$ choices for the second column vector (all but vectors which are multiples of the first column vector). Also, for $h \in H$, we have

$$h = \begin{pmatrix} 1+\varpi_v x_{11} & \varpi_v x_{12} \\ \varpi_v x_{21} & 1+\varpi_v x_{22} \end{pmatrix}$$
$$\Rightarrow |\det(h)|_v = |1 + \varpi_v(x_{11}+x_{22}) + \varpi_v^2(x_{11}x_{22} - x_{12}x_{21})|_v = 1$$

So

$$\begin{aligned} |\omega|_v(H) &= \int_{\mathcal{O}_{F_v}^4} d(\varpi_v x_{11}) \wedge \cdots \wedge d(\varpi_v x_{22}) \\ &= q_v^{-4} \int_{\mathcal{O}_{F_v}^4} dx_{11} \wedge \cdots \wedge dx_{22} = q_v^{-4} \end{aligned}$$

and the measure is $|\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = (1-q_v^{-1})(1-q_v^{-2})$.

When F is a number field, then the *Dedekind zeta function* of F, defined as

$$\zeta_F(s) := \sum_{0 \neq I \subseteq \mathcal{O}_F} \frac{1}{N_{F/\mathbb{Q}}(I)^s}$$

admits an Euler product for $\Re s > 1$:

$$\zeta_F(s) = \prod_{\mathfrak{p} \subseteq \mathcal{O}_F} \frac{1}{1 - N_{F/\mathbb{Q}}(\mathfrak{p})^s}.$$

Then the product is

$$\prod_{v \nmid \infty} \left(1 - \frac{1}{q_v}\right)\left(1 - \frac{1}{q_v^2}\right)$$

and this diverges since $\prod_{v\nmid\infty}(1-q_v^{-1})$ does and $\prod_{v\nmid\infty}(1-q_v^{-2}) = \zeta_F(2)^{-1}$ does not. However, the normalized product

$$\prod_{v\nmid\infty}(1-q_v^{-1})^{-1}|\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = \prod_{v\nmid\infty}(1-q_v^{-2})$$

conveges to $\zeta_F(2)^{-1}$.

Now assume that F is a function field.

***Problem* 3.15** (Note that this is a theorem of Maschke.) It is enough to show the following:

Claim. Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of finite group G, and let U be a subrepresentation of ρ, i.e. invariant under ρ. Then there exists $W \leq V$ such that $U \cap W = \{0\}$ and $U \oplus W = V$.

Applying the above claim repeatedly shows that any representation of a finite group is completely decomposable. To show the lemma, let W' be *any* subspace of V such that $U \cap W' = \{0\}$ and $U \oplus W' = V$. Let $\pi' : V \to U$ be a corresponding projection. Then define $\pi : V \to V$ as

$$\pi(v) = \frac{1}{|G|}\sum_{g \in G} g^{-1}\pi'(gv)$$

whose image is in U ($gv := \rho(g)v$). Our claim is that $W = \ker \pi$ is the desired subspace: W is ρ-invariant and $U \oplus W = V$. First of all, since $\pi'|_U$ is identity on U and U is ρ-invariant, $\pi|_U$ is also an identity map on U. Then we have $W \cap U = 0$, and by dimension counting we get $V = U \oplus W$. Hence we only need to show that W is ρ-invariant: for $h \in G$ and $v\pi W = \ker \pi$,

$$\begin{aligned}\pi(hv) &= \frac{1}{|G|}\sum_{g\in G} g^{-1}\pi'(ghv) \\ &= \frac{1}{|G|}\sum_{g'\in G} hg'^{-1}\pi'(g'v) \quad (g' = gh) \\ &= h\pi(v) = 0\end{aligned}$$

so $hv \in W$.

***Problem* 3.16** Assume that the representation $\rho : B(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ is completely reducible. Since the representation is 2-dimensional, it should be decomposed as $\chi_1 \oplus \chi_2$ for some characters $\chi_1, \chi_2 : \mathbb{C}^\times \to \mathbb{C}^\times$. In other words, there exists $g_0 \in \mathrm{GL}_2(\mathbb{C})$ such that

$$\rho(g) = g_0 \begin{pmatrix} \chi_1(g) & 0 \\ 0 & \chi_2(g)\end{pmatrix} g_0^{-1}.$$

This implies $\rho(gh) = \rho(hg)$, which is not true since $B(\mathbb{C})$ is not commutative.

***Problem* 3.17** For any $g \in G$,

$$\begin{aligned}((f_1 * f_2) * f_3)(g) &= \int_G (f_1 * f_2)(gh_1^{-1}) f_3(h_1) d_r h_1 \\ &= \int_G \int_G f_1(gh_1^{-1}h_2^{-1}) f_2(h_2) d_r h_2 f_3(h_1) d_r h_1 \\ &= \int_G \int_G f_1(gh_1^{-1}h_2^{-1}) f_2(h_2) f_3(h_1) d_r h_2 d_r h_1 \\ &= \int_G \int_G f_1(gh_3^{-1}) f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_3 d_r h_1 \quad (h_3 = h_2 h_1,\ d_r h_3 = d_r h_2) \\ &= \int_G \int_G f_1(gh_3^{-1}) f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_1 d_r h_3 \quad \text{(Fubini's theorem)} \\ &= \int_G f_1(gh_3^{-1}) \left(\int_G f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_1\right) d_r h_3 \\ &= \int_G f_1(gh_3^{-1})(f_2 * f_3)(h_3) d_r h_3 \\ &= (f_1 * (f_2 * f_3))(g).\end{aligned}$$

***Problem* 3.18** NOT FINISHED

***Problem* 3.19**

$$\begin{aligned}
\pi(f_1 * f_2)\varphi &= \int_G (f_1 * f_2)(g)\pi(g)\varphi d_r g \\
&= \int_G \int_G f_1(gh^{-1})f_2(h)\pi(g)\varphi d_r h d_r g \\
&= \int_G \int_G f_1(gh^{-1})f_2(h)\pi(g)\varphi d_r g d_r h \quad \text{(Fubini's theorem)} \\
&= \int_G \int_G f_1(g_1)f_2(h)\pi(g_1 h)\varphi d_r g_1 d_r h \quad (g_1 = gh^{-1}, d_r g_1 = d_r g) \\
&= \int_G \int_G f_1(g_1)f_2(h)\pi(g_1 h)\varphi d_r h d_r g_1 \quad \text{(Fubini's theorem)} \\
&= \int_G f_1(g_1)\pi(g_1)\left(\int_G f_2(h)\pi(h)\varphi d_r h\right) d_r g_1 \\
&= \int_G f_1(g_1)\pi(g_1)\pi(f_2)\varphi d_r g_1 \\
&= (\pi(f_1) \circ \pi(f_2))\varphi
\end{aligned}$$

4. Chapter 4

***Problem* 4.1** NOT FINISHED

***Problem* 4.2** NOT FINISHED

***Problem* 4.3** NOT FINISHED

***Problem* 4.4** NOT FINISHED

***Problem* 4.5** By Schur's lemma, any elements in a center $z \in Z_G(F)$ acts as a (nonzero) scalar, let's say, $\omega_\pi(z) \in \mathbb{C}^\times$. Then $\omega_\pi : Z_G(F) \to \mathbb{C}^\times$ is a character since $\omega_G = \pi|_{Z_F(G)}$.

Let $\chi : G(F) \to \mathbb{C}^\times$ be a quasi-character. The representation $\pi \otimes \chi$ is defined as $(\pi \otimes \chi)(g)v = \chi(g) \cdot \pi(g)v$, and it's restriction on the center becomes $\chi|_{Z_G(F)} \cdot \omega_\pi$, which is the centeral character $\omega_{\pi\otimes\chi}$ of $\pi \otimes \chi$.

***Problem* 4.6** Let $G = \mathbb{G}_a$ and $G(\mathbb{R}) = (\mathbb{R}, +)$. Consider a 1-dimensional representation $\chi_\alpha : G(\mathbb{R}) \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, $t \mapsto e^{\alpha t}$, where $\Re(\alpha) \neq 0$. Then this is irreducible since 1-dimensional, but not unitary since $|e^{\alpha t}| = e^{\Re(\alpha)t} \neq 1$ for $t \neq 0$.

***Problem* 4.7** NOT FINISHED

***Problem* 4.8** NOT FINISHED

***Problem* 4.9** NOT FINISHED

***Problem* 4.10** NOT FINISHED

***Problem* 4.11** NOT FINISHED

***Problem* 4.12** NOT FINISHED

***Problem* 4.13** NOT FINISHED

5. Chapter 5

***Problem* 5.1** NOT FINISHED

***Problem* 5.2** One direction is clear. For the other direction, assume that (π, V) is admissible and let U be an open subgroup of G. Then we can choose compact open subgroup K such that $K \leq U \leq G$, and we have $V^U \leq V^K$. Now $\dim V^K < \infty$ gives $\dim V^U < \infty$.

***Problem* 5.3** Let $v \in V_{\mathrm{sm}}$, so that $v \in V^K$ for some open compact subgroup $K \leq G$. Then for $g \in G$ and $k \in K$, we have $\pi(k)v = v \Rightarrow \pi(gkg^{-1})\pi(g)v = \pi(g)v$. Hence $\pi(g)v$ is fixed by gkg^{-1} for all $k \in K$, hence $\pi(g)v \in V^{gKg^{-1}} \subseteq V_{\mathrm{sm}}$. Hence V_{sm} is preserved by G.

Now let $H = \mathrm{Stab}(v) \leq G$ be a stabilizer of $v \in V_{\mathrm{sm}}$. There exists an open compact subgroup K with $v \in V^K \leftrightarrow K \leq \mathrm{Stab}(v)$, so $\mathrm{Stab}(v)$ is a union of open cosets homeomorphic to K, which is also open. Hence $(\pi_{V_{\mathrm{sm}}}, V_{\mathrm{sm}})$ is smooth.

***Problem* 5.4** NOT FINISHED

***Problem* 5.5** NOT FINISHED

***Problem* 5.6** NOT FINISHED

***Problem* 5.7** NOT FINISHED

***Problem* 5.8** The proof is essentially same as that of Problem 4.5, and we also use Schur's lemma (Problem 5.6).

***Problem* 5.9** Take $x = p^k$. Then $|p^k|_p = p^{-k}$ and $|p^k|_\infty = p^k$, so we have $c_1 \cdot p^k \leq p^{-k} \Leftrightarrow c_1 \leq p^{-2k}$ for all k. Now taking limit $k \to \infty$ gives $c_1 = 0$, which gives a contradiction. We can do similarly for the other direction with $k \to -\infty$.

***Problem* 5.10** NOT FINISHED

***Problem* 5.11** NOT FINISHED

***Problem* 5.12** See the argument in Problem 5.3 for showing G-invariance. To show denseness, let $v \in V$ and let $\epsilon > 0$. By continuity of $G \times V \to V$, there exists open neighborhood U of 1 such that $\|\pi(u)v - v\| < \epsilon$ for all $u \in U$. Now we can choose open compact subgroup K of G lies in U (by totally connectedness), and we have

$$\begin{aligned}\|e_K v - v\| &= \left\| \frac{1}{\mathrm{meas}_{dg}(K)} \int_G (\mathbb{1}_K(g)\pi(g)v - v) dg \right\| \\ &\leq \frac{1}{\mathrm{meas}_{dg}(K)} \int_K ||\pi(k)v - v|| dk < \epsilon\end{aligned}$$

and from $e_K v \in V^K$, we get denseness of V_{sm} in V.

***Problem* 5.13** NOT FINISHED

***Problem* 5.14** NOT FINISHED

6. Chapter 6

***Problem* 6.1** NOT FINISHED

***Problem* 6.2** NOT FINISHED

***Problem* 6.3** NOT FINISHED

***Problem* 6.4** NOT FINISHED

***Problem* 6.5** NOT FINISHED

***Problem* 6.6** NOT FINISHED

***Problem* 6.7** NOT FINISHED

***Problem* 6.8** NOT FINISHED

***Problem* 6.9** NOT FINISHED

***Problem* 6.10** NOT FINISHED

7. Chapter 7

***Problem* 7.1** NOT FINISHED

***Problem* 7.2** NOT FINISHED

***Problem* 7.3** NOT FINISHED

***Problem* 7.4** NOT FINISHED

***Problem* 7.5** It is clear that $V \mapsto V^K$ is left exact, i.e. it preserves injectivity. Hence we'll only show right exactness. For $V \twoheadrightarrow V/W$, we'll show that $V^K \to (V/W)^K$ is surjective. Let $[v] \in (V/W)^K$, so that $\pi(k)(v) - v \in W$ for all $k \in K$. Define v_0 as

$$v_0 = \frac{1}{|K|} \int_K \pi(k) v dk,$$

an average of v over K. We can see that

$$v_0 - v = \frac{1}{|K|} \int_K (\pi(k)v - v) dk \in W$$

so $[v_0] = [v]$ in V/W. Also,

$$\pi(k')v_0 = \frac{1}{|K|} \int_K \pi(k'k) v dk = \frac{1}{|K|} \int_K \pi(k) dk = v_0$$

so $v_0 \in V^K$. Hence $[v]$ is an image of v_0 and $V^K \to (V/W)^K$ is surjective.

***Problem* 7.6** NOT FINISHED

***Problem* 7.7** NOT FINISHED

8. Chapter 8

***Problem* 8.1** NOT FINISHED

***Problem* 8.2** NOT FINISHED

***Problem* 8.3** NOT FINISHED

***Problem* 8.4** NOT FINISHED

***Problem* 8.5** NOT FINISHED

***Problem* 8.6** NOT FINISHED

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***Problem* 8.9** NOT FINISHED

***Problem* 8.10** NOT FINISHED

***Problem* 8.11** NOT FINISHED

***Problem* 8.12** NOT FINISHED

***Problem* 8.13** NOT FINISHED

***Problem* 8.14** NOT FINISHED

***Problem* 8.15** NOT FINISHED

9. Chapter 9

***Problem* 9.1** Let $\mathcal{B} = \{\varphi_i\}$ and $\mathcal{B}' = \{\varphi_i'\}$ be two orthonormal basis. Let $C : V \to V$ be a transition operator from $\mathcal{B}$ to $\mathcal{B}'$, so that $\varphi_i' = \sum_k c_{ik}\varphi_k$ for all i. From $\langle \varphi_i', \varphi_j' \rangle = \delta_{ij}$, one can check that C is an isometry (i.e. $C^*C = I$), and since C is surjective it is a unitary operator. In other words, we also have $CC^* = I$ which implies

$$\sum_i c_{ij}\overline{c_{ik}} = \delta_{j,k}.$$

Now, we have

$$\sum_i \|A\varphi_i'\|_2^2 = \sum_i \langle A\varphi_i', A\varphi_i' \rangle = \sum_{i,j,k} c_{ij}\overline{c_{ik}} \langle A\varphi_j, A\varphi_k \rangle = \sum_{j,k} \delta_{jk} \langle A\varphi_j, A\varphi_k \rangle = \sum_j \|A\varphi_i\|_2^2$$

so $\|A\|_{\mathrm{HS}}$ does not depends on the choice of basis. We can prove the same property of $\|A\|_{\mathrm{tr}}$ and $\operatorname{tr} A$ by the same way (using the fact that C is unitary).

***Problem* 9.2** NOT FINISHED

***Problem* 9.3** NOT FINISHED

***Problem* 9.4** NOT FINISHED

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***Problem* 9.8** NOT FINISHED

***Problem* 9.9** NOT FINISHED

***Problem* 9.10** NOT FINISHED

***Problem* 9.11** NOT FINISHED

10. Chapter 10

***Problem* 10.1** NOT FINISHED

***Problem* 10.2** NOT FINISHED

***Problem* 10.3** This follows from the direct computation:

$$\begin{aligned}
&E(x, I(\sigma,\lambda)(g)\varphi, \lambda) \\
&= \sum_{\delta\in P(F)\backslash G(F)} I(\sigma,\lambda)(g)\varphi(\delta x)e^{\langle H_P(\delta x),\lambda+\rho_P\rangle} \\
&= \sum_{\delta\in P(F)\backslash G(F)} \varphi(\delta xg)e^{\langle H_P(\delta xg),\lambda+\rho_P\rangle}e^{-\langle H_P(\delta x),\lambda+\rho_P\rangle}e^{\langle H_P(\delta x),\lambda+\rho_P\rangle} \\
&= \sum_{\delta\in P(F)\backslash G(F)} \varphi(\delta xg)e^{\langle H_P(\delta xg),\lambda+\rho_P\rangle} \\
&= E(xg,\varphi,\lambda).
\end{aligned}$$

***Problem* 10.4** NOT FINISHED

***Problem* 10.5** NOT FINISHED

***Problem* 10.6** NOT FINISHED

***Problem* 10.7** NOT FINISHED

***Problem* 10.8** NOT FINISHED

***Problem* 10.9** NOT FINISHED

11. Chapter 11

Problem **11.1** NOT FINISHED

Problem **11.2** NOT FINISHED

Problem **11.3** NOT FINISHED

Problem **11.4** NOT FINISHED

Problem **11.5** NOT FINISHED

Problem **11.6** NOT FINISHED

Problem **11.7** NOT FINISHED

Problem **11.8** NOT FINISHED

Problem **11.9** NOT FINISHED

12. Chapter 12

***Problem* 12.1** Let G be a connected Lie group and $\phi : \mathrm{Gal}_F \to G$ be a continuous homomorphism. Since $\ker \phi = \phi^{-1}(1)$, it is a closed subgroup of Gal_F and the image $\operatorname{im} \phi \cong \mathrm{Gal}_F / \ker \phi$ is also a profinite group with quotient topology. Hence $\operatorname{im} \phi \leq G$ is a discrete compact subgroup of G, so is finite. (Note that a group is profinite if and only if it is Hausdorff, compact, and totally disconnected.)

***Problem* 12.2** The (geometric) Frobenius element in $\mathrm{Gal}_{\mathbb{Q}}$ is defined as a restriction of inverse image of $\mathrm{Fr}_p \in \mathrm{Gal}_{\mathbb{F}_p}$ via $\mathrm{Gal}_{\mathbb{F}_p} \twoheadleftarrow \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{Gal}_{\mathbb{Q}}$. It gives a well-defined conjugacy class in $\mathrm{Gal}_{\mathbb{Q}}$ and act as a p^{-1}-power map on roots of unity. Hence $a_{\mathrm{Fr}_p, n} = p^{-1}$ for all $n \geq 1$ and $\chi_\ell(\mathrm{Fr}_p) = p^{-1}$.

***Problem* 12.3** NOT FINISHED

***Problem* 12.4** NOT FINISHED

***Problem* 12.5** NOT FINISHED

***Problem* 12.6** NOT FINISHED

***Problem* 12.7** Let $V_i = \oplus_{n \geq 0} V_n^{(i)} \otimes \mathrm{Sym}^n$ be the decompositions of V_1 and V_2 into Sym^n-isotypic components for $i = 1, 2$. Then that for $V_1 \oplus V_2$ is $\oplus_{n \geq 0}(V_n^{(1)} \oplus V_n^{(2)}) \otimes \mathrm{Sym}^n$, and the corresponding L-function becomes

$$L(s, \rho_1 \oplus \rho_2) = \prod_{n \geq 0} \det \left(1 - (\rho_1 \oplus \rho_2)(\mathrm{Fr}) q^{-(s+n/2)}|_{(V_n^{(1)} \oplus V_n^{(2)})^{I_F}}\right)^{-1}.$$

By choosing a basis of $V_n^{(1)}$ and $V_n^{(2)}$, we can express

$$(1 - (\rho_1 \oplus \rho_2)(\mathrm{Fr}) q^{-(s+n/2)})|_{(V_n^{(1)} \oplus V_n^{(2)})^{I_F}}$$

as a following block diagonal matrix form

$$1 - \begin{pmatrix} \rho_1(\mathrm{Fr}) q^{-(s+n/2)} & 0 \\ 0 & \rho_2(\mathrm{Fr}) q^{-(s+n/2)} \end{pmatrix} = \begin{pmatrix} 1 - \rho_1(\mathrm{Fr}) q^{-(s+n/2)} & 0 \\ 0 & 1 - \rho_2(\mathrm{Fr}) q^{-(s+n/2)} \end{pmatrix}$$

which gives

$$\begin{aligned} &\det \left(1 - (\rho_1 \oplus \rho_2)(\mathrm{Fr}) q^{-(s+n/2)}|_{(V_n^{(1)} \oplus V_n^{(2)})^{I_F}}\right) \\ &= \det \left(1 - \rho_1(\mathrm{Fr}) q^{-(s+n/2)}|_{V_n^{(1) I_F}}\right) \det \left(1 - \rho_2(\mathrm{Fr}) q^{-(s+n/2)}|_{V_n^{(2) I_F}}\right) \end{aligned}$$

and this implies $L(s, \rho_1 \oplus \rho_2) = L(s, \rho_1) L(s, \rho_2)$.

***Problem* 12.8** NOT FINISHED

***Problem* 12.9** NOT FINISHED

***Problem* 12.10** NOT FINISHED

***Problem* 12.11** NOT FINISHED

***Problem* 12.12** NOT FINISHED

13. Chapter 13

***Problem* 13.1** NOT FINISHED

***Problem* 13.2** NOT FINISHED

***Problem* 13.3** NOT FINISHED

***Problem* 13.4** See the solution for Exercise 12.1. (Gal_F is a profinite group and $\mathrm{GL}_n(\mathbb{C})$ is a connected Lie group.)

***Problem* 13.5** NOT FINISHED

14. Chapter 14

***Problem* 14.1** NOT FINISHED

***Problem* 14.2** NOT FINISHED

***Problem* 14.3** NOT FINISHED

***Problem* 14.4** NOT FINISHED

***Problem* 14.5** NOT FINISHED

***Problem* 14.6** NOT FINISHED

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***Problem* 14.8** NOT FINISHED

***Problem* 14.9** NOT FINISHED

15. Chapter 15

***Problem* 15.1** NOT FINISHED

***Problem* 15.2** NOT FINISHED

***Problem* 15.3** NOT FINISHED

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***Problem* 15.7** NOT FINISHED

***Problem* 15.8** NOT FINISHED

16. Chapter 16

***Problem* 16.1** NOT FINISHED

***Problem* 16.2** NOT FINISHED

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***Problem* 16.6** NOT FINISHED

***Problem* 16.7** NOT FINISHED

17. Chapter 17

***Problem* 17.1** NOT FINISHED

***Problem* 17.2** NOT FINISHED

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***Problem* 17.9** NOT FINISHED

***Problem* 17.10** NOT FINISHED

***Problem* 17.11** For well-definedness, we only need to check that $g_2 \in C_{B_\sigma(\gamma),G}(R)$ when $(g_1, g_2) \in H_\gamma(R)$. (Note that $(g_1, g_2) \in H_\gamma(R)$ automatically implies $g_2 \in G^\sigma(R)$.) We have

$$g_1^{-1}\gamma g_2 = \gamma \Rightarrow g_1^{-1}\sigma(\gamma)g_2 = \sigma(g_1^{-1}\gamma g_2) = \sigma(\gamma)$$

and we can "cancel out" g_1 from two equations to get $g_2^{-1}\sigma(\gamma)^{-1}\gamma g_2 = \sigma(\gamma)^{-1}\gamma$, i.e. $g_2 \in C_{\sigma(\gamma)^{-1}\gamma,G}(R) = C_{B_\sigma(\gamma),G}(R)$. It is obvious that the map is a homomorphism. The map is injective since when $g_2 = 1$, $g_1^{-1}\gamma g_2 = g_1^{-1}\gamma = \gamma$ implies $g_1 = 1$. Also, surjectivity follows from $(\gamma g_2\gamma^{-1}, g_2) \mapsto g_2$.

***Problem* 17.12** For well-definedness, wee need to show that a) the map $\sigma \mapsto h\sigma(h^{-1})$ is 1-cocycle, b) the map lies in the kernel of $\mathrm{H}^1(k, I) \to \mathrm{H}^1(k, H)$, and c) it does not depend on the choice of the right coset representative. a) follows from the direct computation, and b) is also clear from the definition of cohomologousness and neutral element. For c), assume that $h_2 = ih_1$ for $h_1, h_2 \in H(k^{\mathrm{sep}})$ and $i \in H(k^{\mathrm{sep}})$. Then from $(ih_1)\sigma(ih_1)^{-1} = ih_1\sigma(h_1)^{-1}\sigma(i)^{-1}$, we have $c_1(\sigma) = i^{-1}c_2(\sigma)\sigma(i)$ for $c_i(\sigma) := h_i\sigma(h_i)^{-1}$, hence they lies in the same equivalence class in $\mathrm{H}^1(k, I)$.

For surjectivity, if $c \in \mathcal{D}(k, I, H)$, then $c : \mathrm{Gal}_k \to I(k^{\mathrm{sep}}) \to H(k^{\mathrm{sep}})$ that is cohomologous to the neutral element, and this implies that there exist $h \in H(k^{\mathrm{sep}})$ such that $c(\sigma) = h\sigma(h)^{-1}$.

Now, let's assume that h_1 and h_2 maps to the same element in $\mathcal{D}(k, I, H)$, so that there exists $i \in I(k^{\mathrm{sep}})$ s.t. $h_2\sigma(h_2)^{-1} = ih_1\sigma(h_1)^{-1}\sigma(i)^{-1} = ih_1\sigma(ih_1)^{-1}$. This gives $\sigma((ih_1)^{-1}h_2) = (ih_1)^{-1}h_2$ for all $\sigma \in \mathrm{Gal}_k$, hence $h_0 := (ih)^{-1}h_2$ is in $H(k^{\mathrm{sep}})^{\mathrm{Gal}_k} = H(k)$, and implies that $I(k^{\mathrm{sep}})h_1$ and $I(k^{\mathrm{sep}})h_2$ are in the same $H(k)$-orbit (under the action by right multiplication).

***Problem* 17.13** First, $I_{h,k^{\mathrm{sep}}}$ is a form of $I_{k^{\mathrm{sep}}}$: for any k^{sep}-algebra R, if we denote h_R for the image of h in $H(R)$ under the natural map $H(k^{\mathrm{sep}}) \to H(R)$, then $\varphi(R) : I_{k^{\mathrm{sep}}}(R) \to I_{h,k^{\mathrm{sep}}}(R)$ is given by $\varphi(R)(i) := h_R^{-1}ih_R$ gives an isomorphism. Under this φ, we can check that $(\varphi^{-1} \circ \sigma \circ \varphi \circ \sigma^{-1})(i) = (h\sigma^{-1}(h))i(h\sigma^{-1}(h))^{-1}$, hence it is an inner automorphism and I_h is an inner form of I.

***Problem* 17.14** NOT FINISHED

***Problem* 17.15** For each $x \in \Omega$, choose a compact open neighborhood U_x (note that X_α are locally compact). Then $\cup_{x\in\Omega}U_x = \Omega$, and compactness of Ω implies that $\Omega = \cup_{1\le i\le n} U_i$ for some $U_i = U_{x_i}$. For each i, U_i is a form of

$$U_i = U_{A_i} \times \prod_{\alpha \in A - A_i} K_\alpha$$

for some finite set A_i and compact open $U_{A_i} \subseteq \prod_{\alpha\in A_i} X_\alpha$. If we put $A' = \cup_{1\le i\le n} A_i$, then we can rewrite U_i as

$$U_i = \left(U_{A_i} \times \prod_{\alpha\in A'-A_i} K_\alpha\right) \times \prod_{\alpha\in A-A'} K_\alpha =: U_i' \times \prod_{\alpha\in A-A'} K_\alpha$$

and this gives $\Omega = \Omega_{A'} \times \prod_{\alpha\in A-A'} K_\alpha$ for $\Omega_{A'} = \cup_{1\le i\le n} U_i' \subseteq \prod_{\alpha\in A'} X_\alpha$.

***Problem* 17.16** For $\sigma, \sigma' \in \mathrm{Gal}_k$, we have

$$c_1(\sigma\sigma')c_2(\sigma\sigma') = c_1(\sigma)\sigma(c_1(\sigma'))c_2(\sigma)\sigma(c_2(\sigma')) = c_1(\sigma)c_2(\sigma)\sigma(c_1(\sigma')c_2(\sigma'))$$

so $\sigma \mapsto c_1(\sigma)c_2(\sigma)$ is a 1-cocycle. This gives a well-defined abelian group structure on $\mathrm{H}^1(k, G)$ as follows: if $c_1 \sim c_1'$ and $c_2 \sim c_2'$, there exists $g_1, g_2 \in G$ such that $c_i'(\sigma) = c_i(\sigma)g_i^{-1}\sigma(g_i)$ for all $\sigma \in \mathrm{Gal}_k$ and $i = 1, 2$. Then

$$(c_1'c_2')(\sigma) = c_1(\sigma)g_1^{-1}\sigma(g_1)c_2(\sigma)g_2^{-1}\sigma(g_2) = (c_1c_2)(\sigma)(g_1g_2)^{-1}\sigma(g_1g_2)$$

so the multiplication is well-defined. It is easy to check that the multiplication is commutative and associative. Also, for $[c] \in \mathrm{H}^1(k, G)$, its inverse is given by $[c]^{-1} := [c^{-1}]$, $c^{-1}(\sigma) := c(\sigma)^{-1}$.

***Problem* 17.17** NOT FINISHED

***Problem* 17.18** NOT FINISHED

18. Chapter 18

***Problem* 18.1** NOT FINISHED

***Problem* 18.2** NOT FINISHED

***Problem* 18.3** NOT FINISHED

***Problem* 18.4** NOT FINISHED

***Problem* 18.5** NOT FINISHED

***Problem* 18.6** NOT FINISHED

***Problem* 18.7** NOT FINISHED

***Problem* 18.8** NOT FINISHED

19. Chapter 19

***Problem* 19.1** NOT FINISHED

***Problem* 19.2** NOT FINISHED

***Problem* 19.3** NOT FINISHED

***Problem* 19.4** NOT FINISHED

***Problem* 19.5** NOT FINISHED

***Problem* 19.6** NOT FINISHED

***Problem* 19.7** NOT FINISHED

***Problem* 19.8** NOT FINISHED

20. Appendix B

***Problem* 20.1** Since tr is additive, it is enough to show when $F = \mathbb{R}, \mathbb{Q}_p, \mathbb{F}_p((t^{-1}))$, or $\mathbb{F}_p((t))_\varpi$ for some $\varpi \in \mathbb{F}_p((t))$.

- $F = \mathbb{R}$. $\psi_{\mathbb{R}}(a+b) = e^{-2\pi i(a+b)} = e^{-2\pi i a}e^{-2\pi i b} = \psi_{\mathbb{R}}(a)\psi_{\mathbb{R}}(b)$.
- $F = \mathbb{Q}_p$. Note that $\psi_{\mathbb{Q}_p}(a)$ is well-defined, since differences between two different choices of $\mathrm{pr}(a)$ are integers, and $e^{2\pi i n} = 1$ for all $n \in \mathbb{Z}$. Now, $\psi_{\mathbb{Q}_p}$ becomes a character since $\mathrm{pr}(a+b) - \mathrm{pr}(a) - \mathrm{pr}(b)$ is integer for any $a, b \in \mathbb{Q}_p$ (and for any choices of $\mathrm{pr}(a), \mathrm{pr}(b)$).
- $F = \mathbb{F}_p((t^{-1}))$. First, it is well-defined since $\exp(2\pi i a_1/p)$ does not depend on the choice of a representative $a_1 \in \mathbb{F}_p$ in $\mathbb{Z}$. For $f = \sum_n a_n t^n$ and $g = \sum_n b_n t^n$, we have

$$\psi_{\mathbb{F}_p(t^{-1}))}(f+g) = e^{2\pi i(a_1+b_1)/p} = e^{2\pi i a_1/p}e^{2\pi i b_1/p} = \psi_{\mathbb{F}_p((t^{-1}))}(f)\psi_{\mathbb{F}_p((t^{-1}))}(g)$$

  so it is a character.
- $F = \mathbb{F}_p((t))_\varpi$ for $\varpi \in \mathbb{F}_p((t))$. This case is similar as the case of $F = \mathbb{F}_p((t^{-1}))$. Note that $\mathrm{tr}_{k/\mathbb{F}_p}$ is additive.

***Problem* 20.2** NOT FINISHED(Injectivity) Since ψ is nontrivial, there exists $x_0 \in F$ such that $\psi(x_0) \neq 1$. Now if $\alpha \neq 0$, then for $\psi_\alpha(x) := \psi(\alpha x)$, $\psi_\alpha(x_0/\alpha) = \psi(x_0) \neq 1$, so ψ_α is not trivial.
(Surjectivity) Let $\phi : F \to \mathbb{C}^\times$ be a nontrivial character.

***Problem* 20.3** NOT FINISHED

***Problem* 20.4** NOT FINISHED

***Problem* 20.5** NOT FINISHED

***Problem* 20.6** NOT FINISHED

***Problem* 20.7** NOT FINISHEDWe have

$$\hat{\mathbb{1}}_{\mathcal{O}_F}(x) = \int_F \mathbb{1}_{\mathcal{O}_F}(y)\psi_F(xy)dy = \int_{\mathcal{O}_F} \psi_F(xy)dy$$

If $x \in \mathcal{O}_F$, then $xy \in \mathcal{O}_F$ for all $y \in \mathcal{O}_F$ and $\psi_F(xy) = 1$ (note that $\mathcal{O}_F \subseteq \ker \psi_F$). Hence $\int_{\mathcal{O}_F} \psi_F(xy)dy = \int_{\mathcal{O}_F} dy = dy(\mathcal{O}_F) = 1$. If $x \notin \mathcal{O}_F$, choose

$$\int_{\mathcal{O}_F} \psi_F(xy)dy = \int_{\mathcal{O}_F} \psi_F(x(y+a))dy = \psi_F(xa)\int_{\mathcal{O}_F} \psi_F(xy)dy$$