Irrationality Proofs Using Modular Forms

Re-TEXed by Seewoo Lee*

Frits Beukers

Last updated: November 13, 2024

0 Introduction

In the years following Apery's discovery of his irrationality proofs for $\zeta(2)$, $\zeta(3)$ (see [6]), it has become clear that these proofs do not only have significance as irrationality proofs, but the numbers that occur in them serve as interesting examples for several phenomena in algebraic geometry and modular form theory. See [4, 1, 2] for congruences of the Apéry numbers and [3, 5] for geometrical and modular interpretations.[1] Furthermore, it turns out that Apery's proofs themselves are in fact simple consequences of elementary complex analysis on spaces of certain modular forms. In the present paper we describe this analysis together with some generalisations in Theorems 1 to 5. For example, we prove that $8\zeta(3) - 5\sqrt{5}L(3) \notin \mathbb{Q}(\sqrt{5})$, where $L(3) = \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) n^{-3}$. Although the use of modular forms in irrationality proofs looks promising at first sight, the yield of new irrationality results thus far is disappointingly low. However, in methods such as this it is easy to overlook some simple tricks that may give new interesting results.

The first section of this paper describes the general framework of the proofs. This section may seem vague at first sight, but in combination with the proof of Theorem 1 we hope that things will be clear. We have given the proof of Theorem 1 as extensively as possible in order to set it as an example for the other proofs, where we omit some minor details now and then.

*seewoo5@berkeley.edu. Some notations in the paper are "modernized" or changed a bit.

[1]The original citations were in a different order, but it seems that this is the correct order.

1 Preliminaries

In this section we shall describe the general principles which are used in the arguments of the following sections.

Let $t(q) = \sum_{n=0}^{\infty} t_n q^n$ a power series convergent for all $|q| < 1$. Let $w(q)$ be another analytic function on $|q| < 1$. We like to study w as function of t. In general it will be a multivalued function over which we have no control. However, we shall introduce some assumptions. First, $t_0 = 0, t_1 \neq 0$. Let now $q(t)$ be the local inverse of $t(q)$ with $q(0) = 0$. Choose $w(q(t))$ for the value of w around $t = 0$. In order to determine the radius of convergence of the powerseries $w(q(t)) = \sum_{n=0}^{\infty} w_n t^n$ we introduce branching values of t. We say that t branches above t_0, if either t_0 is not in the image of t, or if $t'(q_0) = 0$ for some q_0 with $t(q_0) = t_0$. In other words, t branches above t_0, if the map $t : \{|q| < 1\} \to \mathbb{C}$ is not a local covering above t_0. We call such a t_0 a branching value of t. Now assume, that t has a discrete set of branching values $t_1, t_2, \ldots$ where we have excluded zero as a possible value and suppose $|t_1| < |t_2| < \ldots$. It is clear now that the radius of convergence is in general $|t_1|$. We shall be interested in cases where the radius of convergence is larger than $|t_1|$. Let γ be a closed contour in the complex t-plane beginning and ending at the origin, not passing through any t_i and which encircles the point t_1 exactly once. Suppose that analytic continuation of $w(q(t))$ along γ again yields the same branch of $w(q(t))$. Then $w(q(t))$ can be continued analytically to the disc $|t| < |t_2|$ with exception of the possible isolated singularity t_1. If $w(q(t))$ remains bounded around we can conclude that the radius of convergence is at least t_2. Our irrationality proofs consist exactly of the construction of such instances. The point of having a radius of convergence as large as possible consists of the following Proposition.

Proposition 1.1. Let $f_0(t), f_1(t), \ldots, f_k(t)$ be powerseries in t. Suppose that for any $n \in \mathbb{N}$, $i = 0, 1, \ldots, k$ the n-th coefficient in the Taylor series of is rational and has denominator dividing $d^n[1, \ldots, n]^r$ where r, d are certain fixed positive integers and $[1, \ldots, n]$ is the lowest common multiple of $1, \ldots, n$. Suppose there exist real numbers $\theta_1, \ldots, \theta_k$ such that $f_0(t) + \theta_1 f_1(t) + \cdots + \theta_k f_k(t)$ has radius of convergence ρ and infinitely many nonzero Taylor coefficients. If $\rho > de^r$, then at least one of $\theta_1, \ldots, \theta_k$ is irrational.

Remark 1.2. Note that if $k = 1$ we have an honest irrationality proof.

Proof. Choose $\epsilon > 0$ such that $\rho - \epsilon > de^{r(1+\epsilon)}$. Let $f_i = \sum_{n=0}^{\infty} a_{i,n} t^n$. Since the radius of the convergence of $f_0 + \theta_1 f_1 + \cdots + \theta_k f_k$ is ρ, we have for sufficiently large n, $|a_{0,n}+\theta_1 a_{1,n}+\cdots+\theta_k a_{k,n}| \le (\rho-\epsilon)^{-n}$. Suppose $\theta_1,\ldots,\theta_k$ are all rational and have common denominator D. Then $A_n = Dd^n[1,\ldots,n]^r|a_{0,n}+\theta_1 a_{1,n}+\cdots+\theta_k a_{k,n}|$ is an integer smaller than $Dd^n[1,\ldots,n]^r(\rho-\epsilon)^{-n}$. By the prime number theorem we have $[1,\ldots,n] < e^{(1+\epsilon)n}$ for sufficiently large n, hence $A_n < D(de^{r(1+\epsilon)}/(\rho-\epsilon))^n$. Since $de^{r(1+\epsilon)}(\rho-\epsilon)^{-1} < 1$ this implies that $A_n = 0$ for sufficiently large n, in contradiction with the assumption $A_n \neq 0$ for infinitely many n. Thus our proposition follows. □

The construction of the functions $t(q)$ and $w(q)$ will proceed using modular forms and functions. The values for which we obtain irrationality results are in fact values at integral points of Dirichlet series associated to modular forms.

Proposition 1.3. Let $F(\tau) = \sum_{n=1}^{\infty} a_n q^n$, $q = e^{2\pi i\tau}$ be a Fourier series convergent for $|q| < 1$, such that for some $k, n \in \mathbb{N}$,

$$F(-1/N\tau) = \varepsilon(-i\tau\sqrt{N})^k F(\tau)$$

where $\varepsilon = \pm 1$. Let $f(\tau)$ be the Fourier series

$$f(\tau) = \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} q^n.$$

Let

$$L(F,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and finally,

$$h(\tau) = f(\tau) - \sum_{0\le r<\frac{1}{2}(k-2)} \frac{L(F,k-r-1)}{k!}(2\pi i\tau)^r.$$

Then

$$h(\tau) - D = (-1)^{k-1}\varepsilon(-i\tau\sqrt{N})^{k-2} h(-1/N\tau)$$

where $D = 0$ if k is odd and $D = L(F,\frac{1}{2}k)(2\pi i\tau)^{\frac{1}{2}k-1}/(\frac{1}{2}k-1)!$ if k is even. Moreover, $L(F,\frac{1}{2}k) = 0$ if $\varepsilon = -1$.

Proof. We apply a lemma of Hecke, see [7, Section 5] with $G(\tau) = \varepsilon F(\tau)/(i\sqrt{N})^k$ to obtain

$$f(\tau) - \varepsilon(-1)^{k-1}(-i\tau\sqrt{N})^{k-2} f(-1/N\tau) = \sum_{r=0}^{k-2} \frac{L(F,k-r-1)}{r!}(2\pi i\tau)^r.$$

Split the summation on the right hand side into summations over $r < \frac{1}{2}k - 1$, $r > \frac{1}{2}k - 1$ and, possibly, $r = \frac{1}{2}k - 1$. For the region $r > \frac{1}{2}k - 1$ we apply the functional equation

$$\frac{L(F, k-r-1)}{r!} = \varepsilon(-1)^k(-i\sqrt{N})^{k-2}\left(-\frac{1}{N}\right)^{k-r-2}(2\pi i)^{k-2r-2}\frac{L(F, r+1)}{(k-r-2)!}$$

and substitute r by $k-2-r$. □

References

[1] Beukers, F. Irrationality of π^2, periods of an elliptic curve and $\Gamma_1(5)$. *Diophantine approximations and transcendental numbers (Luminy, 1982) 31* (1982), 47–66.

[2] Beukers, F. Some congruences for the Apéry numbers. *Journal of Number Theory 21*, 2 (1985), 141–155.

[3] Beukers, F., and Peters, C. A. M. A family of $K3$ surfaces and $\zeta(3)$. *J. Reine Angew. Math. 351* (1984), 42–54.

[4] Gessel, I. Some congruences for apery numbers. *Journal of Number Theory 14*, 3 (1982), 362–368.

[5] Stienstra, J., and Beukers, F. On the Picard-Fuchs equation and the formal Brauer group of certain elliptic $K3$-surfaces. *Mathematische Annalen 271* (1985), 269–304.

[6] Van der Poorten, A. A proof that Euler missed...Apéry's proof of the irrationality of $\zeta(3)$. *Math. Intelligencer 1*, 4 (1979), 195–203.

[7] Weil, A. Remarks on Hecke's lemma and its use. In *Algebraic number theory (Kyoto Internat. Sympos., Res. Inst. Math. Sci., Univ. Kyoto, Kyoto, 1976)*. Japan Soc. Promotion Sci., Tokyo, 1977, pp. 267–274.