SOLUTION FOR “AN INTRODUCTION TO AUTOMORPHIC REPRESENTATION”

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Abstract. This is a solution for the exercises in J. Getz and H. Hahn’s book *An Introduction to Automorphic Representation with a view toward Trace Formulae*.

1. Chapter 1

***Problem* 1.1** *** By Yoneda lemma, the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes corresponds to the k-algebra morphism $\phi : A \to B$. This induces a map on the underlying topological spaces by sending a prime ideal $\mathfrak{p} \subset B$ to $\phi^{-1}(\mathfrak{p}) \subset A$, which is also prime.

***Problem* 1.2** ***

***Problem* 1.3** By Yoneda lemma, we have

$$\mathrm{Mor}(\mathrm{Spec}(B), \mathrm{Spec}(A)) \simeq \mathrm{Nat}(h^B, h^A) \simeq h^B(A) = \mathrm{Hom}_k(A, B)$$

which gives an equivalence between $\mathbf{AffSch}_k^{\mathrm{op}}$ and $\mathbf{Alg}_k$.

***Problem* 1.4**
- Nonreduced: $\mathrm{Spec}(\mathbb{C}[x]/(x^2))$
- Reducible: $\mathrm{Spec}(\mathbb{C}[x, y]/(x, y))$
- Reduced and irreducible (i.e. integral): $\mathrm{Spec}(\mathbb{C}[x])$

***Problem* 1.5** We can assume that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A/I)$ for some k-algebra A and an ideal I of A. Then it is enough to show that the map $\mathrm{Hom}(A/I, R) \to \mathrm{Hom}(A, R)$, given by composing with the natural map $\pi : A \to A/I$, is injective. This follows from the surjectivity of π.

***Problem* 1.6** Let $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B), Z = \mathrm{Spec}(C)$. Then the statement is equivalent to

$$\mathrm{Hom}(A \otimes_B C, R) \simeq \mathrm{Hom}(A, R) \times_{\mathrm{Hom}(B,R)} \mathrm{Hom}(C, R).$$

We can define a bijection as follows. First, consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & A \otimes_B C \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\iota_C} \\ B & \xrightarrow{\gamma} & C \end{array}$$

Using the maps above, we define a map from LHS to RHS as $\phi \mapsto (\phi\iota_A, \phi\iota_C)$. Since $\iota_A\alpha = \iota_C\gamma$, we have $\phi\iota_A\alpha = \phi\iota_C\gamma$ and the map is well-defined. For the other direction, for given $(f, g) : A \times C \to R$ with $f\alpha = g\gamma$, universal property of the tensor product gives a unique map $\phi : A \otimes_B C \to R$ with $f = \phi\iota_A$ and $g = \phi\iota_C$. We can check that these maps are inverses for each other.

***Problem* 1.7** ***

***Problem* 1.8** *** We define an $\mathbb{R}$-algebra A as

$$A = \mathbb{R}[(x_{ij}, y_{ij})_{1 \le i,j \le n}]/I$$

where I is an ideal generated by elements of the form

$$\begin{aligned} &\left(\sum_{k=1}^n (x_{ik}^2 + y_{ik}^2)\right) - 1, \\ &\sum_{k=1}^n (x_{ik}x_{jk} - y_{ik}y_{jk}), \quad i \neq j \\ &\sum_{k=1}^n (x_{ij}y_{jk} + y_{ik}x_{jk}), \quad i \neq j \end{aligned}$$

for $1 \le i, j \le n$.

***Problem* 1.9** ***

***Problem* 1.10** ***

***Problem* 1.11** ***

***Problem* 1.12** ***

***Problem* 1.13** ***

***Problem* 1.14** ***

***Problem* 1.15** ***