Recent Progress on the Gan-Gross-Prasad Conjectures (after Jacquet–Rallis, Waldspurger, W. Zhang, etc.)

Progrès Récents sur les Conjectures de Gan-Gross-Prasad (d'après Jacquet–Rallis, Waldspurger, W. Zhang, etc.)

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Re-TEXed by Seewoo Lee*

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Last updated: September 1, 2023

Introduction

The Gan-Gross-Prasad [18] conjectures have two aspects: local and global. Locally, these relate to certain branching laws between representations of real or p-adic Lie groups while globally, they characterize the non-vanishing of certain explicit integrals of automorphic forms that are commonly called (automorphic) periods. What makes these predictions interesting is that they involve fine arithmetic invariants: local epsilon factors on the one hand and values of automorphic L-functions at their center of symmetry on the other. These conjectures, which relate to all the classical groups (hermitian or skew-hermitian unitary spaces, symplectic and special orthogonal; this last case had moreover been considered long before by Gross and Prasad [26, 27]), have known many recent advances. More precisely, the local conjecture is now demonstrated in almost all cases after the seminal work of Waldspurger [63, 64, 65, 66] and Mœglin-Waldspurger [46] followed by the author [6, 7, 9, 8], Gan-Ichino [20], Hiraku Atobe [4] and finally Hongyu He [30]. The global conjecture has been established for unitary

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groups of hermitian spaces under certain local restrictions in a breakthrough by Wei Zhang [77] following the work of Jacquet-Rallis [36] and Zhiwei Yun [75]. Similar results have been obtained for unitary groups of skew-hermitian spaces by Hang Xue [70] following Yifeng Liu [41]. There is also a refinement of the global conjecture, initially due to Ichino-Ikeda [34] in the case of orthogonal groups then extended to unitary and symplectic groups by Neal Harris [29] and Hang Xue [71, 73], under the form of an identity explicitly linking periods and central values of automorphic L-functions. This refinement is now also proven for unitary groups under certain local assumptions after [76], the author [10], and Hang Xue [71, 72].

In this text, we propose the precise statements of these conjectures and the recent results mentioned above as well as to give brief overviews of the proofs that it would be very difficult to fully describe here as the techniques used vary (relative trace formulae, theta correspondence, endoscopy theory...). Moreover, as we have already explained, these conjectures relate to all the types of classical groups each having its own specificities. For reasons of space, we will focus on the case of unitary groups for which the results obtained are the most exhaustive. Finally, we also refer to [17] for a very good introduction to this subject (dating from 2013, this article unfortunately does not mention the most recent advances).

The arithmetic applications of these conjectures will not be discussed here but let us cite recent works [28], [50] as examples of such applications.

We finish this introduction by giving two examples of previous results which are special cases of the Gan-Gross-Prasad conjectures.

Branching law from $\mathrm{U}(n+1)$ *to* $\mathrm{U}(n)$. We begin by giving a classical example of a branching law (due to H. Weyl [69]) constituting a particular case of local conjectures. For any integer $k \geq 1$, we denote

$$\mathrm{U}(k) := \{g \in \mathrm{GL}_k(\mathbb{C}) : {}^t\bar{g}g = \mathbf{I}_k\}$$

the real compact unitary group of rank k. Let $n \geq 1$ be an integerl. We have a natural embedding

$$\mathrm{U}(n) \hookrightarrow \mathrm{U}(n+1), g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

Let π be an irreducible complex representation of $\mathrm{U}(n+1)$. Such a representation is necessarily of finite dimension (because $\mathrm{U}(n+1)$ is compact) and we are interested in the restriction of π to $\mathrm{U}(n)$. The explicit description of this restriction, or rather of its decomposition into irreducible representations, what are the consitities is called a branching law. Obviously, any comprehensible answer to this problem requires knowing how to independently parameterize (or name) the irreducible representations (up to isomorphism) of $\mathrm{U}(n)$ and $\mathrm{U}(n+1)$. Such a parametrization is precisely provided by the Cartan–Weyl highest weight theory. In the cases that interest us this theory provides natural bijections

$$\begin{aligned}
\mathrm{Irr}(\mathrm{U}(n+1)) &\simeq \{\underline{\alpha} = (\alpha_1, \dots, \alpha_{n+1}) \in \mathbb{Z}^{n+1} : \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n+1}\} \\
\pi_{\underline{\alpha}} &\leftrightarrow \underline{\alpha} \\
\mathrm{Irr}(\mathrm{U}(n)) &\simeq \{\underline{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^{n} : \beta_1 \geq \beta_2 \geq \cdots \geq \beta_n\} \\
\sigma_{\underline{\beta}} &\leftrightarrow \underline{\beta}
\end{aligned}$$

where $\mathrm{Irr}(\mathrm{U}(n+1))$ and $\mathrm{Irr}(\mathrm{U}(n))$ are the set of isomorphism classes of irreducible complex representations of $\mathrm{U}(n+1)$ and $\mathrm{U}(n)$, respectively. Using these parametrizations, the solution to the initial problem is formulated as follows (see [24] Chap. 8 for example): for all $n+1$-tuple $\underline{\alpha} = (\alpha_1, \dots, \alpha_{n+1}) \in \mathbb{Z}^{n+1}$ with $\alpha_1 \geq \alpha_2 \geq \dots \alpha_{n+1}$, we have

$$\pi_{\underline{\alpha}} = \bigoplus_{\substack{\underline{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n \\ \alpha_1 \geq \beta_1 \geq \cdots \geq \alpha_n \geq \beta_n \geq \alpha_{n+1}}} \sigma_{\underline{\beta}}.$$

In other words, for any pair of irreducible representations $(\pi_{\underline{\alpha}}, \sigma_{\underline{\beta}}) \in \mathrm{Irr}(\mathrm{U}(n+1)) \times \mathrm{Irr}(\mathrm{U}(n))$ the space of intertwining maps

$$\mathrm{Hom}_{\mathrm{U}(n)}(\pi_{\underline{\alpha}}, \sigma_{\underline{\beta}})$$

has dimension at most 1 and is non-zero if and only if $\underline{\alpha}$ and $\underline{\beta}$ satisfy the branching condition $\alpha_1 \geq \beta_1 \geq \cdots \geq \beta_n \geq \alpha_{n+1}$. In this form the local Gan-Gross-Prasad conjecture generalizes to pairs of real unitary groups $\mathrm{U}(p,q) \subset \mathrm{U}(p+1,q)$ or p-adic $\mathrm{U}(W) \subset \mathrm{U}(V)$ or more generally. More precisely, we will see in the section 1.3 that for irreducible representations π and σ (in a sense to be specified) of $\mathrm{U}(p+1,q)$ and $\mathrm{U}(p,q)$ the intertwining space $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi, \sigma)$ is always of dimension at most one and the same is true if we consider p-adic unitary groups. The local Gan-Gross-Prasad conjecture then gives (in almost all cases) a necessary and sufficient condition, generalizing the above branching relation, for this space to be nonzero.

Waldspurger's formula for the Maass forms of level 1. Let us now state a particular case of a result of Waldspurger [61] whose global conjectures give a generalization. Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the Poincaré upper half plane and $f : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$ a Maass eigenform of level 1. Let's recall what this means: f is a C^∞ (and even real analytic) which is an eigenvector for the hyperbolic Laplacian $\Delta ::= -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ with an eigenvalue λ (i.e. $\Delta f = \lambda f$), invariant under the $\mathrm{SL}_2(\mathbb{Z})$-action (given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z := \frac{az+b}{cz+d}$), has a moderate growth in the sense that $|f(x+iy)| \ll Cy^N$ for some N as $y \to \infty$ and eigenform for all Hecke operators T_p for prime p, defined by

$$(T_p f)(z) = f\left(\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) z\right) + \sum_{u=0}^{p-1} f\left(\left(\begin{smallmatrix} 1 & u \\ & p \end{smallmatrix}\right) z\right).$$

Since $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right) z = z + 1$, such a function admits a Fourier expansion of the form

$$f(x+iy) = \sum_{n \in \mathbb{Z}} a_n(y) e^{2\pi i n x}, \quad x + iy \in \mathbb{H}.$$

Moreover, the differential equation satisfied by f as well as the moderate growth implies that the functions $a_n(y)$ are, for $n \neq 0$, of the form $a_n(y) = a_n \sqrt{y} K_\nu(2\pi|n|y)$ for $a_n \in \mathbb{C}$ and K_ν is the Bessel function of second kind with parameter $\nu \in \mathbb{C}$ satisfying $\lambda = \frac{1}{4} - \nu^2$. We assume that f is even (i.e. $f(-\bar{z}) = f(z)$) and cuspidal (i.e. $a_0(y) = 0$). We then have $a_{-n} = a_n$ for $n \neq 0$ and we define the complete L-function of f by

$$L(s, f) = \pi^{-s} \Gamma\left(\frac{s+\nu}{2}\right) \Gamma\left(\frac{s-\nu}{2}\right) \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \Re(s) \gg 1.$$

For a quadratic Dirichlet character χ with $\chi(-1) = -1$ we also define a completed L-function twisted by χ by the following way

$$L(s, f \times \chi) = \pi^{-s} \Gamma\left(\frac{s-1+\nu}{2}\right) \Gamma\left(\frac{s-1-\nu}{2}\right) \sum_{n=1}^{\infty} \frac{\chi(n) a_n}{n^s}, \quad \Re(s) \gg 1.$$

Then $L(s, f)$ and $L(s, f \times \chi)$ admit analytic continuations to $\mathbb{C}$ and satisfy the functional equations $L(1-s, f) = L(s, f)$ and $L(1-s, f \times \chi) = L(s, f \times \chi)$. Let F be an imaginary quadratic extension of $\mathbb{Q}$ with fundamental discriminant d (i.e. if $F = \mathbb{Q}(\sqrt{d_0})$ with d_0 a square-free integer then $d = d_0$ if d_0 is congruent to 1 modulo 4, $4d_0$ otherwise). We call Heegner point (relative to F) the unique root z_d in $\mathbb{H}$ of a quadratic equation of the form $aX^2 + bX + c$ with $a, b, c \in \mathbb{Z}$ satisfying $b^2 - 4ac = d$. We then have the following formula, which is a special case of a result of Waldspurger [61]

$$\left(\sum_{z_d / \mathrm{SL}_2(\mathbb{Z})} f(z_d)\right)^2 = \frac{\sqrt{|d|}}{2} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \times \chi_d\right), \tag{1}$$

where the sum is over the set of orbits of Heegner points under $\mathrm{SL}_2(\mathbb{Z})$-action and χ_d denotes the unique quadratic Dirichlet character of conductor $|d|$ with $\chi_d(-1) = -1$.

Applied to this particular case, the global Gan-Gross-Prasad conjecture predicts the equivalence

$$\sum_{z_d/\mathrm{SL}_2(\mathbb{Z})} f(z_d) \neq 0 \Leftrightarrow L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \times \chi\right) \neq 0,$$

while the refinement of the global conjecture by Ichino and Ikeda makes it possible to derive formula (1) directly.

1 The Local Conjectures

1.1 The groups

Let E/F be a quadratic extension of local fields of characteristic zero. We therefore have either $E/F = \mathbb{C}/\mathbb{R}$ or that E and F are finite extensions of the field of p-adic numbers $\mathbb{Q}_p$ for a certain prime number p ($\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ by the p-adic absolute value $|\cdot|_p$ defined by $|p^k \frac{a}{b}|_p = p^{-k}$ for a and b integers prime to p). We denote by σ the unique non-trivial element of the Galois group $\mathrm{Gal}(E/F)$ and $\mathrm{sgn}_{E/F}$ the quadratic character of F associated with the extension E/F by the class field theory (it is therefore the unique quadratic character with kernel $\mathrm{N}_{E/F}(E^\times)$, the image of the norm map). Finally, we will fix two non-trivial additive characters $\psi_0 : F \longrightarrow \mathbb{S}^1$ and $\psi : E \longrightarrow \mathbb{S}^1$ with the property that ψ is trivial on F.

Let V be a finite dimensional vector space of dimension n over E and $\varepsilon \in \{\pm 1\}$. We assume V is equipped with a non-degenerate ε-hermitian form

$$\langle -, - \rangle : V \times V \longrightarrow E.$$

By definition a ε-hermitian form satisfies

$$\langle \lambda v + \mu w, u \rangle = \lambda \langle v, u \rangle + \mu \langle w, u \rangle$$
$$\langle v, u \rangle = \varepsilon \langle u, v \rangle^\sigma$$

for all $u, v, w \in V$ and $\lambda, \mu \in E$. Depending on whether $\varepsilon = 1$ or -1 we call it hermitian or skew-hermitian. Let W be a non-detenerate subspace of V with

$$\dim(V) - \dim(W) = \begin{cases} 1 & \text{if } \varepsilon = 1 \\ 0 & \text{if } \varepsilon = -1. \end{cases}$$

Let $\mathrm{U}(V) \subset \mathrm{GL}(V)$ and $\mathrm{U}(W) \subset \mathrm{GL}(W)$ be the algebraic subgroups (defined over F) of linear automorphisms of V and W preserving the form $\langle -, - \rangle$. Then $\mathrm{U}(V)$

and $\mathrm{U}(W)$ are unitary groups and we have a natural embeddign $\mathrm{U}(W) \hookrightarrow \mathrm{U}(V)$ where $\mathrm{U}(W)$ acts trivially on $W^\perp$ (which of dimension at most 1). In the following we will (abusively) identify an algebraic group defined on F with the group of F-points corresponding to it.

The following discussion also extends to the case where $E = F \times F$ equipped with the involution $\sigma(x, y) = (y, x)$, a case which it will be necessary to include anyway when we will deal with the global conjecture. In such a situation, a non-degenerate form $\langle -, - \rangle$ as above identifies V and W to direct sums $V_0 \oplus V_0^\vee$ and $W_0 \oplus W_0^\vee$ where $W_0 \subset V_0$ are the finite dimensional vector spaces over F and $V_0^\vee$, $W_0^\vee$ denote their duals. We then have a natural identifications $\mathrm{U}(V) \simeq \mathrm{GL}(V_0)$ and $\mathrm{U}(W) \simeq \mathrm{GL}(W_0)$.

In all cases, we put $G = \mathrm{U}(W) \times \mathrm{U}(V)$, $H = \mathrm{U}(W)$ and we embed H into G diagonally. The groups H and G inherit from the field F topologies which make them Lie groups in the archimedean case (i.e. when $F = \mathbb{R}$) and locally profinite groups in the non-archimedean case (i.e. when F is a finite extension of $\mathbb{Q}_p$; recall that a topological group is locally profinite if it has a basis of neighborhoods of the identity element consist of compact subgroups).

1.2 The restriction problem

Let $(\pi, \mathcal{V})$ be a smooth and irreducible complex representation of G. In the p-adic case, this means that π is a representation of G on a $\mathbb{C}$-vector space $\mathcal{V}$ (typically of infinite dimension) all of whose vectors have a open stabilizer, irreducibility is then an algebraic notion (i.e. no non-trivial subspace stable under G). In the archimedean case, this means that $\mathcal{V}$ is a Fréchet space and that π is a smooth representation (in the C^∞ sense), admissible (i.e. the irreducible representations of a maximal compact subgroup appear with finite multiplicities) on $\mathcal{V}$ satisfying a certain condition of "moderate growth" (which was introduced by Casselman and Wallach, see [11] and [67] Chap. 11); irreducibility is then a topological notion (ie no non-trivial closed subspace stable by G). In any case, such an irreducible representation decomposes as a tensor product $\pi = \pi_W \boxtimes \pi_V$ where π_W and π_V are irreducible (smooth) representations of $\mathrm{U}(W)$ and $\mathrm{U}(V)$ respectively (and where the tensor product is a topological tensor product in the archimedean case). We will denote as $\mathrm{Irr}(G)$ for the set of isomorphism classes of smooth irreducible representations of G.

To define the restriction problem that will interest us, we must also introduce a certain "small" representation ν of H. In the hermitian case (i.e. if $\varepsilon = 1$), ν is the trivial representation that we will denote as 1 or simply $\mathbb{C}$ in the following. In the skew-hermitian case (i.e. if $\varepsilon = -1$), we have an inclusion

$$\mathrm{U}(W) \subset \mathrm{Sp}(\mathrm{Res}_{E/F} W)$$

where $\mathrm{Res}_{E/F} W$ denotes the restriction of the scalars from E to F of W equipped with the symplectic form $\mathrm{Tr}_{E/F} \circ \langle -, - \rangle$ and $\mathrm{Sp}(\mathrm{Res}_{E/F} W)$ denotes the corresponding symplectic group. Let $\mathrm{Mp}(\mathrm{Res}_{E/F} W)$ be the metaplectic group associated with this symplectic space (it is a $\mathbb{Z}/2\mathbb{Z}$-extension of $\mathrm{Sp}(\mathrm{Res}_{E/F} W)$). The metaplectic covering splits over $\mathrm{U}(W)$ but this splitting is not unique (because there are non-trivial characters $\mathrm{U}(W) \to \{\pm 1\}$). We can, however, fix such a splitting by choosing a character $\mu : E^\times \to \mathbb{S}^1$ with $\mu|_{F^\times} = \mathrm{sgn}_{E/F}$ from now on. Let $\omega_{\psi_0, W}$ be the Weil representation of $\mathrm{Mp}(\mathrm{Res}_{E/F} W)$ associated to the character ψ_0 (c.f. [43] Chap. 2. II). Then $\nu = \omega_{\psi_0, W, \mu}$ is the restriction of this Weil representation to $\mathrm{U}(W)$ via the splitting that we have just fixed.

For every case, the space of intertwining maps which is of our interest is the following

$$\mathrm{Hom}_H(\pi, \nu) \tag{2}$$

where implicitly we only consider the continuous maps in the archimedean case (for the underlying Fréchet topologies). We denote $m(\pi)$ for the dimension of this space

$$m(\pi) := \dim \mathrm{Hom}_H(\pi, \nu).$$

Note that in the hermitian case we have identifications

$$\mathrm{Hom}_H(\pi, \nu) = \mathrm{Hom}_{\mathrm{U}(W)}(\pi_W \boxtimes \pi_V, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(W)}(\pi_V, \pi_W^\vee)$$

where $\pi_W^\vee$ denotes the (smooth) contragredient representation of π_W.

An element of space (2) is called a Bessel functional if $\varepsilon = 1$ and a Fourier-Jacobi functional if $\varepsilon = -1$. We will then talk in parallel about the Bessel and Fourier-Jacobi cases of the conjecture.

1.3 Multiplicity 1

The following theorem is due to Aizenbud–Gourevitch–Rallis–Schiffmann [2] and Sun [57] in the p-adic case and to Sun–Zhu [58] in the archimedean case.

Theorem 1.1. For any smooth irreducible representations π of G we have

$$m(\pi) \leq 1.$$

The local Gan–Gross–Prasad conjecture then essentially provides an answer to the following simple question: when do we have $m(\pi) = 1$? Just as for the law of branching between real compact unitary groups discussed in the introduction, any comprehensible answer to this question requires knowing how to parameterize the (isomorphism classes of) irreducible representations of G. Such a parameterization is precisely the object of the local Langlands correspondence (for unitary groups) whose main properties we now recall.

1.4 Local Langlands correspondence for unitary groups

In this section we consider a hermitian or skew-hermitian space V of finite dimension n over E and we denote by $\mathrm{U}(V)$ the corresponding unitary group.

1.4.1 Weil-Deligne group

Let W_F be the Weil group of F. If F is non-archimedean, we have the following commutative diagram where each row are exact

$$\begin{tikzcd}
1 \arrow[r] & I_F \arrow[r] & \mathrm{Gal}(\overline{F}/F) \arrow[r] & \mathrm{Gal}(\overline{k_F}/k_F) \simeq \widehat{\mathbb{Z}} \arrow[r] & 1 \\
1 \arrow[r] & I_F \arrow[r] \arrow[u, equal] & W_F \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

where $\overline{F}$ is an algebraic closure of F, k_F is the residue field of F, the isomorphism $\mathrm{Gal}(\overline{k_F}/k_F) \simeq \widehat{\mathbb{Z}}$ correspond to the choice of the geometric Frobeinus Frob_F as a topological generator of $\mathrm{Gal}(\overline{k_F}/k_F)$ and I_F is the inertia subgroup (i.e. the kernel of the arrow $\mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(\overline{k_F}/k_F)$). We then equip W_F with the topology that mesk I_F as an open subgroup (the topology induced from that of $\mathrm{Gal}(\overline{F}/F)$). If F is archimedean, we have

$$W_F = \begin{cases} \mathbb{C}^\times \cup \mathbb{C}^\times j & \text{if } F = \mathbb{R} \\ \mathbb{C}^\times & \text{if } F = \mathbb{C}, \end{cases}$$

where $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for all $z \in \mathbb{C}^\times$. The Weil-Deligne group WD_F of F is defined by

$$\mathrm{WD}_F = \begin{cases} W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } F \text{ is non-archimedean} \\ W_F & \text{if } F \text{ is archimedean.} \end{cases}$$

1.4.2 Langlands parameters

Langlands associates with $\mathrm{U}(V)$, and more generally with any connected reductive group over F, an *L-group* ${}^L\mathrm{U}(V)$ that is a semi-direct product of a complex reductive group $\widehat{\mathrm{U}(V)}$ with the Weil group W_F: ${}^L\mathrm{U}(V) = \widehat{\mathrm{U}(V)} \rtimes W_F$. Here, the L-group is explicitly described as follows: we have $\widehat{\mathrm{U}(V)} = \mathrm{GL}_n(\mathbb{C})$ and the action of W_F factors through $W_F \longrightarrow W_F/W_E = \mathrm{Gal}(E/F)$ with σ acts as $\sigma(g) = J^t g^{-1} J^{-1}$, where

$$J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

A Langlands parameter for $\mathrm{U}(V)$ is then a $\widehat{\mathrm{U}(V)}$-conjugacy class of "admissible" homomorphisms (i.e. satisfying certain properties of continuity, semi-simplicity and algebraicity)

$$\phi : \mathrm{WD}_F \longrightarrow {}^L\mathrm{U}(V)$$

commuting with projections on W_F. We denote $\Phi(\mathrm{U}(V))$ the set of Langlands parameters for $\mathrm{U}(V)$. For the unitary groups we have the following more explicit description (c.f. [18] Theorem 8.1): the restriction to WD_E induces a bijection between $\Phi(\mathrm{U}(V))$ and the set of isomorphism classes of the complex continuous semi-simple and algebraic representations on $\mathrm{SL}_2(\mathbb{C})$ of dimension n of WD_E which are $(-1)^{n+1}$-conjugate dual. Let's recall what this last term means. Fix $c \in W_F \backslash W_E$ maps to σ. A representation $\varphi : \mathrm{WD}_E \longrightarrow \mathrm{GL}(M)$ is called *conjugate dual* if there exists a non-degenerate bilinear form

$$B : M \times M \longrightarrow \mathbb{C}$$

satisfying

$$B(\varphi(\tau)u, \varphi(c\tau c^{-1})v) = B(u, v), \quad \forall u, v \in M, \tau \in \mathrm{WD}_E.$$

It is equivalent to ask if M is isomorphic to $(M^c)^\vee$ where M^c is the c-conjugate of M and $(-)^\vee$ is the contragredient representation. We further say that $\varphi : \mathrm{WD}_E \longrightarrow \mathrm{GL}(M)$ is ε-conjugate-dual, where $\varepsilon \in \{\pm 1\}$, if we can choose a bilinear form satisfying the additional condition

$$B(u, \varphi(c^2)v) = \varepsilon B(v, u), \quad \forall u, v \in M.$$

We will call such a form an ε-conjugate-dual form.

To state the Langlands correspondence in its most complete version, it is necessary to introduce for all $\phi \in \Phi(\mathrm{U}(V))$ a certain finite group S_ϕ. The latter is defined as the group of connected components of the centralizer in $\widehat{\mathrm{U}(V)}$ of the image of ϕ. If we identify ϕ with a $(-1)^{n+1}$-conjugate-dual representation $\varphi : \mathrm{WD}_E \to \mathrm{GL}(M)$, we have the following more concrete description of S_ϕ. Let B be a conjugate-dual form of sign $(-1)^{n+1}$ as above and denote $\mathrm{Aut}(\varphi, B)$ the group of linear automorphisms of M commutes with the image of φ and preserve the form B. We then have (canonically)

$$S_\phi = \mathrm{Aut}(\varphi, B)/\mathrm{Aut}(\varphi, B)^\circ$$

where we denote as $\mathrm{Aut}(\varphi, B)^\circ$ for the connected component of the identity element. Moreover, this group is always abelian and isomorphic to a product of finitely many copies of $\mathbb{Z}/2\mathbb{Z}$.

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