

Sur un résultat de Waldspurger

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Last updated: September 29, 2024

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Note from the translator:

Most of the translations are done by Google Translator and ChatGPT, where I fixed some grammatical errors and made the translation more readable. Here are some notations that I used in this translation, which are different from the original paper or do not occur.

- For a number field F , we denote the ring of adèles over F as \mathbb{A}_F , instead of F_A as in the original paper.
- G would be the group of invertible 2 by 2 matrices (view as an algebraic group) and T be a maximal torus of G , chosen as a subgroup of diagonal matrices in G (the original paper uses A), and Z be the center of G . We will denote inner forms of G as G' and the corresponding maximal tori and centers as T' and Z' (the original paper uses T without $'$). Also, elements, functions or anything that are related to the inner form G' would be denoted with $'$ ($g' \in G', f' : G' \rightarrow \mathbb{C}, \dots$).
- For the above G and T , we will use notations $[G] = G(\mathbb{A}_F)/G(F)Z(\mathbb{A}_F)$ and $[T] = T(\mathbb{A}_F)/T(F)Z(\mathbb{A}_F)$, and similarly for G' and T' as $[G']$ and $[T']$.

Also, all the footnotes are added by myself. If you find any typos or errors, please contact seewoo5@berkeley.edu.

0 Introduction

0.1

We present a new proof of a remarkable result by Waldspurger [5, Theorem 2]. While Waldspurger's original proof relies on the properties of Weil's representation, our approach is based on a variant of the trace formula. We believe that this alternative perspective offers some interest.

We begin by recalling the statement of the result. Let F be a number field, and E a quadratic extension of F . Denote by η the character of the idèle class group of F associated with E . Consider the group $\mathrm{GL}(2)$ as an algebraic group G defined over F , and let Z be its center. Let T be a maximal torus in G , namely the group of diagonal matrices. Let π be an automorphic representation of $G(\mathbb{A}_F)$ that is trivial on the center $Z(\mathbb{A}_F)$.

We say that π satisfies Waldspurger's first condition (denoted **W1**) if there exist automorphic forms ϕ_1 and ϕ_2 in the space of π such that the following integrals are nonzero:

$$\int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} \phi_1(a) da, \quad \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} \phi_2(a) \eta(\det a) da. \quad (0.1.1)$$

Next, define the set $X(E : F)$, or simply X , as the set of isomorphism classes of pairs (G', T') , where G' is an inner form of G and T' is a maximal torus in G' isomorphic to E^\times over F . Such a pair arises from another pair (H, L) , where H is a quaternion algebra over F and L is a subfield of H isomorphic to E over F , by taking G' as the multiplicative group of H and T' as that of L . We denote the center of G' by Z' .

We identify the set X with a set of representatives for each isomorphism class in X . Let $X(\pi)$ be the set of triples (G', T', π') , where the pair (G', T') lies in X and π' is a cuspidal automorphic representation of $G'(\mathbb{A}_F)$ associated with π through the condition outlined in [1]. This condition can be stated as follows: there exists a finite set of places S of F such that, for $v \notin S$, the groups G_v and G'_v are isomorphic, and the representations π_v and π'_v are equivalent under this isomorphism.¹ We say that π satisfies Waldspurger's second condition (denoted **W2**) if there exists a triple (G', T', π') in $X(\pi)$ and an automorphic form ϕ' in the space of π' such that the following integral is nonzero:

$$\int_{T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)} \phi'(t) dt. \quad (0.1.2)$$

¹Jacquet–Langlands correspondence.

We now state the result:

Theorem 0.1.1 (Waldspurger). Conditions **W1** and **W2** are equivalent.

0.2

We now outline the main ideas behind our proof. First, we identify the set of double cosets $T \backslash G / T$ with the disjoint union of the double cosets $T' \backslash G / T'$ (§1). For this identification, we restrict to “regular” double cosets. This motivates the introduction of a compactly supported smooth function f on $G(\mathbb{A}_F) / Z(\mathbb{A}_F)$, and for each (G', T') , a corresponding compactly supported smooth function f' on $G'(\mathbb{A}_F) / Z'(\mathbb{A}_F)$. In fact, f' will be zero for almost all (G', T') . We associate a cuspidal kernel K_c to f , and similarly, a cuspidal kernel K'_c to each f' . The relevant conditions imposed on these functions are (§7 and §10):

$$\int_{[T]} \int_{[T]} K_c(a, b) \eta(\det b) db da = \sum_{(G', T')} \int_{[T']} \int_{[T']} K'_c(s, t) dt ds. \quad (0.2.1)$$

The relation between f and f' is as follows. These functions are products of local functions. If v is a place of F that splits in E , then for all (G', T') , the groups G'_v and G_v are isomorphic, and we define f_v and f'_v to be identical. In this case, the set $X(E_v : F_v)$ consists of two elements (G'_{v_i}, T'_{v_i}) , $i = 1, 2$, with G'_{v_1} being split. We can still identify the regular double cosets of T_v with the disjoint union of the regular double cosets of T'_1 and T'_2 . We show that for a given function f , there are functions f'_i on G'_i such that:

$$\int_{T_v / Z_v} \int_{T_v / Z_v} f(agb) \eta_v(\det b) db da = \int_{T'_{v_1} / Z'_{v_1}} \int_{T'_{v_1} / Z'_{v_1}} f'_1(sg't) dt ds,$$

whenever g corresponds to g' (§2 and §4). If v is unramified and f_v is a Hecke function², then we can set $f'_1 = f_v$ and $f'_2 = 0$ (§5). The condition simplifies to $f'_v = f'_i$ if $G'_v = G'_i$. Waldspurger’s result then follows easily from identity (0.2.1). Auxiliary results can be found in Section 6.

The proof of formula (0.2.1) relies on a generalization of the trace formula, which can be stated as follows. Let G be a semisimple group defined over F , and let A and B be subgroups of G defined over F . Let λ and μ be characters of $[A]$ and $[B]$, respectively.³ Let f be a compactly supported smooth function on $[G]$,

²The characteristic function of the spherical subgroup.

³i.e. the characters of $A(\mathbb{A}_F)$ and $B(\mathbb{A}_F)$ that are trivial on $A(F)$ and $B(F)$, respectively.

and consider the integral:

$$\int_{[A]} \int_{[B]} K_c(a, b) \lambda(a) \mu(b) db da$$

where K_c is the cuspidal kernel attached to f . This kernel has a complex expression, which involves the sum:

$$\sum_{\zeta \in G(F)} f(x^{-1} \zeta y).$$

Choose a system of representatives for the double cosets of the groups $A(F)$ and $B(F)$. For an element η of $G(F)$ let H_η the subgroup of $H = A \times B$ of tuples (α, β) such that $\alpha^{-1} \eta \beta = \eta$. Then any element of $G(F)$ can be uniquely written in the form:

$$\zeta = \alpha^{-1} \eta \beta, \quad \eta \in A(F) \backslash G(F) / B(F), \quad (\alpha, \beta) \in H_\eta(F) \backslash (A(F) \times B(F)).$$

By formal computation, we arrive at the following expression for the integral:

$$\sum_{\eta} \text{vol}(H_\eta(F) \backslash H(\mathbb{A}_F)) \int_{[A]} \int_{[B]} f(a^{-1} \eta b) \lambda(a) \mu(b) db da$$

where sum on the left hand side is over all η such that $\lambda(a) \mu(b) = 1$ if $a^{-1} \eta b = 1$. Note that we have ignored convergence issues and other terms in the expression of K_c .

0.3

I extend my sincere thanks to the Institute for Advanced Study and its permanent members for their hospitality, as the majority of this work was completed during my stay there during the special year 1983-1984 on L -functions. I am particularly grateful to Langlands for his interest in this work. Lastly, I owe a great debt of gratitude to Piatetski-Shapiro, whose deep understanding of Waldspurger's work proved invaluable. A conversation with him was, in fact, the starting point of this work.

1 Double cosets

1.1

In this paragraph F will be any field of characteristic zero and E be a quadratic extension of F . We will denote $N(E : F)$ or simply N the subgroup of norms of

E in the multiplicative group of F . The set $X(E : F)$ or simply X is defined as before. Consider one of its elements (G', T') . There exists a quaternion algebra H over F and a subfield L of H isomorphic to E such that G is the multiplicative group of H and T is that of L . We are going to parametrize the double cosets $T' \backslash G' / T'$. For this purpose let's choose an element ε of the normalizer $N(T')$ of T' which is not in T' . Then every h in H can be uniquely written as:

$$h = h_1 + \varepsilon h_2, \quad h_i \in L. \quad (1.1.1)$$

On the other hand, if $z \rightarrow \bar{z}$ denotes the unique non-trivial F -automorphism of L then:

$$\varepsilon z \varepsilon^{-1} = \bar{z}. \quad (1.1.2)$$

The square $c = \varepsilon^2$ is in Z' , or, in other words, in F . Moreover the class of c modulo N is determined by the isomorphism class of the pair (G', T') and, conversely, determines it.

Define two involutions j^+ and j^- of H by the following fomulae:

$$j^\pm(h) = \bar{h}_1 \pm \varepsilon h_2, \quad h = h_1 + \varepsilon h_2. \quad (1.1.3)$$

It is easy to verify that these are the only involutions of H which induce the unique non-trivial F -automorphism of L . For h in G we will set⁴

$$X'(h) = \frac{\frac{1}{2} \text{tr}(h j^+(h))}{\frac{1}{2} \text{tr}(h j^-(h))}. \quad (1.1.4)$$

As the denominator of this faction is nothing but the reduced norm of h , $X(h)$ is a well-defined element of F only depends on the double coset of h modulo T' . We also introduce the function $P'(h : T')$ or simply $P'(h)$ defined by

$$X'(h) = \frac{1 + P'(h)}{1 - P'(h)} \quad (1.1.5)$$

or

$$P'(h) = c h_2 \bar{h}_2 (h_1 \bar{h}_1)^{-1}, \quad c = \varepsilon^2. \quad (1.1.6)$$

Thus P' is a function **with values in the projective line** which is constant on the double cosets of T' in G' . Note that according to the previous formula, if $P'(h)$ is neither zero nor infinite, then it is an element of the class cN' determined by the tuple (G', T') . Moreover $P'(h)$ can't be 1, otherwise $X'(h)$ would be infinity. We will say that h (or its double coset) is T' -singular if $P'(h)$ is zero or infinity, T' -regular otherwise.

⁴tr is a trace map from H to F , given as $\text{tr}(h_1 + \varepsilon h_2) = h_1 + \bar{h}_1$.

Proposition 1.1. Two elements h and h' in G' are in the same double coset of T' if and only if $P'(h) = P'(h')$. Moreover, if x is in cN and not 1 then there exists a h in G' such that $P'(h) = x$.

The proof is left to the reader.

1.2

The following proposition justifies the use of the adjective T' -regular.

Proposition 1.2. Suppose h is T' -regular. The relations

$$sht = hz, \quad s \in T', \quad t \in T', \quad z \in Z'$$

imply

$$s \in Z', \quad t \in Z', \quad st = z.$$

The proof is left to the reader.

1.3

The above applies *mutatis mutandis* to a tuple of the form (G, T) where G is the group $GL(2)$ and T is a maximal split torus, say the group diagonal matrices in G . Then H is the algebra of 2 by 2 matrices, L the subalgebra of diagonal matrices and we can take⁵

$$\epsilon = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad c = 1.$$

The functions X and $P(\cdot : T)$ (or simply P) are defined as above. In particular:

$$P(h) = bc(ad)^{-1}, \quad h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

They are constant on the double cosets of T in G . Again P cannot be 1. We will also say that an element h of G is T -singular if $P(h)$ is zero or infinity, T -regular otherwise. There are now 6 T -singular double cosets: the cosets on which P takes the value zero:

$$T, \quad Tn_+T, \quad Tn_-T, \quad \text{where } n_+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad (1.3.1)$$

⁵In this case, the non-trivial automorphism on L is a map that swaps two diagonal elements, i.e. $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mapsto \begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix}$.

and the classes on which P takes the infinite value

$$\varepsilon T, \quad T \varepsilon n_+ T, \quad T \varepsilon n_- T. \quad (1.3.2)$$

Thus these classes cannot be distinguished from each other with P .

1.4

However, P distinguishes T -regular cosets:

Proposition 1.3. Let h and h' be T -regular element of G . Then h and h' are in the same coset if and only if $P(h) = P(h')$. If $x \in F$ is neither 1 nor 0, there exists an T -regular element h such that $P(h) = x$.

The proof is left to the reader.

2 Orbital integrals: compact torus

2.1

Let's keep the notations of section 1 but now assume that F is a local field. Then T'/Z' is compact. Choose a non-trivial additive character Ψ of F . Endow the additive group F with the self-dual measure dx for the character Ψ , the multiplicative group F^\times with the measure $L(1, 1_F)|x|^{-1}dx$ (Tamagawa measure relative to Ψ). Also, on the multiplicative group E^\times , we give a Tamagawa measure corresponds to the character $\Psi \circ \text{tr}$. Then these induce measures on T' and Z' . Equip T'/Z' with the quotient measure. Let f' be a compactly supported smooth function on G'/Z' . Let

$$H'(g' : f' : T') = \int_{T'/Z'} \int_{T'/Z'} f(sg't) ds dt. \quad (2.1.1)$$

It is clear that $H'(g' : f' : T')$ only depends on the double coset of g' modulo T' . Let x be an element of F^\times . Define $H'(x : f' : T')$ as $H'(g' : f' : T')$ if there's g' in G' such that $P'(g' : T') = x$, and 0 otherwise. Then we obtain a function $H'(f' : T')$ on F^\times and we are going to characterize the functions H' on F^\times which are of the form $H' = H'(f' : T')$ for an appropriate function f' .

2.2

Consider a function $H' = H'(f' : T')$. By definition H' vanishes, therefore is smooth, on the complement of cN . Consider a point x of the form $P'(h' : T')$.

As the norm is a submersive map from E^\times to F^\times , the map $g' \rightarrow P'(g' : T')$ is a fortiori submersive at the point h . It follows that H' is smooth at point x . Finally suppose that 1 is in cN (that is, the group G' splits); then we can assume that $c = 1$. We are going to show that H' is zero near 1.

Since f' is compactly supported modulo Z' , there exists a compact subset C of G' such that $H'(g' : f' : T') \neq 0$ implies $g' \in T'CT'$. Hence it suffice to show the existence of a number K such that $|P'(g' : T') - 1| > K$ for $g' \in T'CT'$. Suppose there is no such number. Then there would exist a sequence g'_i of elements in $T'CT'$ such that $P'(g'_i : T')$ tends to 1. By enlarging C and multiplying the elements of the sequence by elements of T' , we can assume that

$$g'_i = 1 + \varepsilon t'_i = c_i z'_i$$

with t'_i in T' , c_i in C and z'_i in Z' . So

$$P'(g'_i : T') = t'_i \bar{t}'_i = 1 + a_i$$

and a_i tends to zero. On the other hand, we have:

$$\det g'_i = -a_i = (z'_i)^2 \det c_i.$$

So z'_i tends to zero, and same for g'_i . Since the projection of g'_i onto L is 1, we get a contradiction.

2.3

Let's examine the behavior of the function H' near zero and near infinity. We will show that there is a neighborhood U of 0 in F and a smooth function A' on U such that

$$H'(x) = A'(x)(1 + \eta(cx)), \quad x \in U \tag{2.3.1}$$

$$2A'(0) = \text{vol}(T'/Z') \int_{T'/Z'} f'(t) dt. \tag{2.3.2}$$

Similarly we will show that there exists a neighborhood U of 0 in F a smooth function B' on U such that

$$H'(x) = B'(x^{-1})(1 + \eta(cx)), \quad x^{-1} \in U \tag{2.3.3}$$

$$2B'(0) = \text{vol}(T'/Z') \int_{T'/Z'} f'(\varepsilon t) dt. \tag{2.3.4}$$

Since $P'(\varepsilon g' : T') = P'(g' : T')^{-1}$ we have

$$\int_{T'/Z'} \int_{T'/Z'} f'(s\varepsilon g't) ds dt = H'(x^{-1} : f' : T')$$

or

$$H'(x^{-1} : f' : T') = H'(x : f'_0 : T'), \quad f'_0(g') = f'(\varepsilon g').$$

Hence it suffices to prove the assertions near zero. We will consider non-archimedean case first. Take a x in cN . Then $x = cl\bar{l}$ for some $l \in L$, hence $x = P(h)$ with $h = 1 + \varepsilon l$. Then we can write

$$H'(x : f' : T') = \int_{T'/Z'} \int_{T'/Z'} f(t_1(1 + \varepsilon l)t_2) dt_1 dt_2$$

or with a change of variable

$$H'(x : f' : T') = \int_{T'/Z'} \int_{T'/Z'} f\left(\left(1 + \varepsilon l \frac{\bar{t}_1}{t_1}\right)t_2\right) dt_1 dt_2. \quad (2.3.5)$$

Since f' is smooth there exists an ideal V of E such that for l in V we have

$$f'(g') = f((1 + \varepsilon l)g')$$

for all g' . Then there exists an ideal U in F such that $l\bar{l} \in U$ is equivalent to $l \in V$. For x in cU we therefore have $H'(x) = 0$ if x is not in cN ; if x is in cN then $x = cl\bar{l}$ with l in V and by (2.3.5)

$$H'(x) = \text{vol}(T'/Z') \int_{T'/Z'} f'(t) dt.$$

Our assertion follows immediately from this.

Let move on to the archimedean case, i.e. $F = \mathbb{R}$ and $L = \mathbb{C}$. Let $K(x) = H'(cx)$. Let V be a disk $\{z : z\bar{z} < a\}$ in L such that $1 + \varepsilon V$ is contained in G . Then the right hand side of (2.3.5) defines a smooth function on V , say $C(l)$, depending only on the norm of l . We have

$$K(x) = \begin{cases} 0 & x < 0 \\ C(l) & x > 0 \text{ and } x = l\bar{l} \text{ for } l \in V. \end{cases}$$

In particular, the restriction of C to the real axis is even and smooth and we have

$$K(x) = \begin{cases} 0 & x < 0 \\ C(y) & 0 < x < a \text{ and } x = y^2, y \in \mathbb{R}. \end{cases}$$

Our assertion follows from the existence of a smooth function D on F such that $D(x) = K(x)$ for $a > x > 0$, which is a consequence of Whitney's theorem.⁶

⁶Whitney's approximation theorem: smooth function on \mathbb{R} can be approximated by analytic functions.

2.4

The following properties characterizes $H'(f' : T')$:

Proposition 2.1. Let H' be a function on F^\times . There exists a compactly supported smooth function f' on G'/Z' with $H' = H'(f' : T')$ if and only if the following conditions hold:

- (1) H' vanishes on the complement of cN ,
- (2) H' vanishes on a neighborhood of 1,
- (3) There exists a smooth function A' on a neighborhood of 0 in F such that, for x near 0, we have

$$H'(x) = A'(x)(1 + \eta(cx)),$$

- (4) There exists a smooth function B' on a neighborhood of 0 in F such that, when $|x|$ is sufficiently large,

$$H'(x) = B'(x^{-1})(1 + \eta(cx)).$$

If f' , A' and B' satisfy these conditions then

$$2A'(0) = \text{vol}(T'/Z') \int_{T'/Z'} f'(t) dt, \quad 2B'(0) = \text{vol}(T'/Z') \int_{T'/Z'} f'(\varepsilon t) dt.$$

We have just shown that the conditions (1) to (4) are necessary. We will leave it to the reader to show that they are also sufficient. The last assertion of the proposition has been proved above.

3 Orbital integrals: split torus

3.1

In this section, F is a local field, E a quadratic extension, η the quadratic character of F^\times attached to E , G the group $\text{GL}(2)$ and T the subgroup of diagonal matrices. We give the Tamagawa measure on F^\times and product measure of itself on $(F^\times)^2$. This induces a measure on T and we give T/Z the quotient measure. For a compactly supported smooth function f on G/Z and T -regular element $g \in G$, we set

$$H(g : f : A) = H(g : f : 1) = \int_{T/Z} \int_{T/Z} f(agb) da db \quad (3.1.1)$$

$$H(g : f : \eta) = \int_{T/Z} \int_{T/Z} f(agb) \eta(\det b) da db. \quad (3.1.2)$$

The first integral depends only on $P(g : T)$ and we denote $H(x : f : T)$ or $H(x : f : 1)$ for its value at a point g such that $P(g : A) = x$. We also set $H(1 : f : T) = H(1 : f : 1) = 0$. For x in F neither 0 nor 1, define a matrix $g(x)$ by

$$g(x) = \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix}. \quad (3.1.3)$$

Then $P(g(x)) = x$ so $g(x)$ defines a section of the space of double cosets of A in G . We set $H(x : f : \eta) = H(g(x) : f : \eta)$ for $x \neq 0, 1$, and $H(x : f : \eta) = 0$ when $x = 1$. Let

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (3.1.4)$$

Let N_+ be the group of strict upper triangular matrices and N_- the group of strict lower triangular matrices. Then we have a covering of G by two open sets

$$G = TN_+N_- \cup TN_+wN_+. \quad (3.1.5)$$

We can therefore write f as a sum $f_1 + f_2$ where f_1 is supported on the first open set and f_2 on the second. Let

$$\phi(g) = \int_{T/Z} f(ag) da, \quad (3.1.6)$$

and define ϕ_1 and ϕ_2 in a similar way. Then ϕ is left invariant under T and compactly supported modulo T . The same holds for ϕ_1 and ϕ_2 . Moreover, the functions Φ_1 and Φ_2 defined by

$$\Phi_1(u, v) = \phi_1 \left(\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \right), \quad (3.1.7)$$

$$\Phi_2(u, v) = \phi_1 \left(\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} w \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \right) \quad (3.1.8)$$

are compactly supported on $F \times F$. Since

$$\begin{aligned} g(x) \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} a(1-x) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a^{-1}(1-x)^{-1}x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-x & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & a(1-x)^{-1} \\ 0 & 1 \end{bmatrix} w \begin{bmatrix} 1 & a^{-1} \\ 0 & 1 \end{bmatrix}, \end{aligned} \quad (3.1.9)$$

for $x \neq 0, 1$, we have

$$\begin{aligned} H(x : f : T) &= \int_{T/Z} \Phi_1(a^{-1}(1-x)^{-1}x, a) d^\times a \\ &\quad + \int_{T/Z} \Phi_1(a(1-x)^{-1}, a^{-1}) d^\times a. \end{aligned} \quad (3.1.10)$$

To prove the convergence of our integral, we can assume f positive. Then the RHS of (3.1.10) are compactly supported in F^\times and therefore the integral converges. Similarly, for x different from 0 and 1,

$$\begin{aligned} H(x : f : \eta) &= \int_{T/Z} \Phi_1(a^{-1}(1-x)^{-1}x, a) \eta(a) d^\times a \\ &\quad + \int_{T/Z} \Phi_2(a(1-x)^{-1}, a^{-1}) \eta(a) d^\times a. \end{aligned} \quad (3.1.11)$$

3.2

We are going to study the properties of the functions $H(f : \eta)$. The equation (3.1.11) already shows that $H(x : f : \eta)$ is a smooth function at any point x not equal to 0 or 1. On the other hand, if Φ_1 and Φ_2 are supported on the set of (x, y) such that $|x| < C$ and $|y| < C$, then in the second integral we have, on the support of Φ_2 , $|a(1-x)^{-1}| < C$ and $|a^{-1}| < C$ which gives $C^{-2} < |1-x|$ if the second integral is not zero. Similarly if the first integral is not zero we have $|(1-x)^{-1}x| < C^2$, which also implies $D < |1-x|$ for a suitable constant $D > 0$. It follows that $H(x : f : \eta)$ is zero in the neighborhood of 1. Therefore the equation (3.1.11) holds for all nonzero x .

Now, let's study $H(f : \eta)$ in the neighborhood of 0. In (3.1.11) the second integral is obviously a smooth function on x at 0. To study the first integral we will use the following lemma, the proof of which is left to the reader:

Lemma 3.1. Let Φ be a two variable Schwartz-Bruhat function on $F \times F$. There exist two Schwartz-Bruhat functions $A_1(x)$ and $A_2(x)$ on F such that for all $x \neq 0$ in F , we have

$$\int_{F^\times} \Phi(a^{-1}x, a) \eta(a) d^\times a = A_1(x) + A_2(x) \eta(x).$$

If Φ is real valued and compactly supported, then we can take A_1 and A_2 to be compactly supported.

Let's go back to the first integral of (3.1.11). With the notations used in the lemma, the integral is equal to

$$A_1(x(1-x)^{-1}) + A_2(x(1-x)^{-1})\eta(x(1-x)^{-1}). \quad (3.2.1)$$

If x is close enough to 0 then $1-x$ is a norm and $\eta(x(1-x)^{-1}) = \eta(x)$. On the other hand, $A_i(x(1-x)^{-1})$ are smooth functions of x in a neighborhood of 0 for $i = 1, 2$. As the second integral of (3.1.11) is obviously smooth at point 0 we conclude that, in a neighborhood of 0, $H(x : f : \eta)$ has the following form:

$$H(x : f : \eta) = A_1(x) + A_2(x)\eta(x) \quad (3.2.2)$$

where A_i are smooth for $i = 1, 2$.

To study $H(x : f : \eta)$ for large $|x|$ note that

$$\varepsilon g(x) = g(x^{-1}) \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \quad \text{if } \varepsilon = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then we get

$$H(x^{-1} : f : \eta) = H(x : f_0 : \eta)\eta(x), \quad f_0(g) = f(\varepsilon g). \quad (3.2.3)$$

Then there exists functions B_i , $i = 1, 2$, defined on the neighborhood of 0 and smooth, such that

$$H(x : f : \eta) = B_1(x^{-1}) + B_2(x^{-1})\eta(x) \quad (3.2.4)$$

for x with sufficiently large $|x|$.

3.3

In summary,

Proposition 3.1. Let H be a function on F^\times such that there exists a compactly supported smooth function f on G/Z with $H(x : f : \eta) = H(x)$. We have:

1. H is smooth on F^\times ,
2. H vanishes on a neighborhood of 1,
3. there exists a neighborhood U of 0 and smooth functions A_i on U for $i = 1, 2$ such that, for $x \in U$, we have

$$H(x) = A_1(x) + A_2(x)\eta(x),$$

4. there exists a neighborhood U of 0 and smooth functions B_i on U for $i = 1, 2$ such that, for x with sufficiently large $|x|$, we have

$$H(x) = B_1(x^{-1}) + B_2(x^{-1})\eta(x).$$

3.4

We are going to discuss the significance of the zero sets of the functions A_i and B_i of the proposition 3.1. For this purpose let us first recall that, if ϕ is a Schwartz-Bruhat function on F , then the integral

$$\int_{F^\times} \phi(x) |x|^s d^\times x,$$

or rather its analytic continuation, has a pole at the point $s = 0$; the residue at this point has the form $C\phi(0)$, where the constant C depends on the choice of the Haar measure on the group F^\times . On the other hand the integral

$$\int_{F^\times} \phi(x) |x|^s \eta(x) d^\times x$$

admits analytic continuation at $s = 0$ and its value at this point will again be denoted as an integral

$$\int_{F^\times} \phi(x) \eta(x) d^\times x.$$

Define the following quantities:

$$H(n_+ : f : \eta) = \int_{F^\times} \int_{F^\times} f \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) \eta(b) d^\times a d^\times b, \quad (3.4.1)$$

$$H(n_- : f : \eta) = \int_{F^\times} \int_{F^\times} f \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \right) \eta(b) d^\times a d^\times b, \quad (3.4.2)$$

$$H(\varepsilon n_+ : f : \eta) = \int_{F^\times} \int_{F^\times} f \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \varepsilon \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) \eta(b) d^\times a d^\times b, \quad (3.4.3)$$

$$H(\varepsilon n_- : f : \eta) = \int_{F^\times} \int_{F^\times} f \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \varepsilon \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \right) \eta(b) d^\times a d^\times b. \quad (3.4.4)$$

In general, these integrals are divergent, but they can be interpreted as above. For example, the first integral is the value at point 0 of the meromorphic function which, for $\Re s > 0$, is given by the convergent integral

$$\int_{F^\times} \int_{F^\times} f \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) \eta(b) |b|^s d^\times a d^\times b. \quad (3.4.5)$$

However, note that if f is supported in the open set $TN_+ w N_+$ then the integrals (3.4.1) and (3.4.2) are convergent. In fact the integral (3.4.1) is zero since TN_+ and $TN_+ w N_+$ are disjoint. **On the other hand the intersection of TN_- with**

a compact set contained in TN_+wN_+ is a disjoint compact of T . It follows that in (3.4.2) the integrand is compactly support in $F^\times \times F^\times$ and the integral clearly converges. Similarly the integrals (3.4.3) and (3.4.4) converge if f is supported in TN_+N_+ .

Proposition 3.2. With the notations in the proposition 3.1 we have:

$$H(n_+ : f : \eta) = A_2(0) \quad (3.4.6)$$

$$H(n_- : f : \eta) = A_1(0) \quad (3.4.7)$$

$$H(\varepsilon n_+ : f : \eta) = B_1(0) \quad (3.4.8)$$

$$H(\varepsilon n_- : f : \eta) = B_2(0). \quad (3.4.9)$$

Proof. We will prove (3.4.6) and (3.4.7). By the decomposition (3.1.5), it is sufficient to prove when f is supported on TN_+N_- or AN_+wN_+ . Then we have

$$H(x : f : \eta) = \int_{F^\times} \Phi(a(1-x)^{-1}, a^{-1}) \eta(a) d^\times a \quad (3.4.10)$$

where

$$\Phi(u, v) = \phi \left(\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} w \begin{bmatrix} 1 & a^{-1} \\ 0 & 1 \end{bmatrix} \right), \quad (3.4.11)$$

$$\phi(g) = \int_{T/Z} f(ag) da. \quad (3.4.12)$$

Since H is smooth at zero we have $A_2 = 0$ and $A_1 = H(f : \eta)$. Hence

$$A_1(0) = \int_{F^\times} \phi \left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} w \begin{bmatrix} 1 & a^{-1} \\ 0 & 1 \end{bmatrix} \right).$$

Using the fact that f is invariant under the center, we can rite the integral as follows:

$$A_1(0) = \int_{F^\times} \int_F f \left(b \begin{bmatrix} a^2 & 0 \\ a & 1 \end{bmatrix} \right) \eta(a) db d^\times a.$$

Change of variable shows that this is nothing but $H(n_- : f : \eta)$. (3.4.6) also holds as both A_2 and $H(n_+ : f : \eta)$ vanish.

Now assume that f is supported in TN_+N_- . Then we have

$$H(x : f : \eta) = \int_{F^\times} \Phi(a^{-1}(1-x)^{-1}x, a) \eta(a) d^\times a \quad (3.4.13)$$

where

$$\Phi(u, v) = \phi \left(\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \right), \quad (3.4.14)$$

$$\phi(g) = \int_{T/Z} f(ag) da. \quad (3.4.15)$$

Recall the definitions of A_1 and A_2 :

$$H(x) = A_1(x) + A_2(x)\eta(x), \quad (3.4.16)$$

on the other hand, lemma (3.1) implies

$$\int_{F^\times} \Phi(a^{-1}x, a)\eta(a)d^\times a = C_1(x) + C_2(x)\eta(x). \quad (3.4.17)$$

Comparing with (3.4.4) we get $C_i((1-x)^{-1}x) = A_i(x)$. Hence C_i and A_i vanish on the same set. By taking the Mellin transform of the above equation we get

$$\int_{F^\times} \int_{F^\times} \Phi(x, a)|x|^s \eta(a)|a|^s d^\times a d^\times x = \int_{F^\times} C_1(x)|x|^s d^\times x + \int_{F^\times} C_2(x)|x|^s \eta(x) d^\times x.$$

Comparing the residue of both sides at $s = 0$ we get

$$\int_{F^\times} \Phi(0, a)\eta(a)d^\times a = C_1(0).$$

By (3.4.14) and (3.4.15) the left hand side is nothing but $H(n_- : f : \eta)$. On the other hand the right hand side equals to $A_1(0)$, hence (3.4.7) is proven. (3.4.6) can be proved in a similar way. Equations (3.4.8) and (3.4.9) follow from the equations (3.4.6) and (3.4.7) applied to the function f_0 defined as $f_0(g) = f(\varepsilon g)$. \square

4 Matching functions

4.1

In this section, E is a local field, which is a quadratic extension of F , and η is a quadratic character attached to E . We consider again the pair (G, T) of the group $\mathrm{GL}(2)$ and the subgroup of diagonal matrices, and the set $X = X(E : F)$ that is reduced to two elements (G'_i, T'_i) , $i = 1, 2$, with G_1 split. Let f be a compactly supported smooth function on G/Z and $f'_i, i = 1, 2$ compactly supported smooth function on G'_i/Z'_i . We say that f and the pair (f'_1, f'_2) are matched if the following

condition is satisfied: For all $x \in F$ not equal to 1 or 0, choose i and $g' \in G'_i$ such that $x = P'(g' : T'_i)$ ($i = 1$ if x is a norm of E , $i = 2$ otherwise). Then

$$H(x : f : \eta) = H'(g' : f'_i : T'_i).$$

Proposition 4.1. For given f there exists a pair of functions (f'_1, f'_2) that matches f . Moreover, we have

$$\begin{aligned} \text{vol}(T'_i/Z'_i) \int_{T'_i} f'_i(t'_i) dt'_i &= H(n_+ : f : \eta) \pm H(n_- : f : \eta) \\ \text{vol}(T'_i/Z'_i) \int_{T'_i} f'_i(\varepsilon t'_i) dt'_i &= H(\varepsilon n_+ : f : \eta) \pm H(\varepsilon n_- : f : \eta) \end{aligned}$$

with sign $+$ for $i = 1$ and $-$ for $i = 2$.

Proof. This follows from the propositions 2.1, 3.1, and 3.2. □

4.2

When $F = \mathbb{R}$ we denote as K for the orthogonal subgroup in G . We denote by U the set of pairs (f'_1, f'_2) that match a compactly supported smooth function f on G/Z , that is K -finite if $F = \mathbb{R}$.

Let U_1 (resp. U_2) be a projection of U to the first (resp. second) component. Then the sets U_i is *dense* in the following sense.

Proposition 4.2. Let ϕ' be a continuous function on G'_i/Z'_i that is T'_i bi-invariant. If $\int_{G'_i/Z'_i} \phi(g') f'_i(g') dg' = 0$ for all $f'_i \in U_i$, then ϕ' is a zero function.

The proof will occupy the rest of this section.

4.3

Let f be a compactly supported smooth function on G/Z . Then

$$\int_{G/Z} f(g) dg = c = \int_{x \in F^\times} H(x : f : A) |1 - x|^{-2} dx. \quad (4.3.1)$$

Here c is a constant that does not depend on f . We also need an estimate on the functions $H(x : f : T)$ for compactly supported continuous functions f on G/Z .

Lemma 4.1. Let f be a compactly supported function on G/Z and $H(x) = H(x : f : T)$. Then H vanishes in a neighborhood of 1 and is $O(\log |x|)$ for small or large $|x|$.

Proof. The proof is similar to that given in (3.2) except that the lemma of (3.2) is replaced by the following assertion: if Φ is a two-variable Schwartz-Bruhat function, then there exist two single variable Schwartz-Bruhat functions A_i , $i = 1, 2$, such that

$$\int_{F^\times} \Phi(a^{-1}xa) d^\times a = A_1(x) + A_2(x) \log |x|.$$

We need integration formulas for groups G'_i analogous to (1)

$$\int_{G'_1/Z'_1} f'_1(g) dg = c_1 \int_0^\infty H'(x : f'_1 : T'_1) |1-x|^{-2} dx, \quad (4.3.2)$$

$$\int_{G'_2/Z'_2} f'_2(g) dg = c_2 \int_{-\infty}^0 H'(x : f'_2 : T'_2) |1-x|^{-2} dx, \quad (4.3.3)$$

where c_i is a constant and f'_i is a compactly supported continuous function on G'_i/Z'_i . \square

4.4

Proof of the proposition 4.2. Suppose $i = 1$. Let $H'(x) = H'(x : \phi' : T'_1)$. In particular $H'(x) = 0$ if x is not a norm. We will compute integrals up to multiplicative constants. Suppose f and (f'_1, f'_2) match with f' being K -finite if $F = \mathbb{R}$. By (4.3.2),

$$\int_{G'_1/Z'_1} \phi'(g'_1) f'_1(g'_1) dg'_1 = \int_0^\infty H'(x) H'(x : f'_1 : T'_1) |1-x|^{-2} dx.$$

Define ϕ_0 as a function on G by

$$\phi_0(g) = H'(x) \eta(\det b) \quad \text{if } g = ag(x)b.$$

By the properties of H' and the integration formula (4.3.1) ϕ_0 is locally integrable and

$$\int_{G/Z} \phi_0(g) f(g) dg = \int_{F^\times} H'(x) H(x : f : \eta) |1-x|^{-2} dx.$$

From $H(x : f : \eta) = H'(x : f'_1 : T'_1)$ if x is a norm we get

$$\int_{G/Z} \phi_0(g) f(g) dg = \int_{G'_1/Z'_1} \phi'(g'_1) f'_1(g'_1) dg'_1.$$

By assumption, the second integral vanishes. Therefore ϕ_0 is orthogonal to any smooth function (resp. any K -finite smooth function if F is real), and it follows that ϕ_0 is zero. It is therefore the same for H' : since ϕ' is T'_1 bi-invariant, it is completely determined by H' and we get $\phi' = 0$.

5 Orbital integrals: unramified case

5.1

In this section, F is a non-archimedean local field and E is an unramified quadratic extension of F . Assume that the residual characteristic of F is not 2 and the order of character ψ is 0.⁷ Consider a tuple (G, T) of a group $\mathrm{GL}(2)$ and a diagonal subgroup T . We denote by R the ring of integers of F , \mathfrak{p} a maximal ideal of R , ω a uniformizer and $K = \mathrm{GL}(2, R)$. The set $X = X(E : F)$ reduces to a set of two elements (G'_1, T'_1) and (G'_2, T'_2) . Now suppose $G'_1 = G$ and T'_1 is contained in the subgroup ZK . We will simply write T' for T'_1 . The measures of $T \cap K/Z \cap K$ and $T' \cap K/Z \cap K$ are therefore equal to 1. The goal of this section is to prove the following proposition.

Proposition 5.1. Let f be a K bi-invariant compactly supported function on G/Z . The f matches with the pair $(f, 0)$. Also,

$$H(n_+ : f : \eta) = H(n_- : f : \eta) = \frac{1}{2} \mathrm{vol}(T'/Z) \int_{T'/Z} f(t') dt'.$$

It will be convenient to consider functions with compact support on G rather than functions with compact support on G/Z . Of course the measures of the sets $T \cap K$, $T' \cap K$, and $Z \cap K$ are therefore equal to 1. If f is a K bi-invariant function with compact support on G , then we set

$$H'(g : f : T') = \int_{T'/Z} \int_{T'} f(s'gt') ds' dt'. \quad (5.1.1)$$

Since T' is contained in ZK this reduces to

$$H'(g : f : T') = \int_Z f(zg) dz. \quad (5.1.2)$$

Likewise, we define

$$H(g : f : \eta) = \int_{T/Z} \int_T f(agb) \eta(\det b) da db. \quad (5.1.3)$$

We write $H(x : f : \eta)$ again for $H(g(x) : f : \eta)$. Then we have to prove the following identities

$$H(x : f : \eta) = \int_Z f(zg) dz \quad \text{if } v(x) \text{ is even and } P(g : T') = x \quad (5.1.4)$$

⁷Order of a character ψ is a smallest integer $r \geq 0$ such that $\omega^r R \subseteq \ker \psi$.

$$H(x : f : \eta) = 0 \quad \text{if } v(x) \text{ is odd.} \quad (5.1.5)$$

By linearity we can assume that f is either the characteristic function f_0 of K , or the characteristic function f_m of the set

$$K \begin{bmatrix} \varpi^m & 0 \\ 0 & 1 \end{bmatrix} K, \quad m > 0. \quad (5.1.6)$$

Note that $f_m(g) \neq 0$ if and only if the following conditions hold:

- the entries of g are integers,
- $v(\det g) = m$,
- at least one of the entries of g is a unit.

Note that the condition trivially holds when $m = 0$.

5.2

We will first compute $H(x : f_m : \eta)$ which, for simplicity, denote by $H(x : m)$. Let's assume $m > 0$ first.

Proposition 5.2. Suppose $m > 0$. Then $H(x : m)$ is given by the following formulas:

1. if $v(x)$ is odd then $H(x : m) = 0$.
2. if $v(x)$ is even then $H(x : m) = 0$, unless $v(x) = 0$ and $v(1 - x) = m$ in which case $H(x : m) = 1$.

Proof. We will use the following lemma:

Lemma 5.1. Let

$$S = \sum_{i,j} (-1)^{i+j}$$

where the summation is over all the pairs of integers (i, j) on the edge of the rectangle defined by the inequalities

$$0 \leq i \leq P, \quad 0 \leq j \leq Q.$$

Then S is given by the following formulas:

1. if $PQ > 0$ then $S = 0$,

2. if $P = 0$ and $Q > 0$, then $S = 1$ if Q is even and $S = 0$ if Q is odd,
3. if $Q = 0$ and $P > 0$, the $S = 1$ if P is even and $S = 0$ if P is odd,
4. if $P = Q = 0$ then $S = 1$.

We not prove the proposition. We write $\text{Mat}[a, b, c, d]$ for the matrices with entries a, b, c, d . With this notation we have

$$H(x : m) = \sum_{i,j,k} f_m(\text{Mat}[\varpi^{i+k}, x\varpi^{j+k}, \varpi^i, \varpi^j])(-1)^{i+j}, \quad (5.2.1)$$

where the sum is over all triples of integers (i, j, k) . As the determinant of the matrices in (5.2.1) has valuations equal to $i + j + k + v(1 - x)$, from the condition $v(\det g) = m$ we can restrict ourselves to triples (i, j, k) with

$$i + j + k + v(1 - x) = m.$$

This allows us to eliminate k and, by the previous conditions on the integrality of entries of g ,

$$H(x : m) = \sum_{i,j} (-1)^{i+j} \quad (5.2.2)$$

where the summation is over all the pairs of integers (i, j) such that

$$0 \leq i \leq m - v(1 - x) + v(x) \quad (5.2.3)$$

$$0 \leq j \leq m - v(1 - x) \quad (5.2.4)$$

$$ij(m - v(1 - x) + v(x) - i)(m - v(1 - x) - j) = 0. \quad (5.2.5)$$

The sum is empty and $H(x : m)$ is zero unless

$$m - v(1 - x) \geq 0 \quad \text{and} \quad m - v(1 - x) + v(x) \geq 0. \quad (5.2.6)$$

Suppose (5.2.6) holds. Then we can apply the lemma, and we have $H(x : m) = 0$ unless

$$(m - v(1 - x))(m - v(1 - x) + v(x)) = 0. \quad (5.2.7)$$

Then the proposition follows from elementary calculations. \square

5.3

Let's compute $H(x : 0)$.

Proposition 5.3. $H(x : 0)$ is given by the following formulas:

1. if $v(x)$ is odd then $H(x : 0) = 0$,
2. if $v(x)$ is even then $H(x : 0) = 1$, unless $v(x) = 0$ and $v(1 - x) > 0$ in which case $H(x : 0) = 0$.

Proof. We have

$$H(x : 0) = \sum_{i,j,k} f_0(\text{Mat}[\omega^{i+k}, x\omega^{j+k}, \omega^i, \omega^j])(-1)^{i+j}, \quad (5.3.1)$$

where the sum is over all triples of integers (i, j, k) . As above, based on the conditions on g to be $f_0(g) \neq 0$, we can eliminate k and write

$$H(x : 0) = \sum_{i,j} (-1)^{i+j} \quad (5.3.2)$$

where the sum is over all pairs of integers (i, j) such that

$$0 \leq i \leq v(x) - v(1 - x) \quad (5.3.3)$$

$$0 \leq j \leq -v(1 - x). \quad (5.3.4)$$

Then the proposition follows from elementary calculations. \square

5.4

We will compute $\int_Z f_m(zg)dz$. It only depends on $x = P(g : T')$ and we denote $H'(x : m : T')$ for its value. Recall that by definition x is a norm, in other words the valuation of x is even. We will start with the case $m > 0$.

Proposition 5.4. Suppose $m > 0$. Then $H'(x : m : T') = 0$, unless $v(x) = 0$ and $v(1 - x) = m$ in which case $H'(x : m : T') = 1$.

Proof. We can assume that E is an extension generated by the square root of τ , where τ is a unit. Then we can take for T' the multiplicative group of the following algebra

$$L = \left\{ \begin{bmatrix} a & b \\ b\tau & a \end{bmatrix} \right\} \quad (5.4.1)$$

and ε is a matrix

$$\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (5.4.2)$$

Let's compute $H'(x : m : T')$ for $x = P'(g : T')$. We can assume that

$$g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} u & v \\ v\tau & u \end{bmatrix}. \quad (5.4.3)$$

so $\det g = 1 - x$ and $x = y^2 - v^2\tau$. We have

$$H'(x : m : T') = \sum_k f_m(\omega^k g) \quad (5.4.4)$$

and

$$\omega^k g = \begin{bmatrix} \omega^k(1+u) & \omega^k v \\ -\omega^k v\tau & \omega^k(1-u) \end{bmatrix}. \quad (5.4.5)$$

By the integrality condition of entries of g , this sum consists of a single term where k is determined by the equation

$$k = \frac{1}{2}(m - v(1 - x)). \quad (5.4.6)$$

In particular $H'(x : m : T') = 0$ or 1 . By the integrality conditions again and $\det g = m$, $H(x : m : T') = 1$ if and only if the followings hold:

- $m = v(1 - x) \pmod{2}$,
- the entries of the matrix in (5.4.5) with k given by (5.4.6) are integers,
- at least one of the entries of this matrix is a unit.

First assume $v(x) < 0$. We have $v(1 - x) = v(x)$. Since $x = u^2 - v^2\tau$ and τ is not a square $v(x)$ is even. Then $H'(x : m : T') = 0$ unless m is even. So let's assume that this is the case. We can write

$$u = u_0\omega^{v(x)/2}, \quad v = v_0\omega^{v(x)/2},$$

where u_0, v_0 are integrals, with at least one being a unit. Then the entries of the matrix (5.4.5) are:

$$\omega^{(m-v(x))/2}(1 + u_0\omega^{v(x)/2}), \quad \omega^{m/2}v_0$$

$$-\omega^{m/2}v_0\tau, \quad \omega^{(m-v(x))/2}(1-u_0\omega^{v(x)/2}).$$

All of these are in \mathfrak{p} so $H'(x : m : T') = 0$.

Suppose $v(x) > 0$. We have $v(1-x) = 0$. From $m \equiv v(1-x) \pmod{2}$ we have $H'(x : m : T') = 0$ unless m is even. Assume m is even. Since $k = m/2$, u and v are integral. The entries of the matrix in (5.4.5) are

$$\begin{aligned} &\omega^{m/2}(1+u), \quad \omega^{m/2}v, \\ &-\omega^{m/2}v\tau, \quad \omega^{m/2}(1-u). \end{aligned}$$

All of these are in \mathfrak{p} so $H'(x : m : T') = 0$.

Lastly, assume $v(x) = 0$. We have $v(1-x) \geq 0$. If $m - v(1-x)$ is odd, then $H'(x : m : T') = 0$. Let's assume that $m - v(1-x)$ is even. Then the entries of the matrix in (5.4.5) are

$$\begin{aligned} &\omega^{(m-v(1-x))/2}(1+u), \quad \omega^{(m-v(1-x))/2}v \\ &-\omega^{(m-v(1-x))/2}v\tau, \quad \omega^{(m-v(1-x))/2}(1-u). \end{aligned}$$

Since x is a unit, u and v are both integral and at least one is a unit. If $1+u$ and $1-u$ are both in \mathfrak{p} we would have $2 \in \mathfrak{p}$, a contradiction. So at least one of the numbers $1+u$ and $1-u$ is a unit. If $H(x : m : T')$ is not zero, integrality condition gives $m = v(1-x)$. The entries of the matrix (5.4.5) are therefore reduced to

$$1+u, \quad v, \quad -v\tau, \quad 1-u.$$

These are all integral and at least one of them is a unit. Hence $H'(x : m : T') = 1$.

So we have computed H' completely and the proposition follows. \square

5.5

Let's compute $H'(x : 0 : T')$. Recall that $v(x)$ is even.

Proposition 5.5. $H'(x : 0 : T') = 1$, unless $v(x) = 0$ and $v(1-x) > 0$ in which case $H'(x : 0 : T') = 0$.

Proof. As above we have

$$H'(x : m : T') = \sum_k f_0(\omega^k g). \quad (5.5.1)$$

The sum has at most one term whose index k is given by

$$k = -\frac{v(1-x)}{2}. \quad (5.5.2)$$

In particular, $H'(x : 0 : T') = 0$ or 1 . Moreover $H'(x : 0 : T') = 1$ if and only if $v(1 - x)$ is even and the matrix

$$\omega^k g = \begin{bmatrix} \omega^k(1 + u) & \omega^k v \\ -\omega^k v \tau & \omega^k(1 - u) \end{bmatrix} \quad (5.5.3)$$

with k given by (5.5.2) is in $\text{GL}(2, R)$.

Assume that $v(x) < 0$ and $v(1 - x)$ is even. Then $v(1 - x) = v(x)$, $v(x)$ is even and

$$u = u_0 \omega^{v(x)/2}, \quad v = v_0 \omega^{v(x)/2}$$

where u_0 and v_0 are integral, at least one being unit. Then the entries of the matrix (5.5.3) are the numbers

$$\omega^{-v(x)/2} + u_0, \quad v_0, \quad -v_0 \tau, \quad \omega^{-v(x)/2} - u_0.$$

They are integral. As the determinant of the matrix (5.5.3) is a unit according to the choice of k the matrix (5.5.3) is in $\text{GL}(2, R)$ and $H'(x : 0 : T') = 1$.

Suppose $v(x) \geq 0$ and $v(1 - x) = 0$ (of course $v(x) > 0$ leads to $v(1 - x) = 0$). Then $k = 0$ and the entries of the matrices (5.5.3) reduce to the numbers

$$\begin{aligned} &\omega^{-v(1-x)/2}(1 + u), \quad \omega^{-v(1-x)/2}v \\ &\omega^{-v(1-x)/2}v\tau, \quad \omega^{-v(1-x)/2}(1 - u). \end{aligned}$$

As $1 + u$ or $1 - u$ is a unit at least one of these numbers is not integral so (5.5.3) is not in $\text{GL}(2, R)$ and $H'(x : 0 : T') = 0$. \square

5.6

By comparing the propositions (5.2), (5.3), (5.4) and (5.5) we see that we have proved the identities (5.1.4) and (5.1.5). This therefore completes the proof of the first assertion of proposition (5.1). The second then follows from proposition (4.1).

5.7

To establish the convergence of the global orbital integrals we will need an additional result, the proof of which we will leave to the reader.

Lemma 5.2. Suppose h in KZ and let $x = P(h : T)$. Suppose $v(x) = 0$ and $v(1 - x) = 0$. Then the relation

$$ahb \in KZ, \quad a \in T, \quad b \in T$$

leads to

$$a \in Z(K \cap T), \quad b \in Z(K \cap T).$$

The lemma implies the following proposition.

Proposition 5.6. Let f be the characteristic function of KZ . Suppose E is an unramified quadratic extension and T' contained in KZ . Let h be an element of KZ and $x = P(h : T)$. If x and $1 - x$ are units then

$$H(h : f : T) = 1, \quad H(h : f : \eta) = 1.$$

6 Review on local representations of $GL(2)$

6.1

Let F be a local field and E a quadratic extension of F . We will consider again the set X which is reduced to two elements (G'_1, T'_1) and (G'_2, T'_2) , with say G'_1 splits. It will be convenient to use the following result:

Proposition 6.1. Let π' be an irreducible unitary representation of G'_i/Z'_i . Then the dimension of the space of continuous and T'_i -invariant linear functionals on the space of smooth vectors of π' is at most one. Moreover such a functional is given by the inner product with a smooth T'_i -invariant vector.

If F is non-archimedean then the assertion on the dimension is proven in [6], Proposition 9. It is well-known for $F = \mathbb{R}$. The rest of the proposition is obvious.

6.2

Likewise:

Proposition 6.2. Let π' be an infinite-dimensional irreducible unitary representation of G'_1/Z'_1 . Then the dimension of the space of T -invariant linear functionals (resp. invariant under the character $\eta \circ \det$) on the space of smooth functions of π' is one.

These are the propositions 9 and 10 of [4].

6.3

For $i = 1, 2$, consider irreducible unitary representations π'_i of G'_i/Z'_i . Assume that the tuple (π'_1, π'_2) satisfies the conditions of the theorem (15.1) of [1]; in particular π'_1 is a discrete series representation.

Proposition 6.3. The representations π'_i cannot both have a nonzero invariant vector under the group T'_i .

If F is non-archimedean then our assertion is found in Theorem 2 of [6]. It is well-known for $F = \mathbb{R}$.

7 Global orbital integrals: split torus

7.1

In the rest of this work F will be a number field and E a quadratic extension of F , η the quadratic character of the ideles of F attached to E . In this and the next section we will consider the pair (G, T) and a compactly supported smooth function f on $G(\mathbb{A}_F)/Z(\mathbb{A}_F)$. Denote K_c for the cuspidal kernel attached to f . Let ϕ_j be an orthonormal basis of the space of cusp forms of the group G/Z .

Then, by definition

$$K_c(x, y) = \sum_j \rho(f) \phi_j(x) \bar{\phi}_j(y) \quad (7.1.1)$$

and

$$\rho(f) \phi(x) = \int f(g) \phi(xg) dg. \quad (7.1.2)$$

In this and the following section we will give a nice expression of the integral

$$\int_{[T]} \int_{[T]} K_c(a, b) \eta(\det b) da db \quad (7.1.3)$$

We have chosen a nontrivial character ψ of \mathbb{A}_F/F . Then for each place v we have the Tamagawa measure attached to ψ_v induced on A_v and Z_v . We therefore have the product measure on $A(\mathbb{A}_F/F)$ and the quotient measure on $T(\mathbb{A}_F)/Z(\mathbb{A}_F)$. We will denote by S a finite set of places containing the infinite places, the ramified places in E and the places of residual characteristic 2. For each place v of F we will denote K_v for the usual maximal compact subgroup. In particular

$K_v = \text{GL}(2, R_v)$ if v is finite. We will take the function f product of local functions f_v which are K_v -finite at all places. We will assume that f_v is bi- K_v -invariant for all v not in S . We have a decomposition of K_c as a following sum:

$$K_c(x, y) = \sum_{\gamma \in G(F)/Z(F)} f(x^{-1}\gamma y) - K_{\text{sp}}(x, y) - K_{\text{ei}}(x, y), \quad (7.1.4)$$

where K_{sp} denotes the special kernel and K_{ei} the Eisenstein kernel (the definition will be recalled later). We can write the first term of this sum as the sum of two other terms K_r and K_s where

$$K_r(x, y) = \sum_{\gamma \in G(F)/Z(F)} f(x^{-1}\gamma y), \quad \gamma \text{ is } T\text{-regular} \quad (7.1.5)$$

$$K_s(x, y) = \sum_{\gamma \in G(F)/Z(F)} f(x^{-1}\gamma y), \quad \gamma \text{ is } T\text{-singular} \quad (7.1.6)$$

Then K_c can be written as

$$K_c = K_r + K_s - K_{\text{sp}} - K_{\text{et}}. \quad (7.1.7)$$

7.2

We first consider the integral of K_r . Any T -regular element γ of $G(F)/Z(F)$ can be uniquely written in the form

$$\gamma = \alpha g(\xi) \beta, \quad \alpha, \beta \in T(F)/Z(F) \text{ and } \xi \neq 0, 1 \quad (7.2.1)$$

(cf. (3.1.3) for the notation and §1) This implies

$$\int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} K_r(a, b) \eta(\det b) da db = \sum_{\xi \in F^\times - \{1\}} H(\xi : f : \eta), \quad (7.2.2)$$

where

$$H(\xi : f : \eta) = \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} f(ag(\xi)b) \eta(\det b) da db. \quad (7.2.3)$$

Let's justify our formal computations. Assume that the support of f has only a finite number of regular classes. The function X introduced in the section 1 (equation (1.1.4)) defines a continuous function of the group $G(\mathbb{A}_F)/Z(\mathbb{A}_F)$ over \mathbb{A}_F . Hence it only takes a finite number of values on the intersection of the support of f with the set of rational points: the same is therefore true for the

function $P(\cdot : A)$, which gives us our assertion. On the other hand, each of the integrals (7.2.3) converges absolutely: it suffices to prove it for the integral

$$H(\xi : f : T) = \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} f(ag(\xi)b)dad b. \quad (7.2.4)$$

Each of the local integrals $H(\xi : f_v : T_v)$ converges; almost all are equal to 1 (cf. (5.7)). Hence (7.2.4) converges. It is true for (3) and (3) is the product of the corresponding local integrals

$$H(\xi : f : \eta) = \prod_v H(\xi : f_v : \eta_v). \quad (7.2.5)$$

All but finitely many local integrals in the product are equal to 1 (cf. (5.7)).

7.3

Consider the integral of K_s . It is not absolutely convergent, but it is weakly convergent in the following sense. Let c be a number greater than 1. Define

$$\int_{c^{-1}}^c \int_{c^{-1}}^c K_s(a, b) \eta(\det b) dad b, \quad (7.3.1)$$

the integral of $K_s(a, b) \eta(\det b)$ over the set of pairs (a, b) satisfying $c^{-1} < |a_1/a_2| < c$, $c^{-1} < |b_1/b_2| < c$; where a_1 and a_2 are the diagonal entries a (and similar for b_1 and b_2). The integral exists since it is over a compact set. We will see that the integral (7.3.1) converges as c goes to infinity. Then we define the weak integral of $K_s(a, b) \eta(\det b)$ as the limit. We have seen in (1.3) that there are 6 singular double cosets in T , namely the double cosets of the following elements: $e, n_+, n_-, \varepsilon, \varepsilon n_+, \varepsilon n_-$. Let's number them from 1 to 6. Then we have a decomposition of K_s into 6 terms K_i , $1 \leq i \leq 6$, where K_i is the sum of the $f(x^{-1}\gamma y)$ for all γ in the i -th double coset. Let's study the integral of K_1 for example. We have

$$K_1(x, y) = \sum_{\alpha \in T(F)/Z(F)} f(x^{-1}\alpha y).$$

We have

$$\int_{c^{-1}}^c \int_{c^{-1}}^c K_1(a, b) \eta(\det b) dad b = \int_{c^{-1}}^c \int_{c^{-1}}^c f(ab) \eta(\det b) dad b,$$

in the left integral a and b vary in the compact subset of $[T]$ defined above; in the right integral b still varies in the compact subset of $[T]$ defined by $c^{-1} < |b_1/b_2| <$

c , but a varies in the subset of $T(\mathbb{A}_F)/Z(\mathbb{A}_F)$ defined by $c^{-1} < |a_1/a_2| < c$.⁸ Apply change of variable from a to ab^{-1} in the left integral. We get a double integral, with the inner integral only depending on $|b_1/b_2|$. This inner integral is written as⁹

$$\int_{c^{-1}}^c \eta(\det b) db.$$

It is 0 because the restriction of η to the group of idèles of absolute value 1 is nontrivial. The integral of K_1 is therefore weakly convergent and its value is 0. The same holds for the integral of K_4 .

Let's examie the integrals of the other terms, K_2 for example. We have

$$K_2(x, y) = \sum_{\alpha, \beta \in T(F)/Z(F)} f(x^{-1}\alpha n_+\beta y). \quad (7.3.2)$$

It follows that

$$\int_{c^{-1}}^c \int_{c^{-1}}^c K_2(a, b) \eta(a, b) da db = \int_{c^{-1}}^c \int_{c^{-1}}^c \sum_{\beta \in T(F)/Z(F)} f(an_+\beta b) \eta(\det b) da db;$$

in the right integral b still varies in the compact subset of $[T]$ defined by $c^{-1} < |b_1/b_2| < c$, but a varies the subset of $T(\mathbb{A}_F)/Z(\mathbb{A}_F)$ defined by $c^{-1} < |a_1/a_2| < c$. Let's introduce the function ϕ on $\mathbb{A}_F^\times \times \mathbb{A}_F$ defined by

$$\phi(x, y) = f\left(\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right). \quad (7.3.3)$$

It has a compact support. **Write our integral as**

$$\int_{\mathbb{A}_F^\times/F^\times} \sum_{\zeta \in F^\times} \int_{\mathbb{A}_F^\times} \phi(ab^{-1}, b\zeta) \eta(b) da db, \quad c^{-1} < |a| < c, \quad c^{-1} < |b| < c.$$

Using the Poisson summation formula with respect to the second variable and taking the Fourier transform with respect to the second variable we obtain for this integral the expression

$$\int_{\mathbb{A}_F^\times/F^\times} \sum_{\zeta \in F^\times} \int_{\mathbb{A}_F^\times} \phi(ab^{-1}, b\zeta) \eta(b) da db + \int_{\mathbb{A}_F^\times/F^\times} \sum_{\zeta \in F^\times} \int_{\mathbb{A}_F^\times} \hat{\phi}(ab, b\zeta) |b| \eta(b) da db$$

⁸The integral with respect to a over $[T] = T(\mathbb{A}_F)/T(F)Z(\mathbb{A}_F)$ and the summation over $T(F)$ are combined as an integral over $T(\mathbb{A}_F)/Z(\mathbb{A}_F)$.

⁹Integration over the elements $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in [T]$ satisfying $c^{-1} < |b_1/b_2| < c^2$.

with $c^{-1} < |a| < c$ and $1 < |b| < c$. It is obvious that the integrals extend to the domain

$$a \in \mathbb{A}_F^\times, \quad b \in \mathbb{A}_F^\times/F^\times, \quad 1 < |b|,$$

that converges absolutely. Moreover in the integrals with extended domains we can change a variable a into $ab^{\pm 1}$. We conclude that the integral of K_2 is weakly convergent and that its value is the following sum:

$$\int_{\mathbb{A}_F^\times/F^\times} \int_{\mathbb{A}_F^\times} \sum_{\zeta \in F^\times} \phi(a, b\zeta) \eta(b) da db + \int_{\mathbb{A}_F^\times/F^\times} \int_{\mathbb{A}_F^\times} \sum_{\zeta \in F^\times} \hat{\phi}(a, b\zeta) |b| \eta(b) da db,$$

with $1 < |b|$. The integral is nothing but the value of the analytical continuation of the following integral at $s = 0$:

$$\int_{\mathbb{A}_F^\times} \int_{\mathbb{A}_F^\times} \phi(a, b) |b|^s \eta(b) da db. \quad (7.3.4)$$

The value will be denoted as an integral

$$\int_{\mathbb{A}_F^\times} \int_{\mathbb{A}_F^\times} \phi(a, b) \eta(b) da db. \quad (7.3.5)$$

With this convention we can write that the weak integral of K_2 as

$$H(n_+ : f : \eta) = \int_{\mathbb{A}_F^\times} \int_{\mathbb{A}_F^\times} f \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) \eta(b) da db. \quad (7.3.6)$$

An analogous result is valid for the integrals of the other K_i . Finally we see that the weak integral of K_s exists and is equal to the sum

$$H(n_+ : f : \eta) + H(n_- : f : \eta) + H(n\varepsilon_+ : f : \eta) + H(\varepsilon n_- : f : \eta), \quad (7.3.7)$$

where the first term is defined by (7.3.6) and the others are defined similarly:

$$H(n_- : f : \eta) = \iint f \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \right) \eta(b) da db, \quad (7.3.8)$$

$$H(\varepsilon n_+ : f : \eta) = \iint f \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \varepsilon \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right) \eta(b) da db, \quad (7.3.9)$$

$$H(\varepsilon n_- : f : \eta) = \iint f \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \varepsilon \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \right) \eta(b) da db. \quad (7.3.10)$$

7.4

Let's move on to the integral of K_{sp} . Recall the definition of K_{sp} :

$$K_{\text{sp}}(x, y) = \text{vol}([G])^{-1} \sum_{\chi} \int f(g) \chi(\det g) dg \cdot \chi(\det x) \chi(\det y^{-1})$$

where the sum over all the quadratic characters χ of the group of idèle classes of F and $\text{vol}([G])$ is the volume of the quotient $[G]$. If χ is such a character then either χ or $\chi\eta$ has a non-trivial restriction to the groups of idèle classes of norm 1. Reasoning as for K_1 we immediately see that K_{sp} is weakly integrable and the integral vanishes.

8 Eisenstein kernel

8.1

We continue with the same notations in section 7. We will see that the integral

$$\int_{[T]} \int_{[T]} K_{\text{ei}}(a, b) \eta(\det b) da db \quad (8.1.1)$$

weakly converges. For the value of the integral, that is a classic application of the trace formula, we will only need a fairly small result. Choose a place u not in S that splits in E . Fix the components of f at the other places and view the integral (8.1.1) as a function of f_u . Denote \hat{f}_u for the Satake transform of f_u . We prove the following result:

Proposition 8.1. There exists an integrable function ϕ on \mathbb{R} and a constant c such that

$$\int_{[T]} \int_{[T]} K_{\text{ei}}(a, b) \eta(\det b) da db = \int_{-\infty}^{\infty} \phi(t) \hat{f}_u(q_u^{-it}) dt + c \hat{f}_u(q^{-1}). \quad (8.1.2)$$

8.2

We need standard results on the Mellin transform of an Eisenstein series. We fix a subgroup C of \mathbb{A}_F^\times isomorphic to the group of real numbers > 0 such that \mathbb{A}_F^\times is the product of C and \mathbb{A}_F^1 , the group of idèles of norm 1. The group C is equipped with the pullback measure of the measure $t^{-1}dt$ by the map $c \rightarrow |c|$ and \mathbb{A}_F^1 admits the quotient measure. We assume that all characters on the idèle

class group are trivial on C . Let χ be such a character and $V(\chi)$ the space of functions ϕ on K (the product of K_v) such that

$$\phi\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} k\right) = \chi(ab^{-1})\phi(k) \quad (8.2.1)$$

if

$$\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in K.$$

Now consider a function ϕ on $K \times \mathbb{C}$ such that for each $u \in \mathbb{C}$ the function $\phi(\cdot, u)$ is in $V(\chi)$. The function is assumed to be holomorphic, or at least meromorphic with respect to u ; for example it can be independent of u . We will extend ϕ into a function $\phi(g, u, \chi)$ on $G(\mathbb{A}_F)$ such that

$$\phi\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} g, u, \chi\right) = \chi(ab^{-1})|ab^{-1}|^{u+1/2}\phi(g, u, \chi). \quad (8.2.2)$$

Then the Eisenstein series is the analytic continuation of the series

$$E(g, \phi, u, \chi) = \sum_{\gamma \in G(F)/T(F)N_+(F)} \phi(\gamma g, u, \chi). \quad (8.2.3)$$

The series converges absolutely if $\Re u > 1/2$. The constant term of E along N_+ , the group of strictly upper triangular matrices, is by definition the integral

$$E_{N_+}(g, \phi, u, \chi) = \int_{N_+(\mathbb{A}_F)/N_+(F)} E(n g, \phi, u, \chi) dn. \quad (8.2.4)$$

It has a form of

$$E_{N_+}(g, \phi, u, \chi) = \phi(g, u, \chi) + M(u, \chi)\phi(g, -u, \chi^{-1}) \quad (8.2.5)$$

where $M(u, \chi)$ is the intertwining operator from $V(\chi)$ to $V(\chi^{-1})$. We also need another Fourier coefficient of E , namely

$$W(g, \phi, u, \chi) = \int_{\mathbb{A}_F/F} E\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g, \phi, u, \chi\right) \psi(-x) dx, \quad (8.2.6)$$

where ψ is a fixed character on the group \mathbb{A}_F/F . Then the Fourier series of E is written as

$$E(g, \phi, u, \chi) = \phi(g, u, \chi) + M(u, \chi)\phi(g, -u, \chi^{-1}) + \sum_{\alpha \in T(F)/Z(F)} W(\alpha g, \phi, u, \chi). \quad (8.2.7)$$

We can also consider a Fourier series for the group N_- of strictly lower triangular matrices. Since

$$N_- = wN_+w^{-1}, \quad w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad (8.2.8)$$

the Fourier series is written as

$$E(g, \phi, u, \chi) = \phi(wg, u, \chi) + M(u, \chi)\phi(wg, -u, \chi^{-1}) + \sum_{\alpha \in A(F)/Z(F)} W(\alpha wg, \phi, u, \chi). \quad (8.2.9)$$

The Mellin transform $L(s, \lambda : \phi : u, \chi)$ of E is defined by the following integral (or its analytic continuation)

$$L(s, \lambda : \phi : u, \chi) = \int_{\mathbb{A}_F^\times/F^\times} \left(E \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) - E_{N_+} \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) \right) |a|^{s-1/2} \lambda(a) da. \quad (8.2.10)$$

Let's ignore the variables of E for simplification. By replacing E with its Fourier series, we immediately obtain the following in the Mellin transform

$$\int_{\mathbb{A}_F^\times/F^\times} W \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) |a|^{s-1/2} \lambda(a) da. \quad (8.2.11)$$

We can also write the Mellin transform of E as follows:

$$L(s, \dots) = \int_1^\infty (E - E_{N_+}) + \int_0^1 (E - E_{N_-}) + \int_1^\infty E_{N_+} + \int_0^1 E_{N_-} \quad (8.2.12)$$

In each of these integrals, the function is evaluated at the point $\text{diag}(a, 1)$ and integrated against $|a|^{s-1/2} \lambda(a)$ on a subset of the idèle class group. For the first integral, for example, we integrate over the subset of a with $1 < |a|$. Using the Fourier series of E we easily obtain another expression for the Mellin transform:

$$\begin{aligned} & \int_1^\alpha W \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) |a|^{s-1/2} \lambda(a) da \\ & + \int_1^a W \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} w \right) |a|^{s-1/2} \lambda(a) da \\ & + \int_1^\alpha (|a|^{s+u} (\lambda\chi)(a) \phi(e) + |a|^{s-u} (\lambda\chi^{-1})(a) M(u, \chi) \phi(e)) da \\ & + \int_0^1 (|a|^{s-u-1} (\lambda\chi^{-1})(a) \phi(w) + |a|^{s+u-1} (\lambda\chi)(a) M(u, \chi) \phi(w)) da. \end{aligned} \quad (8.2.13)$$

The first two integrals converge for all s and the last two for $\Re s > 1/2$. The last two integrals can be easily computed. In particular, for $s = 1/2$ and $u \in i\mathbb{R}$, we obtain the following expression for the Melline transform of E at the point $s = 1/2$:

$$\begin{aligned}
L(1/2, \lambda : \phi : u, \chi) &= \int_1^\infty W \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) \lambda(a) da \\
&+ \int_1^\infty W \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} w \right) \lambda(a) da \\
&- \frac{1}{u + 1/2} (\phi(w) \delta(\lambda \chi^{-1}) + \phi(e) \delta(\lambda \chi)) \\
&+ \frac{1}{u - 1/2} (M(u, \chi) \phi(w) \delta(\lambda \chi) + M(u, \chi) \phi(e) \delta(\lambda \chi^{-1})),
\end{aligned} \tag{8.2.14}$$

where

$$\delta(\chi) = \int_{\mathbb{A}_F^1/F^\times} \chi(a) da$$

for a character χ on the idèle class group. We have to compute the difference between the Mellin transform and the following integral

$$\int_{c^{-1}}^c E \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) \lambda(a) da. \tag{8.2.15}$$

Recall that this notation means that the integral is taken over the compact subset of idèle classes a such that $c^{-1} < |a| < c$. Instead of (8.2.12) we have for the integral (8.2.15) the expression:

$$\int_1^c (E - E_{N_+}) + \int_{c^{-1}}^1 (E - E_{N_-}) + \int_1^c E_{N_+} + \int_{c^{-1}}^1 E_{N_-}. \tag{8.2.16}$$

Replacing E again by its Fourier series we obtain for (8.2.15) the expression:

$$\begin{aligned}
&\int_1^c W \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) \lambda(a) da \\
&+ \int_1^c W \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} w \right) \lambda(a) da \\
&+ \int_1^c (|a|^{1/2+u} (\lambda \chi)(a) \phi(e) + |a|^{1/2-u} (\lambda \chi^{-1})(a) M(u, \chi) \phi(e)) da \\
&+ \int_{c^{-1}}^1 (|a|^{-u-1/2} (\lambda \chi^{-1})(a) \phi(w) + |a|^{u-1/2} (\lambda \chi)(a) M(u, \chi) \phi(w)) da
\end{aligned} \tag{8.2.17}$$

Calculating the last two integrals and comparing to (8.2.14) we finally get the expression we had in mind:

$$\begin{aligned}
& \int_{c^{-1}}^c E \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) \lambda(a) da \\
&= L(1/2, \lambda : \phi : u, \chi) \\
&+ \frac{c^{u+1/2}}{u+1/2} \delta(\chi\lambda) \phi(e) + \frac{c^{-u+1/2}}{-u+1/2} \delta(\chi^{-1}\lambda) M(u, \chi) \phi(e) \\
&+ \frac{c^{u+1/2}}{u+1/2} \delta(\chi^{-1}\lambda) \phi(w) + \frac{c^{-u+1/2}}{-u+1/2} \delta(\chi\lambda) M(u, \chi) \phi(w) + R(c)
\end{aligned} \tag{8.2.18}$$

where $R(c)$ is defined as

$$-R(c) = \int_c^\infty W \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right) \lambda(a) da + \int_c^\infty W \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} w \right) \lambda(a) da. \tag{8.2.19}$$

It is clear that $R(c)$ tends to zero as c goes to infinity.

8.3

We need precise estimates for $R(c)$. Recall that R depends not only on c , but also on u , λ and ϕ . Our estimates will be a consequence of the following lemma:

Lemma 8.1. Assume that ϕ is independent of u . There exists a Schwartz-Bruhat function Φ such that, for $u \in i\mathbb{R}$, we have

$$\left| W \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \phi, u, \chi \right) \right| \leq \Phi(a) |a|^{-1/2} |L(2u+1, \chi^{2S})|^{-1}$$

Here $L(s, \chi^S)$ is defined as a product of local factors $L(s, \chi_v)$ for $v \notin S$. We also assume that ϕ is invariant under K_v for $v \notin S$.

Proof. There is a two-variable Schwartz-Bruhat function Φ such that

$$\phi(g, u, \chi) = \int \Phi((0, t)g) \chi^2(t) |t|^{2u+1} dt \times \chi(\det g) |\det g|^{u+1/2} \times L(2u+1, \chi^{2S})^{-1}.$$

A formal computation (see [1], chapter 3 for details) gives

$$W \left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \dots \right) = L(2u+1, \chi^{2S}) \chi(a) |a|^{u+1/2} \int \hat{\Phi}(at, t^{-1}) \chi^2(t) |t|^{2u+1} dt,$$

where $\hat{\Phi}$ is a Fourier transform with respect to the second variable. Hence it is suffice to prove the following assertion: given a Schwartz-Bruhat function $\Phi \geq 0$ with two variables, there exists a Schwartz-Bruhat function with one variable $\phi \geq 0$ such that for idèle a we have

$$\int \Phi(at, t^{-1}) dt \leq \phi(a) |a|^{-1}.$$

Consider the analogous local problem. To be precise, let us first consider the case where the local field F is non-archimedean and the function Φ is the characteristic function of the integers. Then the integral is 0 except for the set defined by the inequalities $|a| \leq |t| \leq 1$. The integral is therefore 0 unless a is integral. In this case the integral is $1 + v(a)$. Since $q \geq a$ this is smaller than $q^{v(a)}$. So our integral is at most $\phi(a) |a|^{-1}$, where ϕ is the characteristic function of the integers. Then the integral, considered as a function of a , has the form

$$\int \Phi(at, t^{-1}) dt = \phi_1(a) + \phi_2(a) \log |a|$$

for some Schwartz-Bruhat functions ϕ_i (cf. (4.3)). It is clear that the right hand side is bounded by $\phi(a) |a|^{-1}$, where ϕ is a suitable Schwartz-Bruhat function. By multiplying these local inequalities we easily obtain the required global inequality. \square

It is well known that the function $L(2u + 1, \chi^{2S})^{-1}$ has polynomial growth on the line $\Re(u) = 0$. On the other hand, if ϕ is a Schwartz-Bruhat function, there exists for all $N > 0$ a constant $C(N)$ such that

$$\int_c^\infty \phi(a) |a|^{-1} da \leq C(N) c^{-N}.$$

By comparing with definition (8.2.19) of R we immediately obtain the following:

Lemma 8.2. For all N there exist constants $C(N)$ and M such that for all imaginary u we have

$$|R(c, u)| \leq C(N) |c|^{-N} |u|^M.$$

In the same way using the expression for the Mellin transform and the fact that the operator $M(u, \chi)$ is unitary on the imaginary axis we obtain the following estimate:

Lemma 8.3. On the imaginary axis $M(u, \chi)\phi(k)$ and $L(1/2, \lambda : \phi : u, \chi)$ have polynomial growth.

8.4

Let's study the integral of the kernel K_{et} . Let's recall its definition. For any character χ choose an orthonormal basis ϕ_i of the Hilbert space $V(\chi)$; denote by $\rho(u, \chi)$ the representation of $G(\mathbb{A}_F)$ by right translations in the space of functions ϕ such that

$$\phi \left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} g \right) = \chi(ab^{-1})|ab^{-1}|^{u+1/2} \phi(g). \quad (8.4.1)$$

We can identify the space of $\rho(u, \chi)$ with $V(\chi)$ and set:

$$F(u, \chi : i, j) = (\rho(u, \chi) \phi_i, \phi_j). \quad (8.4.2)$$

We will write $E_{\text{ei}}(x, i, \dots)$ for $E_{\text{ei}}(x, \phi_i, \dots)$. With these notations

$$K_{\text{ei}}(x, y) = \sum_{\chi} K_{\chi}(x, y) \quad (8.4.3)$$

where, for each character of the idèle class group,

$$K_{\chi}(x, y) = \frac{1}{2i\pi} \sum_{i,j} \int_{-i\infty}^{i\infty} F(u, \chi : i, j) E(x, j, u, \chi) \overline{E(y, i, u, \chi)} du. \quad (8.4.4)$$

For a given f the sum (8.4.3) and (8.4.4) are finite. Define

$$I(c, \chi) = \int_{c^{-1}}^c \int_{c^{-1}}^c K_{\chi}(a, b) \eta(\det b) da db. \quad (8.4.5)$$

We can obviously change the order of integrations for u and the tuple (a, b) . Using (8.2.12) we obtain the following expression on $I(c, \chi)$:

$$\begin{aligned} & \frac{1}{i\pi} \sum_{i,j} \int_{-i\infty}^{i\infty} F(u, \chi : i, j) \\ & \times \left(L(1/2, 1 : j, u, \chi) + R(c, u) + \frac{c^{u+1/2}}{u+1/2} \delta(\chi) \phi_j(e) + \frac{c^{-u+1/2}}{-u+1/2} \delta(\chi^{-1}) M(u, \chi) \phi_j(c) \right. \\ & \left. + \frac{c^{u+1/2}}{u+1/2} \delta(\chi^{-1}) \phi_j(w) + \frac{c^{-u+1/2}}{-u+1/2} \delta(\chi) M(u, \chi) \phi_j(w) \right) \\ & \times \left(\overline{L(1/2, \eta, i, u, \chi)} + R'(c, u) + \frac{c^{-u+1/2}}{-u+1/2} \delta(\eta\chi) \overline{\phi_i}(e) + \frac{c^{u+1/2}}{u+1/2} \delta(\chi^{-1}\eta) M(u, \chi) \overline{\phi_i}(e) \right. \\ & \left. + \frac{c^{-u+1/2}}{-u+1/2} \delta(\chi^{-1}\eta) \overline{\phi_i}(w) + \frac{c^{u+1/2}}{u+1/2} \delta(\chi\eta) M(u, \chi) \overline{\phi_i}(w) \right) du \end{aligned} \quad (8.4.6)$$

For each (i, j) , the terms $R(c, u)$ and $R'(c, u)$ satisfy the conclusions of the Lemma 8.3. For each f , $F(u, \chi : i, j)$ is zero for all but finitely many (i, j) . In particular, $F(u, \chi : i, j)$ is zero unless ϕ_i and ϕ_j are both invariant under all K_v with v not in S . Moreover, on the imaginary axis, $F(u, \chi : i, j)$ rapidly decreases (faster than the inverse of a polynomial in u). On the other hand, according to (8.3), the terms $L(\dots)$ and the terms containing the powers of c moderately grows (at most polynomial in u). It follows that when we expand expression (8.4.6) we find a number of terms which tend to zero as c tends to infinity, and we can ignore these terms. The remaining terms become an integral independent of c :

$$\sum_{i,j} \int_{-i\infty}^{i\infty} F(u, \chi : i, j) L(1/2, 1 : j : u, \chi) \overline{L(1/2, \eta, i, u, \chi)} du. \quad (8.4.7)$$

The other terms does not vanish only if $\chi = 1$ or $\chi = \eta$. Each of these terms is of one of the following types:

$$\int F(u, 1 : i, j) \overline{L(1/2, \eta : i : u, 1)} \frac{c^{1/2+u}}{1/2+u} (\phi_j(e) + \phi_j(w)) du, \quad (8.4.8)$$

$$\int F(u, \eta : i, j) L(1/2, 1 : j : u, \eta) \frac{c^{1/2-u}}{1/2-u} (\overline{\phi_i}(e) + \overline{\phi_i}(w)) du, \quad (8.4.9)$$

$$\int F(u, 1 : i, j) \overline{L(1/2, \eta : i : u, 1)} \frac{c^{1/2-u}}{1/2-u} M(u, 1) (\phi_j(e) + \phi_j(w)) du, \quad (8.4.10)$$

$$\int F(u, \eta : i, j) L(1/2, 1 : j : u, \eta) \frac{c^{1/2+u}}{1/2+u} M(u, \eta) (\overline{\phi_i}(e) + \overline{\phi_i}(w)) du. \quad (8.4.11)$$

Integral (8.4.7) obviously has the properties required by Proposition 8.1. To prove the proposition, it suffices to show that each of the expressions (8.4.8) to (8.4.11) converges when c tends to infinity and that, moreover, the limit is zero if the Satake transform of the function f_u is zero at q^{-1} . This last condition implies that the integral of f_u over G_u/Z_u is zero and $F(u, 1 : u, j)$ and $F(u, \eta : i, j)$ cancel out at points $u = 1/2$ and $u = -1/2$.

8.5

Let's study (8.4.8). We will move the contour of integration from line $\Re u = 0$ to line $\Re u = -1/2$; but for the latter one, we will replace the segment joining the point $-1/2 - i\varepsilon$ and the point $-1/2 + i\varepsilon$ by the semi-circle centered at $-1/2$ and of radius ε which passes through the points $-1/2 - \varepsilon i$, $\varepsilon - 1/2$ and $-1/2 + i\varepsilon$. Let's prove that such a transformation of cantour is valid. The factor

$$F(u) = F(u, 1 : i, j) (\phi_j(e) + \phi_j(w))$$

and its derivatives, are holomorphic and rapidly decreasing on the vertical strip $-1/2 \leq \Re u \leq 0$. The exponential function remains bounded. The factor $(1/2 + u)^{-1}$ also remains bounded at infinity on this vertical strip. Now we study the Mellin transform. Recall that we have an integral representation of $\phi_i(g, u, 1)$:

$$\phi_i(g, u, 1) = \int \Phi((0, t)g) |t|^{2u+1} dt \times |\det g|^{u+1/2} L(2u+1, 1^S)^{-1}.$$

A formal computation gives the following expression of the Mellin transform (denoted as $L(u)$ in short):

$$L(u) = L(2u+1, 1^S)^{-1} \iint \hat{\Phi}(a, b) |a|^{1/2+u} \eta(a) |b|^{1/2-u} \eta(b) da db, \quad (8.5.1)$$

where $\hat{\Phi}$ is the Fourier transform of Φ with respect to the second variable. By taking the complex conjugation of the second variable we obtain

$$\overline{L(-\bar{u})} = L(-2u+1, 1^S)^{-1} T(u), \quad (8.5.2)$$

where

$$T(u) = \iint \Phi_1(a, b) |a|^{1/2-u} \eta(a) |b|^{1/2+u} \eta(b) da db. \quad (8.5.3)$$

Here Φ_1 is a Schwartz-Bruhat function; the *Tate's double integral* $T(u)$, as well as all its derivatives, is bounded on the vertical strip $-1/2 \leq \Re u \leq 0$. At last, on the vertical strip we have $1 \leq \Re(-2u+1) \leq 2$ and the function $L(-2u+1, 1^S)^{-1}$ is holomorphic and bounded by a polynomial in $\Im u$. We can write the integral as

$$\int F(u) L(1-2u, 1^S)^{-1} T(u) c^{1/2+u} (1/2+u)^{-1} du.$$

Hence our transformation of the contour of the integration is valid. By replacing u with $u - 1/2$, we obtain the following expression for (8.4.8):

$$\int F(u - 1/2) L(2-2u, 1^S)^{-1} T(u - 1/2) c^u u^{-1} du \quad (8.5.4)$$

In (8.5.4) the contour of the integration is the line $\Re u = 0$, except that the segment joining the point $-i\varepsilon$ to the point $i\varepsilon$ is replaced by the semicircle centered at 0 which goes through the points $-i\varepsilon, \varepsilon, i\varepsilon$. Now let $\varepsilon \rightarrow 0$. Then the integral over the semicircle tends to

$$i\pi F(-1/2) L(2, 1^S)^{-1} T(-1/2)$$

while the integral on the linear part of the contour tends to the Cauchy's principal value. In terms of the real variable t the integral (8.5.4) is also equal to

$$\int_{-\infty}^{\infty} F(it - 1/2)L(2 - 2it, 1^S)^{-1}T(it - 1/2)c^{it}t^{-1}dt + i\pi F(-1/2)L(2, 1^S)^{-1}T(-1/2). \quad (8.5.5)$$

For real r the function $L(-2it + 2, 1^S)$ is given by an absolutely and uniformly convergent infinite product (or a Dirichlet series). Its derivatives are therefore bounded and its inverse is also bounded. The derivatives of the factor $L(-2it + 2, 1^S)^{-1}$ are therefore bounded. In (8.5.5) the product of the first three terms is therefore a Schwartz function of t . When c tends to infinity the Cauchy integral tends to $i\pi$ times the value of the Schwartz function at point 0. In total we see that (8.5.5), i.e. the term (8.4.8), converges as $c \rightarrow \infty$, to

$$2i\pi F(-1/2)L(2, 1^S)^{-1}T(-1/2),$$

where the limit vanishes if and only if $F(-1/2, 1 : i, j)$ does. This is what we had to prove. Similar argument holds for (8.4.9).

8.6

Let's move on to (8.4.10). We'll simplify the notation as

$$F(u) = F(u, 1 : i, j).$$

We are going to use a slightly different expression from the one we have used so far for the Mellin transform.

Write ϕ for ϕ_j and suppose that ϕ is a product of local functions ϕ_v . We can also assume that, for each place v , ϕ_v is either K_v invariant, or has a vanishing integral over K_v . Let S_0 denote the set of places where this last condition is satisfied. Then S_0 is finite and contains S . We can find an integral representation for $\phi(g, u, 1)$ of the form

$$\phi(g, u, 1) = \int \Phi((0, t)g)|t|^{2u+1}dt \times |\det g|^{u+1/2}L(2u + 1, 1^{S_0})^{-1} \quad (8.6.1)$$

We can conclude that, as before, the Mellin transform in (8.4.10) can be written as

$$L(-2u + 1, 1^{S_0})T(u) \quad (8.6.2)$$

where $T(u)$ is defined as a Tate's double integral, holomorphic in u . On the other hand we can write the intertwining operator $M(u, 1)$ as a product

$$M(u, 1) = L(2u, 1)L(2u + 1, 1)^{-1}N(u, 1) \quad (8.6.3)$$

where N is the normalized intertwining operator. Now the quotient of $L(2u, 1)$ by $L(-2u + 1, 1)$ is an exponential function ab^u . It follows that the product of factors (8.6.2) and (8.6.3) reduces to

$$ab^u L(-2u + 1, 1_{S_0})L(2u + 1, 1_{S_0})^{-1}L(2u + 1, 1^{S_0})^{-1}T(u)N(u, 1). \quad (8.6.4)$$

Then (8.4.10) is given by the following integral

$$\int F(u)L(2u + 1, 1^{S_0})^{-1}T(u)c^{1/2-u}(1/2 - u)^{-1}A(u)du, \quad (8.6.5)$$

with

$$A(u) = ab^u L(-2u + 1, 1_{S_0})L(2u + 1, 1_{S_0})^{-1}N(u, 1)(\phi(e) + \phi(w)).$$

We are going to move the contour of integration. The present contour is the line $\Re u = 0$. The new contour will be the line $\Re u = 1/2$, except that the segment joining the points $1/2 - i\varepsilon$ and $1/2 + i\varepsilon$ will be replaced by the semicircle passing through the points $1/2 - i\varepsilon, 1/2 - \varepsilon, 1/2 + i\varepsilon$. Remaining part of the proof will then be the same as in the previous case, except that we have to show that the factor $A(u)$ is holomorphic and has a moderate growth on the strip $0 \leq \Re u \leq 1/2$. The ratio of the factors L which appears in A is the product of the ratios

$$L(-2u + 1, 1_v)L(2u + 1, 1_v)^{-1}$$

for all v in S_0 . If v is finite, then the ratio is a rational function in q_v^{-u} and has a moderate growth. If v is infinite, then the Stirling's formula implies that the ratio has a moderate growth. Recall that ϕ_v equals to 1 on K_v for all v not in S_0 . For such v , we have $N(u, 1_v)\phi_v(k_v) = 1$ for all u . So $N(u, 1)\phi(e)$ is in fact the product over all v in S_0 of

$$N(u, 1_v)\phi_v(e).$$

If v is finite this is still has a moderate growth. If v is infinite, this is a polynomial in u , and so A has a moderate growth. At last, let's prove that A is holomorphic at the poles of the factor $L(-2u + 1, 1_{S_0})$ on the strip. Let's prove, for example, holomorphy at $1/2$ of

$$L(-2u + 1, 1_{S_0})L(2u + 1, 1_{S_0})^{-1}N(u, 1)\phi(e).$$

The previous product can be written as

$$\prod_{v \in S_0} L(-2u + 1, 1_v) L(2u + 1, 1_v)^{-1} N(u, 1_v) \phi_v(e).$$

Take a v in S_0 . As the integral of φ_v over K_v is zero, $N(u, 1_v) \phi_v(e)$ vanishes at the point $u = 1/2$ and this zero cancel out the pole of the factor $L(-2u + 1, 1_v)$ at the same point. The product is therefore holomorphic at the point $1/2$ and this concludes our proof for the term (8.4.10). A similar argument applies to the term (8.4.11). Hence we complete the proof of the assertions in (8.1).

9 Global orbital integrals: compact torus

9.1

In this section F is again a number field and E a quadratic extension of F . We will fix an element (G', T') of the set $X(E : F)$ and an element ε of $N(T') - T'$. Then the square c of ε is an element of F^\times and the class cN of the group of norms N of E determines the isomorphism class of (G', T') . Let f' be a compactly supported smooth function on the group $G'(\mathbb{A}_F)/Z'(\mathbb{A}_F)$. We have a cuspidal kernel K'_c attached to f' . Let ϕ'_i be an orthonormal basis of the space of automorphic forms which are cuspidal and orthogonal to the functions $sg \mapsto \chi(\det g)$, where χ is a character of idèle class group whose square is trivial. By definition:

$$K'_c(x, y) = \sum_j \rho(f') \phi'_j(x) \overline{\phi'_j}(y). \quad (9.1.1)$$

We will give a useful expression of the integral

$$\int_{[T']} \int_{[T']} K'_c(s, t) ds dt. \quad (9.1.2)$$

Of course $\psi \circ \text{tr}$ is a character of \mathbb{A}_E/E and we therefore have for each place v of E the Tamagawa measure on the group E_v^\times and an induced measure on T'_v . We also have the product measure on the group $T'(\mathbb{A}_E)$ and the quotient measure on $T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)$. We will denote by S a finite set of places of F containing the places at infinity, the ramified places in E , the places where G' does not split, the places where ψ'_v is not of order 0 and places of residual characteristic 2. We choose for all v a maximal compact subgroup K'_v of G'_v such that T'_v is contained in $K'_v Z'_v$ if v does not split in E and $G'(\mathbb{A}_F)$ be the restricted product of G'_v with respect to K'_v . We suppose that f' is the product of compactly supported smooth

local functions f'_v on G'_v/Z'_v . We assume f'_v bi- K'_v -invariant for each v not in S . For v that does not split in E we replace f_v for (9.1.1) as f'_v defined by

$$f'_{v0}(g') = \frac{1}{\text{vol}(T_v)} \int_{T'_v} \int_{T'_v} f'_v(s_v g' t_v) ds_v dt_v.$$

We can therefore assume that each v which does not split in E the function f'_v is bi- T'_v -invariant, in particular bi- K'_v -finite. At last we assume that f'_v is bi- K'_v -finite at v 's split in E . Then we have:

$$K'_c(x, y) = \sum_{\gamma \in G'(F)/Z'(F)} f'(x^{-1}\gamma y) - K'_{\text{sp}}(x, y) - K'_{\text{ei}}(x, y). \quad (9.1.3)$$

where K'_{sp} denote the special kernel and K'_{ei} is the Eisenstein kernel. The Eisenstein kernel is zero if G' does not split. The kernel K'_{sp} is defined by the following sum

$$K'_{\text{sp}}(x, y) = \sum_{\chi} \frac{1}{\text{vol}([G'])} \int f'(\det g') dg' \cdot \chi(\det x) \chi^{-1}(\det y) \quad (9.1.4)$$

where the sum is over all the quadratic characters χ of the idèle class group of F . We define two other kernels

$$K'_r(x, y) = \sum_{\gamma \text{ is } T'\text{-regular}} f(x^{-1}\gamma y) \quad (9.1.5)$$

$$K'_s(x, y) = \sum_{\gamma \text{ is } T'\text{-singular}} f(x^{-1}\gamma y), \quad (9.1.6)$$

then K'_c can be written as a sum

$$K'_c = K'_r + K'_s - K'_{\text{sp}} - K'_{\text{ei}}. \quad (9.1.7)$$

Since $[T']$ is compact, (9.1.2) is simply an integral of each term in (9.1.7).

9.2

Let's consider K'_r first. Each T' -regular element $\gamma \in G'(F)/Z'(F)$ can be uniquely written as a form

$$\gamma = \sigma^{-1} \mu \tau, \quad (9.2.1)$$

where σ and τ are in $T'(F)/Z'(F)$ and μ is a representative of a T' -regular double coset of $T'(F)$ in $G'(F)$ (Proposition (1.2)). Then we get¹⁰

$$\iint K'_r(s, t) ds dt = \sum_{\mu} \iint f'(s^{-1} \mu t) ds dt, \quad (9.2.2)$$

¹⁰Of course, the summation on the RHS is over $T'(F)$ -regular double coset representatives of $T'(F)$ in $G'(F)$.

where the integral on the RHS is over $T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)$. The double integral of the right-hand side only depends on $\zeta = P'(\mu : T')$ and we will note $H'(\zeta : f' : T')$ its value. Then we can write as

$$\iint K'_r(s, t) ds dt = \sum_{\zeta \in cN - \{1\}} H(\zeta : f' : T'), \quad (9.2.3)$$

since the function P' parametrizes the regular double cosets and its values, on the regular elements, are all the points of the class cN associated with the pair (G', T') minus identity (Proposition (1.1)). Of course the orbital integral $H'(\zeta : f' : T')$ is the product of the local orbital integrals:

$$H'(\zeta : f' : T') = \prod_v H'(\zeta : f'_v : T'_v). \quad (9.2.4)$$

Almost all the factors are equal to 1. Indeed when v be a place of F which is not in S ; suppose that f'_v is the characteristic function of $Z'_v K'_v$. If v does not split in E , then T'_v is contained in $Z'_v K'_v$ and the integral is 1. If v splits in E then the local integral is still 1 by Proposition (5.7).

9.3

Let's consider the integral of the term K'_s . There are only two singular double cosets, $T'(F)$ and $\varepsilon T'(F)$. Then we get

$$\iint K'_s(s, t) = \text{vol}([T']) \int_{T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)} f'(t) dt + \text{vol}([T']) \int_{T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)} f'(\varepsilon t) dt. \quad (9.3.1)$$

9.4

Consider K'_{sp} . By (9.1.4), we have

$$\iint K'_{\text{sp}}(s, t) ds dt = \sum_{\chi} \frac{1}{\text{vol}([G'])} \int f'(g) \chi(\det g) dg \int \chi(\det s) ds \int \chi^{-1}(\det t) dt, \quad (9.4.1)$$

each of the integral of characters over $[T']$ is 0 unless $s \mapsto \chi(\det s)$ is trivial over $T'(\mathbb{A}_F)$; this is the case if and only if $\chi = 1$ or $\chi = \eta$. The integral of K_{sp} therefore reduces to two terms:

$$\iint K'_{\text{sp}}(s, t) ds dt = \frac{\text{vol}([T'])^2}{\text{vol}([G'])} \left(\int f'(g) \chi(\det g) dg + \int f'(g) dg \right) \quad (9.4.2)$$

In particular, let's choose as in (8.1) a place z of F not in S , fix the components of f at places other than z and look at the integral as a function of \hat{f}'_z . Then **the integral (5)** is of the form $c \hat{f}'_z(q_z^{-1})$ for a constant c .

9.5

Consider K_{ei} . It vanishes if G' does not split over F . Suppose G' splits and recall the notations in (8.4). we have:

$$K'_{\text{ei}}(x, y) = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} A(x, y, u) \quad (9.5.1)$$

where

$$A(x, y, u) = \sum_{\chi, j} (\rho(f')E)(x, j, u, \chi) \overline{E(y, j, u, \chi)}. \quad (9.5.2)$$

Note that the maximal compact subgroup implicit in the definition of Eisenstein series is now the product of the groups K'_v , with $K'_v Z'_v = T'_v$ if v is infinite. In particular the series (9.5.2) is finite. As we integrate over a compact set we get:

$$\int_{[T']} \int_{[T']} K'_{\text{ei}}(s, t) ds dt = \frac{1}{i\pi} \int_{-i\infty}^{i\infty} A(u : f') du, \quad (9.5.3)$$

where

$$A(u : f') = \sum_{\chi, j} \int_{[T']} (\rho(f')E)(s, j, u, \chi) ds \int_{[T']} \overline{E(t, j, u, \chi)} dt. \quad (9.5.4)$$

Now $(\rho(f')E)(x, j, u, \chi)$ is zero unless ϕ'_j is K'_v -invariant for all places v not in S . In particular, let's choose as above a place z of F which is not in S and splits in E . Then $f' = f^{z'} f'_z$ where $f^{z'}$ is the product of f'_v for $v \neq z$ and

$$(\rho(f')E)(x, j, u, \chi) = \hat{f}'_z(q_z^{-2iu}) (\rho(f^{z'})E)(x, j, u, \chi). \quad (9.5.5)$$

Hence we get

$$\int_{[T']} \int_{[T']} K'_{\text{ei}}(s, t) ds dt = \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \hat{f}'_z(q_z^{-2iu}) A(u : f^{z'}) du, \quad (9.5.6)$$

where $A(u : f^{z'})$ is integrable, a result which will be sufficient for our purpose.

10 Fundamental lemma

10.1

In this section, we again assume that F is a number field, E is a quadratic extension of G , and η is a quadratic character attached to E . We consider the pair (G, T) of a group $\text{GL}(2)$ and a subgroup T of diagonal matrices; fix an element

ε in the normalizer of T which is not in T . We denote by K_v the usual maximal compact subgroup of G_v and we assume ε contained in K_v for all v . We define a finite set S of places of F , containing the infinite places, the places which ramify in E , the places where ψ has no order 0 and the places of residual characteristic 2. It will be convenient to assume that S has an even number of elements. Let $X(S)$ be the set of pairs (G', T') in $X(E : F)$ such that G' splits outside S . For each (G', T') in $X(S)$ and each place v , we choose a maximal compact subgroup K'_v of G'_v so that $G'(\mathbb{A}_F)$ is the restricted product of G'_v with respect to K'_v . We assume that if v does not split in E then T'_v is contained in $K'_v Z'_v$. For all v not in S the measures of $T'_v \cap K'_v / K'_v \cap Z'_v$ and $T_v \cap K_v / K_v \cap Z_v$ are 1. We fix an element in the normalizer of T' which is not in T' , and assume that ε' is in K'_v for all v not in S . Let f be a compactly supported smooth function on $G(\mathbb{A}_F)/Z(\mathbb{A}_F)$ and, for each (G', T') in $X(S)$, f' be a compactly supported smooth function on $G'(\mathbb{A}_F)/Z'(\mathbb{A}_F)$. Of course, these functions are assumed to be products of local functions. We also make the following assumptions:

1. Let $v \in S$ be a place that does not split in E . Then f'_v is T'_v -bi-invariant. Moreover if x is an element of F_v that is not 1 or zero, $(G', T') \in X(S)$ and $g' \in G'_v$ with $x = P'(g' : T'_v)$ then

$$H(x : f_v : \eta_v) = H(g' : f'_v : T'_v).$$

2. Let $v \in S$ be a place that splits in E . Then f_v is K_v -finite and f'_v is K'_v -finite. Let g be a A_v -regular element in G_v . If $(G', T') \in X(S)$ and $g' \in G_v$ satisfy

$$P(g : T_v) = P(g' : T'_v)$$

then

(a)

$$H(g : f_v : T_v) = H(g' : f'_v : T'_v)$$

(b)

$$\int_{T_v} f_v(a_v) da_v = \int_{T'_v} f'_v(t'_v) dt'_v$$

(c)

$$\int_{T_v} f_v(\varepsilon a_v) da_v = \int_{T'_v} f'_v(\varepsilon' t'_v) dt'_v.$$

3. If v is not in S then f_v is K_v -bi-invariant, f'_v is K'_v -bi-invariant and the isomorphism between (G_v, K_v) and (G'_v, K'_v) induces a map from f_v to f'_v .

4. *Remark.* In the situation of assumption (2) there is an isomorphism between (G_v, T_v) and (G'_v, T'_v) . Condition (2) is satisfied if we take for f'_v the image of f_v under the isomorphism. Indeed this is clear for (2.a) and (2.b). For (2.c), the integral of the right hand side does not change if we replace ε' by the image of ε under the isomorphism in question and then our assertion is obvious.

For given function f , we have a cuspidal kernel K_c for the group G associated to it. Similarly, for each (G', T') , there is a cuspidal kernel K'_c for the group G' attached to the function g' . In this section we will prove the following result:

Theorem 10.1.1. With the previous assumptions, we have

$$\iint K_c(a, b) \eta(\det b) da db = \sum_{(G', T') \in X(S)} \iint K'_c(s, t) ds dt. \quad (10.1.1)$$

10.2

To prove the identity, as in the section 7 and 9, we write

$$K_c = K_r + K_s - K_{sp} - K_{ei}, \quad (10.2.1)$$

$$K'_c = K'_r + K'_s - K'_{sp} - K'_{ei}. \quad (10.2.2)$$

We will first prove the following identities:

$$\iint K_r(a, b) \eta(\det b) da db = \sum_{(G', T') \in X(S)} \iint K'_r(s, t) ds dt, \quad (10.2.3)$$

$$\iint K_s(a, b) \eta(\det b) da db = \sum_{(G', T') \in X(S)} \iint K'_s(s, t) ds dt. \quad (10.2.4)$$

We first assume these identities and will show how the theorem follows from. Consider the difference

$$\iint K_c(a, b) \eta(\det b) da db - \sum_{(G', T') \in X(S)} \iint K'_c(s, t) ds dt. \quad (10.2.5)$$

Considering (10.2.3) and (10.2.4), we write

$$\begin{aligned} & - \iint K_{sp}(a, b) \eta(\det b) da db + \sum_{(G', T') \in X(S)} \iint K'_{sp}(s, t) ds dt \\ & - \iint K_{ei}(a, b) \eta(\det b) da db + \sum_{(G', T') \in X(S)} \iint K'_{ei}(s, t) ds dt. \end{aligned}$$

Recall that these are weak integrals for the group G .

Now choose a place z of E which is not in S and splits in E . Fix local factors of f and f' at the other places. At the place z the Satake transforms of f_z and f'_z are the same. Hence we can regard our integrals as functions of \hat{f}_z . Then by (8.1), (9.3) and (9.4) the above sum has the form of

$$\int_{-\infty}^{\infty} \phi(t) \hat{f}_z(q_z^{-2it}) dt + c \hat{f}_z(q_z^{-1}), \quad (10.2.6)$$

where ϕ is integrable. We finish the proof as in [2] by using the fact that the integrals of K_c and K'_c also have the form

$$\sum_t a_t \hat{f}_z(t), \quad (10.2.7)$$

where the complex numbers t are either on the unit circle or on the real axis between q_z^{-1} and q_z and the series $\sum_t a_t$ absolutely converges. The uniqueness of the decomposition of a measure into an atomic measure and a continuous measure implies that the difference (10.2.5) is zero.

10.3

Let's prove (10.2.3). LHS can be written as

$$\sum_{\zeta} H(\zeta : f : \eta), \quad \zeta \neq 0, 1,$$

where RHS can be written as a double sum

$$\sum_{(G', T')} \sum_{\zeta} H'(\zeta : f' : T')$$

the inner sum is over all $1 \neq \zeta \in cN$, determined by the pair (G', T') . We can combine the two sums and write RHS as a sum

$$\sum_{\zeta} H(\zeta : f' : T'), \quad \zeta \in N(S) - \{1\},$$

where $N(S)$ is the union of the classes cN corresponds to the elements of $X(S)$. According to the class field theory the elements of $F^\times - N(S)$ are exactly the ζ in F^\times which satisfy the following condition: there exists a place v of F , which is not in S , **inert** in E and not a norm of the quadratic extension E_v of F_v . By Proposition (5.1) we have, for such a ζ , $H(\zeta : f_v : \eta_v) = 0$ if v is the place in question. This results in $H(\zeta : f : \eta) = 0$. Therefore it is sufficient to show the

equality of the orbital integrals $H(\zeta : f : \eta)$ and $H(\zeta : f' : T')$ when ζ is in $N(S)$. Decompose these integrals into products of local integrals $H(\zeta : f_v : \eta_v)$ and $H(\zeta : f'_v : T'_v)$ respectively. For v in S the equality of these integrals results from hypotheses (1) and (2). For v not in S the equality follows from hypothesis (3) and proposition (5.1). This proves the equality of the global orbital integrals, and the formula (3).

10.4

Let's prove (10.2.4). We can use (7.3.7) to compute LHS and (9.3.1) to compute RHS. The equality (10.2.4) will then be a consequence of the following two identities

$$H(n_+ : f : \eta) + H(n_- : f : \eta) = \sum_{(G', T') \in X(S)} \text{vol}([T']) \int_{[T']} f'(t') dt', \quad (10.4.1)$$

$$H(\varepsilon n_+ : f : \eta) + H(\varepsilon n_- : f : \eta) = \sum_{(G', T') \in X(S)} \text{vol}([T']) \int_{[T']} f'(\varepsilon' t') dt'. \quad (10.4.2)$$

The second identity results from the first identity applied to the function f_1 defined by $f'_1(g) = f'(\varepsilon' g)$. It is indeed easy to verify that the conditions (10.1.1) to (10.1.3) are satisfied by f_1 and f'_1 . Therefore let's prove the first identity.

Let's compute (10.4.1). Let a and b be idèles of E and F . The analytic continuation of the Tate integral is

$$\int \phi(t) |t|^s \eta(t) dt \quad (10.4.3)$$

where ϕ is a Schwartz-Bruhat function, whose value at $s = 0$ is

$$L(0, \eta) \prod_{v \in W} \int_{T_v} \phi_v(t_v) \eta(t_v) dt_v L(0, \eta_v)^{-1} \prod_{v \in V} \phi_v(0) |a_v|^{1/2},$$

where W is a set of places of F that do not split in E and V is a set of places split in E .

Apply this formula to the functions ϕ_+ and ϕ_- defined as

$$\begin{aligned} \phi_+(x) &= \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} f \left(a \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right) da, \\ \phi_-(x) &= \int_{T(\mathbb{A}_F)/Z(\mathbb{A}_F)} f \left(a \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \right) da. \end{aligned}$$

The local components of ϕ_+ and ϕ_- are defined analogously in terms of the local decomponents of f . Then the right hand side of (10.4.1) is nothing but the sum of the values of the Tate integrals (10.4.3) of ϕ_+ and ϕ_- at the point $s = 0$. Moreover we obviously have for each v in V :

$$\phi_{+v}(0) = \phi_{-v}(0) = \int_{T_v/Z_v} f_v(a_v) da_v.$$

On the other hand, for each v in W the values at point 0 of the Tate integrals of ϕ_{+v} and ϕ_{-v} are nothing but the singular orbital integrals of the points n_+ and n_- . To simplify the notations, define

$$\begin{aligned} M_v &= \int_{T_v/Z_v} f_v(a_v) da_v, \quad v \in V, \\ M_{v\pm} &= 2H(n_{\pm} : f_v : \eta_v), \quad v \in W. \end{aligned}$$

Then the LHS of (10.4.1) can be written as

$$L(0, \eta) \prod_{v \in W} \frac{1}{2L(0, \eta_v)} \prod_{v \in V} |a_v|^{1/2} \times \prod_{v \in V} M_v \left(\prod_{v \in W} M_{v+} + \prod_{v \in W} M_{v-} \right). \quad (10.4.4)$$

Note that all but finitely many factors of each product are equal to 1.

Let's move on to RHS of (10.4.1). The integral is obviously the product of similar local integrals:

$$\int_{T'(\mathbb{A}_F)/Z'(\mathbb{A}_F)} f'(t') dt' = \prod_v \int_{T'_v/Z'_v} f'_v(t'_v) dt'_v.$$

If $v \in V$, the local integral is M_v by the assumption (2.b). If $v \in W$ the integral equals to

$$\frac{1}{2\text{vol}(T'_v/Z'_v)} (M_{v-} + \eta_v(c) M_{v+})$$

by Proposition (4.1) and Proposition (5.1). The volume that appears in this formula is $|b_w|^{1/2} |a_v|^{-1/2}$, where w is the only place of E above v . Hence RHS of (10.4.1) is equal to the product

$$2L(1, \eta) \sum_{c \in N(S)/N} \prod_{v \in W} \frac{1}{2L(0, \eta_v)} \prod_{v \in W} \left| \frac{a_v}{b_w} \right|^{1/2} \prod_{v \in V} M_v \prod_{v \in W} \frac{1}{2} (M_{v-} + \eta_v(c) M_{v+}). \quad (10.4.5)$$

Comparing with (10.4.4), we can find that it suffices to prove the following identities:

$$L(0, \eta) \prod_{v \in V} |a_v|^{1/2} = L(1, \eta) \prod_{v \in W} \left| \frac{a_v}{b_w} \right|^{1/2}, \quad (10.4.6)$$

$$\prod_{v \in W} M_{v+} + \prod_{v \in W} M_{v-} = 2 \sum_{c \in N(S)/N} \prod_{v \in W} \frac{1}{2} (M_{v-} + \eta_v(c) M_{v+}). \quad (10.4.7)$$

The equality (10.4.6) immediately follows from the functional equations of the terms $L(s, 1_E)$ and $L(s, 1_F)$ and their relation to $L(s, \eta)$.

Let's move on to (10.4.7). For $v \in W - S$ we have $\eta_v(c) = 1$ by the definition of $N(S)$ and $M_{v+} = M_{v-}$ (Proposition (5.1)); moreover for almost all $v \in W - V$, $M_{v+} = M_{v-} = 1$. For $U = W \cap S$, we see that the identity (10.4.7) reduces to

$$\prod_{v \in U} M_{v+} + \prod_{v \in U} M_{v-} = 2 \sum_{c \in N(S)/N} \prod_{v \in U} \frac{1}{2} (M_{v-} + \eta_v(c) M_{v+}).$$

Let $H = \{1, -1\}^U$. For a place $v \in U$ define a character χ_v of H by the formula $\chi_v(h) = h_v$. Let H' be a subgroup of H defined by the equation $\prod_{v \in U} \chi_v(h) = 1$. Then the map $c \mapsto (\eta_v(c))$ defines a bijection between $N(S)/N$ and H' . Then the formula that we want to prove can be written as

$$\prod_{v \in U} M_{v+} + \prod_{v \in U} M_{v-} = 2 \sum_{h \in H'} \prod_{v \in U} \frac{1}{2} (M_{v-} + \chi_v(h) M_{v+}).$$

Since $|H| = 2|H'|$, RHS can be written as

$$\frac{1}{|H'|} \sum_Y \sum_{h \in H'} \prod_{v \in Y} \chi_v(h) \prod_{v \in Y} M_{v+} \prod_{v \in U-Y} M_{v-},$$

where the outer sum is over the subset Y of U . The character $\prod_{v \in Y} \chi_v$ is nontrivial on H' , unless Y is empty or equal to U . Therefore only the terms corresponding to the empty set and U contributes to the sum; this gives us our equality.

11 Waldspurger's result

11.1

We can finally prove Waldspurger's result using the identities from section 10. We will denote by S a finite set of places of F satisfying the conditions of section 10. By taking S sufficiently large, we can only consider the cuspidal representations of G unramified outside of S , pairs (G', T') belong to $X(S)$ and, for such a pair, cuspidal representations of G' unramified outside S . We will denote by K (resp. K^S) for the product of the compact subgroups K_v for all v (resp. all v not in S) and G^S the restricted product of G_v for v not in S . For a pair (G', T') in $X(S)$, we denote as K', K'^S and G'^S for the analogous groups.

To describe Waldspurger's first condition, note that if

$$\int \phi_1(a)da \quad \text{and} \quad \int \phi_2(b)\eta(\det b)db$$

is nonzero for some pair (ϕ_1, ϕ_2) of smooth vectors, then it is also nonzero for some K -finite vectors (ϕ_1, ϕ_2) ; moreover, if S is large enough, then ϕ and ϕ' are K^S -invariant. Similarly if there exists a pair (G', T') in X , a cuspidal representation π' and a smooth vector ϕ' in the space of π' such that the integral $\int \phi'(t')dt'$ is nonzero, then we can take ϕ' to be K' -finite; moreover, if S is large enough, (G', T') is in $X(S)$ and ϕ' is invariant under K'^S .

11.2

Consider a set S and functions f and f' satisfying the conditions in section 10. In particular f_v (resp. f'_v) is bi-invariant under K_v (resp. K'_v). Consider the kernel K_c . We can write as

$$K_c = \sum_{\pi} K_{\pi} \tag{11.2.1}$$

where, for each cuspidal automorphic representation π (unramified outside of S), we have

$$K_{\pi}(x, y) = \sum_j \rho(f) \phi_j(x) \overline{\phi_j}(y), \tag{11.2.2}$$

where $\{\phi_j\}$ is an orthonormal basis of the subspace of K^S -invariant vectors in the space of π . The series (11.2.1) not only converges in the Hilbert space of square integrable functions on the quotient $[G]$, but also in the space of rapidly decreasing functions on the quotient $[G]$. Moreover, since f_v is K_v -finite for all v infinite, the series (11.2.2) has only a finite number of nonzero terms for given f . Denote by $H(S)$ the Hecke algebra of the group G^S relative to the subgroup K^S . Write $f = f_S f^S$, where f_S (resp. f^S) is the product of f_v for v in S (resp. not in S). Let Λ_{π} be the character of $H(S)$ attached to a representation π . Then we have

$$\iint K_c(a, b) \eta(\det b) da db = \sum_{\pi} a(\pi, f_S) \Lambda_{\pi}(f^S), \tag{11.2.3}$$

where $a(\pi, f_S)$ is

$$a(\pi, f_S) = \sum_j \int \rho(f_S) \phi_j(a) da \int \overline{\phi_j}(b) \eta(\det b) db. \tag{11.2.4}$$

11.3

Consider $(G', T') \in X(S)$. We have a similar decomposition

$$K'_c = \sum_{\pi'} K_{\pi'}, \quad (11.3.1)$$

$$K_{\pi'}(x, y) = \sum_j \rho(f') \phi'_j(x) \overline{\phi'_j}(y), \quad (11.3.2)$$

where $\{\phi'_j\}$ is an orthonormal basis of the subspace of K'^S -invariant vectors in the space of π' . The series (11.3.1) also converges in the space of rapidly decreasing functions and the series (11.3.2) has only finitely many nonzero terms. By integrating term by term, we get

$$\iint K'_\pi(s, t) ds dt = a(\pi', f'_S) \Lambda'_\pi(f'^S) \quad (11.3.3)$$

where $a(\pi', f'_S)$ is given by

$$a(\pi', f'_S) = \sum_j \int \rho(f) \phi'_j(s) ds \int \overline{\phi'_j}(t) dt. \quad (11.3.4)$$

Then the whole integral of K'_c becomes

$$\iint K'_c(s, t) ds dt = \sum_{\pi'} a(\pi', f'_S) \Lambda_{\pi'}(f'^S).$$

11.4

Now let's use our fundamental lemma. Note that if π' is a cuspidal representation of G' and π the corresponding cuspidal representation of G , then $\Lambda_\pi(f_S) = \Lambda_{\pi'}(f'_S)$. Since π' determines π we can write $a(\pi, f'_S)$ for $a(\pi', f'_S)$. On the other hand, for a representation π of the group G , it will be convenient to set $a(\pi, f'_S) = 0$ if there is no representation π' of G' corresponding to π . Then our fundamental lemma is written as

$$\sum_{\pi} a(\pi, f_S) \Lambda_\pi(f^S) = \sum_{\pi} \sum_{(G', T')} a(\pi, f'_S) \Lambda_\pi(f^S) \quad (11.4.1)$$

Here f^S is an arbitrary element of $H(S)$. Let v be a place in S . Then the function f_v is an arbitrary K_v -finite function. The function f'_v matches with f_v by the conditions (10.1.1) and (10.1.2). If v splits then f'_v is in fact can be chosen as an arbitrary K'_v -finite function. If v does not split, then f'_v is no longer arbitrary

but satisfies a density condition: if a function h is bi- T_v -invariant on G_v/Z_v and orthogonal to all f'_v then h is zero (Proposition 4.2). Suppose that π satisfies Waldspurger's first condition; then there exist K -finite vectors ϕ_1 and ϕ_2 in the space of π such that:

$$\int \phi_1(a)da \neq 0 \quad \text{and} \quad \int \phi_2(b)\eta(\det b)db \neq 0,$$

and we can assume ϕ_1 and ϕ_2 to be K^S -invariant. Choose a basis $\{\widetilde{\phi}_j\}$ such that $\widetilde{\phi}_1 = \phi_2/\|\phi_2\|$. There exists f_S such that $\rho(f_S)\widetilde{\phi}_1 = \phi_1$ and $\rho(f_S)\widetilde{\phi}_j = 0$ if $j \neq 1$. Then we have

$$a(\pi, f_S) = \int \phi_1(a)da \int \widetilde{\phi}_1(b)\eta(\det b)db \neq 0.$$

According to the principle of *infinite* linear independence of the characters of $H(S)$ [2], there exists a pair (G', T') such that $a(\pi, f'_S)$ is nonzero. It clearly follows that there exists ϕ' in the space of π' such that $\int \phi'(t')dt'$ is nonzero. Thus π satisfies the second condition of Waldspurger.

Now assume that there exists a pair (G', T') , a representation π' and a K' -finite vector ϕ' in the space of π' such that the integral $\int \phi'(t')dt'$ is nonzero; we can assume that (G', T') is in $X(S)$ and ϕ' is invariant under K'^S . We will see that we can choose f'_S so that $a(\pi', f'_S)$ is nonzero. The integral over T' defines a continuous linear functional on the space of smooth vectors of π' fixed by K'^S . Let's write it as the dot product with a *generalized* vector $e_{T'}$:

$$\int \phi'(t')dt' = (\phi', e_{T'}).$$

If h is a compactly supported smooth function on G_S/Z_S , then $\pi'(h)(e_{T'})$ is defined: it is a smooth vector satisfies $(\phi, \pi'(h)e_{T'}) = (\pi'(h^*)\phi, e_{T'})$ for any vector ϕ , smooth or not. With this notation we have

$$a(\pi, f'_S) = (\pi'(f'_S)e_{T'}, e_{T'}).$$

The subspace of K'^S -invariant vectors of π' is isomorphic to the tensor product of the spaces of π'_v with v in S . For each v in S , there exists a continuous linear functional on the space of smooth vectors of π' nonzero at e'_v which is invariant under T'_v . This functional is unique up to scalar (cf. (6.1) and (6.2)), and we can therefore write:

$$a(\pi', f'_S) = (\pi'(f'_S)e_{T'}, e_{T'}) = C \prod_{v \in S} (\pi'_v(f'_v)e'_v, e'_v),$$

where C is a nonzero constant. We want to show that we can choose f'_v so that $(\pi_v(f'_v)e'_v, e'_v)$ is nonzero. It is obvious when v splits since f'_v is then arbitrary K'_v -finite. If v splits e'_v is in fact an ordinary vector since T'_v is compact. Then $(\pi_v(f'_v)e'_v, e'_v)$ is the scalar product of the function f'_v with the continuous matrix coefficient $(\pi'(g)e'_v, e'_v)$; therefore it cannot be zero for any f'_v according to the density property of f'_v . On the other hand if (G'', T'') is another element of $X(S)$, then $a(\pi, f''_S) = 0$; otherwise there would exist at least one place v in S where the groups G'_v and G''_v are not isomorphic and the representations π'_v and π''_v admit nonzero vectors that are invariant under T'_v and T''_v respectively. But this is impossible (Proposition (6.3)). The coefficient of Λ_π in the RHS of (11.4.1) is therefore nonzero, for a suitable choice of f'_S (i.e. f_S). This results that $a(\pi, f_S)$ is not zero. This obviously implies that π satisfies Waldspurger's first condition. Therefore this completes the proof of the equivalence of the two Waldspurger conditions.

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