SOLUTION FOR “AN INTRODUCTION TO AUTOMORPHIC REPRESENTATION”

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Abstract. This is a solution for the exercises in J. Getz and H. Hahn’s book *An Introduction to Automorphic Representation with a view toward Trace Formulae*.

1. Chapter 1

***Problem* 1.1** NOT FINISHEDBy Yoneda lemma, the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes corresponds to the k-algebra morphism $\phi : A \to B$. This induces a map on the underlying topological spaces by sending a prime ideal $\mathfrak{p} \subset B$ to $\phi^{-1}(\mathfrak{p}) \subset A$, which is also prime.

***Problem* 1.2** NOT FINISHED

***Problem* 1.3** By Yoneda lemma, we have

$$\mathrm{Mor}(\mathrm{Spec}(B), \mathrm{Spec}(A)) \simeq \mathrm{Nat}(h^B, h^A) \simeq h^B(A) = \mathrm{Hom}_k(A, B)$$

which gives an equivalence between $\mathbf{AffSch}_k^{\mathrm{op}}$ and $\mathbf{Alg}_k$.

***Problem* 1.4**
- Nonreduced: $\mathrm{Spec}(\mathbb{C}[x]/(x^2))$
- Reducible: $\mathrm{Spec}(\mathbb{C}[x, y]/(x, y))$
- Reduced and irreducible (i.e. integral): $\mathrm{Spec}(\mathbb{C}[x])$

***Problem* 1.5** We can assume that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A/I)$ for some k-algebra A and an ideal I of A. Then it is enough to show that the map $\mathrm{Hom}(A/I, R) \to \mathrm{Hom}(A, R)$, given by composing with the natural map $\pi : A \to A/I$, is injective. This follows from the surjectivity of π.

***Problem* 1.6** Let $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B), Z = \mathrm{Spec}(C)$. Then the statement is equivalent to

$$\mathrm{Hom}(A \otimes_B C, R) \simeq \mathrm{Hom}(A, R) \times_{\mathrm{Hom}(B,R)} \mathrm{Hom}(C, R).$$

We can define a bijection as follows. First, consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & A \otimes_B C \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\iota_C} \\ B & \xrightarrow{\gamma} & C \end{array}$$

Using the maps above, we define a map from LHS to RHS as $\phi \mapsto (\phi\iota_A, \phi\iota_C)$. Since $\iota_A\alpha = \iota_C\gamma$, we have $\phi\iota_A\alpha = \phi\iota_C\gamma$ and the map is well-defined. For the other direction, for given $(f, g) : A \times C \to R$ with $f\alpha = g\gamma$, universal property of the tensor product gives a unique map $\phi : A \otimes_B C \to R$ with $f = \phi\iota_A$ and $g = \phi\iota_C$. We can check that these maps are inverses for each other.

***Problem* 1.7** NOT FINISHED

***Problem* 1.8** NOT FINISHEDWe define an $\mathbb{R}$-algebra A as

$$A = \mathbb{R}[(x_{ij}, y_{ij})_{1\le i,j\le n}]/I$$

where I is an ideal generated by elements of the form

$$\left(\sum_{k=1}^n (x_{ik}^2 + y_{ik}^2)\right) - 1,$$
$$\sum_{k=1}^n (x_{ik}x_{jk} - y_{ik}y_{jk}), \quad i \neq j$$
$$\sum_{k=1}^n (x_{ij}y_{jk} + y_{ik}x_{jk}), \quad i \neq j$$

for $1 \le i, j \le n$. Then we can identify $\mathrm{U}_n(R)$ with $\mathrm{Hom}(A, R)$ as follows: for given $\phi : A \to R$, let $\alpha_{ij} = \phi(x_{ij})$ and $\beta_{ij} = \phi(y_{ij})$. Then a matrix $g = (g_{ij})_{1\le i,j\le n}$ with $g_{ij} = 1 \otimes \alpha_{ij} + \sqrt{-1} \otimes \beta_{ij}$ becomes an element of $U_n(R)$ by the relations of x_{ij} and y_{ij}s defined by the ideal I. Similarly, for given $g = (g_{ij}) \in \mathrm{U}_n(R)$, we can write $g_{ij} = (a_{ij} + \sqrt{-1}b_{ij}) \otimes r_{ij} = 1 \otimes a_{ij}r_{ij} + \sqrt{-1} \otimes b_{ij}r_{ij}$ and we have a corresponding map $\phi : A \to R$ sending x_{ij} to $a_{ij}r_{ij}$ and y_{ij} to $b_{ij}r_{ij}$.

The group $\mathrm{U}_n(\mathbb{R})$ is a compact group (as a topological subgroup of $\mathrm{GL}_n(\mathbb{C})$) since it is closed (it is an inverse image of point I of a continuous map $g \to g\overline{g}^t$) and bounded (each row and column vectors have norm 1).

At last, NOT FINISHED

***Problem* 1.9** Consider the following short exact sequence:

$$0 \to \ker(\epsilon)/\ker(\epsilon)^2 \to \mathcal{O}(G)/\ker(\epsilon)^2 \to k \to 0.$$

The map $O(G)/\ker(\epsilon)^2 \to k$ is defined as a composition of the natural map $\mathcal{O}(G)/\ker(\epsilon)^2 \to \mathcal{O}(G)/\ker(\epsilon)$ followed by ϵ. Then we have a section $k \to \mathcal{O}(G)/\ker(\epsilon)$ which is the composition $k \to \mathcal{O}(G) \to \mathcal{O}(G)/\ker(\epsilon)^2$ and the above sequence splits.

***Problem* 1.10** Let $g = (g_{ij}) \in GL_n(R)$ and $J = (\alpha_{ij}) \in \mathrm{GL}_n(k)$. Then $g^tJg = J$ is equivalent to

$$\sum_{k,l=1}^{n} \alpha_{kl} g_{ki} g_{lj} = \alpha_{ij}$$

for all $1 \le i, j \le n$. Hence G is an affine algebraic group with a coordinate ring

$$A = k[(X_{ij})_{1\le i,j\le n}]/\left(\sum_{k,l=1}^{n} \alpha_{kl} X_{ki} X_{lj} - \alpha_{ij}, 1 \le i, j \le n\right).$$

Since $\mathrm{Lie}\, G = \ker(G(k[t]/t^2) \to G(k))$, the elements of $\mathrm{Lie}\, G$ have a form of $I + tX$ for some $X \in \mathrm{M}_n(k)$. Then the defining equation $g^tJg = J$ is equivalent to

$$(I + tX)^tJ(I + tX) = J \Leftrightarrow J + tX^tJ + tJX + t^2X^tJX = J + t(X^tJ + JX) = J,$$

(here every elements are in $\mathrm{GL}_n(k[t]/t^2)$) so we should have $X^tJ + JX = 0$. In other words, we have a map

$$\mathrm{Lie}\, G \xrightarrow{\sim} \{X \in \mathfrak{gl}_n(k) : X^tJ + JX\}, \quad I + tX \to X.$$

***Problem* 1.11** NOT FINISHED

***Problem* 1.12** NOT FINISHED

***Problem* 1.13** Using the equivalence of $\mathbf{Spl}_k$ and **RRD**, it is enough to check that the dual of the root datum of GL_n is isomorphic to itself in **RRD**. Recall that the root datum of GL_n with torus T of diagonal elements is given as follows: (Example 1.12)

- $X^*(T) = \{\alpha_{k_1,\ldots,k_n} : \mathrm{diag}(t_1, \ldots, t_n) \mapsto \prod_{1\le j\le n} t_j^{k_j},\, k_1, \ldots, k_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $X_*(T) = \{\beta_{k_1,\ldots,k_n} : t \mapsto \mathrm{diag}(t^{k_1}, \ldots, t^{k_n}),\, t_1, \ldots, t_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $\Phi(\mathrm{GL}_n, T) = \{e_{ij},\, 1 \le i \ne j \le n\}$, $e_{ij}(\mathrm{diag}(t_1, \ldots, t_n)) = t_it_j^{-1}$
- $\Phi^\vee(\mathrm{GL}_n, T) = \{e_{ij},\, 1 \le i \ne j \le n\}$, $e_{ij}^\vee(t) = \mathrm{diag}(1, \ldots, t, \ldots, t^{-1}, \ldots, 1)$ (t in the i-th entry, t^{-1} in the j-th entry, 1 for other entries)

Then we define a map $f : X_*(T) \to X^*(T)$ and $\iota : \Phi(\mathrm{GL}_n, T) \to \Phi^\vee(\mathrm{GL}_n, T)$ as

$$f(\beta_{k_1,\ldots,k_n}) = \alpha_{k_1,\ldots,k_n}, \quad \iota(e_{ij}) = e_{ij}^\vee.$$

and define $f^\vee : X^*(T) \to X_*(T)$ and $\iota^\vee : \Phi^\vee(\mathrm{GL}_n, T) \to \Phi(\mathrm{GL}_n, T)$ similarly. Then these maps are inverse to each other and gives an isomorphism between two root data

$$(X^*(T), X_*(T), \Phi(\mathrm{GL}_n, T), \Phi^\vee(\mathrm{GL}_n, T)) \simeq (X_*(T), X^*(T), \Phi^\vee(\mathrm{GL}_n, T), \Phi(\mathrm{GL}_n, T))$$

(they are central isogenies) so we get $\widehat{\mathrm{GL}_n} = \mathrm{GL}_{n\mathbb{C}}$.

***Problem* 1.14** We will show that complex dual of SL_n is PGL_n, and vice versa. Let's compute root datum for SL_n. We choose a maximal torus $T = T_{\mathrm{SL}_n} \leq \mathrm{SL}_n$ of diagonal matrices, so that

$$T(R) = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} : t_k \in R, \prod_{1 \leq k \leq n} t_k = 1 \right\}.$$

Then the characters $X^*(T)$ is almost same as the GL_n case, but we get a quotient of it. For given $\lambda = (\lambda_1, \dots, \lambda_n), \lambda' = (\lambda'_1, \dots, \lambda'_n) \in \mathbb{Z}^n$, two characters $\alpha_\lambda, \alpha_{\lambda'}$ are the same when $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$. Hence we have

$$X^*(T) \simeq \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n\} / \sim$$

where $\lambda \sim \lambda'$ if $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$. Similarly, cocharacter $\beta_\lambda(t) = \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_1})$ is well-defined only when $\sum_{k=1}^n \lambda_k = 0$, so that

$$X_*(T) \simeq \left\{ \lambda = (\lambda_1, \dots, \lambda_n) : \sum_{k=1}^n \lambda_k = 0 \right\} =: H \subset \mathbb{Z}^n.$$

The set of roots and coroots for SL_n is the same as that of GL_n: only Cartan subalgebra $\mathfrak{t}$ is changed from diagonal matrices in $\mathfrak{gl}_n$ to traceless diagonal matrices.

For PGL_n, we choose the maximal torus $T' = T_{\mathrm{PGL}_n}$ of diagonal matrices, and characters of T' has a form of $\alpha'_\lambda : \mathrm{diag}(t_1, \dots, t_n) \mapsto \prod_{k=1}^n t_k^{\lambda_k}$ for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, and we should have $\sum_{k=1}^n \lambda_k = 0$ for the character to be well-defined on T'. Hence we have $X^*(T') \simeq H \subset \mathbb{Z}^n$. Similarly, any cocharacter on T' has a form of $\beta'_\lambda : t \mapsto \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_n})$, and two different $\lambda, \lambda \in \mathbb{Z}^n$ define same cocharacter as a map to T' if $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$, so $X_*(T)'$ is isomorphic to the quotient of $\mathbb{Z}^n$ by $\mathbb{Z} \cdot (1, \dots, 1)$. The set of roots and coroots are the same as GL_n and SL_n. Note that the Lie algebra $\mathfrak{pgl}_n$ of PGL_n can be thought as a quotient of $\mathfrak{gl}_n(R)$ by $R \cdot I_n$, where I_n is the $n \times n$ identity matrix.

Observe that we can natually identify $X^*(T) \simeq \mathbb{Z}^n / (\mathbb{Z} \cdot (1, \cdots, 1)) \simeq X_*(T')$ and $X_*(T) \simeq H \simeq X^*(T')$. We can define a map between two root data of SL_n and $\widehat{\mathrm{PGL}_n}$ as follows:

$$\begin{aligned} f : X^*(T) \to X_*(T'), &\quad \alpha_\lambda \mapsto \beta'_\lambda \\ f^\vee : X_*(T) \to X^*(T'), &\quad \beta_\lambda \mapsto \alpha'_\lambda \\ \iota : \Phi(\mathrm{SL}_n, T) \to \Phi^\vee(\mathrm{PGL}_n, T'), &\quad e_{ij} \mapsto e_{ij}^\vee \\ \iota^\vee : \Phi^\vee(\mathrm{SL}_n, T) \to \Phi(\mathrm{PGL}_n, T'), &\quad e_{ij}^\vee \mapsto e_{ij} \end{aligned}$$

and this gives $\mathrm{SL}_n \simeq \widehat{\mathrm{PGL}_n}$. Similarly, we have $\mathrm{PGL}_n \simeq \widehat{\mathrm{SL}_n}$.

***Problem* 1.15** NOT FINISHED

2. Chapter 2

***Problem* 2.1** NOT FINISHED

***Problem* 2.2** NOT FINISHED

***Problem* 2.3** It is compact since it is an intersection of closed subset $G(F)$ of $\mathrm{GL}_n(F)$ ($G \hookrightarrow \mathrm{GL}_n$ is closed immersion) and intersection of closed set with compact set is again compact. Openness follows from continuity of $G(F) \hookrightarrow \mathrm{GL}_n(F)$: $\rho(G(F)) \cap K$ is an inverse image of K under $G(F) \hookrightarrow \mathrm{GL}_n(F)$.

***Problem* 2.4** NOT FINISHED

***Problem* 2.5** Using the anti-equivalence of category $\mathbf{AffSch}_k$ and $\mathbf{Alg}_k$, we can reformulate the situation in terms of algebra as follows. Let $A = \mathcal{O}(Y)$ be $\mathfrak{o}$-algebra and $A_F := A \otimes_{\mathfrak{o}} F$. Let $X = \mathrm{Spec}(A_F/I)$ and $\mathcal{X}$ be schematic closure of X in Y, so that $\mathcal{O}(\mathcal{X}) = \mathrm{Im}(\pi^I \iota)$ where $\iota : A \hookrightarrow A_F$ and $\pi^I : A_F \twoheadrightarrow A_F/I$. Let $\mathcal{Z} = \mathrm{Spec}\, A/J$ (we have closed immersion $\mathcal{Z} \hookrightarrow \mathcal{Y}$), and we assume that the map on generic fibre, which corresponds to $A_F \twoheadrightarrow (A/J)_F$, induces an isomorphism $A_F/I = \mathcal{O}(X) \simeq \mathcal{O}(\mathcal{Z}) = (A/J)_F$. This means that there exists an isomorphism $\phi : A_F/I \to (A/J)_F$ such that the following diagram commutes:

$$\begin{tikzcd} (A/J)_F & \\ A_F/I \arrow[u, "\phi"] & A_F \arrow[l, two heads, "\pi^I"] \arrow[ul, two heads, "\pi^J_F"'] \end{tikzcd}$$

Now our goal is to show that there exists a unique map

$$f : \mathcal{O}(\mathcal{X}) = \mathrm{Im}(\pi^I \iota) \to \mathcal{O}(\mathcal{Z}) = A/J$$

such that the following diagram commutes:

$$\begin{tikzcd} A/J & \\ \mathrm{Im}(\pi^I\iota) \arrow[u, "f"] & A \arrow[l, two heads, "\pi^I\iota"] \arrow[ul, two heads, "\pi^J"'] \end{tikzcd}$$

The only way to define f that the above diagram commutes is following: for $x \in \mathrm{Im}(\pi^I \iota)$, choose $a \in A$ with $x = \pi^I \iota(a)$ and define $f(x) := \pi^J(a)$. Then we only need to show that the map is well-defined regardless of the choice of a. Let $a_1, a_2 \in A$ such that $\pi^I \iota(a_1) = \pi^I \iota(a_2) = x$. Since $\iota^J : A/J \hookrightarrow (A/J)_F$ is an injection, it is enough to show that $\iota^J \pi^J(a_1) = \iota^J \pi^J(a_2)$. By the commutativity of the following diagram

$$\begin{tikzcd} A/J \arrow[d, hook, "\iota^J"'] & A \arrow[l, two heads, "\pi^J"'] \arrow[d, hook, "\iota"] \\ (A/J)_F & A_F \arrow[l, two heads, "\pi^J_F"'] \end{tikzcd}$$

we have $\iota^J \pi^J = \pi^J_F \iota = \phi \pi^I \iota$, and this proves

$$\iota^J \pi^J(a_1) = \phi \pi^I \iota(a_1) = \phi(x) = \phi \pi^I \iota(a_2) = \iota^J \pi^J(a_2),$$

i.e. the map is well-defined.

***Problem* 2.6** NOT FINISHED

***Problem* 2.7** NOT FINISHED

***Problem* 2.8** Note that the coordinate ring of $\mathrm{GL}_{n,\mathbb{Q}}$ is

$$B = \mathcal{O}(\mathrm{GL}_{n,\mathbb{Q}}) = \mathbb{Q}[x_{ij}, y]_{1\leq i,j\leq n}/(\det(x_{ij})y - 1).$$

To show that $\mathcal{G}$ is a model of $\mathrm{GL}_{n,\mathbb{Q}}$ over $\mathbb{Z}$, we need to show that $A \hookrightarrow B$ and $A \otimes_{\mathbb{Z}} \mathbb{Q} \simeq B$. Latter isomorphism easily follows from

$$A \otimes \mathbb{Q} = \mathbb{Q}[x_{ij}, t_{ij}, y]/(\det(x_{ij})y - 1, \{x_{ij} - \delta_{ij} - mt_{ij}\}) \simeq B$$

since we can invert $m > 1$ in $\mathbb{Q}$ and get an isomorphism $A \otimes \mathbb{Q} \to B$ via $t_{ij} \mapsto (1 - x_{ij})/m$. Shoing $A \hookrightarrow B \simeq A \otimes_{\mathbb{Z}} \mathbb{Q}$ is equivalent to showing that A is a torsion-free $\mathbb{Z}$-module. Assume that we have $z \in \mathbb{Z}[x_{ij}, t_{ij}, y]$ and $0 \neq a \in \mathbb{Z}$ such that $az = 0$ in A. Then there exists $\alpha, \beta_{ij} \in \mathbb{Z}$ for $1 \leq i, j \leq n$ s.t.

$$\begin{aligned} az &= \alpha(\det(x_{ij})y - 1) + \sum_{ij} \beta_{ij}(x_{ij} - \delta_{ij} - mt_{ij}) \\ \Leftrightarrow z &= \frac{\alpha}{a}\det(x_{ij})y + \sum_{i,j}\frac{\beta_{ij}}{a}x_{ij} - \sum_{i,j}\frac{m\beta_{ij}}{a}t_{ij} - \frac{\alpha + \sum_i \beta_{ii}}{a} \end{aligned}$$

which implies $a|\alpha$ and $a|\beta_{ij}$, i.e. $z = 0$ in A. Hence $\mathcal{G}$ is a model of $\mathrm{GL}_{n,\mathbb{Q}}$ over $\mathbb{Z}$.

The set of $\mathbb{Z}$-points $\mathcal{G}(\mathbb{Z}) = \mathrm{Hom}(A, \mathbb{Z})$ can be identified with the set via map

$$\begin{aligned} \mathrm{Hom}(A, \mathbb{Z}) &\to \{g \in \mathrm{GL}_n(\mathbb{Z}) : g \equiv I_n \pmod{m\mathrm{M}_n(\mathbb{Z})}\} \\ \phi &\mapsto (g_{ij} = \phi(x_{ij})) \end{aligned}$$

since $\phi(x_{ij}) = \delta_{ij} + m\phi(t_{ij}) \Rightarrow g - I_n \in m\mathrm{M}_n(\mathbb{Z})$.

***Problem* 2.9** NOT FINISHEDIt is not hard to prove that if Z_1, Z_2 are dense subsets of a topological space Y_1, Y_2 respectively, then $Z_1 \times Z_2$ is dense in $Y_1 \times Y_2$. Combining with Exercise 1.6 and Theorem 2.2.1 (b), we get the desired results for both weak and strong approximation.

***Problem* 2.10** By Exercise 2.7 and 2.9, $\mathrm{M}_n \simeq \mathbb{G}_a^{n^2}$ admits weak approximation over F. With embedding $\mathrm{GL}_n \hookrightarrow \mathrm{M}_n$ with $\mathrm{GL}_n(F) = \mathrm{M}_n(F) \cap \mathrm{GL}_n(F_S) \subset \mathrm{M}_n(F_S)$, we also have $\mathrm{GL}_n(F)$ dense in $\mathrm{GL}_n(F_S)$.

***Problem* 2.11** NOT FINISHED

***Problem* 2.12** NOT FINISHED

***Problem* 2.13** NOT FINISHED

***Problem* 2.14** NOT FINISHED

***Problem* 2.15** NOT FINISHED

***Problem* 2.16** NOT FINISHEDThe center Z_{GL_2} of GL_2 is $Z_{\mathrm{GL}_2}(R) = R^\times I_2$. Hence the largest $\mathbb{F}_p(t)$ split torus in $\mathrm{Res}_{F/\mathbb{F}_p(t)} Z_{\mathrm{GL}_2}$ is just $\mathrm{Res}_{F/\mathbb{F}_p(t)} Z_{\mathrm{GL}_2}$ itself which has degree $d = [F : \mathbb{F}_q(t)]$.

***Problem* 2.17** NOT FINISHED

***Problem* 2.18** NOT FINISHED

***Problem* 2.19** Let $N = p_1^{e_1} \cdots p_r^{e_r}$ be a prime factorization of N. Define $K_N \leq \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^\infty)$ as

$$K_N = \prod_{i=1}^r (I_n + p_i^{e_i}\mathrm{M}_n(\mathbb{Z}_{p_i})) \times \prod_{p \neq p_i} \mathrm{GL}_n(\mathbb{Z}_p).$$

Then K_N is an open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^\infty)$ such that $K_N \cap \mathrm{GL}_n(\mathbb{Q}) = \Gamma(N)$.

($\Rightarrow$) Let H be a congruence subgroup of $\mathrm{GL}_n(\mathbb{Q})$, which means that there exists an open compact subgroup $K_H \leq \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ such that $H = K_H \cap \mathrm{GL}_n(\mathbb{Q})$. Then we can find an open compact neighborhood $U \leq K_H$ of I_n which has a form of

$$U = \prod_{p \in S} (I_n + p^{e_p} \mathrm{M}_n(\mathbb{Z}_p)) \times \prod_{p \notin S} \mathrm{GL}_n(\mathbb{Z}_p)$$

for some finite set of primes S (Note that $\{I_n + p^k \mathrm{M}_n(\mathbb{Z}_p)\}_{k \geq 1}$ is a decreasing sequence of open compact neighborhoods of I_n, which is also a subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$). Then $U = K_N$ for $N = \prod_{p \in S} p^{e_p}$, i.e. U is also an open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$, and it is a finite index subgroup of K_H since K_H is open and compact (consider all the cosets of K_N in K_H, which are all homeomorphic to K_N). Then $[H : \Gamma(N)] = [K_H : K_N]$ implies that H contains $\Gamma(N)$ as a finite index subgroup.

($\Leftarrow$) Let H be a subgroup of $\mathrm{GL}_n(\mathbb{Q})$ contains $\Gamma(N)$ with $[H : \Gamma(N)] < \infty$. Let K_H be an image of H in $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ under the diagonal embedding $\mathrm{GL}_n(\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ so that $K_H \cap \mathrm{GL}_n(\mathbb{Q}) = H$. Then K_H contains K_N and $[K_H : K_N] = [H : \Gamma(N)]$, so K_N is a finite index subgroup of K_H. for coset representatives $g_1, g_2, \ldots, g_t$ of K_H/K_N, $K_H = \cup_{j=1}^{t} g_j K_N$ and by openness (resp. compactness) of K_N, K_H is also open (resp. compact) subgroup.

3. Chapter 3

***Problem* 3.1** NOT FINISHED

***Problem* 3.2** Since G is compact, the image of the modular quasi-character $\delta_G : G \to \mathbb{R}_{>0}^\times$ is a compact subgroup of $\mathbb{R}_{>0}^\times$. Then it should be trivial - otherwise, there exists $g \in G$ with $\delta_G(g) > 1$ (we can choose g or g^{-1}), and then $\delta_G(g^n) = \delta_G(g)^n \to \infty$ as $n \to \infty$, i.e. the image is not bounded. Hence G is unimodular.

***Problem* 3.3** NOT FINISHED

***Problem* 3.4** NOT FINISHED

***Problem* 3.5** NOT FINISHED

***Problem* 3.6** NOT FINISHED

***Problem* 3.7** Let k be a residue field and ϖ be a uniformizer of F. We have $\mathcal{O}_F^\times = \coprod_{a\in k^\times}(a + \varpi\mathcal{O}_F)$ and

$$\begin{aligned}
d^\times x(\mathcal{O}_F^\times) &= \int_{\mathcal{O}_F^\times} \frac{dx}{|x|} \\
&= \int_{\mathcal{O}_F^\times} dx \\
&= dx(\mathcal{O}_F^\times) \\
&= \sum_{a\in k^\times} dx(a + \varpi\mathcal{O}_F) \\
&= \sum_{a\in k^\times} q^{-1} dx(\mathcal{O}_F) \\
&= (q-1)q^{-1} dx(\mathcal{O}_F) = (1 - q^{-1}) dx(\mathcal{O}_F).
\end{aligned}$$

***Problem* 3.8** NOT FINISHED

***Problem* 3.9** NOT FINISHED

***Problem* 3.10** NOT FINISHED

***Problem* 3.11** Let $x, g, y \in \mathrm{GL}_n(F)$ with $y = xg$ (regard g as a constant matrix). Then we have $y_{ij} = \sum_{1\le k\le n} x_{ik}g_{kj}$ and $dy_{ij} = \sum_{1\le k\le n} g_{kj}dx_{ik}$. This gives

$$\begin{aligned}
&dy_{11} \wedge dy_{12} \wedge \cdots \wedge dy_{1n} \\
&= (g_{11}dx_{11} + g_{21}dx_{12} + \cdots + g_{n1}dx_{1n}) \wedge \cdots \wedge (g_{1n}dx_{11} + \cdots + g_{nn}dx_{1n}) \\
&= |\det(g^t)| dx_{11} \wedge \cdots \wedge dx_{1n} \\
&= |\det(g)| dx_{11} \wedge \cdots \wedge dx_{1n}
\end{aligned}$$

and along with $\det(xg) = \det(x)\det(g)$, we have

$$\frac{\wedge_{i,j} dy_{ij}}{|\det(y)|^n} = \frac{|\det(g)|^n \wedge_{i,j} dx_{ij}}{|\det(xg)|^n} = \frac{\wedge_{i,j} dx_{ij}}{|\det(x)|^n}$$

so $d(x_{ij})$ is right Haar measure. Since GL_n is reductive, it is unimodular and so $d(x_{ij})$ is also a left Haar measure.

***Problem* 3.12** NOT FINISHED

***Problem* 3.13** NOT FINISHED

***Problem* 3.14** NOT FINISHEDConsider a reduction map $\mathrm{GL}_n(\mathcal{O}_{F_v}) \twoheadrightarrow \mathrm{GL}_n(k_v)$ where k_v is a residue field of F_v with $\#k_v = q_v$, which is surjective. The kernel H

of the map is $1 + \varpi_v \mathrm{M}_n(\mathcal{O}_{F_v})$ where ϖ_v is a uniformizer of F_v. Then we have

$$|\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = |\omega|_v(H) \cdot \#\,\mathrm{GL}_n(k_v).$$

The order of $\mathrm{GL}_n(k_v)$ is $(q_v^2-1)(q_v^2-q_v)$: there are q_v^2-1 choices for the first column vector (all but zero vector), and $q_v^2-q_v$ choices for the second column vector (all but vectors which are multiples of the first column vector). Also, for $h \in H$, we have

$$h = \begin{pmatrix} 1+\varpi_v x_{11} & \varpi_v x_{12} \\ \varpi_v x_{21} & 1+\varpi_v x_{22} \end{pmatrix}$$
$$\Rightarrow |\det(h)|_v = |1+\varpi_v(x_{11}+x_{22}) + \varpi_v^2(x_{11}x_{22} - x_{12}x_{21})|_v = 1$$

So

$$\begin{aligned} |\omega|_v(H) &= \int_{\mathcal{O}_{F_v}^4} d(\varpi_v x_{11}) \wedge \cdots \wedge d(\varpi_v x_{22}) \\ &= q_v^{-4} \int_{\mathcal{O}_{F_v}^4} dx_{11} \wedge \cdots \wedge dx_{22} = q_v^{-4} \end{aligned}$$

and the measure is $|\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = (1-q_v^{-1})(1-q_v^{-2})$.

When F is a number field, then the *Dedekind zeta function* of F, defined as

$$\zeta_F(s) := \sum_{0 \neq I \subseteq \mathcal{O}_F} \frac{1}{N_{F/\mathbb{Q}}(I)^s}$$

admits an Euler product for $\Re s > 1$:

$$\zeta_F(s) = \prod_{\mathfrak{p} \subseteq \mathcal{O}_F} \frac{1}{1 - N_{F/\mathbb{Q}}(\mathfrak{p})^s}.$$

Then the product is

$$\prod_{v \nmid \infty} \left(1 - \frac{1}{q_v}\right)\left(1 - \frac{1}{q_v^2}\right)$$

and this diverges since $\prod_{v\nmid\infty}(1-q_v^{-1})$ does and $\prod_{v\nmid\infty}(1-q_v^{-2}) = \zeta_F(2)^{-1}$ does not. However, the normalized product

$$\prod_{v\nmid\infty}(1-q_v^{-1})^{-1}|\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = \prod_{v\nmid\infty}(1-q_v^{-2})$$

conveges to $\zeta_F(2)^{-1}$.

Now assume that F is a function field.

***Problem* 3.15** (Note that this is a theorem of Maschke.) It is enough to show the following:

Claim. Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of finite group G, and let U be a subrepresentation of ρ, i.e. invariant under ρ. Then there exists $W \leq V$ such that $U \cap W = \{0\}$ and $U \oplus W = V$.

Applying the above claim repeatedly shows that any representation of a finite group is completely decomposable. To show the lemma, let W' be *any* subspace of V such that $U \cap W' = \{0\}$ and $U \oplus W' = V$. Let $\pi' : V \to U$ be a corresponding projection. Then define $\pi : V \to V$ as

$$\pi(v) = \frac{1}{|G|}\sum_{g \in G} g^{-1}\pi'(gv)$$

whose image is in U ($gv := \rho(g)v$). Our claim is that $W = \ker \pi$ is the desired subspace: W is ρ-invariant and $U \oplus W = V$. First of all, since $\pi'|_U$ is identity on U and U is ρ-invariant, $\pi|_U$ is also an identity map on U. Then we have $W \cap U = 0$, and by dimension counting we get $V = U \oplus W$. Hence we only need to show that W is ρ-invariant: for $h \in G$ and $v\pi W = \ker \pi$,

$$\begin{aligned}
\pi(hv) &= \frac{1}{|G|}\sum_{g\in G} g^{-1}\pi'(ghv) \\
&= \frac{1}{|G|}\sum_{g'\in G} hg'^{-1}\pi'(g'v) \quad (g' = gh) \\
&= h\pi(v) = 0
\end{aligned}$$

so $hv \in W$.

***Problem* 3.16** Assume that the representation $\rho : B(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ is completely reducible. Since the representation is 2-dimensional, it should be decomposed as $\chi_1 \oplus \chi_2$ for some characters $\chi_1, \chi_2 : \mathbb{C}^\times \to \mathbb{C}^\times$. In other words, there exists $g_0 \in \mathrm{GL}_2(\mathbb{C})$ such that

$$\rho(g) = g_0 \begin{pmatrix} \chi_1(g) & 0 \\ 0 & \chi_2(g) \end{pmatrix} g_0^{-1}.$$

This implies $\rho(gh) = \rho(hg)$, which is not true since $B(\mathbb{C})$ is not commutative.

***Problem* 3.17** For any $g \in G$,

$$\begin{aligned}
((f_1 * f_2) * f_3)(g) &= \int_G (f_1 * f_2)(gh_1^{-1}) f_3(h_1) d_r h_1 \\
&= \int_G \int_G f_1(gh_1^{-1}h_2^{-1}) f_2(h_2) d_r h_2 f_3(h_1) d_r h_1 \\
&= \int_G \int_G f_1(gh_1^{-1}h_2^{-1}) f_2(h_2) f_3(h_1) d_r h_2 d_r h_1 \\
&= \int_G \int_G f_1(gh_3^{-1}) f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_3 d_r h_1 \quad (h_3 = h_2 h_1,\ d_r h_3 = d_r h_2) \\
&= \int_G \int_G f_1(gh_3^{-1}) f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_1 d_r h_3 \quad \text{(Fubini's theorem)} \\
&= \int_G f_1(gh_3^{-1}) \left(\int_G f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_1 \right) d_r h_3 \\
&= \int_G f_1(gh_3^{-1}) (f_2 * f_3)(h_3) d_r h_3 \\
&= (f_1 * (f_2 * f_3))(g).
\end{aligned}$$

***Problem* 3.18** NOT FINISHED

***Problem* 3.19**

$$\begin{aligned}
\pi(f_1 * f_2)\varphi &= \int_G (f_1 * f_2)(g)\pi(g)\varphi d_r g \\
&= \int_G \int_G f_1(gh^{-1})f_2(h)\pi(g)\varphi d_r h d_r g \\
&= \int_G \int_G f_1(gh^{-1})f_2(h)\pi(g)\varphi d_r g d_r h \quad \text{(Fubini's theorem)} \\
&= \int_G \int_G f_1(g_1)f_2(h)\pi(g_1 h)\varphi d_r g_1 d_r h \quad (g_1 = gh^{-1}, d_r g_1 = d_r g) \\
&= \int_G \int_G f_1(g_1)f_2(h)\pi(g_1 h)\varphi d_r h d_r g_1 \quad \text{(Fubini's theorem)} \\
&= \int_G f_1(g_1)\pi(g_1)\left(\int_G f_2(h)\pi(h)\varphi d_r h\right) d_r g_1 \\
&= \int_G f_1(g_1)\pi(g_1)\pi(f_2)\varphi d_r g_1 \\
&= (\pi(f_1) \circ \pi(f_2))\varphi
\end{aligned}$$

4. Chapter 4

***Problem* 4.1** NOT FINISHED

***Problem* 4.2** NOT FINISHED

***Problem* 4.3** NOT FINISHED

***Problem* 4.4** NOT FINISHED

***Problem* 4.5** By Schur's lemma, any elements in a center $z \in Z_G(F)$ acts as a (nonzero) scalar, let's say, $\omega_\pi(z) \in \mathbb{C}^\times$. Then $\omega_\pi : Z_G(F) \to \mathbb{C}^\times$ is a character since $\omega_G = \pi|_{Z_F(G)}$.

Let $\chi : G(F) \to \mathbb{C}^\times$ be a quasi-character. The representation $\pi \otimes \chi$ is defined as $(\pi \otimes \chi)(g)v = \chi(g) \cdot \pi(g)v$, and it's restriction on the center becomes $\chi|_{Z_G(F)} \cdot \omega_\pi$, which is the centeral character $\omega_{\pi\otimes\chi}$ of $\pi \otimes \chi$.

***Problem* 4.6** Let $G = \mathbb{G}_a$ and $G(\mathbb{R}) = (\mathbb{R}, +)$. Consider a 1-dimensional representation $\chi_\alpha : G(\mathbb{R}) \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, $t \mapsto e^{\alpha t}$, where $\Re(\alpha) \neq 0$. Then this is irreducible since 1-dimensional, but not unitary since $|e^{\alpha t}| = e^{\Re(\alpha)t} \neq 1$ for $t \neq 0$.

***Problem* 4.7** NOT FINISHED

***Problem* 4.8** NOT FINISHED

***Problem* 4.9** NOT FINISHED

***Problem* 4.10** NOT FINISHED

***Problem* 4.11** NOT FINISHED

***Problem* 4.12** NOT FINISHED

***Problem* 4.13** NOT FINISHED

5. Chapter 5

***Problem* 5.1** NOT FINISHED

***Problem* 5.2** One direction is clear. For the other direction, assume that (π, V) is admissible and let U be an open subgroup of G. Then we can choose compact open subgroup K such that $K \leq U \leq G$, and we have $V^U \leq V^K$. Now $\dim V^K < \infty$ gives $\dim V^U < \infty$.

***Problem* 5.3** Let $v \in V_{\mathrm{sm}}$, so that $v \in V^K$ for some open compact subgroup $K \leq G$. Then for $g \in G$ and $k \in K$, we have $\pi(k)v = v \Rightarrow \pi(gkg^{-1})\pi(g)v = \pi(g)v$. Hence $\pi(g)v$ is fixed by gkg^{-1} for all $k \in K$, hence $\pi(g)v \in V^{gKg^{-1}} \subseteq V_{\mathrm{sm}}$. Hence V_{sm} is preserved by G.

Now let $H = \mathrm{Stab}(v) \leq G$ be a stabilizer of $v \in V_{\mathrm{sm}}$. There exists an open compact subgroup K with $v \in V^K \leftrightarrow K \leq \mathrm{Stab}(v)$, so $\mathrm{Stab}(v)$ is a union of open cosets homeomorphic to K, which is also open. Hence $(\pi_{V_{\mathrm{sm}}}, V_{\mathrm{sm}})$ is smooth.

***Problem* 5.4** NOT FINISHED

***Problem* 5.5** NOT FINISHED

***Problem* 5.6** NOT FINISHED

***Problem* 5.7** NOT FINISHED

***Problem* 5.8** The proof is essentially same as that of Problem 4.5, and we also use Schur's lemma (Problem 5.6).

***Problem* 5.9** Take $x = p^k$. Then $|p^k|_p = p^{-k}$ and $|p^k|_\infty = p^k$, so we have $c_1 \cdot p^k \leq p^{-k} \Leftrightarrow c_1 \leq p^{-2k}$ for all k. Now taking limit $k \to \infty$ gives $c_1 = 0$, which gives a contradiction. We can do similarly for the other direction with $k \to -\infty$.

***Problem* 5.10** NOT FINISHED

***Problem* 5.11** NOT FINISHED

***Problem* 5.12** See the argument in Problem 5.3 for showing G-invariance. To show denseness, let $v \in V$ and let $\epsilon > 0$. By continuity of $G \times V \to V$, there exists open neighborhood U of 1 such that $\|\pi(u)v - v\| < \epsilon$ for all $u \in U$. Now we can choose open compact subgroup K of G lies in U (by totally connectedness), and we have

$$\begin{aligned}
\|e_K v - v\| &= \left\| \frac{1}{\mathrm{meas}_{dg}(K)} \int_G (\mathbb{1}_K(g)\pi(g)v - v)dg \right\| \\
&\leq \frac{1}{\mathrm{meas}_{dg}(K)} \int_K ||\pi(k)v - v|| dk < \epsilon
\end{aligned}$$

and from $e_K v \in V^K$, we get denseness of V_{sm} in V.

***Problem* 5.13** NOT FINISHED

***Problem* 5.14** NOT FINISHED

6. Chapter 6

***Problem* 6.1** NOT FINISHED

***Problem* 6.2** NOT FINISHED

***Problem* 6.3** NOT FINISHED

***Problem* 6.4** NOT FINISHED

***Problem* 6.5** NOT FINISHED

***Problem* 6.6** NOT FINISHED

***Problem* 6.7** NOT FINISHED

***Problem* 6.8** NOT FINISHED

***Problem* 6.9** NOT FINISHED

***Problem* 6.10** NOT FINISHED

7. Chapter 7

***Problem* 7.1** NOT FINISHED

***Problem* 7.2** NOT FINISHED

***Problem* 7.3** NOT FINISHED

***Problem* 7.4** NOT FINISHED

***Problem* 7.5** It is clear that $V \mapsto V^K$ is left exact, i.e. it preserves injectivity. Hence we'll only show right exactness. For $V \twoheadrightarrow V/W$, we'll show that $V^K \to (V/W)^K$ is surjective. Let $[v] \in (V/W)^K$, so that $\pi(k)(v) - v \in W$ for all $k \in K$. Define v_0 as

$$v_0 = \frac{1}{|K|} \int_K \pi(k) v dk,$$

an average of v over K. We can see that

$$v_0 - v = \frac{1}{|K|} \int_K (\pi(k)v - v) dk \in W$$

so $[v_0] = [v]$ in V/W. Also,

$$\pi(k')v_0 = \frac{1}{|K|} \int_K \pi(k'k) v dk = \frac{1}{|K|} \int_K \pi(k) dk = v_0$$

so $v_0 \in V^K$. Hence $[v]$ is an image of v_0 and $V^K \to (V/W)^K$ is surjective.

***Problem* 7.6** NOT FINISHED

***Problem* 7.7** NOT FINISHED