

# **SOLUTION FOR “AN INTRODUCTION TO AUTOMORPHIC REPRESENTATION”**

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ABSTRACT. This is a solution for the exercises in J. Getz and H. Hahn’s book  
*An Introduction to Automorphic Representation with a view toward Trace  
Formulae.*

## 1. CHAPTER 1

**Problem 1.1 NOT FINISHED** By Yoneda lemma, the morphism  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  of affine schemes corresponds to the  $k$ -algebra morphism  $\phi : A \rightarrow B$ . This induces a map on the underlying topological spaces by sending a prime ideal  $\mathfrak{p} \subset B$  to  $\phi^{-1}(\mathfrak{p}) \subset A$ , which is also prime.

**Problem 1.2 NOT FINISHED**

**Problem 1.3** By Yoneda lemma, we have

$$\text{Mor}(\text{Spec}(B), \text{Spec}(A)) \simeq \text{Nat}(h^B, h^A) \simeq h^B(A) = \text{Hom}_k(A, B)$$

which gives an equivalence between  $\mathbf{AffSch}_k^{\text{op}}$  and  $\mathbf{Alg}_k$ .

**Problem 1.4** • Nonreduced:  $\text{Spec}(\mathbb{C}[x]/(x^2))$

- Reducible:  $\text{Spec}(\mathbb{C}[x, y]/(x, y))$
- Reduced and irreducible (i.e. integral):  $\text{Spec}(\mathbb{C}[x])$

**Problem 1.5** We can assume that  $Y = \text{Spec}(A)$  and  $X = \text{Spec}(A/I)$  for some  $k$ -algebra  $A$  and an ideal  $I$  of  $A$ . Then it is enough to show that the map  $\text{Hom}(A/I, R) \rightarrow \text{Hom}(A, R)$ , given by composing with the natural map  $\pi : A \rightarrow A/I$ , is injective. This follows from the surjectivity of  $\pi$ .

**Problem 1.6** Let  $X = \text{Spec}(A), Y = \text{Spec}(B), Z = \text{Spec}(C)$ . Then the statement is equivalent to

$$\text{Hom}(A \otimes_B C, R) \simeq \text{Hom}(A, R) \times_{\text{Hom}(B, R)} \text{Hom}(C, R).$$

We can define a bijection as follows. First, consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & A \otimes_B C \\ \alpha \uparrow & & \uparrow \iota_C \\ B & \xrightarrow{\gamma} & C \end{array}$$

Using the maps above, we define a map from LHS to RHS as  $\phi \mapsto (\phi \iota_A, \phi \iota_C)$ . Since  $\iota_A \alpha = \iota_C \gamma$ , we have  $\phi \iota_A \alpha = \phi \iota_C \gamma$  and the map is well-defined. For the other direction, for given  $(f, g) : A \times C \rightarrow R$  with  $f \alpha = g \gamma$ , universal property of the tensor product gives a unique map  $\phi : A \otimes_B C \rightarrow R$  with  $f = \phi \iota_A$  and  $g = \phi \iota_C$ . We can check that these maps are inverses for each other.

**Problem 1.7 NOT FINISHED**

**Problem 1.8 NOT FINISHED** We define an  $\mathbb{R}$ -algebra  $A$  as

$$A = \mathbb{R}[(x_{ij}, y_{ij})_{1 \leq i, j \leq n}] / I$$

where  $I$  is an ideal generated by elements of the form

$$\begin{aligned} & \left( \sum_{k=1}^n (x_{ik}^2 + y_{ik}^2) \right) - 1, \\ & \sum_{k=1}^n (x_{ik} x_{jk} - y_{ik} y_{jk}), \quad i \neq j \\ & \sum_{k=1}^n (x_{ij} y_{jk} + y_{ik} x_{jk}), \quad i \neq j \end{aligned}$$

for  $1 \leq i, j \leq n$ . Then we can identify  $U_n(R)$  with  $\text{Hom}(A, R)$  as follows: for given  $\phi : A \rightarrow R$ , let  $\alpha_{ij} = \phi(x_{ij})$  and  $\beta_{ij} = \phi(y_{ij})$ . Then a matrix  $g = (g_{ij})_{1 \leq i, j \leq n}$  with  $g_{ij} = 1 \otimes \alpha_{ij} + \sqrt{-1} \otimes \beta_{ij}$  becomes an element of  $U_n(R)$  by the relations of  $x_{ij}$  and  $y_{ij}$ s defined by the ideal  $I$ . Similarly, for given  $g = (g_{ij}) \in U_n(R)$ , we can write  $g_{ij} = (a_{ij} + \sqrt{-1}b_{ij}) \otimes r_{ij} = 1 \otimes a_{ij}r_{ij} + \sqrt{-1} \otimes b_{ij}r_{ij}$  and we have a corresponding map  $\phi : A \rightarrow R$  sending  $x_{ij}$  to  $a_{ij}r_{ij}$  and  $y_{ij}$  to  $b_{ij}r_{ij}$ .

The group  $U_n(\mathbb{R})$  is a compact group (as a topological subgroup of  $\text{GL}_n(\mathbb{C})$ ) since it is closed (it is an inverse image of point  $I$  of a continuous map  $g \rightarrow g\bar{g}^t$ ) and bounded (each row and column vectors have norm 1).

At last, **NOT FINISHED**

**Problem 1.9** Consider the following short exact sequence:

$$0 \rightarrow \ker(\epsilon)/\ker(\epsilon)^2 \rightarrow \mathcal{O}(G)/\ker(\epsilon)^2 \rightarrow k \rightarrow 0.$$

The map  $\mathcal{O}(G)/\ker(\epsilon)^2 \rightarrow k$  is defined as a composition of the natural map  $\mathcal{O}(G)/\ker(\epsilon)^2 \rightarrow \mathcal{O}(G)/\ker(\epsilon)$  followed by  $\epsilon$ . Then we have a section  $k \rightarrow \mathcal{O}(G)/\ker(\epsilon)$  which is the composition  $k \rightarrow \mathcal{O}(G) \rightarrow \mathcal{O}(G)/\ker(\epsilon)^2$  and the above sequence splits.

**Problem 1.10** Let  $g = (g_{ij}) \in \text{GL}_n(R)$  and  $J = (\alpha_{ij}) \in \text{GL}_n(k)$ . Then  $g^t J g = J$  is equivalent to

$$\sum_{k,l=1}^n \alpha_{kl} g_{ki} g_{lj} = \alpha_{ij}$$

for all  $1 \leq i, j \leq n$ . Hence  $G$  is an affine algebraic group with a coordinate ring

$$A = k[(X_{ij})_{1 \leq i, j \leq n}] / \left( \sum_{k,l=1}^n \alpha_{kl} X_{ki} X_{lj} - \alpha_{ij}, 1 \leq i, j \leq n \right).$$

Since  $\text{Lie } G = \ker(G(k[t]/t^2) \rightarrow G(k))$ , the elements of  $\text{Lie } G$  have a form of  $I + tX$  for some  $X \in M_n(k)$ . Then the defining equation  $g^t J g = J$  is equivalent to

$$(I + tX)^t J (I + tX) = J \Leftrightarrow J + tX^t J + tJX + t^2 X^t JX = J + t(X^t J + JX) = J,$$

(here every elements are in  $\text{GL}_n(k[t]/t^2)$ ) so we should have  $X^t J + JX = 0$ . In other words, we have a map

$$\text{Lie } G \xrightarrow{\sim} \{X \in \mathfrak{gl}_n(k) : X^t J + JX\}, \quad I + tX \rightarrow X.$$

**Problem 1.11** **NOT FINISHED**

**Problem 1.12** **NOT FINISHED**

**Problem 1.13** Using the equivalence of **Spl<sub>k</sub>** and **RRD**, it is enough to check that the dual of the root datum of  $\text{GL}_n$  is isomorphic to itself in **RRD**. Recall that the root datum of  $\text{GL}_n$  with torus  $T$  of diagonal elements is given as follows: (Example 1.12)

- $X^*(T) = \{\alpha_{k_1, \dots, k_n} : \text{diag}(t_1, \dots, t_n) \mapsto \prod_{1 \leq j \leq n} t_j^{k_j}, k_1, \dots, k_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $X_*(T) = \{\beta_{k_1, \dots, k_n} : t \mapsto \text{diag}(t^{k_1}, \dots, t^{k_n}), t_1, \dots, t_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $\Phi(\text{GL}_n, T) = \{e_{ij}, 1 \leq i \neq j \leq n\}, e_{ij}(\text{diag}(t_1, \dots, t_n)) = t_i t_j^{-1}$
- $\Phi^\vee(\text{GL}_n, T) = \{e_{ij}, 1 \leq i \neq j \leq n\}, e_{ij}^\vee(t) = \text{diag}(1, \dots, t, \dots, t^{-1}, \dots, 1)$   
( $t$  in the  $i$ -th entry,  $t^{-1}$  in the  $j$ -th entry, 1 for other entries)

Then we define a map  $f : X_*(T) \rightarrow X^*(T)$  and  $\iota : \Phi(\text{GL}_n, T) \rightarrow \Phi^\vee(\text{GL}_n, T)$  as

$$f(\beta_{k_1, \dots, k_n}) = \alpha_{k_1, \dots, k_n}, \quad \iota(e_{ij}) = e_{ij}^\vee.$$

and define  $f^\vee : X^*(T) \rightarrow X_*(T)$  and  $\iota^\vee : \Phi^\vee(\mathrm{GL}_n, T) \rightarrow \Phi(\mathrm{GL}_n, T)$  similarly. Then these maps are inverse to each other and gives an isomorphism between two root data

$$(X^*(T), X_*(T), \Phi(\mathrm{GL}_n, T), \Phi^\vee(\mathrm{GL}_n, T)) \simeq (X_*(T), X^*(T), \Phi^\vee(\mathrm{GL}_n, T), \Phi(\mathrm{GL}_n, T))$$

(they are central isogenies) so we get  $\widehat{\mathrm{GL}_n} = \mathrm{GL}_{n\mathbb{C}}$ .

**Problem 1.14** We will show that complex dual of  $\mathrm{SL}_n$  is  $\mathrm{PGL}_n$ , and vice versa. Let's compute root datum for  $\mathrm{SL}_n$ . We choose a maximal torus  $T = T_{\mathrm{SL}_n} \leq \mathrm{SL}_n$  of diagonal matrices, so that

$$T(R) = \left\{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & \ddots \\ & & & t_n \end{pmatrix} : t_k \in R, \prod_{1 \leq k \leq n} t_k = 1 \right\}.$$

Then the characters  $X^*(T)$  is almost same as the  $\mathrm{GL}_n$  case, but we get a quotient of it. For given  $\lambda = (\lambda_1, \dots, \lambda_n), \lambda' = (\lambda'_1, \dots, \lambda'_n) \in \mathbb{Z}^n$ , two characters  $\alpha_\lambda, \alpha_{\lambda'}$  are the same when  $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$ . Hence we have

$$X^*(T) \simeq \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n\} / \sim$$

where  $\lambda \sim \lambda'$  if  $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$ . Similarly, cocharacter  $\beta_\lambda(t) = \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_n})$  is well-defined only when  $\sum_{k=1}^n \lambda_k = 0$ , so that

$$X_*(T) \simeq \left\{ \lambda = (\lambda_1, \dots, \lambda_n) : \sum_{k=1}^n \lambda_k = 0 \right\} =: H \subset \mathbb{Z}^n.$$

The set of roots and coroots for  $\mathrm{SL}_n$  is the same as that of  $\mathrm{GL}_n$ : only Cartan subalgebra  $\mathfrak{t}$  is changed from diagonal matrices in  $\mathfrak{gl}_n$  to traceless diagonal matrices.

For  $\mathrm{PGL}_n$ , we choose the maximal torus  $T' = T_{\mathrm{PGL}_n}$  of diagonal matrices, and characters of  $T'$  has a form of  $\alpha'_\lambda : \mathrm{diag}(t_1, \dots, t_n) \mapsto \prod_{k=1}^n t_k^{\lambda_k}$  for  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ , and we should have  $\sum_{k=1}^n \lambda_k = 0$  for the character to be well-defined on  $T'$ . Hence we have  $X^*(T') \simeq H \subset \mathbb{Z}^n$ . Similarly, any cocharacter on  $T'$  has a form of  $\beta'_\lambda : t \mapsto \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_n})$ , and two different  $\lambda, \lambda' \in \mathbb{Z}^n$  define same cocharacter as a map to  $T'$  if  $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$ , so  $X_*(T)'$  is isomorphic to the quotient of  $\mathbb{Z}^n$  by  $\mathbb{Z} \cdot (1, \dots, 1)$ . The set of roots and coroots are the same as  $\mathrm{GL}_n$  and  $\mathrm{SL}_n$ . Note that the Lie algebra  $\mathfrak{pgl}_n$  of  $\mathrm{PGL}_n$  can be thought as a quotient of  $\mathfrak{gl}_n(R)$  by  $R \cdot I_n$ , where  $I_n$  is the  $n \times n$  identity matrix.

Observe that we can natually identify  $X^*(T) \simeq \mathbb{Z}^n / (\mathbb{Z} \cdot (1, \dots, 1)) \simeq X_*(T')$  and  $X_*(T) \simeq H \simeq X^*(T')$ . We can define a map between two root data of  $\mathrm{SL}_n$  and  $\widehat{\mathrm{PGL}_n}$  as follows:

$$\begin{aligned} f : X^*(T) &\rightarrow X_*(T'), & \alpha_\lambda &\mapsto \beta'_\lambda \\ f^\vee : X_*(T) &\rightarrow X^*(T'), & \beta_\lambda &\mapsto \alpha'_\lambda \\ \iota : \Phi(\mathrm{SL}_n, T) &\rightarrow \Phi^\vee(\mathrm{PGL}_n, T'), & e_{ij} &\mapsto e_{ij}^\vee \\ \iota^\vee : \Phi^\vee(\mathrm{SL}_n, T) &\rightarrow \Phi(\mathrm{PGL}_n, T'), & e_{ij}^\vee &\mapsto e_{ij} \end{aligned}$$

and this gives  $\mathrm{SL}_n \simeq \widehat{\mathrm{PGL}_n}$ . Similarly, we have  $\mathrm{PGL}_n \simeq \widehat{\mathrm{SL}_n}$ .

**Problem 1.15** NOT FINISHED

## 2. CHAPTER 2

**Problem 2.1** NOT FINISHED

**Problem 2.2** NOT FINISHED

**Problem 2.3** It is compact since it is an intersection of closed subset  $G(F)$  of  $\mathrm{GL}_n(F)$  ( $G \hookrightarrow \mathrm{GL}_n$  is closed immersion) and intersection of closed set with compact set is again compact. Openness follows from continuity of  $G(F) \hookrightarrow \mathrm{GL}_n(F)$ :  $\rho(G(F)) \cap K$  is an inverse image of  $K$  under  $G(F) \hookrightarrow \mathrm{GL}_n(F)$ .

**Problem 2.4** NOT FINISHED

**Problem 2.5** Using the anti-equivalence of category  $\mathbf{AffSch}_k$  and  $\mathbf{Alg}_k$ , we can reformulate the situation in terms of algebra as follows. Let  $A = \mathcal{O}(Y)$  be  $\mathfrak{o}$ -algebra and  $A_F := A \otimes_{\mathfrak{o}} F$ . Let  $X = \mathrm{Spec}(A_F/I)$  and  $\mathcal{X}$  be schematic closure of  $X$  in  $Y$ , so that  $\mathcal{O}(\mathcal{X}) = \mathrm{Im}(\pi^I \iota)$  where  $\iota : A \hookrightarrow A_F$  and  $\pi^I : A_F \twoheadrightarrow A_F/I$ . Let  $Z = \mathrm{Spec} A/J$  (we have closed immersion  $Z \hookrightarrow Y$ ), and we assume that the map on generic fibre, which corresponds to  $A_F \twoheadrightarrow (A/J)_F$ , induces an isomorphism  $A_F/I = \mathcal{O}(X) \simeq \mathcal{O}(Z) = (A/J)_F$ . This means that there exists an isomorphism  $\phi : A_F/I \xrightarrow{\sim} (A/J)_F$  such that the following diagram commutes:

$$\begin{array}{ccc} (A/J)_F & & \\ \uparrow \phi & \nwarrow \pi_F^J & \\ A_F/I & \xleftarrow{\pi^I} & A_F \end{array}$$

Now our goal is to show that there exists a unique map

$$f : \mathcal{O}(\mathcal{X}) = \mathrm{Im}(\pi^I \iota) \rightarrow \mathcal{O}(Z) = A/J$$

such that the following diagram commutes:

$$\begin{array}{ccc} A/J & & \\ \uparrow f & \nwarrow \pi^J & \\ \mathrm{Im}(\pi^I \iota) & \xleftarrow{\pi^I \iota} & A \end{array}$$

The only way to define  $f$  that the above diagram commutes is following: for  $x \in \mathrm{Im}(\pi^I \iota)$ , choose  $a \in A$  with  $x = \pi^I \iota(a)$  and define  $f(x) := \pi^J(a)$ . Then we only need to show that the map is well-defined regardless of the choice of  $a$ . Let  $a_1, a_2 \in A$  such that  $\pi^I \iota(a_1) = \pi^I \iota(a_2) = x$ . Since  $\iota^J : A/J \hookrightarrow (A/J)_F$  is an injection, it is enough to show that  $\iota^J \pi^J(a_1) = \iota^J \pi^J(a_2)$ . By the commutativity of the following diagram

$$\begin{array}{ccc} A/J & \xleftarrow{\pi^J} & A \\ \iota^J \downarrow & & \downarrow \iota \\ (A/J)_F & \xleftarrow{\pi_F^J} & A_F \end{array}$$

we have  $\iota^J \pi^J = \pi_F^J \iota = \phi \pi^I \iota$ , and this proves

$$\iota^J \pi^J(a_1) = \phi \pi^I \iota(a_1) = \phi(x) = \phi \pi^I \iota(a_2) = \iota^J \pi^J(a_2),$$

i.e. the map is well-defined.

**Problem 2.6** NOT FINISHED

**Problem 2.7** NOT FINISHED

**Problem 2.8** Note that the coordinate ring of  $\mathrm{GL}_{n,\mathbb{Q}}$  is

$$B = \mathcal{O}(\mathrm{GL}_{n,\mathbb{Q}}) = \mathbb{Q}[x_{ij}, y]_{1 \leq i, j \leq n} / (\det(x_{ij})y - 1).$$

To show that  $\mathcal{G}$  is a model of  $\mathrm{GL}_{n,\mathbb{Q}}$  over  $\mathbb{Z}$ , we need to show that  $A \hookrightarrow B$  and  $A \otimes_{\mathbb{Z}} \mathbb{Q} \simeq B$ . Latter isomorphism easily follows from

$$A \otimes \mathbb{Q} = \mathbb{Q}[x_{ij}, t_{ij}, y] / (\det(x_{ij})y - 1, \{x_{ij} - \delta_{ij} - mt_{ij}\}) \simeq B$$

since we can invert  $m > 1$  in  $\mathbb{Q}$  and get an isomorphism  $A \otimes \mathbb{Q} \rightarrow B$  via  $t_{ij} \mapsto (1 - x_{ij})/m$ . Showing  $A \hookrightarrow B \simeq A \otimes_{\mathbb{Z}} \mathbb{Q}$  is equivalent to showing that  $A$  is a torsion-free  $\mathbb{Z}$ -module. Assume that we have  $z \in \mathbb{Z}[x_{ij}, t_{ij}, y]$  and  $0 \neq a \in \mathbb{Z}$  such that  $az = 0$  in  $A$ . Then there exists  $\alpha, \beta_{ij} \in \mathbb{Z}$  for  $1 \leq i, j \leq n$  s.t.

$$\begin{aligned} az &= \alpha(\det(x_{ij})y - 1) + \sum_{ij} \beta_{ij}(x_{ij} - \delta_{ij} - mt_{ij}) \\ \Leftrightarrow z &= \frac{\alpha}{a} \det(x_{ij})y + \sum_{i,j} \frac{\beta_{ij}}{a} x_{ij} - \sum_{i,j} \frac{m\beta_{ij}}{a} t_{ij} - \frac{\alpha + \sum_i \beta_{ii}}{a} \end{aligned}$$

which implies  $a|\alpha$  and  $a|\beta_{ij}$ , i.e.  $z = 0$  in  $A$ . Hence  $\mathcal{G}$  is a model of  $\mathrm{GL}_{n,\mathbb{Q}}$  over  $\mathbb{Z}$ .

The set of  $\mathbb{Z}$ -points  $\mathcal{G}(\mathbb{Z}) = \mathrm{Hom}(A, \mathbb{Z})$  can be identified with the set via map

$$\begin{aligned} \mathrm{Hom}(A, \mathbb{Z}) &\rightarrow \{g \in \mathrm{GL}_n(\mathbb{Z}) : g \equiv I_n \pmod{mM_n(\mathbb{Z})}\} \\ \phi &\mapsto (g_{ij} = \phi(x_{ij})) \end{aligned}$$

since  $\phi(x_{ij}) = \delta_{ij} + m\phi(t_{ij}) \Rightarrow g - I_n \in mM_n(\mathbb{Z})$ .

**Problem 2.9 NOT FINISHED** It is not hard to prove that if  $Z_1, Z_2$  are dense subsets of a topological space  $Y_1, Y_2$  respectively, then  $Z_1 \times Z_2$  is dense in  $Y_1 \times Y_2$ . Combining with Exercise 1.6 and Theorem 2.2.1 (b), we get the desired results for both weak and strong approximation.

**Problem 2.10** By Exercise 2.7 and 2.9,  $M_n \simeq \mathbb{G}_a^{n^2}$  admits weak approximation over  $F$ . With embedding  $\mathrm{GL}_n \hookrightarrow M_n$  with  $\mathrm{GL}_n(F) = M_n(F) \cap \mathrm{GL}_n(F_S) \subset M_n(F_S)$ , we also have  $\mathrm{GL}_n(F)$  dense in  $\mathrm{GL}_n(F_S)$ .

**Problem 2.11 NOT FINISHED**

**Problem 2.12 NOT FINISHED**

**Problem 2.13 NOT FINISHED**

**Problem 2.14 NOT FINISHED**

**Problem 2.15 NOT FINISHED**

**Problem 2.16 NOT FINISHED** The center  $Z_{\mathrm{GL}_2}$  of  $\mathrm{GL}_2$  is  $Z_{\mathrm{GL}_2}(R) = R^\times I_2$ . Hence the largest  $\mathbb{F}_p(t)$  split torus in  $\mathrm{Res}_{F/\mathbb{F}_p(t)} Z_{\mathrm{GL}_2}$  is just  $\mathrm{Res}_{F/\mathbb{F}_p(t)} Z_{\mathrm{GL}_2}$  itself which has degree  $d = [F : \mathbb{F}_q(t)]$ .

**Problem 2.17 NOT FINISHED**

**Problem 2.18 NOT FINISHED**

**Problem 2.19** Let  $N = p_1^{e_1} \cdots p_r^{e_r}$  be a prime factorization of  $N$ . Define  $K_N \leq \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^\infty)$  as

$$K_N = \prod_{i=1}^r (I_n + p_i^{e_i} M_n(\mathbb{Z}_{p_i})) \times \prod_{p \neq p_i} \mathrm{GL}_n(\mathbb{Z}_p).$$

Then  $K_N$  is an open compact subgroup of  $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^\infty)$  such that  $K_N \cap \mathrm{GL}_n(\mathbb{Q}) = \Gamma(N)$ .

( $\Rightarrow$ ) Let  $H$  be a congruence subgroup of  $\mathrm{GL}_n(\mathbb{Q})$ , which means that there exists an open compact subgroup  $K_H \leq \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$  such that  $H = K_H \cap \mathrm{GL}_n(\mathbb{Q})$ . Then we can find an open compact neighborhood  $U \leq K_H$  of  $I_n$  which has a form of

$$U = \prod_{p \in S} (I_n + p^{e_p} M_n(\mathbb{Z}_p)) \times \prod_{p \notin S} \mathrm{GL}_n(\mathbb{Z}_p)$$

for some finite set of primes  $S$  (Note that  $\{I_n + p^k M_n(\mathbb{Z}_p)\}_{k \geq 1}$  is a decreasing sequence of open compact neighborhoods of  $I_n$ , which is also a subgroup of  $\mathrm{GL}_n(\mathbb{Z}_p)$ ). Then  $U = K_N$  for  $N = \prod_{p \in S} p^{e_p}$ , i.e.  $U$  is also an open compact subgroup of  $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ , and it is a finite index subgroup of  $K_H$  since  $K_H$  is open and compact (consider all the cosets of  $K_N$  in  $K_H$ , which are all homeomorphic to  $K_N$ ). Then  $[H : \Gamma(N)] = [K_H : K_N]$  implies that  $H$  contains  $\Gamma(N)$  as a finite index subgroup.

( $\Leftarrow$ ) Let  $H$  be a subgroup of  $\mathrm{GL}_n(\mathbb{Q})$  contains  $\Gamma(N)$  with  $[H : \Gamma(N)] < \infty$ . Let  $K_H$  be an image of  $H$  in  $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$  under the diagonal embedding  $\mathrm{GL}_n(\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$  so that  $K_H \cap \mathrm{GL}_n(\mathbb{Q}) = H$ . Then  $K_H$  contains  $K_N$  and  $[K_H : K_N] = [H : \Gamma(N)]$ , so  $K_N$  is a finite index subgroup of  $K_H$ . for coset representatives  $g_1, g_2, \dots, g_t$  of  $K_H/K_N$ ,  $K_H = \cup_{j=1}^t g_j K_N$  and by openness (resp. compactness) of  $K_N$ ,  $K_H$  is also open (resp. compact) subgroup.

## 3. CHAPTER 3

**Problem 3.1** NOT FINISHED

**Problem 3.2** Since  $G$  is compact, the image of the modular quasi-character  $\delta_G : G \rightarrow \mathbb{R}_{>0}^\times$  is a compact subgroup of  $\mathbb{R}_{>0}^\times$ . Then it should be trivial - otherwise, there exists  $g \in G$  with  $\delta_G(g) > 1$  (we can choose  $g$  or  $g^{-1}$ ), and then  $\delta_G(g^n) = \delta_G(g)^n \rightarrow \infty$  as  $n \rightarrow \infty$ , i.e. the image is not bounded. Hence  $G$  is unimodular.

**Problem 3.3** NOT FINISHED

**Problem 3.4** NOT FINISHED

**Problem 3.5** NOT FINISHED

**Problem 3.6** NOT FINISHED

**Problem 3.7** Let  $k$  be a residue field and  $\varpi$  be a uniformizer of  $F$ . We have  $\mathcal{O}_F^\times = \coprod_{a \in k^\times} (a + \varpi \mathcal{O}_F)$  and

$$\begin{aligned} d^\times x(\mathcal{O}_F^\times) &= \int_{\mathcal{O}_F^\times} \frac{dx}{|x|} \\ &= \int_{\mathcal{O}_F^\times} dx \\ &= dx(\mathcal{O}_F^\times) \\ &= \sum_{a \in k^\times} dx(a + \varpi \mathcal{O}_F) \\ &= \sum_{a \in k^\times} q^{-1} dx(\mathcal{O}_F) \\ &= (q-1)q^{-1} dx(\mathcal{O}_F) = (1 - q^{-1}) dx(\mathcal{O}_F). \end{aligned}$$

**Problem 3.8** NOT FINISHED

**Problem 3.9** NOT FINISHED

**Problem 3.10** NOT FINISHED

**Problem 3.11** Let  $x, g, y \in \mathrm{GL}_n(F)$  with  $y = xg$  (regard  $g$  as a constant matrix). Then we have  $y_{ij} = \sum_{1 \leq k \leq n} x_{ik} g_{kj}$  and  $dy_{ij} = \sum_{1 \leq k \leq n} g_{kj} dx_{ik}$ . This gives

$$\begin{aligned} dy_{11} \wedge dy_{12} \wedge \cdots \wedge dy_{1n} &= (g_{11} dx_{11} + g_{21} dx_{12} + \cdots + g_{n1} dx_{1n}) \wedge \cdots \wedge (g_{1n} dx_{11} + \cdots + g_{nn} dx_{1n}) \\ &= |\det(g^t)| dx_{11} \wedge \cdots \wedge dx_{1n} \\ &= |\det(g)| dx_{11} \wedge \cdots \wedge dx_{1n} \end{aligned}$$

and along with  $\det(xg) = \det(x) \det(g)$ , we have

$$\frac{\wedge_{i,j} dy_{ij}}{|\det(y)|^n} = \frac{|\det(g)|^n \wedge_{i,j} dx_{ij}}{|\det(xg)|^n} = \frac{\wedge_{i,j} dx_{ij}}{|\det(x)|^n}$$

so  $d(x_{ij})$  is right Haar measure. Since  $\mathrm{GL}_n$  is reductive, it is unimodular and so  $d(x_{ij})$  is also a left Haar measure.

**Problem 3.12** NOT FINISHED

**Problem 3.13** NOT FINISHED

**Problem 3.14** NOT FINISHED Consider a reduction map  $\mathrm{GL}_n(\mathcal{O}_{F_v}) \twoheadrightarrow \mathrm{GL}_n(k_v)$  where  $k_v$  is a residue field of  $F_v$  with  $\#k_v = q_v$ , which is surjective. The kernel  $H$



of the map is  $1 + \varpi_v M_n(\mathcal{O}_{F_v})$  where  $\varpi_v$  is a uniformizer of  $F_v$ . Then we have

$$|\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = |\omega|_v(H) \cdot \#\mathrm{GL}_n(k_v).$$

The order of  $\mathrm{GL}_n(k_v)$  is  $(q_v^2 - 1)(q_v^2 - q_v)$ : there are  $q_v^2 - 1$  choices for the first column vector (all but zero vector), and  $q_v^2 - q_v$  choices for the second column vector (all but vectors which are multiples of the first column vector). Also, for  $h \in H$ , we have

$$h = \begin{pmatrix} 1 + \varpi_v x_{11} & \varpi_v x_{12} \\ \varpi_v x_{21} & 1 + \varpi_v x_{22} \end{pmatrix} \\ \Rightarrow |\det(h)|_v = |1 + \varpi_v(x_{11} + x_{22}) + \varpi_v^2(x_{11}x_{22} - x_{12}x_{21})|_v = 1$$

So

$$|\omega|_v(H) = \int_{\mathcal{O}_{F_v}^4} d(\varpi_v x_{11}) \wedge \cdots \wedge d(\varpi_v x_{22}) \\ = q_v^{-4} \int_{\mathcal{O}_{F_v}^4} dx_{11} \wedge \cdots \wedge dx_{22} = q_v^{-4}$$

and the measure is  $|\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = (1 - q_v^{-1})(1 - q_v^{-2})$ .

When  $F$  is a number field, then the *Dedekind zeta function* of  $F$ , defined as

$$\zeta_F(s) := \sum_{0 \neq I \subseteq \mathcal{O}_F} \frac{1}{N_{F/\mathbb{Q}}(I)^s}$$

admits an Euler product for  $\Re s > 1$ :

$$\zeta_F(s) = \prod_{\mathfrak{p} \subseteq \mathcal{O}_F} \frac{1}{1 - N_{F/\mathbb{Q}}(\mathfrak{p})^s}.$$

Then the product is

$$\prod_{v \nmid \infty} \left(1 - \frac{1}{q_v}\right) \left(1 - \frac{1}{q_v^2}\right)$$

and this diverges since  $\prod_{v \nmid \infty} (1 - q_v^{-1})$  does and  $\prod_{v \nmid \infty} (1 - q_v^{-2}) = \zeta_F(2)^{-1}$  does not. However, the normalized product

$$\prod_{v \nmid \infty} (1 - q_v^{-1})^{-1} |\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = \prod_{v \nmid \infty} (1 - q_v^{-2})$$

converges to  $\zeta_F(2)^{-1}$ .

Now assume that  $F$  is a function field.

**Problem 3.15** (Note that this is a theorem of Maschke.) It is enough to show the following:

**Claim.** Let  $\rho : G \rightarrow \mathrm{GL}(V)$  be a complex representation of finite group  $G$ , and let  $U$  be a subrepresentation of  $\rho$ , i.e. invariant under  $\rho$ . Then there exists  $W \leq V$  such that  $U \cap W = \{0\}$  and  $U \oplus W = V$ .

Applying the above claim repeatedly shows that any representation of a finite group is completely decomposable. To show the lemma, let  $W'$  be *any* subspace of  $V$  such that  $U \cap W' = \{0\}$  and  $U \oplus W' = V$ . Let  $\pi' : V \rightarrow U$  be a corresponding projection. Then define  $\pi : V \rightarrow V$  as

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi'(gv)$$

whose image is in  $U$  ( $gv := \rho(g)v$ ). Our claim is that  $W = \ker \pi$  is the desired subspace:  $W$  is  $\rho$ -invariant and  $U \oplus W = V$ . First of all, since  $\pi'|_U$  is identity on  $U$  and  $U$  is  $\rho$ -invariant,  $\pi|_U$  is also an identity map on  $U$ . Then we have  $W \cap U = 0$ , and by dimension counting we get  $V = U \oplus W$ . Hence we only need to show that  $W$  is  $\rho$ -invariant: for  $h \in G$  and  $v \in W = \ker \pi$ ,

$$\begin{aligned} \pi(hv) &= \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi'(ghv) \\ &= \frac{1}{|G|} \sum_{g' \in G} h g'^{-1} \pi'(g'v) \quad (g' = gh) \\ &= h \pi(v) = 0 \end{aligned}$$

so  $hv \in W$ .

**Problem 3.16** Assume that the representation  $\rho : B(\mathbb{C}) \rightarrow \text{GL}_2(\mathbb{C})$  is completely reducible. Since the representation is 2-dimensional, it should be decomposed as  $\chi_1 \oplus \chi_2$  for some characters  $\chi_1, \chi_2 : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ . In other words, there exists  $g_0 \in \text{GL}_2(\mathbb{C})$  such that

$$\rho(g) = g_0 \begin{pmatrix} \chi_1(g) & 0 \\ 0 & \chi_2(g) \end{pmatrix} g_0^{-1}.$$

This implies  $\rho(gh) = \rho(hg)$ , which is not true since  $B(\mathbb{C})$  is not commutative.

**Problem 3.17** For any  $g \in G$ ,

$$\begin{aligned} ((f_1 * f_2) * f_3)(g) &= \int_G (f_1 * f_2)(gh_1^{-1}) f_3(h_1) d_r h_1 \\ &= \int_G \int_G f_1(gh_1^{-1} h_2^{-1}) f_2(h_2) d_r h_2 f_3(h_1) d_r h_1 \\ &= \int_G \int_G f_1(gh_1^{-1} h_2^{-1}) f_2(h_2) f_3(h_1) d_r h_2 d_r h_1 \\ &= \int_G \int_G f_1(gh_3^{-1}) f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_3 d_r h_1 \quad (h_3 = h_2 h_1, d_r h_3 = d_r h_2) \\ &= \int_G \int_G f_1(gh_3^{-1}) f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_1 d_r h_3 \quad (\text{Fubini's theorem}) \\ &= \int_G f_1(gh_3^{-1}) \left( \int_G f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_1 \right) d_r h_3 \\ &= \int_G f_1(gh_3^{-1}) (f_2 * f_3)(h_3) d_r h_3 \\ &= (f_1 * (f_2 * f_3))(g). \end{aligned}$$

**Problem 3.18** NOT FINISHED

**Problem 3.19**

$$\begin{aligned}
\pi(f_1 * f_2)\varphi &= \int_G (f_1 * f_2)(g)\pi(g)\varphi d_r g \\
&= \int_G \int_G f_1(gh^{-1})f_2(h)\pi(g)\varphi d_r h d_r g \\
&= \int_G \int_G f_1(gh^{-1})f_2(h)\pi(g)\varphi d_r g d_r h \quad (\text{Fubini's theorem}) \\
&= \int_G \int_G f_1(g_1)f_2(h)\pi(g_1h)\varphi d_r g_1 d_r h \quad (g_1 = gh^{-1}, d_r g_1 = d_r g) \\
&= \int_G \int_G f_1(g_1)f_2(h)\pi(g_1h)\varphi d_r h d_r g_1 \quad (\text{Fubini's theorem}) \\
&= \int_G f_1(g_1)\pi(g_1) \left( \int_G f_2(h)\pi(h)\varphi d_r h \right) d_r g_1 \\
&= \int_G f_1(g_1)\pi(g_1)\pi(f_2)\varphi d_r g_1 \\
&= (\pi(f_1) \circ \pi(f_2))\varphi
\end{aligned}$$

## 4. CHAPTER 4

**Problem 4.1** NOT FINISHED

**Problem 4.2** NOT FINISHED

**Problem 4.3** NOT FINISHED

**Problem 4.4** NOT FINISHED

**Problem 4.5** By Schur's lemma, any elements in a center  $z \in Z_G(F)$  acts as a (nonzero) scalar, let's say,  $\omega_\pi(z) \in \mathbb{C}^\times$ . Then  $\omega_\pi : Z_G(F) \rightarrow \mathbb{C}^\times$  is a character since  $\omega_G = \pi|_{Z_F(G)}$ .

Let  $\chi : G(F) \rightarrow \mathbb{C}^\times$  be a quasi-character. The representation  $\pi \otimes \chi$  is defined as  $(\pi \otimes \chi)(g)v = \chi(g) \cdot \pi(g)v$ , and it's restriction on the center becomes  $\chi|_{Z_G(F)} \cdot \omega_\pi$ , which is the central character  $\omega_{\pi \otimes \chi}$  of  $\pi \otimes \chi$ .

**Problem 4.6** Let  $G = \mathbb{G}_a$  and  $G(\mathbb{R}) = (\mathbb{R}, +)$ . Consider a 1-dimensional representation  $\chi_\alpha : G(\mathbb{R}) \rightarrow \text{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ ,  $t \mapsto e^{\alpha t}$ , where  $\Re(\alpha) \neq 0$ . Then this is irreducible since 1-dimensional, but not unitary since  $|e^{\alpha t}| = e^{\Re(\alpha)t} \neq 1$  for  $t \neq 0$ .

**Problem 4.7** NOT FINISHED

**Problem 4.8** NOT FINISHED

**Problem 4.9** NOT FINISHED

**Problem 4.10** NOT FINISHED

**Problem 4.11** NOT FINISHED

**Problem 4.12** NOT FINISHED

**Problem 4.13** NOT FINISHED

## 5. CHAPTER 5

**Problem 5.1 NOT FINISHED**

**Problem 5.2** One direction is clear. For the other direction, assume that  $(\pi, V)$  is admissible and let  $U$  be an open subgroup of  $G$ . Then we can choose compact open subgroup  $K$  such that  $K \leq U \leq G$ , and we have  $V^U \leq V^K$ . Now  $\dim V^K < \infty$  gives  $\dim V^U < \infty$ .

**Problem 5.3** Let  $v \in V_{\text{sm}}$ , so that  $v \in V^K$  for some open compact subgroup  $K \leq G$ . Then for  $g \in G$  and  $k \in K$ , we have  $\pi(k)v = v \Rightarrow \pi(gkg^{-1})\pi(g)v = \pi(g)v$ . Hence  $\pi(g)v$  is fixed by  $gkg^{-1}$  for all  $k \in K$ , hence  $\pi(g)v \in V^{gKg^{-1}} \subseteq V_{\text{sm}}$ . Hence  $V_{\text{sm}}$  is preserved by  $G$ .

Now let  $H = \text{Stab}(v) \leq G$  be a stabilizer of  $v \in V_{\text{sm}}$ . There exists an open compact subgroup  $K$  with  $v \in V^K \leftrightarrow K \leq \text{Stab}(v)$ , so  $\text{Stab}(v)$  is a union of open cosets homeomorphic to  $K$ , which is also open. Hence  $(\pi_{V_{\text{sm}}}, V_{\text{sm}})$  is smooth.

**Problem 5.4 NOT FINISHED****Problem 5.5 NOT FINISHED****Problem 5.6 NOT FINISHED****Problem 5.7 NOT FINISHED**

**Problem 5.8** The proof is essentially same as that of Problem 4.5, and we also use Schur's lemma (Problem 5.6).

**Problem 5.9** Take  $x = p^k$ . Then  $|p^k|_p = p^{-k}$  and  $|p^k|_\infty = p^k$ , so we have  $c_1 \cdot p^k \leq p^{-k} \Leftrightarrow c_1 \leq p^{-2k}$  for all  $k$ . Now taking limit  $k \rightarrow \infty$  gives  $c_1 = 0$ , which gives a contradiction. We can do similarly for the other direction with  $k \rightarrow -\infty$ .

**Problem 5.10 NOT FINISHED****Problem 5.11 NOT FINISHED**

**Problem 5.12** See the argument in Problem 5.3 for showing  $G$ -invariance. To show denseness, let  $v \in V$  and let  $\epsilon > 0$ . By continuity of  $G \times V \rightarrow V$ , there exists open neighborhood  $U$  of 1 such that  $\|\pi(u)v - v\| < \epsilon$  for all  $u \in U$ . Now we can choose open compact subgroup  $K$  of  $G$  lies in  $U$  (by totally connectedness), and we have

$$\begin{aligned} \|e_K v - v\| &= \left\| \frac{1}{\text{meas}_{dg}(K)} \int_G (\mathbb{1}_K(g) \pi(g)v - v) dg \right\| \\ &\leq \frac{1}{\text{meas}_{dg}(K)} \int_K \|\pi(k)v - v\| dk < \epsilon \end{aligned}$$

and from  $e_K v \in V^K$ , we get denseness of  $V_{\text{sm}}$  in  $V$ .

**Problem 5.13 NOT FINISHED****Problem 5.14 NOT FINISHED**

## 6. CHAPTER 6

- Problem 6.1* NOT FINISHED  
*Problem 6.2* NOT FINISHED  
*Problem 6.3* NOT FINISHED  
*Problem 6.4* NOT FINISHED  
*Problem 6.5* NOT FINISHED  
*Problem 6.6* NOT FINISHED  
*Problem 6.7* NOT FINISHED  
*Problem 6.8* NOT FINISHED  
*Problem 6.9* NOT FINISHED  
*Problem 6.10* NOT FINISHED

## 7. CHAPTER 7

**Problem 7.1** NOT FINISHED**Problem 7.2** NOT FINISHED**Problem 7.3** NOT FINISHED**Problem 7.4** NOT FINISHED

**Problem 7.5** It is clear that  $V \mapsto V^K$  is left exact, i.e. it preserves injectivity. Hence we'll only show right exactness. For  $V \twoheadrightarrow V/W$ , we'll show that  $V^K \rightarrow (V/W)^K$  is surjective. Let  $[v] \in (V/W)^K$ , so that  $\pi(k)(v) - v \in W$  for all  $k \in K$ . Define  $v_0$  as

$$v_0 = \frac{1}{|K|} \int_K \pi(k)v dk,$$

an average of  $v$  over  $K$ . We can see that

$$v_0 - v = \frac{1}{|K|} \int_K (\pi(k)v - v) dk \in W$$

so  $[v_0] = [v]$  in  $V/W$ . Also,

$$\pi(k')v_0 = \frac{1}{|K|} \int_K \pi(k'k)v dk = \frac{1}{|K|} \int_K \pi(k)v dk = v_0$$

so  $v_0 \in V^K$ . Hence  $[v]$  is an image of  $v_0$  and  $V^K \rightarrow (V/W)^K$  is surjective.

**Problem 7.6** NOT FINISHED**Problem 7.7** NOT FINISHED

## 8. CHAPTER 8

*Problem 8.1* NOT FINISHED  
*Problem 8.2* NOT FINISHED  
*Problem 8.3* NOT FINISHED  
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*Problem 8.13* NOT FINISHED  
*Problem 8.14* NOT FINISHED  
*Problem 8.15* NOT FINISHED



## 9. CHAPTER 9

**Problem 9.1** Let  $\mathcal{B} = \{\varphi_i\}$  and  $\mathcal{B}' = \{\varphi'_i\}$  be two orthonormal basis. Let  $C : V \rightarrow V$  be a transition operator from  $\mathcal{B}$  to  $\mathcal{B}'$ , so that  $\varphi'_i = \sum_k c_{ik} \varphi_k$  for all  $i$ . From  $\langle \varphi'_i, \varphi'_j \rangle = \delta_{ij}$ , one can check that  $C$  is an isometry (i.e.  $C^*C = I$ ), and since  $C$  is surjective it is a unitary operator. In other words, we also have  $CC^* = I$  which implies

$$\sum_i c_{ij} \overline{c_{ik}} = \delta_{j,k}.$$

Now, we have

$$\sum_i \|A\varphi'_i\|_2^2 = \sum_i \langle A\varphi'_i, A\varphi'_i \rangle = \sum_{i,j,k} c_{ij} \overline{c_{ik}} \langle A\varphi_j, A\varphi_k \rangle = \sum_{j,k} \delta_{jk} \langle A\varphi_j, A\varphi_k \rangle = \sum_j \|A\varphi_j\|_2^2$$

so  $\|A\|_{\text{HS}}$  does not depends on the choice of basis. We can prove the same property of  $\|A\|_{\text{tr}}$  and  $\text{tr } A$  by the same way (using the fact that  $C$  is unitary).

**Problem 9.2** NOT FINISHED

**Problem 9.3** NOT FINISHED

**Problem 9.4** NOT FINISHED

**Problem 9.5** NOT FINISHED

**Problem 9.6** NOT FINISHED

**Problem 9.7** NOT FINISHED

**Problem 9.8** NOT FINISHED

**Problem 9.9** NOT FINISHED

**Problem 9.10** NOT FINISHED

**Problem 9.11** NOT FINISHED

## 10. CHAPTER 10

**Problem 10.1** NOT FINISHED

**Problem 10.2** NOT FINISHED

**Problem 10.3** This follows from the direct computation:

$$\begin{aligned}
 & E(x, I(\sigma, \lambda)(g)\varphi, \lambda) \\
 &= \sum_{\delta \in P(F) \setminus G(F)} I(\sigma, \lambda)(g)\varphi(\delta x) e^{\langle H_P(\delta x), \lambda + \rho_P \rangle} \\
 &= \sum_{\delta \in P(F) \setminus G(F)} \varphi(\delta x g) e^{\langle H_P(\delta x g), \lambda + \rho_P \rangle} e^{-\langle H_P(\delta x), \lambda + \rho_P \rangle} e^{\langle H_P(\delta x), \lambda + \rho_P \rangle} \\
 &= \sum_{\delta \in P(F) \setminus G(F)} \varphi(\delta x g) e^{\langle H_P(\delta x g), \lambda + \rho_P \rangle} \\
 &= E(xg, \varphi, \lambda).
 \end{aligned}$$

**Problem 10.4** NOT FINISHED

**Problem 10.5** NOT FINISHED

**Problem 10.6** NOT FINISHED

**Problem 10.7** NOT FINISHED

**Problem 10.8** NOT FINISHED

**Problem 10.9** NOT FINISHED

11. CHAPTER 11

*Problem 11.1* NOT FINISHED

*Problem 11.2* NOT FINISHED

*Problem 11.3* NOT FINISHED

*Problem 11.4* NOT FINISHED

*Problem 11.5* NOT FINISHED

*Problem 11.6* NOT FINISHED

*Problem 11.7* NOT FINISHED

*Problem 11.8* NOT FINISHED

*Problem 11.9* NOT FINISHED

## 12. CHAPTER 12

**Problem 12.1** Let  $G$  be a connected Lie group and  $\phi : \text{Gal}_F \rightarrow G$  be a continuous homomorphism. Since  $\ker \phi = \phi^{-1}(1)$ , it is a closed subgroup of  $\text{Gal}_F$  and the image  $\text{im } \phi \cong \text{Gal}_F / \ker \phi$  is also a profinite group with quotient topology. Hence  $\text{im } \phi \leq G$  is a discrete compact subgroup of  $G$ , so is finite. (Note that a group is profinite if and only if it is Hausdorff, compact, and totally disconnected.)

**Problem 12.2** The (geometric) Frobenius element in  $\text{Gal}_{\mathbb{Q}}$  is defined as a restriction of inverse image of  $\text{Fr}_p \in \text{Gal}_{\mathbb{F}_p}$  via  $\text{Gal}_{\mathbb{F}_p} \leftarrow \text{Gal}_{\mathbb{Q}_p} \rightarrow \text{Gal}_{\mathbb{Q}}$ . It gives a well-defined conjugacy class in  $\text{Gal}_{\mathbb{Q}}$  and act as a  $p^{-1}$ -power map on roots of unity. Hence  $a_{\text{Fr}_p, n} = p^{-1}$  for all  $n \geq 1$  and  $\chi_{\ell}(\text{Fr}_p) = p^{-1}$ .

**Problem 12.3** NOT FINISHED

**Problem 12.4** NOT FINISHED

**Problem 12.5** NOT FINISHED

**Problem 12.6** NOT FINISHED

**Problem 12.7** Let  $V_i = \bigoplus_{n \geq 0} V_n^{(i)} \otimes \text{Sym}^n$  be the decompositions of  $V_1$  and  $V_2$  into  $\text{Sym}^n$ -isotypic components for  $i = 1, 2$ . Then that for  $V_1 \oplus V_2$  is  $\bigoplus_{n \geq 0} (V_n^{(1)} \oplus V_n^{(2)}) \otimes \text{Sym}^n$ , and the corresponding  $L$ -function becomes

$$L(s, \rho_1 \oplus \rho_2) = \prod_{n \geq 0} \det \left( 1 - (\rho_1 \oplus \rho_2)(\text{Fr}) q^{-(s+n/2)} |_{(V_n^{(1)} \oplus V_n^{(2)})_{I_F}} \right)^{-1}.$$

By choosing a basis of  $V_n^{(1)}$  and  $V_n^{(2)}$ , we can express

$$(1 - (\rho_1 \oplus \rho_2)(\text{Fr}) q^{-(s+n/2)}) |_{(V_n^{(1)} \oplus V_n^{(2)})_{I_F}}$$

as a following block diagonal matrix form

$$1 - \begin{pmatrix} \rho_1(\text{Fr}) q^{-(s+n/2)} & 0 \\ 0 & \rho_2(\text{Fr}) q^{-(s+n/2)} \end{pmatrix} = \begin{pmatrix} 1 - \rho_1(\text{Fr}) q^{-(s+n/2)} & 0 \\ 0 & 1 - \rho_2(\text{Fr}) q^{-(s+n/2)} \end{pmatrix}$$

which gives

$$\begin{aligned} & \det \left( 1 - (\rho_1 \oplus \rho_2)(\text{Fr}) q^{-(s+n/2)} |_{(V_n^{(1)} \oplus V_n^{(2)})_{I_F}} \right) \\ &= \det \left( 1 - \rho_1(\text{Fr}) q^{-(s+n/2)} |_{V_n^{(1)} I_F} \right) \det \left( 1 - \rho_2(\text{Fr}) q^{-(s+n/2)} |_{V_n^{(2)} I_F} \right) \end{aligned}$$

and this implies  $L(s, \rho_1 \oplus \rho_2) = L(s, \rho_1) L(s, \rho_2)$ .

**Problem 12.8** NOT FINISHED

**Problem 12.9** NOT FINISHED

**Problem 12.10** NOT FINISHED

**Problem 12.11** NOT FINISHED

**Problem 12.12** NOT FINISHED

## 13. CHAPTER 13

**Problem 13.1** NOT FINISHED

**Problem 13.2** NOT FINISHED

**Problem 13.3** NOT FINISHED

**Problem 13.4** See the solution for Exercise 12.1. ( $\text{Gal}_F$  is a profinite group and  $\text{GL}_n(\mathbb{C})$  is a connected Lie group.)

**Problem 13.5** NOT FINISHED

## 14. CHAPTER 14

*Problem 14.1* NOT FINISHED

*Problem 14.2* NOT FINISHED

*Problem 14.3* NOT FINISHED

*Problem 14.4* NOT FINISHED

*Problem 14.5* NOT FINISHED

*Problem 14.6* NOT FINISHED

*Problem 14.7* NOT FINISHED

*Problem 14.8* NOT FINISHED

*Problem 14.9* NOT FINISHED

## 15. CHAPTER 15

*Problem 15.1* NOT FINISHED

*Problem 15.2* NOT FINISHED

*Problem 15.3* NOT FINISHED

*Problem 15.4* NOT FINISHED

*Problem 15.5* NOT FINISHED

*Problem 15.6* NOT FINISHED

*Problem 15.7* NOT FINISHED

*Problem 15.8* NOT FINISHED

## 16. CHAPTER 16

*Problem 16.1* NOT FINISHED

*Problem 16.2* NOT FINISHED

*Problem 16.3* NOT FINISHED

*Problem 16.4* NOT FINISHED

*Problem 16.5* NOT FINISHED

*Problem 16.6* NOT FINISHED

*Problem 16.7* NOT FINISHED



## 17. CHAPTER 17

- Problem 17.1* NOT FINISHED  
*Problem 17.2* NOT FINISHED  
*Problem 17.3* NOT FINISHED  
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*Problem 17.15* NOT FINISHED  
*Problem 17.16* NOT FINISHED  
*Problem 17.17* NOT FINISHED  
*Problem 17.18* NOT FINISHED