Automorphic forms and L-functions for the unitary group*

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Introduction

Our purpose is to define and analyze L-functions attached to automorphic cusp forms on the unitary group $G = \mathrm{U}_{2,1}$ and a six-dimensional representation

$$\rho : {}^L G \to \mathrm{GL}_6(\mathbb{C})$$

of its L-group.

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The motivation for this work is three fold.

Firstly, we use these L-functions to analyze the lifting of cusp forms from $\mathrm{U}_{1,1}$ to $\mathrm{U}_{2,1}$; here the model for our work is Waldspurger's L-function theoretic characterization of the image of Shimura's map for modular forms of half-integral weight (cf. [Wald]).

A second motivation comes from the need to relate the poles of the L-functions for G, to integrals of cusp forms over cycles coming from $\mathrm{U}_{1,1}$. The prototvpe here is the recent proof of Tate's conjecture for Hilbert modular surfaces due to Harder, Langlands, and Rapaport.

Thirdly, we view this work as a special contribution to the general program of constructing local L and ε factors of Langlands type for representations of arbitrary reductive groups. In [PS1], such a program was sketched generalizing classical methods of Heeke, Rankin–Selberg, and Shimura. Related developments are discussed in [Jacquet], [Novod], [PS2], and [PS3]. For the unitary group $\mathrm{U}_{2,1}$ the present paper extends the developments initiated in [PS3].

Contents

Notation

(i) F is a field (sometimes local, somtimes a global field), E is a quadratic extension of F with Galois involution $z \mapsto \bar{z}$.

(ii) V is a 3-dimensional vector space over E, with basis $\{\ell_{-1}, \ell_0, \ell_1\}$. $(-,-)_V$ is a Hermitian form on V, with matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

with respect to $\{\ell_{-1}, \ell_0, \ell_1\}$.

(iii) $G = \mathrm{U}_{2,1} = \mathrm{U}(V)$ is the unitary group for the form $(-,-)_V$. P=parabolic subgroup stabilizing the isotropic line through $\ell_{-1} = MN$ with

$$M = \left\{ \begin{bmatrix} \delta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \bar{\delta}^{-1} \end{bmatrix} : \delta \in E^{\times}, \beta \in E^{1} = \{z : z\bar{z} = 1\} \right\}$$

and unipotent radical

$$N = \left\{ \begin{bmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{bmatrix} : z, b \in E, z + \bar{z} = -b\bar{b} \right\}.$$

The center of N is

$$Z = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \bar{z} = -z \right\}$$

※ Whittaker Models (Ordinary and Generalized)

Some kind of Whittaker model is needed in order to introduce L-functions on G.

Fix F local (not of characteristic two), and suppose (π, H_π) is an irreducible admissible representation of G. Naively, we should look for functionals on H_π which·transform under N according to a one-dimensional representation. However, since such functionals need not exist in general, and since there are irreducible representations of N which are not 1-dimensional, it is natural to pursue a more general approach.

1.1

Recall N is the maximal unipotent subgroup of G and E is a quadratic extension of F. We fix, once and for all, an element i in E such that $\bar{i} = -i$, so $\Im(z) = (z - \bar{z})/2i$. Regarding E as a 2-dimensional symplectic space over F with skew-form $\langle z_1, z_2 \rangle = \Im(z_1\overline{z_2})$ we have

$$N = \left\{ \begin{bmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{bmatrix} : z, b \in E, z + \bar{z} = -b\bar{b} \right\} \simeq H(E),$$

the Heisenberg group attached to E over F. In particular, N is non-abelian, with commutator subgroup

$$[N,N] = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = Z,$$

the center of N. The maximal abelian subgroup of N is

$$N' = \left\{ \begin{bmatrix} 1 & b & z \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \in N : b \in F \right\}.$$

1.2

The irreducible representations of the Heisenberg group, and hence those of N, are well known:

(i) σ is 1-dimensional.

In this case, σ must be trivial on

$$Z = [N,N]$$

and define a character of N/Z. So

$$N/Z \simeq \left\{ \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{bmatrix} \right\} \simeq E$$

implies σ corresponds to a character of E, i.e.

$$\sigma = \psi_N \left(\begin{bmatrix} 1 & a & z \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{bmatrix} \right) = \psi(\Im a)$$

with ψ a character of F.

(ii) σ is infinite-dimensional.

In this case (by the Stone-von Neumann uniqueness theorem), σ is completely determined by its "central" character. In particular, if

$$\sigma \left(\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \psi(\Im z) I$$

for some (additive) character ψ of F, then

$$\sigma = \rho_\psi = \mathrm{Ind}_{N'}^{N} \psi_{N'},$$

with $\psi_{N'}$ the character of (the maximal abelian subgroup) N' obtained by trivially extending ψ from Z to N'.

1.3

Definition. By a (generalized) Whittaker functional for (π, H_π) we understand N-map from N_π to some irreducible representation of (σ, L_σ) of N (possibly infinite dimensional).

1.4

Remark. The torus

$$T = \left\{ \begin{bmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar{\delta}^{-1} \end{bmatrix} : \delta \in E^\times \right\}$$

acts by conjugation on N, taking

$$\begin{bmatrix} 1 & b & z \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & \delta b & \delta\bar{\delta} z \\ 0 & 1 & -\overline{\delta b} \\ 0 & 0 & 1 \end{bmatrix}.$$

So if ψ_N denotes the 1-dimensional representation of N corresponding to the fixed character of F as in 1.2 (i), Pontrygin duality for $E \simeq N/Z$ implies that any other 1-dimensional representation is trivial or of the form

$$\psi_N^\delta(n) = \psi_N\left(\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} n \begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix}^{-1} \right)$$

for some $\delta \in E^\times$.

1.5

If σ is a one-dimensional representation of N of the form ψ_N, a given irreducible admissible representation (π, H_π) need not possess a nontrivial ψ_N-Whittaker functional $\mathcal{L}$. However, if it does, then by 1.4 it possesses a σ-Whittaker functional for any one-dimensional representation ψ_N^δ, given by the formula

$$\mathcal{L}^\delta(v) = \mathcal{L}\left(\pi\left(\begin{bmatrix}\delta & & \\ & 1 & \\ & & \bar{\delta}^{-1}\end{bmatrix}\right)v\right), \quad v \in H_\pi.$$

In this case, we call (π, H_π) non-degenerate. By a well-known theorem of Shalika and Gelfand-Kazhdan (cf. [Sha1]), the space of such σ-Whittaker functionals is one-dimensional. In particular, the corresponding Whittaker models

$$\mathcal{W}(\pi,\psi) = \{W(g) = \mathcal{L}(\pi(g)v) : v \in H_\pi\}$$

are unique.

1.6

In general, (π, H_π) is not non-degenerate, examples being provided by the Weil representations discussed in §**??**. Thus it is necessary to consider σ-Whittaker models for infinite dimensional σ as well. Such σ, however, are completely determined by their central character ψ_Z, so it is convenient to work with a slight thickening of N. More precisely, consider the stabilizer R in P of the central character ψ_Z of Z. Because $\begin{bmatrix}\delta & & \\ & \beta & \\ & & \bar{\delta}^{-1}\end{bmatrix}$ conjugates $\begin{bmatrix}1&0&z\\0&1&0\\0&0&1\end{bmatrix}$ to $\begin{bmatrix}1&0&\delta\bar{\delta}z\\0&1&0\\0&0&1\end{bmatrix}$,

$$R = \left\{\begin{bmatrix}\delta & * & * \\ 0 & \beta & * \\ 0 & 0 & \delta\end{bmatrix} \in P : \delta,\beta \in E^1\right\} \simeq (E^1 \times E^1) \ltimes N.$$

In particular, each irreducible infinite dimensional representation ρ_ψ of N extends to a like representation ρ_ψ^α of R with α a character of $E^1 \times E^1$.

Theorem (Existence and Uniqueness of Generalized Whittaker Models: [PS3)**.**] Any (π, H_π) possesses a ρ_ψ^α-Whittaker functional for some choice of ρ_ψ^α; moreover, the space of such functionals is at most one dimensional.

We shall discuss this result in more detail in the global context of §**??**.

※ Some Fourier Expansions and Hypercuspidality

Now F is a global field not of characteristic 2, and π is an automorphic cuspidal representation of $G(\mathbb{A})$ which we suppose realized in some subspace of cusp forms H_π in $L^2_0(G(F)\backslash G(\mathbb{A}))$. To attach an L-function to π, it is useful to take forms f in H_π and examine their Fourier coefficients along the maximal unipotent subgroup N. When such coefficients are non-zero, π is non-degenerate, and we are led back to the local Whittaker models of 1.5; in this case, we can (and eventually do) introduce L-functions using Jacquet's generalization of the "Rankin–Selberg method".

On the other hand, if these Fourier coefficients represent zero, then π is hypercuspidal; in this case, looking at Fourier expansions along Z will bring us back to the generalized Whittaker models of 1.6, and ultimately allow us to introduce an L-function for π using the so-called "Shimura method".

Henceforth, let us fix a non-trivial character ψ of $F\backslash\mathbb{A}$, and define characters ψ_N and ψ_Z of $N = N(\mathbb{A})$ and $Z = Z(\mathbb{A})$ by

$$\psi_N\left(\begin{bmatrix} 1 & a & z \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{bmatrix}\right) = \psi(\Im a)$$

and

$$\psi_Z\left(\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \psi(\Im z).$$

2.1

Fix f in H_π. To obatin a Fourier expansion of f "along N", we introduce the familiar ψ-th coefficient

$$W_f^\psi(g) = \int_{N(F)\backslash N(\mathbb{A})} f(ng)\overline{\psi_N(n)}\mathrm{d}n.$$

The transitivity of $T(\mathbb{A}) = \left\{\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix}\right\}$ acting on $Z(\mathbb{A})\backslash N(\mathbb{A})$ implies - as in the local theory - that

$$\begin{aligned} W_f^{\psi^\delta}(g) &= \int_{N(F)\backslash N(\mathbb{A})} f(ng)\overline{\psi_N^\delta(n)}\mathrm{d}n \\ &= \int_{N(F)\backslash N(\mathbb{A})} f(ng)\psi_N\left(\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} n \begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix}^{-1}\right)\mathrm{d}n \\ &= W_f^\psi\left(\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} g\right). \end{aligned}$$

In other words, knowing W_f^{ψ} determines $W_f^{\psi^\delta}$ for all $\psi^\delta, \delta \in E^\times$.

However, through $N(F)\backslash N(\mathbb{A})$ is compact, it is not abelian; to obtain a nice Fourier expansion, we must bring into play the compact abelian group $N(F)Z(\mathbb{A})\backslash N(\mathbb{A})$.

2.2

We compute

$$\begin{aligned} W_f^{\psi}(g) &= \int_{N(F)\backslash N(\mathbb{A})} f(ng)\overline{\psi_N(n)}\mathrm{d}n \\ &= \int_{N(F)Z(\mathbb{A})\backslash N(\mathbb{A})} \int_{Z(F)\backslash Z(\mathbb{A})} f(nzg)\mathrm{d}z\overline{\psi_N(n)}\mathrm{d}n \\ &= \int_{N(F)Z(\mathbb{A})\backslash N(\mathbb{A})} f_{00}(ng)\overline{\psi_N(n)}\mathrm{d}n \end{aligned}$$

with

$$f_{00}(g) = \int_{Z(F)\backslash Z(\mathbb{A})} f(zg)\mathrm{d}z \tag{1}$$

the constant term (in the Fourier expansion) of $f(zg)$ along Z.

Fix g in $G(\mathbb{A})$. As a function on the compact abelian group $N(F)Z(\mathbb{A})\backslash N(\mathbb{A})$, $f_{00}(ng)$ has a Fourier expansion

$$f_{00}(g) = \sum_{\delta\in E^\times} W_F^{\psi^\delta}(g) + \int_{N(F)Z(\mathbb{A})\backslash N(\mathbb{A})} f_{00}(n'g)\mathrm{d}n'. \tag{2}$$

Indeed, the last paragraph says precisely that $W_f^{\psi}(g)$ is the ψ-th Fourier coefficient of $f_{00}(ng)$ along $Z\backslash N \simeq E$. Moreover, the constant term is actually zero since f cuspidal implies

$$\begin{aligned} \int_{N(F)Z(\mathbb{A})\backslash N(\mathbb{A})} f_{00}(n'g)\mathrm{d}n' &= \int_{N(F)Z(\mathbb{A})\backslash N(\mathbb{A})} \int_{Z(F)\backslash Z(\mathbb{A})} f(zn'g)\mathrm{d}z\mathrm{d}n' \\ &= \int_{N(F)\backslash N(\mathbb{A})} f(ng)\mathrm{d}n = 0. \end{aligned}$$

2.3

Let $\mathcal{W}(\pi,\psi)$ denote the space of ψ-th Fourier coefficients $W_f^{\psi}(g)$, $f \in H_\pi$.

Proposition 2.1. The vanishing or nonvanishing of $\mathcal{W}(\pi,\psi)$ is independent of ψ; in particular, $\mathcal{W}(\pi,\psi)=0$ if and only if

$$f_{00}(g)=0 \quad \forall f\in H_\pi.$$

Proof. According to (1) and (2),

$$\begin{aligned} f_{00}(g) &= \sum_{\delta\in E^\times} W_f^{\psi^\delta}(g) \\ &= \sum_{\delta\in E^\times} W_f^{\psi}\left(\begin{bmatrix}\delta & & \\ & 1 & \\ & & \bar{\delta}^{-1}\end{bmatrix} g\right) \end{aligned} \tag{3}$$

with

$$W_f^\psi(g)=\int_{N(F)Z(\mathbb{A})\backslash N(\mathbb{A})} f_{00}(ng)\overline{\psi_N(n)}\mathrm{d}n.$$

□

2.4

Definition 2.2. We call (π,H_π) hypercuspidal if $f\in H_\pi$ implies $f_{00}=0$.

Proposition 2.3. Let $L^2_{0,1}$ be the orthogonal complement in L^2_0 of all cusp forms. Then

(i) $L^2_{0,1}$ has multiplicity 1.

(ii) each $(\pi,H_\pi)\subset L^2_{0,1}$ is non-degenerate, and

(iii) for any $f\in H_\pi\subset L^2_{0,1}$, the constant term

$$f_{00}(g)=\sum_{\delta\in E^\times} W_f^{\psi^\delta}(g)$$

completely determines f.

Proof. We start with (iii). Suppose f and f' are in H_π such that $f_{00}=f'_{00}$. Then $(f-f')_{00}=0$ implies $f-f'=0$ (by the hypothesis $H_\pi\in L^2_{0,1}$). This proves (iii). To prove (i) and (ii), suppose there exists $H'_\pi\subset L^2_{0,1}$ such that the right regular representation restricted to H'_π again realizes π. If $f\in H_\pi$ and $f'\in H'_\pi$, then

$$f_{00}(g)=\sum_{\delta\in E^\times} W_f^{\psi}\left(\begin{bmatrix}\delta & & \\ & 1 & \\ & & \bar{\delta}^{-1}\end{bmatrix} g\right) \tag{4}$$

and

$$f'_{00}(g) = \sum_{\delta \in E^\times} W^{\psi}_{f'}\left(\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} g\right).$$

Note that each W^{ψ}_{f} (or $W^{\psi}_{f'}$) satisfies the condition $W^{\psi}_{f}(ng) = \psi(n) W^{\psi}_{f}(g)$, $n \in N$, i.e. the spaces (W^{ψ}_{f}) and $W^{\psi}_{f'}$ afford Whittaker models for π. But by §2.3 these spaces are nonzero (which proves (ii)) and by the uniqueness of Whittaker models quoted in §1.5, these spaces coincide. Thus by (4), the spaces (f_{00}) and (f'_{00}) coincide; by (iii) the spaces $H_\pi = (f)$ and $H'_\pi = (f')$ also coincide, thereby proving (i). □

2.5

Remark. (i) It is conjectured (c.f. [Flicker]) that multiplicity one holds for the entire space of cusp forms; however, at the present time, we can prove this only for $L^2_{0,1}$.

(ii) Hypercuspforms do exist; again, the examples are provided by the Weil representation discussed in §**??**.

(iii) Although $\mathcal{W}(\pi, \psi) \neq \{0\}$ implies π non-degenerate (in the sense that an abstract functional exists), the converse is not clear. Indeed, the work of [Wald] indicates that characterizing the nonvanishing of a space of Fourier coefficients is a delicate matter.

※ L-functions à la Rankin–Selberg–Jacquet

We are now ready to attach (global) L-functions to (non-degenerate) cuspidal representations π of $G(\mathbb{A})$, The method used goes back to [Rankin] and [Selberg] who used it to analytically continue the convolution of Dirichlet series corresponding to classical holomorphic modular forms. The reformulation of their construction in the language of representation theory was carried out in detail by [Jacquet], for $GL_2 \times GL_2$, and by [PS 1] in general (but without details or exp1icit computation). In this Section (and the next), we carry out the construction for $G = U_{2,1}$.

3.1

Recall V is a 3-dimensional vector space over E/F and $(-,-)_V$ is a hermitian form on V whose matrix with respect to the basis $\{\ell_{-1}, \ell_0, \ell_1\}$ is $\left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]$.

Let $H \subset G$ denote the stabilizer of the anisotropic line $\langle \ell_0 \rangle$. Then H also preserves the orthocomplement of $\langle \ell_0 \rangle$, namely

$$W = \langle \ell_{-1}, \ell_1 \rangle.$$

Regarding W as a 2-dimensional Hermitian space (whose matrix with respect to the basis $\{\ell_{-1}, \ell_1\}$ is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$), we have $H \simeq \mathrm{U}(W) \simeq \mathrm{U}_{1,1}$ via the embedding

$$\begin{bmatrix} * & * \\ * & * \end{bmatrix} \mapsto \begin{bmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{bmatrix}.$$

Let B denote the standard maximal parabolic (Borel) subgroup

$$\left\{ \begin{bmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & \bar{\alpha}^{-1} \end{bmatrix} \right\} \text{ of } H$$

so that $B \simeq T \ltimes Z$, with T the torus

$$T = \left\{ \begin{bmatrix} \alpha & & \\ & 1 & \\ & & \bar{\alpha}^{-1} \end{bmatrix} : \alpha \in E^\times \right\}$$

and Z is the center of N.

3.2

Given an automorphic cuspidal realization (π, H_π) of $G(\mathbb{A})$, and $f \in H_\pi$, we shall analyze a global zeta-integral of the form

$$L^\mu(f, F, s) = \int_{H(F)\backslash H(\mathbb{A})} f(h) E^\mu(h, F, s) \mathrm{d}h. \tag{5}$$

First we need to describe the (as yet) undefined terms μ, F, E etc.

Fix a (not necessarily unitary) character μ of the idèle class group $E^\times \backslash \mathbb{A}_E^\times$ of E, and $s \in \mathbb{C}$, define a character ω_μ^s of the Borel subgroup B by the formula

$$\omega_\mu^s \left(\begin{bmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & \bar{\alpha}^{-1} \end{bmatrix} \right) = \mu(\alpha) |\alpha|_E^s, \quad \alpha \in \mathbb{A}_E^\times. \tag{6}$$

Fixing an arbitrary Schwartz-Bruhat function ϕ in the space $\mathcal{S}(W(\mathbb{A}_E))$, set

$$F_\phi(h) = \int_{\mathbb{A}_E^\times} (h \cdot \phi)(t\ell_{-1})\mu(t)|t|_E^s \mathrm{d}^\times t \tag{7}$$

where $(h \cdot \phi)(w) = \phi(h^{-1} \cdot w)$, and $h \cdot w$ denotes the natural action of $H(\mathbb{A})$ on $W(\mathbb{A}) \subset V(\mathbb{A})$; as usual, this integral converges for $\mathfrak{R}(s)$ sufficiently large, and continues meromorphically to define a function of h on $H(\mathbb{A})$ for all s in $\mathbb{C}$. Note

$$F(bh) = \omega_\mu^s(b)F(h) \text{ for } b \in B(\mathbb{A}), h \in H(\mathbb{A}).$$

Finally, the Eisenstein series $E^\mu(h, F, s)$ is defined by the familiar series

$$E^\mu(h, F, s) = \sum_{\gamma \in B(F)\backslash H(F)} F(\gamma h);$$

it converges initially only for $\mathfrak{R}(s)$ large, but the Selberg–Langlands theory of Eisenstein series implies that the function it defines continues meromorphically in s and defines an automorphic form on $H(\mathbb{A})$.

3.3

Remark. (i) Because $E(h, F, s)$ is a automorphic form on H, and the restriction of the cusp form f from $G(\mathbb{A})$ to $H(\mathbb{A})$ is still rapidly decreasing, the integral defining $L^\mu(f, F, s)$ (cf. (5)) is convergent. The resulting function of s - the global zeta-function of f - has poles which can arise only from poles of $E(h, F, s)$.

(ii) The function $E(h, F, s)$ is essentially the familiar GL_2-Eisenstein series discussed (for example) in [Jacquet]. Indeed, SL_2 is isomorphic to a subgroup of H, namely $\mathrm{SU}_{2,1}$, and $H \simeq \mathrm{SU}_{2,1} \rtimes \mathrm{U}_1$, where $\mathrm{U}_1 \simeq \{\left[\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right] : \alpha \in E^1\}$ and $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right]$ acts on $\mathrm{SU}_{2,1}$ by conjugation. Thus the restriction of $E(h, F, s)$ to $\mathrm{SU}_{2,1}$ is the familiar Eisenstein series on $\mathrm{SL}_2(\mathbb{A})$ with functional equation and non-trivial residue given by the constant function (arising from the pole at $s = 1$). At the $\mathrm{U}_{2,1}$ level, these residues become proportional to $\mu(\det h)$, and the functional equation relates $E(h, F, s)$ to a "partially Fourier transformed" Eisenstein series at $1 - s$.

3.4

From the theory above, we conclude that $L^\mu(f, F, s)$ is meromorphic in $\mathbb{C}$ with funcitonal equation relating values at s and $1 - s$; more significantly for the sequel, the only possible residues of $L^\mu(f, F, s)$ are proportional to

$$\int_{H(F)\backslash H(\mathbb{A})} f(h)\mu(\det h)\mathrm{d}h. \tag{8}$$

In particular, if this last integral vanishes, then the zeta-function $L^\mu(f, F, s)$ is entire.

Regarding $H(F)\backslash H(\mathbb{A})$ as an (algebraic) cycle in $G(F)\backslash G(\mathbb{A})$, we (ultimately) obtain the following statement: The existence of a pole for $L^\mu(f, F, s)$ (and ultimately the L-function $L(s, \pi, \mu)$) is related to the non-vanishing integral of f in H_π (suitably tensored with μ) over the cycle coming from $U_{1,1}$.

We shall return to these considerations in §**??**. For the moment, we content ourselves with a factorization of $L^\mu(f, F, s)$ into local integrals.

3.5

Proposition.

$$L^\mu(f, F, s) = \int_{Z(\mathbb{A})\backslash H(\mathbb{A})} W_f^\psi(h)F(h)\mathrm{d}h.$$

Proof. From the definition of the series E,

$$\begin{aligned} L^\mu(f, F, s) &= \int_{H(F)\backslash H(\mathbb{A})} \sum_{B(F)\backslash H(F)} f(\gamma h)F(\gamma h)\mathrm{d}h \\ &= \int_{B(F)\backslash H(\mathbb{A})} f(h)F(h)\mathrm{d}h. \end{aligned}$$

Recall our subgroups

$$Z = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \subset H = \left\{ \begin{bmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{bmatrix} \right\}$$

Since $F(h)$ is invariant by $Z(\mathbb{A})$ (c.f. (6)),

$$\begin{aligned} L^\mu(f, F, s) &= \int_{B(F)Z(\mathbb{A})\backslash H(\mathbb{A})} F(h) \left(\int_{Z(F)\backslash Z(\mathbb{A})} f(zh)\mathrm{d}z \right) \mathrm{d}h \\ &= \int_{B(F)Z(\mathbb{A})\backslash H(\mathbb{A})} F(h) f_{00}(h)\mathrm{d}h \end{aligned}$$

Now recall the Fourier expansion

$$f_{00}(h) = \sum_{\delta \in E^\times} W_f^\psi \left(\begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} h \right),$$

c.f. (1). Since

$$B_F = \left\{ \begin{bmatrix} \delta & & z \\ 0 & 1 & 0 \\ 0 & 0 & \bar{\delta}^{-1} \end{bmatrix} : \delta \in E^\times, z \in E \right\}$$

we have

$$Z(\mathbb{A})\backslash B(F)Z(\mathbb{A}) = \left\{ \begin{bmatrix} \delta & & \\ & 1 & \\ & & \bar{\delta}^{-1} \end{bmatrix} : \delta \in E^\times \right\}$$

and therefore

$$\begin{aligned} L^\mu(f, F, s) &= \int_{B(F)Z(\mathbb{A})\backslash H(\mathbb{A})} \left(\sum_{b \in Z(\mathbb{A})\backslash B(F)Z(\mathbb{A})} W_f^\psi(bh) \right) F(h) \mathrm{d}h \\ &= \int_{Z(\mathbb{A})\backslash H(\mathbb{A})} W_f^\psi(h) F(h) \mathrm{d}h \end{aligned}$$

as was to be shown. □

Remark. (i) We defined $L^\mu(f, F, s)$ for any $f \in H_\pi$ without assuming H_π orthogonal to all hypercuspforms. However, this last proposition shows tha t$L^\mu(f, F, s)$ is identically 0 if $\mathcal{W}(\pi, \psi) = \{0\}$, i.e. if f is a hypercuspform. This is wy the Rankin–Selberg method fails for arbitrary π, and why (in §**??**) we need ot use Shimura's method.

(ii) The significance of this Proposition is that it allows to factor $L^\mu(f, F, s)$ into local zeta-integrals, one for each place v of F. Note that whenever v splits in E, i.e. whenever $E \otimes_F F_v$ splits as the direct sum of two fields E_{w_1} and E_{w_2} (isomorphic to F_v), we have

$$G_v = G(F_v) \simeq \mathrm{GL}_3(F_v).$$