Low Dimensional Complex Spin Groups

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Spin group is a universal cover of an orthogonal group. In this note, we will show that low dimensional complex spin groups, $\mathrm{Spin}(3, \mathbb{C}), \mathrm{Spin}(4, \mathbb{C}), \mathrm{Spin}(5, \mathbb{C})$ and $\mathrm{Spin}(6, \mathbb{C})$ are isomorphic to familiar groups.

1 $\mathrm{Spin}(3, \mathbb{C}) \simeq \mathrm{SL}(2, \mathbb{C})$

First, we are going to prove the following theorem:

Theorem 1.

$$\mathrm{PSL}(2, \mathbb{C}) \simeq \mathrm{SO}(3, \mathbb{C})$$

Using this Theorem, we can prove that $\mathrm{Spin}(3, \mathbb{C})$ is isomorphic to a well-known group.

Corollary 1.

$$\mathrm{Spin}(3, \mathbb{C}) \simeq \mathrm{SL}(2, \mathbb{C})$$

Proof. First, we have to show that $\mathrm{SL}(2, \mathbb{C})$ is a simply connected group. To prove this, consider a natural action $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}^2 \backslash \{0\}$. Then this is a transitive action and the stabilizer subgroup of $\mathbf{e}_1 = (1, 0)^T$ is

$$\mathrm{Stab}(\mathbf{e}_1) = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\} \simeq \mathbb{C}.$$

Hence we have a diffeomorphism

$$\mathrm{SL}(2, \mathbb{C}) / \mathrm{Stab}(\mathbf{e}_1) \simeq \mathbb{C}^2 \backslash \{0\}.$$

We know that $\mathbb{C}^2 \backslash \{0\}$ is homotopic to S^3, which is simply connected. Also, since $\mathrm{Stab}(\mathbf{e}_1) \simeq \mathbb{C}$ is contractible, $\mathrm{SL}(2, \mathbb{C})$ is homotopic to S^3, so is simply connected. Now we have a 2-cover $\mathrm{SL}(2, \mathbb{C}) \twoheadrightarrow \mathrm{PSL}(2, \mathbb{C}) \simeq \mathrm{SO}(3, \mathbb{C})$, so $\mathrm{SL}(2, \mathbb{C})$ is a universal cover of $\mathrm{SO}(3, \mathbb{C})$. $\square$

So, how we can prove the Theorem 1? Actually, there is a one line proof:

proof of the Theorem 1. The map $\Phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,\mathbb{C})$ defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}(a^2-b^2-c^2+d^2) & -(ab-cd) & \frac{i}{2}(a^2+b^2-c^2-d^2) \\ -(ac-bd) & ad+bc & -i(ac+bd) \\ -\frac{i}{2}(a^2-b^2+c^2-d^2) & i(ab+cd) & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix}$$

is a surjective homomorphism whose kernel is the center of $\mathrm{SL}(2,\mathbb{C})$. □

Our main question is: where does the isomorphism come from? This map seems very complicated and unnatural. It is not even trivial that ϕ is a group homomorphism and surjective. We will construct such homomorphism by using the *adjoint* action of a Lie group on a Lie algebra.

Let $\mathfrak{sl}(2,\mathbb{C})$ be a Lie algebra of $\mathrm{SL}(2,\mathbb{C})$. This is a 3-dimensional complex vector space with a basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $G = \mathrm{SL}(2,\mathbb{C})$. Consider a left G-action on G itself by a conjugation, i.e. $g \in G$ acts on G by $h \mapsto ghg^{-1}$. Then we have an induced action of G on its Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ by conjugation $v \mapsto gvg^{-1}$ again. Hence we get an adjoint representation $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$ of G, and we will analyze this map more rigorously.

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$

Then its action on E, H, F is given by

$$\begin{aligned} E &\mapsto gEg^{-1} = \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix} \\ H &\mapsto gHg^{-1} = \begin{pmatrix} ad+bc & -2ab \\ 2cd & -(ad+bc) \end{pmatrix} \\ F &\mapsto gFg^{-1} = \begin{pmatrix} bd & -b^2 \\ d^2 & -bd \end{pmatrix} \end{aligned}$$

so the automorphism $\phi_g : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C})$ corresponds to the g can be represented as a 3 by 3 matrix

$$\phi_g = \begin{pmatrix} a^2 & -2ab & -b^2 \\ -ac & ad+bc & bd \\ -c^2 & 2cd & d^2 \end{pmatrix}$$

with respect to the ordered basis $\{E, H, F\}$ of $\mathfrak{sl}(2,\mathbb{C})$. We have $\det(\phi_g) = 1$: if we expand determinant by the second row, we have

$$\begin{aligned} \det(\phi_g) &= ac(-2abd^2 + 2cdb^2) + (ad+bc)(a^2d^2 - b^2c^2) - bd(2a^2cd - 2c^2ab) \\ &= -2abcd + (ad+bc)^2 - 2abcd = (ad-bc)^2 = 1. \end{aligned}$$

Now define an inner product on $\mathfrak{sl}(2,\mathbb{C})$ by $\langle v, w\rangle = \mathrm{Tr}(vw)$, which is clearly $\mathbb{C}$-bilinear. The associated quadratic form is

$$Q_1(v) = \langle v, v\rangle = \mathrm{Tr}(v^2) = 2(y^2 + xz)$$

for $v = xE + yH + zF$, and this is a nondegenerated quadratic form. Clearly, this inner product and the quadratic form is invariant under the G-action:

$$\langle \phi_g(v), \phi_g(w)\rangle = \mathrm{Tr}(gvg^{-1}gwg^{-1}) = \mathrm{Tr}(gvwg^{-1}) = \mathrm{Tr}(vw) = \langle v, w\rangle.$$

Hence image of the map $g \mapsto \phi_g$ lies in $\mathrm{SO}(Q_1)$, special orthogonal group which preserves the quadratic form Q_1. If we denote Q_2 as a standard quadratic form defined as $Q_2(xE + yH + zF) = x^2 + y^2 + z^2$, a basis change matrix

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

relates two quadratic forms as $Q_2(v) = Q_1(Bv)$. Since $\mathrm{SO}(Q_2) = \mathrm{SO}(3,\mathbb{C})$, we have an isomorphism $\mathrm{SO}(Q_1) \simeq \mathrm{SO}(3,\mathbb{C})$ given by $A \mapsto B^{-1}AB$. Explicit computation gives us that

$$\begin{aligned} B^{-1}\phi_g B &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & 0 \\ -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a^2 & -2ab & -b^2 \\ -ac & ad+bc & bd \\ -c^2 & 2cd & d^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(a^2 - b^2 - c^2 + d^2) & -(ab - cd) & \frac{i}{2}(a^2 + b^2 - c^2 - d^2) \\ -(ac - bd) & ad + bc & -i(ac + bd) \\ -\frac{i}{2}(a^2 - b^2 + c^2 - d^2) & i(ab + cd) & \frac{1}{2}(a^2 + b^2 + c^2 + d^2) \end{pmatrix} \end{aligned}$$

which gives the previous homomorphism $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,\mathbb{C})$.

Since $\mathrm{SO}(Q_1) \simeq \mathrm{SO}(3,\mathbb{C})$, $\ker\Phi$ is same as $\ker\phi$. If $\phi_g = \mathrm{id}$, then we should have $b = c = 0$ and $a^2 = d^2 = ad = 1$. So the only possible choice is $(a, d) = (1, 1)$ or $(-1, -1)$, which corresponds to elements in the center. Hence we have an induced map $\mathrm{PSL}(2,\mathbb{C}) \to \mathrm{SO}(3,\mathbb{C})$, which is an embedding.

Now we only need to show that this map is an isomorphism. First, both have dimension 3: since $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ is a finite cover, both group have a same dimension, which is 3 since $\dim_{\mathbb{C}} \mathfrak{sl}(2,\mathbb{C}) = 3$. For $\mathrm{SO}(3,\mathbb{C})$, one can check that $\mathfrak{so}(3,\mathbb{C}) = \{X \in M_{3\times 3}(\mathbb{C}) : X^T + X = 0\}$, and this space also has dimension 3 over $\mathbb{C}$. We need the following lemma:

Lemma 1. *Let G be a conneted Lie group of dimension n and H be a Lie subgroup of G with same dimension. Then $G = H$.*

Proof. Since $H \subseteq G$ is a Lie subgroup, G/H has a smooth manifold structure. Since $\dim G = \dim H$, $\dim(G/H) = 0$ and thus G/H is a 0-dimensional smooth manifold, i.e. a set of points endowed wit a discrete topology. Since $G = \coprod_{g\in G/H} gH$, G is not connected if $G \neq H$. $\square$

By lemma, it is enough to show that $\mathrm{SO}(3,\mathbb{C})$ is connected. Actually, we can prove more general result:

Lemma 2. $\mathrm{SO}(n,\mathbb{C})$ *is connected for* $n \geq 1$.

Proof. Clearly, $\mathrm{SO}(1,\mathbb{C}) = \{1\}$ is connected. We will use induction on n. Assume that $\mathrm{SO}(n-1,\mathbb{C})$ is connected for some $n \geq 2$. Let

$$X_n = \{(x_1, \dots, x_n) \in \mathbb{C}^n \,:\, x_1^2 + \cdots + x_n^2 = 1\}.$$

$g \in \mathrm{SO}(n,\mathbb{C})$ acts on X as $v \mapsto gv$. If $ge_1 = e_1$ for $e_1 = (1, 0, \dots, 0)^T$, since $g \in \mathrm{SO}(n,\mathbb{C})$, g should has a form

$$g = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & g' \end{pmatrix}$$

where $g' \in \mathrm{SO}(n-1,\mathbb{C})$. Thus $\mathrm{Stab}(e_1) \simeq \mathrm{SO}(n-1,\mathbb{C})$. Also, we can show that the action is transitive, hence we have $\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n-1,\mathbb{C}) \simeq X_n$. It is known that X_n is connected, so $\mathrm{SO}(n,\mathbb{C})$ is also connected since both $\mathrm{SO}(n-1,\mathbb{C})$ and X_n are connected. (First one is because of the induction hypothesis and the Lemma 3 in the Appendix. For the second one, in general, for any irreducible polynomial $f(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$, zero set of f in $\mathbb{C}^n$ is connected with respect to the usual topology on $\mathbb{C}^n$, which is hard to prove in general.) $\square$

Thus we get $\mathrm{PSL}(2,\mathbb{C}) \simeq \mathrm{SO}(3,\mathbb{C})$ with an explicit isomorphism. Since $\mathrm{SL}(2,\mathbb{C})$ is a double cover of $\mathrm{PSL}(2,\mathbb{C})$ and is simply connected, we just showed that the complex spin group $\mathrm{Spin}(3,\mathbb{C})$ is $\mathrm{SL}(2,\mathbb{C})$.

2 $\mathrm{Spin}(4,\mathbb{C}) \simeq \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$

By the similar way, we can also show the following:

Theorem 2.

$$(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))/\langle(-I,-I)\rangle \simeq \mathrm{SO}(4,\mathbb{C}).$$

Corollary 2.

$$\mathrm{Spin}(4,\mathbb{C}) \simeq \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}).$$

Proof. By the Theorem 2, there exists a surjective homomorphism $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(4,\mathbb{C})$, which is a double cover of $\mathrm{SO}(4,\mathbb{C})$. Since $\mathrm{SL}(2,\mathbb{C})$ is simply connected, $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ is also simply connected. $\square$

To prove the Theorem 2, we may find some appropriate $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ on a 4-dimensional $\mathbb{C}$-vector space. Consider the action of the group on a space $\mathrm{M}_{2\times 2}(\mathbb{C})$ (the space of complex 2×2 matrices) defined as

$$(g,h) \cdot v := gvh^{-1}, \quad (g,h) \in \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}),\ v \in \mathrm{M}_{2\times 2}(\mathbb{C}).$$

Then this is a well-defined action on $\mathrm{M}_{2\times 2}(\mathbb{C})$. With respect to the basis $\{\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{21}, \mathbf{e}_{22}\}$, the map $\phi_{g,h} : \mathrm{M}_{2\times 2}(\mathbb{C}) \to \mathrm{M}_{2\times 2}(\mathbb{C})$ corresponds to a 4×4 matrix

$$A_{g,h} = \begin{pmatrix} a\delta & -a\gamma & b\delta & -b\gamma \\ -a\beta & a\alpha & -b\beta & b\alpha \\ c\delta & -c\gamma & d\delta & -d\gamma \\ -c\beta & c\alpha & -d\beta & d\alpha \end{pmatrix}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$

We can check that $\det(A_{g,h}) = (ad - bc)(\alpha\delta - \beta\gamma) = 1$, so the image of

$$\phi : \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathrm{M}_{2\times 2}(\mathbb{C}))$$

is contained in $\mathrm{SL}(\mathrm{M}_{2\times 2}(\mathbb{C}))$. Now we need a bilinear map and a (non-degenerate) quadratic form on $\mathrm{M}_{2\times 2}(\mathbb{C})$ so that $\phi_{g,h}$ preserves the quadratic form. Define $Q : \mathrm{M}_{2\times 2}(\mathbb{C}) \to \mathbb{C}$ as the *determinant*, i.e. $Q(v) = \det(v)$. Then we have

$$Q(\phi_{g,h}(v)) = \det(gvh^{-1}) = \det(g)\det(v)\det(h)^{-1} = \det(v) = Q(v),$$

so it is preserved by the action. The corresponding inner product can be defined as

$$\langle v, w\rangle = \frac{1}{2}\left(Q(v+w) - Q(v) - Q(w)\right) = \frac{1}{2}(\det(v+w) - \det(v) - \det(w))$$

which satisfies $\langle v, v\rangle = Q(v)$. Simple computation shows that the inner product is given as

$$\langle v, w\rangle = \frac{1}{2}(x_1w_2 - y_1z_2 + x_2w_1 - y_2z_1), \quad v = \begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix},\ w = \begin{pmatrix} x_2 & y_2 \\ z_2 & w_2 \end{pmatrix} \in \mathrm{M}_{2\times 2}(\mathbb{C}).$$

Hence we obtain a map $\phi : \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(4,\mathbb{C})$. If $(g,h) \in \ker\phi$, then $gvh^{-1} = v$ for any $v \in \mathrm{M}_{2\times 2}(\mathbb{C})$. If we put $v = h$, we get $g = h$ and $gvg^{-1} = v$. Thus $g \in Z(\mathrm{M}_{2\times 2}(\mathbb{C})) = \mathbb{C}I_2$. Since $\det(g) = 1$, we should have $g = h = \pm I_2$, and ϕ induces an injection

$$\phi : (\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))/\langle(-I,-I)\rangle \to \mathrm{SO}(4,\mathbb{C}).$$

One can check that both groups has dimension 6 (by computing dimensions of Lie algebra of each groups), so ϕ is an isomorphism by the Lemma 1 and 2.

3 $\mathrm{Spin}(6,\mathbb{C}) \simeq \mathrm{SL}(4,\mathbb{C})$

We are going to see $\mathrm{Spin}(6,\mathbb{C})$ first since $\mathrm{Spin}(5,\mathbb{C})$ uses similar technic but slightly more difficult. For $\mathrm{Spin}(6,\mathbb{C})$, we have the following isomorphism

Theorem 3.

$$\mathrm{SL}(4,\mathbb{C})/\langle -I\rangle \simeq \mathrm{SO}(6,\mathbb{C}).$$

Corollary 3.
$$\mathrm{Spin}(6,\mathbb{C}) \simeq \mathrm{SL}(4,\mathbb{C}).$$

Proof. More generally, we will prove that $\mathrm{SL}(n,\mathbb{C})$ is simply connected for $n \geq 2$. The natural action of $\mathrm{SL}(n,\mathbb{C})$ on $\mathbb{C}^n\backslash\{0\}$ is transitive, so we have a diffeomorphism
$$\mathrm{SL}(n,\mathbb{C})/\mathrm{Stab}(\mathbf{e}_1) \simeq \mathbb{C}^n\backslash\{0\}.$$
We know that $\mathbb{C}^n\backslash\{0\}$ is homotopic to S^{2n-1}, which is simply connected. Also, we have
$$\mathrm{Stab}(\mathbf{e}_1) = \left\{ \begin{pmatrix} 1 & \mathbf{v}^T \\ \mathbf{0} & g \end{pmatrix} : g \in \mathrm{SL}(n-1,\mathbb{C}) \right\} \simeq \mathrm{SL}(n-1,\mathbb{C}) \rtimes \mathbb{C}^{n-1}$$
which is diffeomorphic to $\mathrm{SL}(n-1,\mathbb{C}) \times \mathbb{C}^{n-1}$, so is homotopic to $\mathrm{SL}(n-1,\mathbb{C})$. Thus by induction with the Lemma 4, $\mathrm{SL}(n,\mathbb{C})$ is simply connected for any $n \geq 2$. $\square$

What is a 6-dimensional $\mathbb{C}$-vector space that $\mathrm{SL}(4,\mathbb{C})$ can act on? $\mathrm{SL}(4,\mathbb{C})$ naturally acts on $\mathbb{C}^4$, and this induces an action on $\wedge^2\mathbb{C}^4$, which has a dimension $\binom{4}{2} = 6$. The action is defined as
$$g(v_1 \wedge v_2) = gv_1 \wedge gv_2, \quad g \in \mathrm{SL}(4,\mathbb{C}),\ v_1, v_2 \in \mathbb{C}^4.$$
It is easy to prove that this action has determinant 1. Recall that for any finite d-dimensional $\mathbb{C}$-vector space V, we have the determinant map $\det : \mathrm{GL}(V) \to \mathbb{C}^\times$ defined as $g \mapsto \det(g) = \wedge^d g$, where $\wedge^d g$ is the induced map on $\wedge^d V$ which is isomorphic to $\mathbb{C}$. Now we define $\mathrm{SL}(V) = \ker(\det)$. In our case, for $g \in \mathrm{SL}(4,\mathbb{C}) = \mathrm{SL}(\mathbb{C}^4)$, we have to check that $\wedge^2 g \in \mathrm{SL}(\wedge^2\mathbb{C}^4)$, which trivially follows from $\wedge^4 g \in \ker(\det : \mathrm{GL}(\wedge^4\mathbb{C}^4) \to \mathbb{C}^\times)$: then $\wedge^4 g$ acts on $\wedge^4\mathbb{C}^4$ trivially, and then $\wedge^6(\wedge^2 g) = \wedge^3(\wedge^4 g)$ also acts on $\wedge^6(\wedge^2\mathbb{C}^4)$ trivially.

For a bilinear pairing on $\wedge^2\mathbb{C}^4$, define it as
$$\langle v_1 \wedge v_2, w_1 \wedge w_2 \rangle = v_1 \wedge v_2 \wedge w_1 \wedge w_2 \in \wedge^4\mathbb{C}^4 \simeq \mathbb{C}, \quad v_1, v_2, w_1, w_2 \in \mathbb{C}^4$$
This is a symmetric pairing on $\wedge^2\mathbb{C}^4$ since $(13)(24) \in S_4$ is an even permutation. Actually, this pairing can be considered as a determinant of the matrix with column vectors v_1, v_2, w_1, w_2, and linearly extends to $\wedge^2\mathbb{C}^4$. One can check that this is a nondegenerate paring on $\wedge^2\mathbb{C}^4$, so we get a map
$$\phi : \mathrm{SL}(4,\mathbb{C}) \to \mathrm{SO}(Q, \wedge^2\mathbb{C}^4) \simeq \mathrm{SO}(6,\mathbb{C}).$$
where the nondegenerate quadratic form Q on $\wedge^2\mathbb{C}^4$ corresponds to the above pairing is
$$Q(v) = \langle v, v \rangle = 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}), \quad v = \sum_{i<j} a_{ij}(e_i \wedge e_j).$$

To show that the kernel of ϕ is $\langle -I \rangle$, assume that $g = (b_{ij})_{1 \le i,j \le 4} \in SL(4, \mathbb{C})$ trivially acts on $\wedge^2 \mathbb{C}^4$. From $ge_1 \wedge ge_2 = e_1 \wedge e_2$, we get, for example, the following equations:

$$\begin{aligned} b_{11}b_{22} - b_{21}b_{12} &= 1 \\ b_{11}b_{32} - b_{31}b_{12} &= 0 \\ b_{21}b_{32} - b_{31}b_{22} &= 0 \end{aligned}$$

by comparing coefficients of each basis elements $e_i \wedge e_j$ $(i < j)$. Then

$$b_{31} = b_{31}b_{11}b_{22} - b_{31}b_{21}b_{12} = b_{21}b_{32}b_{11} - b_{11}b_{21}b_{32} = 0.$$

By the similar way, we can prove that all off-diagonal entries of g are zero, and $b_{ii}b_{jj} = 1$ for all $i < j$ gives $b_{11} = b_{22} = b_{33} = b_{44} = \pm 1$.

4 $\mathrm{Spin}(5, \mathbb{C}) \simeq \mathrm{Sp}(4, \mathbb{C})$

This is the most difficult one among $3, 4, 5$ and 6, since we need to consider symplectic structure. The *standard symplectic form* on $\mathbb{C}^4$ is a bilinear map $\omega : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$ which is anti-symmetic and nondegenerate. More explicitly, it is defined as

$$\omega(v_1, v_2) := v_1^T J v_2, \quad J = \begin{pmatrix} O & I_2 \\ -I_2 & O \end{pmatrix}, \ v_1, v_2 \in \mathbb{C}^4.$$

Since ω is anti-symmetric, we have an induced map $\omega : \wedge^2 \mathbb{C}^4 \to \mathbb{C}$. The standard action of $\mathrm{Sp}(4, \mathbb{C})$ on $\mathbb{C}^4$ induces the action on the space $\wedge^2 \mathbb{C}^4$ as before, and its dual action on $V' = \mathrm{Hom}_{\mathbb{C}}(\wedge^2 \mathbb{C}^4, \mathbb{C})$ defined as

$$(g \cdot f)(v_1 \wedge v_2) = f(g^{-1}v_1 \wedge g^{-1}v_2), \quad f : \wedge^2 \mathbb{C}^4 \to \mathbb{C}.$$

We can easily check that ω is fixed by the action (in some sense, $\mathrm{Sp}(4, \mathbb{C})$ is *defined* to be the group that fixes ω), and in fact, it is a unique such element in V' up to constant multiplication. This would be the heart of the our following proof.

Theorem 4.

$$\mathrm{PSp}(4, \mathbb{C}) = \mathrm{Sp}(4, \mathbb{C}) / \langle -I \rangle \simeq \mathrm{SO}(5, \mathbb{C})$$

Corollary 4.

$$\mathrm{Spin}(5, \mathbb{C}) \simeq \mathrm{Sp}(4, \mathbb{C}).$$

Proof. It is enough to show that $\mathrm{Sp}(4, \mathbb{C})$ is simply connected. We use the argument of Eric Wofsey in [2]. Consider the standard action of $\mathrm{Sp}(4, \mathbb{C})$ on $\mathbb{C}^4 \backslash \{0\}$. This action is transitive: choose any nonzero vector $v = (a_{11}, a_{21}, c_{11}, c_{21})^T \in \mathbb{C}^4$. Note that the matrix $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ is in $\mathrm{Sp}(4, \mathbb{C})$ if and only if

$$A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = I.$$

Assume that $(a_{11}, a_{21}) \neq (0,0)$. Then we can find $a_{21}, a_{22} \in \mathbb{C}$ s.t. $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Then we can also find $c_{12}, c_{22} \in \mathbb{C}$ s.t.

$$a_{11}c_{12} + a_{21}c_{22} = a_{12}c_{11} + a_{22}c_{21},$$

which implies $A^T C = C^T A$ for $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right)$. Now take $D = A^{-T} = (A^{-1})^T$ and $B = O$, then we have $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(4, \mathbb{C})$. If $(a_{11}, a_{21}) = (0,0)$, we have $(c_{11}, c_{21}) \neq (0,0)$ and do the similar thing with $D = O$.

Now consider the diagonal action of $\mathrm{Sp}(4, \mathbb{C})$ on $(\mathbb{C}^4 \backslash \{0\}) \times (\mathbb{C}^4 \backslash \{0\})$. We will figure out what is the orbit and the stabilizer of the element $(e_1, e_3) \in (\mathbb{C}^4 \backslash \{0\}) \times (\mathbb{C}^4 \backslash \{0\})$. First, assume that $g \in \mathrm{Sp}(4, \mathbb{C})$ fixes e_1 and e_3. Since ω is preserved under the action, g must also fix their orthogonal complement with respect to the symplectic form, which is $\mathbb{C}e_2 \oplus \mathbb{C}e_4$. So we can see that the stabilizer group of (e_1, e_3) is isomorphic to $\mathrm{Sp}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})$, which is simply connected.

For the orbit of (e_1, e_3), we just saw that $\mathrm{Sp}(4, \mathbb{C})$ acts on $\mathbb{C}^4 \backslash \{0\}$ transitively, so we can map e_1 to the any vector in $\mathbb{C}^4 \backslash \{0\}$. Once we choose the image of e_1, then e_3 may goes to some vector that lies on the affine space

$$S = \{v \in \mathbb{C}^4 \backslash \{0\} \,:\, \omega(ge_1, v) = \omega(e_1, e_3) = 1\}$$

which is just $\mathbb{C}^3$ topologically. Hence our orbit space is a fiber bundle over $\mathbb{C}^4 \backslash \{0\}$ with fiber $\mathbb{C}^3$. Since both $\mathbb{C}^4 \backslash \{0\}$ and $\mathbb{C}^3$ are simply connected, the orbit space should be simply connected, too.

So both stabilizer and the orbit of (e_1, e_3) are simply connected, and so $\mathrm{Sp}(4, \mathbb{C})$ too by the Lemma 4. □

Now consider the non-degenerate bilinear paring on $\wedge^2 \mathbb{C}^4$, defined as

$$\langle v_1 \wedge v_2, v_3 \wedge v_4 \rangle = v_1 \wedge v_2 \wedge v_3 \wedge v_4 \in \wedge^4 \mathbb{C}^4 \simeq \mathbb{C}, \quad v_i \in \mathbb{C}^4 \text{ for } 1 \leq i \leq 4.$$

Then we have an isomorphism

$$\wedge^2 \mathbb{C}^4 \simeq \mathrm{Hom}_{\mathbb{C}}(\wedge^2 \mathbb{C}^4, \mathbb{C}), \quad v_1 \wedge v_2 \mapsto \langle v_1 \wedge v_2, - \rangle$$

which is a $\mathrm{Sp}(4, \mathbb{C})$-equivariant isomorphism, from the fact that $\mathrm{Sp}(4, \mathbb{C}) \subseteq \mathrm{SL}(4, \mathbb{C})$. (This was proven in the previous section.) Hence there exists a nonzero vector v_ω in $\wedge^2 \mathbb{C}^4$ which is fixed by the action that corresponds to ω, and we can compute it explicitly - $\{e_i \wedge e_j\}_{1 \leq i < j \leq 4}$ is a basis of $\wedge^2 \mathbb{C}^4$, and if we write dual basis of $e_i \wedge e_j$ as $(e_i \wedge e_j)^*$, then $\omega = (e_1 \wedge e_3)^* + (e_2 \wedge e_4)^*$ and the corresponding element that fixed by $\mathrm{Sp}(4, \mathbb{C})$-action is

$$v_\omega = -e_1 \wedge e_3 - e_2 \wedge e_4.$$

So we have an induced action on the 5-dimensional vector space $V = \wedge^2 \mathbb{C}^4 / \langle v_\omega \rangle$, which is the desired space that $\mathrm{Sp}(4, \mathbb{C})$ acts on. Now define a bilinear paring on V as

$$\langle v_1 \wedge v_2, v_3 \wedge v_4 \rangle_V := \omega(v_1 \wedge v_3)\omega(v_2 \wedge v_4) - \omega(v_1 \wedge v_4)\omega(v_2 \wedge v_3).$$

We can check that this is a well-defined on V by checking that $\langle v_\omega, e_i \wedge e_j \rangle_V = 0$ for any $1 \leq i < j \leq 4$. Also, this is a $\mathrm{Sp}(4, \mathbb{C})$-invariant bilinear paring since ω does. By the Lemma 5 and the previous section, the action on V has determinant 1, hence the image of the representation $\mathrm{Sp}(4, \mathbb{C}) \hookrightarrow \mathrm{GL}(V)$ lies in $\mathrm{SO}(5, \mathbb{C})$.

To prove that the kernel of the map is $\langle -I \rangle$, assume that $g \in \mathrm{Sp}(4, \mathbb{C})$ is in the kernel, so that $gv_1 \wedge gv_2 = v_1 \wedge v_2$ for all $v_1 \wedge v_2 \in V$. This means that $gv_1 \wedge gv_2 = v_1 \wedge v_2 + \lambda v_\omega$ for some $\lambda \in \mathbb{C}$. Now define $\lambda_{ij} \in \mathbb{C}$ as $ge_i \wedge ge_j = e_i \wedge e_j + \lambda_{ij}(e_1 \wedge e_3 + e_2 \wedge e_4)$. Let $g = (b_{ij})_{1 \leq i,j \leq 4}$. For $(i, j) = (1, 3)$, we get the following equations

$$\begin{aligned}
b_{11}b_{23} - b_{21}b_{13} &= 0 \\
b_{11}b_{33} - b_{13}b_{31} &= \lambda_{13} + 1 \\
b_{11}b_{43} - b_{41}b_{13} &= 0 \\
b_{21}b_{33} - b_{31}b_{23} &= 0 \\
b_{21}b_{43} - b_{41}b_{23} &= \lambda_{13} \\
b_{31}b_{43} - b_{41}b_{33} &= 0
\end{aligned}$$

and by using the same trick as before, we get

$$\begin{aligned}
(\lambda_{13} + 1)(b_{21}, b_{23}, b_{41}, b_{43}) &= (0, 0, 0, 0) \\
\lambda_{13}(b_{11}, b_{13}, b_{31}, b_{33}) &= (0, 0, 0, 0).
\end{aligned}$$

Now we can prove that $\lambda_{13} = 0$ - if not, we must have $b_{11} = b_{31} = b_{13} = b_{33} = 0$, and this gives a contradiction when we do the similar computation for $(i, j) = (1, 2)$ and $(i, j) = (1, 4)$. (I'm not going to write down all the equations since margin is too small to contain.) Hence we must have $\lambda_{13} = 0$ and $b_{21} = b_{23} = b_{41} = b_{43} = 0$. Similar argument shows that the off-diagonal elements should be all zero, and the diagonal entries should satisfy $b_{ii}b_{jj} = 1$, which implies $b_{11} = b_{22} = b_{33} = b_{44} = \pm 1$.

4.1 Appendix

Lemma 3. *Let G be a Lie group and H be a closed Lie subgroup. If both H and G/H are connected, then G is also connected.*

Proof. Assume that G is not connected. Then there exists a proper clopen subset U of G. Since H is connected, U is a union of some cosets of H, and then U/H is a proper clopen subset of G/H, which contradicts to the connectedness of G/H. □

Lemma 4. *Let G be a connected Lie group and H be a closed Lie subgroup. If both H and G/H are simply connected, then G is also simply connected.*

Proof. The canonical projection $G \to G/H$ is an H-fibration, so we obtain a long exact sequence of homotopy groups

$$\cdots \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \cdots.$$

Since both $\pi_1(H)$ and $\pi_1(G/H)$ are trivial, so is $\pi_1(G)$. $\square$

Lemma 5. *Let V be a d-dimensional $\mathbb{C}$-vector space and $\phi : V \to V$ be the invertible linear map, i.e. $\phi \in \mathrm{GL}(V)$. Assume that there exists a nonzero vector $v_0 \in V$ which is fixed by ϕ. If $\phi \in \mathrm{SL}(V)$, then the induced map $\overline{\phi} : V/\langle v_0 \rangle \to V/\langle v_0 \rangle$ also satisfies $\overline{\phi} \in \mathrm{SL}(V/\langle v_0 \rangle)$.*

Proof. Consider a basis $\mathcal{B} = \{v_0, v_1, \ldots, v_{d-1}\}$ which contains $v_0 \neq 0$. Then $\overline{\mathcal{B}} = \{\overline{v_1}, \ldots, \overline{v_{d-1}}\}$ is a basis of $V/\langle v_0 \rangle$. Since $\phi \in \mathrm{SL}(V)$, it acts on $\wedge^d V$ trivially, i.e.

$$\phi(v_0) \wedge \phi(v_1) \wedge \cdots \wedge \phi(v_{d-1}) = v_0 \wedge v_1 \wedge \cdots \wedge v_{d-1}.$$

Since $\phi(v_0) = v_0$, we have

$$v_0 \wedge (\phi(v_1) \wedge \cdots \wedge \phi(v_{d-1}) - v_1 \wedge \cdots \wedge v_{d-1}) = 0,$$

which implies that

$$\phi(v_1) \wedge \cdots \wedge \phi(v_{d-1}) - v_1 \wedge \cdots \wedge v_{d-1} = \sum_{i=1}^{d-1} c_i (v_1 \wedge \cdots \wedge \widehat{v_i} \wedge v_{d-1})$$

for some $c_1, \ldots, c_{d-1} \in \mathbb{C}$. Since the image of RHS in $\wedge^{d-1}(V/\langle v_0 \rangle)$ is 0, we get Now we have to show that $\overline{\phi(v_1)} \wedge \cdots \wedge \overline{\phi(v_{d-1})} = \overline{v_1} \wedge \cdots \wedge \overline{v_{d-1}}$ and so $\overline{\phi} \in \mathrm{SL}(V/\langle v_0 \rangle)$. $\square$

References

[1] P. Deligne, *Notes on Spinors*, Quantum fields and strings: a course for mathematicians 1 (1999):2.

[2] Eric Wofsey, Answer to the MSE question ”$\mathrm{Sp}(4, \mathbb{C})$ *is simply connected”*, `https://math.stackexchange.com/a/2931022/350772`.