LANGLANDS FUNCTORIALITY CONJECTURE - A SURVEY

SEEWOO LEE

1. Introduction

Conjecture 1.1 (Langlands functoriality conjecture). *Let G and G' be reductive groups over a global field F.*

This is an introductory note on Langlands functoriality conjecture view towards classical examples. Here is a list of topics we are going to study:

(1) Automorphic induction
(2) Base change
(3) Rankin-Selberg product
(4) Symmetric power lifting and Selberg's 1/4 conjecture
(5) Jacquet-Langlands correspondence
(6) Theta correspondence and Howe duality

2. Automorphic induction

2.1. **From GL_1/K to $\mathrm{GL}_2/\mathbb{Q}$.** Let K be a quadratic field (over $\mathbb{Q}$) and ξ be a Hecke character for K. By Hecke and Maass, it was proven that one can associate GL_2 automorphic forms. Hecke attached modular forms to Hecke characters for imaginary quadratic fields, and Maass attached Maass forms to those for real quadratic fields. More precisely, they proved the following:

Theorem 2.1 (Hecke)**.** *Let ξ (mod $\mathfrak{m}$) be a primitive Hecke character of $K = \mathbb{Q}(\sqrt{D})$ of discriminant $D < 0$ such that*

$$\xi((a)) = \left(\frac{a}{|a|}\right)^u \quad \text{if } a \equiv 1 \pmod{\mathfrak{m}}$$

where u is a non-negative integer. Then

$$f(z) = \sum_{\mathfrak{a}\subset\mathcal{O}_K} \xi(\mathfrak{a})(N\mathfrak{a})^{\frac{u}{2}} e^{2\pi(N\mathfrak{a})z}$$

is a modular form of weight $k = u+1$ and level $N = |D|\cdot N\mathfrak{m}$ with Nebentypus $\chi \pmod{N}$, which is a Dirichlet character defined as

$$\chi(n) = \chi_D(n)\xi((n)) \quad n \in \mathbb{Z}.$$

Theorem 2.2 (Maass)**.** *Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of discriminant $D > 0$ and $\xi \pmod{\mathfrak{m}}$ a Hecke character such that*

$$\xi((a)) = \frac{a}{|a|} \quad \text{if } a \equiv 1 \pmod{\mathfrak{m}}$$

or

$$\xi((a)) = \frac{a'}{|a'|} \quad \text{if } a \equiv 1 \pmod{\mathfrak{m}}$$

where a' is a conjugate of a over $\mathbb{Q}$. Then

$$u(z) = \sum_{\mathfrak{a}\in\mathcal{O}_K} \xi(\mathfrak{a}) y^{\frac{1}{2}} e^{2\pi i(N\mathfrak{a})z}, \quad z = x + yi$$

is a Maass form of level N, eigenvalue $1/4$, and a Nebentypus $\chi \pmod{N}$ for $N = D\cdot N\mathfrak{m}$.

Both theorem can be proved using converse theorems for L-functions. By showing that the L-function attached to Hecke character ξ satisfies suitable functional equations, converse theorem shows that the L function should coincides with one comes from modular forms or Maass forms.

2.2. **From GL_n/K to GL_{rn}/F.** In view of Langlands functoriality conjecture, Hecke and Maass' results can be considered as a special case when $G = \mathrm{GL}_1/K$ and $G' = \mathrm{GL}_2/\mathbb{Q}$. Automorphic induction, which is a vast generalization of this, is a functoriality from GL_n/K to GL_{rn}/F, where K/F is a degree r extension. On Galois side, this actually corresponds to the *induction* of Galois representation of $G_K = \mathrm{Gal}(\overline{K}/K)$ to $G_F = \mathrm{Gal}(\overline{F}/F)$. In other words, if one has a GL_n/K automorphic representation π with corresponding Galois representation $\sigma = \sigma(\pi) : G_K \to \mathrm{GL}_n(\mathbb{C})$, then automorphic induction predicts the existence of GL_{rn}/F automorphic representation Π that corresponds to the Galois representation

$$\Sigma = \mathrm{Ind}_{G_K}^{G_F}\sigma : G_F \to \mathrm{GL}_{rn}(\mathbb{C})$$

via Langlands correspondence.

Conjecture 2.1 (Automorphic induction)**.** *Let K/F be a degree r extension of number fields. Let π be a GL_n/K automorphic representation. Then there exists a GL_{rn}/F automorphic representation $\Pi = \mathrm{AI}_K^F(\pi)$ such that*

(1) *the Galois representations*

$$\sigma = \sigma(\pi) : G_K \to \mathrm{GL}_n(\mathbb{C}), \quad \Sigma = \Sigma(\Pi) : G_F \to \mathrm{GL}_{rn}(\mathbb{C})$$

corresponds to π and Π via Langlands correspondence satisfies

$$\Sigma \simeq \mathrm{Ind}_{G_K}^{G_F} \sigma.$$

(2) *Local L-functions of π and Π are related as*

$$L(s, \Pi_v) = \prod_{w|v} L(s, \pi_w)$$

for all but finitely many v.

This is open in general, but proven to be true for some cases. We give a sketch of proofs for known cases.

2.2.1. *Local automorphic induction (Henniart-Herb).* In [9], Henniart and Herb proved automorphic induction over local fields of characteristic zero, i.e. for p-adic fields. To give a precise statement, we first define some terminologies. Let F be a characteristic zero local field and E be a finite extension of F of degree d.

2.2.2. *Cyclic Galois extension of prime degree (Arthur-Clozel).*

Theorem 2.3 (Arthur-Clozel, [1])**.**

3. Base change

3.1. **Doi-Naganuma lifting.** In [5], Doi and Naganuma constructed a lifting from the space of elliptic modular forms to the space of Hilbert modular forms of the same (parallel) weight. The proved the following theorem:

Theorem 3.1 (Doi-Naganuma [5])**.** *Let p be a prime such that the real quadratic field $F = \mathbb{Q}(\sqrt{p})$ has class number 1, and let ϕ_p be the Dirichlet character associated to F. Let $f(z) = \sum_{n \geq 1} a_n q^n \in S_k(p, \phi_p)$ be a weight k Hecke eigenform of level p with Nebentypus ϕ_p. Then there exists a Hilbert modular form $h := \mathrm{DN}(f)$ with respect to $\mathrm{GL}_2(\mathcal{O}_F)$, the* Doi-Naganuma lift *of f, that satisfies*

(1) *h is also an Hecke eigenform of weight k,*

(2) $$L(s, \mathrm{DN}(f)) = L(s, f)L(s, f^\rho)$$

where $f^\rho(z) := \sum_{n \geq 1} \overline{a_n} q^n$ is a complex conjugate of $f(z)$,

(3) *has a Fourier expansion*

$$h(z_1, z_2) = -\frac{B_k}{2k}\tilde{a}_0 + \sum_{\substack{\nu \in \mathfrak{d}_F^{-1} \\ \nu >> 0}} \sum_{d \mid \nu} d^{k-1} \tilde{a}_{p\nu\nu'/d^2} q_1^{\nu} q_2^{\nu'}$$

where B_k denotes the k-th Bernoulli number, ν' is a conjugate of ν, and $q_j = e^{2\pi i z_j}$.

4. Rankin-Selberg product

4.1. **Rankin-Selberg convolution of modular forms.** Let f, g be two holomorphic cusp forms of weight k and level 1. Assume that two forms have Fourier expansions

$$f(z) = \sum_{n\geq 1} a_n e^{2\pi i n}, \quad g(z) = \sum_{n\geq 1} b_n e^{2\pi i n}.$$

If f, g are Hecke eigenforms, then their L-functions admit euler products as

$$L(s,f) = \sum_{n\geq 1}\frac{a_n}{n^s} = \prod_p \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}$$
$$L(s,g) = \sum_{n\geq 1}\frac{b_n}{n^s} = \prod_p \frac{1}{(1-\alpha'_p p^{-s})(1-\beta'_p p^{-s})}$$

where $\alpha_p+\beta_p = a_p$, $\alpha'_p+\beta'_p = b_p$, and $\alpha_p\beta_p = \alpha'_p\beta'_p = p^{k-1}$ for all p. Rankin (1939) and Selberg (1940) independently studied the *convolution* of two L-series attached to f and g, which is

$$\begin{aligned} L(s, f\times g) &= \sum_{n\geq 1}\frac{a_n\overline{b_n}}{n^s} \\ &= \prod_p \frac{1}{(1-\alpha_p\alpha'_p p^{-s})(1-\alpha_p\beta'_p p^{-s})(1-\beta_p\alpha'_p p^{-s})(1-\beta_p\beta'_p p^{-s})} \end{aligned}$$

and studied its analytic properties. They proved that the new L-function also satisfy similar properties as original L-functions $L(s,f)$ and $L(s,g)$: it admits a meromorphic continuation, bounded on vertical strips, and satisfy a functional equation.

Theorem 4.1 (Rankin-Selberg convolution)**.** *Let f, g be two holomorphic cusp eigenforms of weight k on $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Let*

$$\Lambda(s, f\times g) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)\zeta(2s-2k+2)L(s, f\times g)$$

be a completed L-function. Then $\Lambda(s, f\times g)$, which is originally defined for large $\Re(s)$, admits a meromorphic continuation to all s except for at most simple poles at $s = k$ and $k-1$. Also, it satisfies a functional equation

$$\Lambda(s, f\times g) = \Lambda(2k-1-s, f\times g).$$

Essence of the proof is using real-analytic Eisenstein series with *unfolding trick.* For $s\in\mathbb{C}$, define a real-analytic Eisenstein series $E_s(z)$ as

$$E_s(z) := \sum_{\gamma\in P\backslash \mathrm{SL}(2,\mathbb{Z})} \Im(\gamma z)^s$$

where $P = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}\subset\Gamma$ is a standard parabolic subgroup of Γ. Clearly, this is a Γ-invariant function, and it converges for $\Re(s) > 1$. Also, using $\Delta y^s = s(1-s)y^s$, one can show that $E_s(z)$ is also a Maass form with eigenvalue $s(1-s)$ (but not a cusp form). By computing its Fourier expansion, we can see that $E_s(z)$, as a function in s, satisfies the functional equation

$$\xi(2s)E_s(z) = \xi(2-2s)E_{1-s}(z)$$

where $\xi(s)$ is the completed zeta function

$$\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

that satisfies $\xi(s) = \xi(1-s)$ for all s.

Proposition 4.1.

$$\langle f \cdot E_s, g\rangle = (4\pi)^{-(s+2k-1)}\Gamma(s+2k-1)\sum_{n\geq 1} L(s+2k-1, f\times g)$$

where $\langle -, -\rangle$ is the Petersson inner product.

Proof. The idea is to unfold the integral. If φ is a P-invariant function on $\mathfrak{H}$, then Fubini's theorem gives

$$\int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\backslash\Gamma} \varphi(\gamma z)\frac{dxdy}{y^2} = \int_{P\backslash\mathfrak{H}} \varphi(z)\frac{dxdy}{y^2} = \int_0^\infty \int_0^1 \varphi(z)\frac{dxdy}{y^2}$$

(the fundamental domain of $P\backslash\mathfrak{H}$ is $\{z = x+iy \in \mathfrak{H} : 0 \leq x < 1\}$.) Once we apply this for $\varphi(z) = y^s f(z)\overline{g(z)}y^{2k}$, we get

$$\begin{aligned}
\langle f \cdot E_s, g\rangle &= \int_0^\infty \int_0^1 y^s f(z)\overline{g(z)}y^{2k}\frac{dxdy}{y^2} \\
&= \sum_{m,n\geq 1} a_m\overline{b_n}y^{s+2k-1}e^{-2\pi(m+n)y}\left(\int_0^1 e^{2\pi i(m-n)x}dx\right)\frac{dy}{y} \\
&= \sum_{n\geq 1} a_n\overline{b_n}\int_0^\infty y^{s+2k-1}e^{-4\pi n y}\frac{dy}{y} \\
&= (4\pi)^{-(s+2k-1)}\Gamma(s+2k-1)L(s+2k-1, f\times g).
\end{aligned}$$

□

4.2. **Modularity of** $\mathrm{GL}(2)\times\mathrm{GL}(2)$**.** It is also possible to construct Rankin-Selberg L-function attached to two Maass cusp forms with similar properties. In general, for given automorphic representations π_1, π_2 on $\mathrm{GL}(2)$, one can define Rankin-Selberg L-function $L(s, \pi_1\times\pi_2)$. According to the philosophy of Langlands, there should exists a $\mathrm{GL}(4)$ automorphic representation whose L-function is $L(s, \pi_1\times\pi_2)$. This is proven by Ramakrishnan in 2000.

Theorem 4.2 (Ramakrishnan, [16])**.** *Let π_1, π_2 be automorphic forms on $\mathrm{GL}(2,\mathbb{A})$. Then there exists an automorphic representation $\pi_1\boxtimes\pi_2$ on $\mathrm{GL}(4,\mathbb{A})$ whose L-function equals the Rankin-Selberg L-function, i.e*

$$L(s, \pi_1\boxtimes\pi_2) = L(s, \pi_1\times\pi_2).$$

As a corollary of Theorem 4.2, he proved multiplicity one result for $\mathrm{SL}(2)$. This was conjectured by Labesse and Langlands before [12].

Theorem 4.3 (Ramakrishnan, [16])**.** *Multiplicity one theorem holds for* $\mathrm{SL}(2)$*. More precisely, any smooth irreducible admissible representation of* $\mathrm{SL}(2,\mathbb{A})$ *occurs with multiplicity at most one in the space of cusp forms* $L_0^2(\mathrm{SL}(2,F)\backslash\mathrm{SL}(2,\mathbb{A}_F))$.

In the context of modular forms, this implies the following. Let f, g be cusp forms of level N and M respectively. Assume that, for all but finitely many p, we have

$$a_p^2 = b_p^2$$

where a_p (resp. b_p) is the p-th Fourier coefficient of f (resp. g). Then there exists a quadratic Dirichlet character χ such that

$$a_p = \chi(p) b_p$$

for all but finitely many p. Also, if N, M are in addition square-free, then $\chi = 1$ and so $f = g$ by strong multiplicity one.

Proof of Theorem 4.3 goes as follows. In [12], the authors proved that the multiplicity one theorem for SL(2) holds if one can show the following.

Theorem 4.4 (Ramakrishnan, [16]). *If two automorphic representation π, π' on* $\mathrm{GL}(2, \mathbb{A}_F)$ *satisfy* $\mathrm{Ad}(\pi) \simeq \mathrm{Ad}(\pi')$*, then $\pi' \simeq \pi \otimes \chi$ for some idele class character χ of F. Here* Ad *is the adjoint lift from* GL(2) *to* GL(3) [7].

He first show this when at least one of π or π' is *dihedral* (i.e. has a form of $\mathrm{AI}_K^F(\mu)$ for some quadratic extension K of F and idele class character μ of K). If both π, π' are not dihedral, then he proved that $\pi \boxtimes \pi'$ is not cuspidal by analyzing poles of L-functions at $s = 1$. which shows that $\pi' \simeq \pi \otimes \chi$ for some χ by cuspidality criterion that is also proved by him in loc. cit.

4.3. $\mathrm{GL}(n) \times \mathrm{GL}(n)$.

5. Symmetric power lifting

Automorphic forms on GL(2) are often *classified* into two kinds of objects: *modular forms* and *Maass forms*[1]. These functions oftenly considered as a starting point for studying automorphic forms and representations for GL(n) and other groups. However, there are not many references for GL(3).

5.1. Automorphic forms on $\mathrm{GL}(3,\mathbb{R})$.

We first introduce the theory of automorphic forms on GL(3) (We follow the Bump's book [4]). We only consider the level 1 automorphic forms. Before we start, let's revisit the GL(2). Modular forms and Maass forms are certain functions defined on the complex upper half plane $\mathfrak{H}$, and one can lift the functions as a function on $\mathrm{GL}(2,\mathbb{R})$ by viewing $\mathfrak{H}$ as a symmetric space

$$\mathfrak{H} \simeq \mathrm{GL}(2,\mathbb{R})/Z(\mathbb{R})\mathrm{O}(2).$$

Here $Z(\mathbb{R}) \simeq \mathbb{R}^\times$ is the center of $\mathrm{GL}(2,\mathbb{R})$ and O(2) is a group of orthogonal matrices. The above isomorphism holds since $\mathrm{GL}(2,\mathbb{R})$ acts on $\mathbb{H}$ transitively and the stabilizer of i is $Z(\mathbb{R})\mathrm{O}(2)$. To develop a theory of automorphic forms on GL(3), it is natural to consider them as a function defined on the symmetric space

$$\mathfrak{H}_3 := \mathrm{GL}(3,\mathbb{R})/Z(\mathbb{R})\mathrm{O}(3).$$

Using Iwasawa decomposition, each coset has a unique representation of the form

$$\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}, \quad y_1, y_2 > 0.$$

Especially, the space is parametrized with 5 real variables and has a real dimension 5, so we can't expect any *holomorphic* automorphic form over GL(3), in constrast to the GL(2) case. Also, we have an involution ι on $\mathfrak{H}_3$ defined as

$$\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -x_1 & x_1x_2 - x_3 \\ 0 & 1 & -x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}.$$

What about the Fourier expansion of GL(3) automorphic forms? In case of GL(2), the algebra of $\mathrm{GL}(2,\mathbb{R})$-invariant differential operators on $\mathfrak{h}_2$ is isomorphic to a polynomial ring of single variable, generated by the following hyperbolic Laplacian

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

A 1-periodic function $f(z) = \sum_{n\geq 0} a_n(y)e^{2\pi i n x}$ become an eigenfunction with respect to Δ if the coefficients $a_n(y)$ satisfy certain degree 2 linear differential equations. More precisely, when $\Delta f = \left(\frac{1}{4} - \nu^2\right) f$, the n-th coefficient $a_n(y)$ satisfies

$$y^2\frac{\partial^2}{\partial y^2}a_n(y) + \left(\frac{1}{4} - \nu^2 - 4\pi^2 n^2 y\right)a_n(y) = 0.$$

Among two linearly independent solutions, only one satisfies the required growth condition (the other one grows exponentially), which can be expressed with a Bessel function of second kind:

$$a_n(y) = c_n\sqrt{y}K_\nu(2\pi|n|y), \quad K_\nu(y) := \frac{1}{2}\int_0^\infty e^{\frac{y(t+t^{-1})}{2}} t^\nu \frac{dt}{t}.$$

[1] and constant functions.

For GL(3), the algebra of $\mathrm{GL}(3,\mathbb{R})$-invariant differential operators on $\mathfrak{h}_3$ is a polynomial ring in *two* variables, with two specific generators Δ_1, Δ_2. Then the automorphic forms of $\mathrm{GL}(3,\mathbb{R})$ would be defined as functions that are eigenforms with respect to these two operators. Then the coefficients of the Fourier expansion (which will be defined explicitly later) of the automorphic forms would satisfy specific differential equations. In fact, for given λ and μ, there are 6 linearly independent functions that are

(1) eigenfunctions with respect to Δ_1, Δ_2, i.e.
$$\Delta_1 F = \lambda F$$
$$\Delta_2 F = \mu F$$
(2) and satisfies the equation
$$F\left(\begin{pmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{pmatrix}\tau\right) = e(x_1+x_2)F(\tau)$$
for all τ and $x_1, x_2, x_3 \in \mathbb{R}$, where $e(x) := \exp(2\pi i x)$.

Among these 6 solutions, only one of them *decays rapidly*, which is the appropriate substitute of $K_\nu(y)$ for GL(3) (This is multiplicity one theorem for GL(3)). It can be written as an inverse Mellin transform of a certain 2-variable function $V(s_1, s_2)$,
$$\begin{aligned}W(y_1,y_2) &= W_{\nu_1,\nu_2}(y_1,y_2)\\ &= \frac{1}{4}\frac{1}{(2\pi i)^2}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty} V(s_1,s_2)(\pi y_1)^{1-s_1}(\pi y_2)^{1-s_2}ds_1 ds_2\end{aligned}$$
where
$$V(s_1,s_2) = \frac{\Gamma\left(\frac{s_1+\alpha}{2}\right)\Gamma\left(\frac{s_1+\beta}{2}\right)\Gamma\left(\frac{s_1+\gamma}{2}\right)\Gamma\left(\frac{s_2-\alpha}{2}\right)\Gamma\left(\frac{s_2-\beta}{2}\right)\Gamma\left(\frac{s_2-\gamma}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)}.$$
Here α, β, γ are auxillary parameters satisfy
$$\begin{aligned}\alpha &= -\nu_1 - 2\nu_2 + 1\\ \beta &= -\nu_1 + \nu_2\\ \gamma &= 2\nu_1 + \nu_2 - 1\\ \lambda &= -1 - \alpha\beta - \beta\gamma - \gamma\alpha\\ \mu &= -\alpha\beta\gamma.\end{aligned}$$
The Whittaker function $W(y_1, y_2)$ also can be written as an integral of Bessel functions
$$W_{\nu_1,\nu_2}(y_1,y_2) = 4y_1^{1-\frac{\beta}{2}}y_2^{1+\frac{\beta}{2}}\int_0^\infty K_{\frac{\gamma-\alpha}{2}}(2\pi y_2\sqrt{1+u^{-2}})K_{\frac{\gamma-\alpha}{2}}(2\pi y_1\sqrt{1+u^2})u^{\frac{-3\beta}{2}}\frac{du}{u}.$$
Definition 5.1 (Automorphic form on $\mathrm{GL}(3,\mathbb{R})$)**.** *Let $\nu_1, \nu_2 \in \mathbb{C}$. An automorphic form of type (ν_1, ν_2) on $\mathrm{GL}(3,\mathbb{R})$ is a function $\phi : \mathfrak{H}_3 \to \mathbb{C}$ such that*

(1) *$\phi(g\tau) = \phi(\tau)$ for all $g \in \mathrm{GL}(3,\mathbb{Z})$ and $\tau \in \mathrm{GL}_3(\mathbb{R})$.*
(2) *ϕ is an eigenfunction of Δ_1, Δ_2 with eigenvalues λ, μ defined above,*

(3) *there exists* n_1, n_2 *such that*

$$y_1^{n_1} y_2^{n_2} \phi\left(\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right)$$

is bounded on the subset of $\mathfrak{H}_2$ *determined by* $y_1, y_2 > 1$.

In addition, for all $\tau \in \mathfrak{H}_3$*, if*

$$\int_0^1 \int_0^1 \phi\left(\begin{pmatrix} 1 & & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} \tau\right) d\xi_1 d\xi_3 = 0$$

and

$$\int_0^1 \int_0^1 \phi\left(\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \\ & & 1 \end{pmatrix} \tau\right) d\xi_1 d\xi_3 = 0$$

then ϕ *is called* cusp form.

Note that, for a given automorphic form ϕ of type (ν_1, ν_2), the *dual* form $\tilde{\phi}(\tau) := \phi({}^{\iota}\tau)$ is also an automorphic form, but of type (ν_2, ν_1).

Any ϕ has a Fourier expansion with double indices

$$\phi(\tau) = \sum_{g \in \Gamma_\infty^2 \backslash \Gamma^2} \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \hat{\phi}_{n_1, n_2}\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} z\right)$$

where Γ_∞^2, Γ^2 are the subgroups of $\mathrm{GL}(3, \mathbb{Z})$ defined as

$$\Gamma^2 = \left\{\begin{pmatrix} * & * & \\ * & * & \\ & & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})\right\}, \Gamma_\infty^2 = \Gamma^2 \cap \left\{\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})\right\}$$

and $\hat{\phi}_{n_1, n_2}(z)$ is

$$\hat{\phi}_{n_1, n_2}(z) = \int_0^1 \int_0^1 \int_0^1 \phi(xz) e^{-2\pi i (n_1 x_1 + n_2 x_2)} dx$$

where

$$x = \begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}$$

and $dx = dx_1 dx_2 dx_3$. One can check that $\hat{\phi}_{n_1, n2}(z)$ is a rapidly decreasing Whittaker function, and multiplicity one theorem gives us that it should be a multiple of (suitable modification of) $W_{\nu_1, \nu_2}(y_1, y_2)$, that is

$$\phi(\tau) = \sum_{g \in \Gamma_\infty^2 \backslash \Gamma^2} \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{a_{n_1, n_2}}{n_1 n_2} W_{1,1}^{(\nu_1, \nu_2)}\left(\begin{pmatrix} n_1 n_2 & & \\ & n_1 & \\ & & 1 \end{pmatrix} g\tau\right).$$

Here

$$W_{1,1}^{(\nu_1, \nu_2)}(\tau) = W_{\nu_1, \nu_2}(y_1, y_2) e(x_1 + x_2).$$

We call $\{a_{n_1, n_2}\}$ the matrix of Fourier coefficients of ϕ. From this, we define the corresponding L-function as

$$L(s, \phi) = \sum_{n \geq 1} \frac{a_{1,n}}{n^s}.$$

As we expect, this function admits an analytic continuation and satisfies certain functional equation.

Theorem 5.1 (L-function of an automorphic form on $\mathrm{GL}(3,\mathbb{R})$)**.** *The L-function $L(s,\phi)$ of an $\mathrm{GL}(3,\mathbb{R})$ automorphic form ϕ admits an analytic continuation for all $\mathbb{C}$ and satisfies the functional equation*

$$\Phi(s)L(s,\phi)=\tilde{\Phi}(1-s)L(1-s,\tilde{\phi})$$

where $\Phi(s),\tilde{\Phi}(s)$ are Gamma factors

$$\Phi(s)=\pi^{-\frac{3s}{2}}\Gamma\left(\frac{s-\alpha}{2}\right)\Gamma\left(\frac{s-\beta}{2}\right)\Gamma\left(\frac{s-\gamma}{2}\right)$$
$$\tilde{\Phi}(s)=\pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+\alpha}{2}\right)\Gamma\left(\frac{s+\beta}{2}\right)\Gamma\left(\frac{s+\gamma}{2}\right)$$

and $L(s,\tilde{\phi})$ is the L-function of the dual automorphic form which equals

$$L(s,\tilde{\phi})=\sum_{n\geq 1}\frac{a_{n,1}}{n^s}.$$

It is also possible to define Hecke operators on the space of $\mathrm{GL}(3,\mathbb{R})$ automorphic forms. We define them via double cosets, and the ring of Hecke operators became commutative. Note that, for each $n\geq 1$, there are *two* Hecke operators T_n,S_n, where

Definition 5.2 (Hecke operators)**.** *Let $\mathcal{H}=\mathbb{Z}[\Gamma\backslash G/\Gamma]$ be a $\mathbb{Z}$-algebra of double cosets where $G=\mathrm{GL}(3,\mathbb{R})$ and $\Gamma=\mathrm{GL}(3,\mathbb{Z})$, which is called* Hecke algebra*. It decomposes as a (internal) tensor product*

$$\mathcal{H}=\bigotimes_p\mathcal{H}_p$$

where $\mathcal{H}_p$ is a subalgebra of $\mathcal{H}$ corresponding to the double cosets whose elementary divisors are powers of a given prime p. For each prime p, $\mathcal{H}_p$ is generated by three elements

$$T_p:=\Gamma\begin{pmatrix}p&&\\&1&\\&&1\end{pmatrix}\Gamma,\quad S_p:=\Gamma\begin{pmatrix}p&&\\&p&\\&&1\end{pmatrix}\Gamma,\quad R_p:=\Gamma\begin{pmatrix}p&&\\&p&\\&&p\end{pmatrix}\Gamma.$$

The whole $\mathcal{H}$ is generated by the operators T_n,S_n,R_n where

$$T_n:=\sum_{n_0^3n_1^2n_2=n}\Gamma\begin{pmatrix}n_0n_1n_2&&\\&n_0n_1&\\&&1\end{pmatrix}\Gamma$$
$$S_n:=\sum_{n_0^3n_1^2n_2=n}\Gamma\begin{pmatrix}n_0^2n_1^2n_2&&\\&n_0^2n_1n_2&\\&&n_0^2n_1\end{pmatrix}\Gamma$$
$$R_n:=\Gamma\begin{pmatrix}n&&\\&n&\\&&n\end{pmatrix}\Gamma$$

which satisfies certain relations given as the formal power series

$$\sum_{n\geq 1}\frac{T_n}{n^s}=\prod_p\frac{1}{1-T_p\cdot p^{-s}+S_p\cdot p^{1-2s}-R_p\cdot p^{3-3s}}.$$

If ϕ is an automorphic form on $\mathrm{GL}(3,\mathbb{R})$*, then the action of Hecke algebra on the form is defined as*

$$(\phi|\Gamma\alpha\Gamma)(\tau) := \sum_i \phi(\alpha_i\tau)$$

where α_i's are the representatives of the double coset $\Gamma\alpha\Gamma$, i.e. $\Gamma\alpha\Gamma = \cup_i \Gamma\alpha_i$.

Note that the Hecke operators also commutes with the differential operators Δ_1 and Δ_2, so the space of automorphic forms of type (ν_1, ν_2) has a basis consisting of simultaneous eigenforms for all Hecke operators. Also, the coefficients of $\phi|T_n$ and $\phi|S_n$ can be expressed as certain sums of coefficients of ϕ - see the equations (9.8) and (9.9) in [4].

In GL(2), L-function attached to an automorphic form admits an Euler product if and only if the form is Hecke eigenform, and the local factors has a form of $P_p(p^{-s})^{-1}$, where $P_p(x)$ is a polynomial of degree 2. The same thing also holds for $\mathrm{GL}(3,\mathbb{R})$, where the local factors are inverses of cubic polynomials in p^{-s}.

Theorem 5.2 (Euler product of L-function)**.** *If ϕ is a normalized Hecke eigenform on* $\mathrm{GL}(3,\mathbb{R})$ *with matrix coefficients $\{a_{n_1,n_2}\}$, then its L-function has an Euler product*

$$L(s,\phi) = \prod_p \frac{1}{1 - a_{1,p}p^{-s} + a_{p,1}p^{-2s} - p^{-3s}}.$$

5.2. **Symmetric square lifting by Gelbart-Jacquet.** Let f be a level 1 Maass cusp form (of weight 0) on $\mathrm{GL}(2,\mathbb{R})$ with eigenvalue $\lambda = \nu(1-\nu)$ which is also a normalized eigenform. Let $\{a_n\}_{n\geq 1}$ be Fourier coefficients of f. Consider the Rankin-Selberg L-function of $f \times f$, i.e.

$$L(s, f\times f) = \zeta(2s)\sum_{n\geq 1}\frac{|a_n|^2}{n^s} = \zeta(2s)\prod_p \frac{1}{(1-\alpha_p^2 p^{-s})(1-p^{-s})^2(1-\beta_p^2 p^{-s})}$$

where $a_p = \alpha_p + \beta_p, \alpha_p\beta_p = 1$. By the theory of Rankin-Selberg convolution, the L-function admits an analytic continuation to $\mathbb{C}$ with functional equation

$$\Lambda(s, f\times f) = G(s)L(s, f\times f) = \Lambda(1-s, f\times f)$$

where $G(s)$ is the Gamma factor

$$G(s) := \pi^{-2s}\Gamma\left(\frac{s+1-2\nu}{2}\right)\Gamma\left(\frac{s}{2}\right)^2\Gamma\left(\frac{s-1+2\nu}{2}\right)$$

If we divide the functional equation of $\zeta(s)$ from both sides, we get

$$\begin{aligned}
&\pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+1-2\nu}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-1+2\nu}{2}\right)\frac{L(s,f\times f)}{\zeta(s)}\\
&= \pi^{-\frac{3(1-s)}{2}}\Gamma\left(\frac{(1-s)+1-2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{(1-s)-1+2\nu}{2}\right)\frac{L(1-s,f\times f)}{\zeta(1-s)}
\end{aligned}$$

One may expect that the degree 3 L-function $L(s, f\times f)/\zeta(s)$ is attached to certain self-dual GL(3) Maass eigenform of type $(2\nu/3, 2\nu/3)$. Indeed, the twisted L-functions by Dirichlet characters admits Euler product, satisfies EBV (entire and bounded in vertical strips) conditions and certain functional equation, so the GL(3) converse theorem gives the desired result. The detailed proof can be found in Chapter 7 of Goldfeld's book [8], where the proof of EBV condition is based on Shimura's brilliant idea that considers Rankin-Selberg product of f with a theta function (see also [22]). Let's write the corresponding GL(3) Maass form as $\phi = \phi(\tau)$. From $L(s,\phi)\zeta(s) = L(s, f \times f)$, the Fourier coefficients matrix $\{b_{n_1,n_2}\}$ of ϕ and the Fourier coefficients of $f(z)$ should be related as

$$a_n^2 = \sum_{d|n} b_{d,1} \Longleftrightarrow b_{n,1} = \sum_{d|n} \mu(d) a_{n/d}^2.$$

Now we will interpret the situation in the context of representation theory. Let $\pi = \otimes_v \pi_v$ be an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$, and let φ_v be the 2-dimensional representations of the Weil-Deligne group $W_v := W_{F_v}$ of F_v attached to π_v via Local Langlands correspondence. The symmetric square representation of $\mathrm{GL}(2,\mathbb{C})$

$$\mathrm{Sym}^2 : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{GL}(3,\mathbb{C}), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

gives a 3-dimensional representation $\mathrm{Sym}^2(\varphi_v) := \mathrm{Sym}^2 \circ \varphi$ of W_v, which should corresponds to an irreducible admissible representation of $\mathrm{GL}(3, F_v)$ via Local Langlands correspondence again. Global Langlands correspondences predicts that the representation $\mathrm{Sym}^2(\pi) := \otimes_v \mathrm{Sym}^2(\pi_v)$ is an automorphic represenetation of $\mathrm{GL}(3,\mathbb{A})$, which is proven by Gelbart-Jacquet.

Theorem 5.3 (Gelbart-Jacquet, [7]). *Let F be a number field and π be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ with $\mathbb{A} = \mathbb{A}_F$. Then $\mathrm{Sym}^2(\pi)$ is an automorphic representation of $\mathrm{GL}(2,\mathbb{A})$.*

In [17], Ramakrishnan proved the following converse of the Gelbart-Jacquet, by using L-functions.

Theorem 5.4 (Ramakrishnan, [17]). *Let F be a number field and Π be a cuspidal automorphic representation of $GL(3,\mathbb{A}_F)$, which is self-dual. Then, up to quadratic twist, it can be realized as an adjoint of a $GL(2,\mathbb{A}_F)$ automorphic representation. More precisely, there exists an automorphic form π of $\mathrm{GL}(2,\mathbb{A}_F)$ and a grössencharacter η of F with $\eta^2 = 1$ such that*

$$\Pi = \mathrm{Ad}(\pi) \otimes \eta$$

where $\mathrm{Ad}(\pi) = \mathrm{Sym}^2(\pi) \otimes \omega_\pi^{-1}$.

5.3. **Higher symmetric power.** Since symmetric power map $\mathrm{Sym}^r : \mathrm{GL}(2) \to \mathrm{GL}(r+1)$ is defined for arbitrary power r, we expect the presence of lifting from GL(2) automorphic representations to $\mathrm{GL}(r+1)$ automorphic representations. Until now, this is proved for $r = 3, 4$ cases.

Theorem 5.5 (Kim-Shahidi, [11]). *Let F be a number field and π be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ with $\mathbb{A} = \mathbb{A}_F$. Then $\mathrm{Sym}^3(\pi)$ is an automorphic representation of $\mathrm{GL}(4,\mathbb{A})$. $\mathrm{Sym}^3(\pi)$ is cuspidal unless π is either dihedral or tetrahedral type. In particular, if $F = \mathbb{Q}$ and π is the automorphic representation attached to nondihedral modular form of level ≥ 2, then $\mathrm{Sym}^3(f)$ is cuspidal.*

Theorem 5.6 (Kim, [10]). *Let F be a number field and π be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ with $\mathbb{A} = \mathbb{A}_F$. Then $\mathrm{Sym}^4(\pi)$ is an automorphic representation of $\mathrm{GL}(5,\mathbb{A})$. If $\mathrm{Sym}^3(\pi)$ is cuspidal, then $\mathrm{Sym}^4(\pi)$ is either cuspidal or induced from cuspidal representation of $\mathrm{GL}(2,\mathbb{A})$ and $\mathrm{GL}(3,\mathbb{A})$.*

To prove Theorem 5.5, Kim and Shahidi first proved the functoriality $\mathrm{GL}(2) \times \mathrm{GL}(3) \to \mathrm{GL}(6)$, i.e. existence of an automorphic representation $\pi_1 \boxtimes \pi_2$ for GL(2) automorphic representation π_1 and GL(3) automorphic representation π_2. Then they obtained the result by applying it for $\pi_1 = \pi$ and $\pi_2 = \mathrm{Ad}(\pi_1)$, where Ad is the automorphic representation of $\mathrm{GL}(3, \mathbb{A})$ obtained with adjoint representation $\mathrm{Ad} : \mathrm{GL}(2) \to \mathrm{PGL}(2) \to \mathrm{GL}(3)$. Note that $\mathrm{Sym}^2(\pi) = \mathrm{Ad}(\pi) \otimes \omega_\pi$, where ω_π is the central character of π.

For Theorem 5.6, Kim first proved exterior square lifting for GL(4), which corresponds to the map $\wedge^2 : \mathrm{GL}(4, \mathbb{C}) \to \mathrm{GL}(6, \mathbb{C})$. Then he obtained the result on the fourth power by applying exterior square to $\mathrm{Sym}^3(\pi) \otimes \omega_\pi^{-1}$, showing that

$$\wedge^2(\mathrm{Sym}^3(\pi) \otimes \omega_\pi^{-1}) = (\mathrm{Sym}^4(\pi) \otimes \omega_\pi^{-1}) \boxplus \omega_\pi.$$

Recently, it is proved that the functoriality holds for arbitrary power when π is a *regular algebraic cuspidal* representation, which corresponds to twists of cuspidal modular forms.

Theorem 5.7 (Newton-Thorne [14, 15])**.** *Let π be a regular algebraic cuspidal representation of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ of level 1, or without complex multiplication. For any $n \geq 1$, $\mathrm{Sym}^n(f)$ is a regular algebraic cuspidal representation of $\mathrm{GL}(n+1, \mathbb{A})$.*

5.4. **Ramanujan's conjecture, Selberg's 1/4 conjecture, and Sato-Tate conjecture.** The importance of symmetric power lifting is due to it's application on the Ramanujan conjecture, Selberg's eigenvalue conjecture, and the Sato-Tate conjecture.

Conjecture 5.1 (Selberg's 1/4 conjecture)**.** *For any Maass form on a congruence subgroup $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$, its eigenvalue is at least $1/4$.*

It is known that the conjecture is false for non-congruence subgroups (See [18] for Sarnak's argument). Also, it is widely believed that the Maass forms with eigenvalue 1/4 are *algebraic*, in a sense that they comes from *even* 2-dimensional Galois representations. Selberg himself proved a weaker bound 3/16 for $\Gamma = \Gamma(N)$ in [19],

Proposition 5.1. *Assume that symmetric power lifting holds for arbitrary power, i.e. for any cuspidal automorphic representation π on $\mathrm{GL}(2, \mathbb{A})$, $\mathrm{Sym}^r(\pi)$ is an automorphic representation of $\mathrm{GL}(r+1, \mathbb{A})$ for any r. Then the Ramanujan's conjecture and Selberg's conjecture are true.*

Proof. By Jacquet-Shalika [REFERENCE], it is proven that the Satake parameters of automorphic forms of $\mathrm{GL}(n, \mathbb{A})$ satisfy

$$q_v^{-1/2} < |\alpha_{i,v}| < q_v^{1/2}$$

for all $1 \leq i \leq n$ and unramified places v (including archimedean places). Now, assume that symmetric power lifting holds for arbitrary power. If $\Pi = \otimes_v \Pi_v = \mathrm{Sym}^r(\pi)$ is the corresponding representation, then the Satake parameters at place v are given as

$$\begin{pmatrix} \alpha_{1,v}^r & & & & \\ & \alpha_{1,v}^{r-1}\alpha_{2,v} & & & \\ & & \ddots & & \\ & & & \alpha_{1,v}\alpha_{2,v}^{r-1} & \\ & & & & \alpha_{2,v}^r \end{pmatrix}$$

and Jacquet-Shalika's bound gives

$$q_v^{-1/2} < |\alpha_{i,v}^r| < q_v^{1/2} \iff q_v^{-1/2r} < |\alpha_{i,v}| < q_v^{1/2r}$$

for all $r \geq 1$. Now taking the limit $r \to \infty$ proves both conjecture. $\square$

Combined with Theorem 5.6, Proposition 5.1 gives the current best bound for the Selberg's conjecture.

Corollary 5.1. *Eigenvalus of Maass forms on a congruence subgroup is at least*

$$\frac{1}{4} - \left(\frac{7}{64}\right)^2 = \frac{975}{4096} \approx 0.238037\ldots$$

Another consequence of symmetric power lifting is Sato-Tate conjecture. It is a conjecture about equidistribution of *Frobenius angle*, which we are going to explain briefly. Let E be an elliptic curve over $\mathbb{Q}$ without CM. For every prime p of good reduction for E, the number of points over $\mathbb{F}_p$ satisfies an inequality

$$|\#E(\mathbb{F}_p) - (p+1)| \leq 2\sqrt{p}$$

which was proven by Hasse in 1933. The quantity $t_p := (p+1) - \#E(\mathbb{F}_p)$ is called *trace*, since it is actually the trace of $\rho_{E,\ell}(\mathrm{Frob}_p)$, where $\rho_{E,\ell}$ is a (family of) ℓ-adic Galois representation attached to E and $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a Frobenius automorphism. Because of Hasse's inequality, we can write t_p as $t_p = 2\sqrt{p}\cos\theta_p$ for some $\theta_p \in [0,\pi]$, where we call θ_p as *Frobenius angle.* Sato-Tate conjecture states that the Frobenius angle is equidistributed over $[0,\pi]$, and it was proven in 2011 by Barnet-Lamb, Geraghty, Harris, and Taylor [2].

Theorem 5.8 (Sato-Tate conjecture, [2])**.** *Let $E/\mathbb{Q}$ be an elliptic curve without CM. The sequence of Frobenius angles $\{\theta_p\}$ is uniformly distributed on the interval $[0,\pi]$. In terms of traces $\{x_p\}$, for every subinterval $[a,b]$ of $[-2,2]$,*

$$\lim_{B\to\infty} \frac{\#\{p \leq B \,:\, x_p \in [a,b]\}}{\#\{p \leq B\}} = \int_a^b \frac{1}{2\pi}\sqrt{4-t^2}dt.$$

A key idea for the proof of Theorem 5.8 is the following equivalence between equidistribution and holomorphicity & nonzeroness of a L-function. Let G be a compact group and $X = \mathrm{conj}(G)$ be a space of conjugacy classes of G. Let K be a number field, and $P = (\mathfrak{p}_1, \mathfrak{p}_2, \ldots)$ be a sequence of all but finitely many primes of K ordered by norm. Let $(x_\mathfrak{p})_{\mathfrak{p}\in P}$ be a sequence in X indexed by P, and for each irreducible representation $\rho : G \to \mathrm{GL}(d,\mathbb{C})$, define the L-function

$$L(s,\rho) := \prod_{\mathfrak{p}\in P} \det(1-\rho(x_\mathfrak{p})N(\mathfrak{p})^{-s})^{-1},$$

for $s \in \mathbb{C}$ with $\Re s > 1$.

Theorem 5.9. *Let G and $(x_\mathfrak{p})$ as above, and assume that $L(s,\rho)$ is meromorphic on $\Re s \geq 1$ with nozeros or poles except possibly at $s=1$, for every irreducible representation ρ of G. The sequence $(x_\mathfrak{p})$ is equidistributed if and only if for each $\rho \neq 1$, the L-function $L(\rho,s)$ extends analytically to a function that is holomorphic and nonvanishing on $\Re s \geq 1$.*

Proof. See Theorem 2.3 of [6]. $\square$

Now let $G = \mathrm{SU}(2)$ (compact group of complex 2 by 2 unitary matrices of determinant 1) and $K = \mathbb{Q}$. The irreducible representations of SU(2) are of the form $\rho_m = \mathrm{Sym}^m \rho_1$ where $\rho_1 : \mathrm{SU}(2) \hookrightarrow \mathrm{GL}(2, \mathbb{C})$ is the representation given by inclusion. In this case, each element of $X = \mathrm{conj}(\mathrm{SU}(2))$ has a representative of the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

with $\theta \in [0, \pi]$, and the L-function of ρ_m can be written as

$$L(s, \rho_m) = \prod_{p \nmid N} \det(1 - \rho_m(x_p) p^{-s})^{-1} = \prod_{p \nmid N} \prod_{k=0}^{m} (1 - e^{i(m-2k)\theta_p} p^{-s})^{-1}$$

where θ_p is Frobenius angle and N is a conductor of elliptic curve E. If we set $\alpha_p := e^{i\theta_p} p^{1/2}$ and

$$L_m^1(s) := \prod_{p \nmid N} \prod_{k=0}^{m} (1 - \alpha_p^{m-k} \bar{\alpha}_p^k p^{-s})^{-1},$$

then we have $L(s, \rho_m) = L_m^1(s - m/2)$. By Theorem 5.9, Sato-Tate theorem (for non-CM elliptic curve) would follow from holomorphicity and nonzeroness of $L_m^1(s - m/2)$ for $\Re s \geq \frac{m}{2} + 1$. Now, the celebrated modularity theorem by several mathematicians (Wiles, Taylor, Brueil, Conrad, ...) states that one can find a modular form f whose L-function $L(s, f)$ coincides with the Hasse-Weil L-function $L(s, E)$ attached to E, and both coincides with $L_1^1(s)$ up to finitely many Euler factors at bad primes. It is easy to show holomorphicity and nonvanishing property of $L(s, f)$, and one essentially have $L_m^1(s) = L(s, \mathrm{Sym}^m f)$. Hence, Sato-Tate conjecture reduces to some analytic properties of $L(s, \mathrm{Sym}^m f)$.

6. Jacquet-Langlands correspondence

6.1. **Quaternionic modular forms and Basis problem.** [13]

6.2. **Jacquet-Langlands correspondence.**

7. Theta correspondence and Howe duality

7.1. Half-integral weight modular forms. The theta series

$$\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}, \qquad q = e^{2\pi i z}$$

is regarded as a powerful tool to study lattices and quadratic forms. For example, power of the theta series $\theta(z)^k$ is a generating function of $r_k(n)$, the number of ways to represent an integral n as a sum of k squares. For *even* k, $\theta(z)^k$ is a weight $k/2$ modular form, which makes us to analyze $\theta(z)^k$ more closely and even find the formula for $r_k(n)$. For example, $\theta(z)^2$ is a weight 1 modular form on $\Gamma_1(4)$ with character (Nebentypus) χ_4, the primitive Dirichlet character of level 4. The space $S_1(\Gamma_1(4), \chi_4)$ of such modular forms has dimension 1, so that $\theta(z)^2$ is actually a non-zero multiple of certain weight 1 Eisenstein series, and this gives a formula

$$r_2(n) = 4\sum_{2\nmid d|n}(-1)^{(d-1)/2}$$

and this gives a one-line proof for the Fermat's theorem on sum of two squares. Similarly, $\theta(z)^4$ is also a modular form (of weight 2 on $\Gamma_0(4)$), and the similar argument gives a formula

$$r_4(n) = 8\sum_{4\nmid d|n} d$$

and Lagrange's four square theorem is a direct consequence of this (see Zagier's article *Elliptic Modular Forms and Their Applications* in the book [3] for details).

How about the *odd* powers of $\theta(z)$? For example, what is $\theta(z)$ itself? Since $\theta(z)^2$ is a weight 1 modular form, we have $\theta(\gamma z)^2 = \chi_4(d)(cz+d)\theta(z)^2$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_1(4)$. So $\theta(z)$ itself satisfies a transformation law $\theta(\gamma z) = j(\gamma, z)\theta(z)$ where $j(\gamma, z)$ is, by definition, $j(\gamma, z) := \theta(\gamma z)/\theta(z)$. Indeed, it can be written as

$$j(\gamma, z) = \begin{cases} \epsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2} & c \neq 0 \\ 1 & c = 0\end{cases}$$

where the branch of $(cz+d)^{1/2}$ is choosen so that its real part is positive. ϵ_d is 1 if $d \equiv 1 \pmod 4$, and i otherwise.

In general, half-integral weight modular forms are defined as follows. Define $\mathcal{G}$ to be the group with elements (γ, ϕ) where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}^+(2, \mathbb{R})$ (group of matrices with positive determinant) and $\phi : \mathfrak{H} \to \mathbb{C}$ is a holomorphic function satisfying $\phi(z)^2 = t\det(\gamma)^{-1/2}(cz+d)$ with $t = t(\gamma, \phi) \in \mathbb{C}$ independent of z satisfying $|t| = 1$. This has a group structure defined as

$$(\gamma_1, \phi_1)(\gamma_2, \phi_2) = (\gamma_1\gamma_2, z \mapsto \phi_1(\gamma_2 z)\phi_2(z))$$

and it is an extension of the group $\mathrm{GL}^+(2, \mathbb{R})$ with fiber $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Some other examples of half-integral weight modular forms are the theta series with characters. For example, an even primitive character ψ of conductor r defines a theta series

$$\theta_\psi(z) = \sum_{n\in\mathbb{Z}} \psi(n) q^{n^2}$$

that belongs to $M_{1/2}(\Gamma_0(4r^2), \psi)$ (Proposition 2.2 of [21], proven by Poisson summation formula). From this, we also have $\theta_{\psi,t}(z) := \theta_\psi(tz) \in M_{1/2}(\Gamma_0(4r^2t), \psi)$. Note that $\theta_\psi(z)$ is different from $(\theta\otimes\psi)(z) = \sum_{n\in\mathbb{Z}}\psi(n)^2 q^{n^2}$, the series obtained by *twisting* θ with character ψ. Serre and Stark proved that every modular form of weight 1/2 is a linear combination of theta series $\theta_{\psi,t}$. Indeed, they find a bases for $M_{1/2}(\Gamma_0(N),\chi)$.

Theorem 7.1 (Serre-Stark, [20]). *Let $\Omega(N,\chi)$ be the set of pairs (ψ,t) where $t\geq 1$ is an integer and ψ is an even primitive character of conductor $r=r(\psi)$ such that*

(1) $4r^2t|N$
(2) $\chi(n)=\psi(n)\left(\frac{n}{t}\right)$ *for all n prime to N (i.e. ψ is the primitive character associated with $\chi\left(\frac{\cdot}{t}\right)$).*

Then $\{\theta_{\psi,t} : (\psi,t)\in\Omega(N,\chi)\}$ forms a basis of $M_{1/2}(\Gamma_0(N),\chi)$.

We have Hecke operators on the space of half-integral modular forms too, but it is quite different from that of integral weights. As in the integral weight cases, we define the Hecke operator as an action of double cosets.

7.2. **Shimura correspondence and Shintani lift.** Let $f(z)=\sum_{n\geq 1}a_nq^n$ be a Hecke eigenform of half-integral weight $k/2$ and level N with character χ, i.e. $T_{\chi,p^2}f=\lambda_p f$ for all p with $p\nmid N$. Then for every square-free integer t, we have

$$\sum_{n\geq 1}\frac{a_{n^2}}{n^s}\prod_p\left(1-\chi(p)\left(\frac{-1}{p}\right)^{\frac{k-1}{2}}p^{\frac{k-1}{2}-1-s}\right)^{-1}=\prod_p(1-\lambda_p p^{-s}+\chi(p)^2p^{k-2-2s})^{-1}$$

In [21], Shimura established a lift from the space of half integral weight modular forms to the space of integral weight modular forms. More precisely, he proved that the RHS of above equation becomes an L-function attached to certain weight $k-1$ modular form.

Theorem 7.2 (Shimura, [21]). *Let $F(z)=\sum_{n\geq 1}A_nq^n$ where*

$$\sum_{n\geq 1}\frac{A_n}{n^s}=\prod_p(1-\lambda_p p^{-s}+\chi(p)^2p^{k-2-2s})^{-1}.$$

Then $F(z)$ is a weight $k-1$ modular form satisfying

$$F(\gamma z)=\chi(d)^2(cz+d)^{k-1}F(z)$$

for all $\gamma\in\Gamma_0(N_0)$, where N_0 is an integer only depends on N and χ. This defines a map

$$S_{k/2}(N,\chi)\to M_{k-1}(N_0,\chi^2),$$

which is called as Shimura correspondence. $F(z)$ is a cusp form if $k\geq 5$.

The proof is based on Weil's converse theorem.

7.3. **Waldspurger's work.**

7.4. **Theta correspondence and Howe duality.**

7.5. **Gan-Gross-Prasad conjecture.**

References

[1] Arthur, J., and Clozel, L. *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula.(AM-120), Volume 120.* Princeton University Press, 2016.

[2] Barnet-Lamb, T., Geraghty, D., Harris, M., and Taylor, R. A family of calabi–yau varieties and potential automorphy ii. *Publications of the Research Institute for Mathematical Sciences 47*, 1 (2011), 29–98.

[3] Bruinier, J. H., Van der Geer, G., Harder, G., and Zagier, D. *The 1-2-3 of modular forms: lectures at a summer school in Nordfjordeid, Norway.* Springer Science & Business Media, 2008.

[4] Bump, D. *Automorphic Forms on* $\mathrm{GL}(3,\mathbb{R})$, vol. 1083. Springer, 2006.

[5] Doi, K., and Naganuma, H. On the functional equation of certain dirichlet series. *Inventiones mathematicae 9*, 1 (1969), 1–14.

[6] Fité, F. Equidistribution, l-functions, and sato–tate groups. *Contemporary Mathematics 649* (2015), 63–88.

[7] Gelbart, S., and Jacquet, H. A relation between automorphic forms on GL(2) and GL(3). *Proceedings of the National Academy of Sciences 73*, 10 (1976), 3348–3350.

[8] Goldfeld, D. *Automorphic forms and L-functions for the group GL (n, R)*, vol. 99. Cambridge University Press, 2006.

[9] Henniart, G., Herb, R., et al. Automorphic induction for $\mathrm{GL}(n)$(over local nonarchimedean fields). *Duke Mathematical Journal 78*, 1 (1995), 131–192.

[10] Kim, H. Functoriality for the exterior square of GL(4) and the symmetric fourth of GL(2). *Journal of the American Mathematical Society 16*, 1 (2003), 139–183.

[11] Kim, H. H., and Shahidi, F. Functorial products for $\mathrm{GL}(2) \times \mathrm{GL}(3)$ and the symmetric cube for GL(2). *Annals of mathematics* (2002), 837–893.

[12] Labesse, J.-P., and Langlands, R. P. L-indistinguishability for sl (2). *Canadian Journal of Mathematics 31*, 4 (1979), 726–785.

[13] Martin, K. The basis problem revisited. *Transactions of the American Mathematical Society 373*, 7 (2020), 4523–4559.

[14] Newton, J., and Thorne, J. A. Symmetric power functoriality for holomorphic modular forms. *Publications mathématiques de l'IHÉS 134*, 1 (2021), 1–116.

[15] Newton, J., and Thorne, J. A. Symmetric power functoriality for holomorphic modular forms ii. *Publications mathématiques de l'IHÉS 134*, 1 (2021), 117–152.

[16] Ramakrishnan, D. Modularity of the rankin-selberg l-series, and multiplicity one for SL(2). *Annals of Mathematics* (2000), 45–111.

[17] Ramakrishnan, D. An exercise concerning the selfdual cusp forms on GL(3). *Indian Journal of Pure and Applied Mathematics 45*, 5 (2014), 777–785.

[18] Sarnak, P. Selberg's eigenvalue conjecture. *Notices of the AMS 42*, 11 (1995), 1272–1277.

[19] Selberg, A. On the estimation of fourier coefficients of modular forms. In *Proc. Sympos. Pure Math.* (1965), vol. 8, Amer. Math. Soc., pp. 1–15.

[20] Serre, J.-P., and Stark, H. M. Modular forms of weight 1/2. In *Modular functions of one variable VI.* Springer, 1977, pp. 27–67.

[21] Shimura, G. On modular forms of half integral weights. *Annals of Mathematics* (1973), 440–481.

[22] Shimura, G. On the holomorphy of certain dirichlet series. *Proceedings of the London mathematical society 3*, 1 (1975), 79–98.