

SOLUTION FOR “AN INTRODUCTION TO AUTOMORPHIC REPRESENTATION”

SEEWOO LEE

ABSTRACT. This is a solution for the exercises in J. Getz and H. Hahn’s book
*An Introduction to Automorphic Representation with a view toward Trace
Formulae*.

1. CHAPTER 1

Problem 1.1 NOT FINISHED By Yoneda lemma, the morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$ of affine schemes corresponds to the k -algebra morphism $\phi : A \rightarrow B$. This induces a map on the underlying topological spaces by sending a prime ideal $\mathfrak{p} \subset B$ to $\phi^{-1}(\mathfrak{p}) \subset A$, which is also prime.

Problem 1.2 NOT FINISHED

Problem 1.3 By Yoneda lemma, we have

$$\text{Mor}(\text{Spec}(B), \text{Spec}(A)) \simeq \text{Nat}(h^B, h^A) \simeq h^B(A) = \text{Hom}_k(A, B)$$

which gives an equivalence between $\mathbf{AffSch}_k^{\text{op}}$ and \mathbf{Alg}_k .

Problem 1.4 • Nonreduced: $\text{Spec}(\mathbb{C}[x]/(x^2))$

- Reducible: $\text{Spec}(\mathbb{C}[x, y]/(x, y))$
- Reduced and irreducible (i.e. integral): $\text{Spec}(\mathbb{C}[x])$

Problem 1.5 We can assume that $Y = \text{Spec}(A)$ and $X = \text{Spec}(A/I)$ for some k -algebra A and an ideal I of A . Then it is enough to show that the map $\text{Hom}(A/I, R) \rightarrow \text{Hom}(A, R)$, given by composing with the natural map $\pi : A \rightarrow A/I$, is injective. This follows from the surjectivity of π .

Problem 1.6 Let $X = \text{Spec}(A), Y = \text{Spec}(B), Z = \text{Spec}(C)$. Then the statement is equivalent to

$$\text{Hom}(A \otimes_B C, R) \simeq \text{Hom}(A, R) \times_{\text{Hom}(B, R)} \text{Hom}(C, R).$$

We can define a bijection as follows. First, consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & A \otimes_B C \\ \alpha \uparrow & & \uparrow \iota_C \\ B & \xrightarrow{\gamma} & C \end{array}$$

Using the maps above, we define a map from LHS to RHS as $\phi \mapsto (\phi \iota_A, \phi \iota_C)$. Since $\iota_A \alpha = \iota_C \gamma$, we have $\phi \iota_A \alpha = \phi \iota_C \gamma$ and the map is well-defined. For the other direction, for given $(f, g) : A \times C \rightarrow R$ with $f \alpha = g \gamma$, universal property of the tensor product gives a unique map $\phi : A \otimes_B C \rightarrow R$ with $f = \phi \iota_A$ and $g = \phi \iota_C$. We can check that these maps are inverses for each other.

Problem 1.7 NOT FINISHED

Problem 1.8 NOT FINISHED We define an \mathbb{R} -algebra A as

$$A = \mathbb{R}[(x_{ij}, y_{ij})_{1 \leq i, j \leq n}] / I$$

where I is an ideal generated by elements of the form

$$\begin{aligned} & \left(\sum_{k=1}^n (x_{ik}^2 + y_{ik}^2) \right) - 1, \\ & \sum_{k=1}^n (x_{ik} x_{jk} - y_{ik} y_{jk}), \quad i \neq j \\ & \sum_{k=1}^n (x_{ij} y_{jk} + y_{ik} x_{jk}), \quad i \neq j \end{aligned}$$

for $1 \leq i, j \leq n$. Then we can identify $U_n(R)$ with $\text{Hom}(A, R)$ as follows: for given $\phi : A \rightarrow R$, let $\alpha_{ij} = \phi(x_{ij})$ and $\beta_{ij} = \phi(y_{ij})$. Then a matrix $g = (g_{ij})_{1 \leq i, j \leq n}$ with $g_{ij} = 1 \otimes \alpha_{ij} + \sqrt{-1} \otimes \beta_{ij}$ becomes an element of $U_n(R)$ by the relations of x_{ij} and y_{ij} s defined by the ideal I . Similarly, for given $g = (g_{ij}) \in U_n(R)$, we can write $g_{ij} = (a_{ij} + \sqrt{-1}b_{ij}) \otimes r_{ij} = 1 \otimes a_{ij}r_{ij} + \sqrt{-1} \otimes b_{ij}r_{ij}$ and we have a corresponding map $\phi : A \rightarrow R$ sending x_{ij} to $a_{ij}r_{ij}$ and y_{ij} to $b_{ij}r_{ij}$.

The group $U_n(\mathbb{R})$ is a compact group (as a topological subgroup of $\text{GL}_n(\mathbb{C})$) since it is closed (it is an inverse image of point I of a continuous map $g \rightarrow g\bar{g}^t$) and bounded (each row and column vectors have norm 1).

At last, **NOT FINISHED**

Problem 1.9 Consider the following short exact sequence:

$$0 \rightarrow \ker(\epsilon)/\ker(\epsilon)^2 \rightarrow \mathcal{O}(G)/\ker(\epsilon)^2 \rightarrow k \rightarrow 0.$$

The map $\mathcal{O}(G)/\ker(\epsilon)^2 \rightarrow k$ is defined as a composition of the natural map $\mathcal{O}(G)/\ker(\epsilon)^2 \rightarrow \mathcal{O}(G)/\ker(\epsilon)$ followed by ϵ . Then we have a section $k \rightarrow \mathcal{O}(G)/\ker(\epsilon)$ which is the composition $k \rightarrow \mathcal{O}(G) \rightarrow \mathcal{O}(G)/\ker(\epsilon)^2$ and the above sequence splits.

Problem 1.10 Let $g = (g_{ij}) \in \text{GL}_n(R)$ and $J = (\alpha_{ij}) \in \text{GL}_n(k)$. Then $g^t J g = J$ is equivalent to

$$\sum_{k,l=1}^n \alpha_{kl} g_{ki} g_{lj} = \alpha_{ij}$$

for all $1 \leq i, j \leq n$. Hence G is an affine algebraic group with a coordinate ring

$$A = k[(X_{ij})_{1 \leq i, j \leq n}] / \left(\sum_{k,l=1}^n \alpha_{kl} X_{ki} X_{lj} - \alpha_{ij}, 1 \leq i, j \leq n \right).$$

Since $\text{Lie } G = \ker(G(k[t]/t^2) \rightarrow G(k))$, the elements of $\text{Lie } G$ have a form of $I + tX$ for some $X \in M_n(k)$. Then the defining equation $g^t J g = J$ is equivalent to

$$(I + tX)^t J (I + tX) = J \Leftrightarrow J + tX^t J + tJX + t^2 X^t JX = J + t(X^t J + JX) = J,$$

(here every elements are in $\text{GL}_n(k[t]/t^2)$) so we should have $X^t J + JX = 0$. In other words, we have a map

$$\text{Lie } G \xrightarrow{\sim} \{X \in \mathfrak{gl}_n(k) : X^t J + JX\}, \quad I + tX \rightarrow X.$$

Problem 1.11 **NOT FINISHED**

Problem 1.12 **NOT FINISHED**

Problem 1.13 Using the equivalence of **Spl_k** and **RRD**, it is enough to check that the dual of the root datum of GL_n is isomorphic to itself in **RRD**. Recall that the root datum of GL_n with torus T of diagonal elements is given as follows: (Example 1.12)

- $X^*(T) = \{\alpha_{k_1, \dots, k_n} : \text{diag}(t_1, \dots, t_n) \mapsto \prod_{1 \leq j \leq n} t_j^{k_j}, k_1, \dots, k_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $X_*(T) = \{\beta_{k_1, \dots, k_n} : t \mapsto \text{diag}(t^{k_1}, \dots, t^{k_n}), t_1, \dots, t_n \in \mathbb{Z}\} \simeq \mathbb{Z}^n$
- $\Phi(\text{GL}_n, T) = \{e_{ij}, 1 \leq i \neq j \leq n\}, e_{ij}(\text{diag}(t_1, \dots, t_n)) = t_i t_j^{-1}$
- $\Phi^\vee(\text{GL}_n, T) = \{e_{ij}, 1 \leq i \neq j \leq n\}, e_{ij}^\vee(t) = \text{diag}(1, \dots, t, \dots, t^{-1}, \dots, 1)$
(t in the i -th entry, t^{-1} in the j -th entry, 1 for other entries)

Then we define a map $f : X_*(T) \rightarrow X^*(T)$ and $\iota : \Phi(\text{GL}_n, T) \rightarrow \Phi^\vee(\text{GL}_n, T)$ as

$$f(\beta_{k_1, \dots, k_n}) = \alpha_{k_1, \dots, k_n}, \quad \iota(e_{ij}) = e_{ij}^\vee.$$

and define $f^\vee : X^*(T) \rightarrow X_*(T)$ and $\iota^\vee : \Phi^\vee(\mathrm{GL}_n, T) \rightarrow \Phi(\mathrm{GL}_n, T)$ similarly. Then these maps are inverse to each other and gives an isomorphism between two root data

$$(X^*(T), X_*(T), \Phi(\mathrm{GL}_n, T), \Phi^\vee(\mathrm{GL}_n, T)) \simeq (X_*(T), X^*(T), \Phi^\vee(\mathrm{GL}_n, T), \Phi(\mathrm{GL}_n, T))$$

(they are central isogenies) so we get $\widehat{\mathrm{GL}}_n = \mathrm{GL}_{n\mathbb{C}}$.

Problem 1.14 We will show that complex dual of SL_n is PGL_n , and vice versa. Let's compute root datum for SL_n . We choose a maximal torus $T = T_{\mathrm{SL}_n} \leq \mathrm{SL}_n$ of diagonal matrices, so that

$$T(R) = \left\{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & \ddots \\ & & & t_n \end{pmatrix} : t_k \in R, \prod_{1 \leq k \leq n} t_k = 1 \right\}.$$

Then the characters $X^*(T)$ is almost same as the GL_n case, but we get a quotient of it. For given $\lambda = (\lambda_1, \dots, \lambda_n), \lambda' = (\lambda'_1, \dots, \lambda'_n) \in \mathbb{Z}^n$, two characters $\alpha_\lambda, \alpha_{\lambda'}$ are the same when $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$. Hence we have

$$X^*(T) \simeq \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n\} / \sim$$

where $\lambda \sim \lambda'$ if $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$. Similarly, cocharacter $\beta_\lambda(t) = \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_n})$ is well-defined only when $\sum_{k=1}^n \lambda_k = 0$, so that

$$X_*(T) \simeq \left\{ \lambda = (\lambda_1, \dots, \lambda_n) : \sum_{k=1}^n \lambda_k = 0 \right\} =: H \subset \mathbb{Z}^n.$$

The set of roots and coroots for SL_n is the same as that of GL_n : only Cartan subalgebra \mathfrak{t} is changed from diagonal matrices in \mathfrak{gl}_n to traceless diagonal matrices.

For PGL_n , we choose the maximal torus $T' = T_{\mathrm{PGL}_n}$ of diagonal matrices, and characters of T' has a form of $\alpha'_\lambda : \mathrm{diag}(t_1, \dots, t_n) \mapsto \prod_{k=1}^n t_k^{\lambda_k}$ for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, and we should have $\sum_{k=1}^n \lambda_k = 0$ for the character to be well-defined on T' . Hence we have $X^*(T') \simeq H \subset \mathbb{Z}^n$. Similarly, any cocharacter on T' has a form of $\beta'_\lambda : t \mapsto \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_n})$, and two different $\lambda, \lambda' \in \mathbb{Z}^n$ define same cocharacter as a map to T' if $\lambda' - \lambda \in \mathbb{Z} \cdot (1, \dots, 1)$, so $X_*(T)'$ is isomorphic to the quotient of \mathbb{Z}^n by $\mathbb{Z} \cdot (1, \dots, 1)$. The set of roots and coroots are the same as GL_n and SL_n . Note that the Lie algebra \mathfrak{pgl}_n of PGL_n can be thought as a quotient of $\mathfrak{gl}_n(R)$ by $R \cdot I_n$, where I_n is the $n \times n$ identity matrix.

Observe that we can natually identify $X^*(T) \simeq \mathbb{Z}^n / (\mathbb{Z} \cdot (1, \dots, 1)) \simeq X_*(T')$ and $X_*(T) \simeq H \simeq X^*(T')$. We can define a map between two root data of SL_n and $\widehat{\mathrm{PGL}}_n$ as follows:

$$\begin{aligned} f : X^*(T) &\rightarrow X_*(T'), & \alpha_\lambda &\mapsto \beta'_\lambda \\ f^\vee : X_*(T) &\rightarrow X^*(T'), & \beta_\lambda &\mapsto \alpha'_\lambda \\ \iota : \Phi(\mathrm{SL}_n, T) &\rightarrow \Phi^\vee(\mathrm{PGL}_n, T'), & e_{ij} &\mapsto e_{ij}^\vee \\ \iota^\vee : \Phi^\vee(\mathrm{SL}_n, T) &\rightarrow \Phi(\mathrm{PGL}_n, T'), & e_{ij}^\vee &\mapsto e_{ij} \end{aligned}$$

and this gives $\mathrm{SL}_n \simeq \widehat{\mathrm{PGL}}_n$. Similarly, we have $\mathrm{PGL}_n \simeq \widehat{\mathrm{SL}}_n$.

Problem 1.15 NOT FINISHED

2. CHAPTER 2

Problem 2.1 NOT FINISHED

Problem 2.2 NOT FINISHED

Problem 2.3 It is compact since it is an intersection of closed subset $G(F)$ of $\mathrm{GL}_n(F)$ ($G \hookrightarrow \mathrm{GL}_n$ is closed immersion) and intersection of closed set with compact set is again compact. Openness follows from continuity of $G(F) \hookrightarrow \mathrm{GL}_n(F)$: $\rho(G(F)) \cap K$ is an inverse image of K under $G(F) \hookrightarrow \mathrm{GL}_n(F)$.

Problem 2.4 NOT FINISHED

Problem 2.5 Using the anti-equivalence of category \mathbf{AffSch}_k and \mathbf{Alg}_k , we can reformulate the situation in terms of algebra as follows. Let $A = \mathcal{O}(Y)$ be \mathfrak{o} -algebra and $A_F := A \otimes_{\mathfrak{o}} F$. Let $X = \mathrm{Spec}(A_F/I)$ and \mathcal{X} be schematic closure of X in Y , so that $\mathcal{O}(\mathcal{X}) = \mathrm{Im}(\pi^I \iota)$ where $\iota : A \hookrightarrow A_F$ and $\pi^I : A_F \twoheadrightarrow A_F/I$. Let $Z = \mathrm{Spec} A/J$ (we have closed immersion $Z \hookrightarrow Y$), and we assume that the map on generic fibre, which corresponds to $A_F \twoheadrightarrow (A/J)_F$, induces an isomorphism $A_F/I = \mathcal{O}(X) \simeq \mathcal{O}(Z) = (A/J)_F$. This means that there exists an isomorphism $\phi : A_F/I \xrightarrow{\sim} (A/J)_F$ such that the following diagram commutes:

$$\begin{array}{ccc} (A/J)_F & & \\ \uparrow \phi & \nwarrow \pi_F^J & \\ A_F/I & \xleftarrow{\pi^I} & A_F \end{array}$$

Now our goal is to show that there exists a unique map

$$f : \mathcal{O}(\mathcal{X}) = \mathrm{Im}(\pi^I \iota) \rightarrow \mathcal{O}(Z) = A/J$$

such that the following diagram commutes:

$$\begin{array}{ccc} A/J & & \\ \uparrow f & \nwarrow \pi^J & \\ \mathrm{Im}(\pi^I \iota) & \xleftarrow{\pi^I \iota} & A \end{array}$$

The only way to define f that the above diagram commutes is following: for $x \in \mathrm{Im}(\pi^I \iota)$, choose $a \in A$ with $x = \pi^I \iota(a)$ and define $f(x) := \pi^J(a)$. Then we only need to show that the map is well-defined regardless of the choice of a . Let $a_1, a_2 \in A$ such that $\pi^I \iota(a_1) = \pi^I \iota(a_2) = x$. Since $\iota^J : A/J \hookrightarrow (A/J)_F$ is an injection, it is enough to show that $\iota^J \pi^J(a_1) = \iota^J \pi^J(a_2)$. By the commutativity of the following diagram

$$\begin{array}{ccc} A/J & \xleftarrow{\pi^J} & A \\ \downarrow \iota^J & & \downarrow \iota \\ (A/J)_F & \xleftarrow{\pi_F^J} & A_F \end{array}$$

we have $\iota^J \pi^J = \pi_F^J \iota = \phi \pi^I \iota$, and this proves

$$\iota^J \pi^J(a_1) = \phi \pi^I \iota(a_1) = \phi(x) = \phi \pi^I \iota(a_2) = \iota^J \pi^J(a_2),$$

i.e. the map is well-defined.

Problem 2.6 NOT FINISHED

Problem 2.7 NOT FINISHED

Problem 2.8 Note that the coordinate ring of $\mathrm{GL}_{n,\mathbb{Q}}$ is

$$B = \mathcal{O}(\mathrm{GL}_{n,\mathbb{Q}}) = \mathbb{Q}[x_{ij}, y]_{1 \leq i, j \leq n} / (\det(x_{ij})y - 1).$$

To show that \mathcal{G} is a model of $\mathrm{GL}_{n,\mathbb{Q}}$ over \mathbb{Z} , we need to show that $A \hookrightarrow B$ and $A \otimes_{\mathbb{Z}} \mathbb{Q} \simeq B$. Latter isomorphism easily follows from

$$A \otimes \mathbb{Q} = \mathbb{Q}[x_{ij}, t_{ij}, y] / (\det(x_{ij})y - 1, \{x_{ij} - \delta_{ij} - mt_{ij}\}) \simeq B$$

since we can invert $m > 1$ in \mathbb{Q} and get an isomorphism $A \otimes \mathbb{Q} \rightarrow B$ via $t_{ij} \mapsto (1 - x_{ij})/m$. Showing $A \hookrightarrow B \simeq A \otimes_{\mathbb{Z}} \mathbb{Q}$ is equivalent to showing that A is a torsion-free \mathbb{Z} -module. Assume that we have $z \in \mathbb{Z}[x_{ij}, t_{ij}, y]$ and $0 \neq a \in \mathbb{Z}$ such that $az = 0$ in A . Then there exists $\alpha, \beta_{ij} \in \mathbb{Z}$ for $1 \leq i, j \leq n$ s.t.

$$\begin{aligned} az &= \alpha(\det(x_{ij})y - 1) + \sum_{ij} \beta_{ij}(x_{ij} - \delta_{ij} - mt_{ij}) \\ \Leftrightarrow z &= \frac{\alpha}{a} \det(x_{ij})y + \sum_{i,j} \frac{\beta_{ij}}{a} x_{ij} - \sum_{i,j} \frac{m\beta_{ij}}{a} t_{ij} - \frac{\alpha + \sum_i \beta_{ii}}{a} \end{aligned}$$

which implies $a|\alpha$ and $a|\beta_{ij}$, i.e. $z = 0$ in A . Hence \mathcal{G} is a model of $\mathrm{GL}_{n,\mathbb{Q}}$ over \mathbb{Z} .

The set of \mathbb{Z} -points $\mathcal{G}(\mathbb{Z}) = \mathrm{Hom}(A, \mathbb{Z})$ can be identified with the set via map

$$\begin{aligned} \mathrm{Hom}(A, \mathbb{Z}) &\rightarrow \{g \in \mathrm{GL}_n(\mathbb{Z}) : g \equiv I_n \pmod{mM_n(\mathbb{Z})}\} \\ \phi &\mapsto (g_{ij} = \phi(x_{ij})) \end{aligned}$$

since $\phi(x_{ij}) = \delta_{ij} + m\phi(t_{ij}) \Rightarrow g - I_n \in mM_n(\mathbb{Z})$.

Problem 2.9 NOT FINISHED It is not hard to prove that if Z_1, Z_2 are dense subsets of a topological space Y_1, Y_2 respectively, then $Z_1 \times Z_2$ is dense in $Y_1 \times Y_2$. Combining with Exercise 1.6 and Theorem 2.2.1 (b), we get the desired results for both weak and strong approximation.

Problem 2.10 By Exercise 2.7 and 2.9, $M_n \simeq \mathbb{G}_a^{n^2}$ admits weak approximation over F . With embedding $\mathrm{GL}_n \hookrightarrow M_n$ with $\mathrm{GL}_n(F) = M_n(F) \cap \mathrm{GL}_n(F_S) \subset M_n(F_S)$, we also have $\mathrm{GL}_n(F)$ dense in $\mathrm{GL}_n(F_S)$.

Problem 2.11 NOT FINISHED

Problem 2.12 NOT FINISHED

Problem 2.13 NOT FINISHED

Problem 2.14 NOT FINISHED

Problem 2.15 NOT FINISHED

Problem 2.16 NOT FINISHED The center Z_{GL_2} of GL_2 is $Z_{\mathrm{GL}_2}(R) = R^\times I_2$. Hence the largest $\mathbb{F}_p(t)$ split torus in $\mathrm{Res}_{F/\mathbb{F}_p(t)} Z_{\mathrm{GL}_2}$ is just $\mathrm{Res}_{F/\mathbb{F}_p(t)} Z_{\mathrm{GL}_2}$ itself which has degree $d = [F : \mathbb{F}_q(t)]$.

Problem 2.17 NOT FINISHED

Problem 2.18 NOT FINISHED

Problem 2.19 Let $N = p_1^{e_1} \cdots p_r^{e_r}$ be a prime factorization of N . Define $K_N \leq \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^\infty)$ as

$$K_N = \prod_{i=1}^r (I_n + p_i^{e_i} M_n(\mathbb{Z}_{p_i})) \times \prod_{p \neq p_i} \mathrm{GL}_n(\mathbb{Z}_p).$$

Then K_N is an open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^\infty)$ such that $K_N \cap \mathrm{GL}_n(\mathbb{Q}) = \Gamma(N)$.

(\Rightarrow) Let H be a congruence subgroup of $\mathrm{GL}_n(\mathbb{Q})$, which means that there exists an open compact subgroup $K_H \leq \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ such that $H = K_H \cap \mathrm{GL}_n(\mathbb{Q})$. Then we can find an open compact neighborhood $U \leq K_H$ of I_n which has a form of

$$U = \prod_{p \in S} (I_n + p^{e_p} M_n(\mathbb{Z}_p)) \times \prod_{p \notin S} \mathrm{GL}_n(\mathbb{Z}_p)$$

for some finite set of primes S (Note that $\{I_n + p^k M_n(\mathbb{Z}_p)\}_{k \geq 1}$ is a decreasing sequence of open compact neighborhoods of I_n , which is also a subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$). Then $U = K_N$ for $N = \prod_{p \in S} p^{e_p}$, i.e. U is also an open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$, and it is a finite index subgroup of K_H since K_H is open and compact (consider all the cosets of K_N in K_H , which are all homeomorphic to K_N). Then $[H : \Gamma(N)] = [K_H : K_N]$ implies that H contains $\Gamma(N)$ as a finite index subgroup.

(\Leftarrow) Let H be a subgroup of $\mathrm{GL}_n(\mathbb{Q})$ contains $\Gamma(N)$ with $[H : \Gamma(N)] < \infty$. Let K_H be an image of H in $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ under the diagonal embedding $\mathrm{GL}_n(\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ so that $K_H \cap \mathrm{GL}_n(\mathbb{Q}) = H$. Then K_H contains K_N and $[K_H : K_N] = [H : \Gamma(N)]$, so K_N is a finite index subgroup of K_H . for coset representatives g_1, g_2, \dots, g_t of K_H/K_N , $K_H = \cup_{j=1}^t g_j K_N$ and by openness (resp. compactness) of K_N , K_H is also open (resp. compact) subgroup.

3. CHAPTER 3

Problem 3.1 NOT FINISHED

Problem 3.2 Since G is compact, the image of the modular quasi-character $\delta_G : G \rightarrow \mathbb{R}_{>0}^\times$ is a compact subgroup of $\mathbb{R}_{>0}^\times$. Then it should be trivial - otherwise, there exists $g \in G$ with $\delta_G(g) > 1$ (we can choose g or g^{-1}), and then $\delta_G(g^n) = \delta_G(g)^n \rightarrow \infty$ as $n \rightarrow \infty$, i.e. the image is not bounded. Hence G is unimodular.

Problem 3.3 NOT FINISHED

Problem 3.4 NOT FINISHED

Problem 3.5 NOT FINISHED

Problem 3.6 NOT FINISHED

Problem 3.7 Let k be a residue field and ϖ be a uniformizer of F . We have $\mathcal{O}_F^\times = \coprod_{a \in k^\times} (a + \varpi \mathcal{O}_F)$ and

$$\begin{aligned} d^\times x(\mathcal{O}_F^\times) &= \int_{\mathcal{O}_F^\times} \frac{dx}{|x|} \\ &= \int_{\mathcal{O}_F^\times} dx \\ &= dx(\mathcal{O}_F^\times) \\ &= \sum_{a \in k^\times} dx(a + \varpi \mathcal{O}_F) \\ &= \sum_{a \in k^\times} q^{-1} dx(\mathcal{O}_F) \\ &= (q-1)q^{-1} dx(\mathcal{O}_F) = (1-q^{-1})dx(\mathcal{O}_F). \end{aligned}$$

Problem 3.8 NOT FINISHED

Problem 3.9 NOT FINISHED

Problem 3.10 NOT FINISHED

Problem 3.11 Let $x, g, y \in \mathrm{GL}_n(F)$ with $y = xg$ (regard g as a constant matrix). Then we have $y_{ij} = \sum_{1 \leq k \leq n} x_{ik} g_{kj}$ and $dy_{ij} = \sum_{1 \leq k \leq n} g_{kj} dx_{ik}$. This gives

$$\begin{aligned} dy_{11} \wedge dy_{12} \wedge \cdots \wedge dy_{1n} &= (g_{11}dx_{11} + g_{21}dx_{12} + \cdots + g_{n1}dx_{1n}) \wedge \cdots \wedge (g_{1n}dx_{11} + \cdots + g_{nn}dx_{1n}) \\ &= |\det(g^t)| dx_{11} \wedge \cdots \wedge dx_{1n} \\ &= |\det(g)| dx_{11} \wedge \cdots \wedge dx_{1n} \end{aligned}$$

and along with $\det(xg) = \det(x)\det(g)$, we have

$$\frac{\wedge_{i,j} dy_{ij}}{|\det(y)|^n} = \frac{|\det(g)|^n \wedge_{i,j} dx_{ij}}{|\det(xg)|^n} = \frac{\wedge_{i,j} dx_{ij}}{|\det(x)|^n}$$

so $d(x_{ij})$ is right Haar measure. Since GL_n is reductive, it is unimodular and so $d(x_{ij})$ is also a left Haar measure.

Problem 3.12 NOT FINISHED

Problem 3.13 NOT FINISHED

Problem 3.14 NOT FINISHED Consider a reduction map $\mathrm{GL}_n(\mathcal{O}_{F_v}) \twoheadrightarrow \mathrm{GL}_n(k_v)$ where k_v is a residue field of F_v with $\#k_v = q_v$, which is surjective. The kernel H

of the map is $1 + \varpi_v M_n(\mathcal{O}_{F_v})$ where ϖ_v is a uniformizer of F_v . Then we have

$$|\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = |\omega|_v(H) \cdot \#\mathrm{GL}_n(k_v).$$

The order of $\mathrm{GL}_n(k_v)$ is $(q_v^2 - 1)(q_v^2 - q_v)$: there are $q_v^2 - 1$ choices for the first column vector (all but zero vector), and $q_v^2 - q_v$ choices for the second column vector (all but vectors which are multiples of the first column vector). Also, for $h \in H$, we have

$$h = \begin{pmatrix} 1 + \varpi_v x_{11} & \varpi_v x_{12} \\ \varpi_v x_{21} & 1 + \varpi_v x_{22} \end{pmatrix} \\ \Rightarrow |\det(h)|_v = |1 + \varpi_v(x_{11} + x_{22}) + \varpi_v^2(x_{11}x_{22} - x_{12}x_{21})|_v = 1$$

So

$$|\omega|_v(H) = \int_{\mathcal{O}_{F_v}^4} d(\varpi_v x_{11}) \wedge \cdots \wedge d(\varpi_v x_{22}) \\ = q_v^{-4} \int_{\mathcal{O}_{F_v}^4} dx_{11} \wedge \cdots \wedge dx_{22} = q_v^{-4}$$

and the measure is $|\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = (1 - q_v^{-1})(1 - q_v^{-2})$.

When F is a number field, then the *Dedekind zeta function* of F , defined as

$$\zeta_F(s) := \sum_{0 \neq I \subseteq \mathcal{O}_F} \frac{1}{N_{F/\mathbb{Q}}(I)^s}$$

admits an Euler product for $\Re s > 1$:

$$\zeta_F(s) = \prod_{\mathfrak{p} \subseteq \mathcal{O}_F} \frac{1}{1 - N_{F/\mathbb{Q}}(\mathfrak{p})^s}.$$

Then the product is

$$\prod_{v \nmid \infty} \left(1 - \frac{1}{q_v}\right) \left(1 - \frac{1}{q_v^2}\right)$$

and this diverges since $\prod_{v \nmid \infty} (1 - q_v^{-1})$ does and $\prod_{v \nmid \infty} (1 - q_v^{-2}) = \zeta_F(2)^{-1}$ does not. However, the normalized product

$$\prod_{v \nmid \infty} (1 - q_v^{-1})^{-1} |\omega|_v(\mathrm{GL}_n(\mathcal{O}_{F_v})) = \prod_{v \nmid \infty} (1 - q_v^{-2})$$

converges to $\zeta_F(2)^{-1}$.

Now assume that F is a function field.

Problem 3.15 (Note that this is a theorem of Maschke.) It is enough to show the following:

Claim. Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a complex representation of finite group G , and let U be a subrepresentation of ρ , i.e. invariant under ρ . Then there exists $W \leq V$ such that $U \cap W = \{0\}$ and $U \oplus W = V$.

Applying the above claim repeatedly shows that any representation of a finite group is completely decomposable. To show the lemma, let W' be *any* subspace of V such that $U \cap W' = \{0\}$ and $U \oplus W' = V$. Let $\pi' : V \rightarrow U$ be a corresponding projection. Then define $\pi : V \rightarrow V$ as

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi'(gv)$$

whose image is in U ($gv := \rho(g)v$). Our claim is that $W = \ker \pi$ is the desired subspace: W is ρ -invariant and $U \oplus W = V$. First of all, since $\pi'|_U$ is identity on U and U is ρ -invariant, $\pi|_U$ is also an identity map on U . Then we have $W \cap U = 0$, and by dimension counting we get $V = U \oplus W$. Hence we only need to show that W is ρ -invariant: for $h \in G$ and $v \in W = \ker \pi$,

$$\begin{aligned} \pi(hv) &= \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi'(ghv) \\ &= \frac{1}{|G|} \sum_{g' \in G} h g'^{-1} \pi'(g'v) \quad (g' = gh) \\ &= h \pi(v) = 0 \end{aligned}$$

so $hv \in W$.

Problem 3.16 Assume that the representation $\rho : B(\mathbb{C}) \rightarrow \text{GL}_2(\mathbb{C})$ is completely reducible. Since the representation is 2-dimensional, it should be decomposed as $\chi_1 \oplus \chi_2$ for some characters $\chi_1, \chi_2 : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$. In other words, there exists $g_0 \in \text{GL}_2(\mathbb{C})$ such that

$$\rho(g) = g_0 \begin{pmatrix} \chi_1(g) & 0 \\ 0 & \chi_2(g) \end{pmatrix} g_0^{-1}.$$

This implies $\rho(gh) = \rho(hg)$, which is not true since $B(\mathbb{C})$ is not commutative.

Problem 3.17 For any $g \in G$,

$$\begin{aligned} ((f_1 * f_2) * f_3)(g) &= \int_G (f_1 * f_2)(gh_1^{-1}) f_3(h_1) d_r h_1 \\ &= \int_G \int_G f_1(gh_1^{-1} h_2^{-1}) f_2(h_2) d_r h_2 f_3(h_1) d_r h_1 \\ &= \int_G \int_G f_1(gh_1^{-1} h_2^{-1}) f_2(h_2) f_3(h_1) d_r h_2 d_r h_1 \\ &= \int_G \int_G f_1(gh_3^{-1}) f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_3 d_r h_1 \quad (h_3 = h_2 h_1, d_r h_3 = d_r h_2) \\ &= \int_G \int_G f_1(gh_3^{-1}) f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_1 d_r h_3 \quad (\text{Fubini's theorem}) \\ &= \int_G f_1(gh_3^{-1}) \left(\int_G f_2(h_3 h_1^{-1}) f_3(h_1) d_r h_1 \right) d_r h_3 \\ &= \int_G f_1(gh_3^{-1}) (f_2 * f_3)(h_3) d_r h_3 \\ &= (f_1 * (f_2 * f_3))(g). \end{aligned}$$

Problem 3.18 NOT FINISHED

Problem 3.19

$$\begin{aligned}
\pi(f_1 * f_2)\varphi &= \int_G (f_1 * f_2)(g)\pi(g)\varphi d_r g \\
&= \int_G \int_G f_1(gh^{-1})f_2(h)\pi(g)\varphi d_r h d_r g \\
&= \int_G \int_G f_1(gh^{-1})f_2(h)\pi(g)\varphi d_r g d_r h \quad (\text{Fubini's theorem}) \\
&= \int_G \int_G f_1(g_1)f_2(h)\pi(g_1h)\varphi d_r g_1 d_r h \quad (g_1 = gh^{-1}, d_r g_1 = d_r g) \\
&= \int_G \int_G f_1(g_1)f_2(h)\pi(g_1h)\varphi d_r h d_r g_1 \quad (\text{Fubini's theorem}) \\
&= \int_G f_1(g_1)\pi(g_1) \left(\int_G f_2(h)\pi(h)\varphi d_r h \right) d_r g_1 \\
&= \int_G f_1(g_1)\pi(g_1)\pi(f_2)\varphi d_r g_1 \\
&= (\pi(f_1) \circ \pi(f_2))\varphi
\end{aligned}$$

4. CHAPTER 4

Problem 4.1 NOT FINISHED

Problem 4.2 NOT FINISHED

Problem 4.3 NOT FINISHED

Problem 4.4 NOT FINISHED

Problem 4.5 By Schur's lemma, any elements in a center $z \in Z_G(F)$ acts as a (nonzero) scalar, let's say, $\omega_\pi(z) \in \mathbb{C}^\times$. Then $\omega_\pi : Z_G(F) \rightarrow \mathbb{C}^\times$ is a character since $\omega_G = \pi|_{Z_F(G)}$.

Let $\chi : G(F) \rightarrow \mathbb{C}^\times$ be a quasi-character. The representation $\pi \otimes \chi$ is defined as $(\pi \otimes \chi)(g)v = \chi(g) \cdot \pi(g)v$, and it's restriction on the center becomes $\chi|_{Z_G(F)} \cdot \omega_\pi$, which is the central character $\omega_{\pi \otimes \chi}$ of $\pi \otimes \chi$.

Problem 4.6 Let $G = \mathbb{G}_a$ and $G(\mathbb{R}) = (\mathbb{R}, +)$. Consider a 1-dimensional representation $\chi_\alpha : G(\mathbb{R}) \rightarrow \text{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, $t \mapsto e^{\alpha t}$, where $\Re(\alpha) \neq 0$. Then this is irreducible since 1-dimensional, but not unitary since $|e^{\alpha t}| = e^{\Re(\alpha)t} \neq 1$ for $t \neq 0$.

Problem 4.7 NOT FINISHED

Problem 4.8 NOT FINISHED

Problem 4.9 NOT FINISHED

Problem 4.10 NOT FINISHED

Problem 4.11 NOT FINISHED

Problem 4.12 NOT FINISHED

Problem 4.13 NOT FINISHED

5. CHAPTER 5

Problem 5.1 NOT FINISHED

Problem 5.2 One direction is clear. For the other direction, assume that (π, V) is admissible and let U be an open subgroup of G . Then we can choose compact open subgroup K such that $K \leq U \leq G$, and we have $V^U \leq V^K$. Now $\dim V^K < \infty$ gives $\dim V^U < \infty$.

Problem 5.3 Let $v \in V_{\text{sm}}$, so that $v \in V^K$ for some open compact subgroup $K \leq G$. Then for $g \in G$ and $k \in K$, we have $\pi(k)v = v \Rightarrow \pi(gkg^{-1})\pi(g)v = \pi(g)v$. Hence $\pi(g)v$ is fixed by gkg^{-1} for all $k \in K$, hence $\pi(g)v \in V^{gKg^{-1}} \subseteq V_{\text{sm}}$. Hence V_{sm} is preserved by G .

Now let $H = \text{Stab}(v) \leq G$ be a stabilizer of $v \in V_{\text{sm}}$. There exists an open compact subgroup K with $v \in V^K \leftrightarrow K \leq \text{Stab}(v)$, so $\text{Stab}(v)$ is a union of open cosets homeomorphic to K , which is also open. Hence $(\pi_{V_{\text{sm}}}, V_{\text{sm}})$ is smooth.

Problem 5.4 NOT FINISHED

Problem 5.5 NOT FINISHED

Problem 5.6 NOT FINISHED

Problem 5.7 NOT FINISHED

Problem 5.8 The proof is essentially same as that of Problem 4.5, and we also use Schur's lemma (Problem 5.6).

Problem 5.9 Take $x = p^k$. Then $|p^k|_p = p^{-k}$ and $|p^k|_\infty = p^k$, so we have $c_1 \cdot p^k \leq p^{-k} \Leftrightarrow c_1 \leq p^{-2k}$ for all k . Now taking limit $k \rightarrow \infty$ gives $c_1 = 0$, which gives a contradiction. We can do similarly for the other direction with $k \rightarrow -\infty$.

Problem 5.10 NOT FINISHED

Problem 5.11 NOT FINISHED

Problem 5.12 See the argument in Problem 5.3 for showing G -invariance. To show denseness, let $v \in V$ and let $\epsilon > 0$. By continuity of $G \times V \rightarrow V$, there exists open neighborhood U of 1 such that $\|\pi(u)v - v\| < \epsilon$ for all $u \in U$. Now we can choose open compact subgroup K of G lies in U (by totally connectedness), and we have

$$\begin{aligned} \|e_K v - v\| &= \left\| \frac{1}{\text{meas}_{dg}(K)} \int_G (\mathbb{1}_K(g) \pi(g)v - v) dg \right\| \\ &\leq \frac{1}{\text{meas}_{dg}(K)} \int_K \|\pi(k)v - v\| dk < \epsilon \end{aligned}$$

and from $e_K v \in V^K$, we get denseness of V_{sm} in V .

Problem 5.13 NOT FINISHED

Problem 5.14 NOT FINISHED