How automorphic forms and elliptic curves fly?

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Abstract. This is an expository note on *murmurations*, which was initially discovered by He, Lee, Oliver, and Pozdnyakov for elliptic curves. We focus on the cases where the mumuration density is computed (under GRH), including the work of Zubrilina, Lee–Oliver–Pozdnyakov, and Sawin–Sutherland.

1 Introduction

1.1 What is the role of Machine Learning?

Although this note focus on the recent results on computing murmuration densities, I make a brief comment on the relation between machine learning and murmuration, which I found that existing literatures are often misleading on distinguishing the machine learning part and the murmuration part. I read few articles on internet which basically say that "AI found new mathematics," which is half true and half false.

One of the main motivation of the paper [7] is to study elliptic curves via machine learning. Especially, they were interested in predicting the rank of elliptic curves (which is widely known to be hard to compute in general) by means of machine learning, where the coefficients $a_p(E)$ of Hasse–Weil L-functions are used as features. Surprisingly, they found that simple logictic regression model can already distinguish between rank 0 and 1 elliptic curves with high accuracy of $> 90\%$. Along the line, they (more precisely, He, Lee, and Oliver) were curious about what was actually going on, and Pozdnyakov (who was an undergraduate student of Lee at that time) figured out the murmuration pattern. This somehow gives an explanation of the high accuracy of the model, since the mumuration patterns for rank 0 and 1 elliptic curves are noticably different. But the correct way to say is that the machine learning experiments *motivated* them to study what models were doing, which is essentially the work of humans, not the ML models. You can find more story in the Quanta Magazine article [4].

1.2 Sato–Tate conjecture and Murmuration

Another confusing part (at least for me) is the difference between murmuration and Sato–Tate conjecture.

2 Murmuration of Dirichlet series

3 Murmuration of modular forms

In this section, we sketch Zubrilina's computation of murmuration density for modular forms.

3.1 Statement

Before we state the result, we define some notations first.

- For $n \in \mathbb{Z}_{\geq 0}$, *Chebyshev polynomial of the second kind* is defined as
$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$
- For $r \in \mathbb{Z}_{\geq 1}$, define
$$\nu(r) := \prod_{p|r}\left(1 + \frac{p^2}{p^4 - 2p^2 - p + 1}\right)$$
- Define constants α, β, γ as
$$\alpha := 2\pi \prod_p \frac{p^4 - 2p^2 - p + 1}{p^4 - 2p^2 + p},$$
$$\beta := 2\pi \prod_p \frac{p^3 + p^2 - 1}{p(p^2 + p - 1)},$$
$$\gamma := 12 \prod_p \frac{p(p+1)}{p^2 + p - 1}.$$

Theorem 3.1 (Zubrilina [16]). Let X, Y, P be parameters going infinite with $X, Y > 0$ and P prime; assume further that $Y = (1 + o(1))X^{1-\delta_2}$ and $P \ll X^{1+\delta_1}$ for some δ_1, δ_2 with $2\delta_1 < \delta_2 < 1$. Let $y = P/X$. Then
$$\frac{\sum^{\square}_{N\in[X,X+Y]} \sum_{f\in H^{\mathrm{new}}(N,k)} \sqrt{P}\lambda_f(P)\varepsilon(f)}{\sum^{\square}_{N\in[X,X+Y]} \sum_{f\in H^{\mathrm{new}}(N,k)} 1} = \mathcal{M}_k(y) + O_\varepsilon\left(X^{-\delta'+\varepsilon} + \frac{1}{P}\right) \tag{1}$$
where
$$\mathcal{M}_k(y) = \frac{\alpha(-1)^{k/2-1}}{k-1} \sum_{1\leq r\leq 2\sqrt{y}} \nu(r)\sqrt{4y - r^2}\,U_{k-2}\left(\frac{r}{2\sqrt{y}}\right) + \frac{\beta}{k-1}\sqrt{y} - \gamma\delta_{k=2}y. \tag{2}$$

3.2 Eichler–Selberg trace formula

To prove Theorem 3.1, one need to understand how to estimate the numerator on the LHS. Recall that $a_f(P) = P^{(k-1)/2}\lambda_f(P)$ is the P-th Fourier coefficient of f, which is also the eigenvalue of the Hecke operator T_P acting on f. Also, $(-1)^{k/2}\varepsilon(f)$ is equal to the eigenvalue of the Atkin–Lehner involution $W_N = T_N$ acting on f. Thus the sum appears in the numerator of LHS of (1) can be interpreted as the trace of the operator $(-1)^{k/2}T_P \circ W_N$ acting on the space of cusp forms of weight k and level N (multiplied by $P^{1-k/2}$). Eichler [5] studied such a sum of traces and proved that it can be expressed in terms of (Hurwitz) class numbers, which is generalized by Selberg [11]. To account the root number $\varepsilon(f)$, i.e. eigenvalue of W_N, we used the following version of Eichler–Selberg trace formula by Skoruppa and Zagier [12].

Theorem 3.2 (Skoruppa–Zagier [12]). For square-free N and prime $P \nmid N$,
$$\begin{aligned}&\sum_{f\in H^{\mathrm{new}}(N,k)} \sqrt{P}\lambda_f(P)\varepsilon(f)\\ &= \frac{H_1(-4PN)}{2} + (-1)^{k/2-1}U_{k-2}\left(\frac{r\sqrt{N}}{2\sqrt{P}}\right) \sum_{0<r\leq 2\sqrt{P/N}} H_1(r^2N^2 - 4PN)\\ &\quad - \delta_{k-2}(P+1)\end{aligned}$$

Here $H_1(-d)$ ($d > 0$) is the Hurwitz class number, the number of equivalence classes of positive definite binary quadratic forms of discriminant $-d$ weighted by the number of automorphisms, i.e. with forms correspond to $x^2 + y^2$ or $x^2 + xy + y^2$ counted with multiplicity $1/2$ and $1/3$ respectively.

Hurwitz class number can be expressed as a sum of usual class numbers as

$$H_1(-d) = \sum_{f^2|d} h(-d/f^2) + O(1)$$

where the "error term" $O(1)$ disappears if $d \neq 3 \cdot \square$ or $4 \cdot \square$. Using this, we can rewrite the Skoruppa–Zagier trace formula as

$$\begin{aligned}&\sum_{f \in H^{\text{new}}(k,N)} \sqrt{P}\lambda_f(P)\varepsilon(f)\\ &= \frac{h(-4PN)}{2} + \frac{h(-PN)}{2} - \delta_{k=2}P + O(1)\\ &+ (-1)^{k/2-1}U_{k-2}\left(\frac{r\sqrt{N}}{2\sqrt{P}}\right) \sum_{1 \le r \le 2\sqrt{P/N}} \sum_{d^2|r^2N-4P} h\left(\frac{N(r^2N-4P)}{d^2}\right)\end{aligned}$$

From this, our new goal is to estimage the average of class numbers over short intervals, i.e. when $N \in [X, X+Y]$ with $Y = o(X)$. The main idea is to use class number formula to write class numbers as special L-values at $s = 1$, e.g.

$$h(-d) = \frac{\sqrt{d}}{\pi} L(1, \chi_d)$$

when $d > 4$ and $-d \equiv 1 (\text{mod } 4)$. Then the sum (average) of the corresponding L-values can be estimated via truncation and Polya–Vinogradov inequality. For example, we have an estimate

$$L(1, \chi_d) = \sum_{n \ge 1} \frac{\chi_d(n)}{n} = \sum_{1 \le n \le T} \frac{\chi_d(n)}{n} + O\left(\frac{\sqrt{d}\log d}{T}\right).$$

With some hard analysis, one get the following estimations.

Proposition 3.3 (Zubrilina [16, Proposition 3.1]). Let P be an odd prime and let $[X, X+Y]$ be an interval with $Y = o(X)$. Then as $X \to \infty$,

$$\begin{aligned}&\frac{\zeta(2)\pi}{XY} \sideset{}{^\square}\sum_{\substack{N \in [X,X+Y]\\ P \nmid N}} \left(\frac{h(-PN)}{2} + \frac{h(-4PN)}{2}\right)\\ &= A\sqrt{y} + O_\varepsilon\left(\frac{1}{P^{3/2}X^{1/2}} + \frac{P^{7/12}}{Y^{5/6}X^{5/12}} + \frac{YP^{1/2}}{X^{3/2}}\right)(XYP)^\varepsilon\end{aligned}$$

Proposition 3.4 (Zubrilina [16, Proposition 3.2]). Let P be an odd prime, $r \in \mathbb{N}$, and $X > Y > 0$ be such that $r^2(X+Y) < 4P$ for each $r > 2\sqrt{P/X}$. Let $y = P/X$. Then

$$\begin{aligned}&\frac{\zeta(2)\pi}{XY} \sum_{r \le 2\sqrt{P/X}} \sideset{}{^\square}\sum_{\substack{N \in [X,X+Y]\\ P \nmid N}} H_1\left(r^2N^2 - 4PN\right)\\ &= \sum_{r \le 2\sqrt{P/X}} \nu(r)\sqrt{4y - r^2}\\ &+ O\left(\frac{P^{11/10}}{Y^{2/5}X^{9/10}} + \frac{YP}{X^2} + \frac{PY^{1/2}}{X^{3/2}} + \frac{P}{X^{1/2}Y^{13/18}} + \frac{P}{XY^{1/9}}\right)(XYP)^\varepsilon\end{aligned}$$

4 Murmuration of elliptic curves

5 Other known cases

After the success of Zubrilina, a lot of people are interested in murmuration density for different objects in number theory. We list the known works here.

5.1 Flying Dirichlet characters

Lee, Oliver, and Pozdnyakov computed murmuration density for Dirichlet characters [8][1]. For complex characters, the corresponding murmuration densities are given by the following theorem.

Theorem 5.1 (Lee–Oliver–Pozdnyakov [8, Theorem 1.1]). Let $\mathcal{D}_+(N)$ (resp. $\mathcal{D}_-(N)$) denote the set of primitive even (resp. odd) Dirichlet characters modulo N. For $x \in \mathbb{R}_{>0}$, let $\lceil x \rceil^{\mathfrak{p}}$ be the smallest prime $\geq x$. For $c > 1$, $\delta > 0$, and $y > 0$, define

$$P_{\pm}(y, X, c) := \frac{\log X}{X} \sum_{\substack{N \in [X, cX] \\ N \text{ prime}}} \sum_{\chi \in \mathcal{D}_{\pm}(N)} \frac{\chi(\lceil yX \rceil^{\mathfrak{p}})}{\tau(\chi)}$$

$$P_{\pm}(y, X, \delta) := \frac{\log X}{X^{\delta}} \sum_{\substack{N \in [X, X+X^{\delta}] \\ N \text{ prime}}} \sum_{\chi \in \mathcal{D}_{\pm}(N)} \frac{\chi(\lceil yX \rceil^{\mathfrak{p}})}{\tau(\chi)}.$$

Then

$$\lim_{X \to \infty} P_{\pm}(y, X, c) = \begin{cases} \int_1^c \cos\left(\frac{2\pi y}{x}\right) \mathrm{d}x & \text{if } + \\ -i \int_1^c \sin\left(\frac{2\pi y}{x}\right) \mathrm{d}x & \text{if } - \end{cases} \tag{3}$$

and assumming RH, if $\frac{1}{2} < \delta < 1$ we have

$$\lim_{X \to \infty} P_{\pm}(y, X, \delta) = \begin{cases} \cos(2\pi y) & \text{if } + \\ -i \sin(2\pi y) & \text{if } - \end{cases} \tag{4}$$

The proof is much simpler than the case of modular forms. The main ingredient of the proof is the following formulas [8, Lemma 2.6]: for two distinct primes p and N,

$$\sum_{\chi \in \mathcal{D}_+(N)} \frac{\chi(p)}{\tau(\chi)} = \left(\frac{N-1}{N}\right) \cos\left(\frac{2\pi p}{N}\right) + \frac{1}{N} \tag{5}$$

$$\sum_{\chi \in \mathcal{D}_-(N)} \frac{\chi(p)}{\tau(\chi)} = -i \left(\frac{N-1}{N}\right) \sin\left(\frac{2\pi p}{N}\right) \tag{6}$$

Combined with the prime number theorem (which gives equidistribution results of primes in $[X, cX]$ normalized by X), we get (3). For (4), we need to estimate the number of primes in the short interval $[X, X + X^{\delta}]$, which is the part that requires RH.

They also proved similar results for real Dirichlet characters, but the proof is more complicated. Let $\mathcal{G}$ be the set of odd square-free integers and let $\chi_d = \left(\frac{d}{\cdot}\right)$. For a compactly supported smooth function $\Phi \geq 0$

[1] which can be thought as automorphic forms on GL_1 over $\mathbb{Q}$.

on $\mathbb{R}$, define

$$M_\Phi(y,X,\delta)=\frac{\log X}{X^{1+\delta}}\sum_{\substack{p\in[yX,yX+X^\delta]\\ p\text{ prime}}}\sum_{d\in\mathcal{G}}\Phi\left(\frac{d}{X}\right)\chi_{8d}(p)\sqrt{p}. \tag{7}$$

Theorem 5.2 (Lee–Oliver–Pozdnyakov [8, Theorem 1.2]). Fix $y>0$ and assume $\frac{3}{4}<\delta<1$. Assumming GRH, we have

$$M_\Phi(y,\delta):=\lim_{X\to\infty}M_\Phi(y,X,\delta)=\frac{1}{2}\sum_{\substack{a\geq 1\\ a\text{ odd}}}\frac{\mu(a)}{a^2}\sum_{m\geq 1}(-1)^m\widetilde{\Phi}\left(\frac{m^2}{2a^2y}\right), \tag{8}$$

where

$$\widetilde{\Phi}(\xi)=\int_{-\infty}^{\infty}(\cos(2\pi\xi x)+\sin(2\pi\xi x))\Phi(x)\mathrm{d}x. \tag{9}$$

The proof is more involved, which is based on the Polya–Vinogradov inequality

$$\left|\sum_{\substack{p\in[yX,yX+X^\delta]\\ p\text{ prime}}}\chi_d(p)\right|\ll(yX)^{\frac{1}{2}+\epsilon}$$

(for non-principal χ_d with $\frac{1}{2}<\delta<1$, which uses GRH [6]) and a summation formula

$$\frac{1}{X}\sum_{\substack{d\in\mathbb{Z}\\ d\text{ odd}}}\left(\sum_{\substack{a^2||d|\\ a\leq A}}\mu(a)\right)\Phi\left(\frac{d}{X}\right)\left(\frac{d}{p}\right)\sqrt{p}=\frac{1}{2}\left(\frac{2}{p}\right)\sum_{\substack{0<a\leq A\\ (a,2p)=1}}\frac{\mu(a)}{a^2}\sum_{k\in\mathbb{Z}}(-1)^k\left(\frac{k}{p}\right)\widetilde{\Phi}\left(\frac{kX}{2a^2p}\right)$$

which can be proved by using Poisson summation formula.

5.2 Flying Hecke characters of imaginary quadratic fields

[15]

5.3 Flying modular forms (in weight direction)

Recall that Zubrilina computed murmuration density for a *fixed weight k and varying level N*. In [1], Bober, Booker, Lee[2], and Lowry-Duda considered the opposite case, where they fix the level $N=1$ and vary the weight k. In this case, the considered family of Hecke newforms whose *analytic conductor*

$$\mathcal{N}(k):=\left(\frac{\exp\psi(k/2)}{2\pi}\right)^2\approx\left(\frac{k-1}{4\pi}\right)^2+O(1)$$

are in certain range, where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the digamma function.

Theorem 5.3 (Bober–Booker–Lee–Lowry-Duda [1, Theorem 1.1]). Fix $\epsilon\in(0,\frac{1}{12})$, $\delta\in\{0,1\}$, and a compact interval $E\subset\mathbb{R}_{>0}$ with $|E|>0$. Let $K,H>0$ with $K^{\frac{5}{6}+\epsilon}<H<K^{1-\epsilon}$, and let $N=\mathcal{N}(K)$. Then as $K\to\infty$, we have

$$\frac{\sum_{\substack{p\text{ prime}\\ p/N\in E}}\log p\sum_{\substack{k\equiv 2\delta\pmod 4\\ |k-K|\leq H}}\sum_{f\in H_k(1)}\lambda_f(p)}{\sum_{\substack{p\text{ prime}\\ p/N\in E}}\log p\sum_{\substack{k\equiv 2\delta\pmod 4\\ |k-K|\leq H}}\sum_{f\in H_k(1)}1}=\frac{(-1)^\delta}{\sqrt{N}}\left(\frac{\nu(E)}{|E|}+o_{E,\epsilon}(1)\right) \tag{10}$$

[2] Min Lee, not Kyu Hwan Lee

where

$$\nu(E) = \frac{1}{\zeta(2)} \sum_{\substack{a,q\in\mathbb{Z}_{>0}\\ (a,q)=1\\ q^2/a^2\in E}} \frac{\mu(q)^2}{\varphi(q)^2\sigma(q)} \left(\frac{a}{q}\right)^{-3} \tag{11}$$

$$= \frac{1}{2} \sum_{t\in\mathbb{Z}} \prod_{p\nmid t} \frac{p^2-p-1}{p^2-p} \int_E \cos\left(\frac{2\pi t}{\sqrt{y}}\right) \mathrm{d}y \tag{12}$$

where the summation $\sum^*$ indicates that the terms occuring at the endpoints of E are halved.

The main tool for the proof is the (original) Eichler–Selberg trace formula that does not include Atkin–Lehner operators (e.g. [3, Theorem 2.1]). Then apply class number formula to replace class numbers with the special values of Dirichlet L-functions at $s = 1$, which can be estimated under GRH.

5.4 Flying Maass forms

Booker, Lee, Lowry-Duda, Seymour-Howell, and Zubrilina computed murmuration densities for weight 0 and level 1 Maass forms [2]. They considered a family of Maass forms where the spectral parameter (R with $\lambda = \frac{1}{4} + R^2$) goes ot ∞, which is equivalent to the *analytic conductor* $\mathcal{N}(R)$ going to ∞.

Theorem 5.4 (Booker–Lee–Lowry-Duda–Seymour-Howell–Zubrilina [2, Theorem 1.1]). Let $E \subset \mathbb{R}_{>0}$ be a fixed compact interval with $|E| > 0$. Let $R, H > 0$ with $R^{\frac{5}{6}+\delta} < H < R^{1-\delta}$ for some $\delta > 0$ and $N = \mathcal{N}(R)$. Assumming GRH for L-functions of Dirichlet characters and Maass forms, as $R \to \infty$ we have

$$\frac{\sum_{\substack{p \text{ prime}\\ p/N\in E}} \log p \sum_{|r(f)-R|\le H} \epsilon(f) a_f(p)}{\sum_{\substack{p \text{ prime}\\ p/N\in E}} \log p \sum_{|r(f)-R|\le H} 1} \to \frac{1}{\sqrt{N}|E|} \sum^*_{\frac{q^2}{a^2}\in E} \frac{\mu(q)^2}{\varphi(q)^2\sigma(q)} \left(\frac{a}{q}\right)^{-3} \tag{13}$$

where the summation $\sum^*$ indicates that the terms occuring at the endpoints of E are halved.

Proof uses an explicit Selberg trace formula due to Strömbergsson in his unpublished work [13], which requires an analytic test function and cannot be compactly supported, where GRH is needed to control the cutoff error term. The remaining proof is similar to the weight aspect case of the modular forms [1].

5.5 General formulation

In fact, all the above works fit into the general framework suggested by Sarnak, in his letter to Sutherland and Zubrilina [9].

5.6 Elliptic curves?

Recently, Will Sawin and Andrew Sutherland announced a murmuration theorem for elliptic curves, which is slightly different from the formulation in [7].

Theorem 5.5 (Sawin–Sutherland [10]). Let $\mathcal{E}(X) := \{y^2 = x^3 + ax + b : a, b \in \mathbb{Z}, p^4 \mid a \Rightarrow p^6 \nmid b, \max\{4|a|^3, 27b^2\} \le X\}$ be the set of naive isomorphism classes of elliptic curves over $\mathbb{Q}$ ordered by height. For any smooth function $W : \mathbb{R}_{>0} \to \mathbb{R}$ with compact support, the limit

$$\lim_{X\to\infty} \frac{1}{|\mathcal{E}(X)|} \sum_{E\in\mathcal{E}(X)} \frac{\varepsilon(E)}{N_E} \sum_{n\ge1} W\left(\frac{n}{N_E}\right) a_n(E) \tag{14}$$

exists and is equal to

$$\int_0^\infty 2\pi W(u) \sum_{n\geq 1} \frac{\prod_{p|n} \ell_{p^{\nu_p(n)}}}{\sqrt{n}} \sqrt{u} J_1(4\pi\sqrt{un}) \mathrm{d}u \tag{15}$$

where $\ell_{p^\nu} = \frac{p^9 - p^8}{p^{10}-1} \mathrm{Tr}(T_p|S_{\nu+2}(1))$.

The difference between the original murmuration observed in HLOP [7] and the one in Sawin–Sutherland is well-explained in [10, Section 1.1]. The original murmuration considered the averages of the form

$$\mathbb{E}_{\substack{N(E)\in[X,X+1000] \\ \mathrm{rank}(E)=r}}[a_p(E)]$$

as a function in p for fixed r. However, subsequent works found that the dyadic intervals like $[X, 2X]$ or slightly smaller intervals like $[X, X+X^{1-\delta}]$ for $\delta > 0$ are more appropriate, since it make analysis more tractable and plots smoother. Hence the reformulated HLOP's murmuration would be

$$\mathbb{E}_{\substack{N(E)\in[X,2X] \\ \mathrm{rank}(E)=r}}[a_p(E)] \tag{16}$$

Also, later study found that the oscillations would converge to a continuous function in p/X, so we can understand (16) as (1) the limit of $X \to \infty$ with fixed p/X value, or (2) the limit of the average over p with p/X lies in a fixed interval.

Another subsequent observation is that considering all elliptic curves with different ranks would be better to study, where we weight $a_p(E)$ by the ϵ factor of E. Also, rather than p/X, the crucial ratio might be $p/N(E)$. In other words, we can consider further averaging over p where p/N lies in a certain interval, such as

$$\mathbb{E}_{N(E)\in[X,2X]}\left[\mathbb{E}_{p\in(C_1N(E),C_2N(E))}[\epsilon(E)a_p(E)]\right]$$

for $0 < C_1 < C_2$. Note that it is slightly easier to work with

$$\mathbb{E}_{N(E)\in[X,2X]}\left[\frac{\log\left(N(E)^{\frac{C_2-C_1}{2}}\right)}{N(E)} \sum_{p\in(C_1N(E),C_2N(E))} \epsilon(E)a_p(E)\right]$$

instead of the previous double expectation, where the term $N/\log(N^{\frac{C_2-C_1}{2}})$ roughly counts the number of primes in the interval (C_1N, C_2N). What Sawin and Sutherland proved is a naive height variation of the above average.

The main idea of the proof is using Voronoi summation formula.

You can find more on the Sutherland's lecture [14] at Tate conference (*The legacy of John Tate, and beyond* at Harvard university). He considered it as *a* murmuration theorem, and might not be *the* murmuration theorem since the density formula is too complicated.

References

[1] Bober, J., Booker, A. R., Lee, M., and Lowry-Duda, D. Murmurations of modular forms in the weight aspect. *arXiv preprint arXiv:2310.07746* (2023).

[2] Booker, A. R., Lee, M., Lowry-Duda, D., Seymour-Howell, A., and Zubrilina, N. Murmurations of Maass forms. *arXiv preprint arXiv:2409.00765* (2024).

[3] Child, K. Twist-minimal trace formula for holomorphic cusp forms. *Research in Number Theory 8*, 1 (2022), 11.

[4] Chiou, L. Elliptic curve murmurations found with AI take flight. *Qunata Magazine* (2024).

[5] Eichler, M. On the class number of imaginary quadratic fields and the sums of divisors of natural numbers. *The Journal of the Indian Mathematical Society* (1955), 153–180.

[6] Granville, A., and Soundararajan, K. Large character sums: pretentious characters and the Pólya-Vinogradov theorem. *Journal of the American Mathematical Society 20*, 2 (2007), 357–384.

[7] He, Y.-H., Lee, K.-H., Oliver, T., and Pozdnyakov, A. Murmurations of elliptic curves. *Experimental Mathematics* (2024), 1–13.

[8] Lee, K.-H., Oliver, T., and Pozdnyakov, A. Murmurations of Dirichlet characters. *International Mathematics Research Notices 2025*, 1 (2025), rnae277.

[9] Sarnak, P. Letter to Drew Sutherland and Nina Zubrilina. https://publications.ias.edu/sites/default/files/Nina%20and%20Drew%20letter_0.pdf. Accessed: 2025-04-12.

[10] Sawin, W., and Sutherland, A. Murmurations for elliptic curves ordered by height. *arXiv preprint arXiv:2504.12295* (2025).

[11] Selberg, A. Harmonic analysis and discontinuous groups in weakly symmetric riemannian spaces with applications to dirichlet series. *The Journal of the Indian Mathematical Society* (1956), 47–87.

[12] Skoruppa, N.-P., and Zagier, D. Jacobi forms and a certain space of modular forms.

[13] Strömbergsson, A. Explicit trace formula for Hecke operators.

[14] Sutherland, A. Sato–Tate distributions and murmurations. https://www.youtube.com/watch?v=EL5MzprelYM. Accessed: 2025-04-12.

[15] Wang, Z. Murmurations of Hecke *L*-Functions of Imaginary Quadratic Fields. *arXiv preprint arXiv:2503.17967* (2025).

[16] Zubrilina, N. Murmurations. *arXiv preprint arXiv:2310.07681* (2023).