Heisenberg's principle and positive functions

Principe d'Heisenberg et fonctions positives

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Abstract

We consider a natural problem concerning Fourier transforms. In one variable, one seeks functions f and $\widehat{f}$, both positive for $|x| \geq a$ and vanishing at 0. What is the lowest bound for a? In higher dimension, the same problem can be posed by replacing the interval by the ball of radius a. We show that there is indeed a strictly positive lower bound, which is estimated as a function of the dimension. In the last section the question, and its solution, are shown to be naturally related to the theory of zeta functions.

Introduction

The inequalities of Heisenberg's experiments, with the notations of the present article, have the form

$$\int x^2 |f(x)|^2 \mathrm{d}x \int y^2 |\widehat{f}(y)|^2 \mathrm{d}y \geq 1/16\pi^2$$

(if f is of norm 1)s, and they are optimal, since equality holds for $f(x) = e^{-\pi x^2}$. In the following form

$$\Delta p \Delta x \geq \hbar$$

they are interpreted by physicists as a relationship between ???;

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1 Statement of the problem and bounding B_1 from below

Consider a pair of functions $(f, \widehat{f})$ on reals: they are Fourier pairs if

$$\begin{cases} \widehat{f}(y) = \int f(x)e^{-2i\pi xy}\mathrm{d}x, & f \in L^1(\mathbb{R}) \\ f(x) = \int \widehat{f}(y)e^{2i\pi xy}\mathrm{d}y, & \widehat{f} \in L^1(\mathbb{R}). \end{cases}$$

So f and $\widehat{f}$ are continuous and converges to 0 at infinity. We are interested in the Fourier pairs $(f, \widehat{f})$ such that

1. f and $\widehat{f}$ are real-valued, even, and not identically zero,
2. $f(0) \leq 0$ and $\widehat{f}(0) \geq 0$,
3. $f(x) \geq 0$ for $x \geq a_f$ and $\widehat{f}(y) \geq 0$ for $y \geq a_{\widehat{f}}$.

Note that the condition 2 and the non-vanishing assumptions on f and $\widehat{f}$ imply a_f and $a_{\widehat{f}} > 0$.

Problem. What is the infimum of the product $a_f a_{\widehat{f}}$ for the Fourier pairs $(f, \widehat{f})$ satisfying 1–3?

We denote the infimum as $B_1 \geq 0$ (note that the pair attaining infimum clearly exists). We will show, which is not obvious a priori, that B_1 is strictly positive.

Until section 3, we will focus on dimension 1. For a Fourier pair $(f, \widehat{f})$ satisfying 1–3 let

$$A(f) = \inf\{x > 0 : f((x, \infty)) \subset \mathbb{R}^+\}$$
$$A(\widehat{f}) = \inf\{y > 0 : \widehat{f}((t, \infty)) \subset \mathbb{R}^+\}.$$

The product $A(f)A(\widehat{f})$ is invariant under scaling, i.e. replacing $f(x)$, $\widehat{f}(y)$ by $f(x/\lambda)$, $\lambda\widehat{f}(\lambda y)$, $\lambda > 0$. Since

$$B_1 = \inf A(f)A(\widehat{f})$$

for all Fourier pairs satisfying 1–3, we only consider pairs satisfying $A(f) = A(\widehat{f})$. Then $f + \widehat{f} \neq 0$ (consider their values at points near $A(f)$), and

$$A(f + \widehat{f}) \leq A(f) = A(\widehat{f}).$$

So $B_1 = \inf A^2(f + \widehat{f})$. Hence we see that

$$B_1 = A^2, \quad A = \inf A(f)$$

where infimum is taken over all functions $f \in L^1(\mathbb{R})$, real-valued and even, not identically zero, equal to their own Fourier transforms, and $f(0) < 0$.

Let

$$\gamma(x) = e^{-\pi x^2}$$

so that $\gamma = \widehat{\gamma}$. If $f(0) < 0$, $f - f(0)\gamma$ satisfies the same conditions as f, and

$$A(f - f(0)\gamma) \le A(f).$$

Finally,

$$A = \inf A(f) \tag{1}$$

where infimum is taken over all $f \in L^1(\mathbb{R})$, real-valued, even, not identically zero, $f = \widehat{f}$, and $f(0) = 0$.

Here is an important result.

Theorem 1.1. Let $\lambda = -\inf\left(\frac{\sin x}{x}\right) = 0.2712\cdots$. Then

$$A \ge \frac{1}{2(1+\lambda)} = 0.4107\cdots$$

so

$$B \ge 0.1687\cdots.$$

Proof. Choose $f = \widehat{f}$, $f(0) = 0$, and $\int_{\mathbb{R}} |f(x)|\mathrm{d}x := \int_{\mathbb{R}} |f| = 1$. Write $A = A(f)$. Put $f = f^+ - f^-$, $|f| = f^+ + f^-$. Since $\int_{\mathbb{R}} f = \widehat{f}(0) = 0$, we have $\int_{\mathbb{R}} f^+ = \int_{\mathbb{R}} f^- = \int_{-A}^{A} f^- = \frac{1}{2}$. So $\int_{-A}^{A} |f| \ge \frac{1}{2}$. From $|f(x)| \le \int |\widehat{f}| = 1$, $2A \ge \frac{1}{2}$ and we obtain a first bound $A \ge \frac{1}{4}$. We will see that this argument extends to higher dimensions.

In dimension 1, we can refine it in the following way. From $f = \widehat{f}$,

$$\begin{aligned} f(x) &= \int f(y)\cos 2\pi yx\mathrm{d}y = \int f(y)(\cos 2\pi yx - 1)\mathrm{d}y \\ &= \int f^-(y)(1 - \cos 2\pi yx)\mathrm{d}y - \int f^+(y)(\cos 2\pi yx - 1)\mathrm{d}y. \end{aligned}$$

This implies, ???

$$f^-(x) \le \int f^+(y)(1 - \cos 2\pi yx)\mathrm{d}y$$

and

$$\frac{1}{4} = \int_0^A f^- \le \int_{-\infty}^{\infty} f^+(y)\left(A - \frac{\sin 2\pi yA}{2\pi y}\right)\mathrm{d}y$$

so

$$\frac{1}{4} \le \frac{A}{2}\sup_{u\in\mathbb{R}}\left(1 - \frac{\sin u}{u}\right) = \frac{A}{2}(1+\lambda)$$

and we obtain the theorem. $\square$

Later, we will need to consider functions that are regular enough. A natural class is the Schwartz space $\mathcal{S}$. It is not obvious that the infimum A defined by (1), taken only over the functions in $\mathcal{S}$, coincides with that over all $f \in L^1(\mathbb{R})$.

Let $\mathcal{B}_1$ be A^2, where A is defined by (1) for $f \in \mathcal{S}$. We will see that B_1 and $\mathcal{B}_1$ are not much different. Clearly, we have

$$B_1 \leq \mathcal{B}_1. \tag{2}$$

Let

$$B_1^- = \inf\{A^2 : f(0) < 0, f = \widehat{f} \text{ even} \neq 0, f \in L^1(\mathbb{R})\}.$$

Hence B_1^- is defined by (1), with additional assumption $f(0) < 0$. Define $\mathcal{B}_1^-$ similarly for $f \in \mathcal{S}$. Clearly,

$$B_1^- \leq \mathcal{B}_1^- \tag{3}$$

$$\mathcal{B}_1 \leq \mathcal{B}_1^-, \quad B_1 \leq B_1^-. \tag{4}$$

To prove $\mathcal{B}_1^- \leq B_1^-$, let $f \in L^1(\mathbb{R})$ be a function satisfying the conditions for (1) but $f(0) < 0$, and let $a = A(f)$. Let $\varphi = \psi * \psi$, where ψ is C^∞, even, positive, and compactly supported near 0, and $g = f * \varphi$. Then $A(g) \leq a + \varepsilon$ and $g(0) < 0$. We have $\widehat{g} = \widehat{f}\widehat{\psi}^2$; by applying the same operation on $\widehat{g}$ we obtain a function $h \in \mathcal{S}$ such that $h = \widehat{h}$, $h(0) < 0$, and $A(h) \leq a + \varepsilon$; from this we get $\mathcal{B}_1^- \leq B_1^-$ and

$$\mathcal{B}_1^- = B_1^-. \tag{5}$$

Note that the argument does not work if $f(0) = 0$. We will show

$$B_1^- \leq 2B_1; \tag{6}$$

combining (4) and (6) we obtain

$$B_1 \leq \mathcal{B}_1 \leq 2B_1. \tag{7}$$

Let f be a function satisfying the conditions for (1) and $a = A(f)$. Since $\widehat{f}(0) = \int f(x)\mathrm{d}x = 0$, f takes a negative value on $[-a, a]$. Let $b > 0$ be such a number, and consider the distribution

$$T = \delta_b + \delta_{-b} + 2\delta_0.$$

It is a positive measure with positive Fourier transform

$$\widehat{T} = 2\cos(2\pi b y) + 2 \geq 0.$$

We have

$$(T * f)(0) = f(b) + f(-b) < 0.$$

Since $b < a$, $g = T * f$ satisfies

$$g(0) < 0, \quad g \geq 0 \text{ on } (2a, \infty).$$

Moreover $\widehat{g} = \widehat{T}\widehat{f}$ is nonnegative on $[0, \infty)$, and $\widehat{g}(0) = 0$. By scaling, we obtain a function h such that

$$\begin{aligned} h &\geq 0 \text{ on } [a\sqrt{2}, \infty), \quad h(0) < 0 \\ \widehat{h} &\geq 0 \text{ on } [a\sqrt{2}, \infty), \quad \widehat{h}(0) = 0. \end{aligned}$$

The functions h and $\widehat{h}$ are real-valued and even. Hence $h + \widehat{h}$ satisfy the conditions defining B_1^-. So $B_1^- \leq (a\sqrt{2})^2 = 2a^2$; by varying f, we obtain (6).

2 The Global Conjectures

References

[1] Aizenbud, A. A partial analog of the integrability theorem for distributions on p-adic spaces and applications. *Israel Journal of Mathematics 193* (2013), 233–262.

[2] Aizenbud, A., Gourevitch, D., Rallis, S., and Schiffmann, G. Multiplicity one theorems. *Annals of Mathematics* (2010), 1407–1434.

[3] Arthur, J. *The Endoscopic classification of representations orthogonal and symplectic groups*, vol. 61. American Mathematical Soc., 2013.

[4] Atobe, H. The local theta correspondence and the local gan–gross–prasad conjecture for the symplectic-metaplectic case. *Mathematische Annalen 371* (2018), 225–295.

[5] Beuzard-Plessis, R. Factorisations de périodes et formules de plancherel. *Peccot lecture series, held at the Collège de France, Paris, April* (2017).

[6] Beuzart-Plessis, R. Expression d'un facteur epsilon de paire par une formule intégrale. *Canadian Journal of Mathematics 66*, 5 (2014), 993–1049.

[7] Beuzart-Plessis, R. Endoscopie et conjecture locale raffinée de gan–gross–prasad pour les groupes unitaires. *Compositio Mathematica 151*, 7 (2015), 1309–1371.

[8] Beuzart-Plessis, R. A local trace formula for the local gan-gross-prasad conjecture for unitary groups: The archimedean case. *preprint* (2015).

[9] Beuzart-Plessis, R. La conjecture locale de gross-prasad pour les représentations tempérées des groupes unitaires. *Mémoires de la Société Mathématique de France 149* (2016), 191p.

[10] Beuzart-Plessis, R. Comparison of local relative characters and the ichino–ikeda conjecture for unitary groups. *Journal of the Institute of Mathematics of Jussieu 20*, 6 (2021), 1803–1854.

[11] Casselman, W. Canonical extensions of harish-chandra modules to representations of g. *Canadian Journal of Mathematics 41*, 3 (1989), 385–438.

[12] Chaudouard, P.-H., and Zydor, M. Le transfert singulier pour la formule des traces de jacquet–rallis. *Compositio Mathematica 157*, 2 (2021), 303–434.

[13] Clozel, L. Changement de base pour les représentations tempérées des groupes réductifs réels. In *Annales scientifiques de l'École Normale Supérieure* (1982), vol. 15, pp. 45–115.

[14] Deligne, P. Les constantes des équations fonctionnelles des fonctions l. In *Modular Functions of One Variable II: Proceedings International Summer School University of Antwerp, RUCA July 17–August 3, 1972* (1973), Springer, pp. 501–597.

[15] Flicker, Y. Z. Twisted tensors and euler products. *Bulletin de la Société Mathématique de France 116*, 3 (1988), 295–313.

[16] Gan, W., and Takeda, S. On the howe duality conjecture in classical theta correspondence. *Advances in the Theory of Automorphic Forms and Their L-functions* (2016), 105–117.

[17] Gan, W. T. Recent progress on the gross–prasad conjecture. *Acta Mathematica Vietnamica 39* (2014), 11–33.

[18] Gan, W. T., Gross, B. H., and Prasad, D. Symplectic local root numbers, central critical l-values, and restriction problems in the representation theory of classical groups. *Astérisque* (2011), No–pp.

[19] Gan, W. T., and Ichino, A. Formal degrees and local theta correspondence. *Inventiones mathematicae 195* (2014), 509–672.

[20] Gan, W. T., and Ichino, A. The gross–prasad conjecture and local theta correspondence. *Inventiones mathematicae 206* (2016), 705–799.

[21] Ginzburg, D., Jiang, D., and Rallis, S. Models for certain residual representations of unitary groups. automorphic forms and l-functions i. global aspects, 125–146. *Contemp. Math 488*.

[22] Ginzburg, D., Jiang, D., and Rallis, S. On the nonvanishing of the central value of the rankin-selberg l-functions. *Journal of the American Mathematical Society 17*, 3 (2004), 679–722.

[23] Ginzburg, D., Jiang, D., and Rallis, S. On the nonvanishing of the central value of the rankin-selberg l-functions, ii, automorphic representations, l-functions and applications: Progress and prospects, 157-191. *Ohio State Univ. Math. Res. Inst. Publ 11* (2005).

[24] Goodman, R., Wallach, N. R., et al. *Symmetry, representations, and invariants*, vol. 255. Springer, 2009.

[25] Grobner, H., Harris, M., and Lin, J. Deligne's conjecture for automorphic motives over cm-fields. *arXiv preprint arXiv:1802.02958* (2018).

[26] Gross, B. H., and Prasad, D. On the decomposition of a representation of SO_n when restricted to SO_{n-1}. *Canadian Journal of Mathematics 44*, 5 (1992), 974–1002.

[27] Gross, B. H., and Prasad, D. On irreducible representations of $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$. *Canadian Journal of Mathematics 46*, 5 (1994), 930–950.

[28] Harris, M., and Taylor, R. *The Geometry and Cohomology of Some Simple Shimura Varieties.(AM-151), Volume 151*, vol. 151. Princeton university press, 2001.

[29] Harris, R. N. The refined gross–prasad conjecture for unitary groups. *International Mathematics Research Notices 2014*, 2 (2014), 303–389.

[30] He, H. On the gan–gross–prasad conjecture for u (p, q). *Inventiones mathematicae 209* (2017), 837–884.

[31] Heiermann, V. A note on standard modules and vogan l-packets. *manuscripta mathematica 150* (2016), 571–583.

[32] Henniart, G. Une preuve simple des conjectures de langlands pour $\mathrm{GL}(n)$ sur un corps p-adique. *Inventiones mathematicae 139* (2000), 439–455.

[33] Ichino, A. Trilinear forms and the central values of triple product l-functions.

[34] Ichino, A., and Ikeda, T. On the periods of automorphic forms on special orthogonal groups and the gross–prasad conjecture. *Geometric and Functional Analysis 19* (2010), 1378–1425.

[35] Jacquet, H., Piatetskii-Shapiro, I. I., and Shalika, J. A. Rankin-selberg convolutions. *American journal of mathematics 105*, 2 (1983), 367–464.

[36] Jacquet, H., and Rallis, S. On the gross-prasad conjecture for unitary groups. *On certain L-functions 13* (2011), 205–264.

[37] Kaletha, T., Minguez, A., Shin, S. W., and White, P.-J. Endoscopic classification of representations: inner forms of unitary groups. *arXiv preprint arXiv:1409.3731* (2014).

[38] Kottwitz, R. E., and Shelstad, D. Foundations of twisted endoscopy. *Astérisque 255* (1999), 1–190.

[39] Langlands, R. P. On the classification of irreducible representations of real algebraic groups. *Representation theory and harmonic analysis on semisimple Lie groups 31* (1989), 101–170.

[40] Lapid, E. M. The relative trace formula and its applications. *Automorphic Forms and Automorphic L-Functions 1468* (2006), 76–87.

[41] Liu, Y. Relative trace formulae toward bessel and fourier–jacobi periods on unitary groups. *Manuscripta Mathematica 145* (2014), 1–69.

[42] Mezo, P. Tempered spectral transfer in the twisted endoscopy of real groups. *Journal of the Institute of Mathematics of Jussieu 15*, 3 (2016), 569–612.

[43] Mœglin, C., Vignéras, M.-F., and Waldspurger, J.-L. *Correspondances de Howe sur un corps p-adique*, vol. 1291. Springer, 2006.

[44] Moeglin, C., and Waldspurger, J.-L. *Stabilisation de la formule des traces tordue*. Springer.

[45] Mœglin, C., and Waldspurger, J.-L. *Décomposition spectrale et séries d'Eisenstein: une paraphrase de l'écriture*, vol. 113. Springer Science & Business Media, 1994.

[46] Mœglin, C., and Waldspurger, J.-L. La conjecture locale de gross-prasad pour les groupes spéciaux orthogonaux: Le cas général par. *Astérisque 347* (2012), 167–216.

[47] Mok, C. P. *Endoscopic classification of representations of quasi-split unitary groups*, vol. 235. American Mathematical Society, 2015.

[48] Prasad, D. On the local howe duality correspondence. *International mathematics research notices 1993*, 11 (1993), 279–287.

[49] Prasad, D. Theta correspondence for unitary groups. *Pacific Journal of Mathematics 194*, 2 (2000), 427–438.

[50] Prasanna, K. A., and Venkatesh, A. Automorphic cohomology, motivic cohomology, and the adjoint l-function. *Astérisque 428* (2021).

[51] Ramakrishnan, D. A mild tchebotarev theorem for $\mathrm{GL}(n)$. *Journal of Number Theory 146* (2015), 519–533.

[52] Rodier, F. Modèle de whittaker et caractères de représentations. In *Non-Commutative Harmonic Analysis: Actes du Colloque d'Analyse Harmonique Non Commutative, Marseille-Luminy, 1 au 5 Juillet 1974* (2006), Springer, pp. 151–171.

[53] Scholze, P. The local langlands correspondence for GL_n over p-adic fields. *Inventiones mathematicae 192* (2013), 663–715.

[54] Shelstad, D. L-indistinguishability for real groups. *Mathematische Annalen 259*, 3 (1982), 385–430.

[55] Shelstad, D. Tempered endoscopy for real groups. i. geometric transfer with canonical factors. *Representation theory of real reductive Lie groups 472* (2008), 215–246.

[56] Shelstad, D. Tempered endoscopy for real groups. ii. spectral transfer factors. *Automorphic forms and the Langlands program 9* (2010), 236–276.

[57] Sun, B. Multiplicity one theorems for fourier-jacobi models. *American Journal of Mathematics 134*, 6 (2012), 1655–1678.

[58] Sun, B., and Zhu, C.-B. Multiplicity one theorems: the archimedean case. *Annals of Mathematics* (2012), 23–44.

[59] Tate, J. Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part* (1979), vol. 2, pp. 3–26.

[60] Vogan, D. A. The local langlands conjecture. *Representation theory of groups and algebras* (1993), 305–379.

[61] Waldspurger, J.-L. Sur les valeurs de certaines fonctions l automorphes en leur centre de symétrie. *Compositio Mathematica 54*, 2 (1985), 173–242.

[62] Waldspurger, J.-L. Démonstration d'une conjecture de dualité de howe dans le cas p-adique, $p \neq 2$. *(No Title)* (1990), 267.

[63] Waldspurger, J.-L. Une formule intégrale reliée à la conjecture locale de gross–prasad. *Compositio Mathematica 146*, 5 (2010), 1180–1290.

[64] Waldspurger, J.-L. Calcul d'une valeur d'un facteur ε par une formule intégrale par. *Astérisque 347* (2012), 1–102.

[65] Waldspurger, J.-L. La conjecture locale de gross-prasad pour les représentations tempérées des groupes spéciaux orthogonaux. *Ast\'{e} risque*, 347 (2012), 103.

[66] Waldspurger, J.-L. Une formule intégrale reliée à la conjecture locale de gross-prasad, 2e partie: extension aux représentations tempérées. *Ast\'{e} risque*, 346 (2012), 171.

[67] Wallach, N. R. Real reductive groups. ii, volume 132 of. *Pure and Applied Mathematics*.

[68] Weil, A. *Adeles and algebraic groups*, vol. 23. Springer Science & Business Media, 2012.

[69] Weyl, H. *The classical groups: their invariants and representations*, vol. 45. Princeton university press, 1946.

[70] Xue, H. The gan–gross–prasad conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$. *Advances in Mathematics 262* (2014), 1130–1191.

[71] Xue, H. Fourier–jacobi periods and the central value of rankin–selberg l-functions. *Israel Journal of Mathematics 212* (2016), 547–633.

[72] XUE, H. Fourier–jacobi periods and local spherical character identities.

[73] XUE, H. Refined global gan–gross–prasad conjecture for fourier–jacobi periods on symplectic groups. *Compositio Mathematica 153*, 1 (2017), 68–131.

[74] XUE, H. On the global gan–gross–prasad conjecture for unitary groups: approximating smooth transfer of jacquet–rallis. *Journal für die reine und angewandte Mathematik (Crelles Journal) 2019*, 756 (2019), 65–100.

[75] YUN, Z., AND GORDON, J. The fundamental lemma of jacquet and rallis. *Duke Math. J. 156*, 1 (2011), 167–227.

[76] ZHANG, W. Automorphic period and the central value of rankin-selberg l-function. *Journal of the American Mathematical Society 27*, 2 (2014), 541–612.

[77] ZHANG, W. Fourier transform and the global gan—gross—prasad conjecture for unitary groups. *Annals of Mathematics 180*, 3 (2014), 971–1049.

[78] ZYDOR, M. La variante infinitésimale de la formule des traces de jacquet—rallis pour les groupes unitaires. *Canadian Journal of Mathematics 68*, 6 (2016), 1382–1435.

[79] ZYDOR, M. La variante infinitésimale de la formule des traces de jacquet-rallis pour les groupes linéaires. *Journal of the Institute of Mathematics of Jussieu 17*, 4 (2018), 735–783.

[80] ZYDOR, M. Les formules des traces relatives de jacquet–rallis grossières. *Journal für die reine und angewandte Mathematik (Crelles Journal) 2020*, 762 (2020), 195–259.