The Littlewood-Richardson rule and decomposition of representations of $\mathfrak{sl}(3, \mathbb{C})$

Seewoo Lee

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Let V_n be a unique irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ with highest weight n. We can show that the representation $V_m \otimes V_n$ decomposes as

$$V_m \otimes V_n = \bigoplus_{k=0}^{m-n} V_{m+n-2k}$$

for $m \geq n$. This can be shown by analyzing multiplicity of weights $V_m \otimes V_n$, or by using the Weyl character formula.

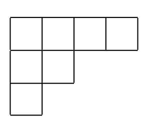
In this note, we introduce the Littlewood-Richardson rule, which gives a formula to express a product of two Schur polynomials into a linear combination of Schur polynomials. Using this, we decompose a representation $V_{\lambda_1} \otimes V_{\lambda_2}$ into irreducibles, where V_{λ_i} is a unique irreducible representation of highest weight λ_i for $i = 1, 2$.

1 Irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$

Here we give a summary of the representation theory of $\mathfrak{sl}(3, \mathbb{C})$.

2 The Littlewood-Richardson rule

In this section, we introduce Schur polynomials and the Littlewood-Richardson rule. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition of $d = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and let $\mathcal{P}_{d,n}$ be a set of such partitions, i.e. partition of d with at most n nonzero numbers. Now the *Young diagram* $Y(\lambda)$ of the partition λ is a subset $\{(i, j) \in \mathbb{N}^2 : j \in [\lambda_i]$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ and $[n] = \{0, 1, \dots, n-1\}$. For example, we can display the Young diagram $Y((4, 2, 1))$ as



A *semistandard Young tableux* T of a partition λ is a function $T : Y(\lambda) \to \mathbb{N}$, $(i,j) \mapsto T_{i,j}$ that satisfies $T_{i,j} \le T_{i,j+1}$ and $T_{i,j} < T_{i+1,j}$ for any $i, j \in \mathbb{N}$. For example,

$$\begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 1 & 2 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$$

depicts a semistandard Young tableau of shape $(4,2,1)$ and entries in $[3]$. Let $\mathrm{SST}(\lambda, n)$ be a set of semistandard Young tableaux of shape λ and entries in $[n]$. The *weight* $\mathrm{wt}(T)$ of $T \in \mathrm{SST}(\lambda, n)$ is $\alpha \in \mathbb{N}^n$ such that α_k counts the occurrences of the entry k in T; the tableau just depicted has weight $(2,3,2)$. Then the *Schur polynomial* $s_\lambda(n)$ of $\lambda \in \mathcal{P}_{d,n}$ is defined by

$$s_\lambda(n) = \sum_{T \in \mathrm{SST}(\lambda,n)} x^{\mathrm{wt}(T)}.$$

We can prove that any Schur polynomials are symmetric, and they forms a basis of the space Λ_n^d of symmetric polynomials of n variables with degree at most d over $\mathbb{Z}$. For example, the set $\mathrm{SST}((2,1),3)$ has the following 8 elements

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

and the Schur polynomial $s_{(2,1)}(3)$ of $\lambda = (2,1)$ is

$$s_{(2,1)}(3) = x_0^2 x_1 + x_0 x_1^2 + x_0^2 x_2 + x_0 x_2^2 + x_1^2 x_2 + x_1 x_2^2 + 2x_0 x_1 x_2.$$

Since Schur polynomials forms a basis, we may express the product $s_{\lambda_1}(n) s_{\lambda_2}(n)$ as a $\mathbb{Z}$-linear combination of Schur polynomials. The Littlewood-Richardson rule gives us a way to compute the coefficient, by counting some combinatorial objects.

3 Decomposition of $V_{\lambda_1} \otimes V_{\lambda_2}$

Now we are going to decompose the tensor product representation as irreducibles, by using the Littlewood-Richardson rule. First, we introduce how to interpret the irreducible representations of given highest weight as partitions. As we said in the section 1, for each given highest weight $\lambda = mL_1 - nL_3$, there exists a unique irreducible representation V_λ, of highest weight λ. Now for each weight of the representation V_λ, we can express it as a sum $b_1 L_1 + b_2 L_2 + b_3 L_3$, where $b_i \ge 0$ for all $i = 1, 2, 3$ and $b_1 + b_2 + b_3 = 2m + n$. For example, we can see the picture for $\lambda = 3L_1 - 2L_3$:

Now we can associate

Example 1. Let $\lambda_1 = L_1$ and $\lambda_2 = -L_3 = L_1 + L_2$. Then each λ_i's correspond to partitions $(1,0) \in \mathcal{P}_{1,3}$ and $(1,1) \in \mathcal{P}_{2,3}$. For each partitions, the corresponding Schur polynomials are

$$\begin{aligned}
s_{(1,0)}(3) &= x_0 + x_1 + x_2 \\
s_{(1,1)}(3) &= x_0x_1 + x_0x_2 + x_1x_2.
\end{aligned}$$

Now the product decomposes as

$$\begin{aligned}
s_{(1,0)}(3)s_{(1,1)}(3) &= (x_0 + x_1 + x_2)(x_0x_1 + x_0x_2 + x_1x_2) \\
&= x_0^2x_1 + x_0x_1^2 + x_0^2x_2 + x_0x_2^2 + x_1^2x_2 + x_1x_2^2 + 3x_0x_1x_2 \\
&= (x_0^2x_1 + x_0x_1^2 + x_0^2x_2 + x_0x_2^2 + x_1^2x_2 + x_1x_2^2 + 2x_0x_1x_2) + x_0x_1x_2 \\
&= s_{(2,1)}(3) + s_{(1,1,1)}(3)
\end{aligned}$$

and each partition corresponds to weights $2L_1 + L_2$ and $L_1 + L_2 + L_3 = 0$. Thus we get

$$V_{L_1} \otimes V_{L_1+L_2} = V_{2L_1+L_2} \oplus V_0.$$

Example 2. Let $\lambda_1 = L_1 - L_3 = 2L_1 + L_2$ and $\lambda_2 = 2L_1 - L_3 = 3L_1 + 2L_2$. Then each λ_i's correspond to partitions $(2,1) \in \mathcal{P}_{3,3}$ and $(3,2) \in \mathcal{P}_{5,3}$. For each partitions, the corresponding Schur polynomials are

$$\begin{aligned}
s_{(2,1)}(3) &= x_0^2x_1 + x_0x_1^2 + x_0^2x_2 + x_0x_2^2 + x_1^2x_2 + x_1x_2^2 + 2x_0x_1x_2 \\
s_{(3,2)}(3) &= (x_0^3x_1^2 + x_0^2x_1^3 + x_1^3x_2^2 + x_1^2x_2^3 + x_2^3x_0^2 + x_2^2x_0^3) \\
&\quad + (x_0^3x_1x_2 + x_1^3x_2x_0 + x_2^3x_0x_1) \\
&\quad + 2(x_0^2x_1^2x_2 + x_1^2x_2^2x_0 + x_2^2x_0^2x_1).
\end{aligned}$$

Now by the Littlewood-Richardson rule, we have

$$s_{(2,1)}(3)s_{(3,2)}(3) = \sum_{\substack{\lambda \supset (2,1),(3,2) \\ \lambda \in \mathcal{P}_{8,3}}} c^{\lambda}_{(2,1),(3,2)} s_\lambda(3)$$

where the coefficients are given by

$$\begin{aligned}
&c^{(6,2)} = 0 \\
&c^{(5,3)} = c^{(5,2,1)} = c^{(4,4)} = c^{(4,2,2)} = c^{(3,3,2)} = 1 \\
&c^{(4,3,1)} = 2.
\end{aligned}$$

Note that the last coefficient $c^{(4,3,1)} = 2$ counts the following two tableau

$$\begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & 0 & 0 \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 0 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad\qquad \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & 0 & 0 \\ \cline{2-4} \multicolumn{1}{c|}{} & 0 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Thus we get the following decomposition

$$V_{2L_1+L_2} \otimes V_{3L_1+2L_2} = V_{5L_1+3L_2} \oplus V_{4L_1+L_2} \oplus V_{4L_1+4L_2} \oplus V_{2L_1} \oplus V_{L_1+L_2} \oplus V^{\oplus 2}_{3L_1+2L_2}.$$

References

[1] Marc A. A. van Leeuwen, *The Littlewood-Richardson Rule, and Related Combinatorics*,