Math 53 (Multivariable Calculus), Section 102 & 108
Week 5, Friday
Sep 23, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

1. Show that the curve with parametric equations $x = t\cos t, y = t\sin t, z = t^2$ lies on the elliptic paraboloid $x^2 + y^2 = z$, and use this fact to sketch the curve.

2. Show that the curve with parametric equations $x(t) = t^2 - 1, y(t) = -t + 1, z(t) = -t^2 + t + 1$ lies on a plane. Find an equation of the plane.

3. Find a vector function that represents the curve of intersection of the hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$.

Solution

1. For a point on a curve, we have

$$x^2 + y^2 = (t\cos t)^2 + (t\sin t)^2 = t^2(\cos^2 t + \sin^2 t) = t^2 = z$$

so it lies on the elliptic paraboloid.

2. Let's assume that the equation of the plane is given by $ax + by + cz + d = 0$ for some a, b, c, d. Then, for all t, we should have

$$a(t^2 - 1) + b(-t + 1) + c(-t^2 + t + 1) + d = (a - c)t^2 + (-b + c)t + (-a + b + c + d) = 0$$

From this, we have $a - c = 0 \Leftrightarrow a = c$, and $-b + c = 0 \Leftrightarrow b = c$. Then we get $-a + b + c + d = 0 = -a + a + a + d = a + d \Leftrightarrow d = -a$. It means that our equation of the plane is

$$ax + by + cz + d = ax + ay + az - a = a(x + y + z - 1) = 0,$$

and any choice of a gives the (essentially) same equation, for example, $x + y + z - 1 = 0$.

3. First, we can express x and y in terms of a single parameter t as $x = \cos t, y = \sin t$. Then we have $z = x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t$, and this gives a vector function

$$\mathbf{r}(t) = (\cos t, \sin t, \cos 2t)$$