Taylor approximation is not a universal problem solver

Seewoo Lee

Abstract

This is an example where Taylor approximation does not work at all.

Define a function $y = f(x)$ as follows:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Q1. Show that the function is continuous.

Q2. Show that the function is differentiable. What is $f'(0)$? To compute this, you can use the following fact without proof: exponential grows faster than any polynomials. In other words, for any $N \geq 0$,

$$\lim_{x \to \infty} \frac{x^N}{e^x} = 0$$

Q3. For $x \neq 0$, show that $f^{(n)}(x)$ has a form of

$$f^{(n)}(x) = e^{-1/x^2} p_n(1/x)$$

for a polynomial $p_n(x)$. (Hint: use mathematical induction on n to find a recursive relation on p_n.) Can you express degree of p_n in terms of n?

Q4. Using Q3, show that $f^{(n)}(0) = 0$ for any $n \geq 1$.

Q5. What is n-th Taylor polynomial of $f(x)$ at $x = 0$? (Answer: zero)

Hence for this specific function, the Taylor approximations are zero and it does not give you any useful approximations of e^{-1/x^2}. The functions where $\lim_{n \to \infty} T_n f(x) = f(x)$ holds is called *real analytic functions*, and the above function is an example of non-real analytic function. Most of the functions we see in the course are real analytic though.