

1. Diagonalize the following matrices. In other words, for each matrix, find an invertible matrix  $P$  and a diagonal matrix  $D$  such that the matrix equals to  $PDP^{-1}$ .

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$B = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

2. Compute  $A^{20}$ .

Use the above diagonalization.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{20} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^{20} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{1}{2}(3^{20} - 1) \\ 0 & 3^{20} - \frac{1}{2} \end{bmatrix}$$

3. Let

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Compute  $C^2, C^3, C^4$ , and  $C^5$ . Can you find a pattern?

$$C^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I, \quad C^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad C^5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = C.$$

It is periodic with the period 4.

4. What is  $C^{2023}$ ? Can you compute it using diagonalization?

Since  $2023 = 4 \times 505 + 3$ ,  $C^{2023} = C^3$ . Using diagonalization, you can check that  $i, -i$  are eigenvalues of  $C$  with eigenvectors  $\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}$ , so

$$C = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$$

$$C^{2023} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^{2023} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(Note that  $i = i, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \dots$ )

5. Consider the recursion of the form

$$\mathbf{x}_{t+1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \mathbf{x}_t, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

What is  $\mathbf{x}_5$ ?

Let  $A$  be the above matrix and diagonalize it. We have

$$\det(A - \lambda I) = \lambda^2 - \frac{7}{6} + \frac{1}{6} = (\lambda - 1) \left( \lambda - \frac{7}{6} \right) \Rightarrow \lambda = 1, \frac{1}{6}.$$

(a) For  $\lambda_1 = 1$ , the corresponding eigenvector  $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$  satisfies

$$(A - \lambda_1 I) \mathbf{v}_1 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -\frac{1}{2}x_1 + \frac{1}{2}y_1 = 0.$$

We can take  $x_1 = 1$  and  $y_1 = 1$ , and we get an eigenvector  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

(b) For  $\lambda_2 = \frac{1}{6}$ , the corresponding eigenvector  $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$  satisfies

$$(A - \lambda_2 I) \mathbf{v}_2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \frac{1}{3}x_2 + \frac{1}{2}y_2 = 0.$$

We can take  $x_2 = 3$  and  $y_2 = -2$ , and we get an eigenvector  $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ .

So we have

$$A = PDP^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}^{-1}$$

and

$$\mathbf{x}_5 = A^5 \mathbf{x}_0 = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6^5} \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} + \frac{3}{5 \times 6^5} \\ \frac{2}{5} - \frac{2}{5 \times 6^5} \end{bmatrix}$$

6. What is the limiting behavior of  $\mathbf{x}_t$ , i.e.  $\mathbf{y} = \lim_{t \rightarrow \infty} \mathbf{x}_t$ ?

$$\begin{aligned} \mathbf{y} &= \lim_{t \rightarrow \infty} \mathbf{x}_t = \lim_{t \rightarrow \infty} A^t \mathbf{x}_0 = \lim_{t \rightarrow \infty} \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}^t \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \end{bmatrix} \end{aligned}$$

7. Check that  $\mathbf{y}$  satisfies

$$\mathbf{y} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \mathbf{y}.$$

Can you explain why?

You can directly check it by multiplying the matrix. Intuitively,  $\mathbf{y}$  is the limiting behavior of  $\mathbf{x}_t$ , so it is *stable* (equilibrium) in the sense that the state does not change for further steps.

8. Let  $A$  be a 3 by 3 matrix with the eigenvalues and eigenvectors.

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 0.7, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = -0.3, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Find  $\lim_{k \rightarrow \infty} A^k$ .

Based on given eigenvalues and eigenvectors, we can diagonalize  $A$  as

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}.$$

Since  $A^k = (PDP^{-1})^k = PD^kP^{-1}$ , we have

$$\begin{aligned}
 \lim_{k \rightarrow \infty} A^k &= P \left( \lim_{k \rightarrow \infty} D^k \right) P^{-1} \\
 &= PDP^{-1} \\
 &= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left( \lim_{k \rightarrow \infty} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}^k \right) \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
 &= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left( \lim_{k \rightarrow \infty} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7^k & 0 \\ 0 & 0 & 0.3^k \end{bmatrix} \right) \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
 &= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
 &= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$