Math 53 (Multivariable Calculus), Section 102 & 108
Week 10, Monday
Oct 24, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Let D be a rectangle defined by the inequalities $0 \leq x \leq a$ and $0 \leq y \leq b$. Assume that it has a uniform density $\rho(x, y) = 1$.

   (a) Guess the center of the mass of D. Check that your guess is correct.

   (b) Find the moments of inertia I_x, I_y, and I_0. Compare them.

   (c) For given h and k, let D' be the translated rectangle

   $$D' = \{(x, y) : -h \leq x \leq a - h, -k \leq y \leq b - k\}.$$

   Find the moment of inertia $I = I(h, k)$ about the origin of D, as a function in h and k. (Hint: you can directly compute it, or you can use the result from (b).)

   (d) When $I(h, k)$ is minimized? Could you guess it before do the computation?

Solution

1. (a) Since the density is uniform, we can guess that the center of the mass is the center of the rectangle, i.e. $(a/2, b/2)$. This is true as follows:

$$\bar{x} = \frac{\iint_D x\rho(x,y)dA}{\iint_D \rho(x,y)dA} = \frac{\int_0^b \int_0^a xdxdy}{ab} \frac{\frac{1}{2}a^2b}{ab} = \frac{1}{2}a$$
$$\bar{y} = \frac{\iint_D y\rho(x,y)dA}{\iint_D \rho(x,y)dA} = \frac{\int_0^b \int_0^a ydxdy}{ab} \frac{\frac{1}{2}ab^2}{ab} = \frac{1}{2}b.$$

(b) By definition, we have

$$I_x = \iint_D y^2\rho(x,y)dA = \int_0^b \int_0^a y^2 dxdy = \frac{1}{3}ab^3$$
$$I_y = \iint_D y^2\rho(x,y)dA = \int_0^b \int_0^a x^2 dxdy = \frac{1}{3}a^3b$$
$$I_0 = \iint_D (x^2+y^2)\rho(x,y)dA = I_x + I_y = \frac{1}{3}ab(a^2+b^2)$$

I_0 is the largest, and $I_x > I_y$ if and only if $b > a$.

(c) One can compute $I(h,k)$ directly:

$$I(h,k) = \int_{-k}^{b-k} \int_{-h}^{a-h} (x^2+y^2)dxdy = \frac{1}{3}((a-h)^3 - (-h)^3)b + \frac{1}{3}((b-k)^3 - (-k)^3)a$$
$$= \frac{1}{3}(a^3 - 3a^2h + 3ah^2)b + \frac{1}{3}a(b^3 - 3b^2k + 3bk^2)$$

Another way is to divide D into four rectangles that origin is one of their vertices, and use (b) to compute moments of inertia of those rectangles.

(d) Partial derivatives of $I(h,k)$ are $I_h = -a^2 + 2ah$ and $I_k = -b^2 + 2bk$, so the unique critical point is $(a/2, b/2)$, where $I(h,k)$ is minimized.

Note that such (h,k) is the same as the center of the mass. In fact, this is true for any lamina: the point where the moments of intertia about axis parallel to z-axis and passes the point is the center of the mass.