Math 53 (Multivariable Calculus), Section 102 & 108
Week 2, Monday
Aug 29, 2022
For the other materials: seewoo5.github.io/teaching/2022Fall

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1. Find dy/dx for a curve parametrized by $x = 1/(1+t), y = \sqrt{1+t}$ (for $t > -1$).
2. Consider an ellipse parametrized as $x = 2\cos t, y = \sin t$.
   (a) Eliminate t to find an equation in x and y.
   (b) Find dy/dx. Can you express it without t (only in x and y)?
   (c) Find the tangent line at $t = \pi/3$.
3. Find dy/dx for a curve parametrized by $x = \sin t \cos t, y = \sin^2 t$. Also, find the points where the tangent line is horizontal or vertical. (Hint: use the double angle formula.)

Solution

1. We have $dx/dt = -1/(1+t)^2$ and $dy/dt = \frac{1}{2} \cdot (1+t)^{-1/2} = 1/(2\sqrt{1+t})$. Hence

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2\sqrt{1+t}}}{-\frac{1}{(1+t)^2}} = -\frac{1}{2}(1+t)^{3/2}.$$

   Or, you can eliminate t by observing $y = 1/\sqrt{x}$ and get $dy/dx = -\frac{1}{2x\sqrt{x}}$, which gives the same result.

2. (a) Using $\cos t = x/2, \sin t = y$ and $\cos^2 t + \sin^2 t = 1$, we get $(x/2)^2 + y^2 = 1$.

   (b) From $dx/dt = -2\sin t = -2y$ and $dy/dt = \cos t = x/2$, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x/2}{-2y} = -\frac{x}{4y}$$

   (c) For $t = \pi/3$, $x = 2\cos(\pi/3) = 1$ and $y = \sin(\pi/3) = \sqrt{3}/2$. Also, by (b), the slope of the line is $dy/dx = -x/(4y) = -1/(4 \cdot \sqrt{3}/2) = -1/(2\sqrt{3})$. Hence the tangent line is

$$y = -\frac{1}{2\sqrt{3}}(x-1) + \frac{\sqrt{3}}{2}.$$

3. We have $dx/dt = (\sin t)' \cos t + \sin t(\cos t)' = \cos^2 t - \sin^2 t$ and $dy/dt = 2\sin t(\sin t)' = 2\sin t \cos t$. Now recall the double angle formula:

$$\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

   Using this, we can simplify the result as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\sin t\cos t}{\cos^2 t - \sin^2 t} = \frac{\sin 2t}{\cos 2t} = \tan 2t.$$

   It has a horizontal tangent line when $\sin 2t = 0$ and $\cos 2t \neq 0$, i.e. for $t = n\pi/2$ where $n = \cdots, -1, 0, 1, \cdots$. These give two different points: $(0,0)$ (when $t = n\pi$) and $(0,1)$ (when $t = n\pi + \pi/2$). Similarly, it has a vertical tangent line when $\cos 2t = 0$, i.e. for $t = n\pi/2 + \pi/4$ where $n = \cdots, -1, 0, 1, \cdots$. These give two different points: $(1/2, 1/2)$ (when $t = n\pi + \pi/4$) and $(-1/2, 1/2)$ (when $t = n\pi + 3\pi/4$).

   In fact, this curve represents a circle: using double angle formula, we have $x = \frac{1}{2}\sin 2t$ and $y = \frac{1-\cos 2t}{2}$ you can check that

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{\sin^2 2t}{4} + \frac{\cos^2 2t}{4} = \frac{1}{4} = \left(\frac{1}{2}\right)^2,$$

   so it is a circle of radius $1/2$ centered at $(0, 1/2)$. You may graphically check that the above points actually have horizontal or vertical tangent lines.