1 Dot products[1], projection of vectors

1. What is the dot product $[1, 2, 3] \cdot [-2, 5, 0]$?

1. ____________

2. What is the dot product $[1, 2, 3] \cdot [-2, 5]$?

2. ____________

3. Compute the cosine of the angle between $[1, 2, 3]$ and $[-2, 5, 0]$.

3. ____________

4. Let $A(0, 1, 1)$, $B(1, 0, 1)$, $C(1, 1, 0)$ be three points in $\mathbb{R}^3$. Check that the triangle ABC is an equilateral triangle. What is the angle $\angle BAC$? Can you compute it using vectors and dot products?

4. ____________

5. Use the vectors to decide whether the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, 4)$, and $R(6, -2, -5)$ is right-angled.

5. ____________

6. Compute the vector projection of $\mathbf{b} = [1, 1, 1]$ to $\mathbf{a} = [2, 0, 0]$.

6. ____________

7. Compute the vector projection of $\mathbf{b} = [1, 1, 1]$ to $\mathbf{a} = [-1, 1, 0]$.

7. ____________

8. What is the projection of the zero vector $\mathbf{0} = [0, 0, 0]$ to $\mathbf{a} = [1, 2, 3]$?

8. ____________

9. Is $\text{proj}_{\mathbf{a}}\mathbf{b}$ and $\text{proj}_{2\mathbf{a}}\mathbf{b}$ are the same for any $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b}$? Can you explain why? How about $\text{proj}_{\mathbf{a}}(2\mathbf{b})$?

9. ____________

10. For $\mathbf{a} = [1, 1]$, find all $\mathbf{b}$ with $\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{1}{2}\mathbf{a}$.

10. ____________

[1] Inner products and dot products are the same thing.

Dot products, projection of vectors

1. $1 \times (-1) + 2 \times 5 + 3 \times 0 = 8$.

2. You cannot take dot product of vectors with different dimensions.

3. $$\cos\theta = \frac{[1,2,3] \cdot [-2,5,0]}{\|[1,2,3]\|\|[-2,5,0]\|} = \frac{8}{\sqrt{14}\sqrt{29}}.$$

4. You can compute the lengths of AB, BC, AC and see they are all equal. For example,

$$\overline{AB} = \sqrt{(0-1)^2 + (1-0)^2 + (1-1)^2} = \sqrt{2}$$

and you can find that all the other sides have the same length $\sqrt{2}$. Since it is an equilateral triangle, all the angles should be $\pi/3 = 60°$. You can also compute that angle $\angle BAC$ using the dot product: it is an angle between two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\begin{aligned}\overrightarrow{AB} &= [1,0,1] - [0,1,1] = [1,-1,0] \\ \overrightarrow{AC} &= [1,1,0] - [0,1,1] = [1,0,-1] \\ \cos\angle BAC &= \frac{[1,-1,0] \cdot [1,0,-1]}{\|[1,-1,0]\| \cdot \|[1,0,-1]\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}\end{aligned}$$

so $\angle BAC = \pi/3$.

5. We'll compute the (cosine of) the angles using dot products. For example, the angle $\angle QPR$ is the angle between $\overrightarrow{PQ} = [2-1, 0-(-3), 4-(-2)] = [1,3,6]$ and $\overrightarrow{PR} = [6-1, (-2)-(-3), (-5)-(-2)] = [5,1,-3]$, and

$$\cos\angle QPR = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|} = \frac{[1,3,6] \cdot [5,1,-3]}{\|[1,3,6]\|\|[5,1,-3]\|} = \frac{-10}{\sqrt{46}\sqrt{35}}$$

and since it is nonzero, the angle $\angle QPR$ is not the right angle. In fact, we are only interested in whether the angle is right or not, so we only need to compute the dot products (not the lengths) and see if they are zero. For the other vectors, we have

$$\begin{aligned}\overrightarrow{QP} \cdot \overrightarrow{QR} &= [-1,-3,-6] \cdot [4,-2,-9] = 56 \neq 0 \Leftrightarrow \cos\angle PQR \neq 0 \Leftrightarrow \angle PQR \neq \frac{\pi}{2} \\ \overrightarrow{RP} \cdot \overrightarrow{RQ} &= [-5,-1,3] \cdot [-4,2,9] = 45 \neq 0 \Leftrightarrow \cos\angle PRQ \neq 0 \Leftrightarrow \angle PRQ \neq \frac{\pi}{2}\end{aligned}$$

so it is not a right-angled triangle.

6. $$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} = \frac{2}{4}[2,0,0] = [1,0,0].$$

In fact, $\mathbf{a}$ is a vector with direction same as x-axis, so the projected vector becomes "x"-component of $\mathbf{b}$.

7. $$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} = \frac{0}{2}[-1,1,0] = [0,0,0].$$

   Note that two vectors are orthogonal (perpendicular) each other since $\mathbf{a}\cdot\mathbf{b} = 0$, hence the projection becomes a zero vector.

8. Projection a zero vector to any vector is again a zero vector.

9. Since $\mathbf{a}$ and $2\mathbf{a}$ have the same direction, projecting down $\mathbf{b}$ to $\mathbf{a}$ and $2\mathbf{a}$ are also same. You can also check this with the formula:

   $$\mathrm{proj}_{\mathbf{2a}}\mathbf{b} = \left(\frac{(2\mathbf{a})\cdot\mathbf{b}}{\|2\mathbf{a}\|^2}\right)(2\mathbf{a}) = \frac{2(\mathbf{a}\cdot\mathbf{b})}{4\|\mathbf{a}\|^2}(2\mathbf{a}) = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} = \mathrm{proj}_{\mathbf{a}}\mathbf{b}.$$

   However, they are not the same as $\mathrm{proj}_{\mathbf{a}}(2\mathbf{b})$ - you are projecting down a vector $2\mathbf{b}$ which has a length twice longer than $\mathbf{b}$, so the projection would be $\mathrm{proj}_{\mathbf{a}}(2\mathbf{b}) = 2 \times \mathrm{proj}_{\mathbf{a}}\mathbf{b}$ (check this with the formula).

10. Let $\mathbf{b} = [x,y]$. Then

    $$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \frac{[1,1]\cdot[x,y]}{\|[1,1]\|^2}[1,1] = \frac{x+y}{2}[1,1]$$

    and we want this to be the same as $\frac{1}{2}\mathbf{a} = \frac{1}{2}[1,1]$. So we get

    $$\frac{x+y}{2} = \frac{1}{2} \Leftrightarrow x+y = 1.$$

    This represents a line. Can you see this through drawings?