1. (Countings) Count the followings.
   (a) How many permutations of the letters ABCDEFGH containing *EF*?
   (b) How many permutations of the letters ABCDEFGH containing *ABC* and *EFG*?
   (c) How many ways are there for 10 dogs and 6 cats to stand in a line so that no two cats stand next to each other?
   (d) How many subsets with an odd number of elements does a set with 8 elements have?
   (e) How many subsets with an even number of elements does a set with 8 elements have?
   (f) (Binomial theorem) Spoiler: (d) and (e) are the same. Is this a coincidence?

2. (Binomial theorem)
   (a) Find the coefficient of $x^3y^2$ in $(x+y)^5$.
   (b) Find the coefficient of $x^2y^2$ in $(2x+3y)^4$.
   (c) Find the coefficient of $x^4y^6$ in $(3x^2-y^3)^4$.
   (d) Find the coefficient of $x$ in $(x+1/x)^7$.
   (e) Find the coefficient of $x$ in $(2x^2-1/x)^5$.

3. Prove that, for $0 \leq k \leq r \leq n$, we have

$$\binom{n}{n-r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k},$$

   (a) by doing algebra, or

   (b) by combinatorial arguments (both sides count the same thing). (Hint: Color $n$ balls with red, green, and blue, $n-r$, $r-k$, $k$-many for each color.)

1. (Countings) Count the followings.
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   (a) Consider EF as a single group. Then the number of permutations of A, B, C, D, EF, G, H is 7!.
   (b) Consider ABC and EFG as groups. Then the number of permutations of ABC, D, EFG, H is 4!.
   (c) $10! \cdot {}_{11}P_6$.
   (d) ${}_8C_1 + {}_8C_3 + {}_8C_5 + {}_8C_7 = 8 + 56 + 56 + 8 = 128$.
   (e) ${}_8C_0 + {}_8C_2 + {}_8C_4 + {}_8C_6 + {}_8C_8 = 1 + 28 + 70 + 28 + 1 = 128$.
   (f) It is not a coincidence. Expanding $0 = (1-1)^8$ using binomial theorem gives the equality of (d) and (e), and the same holds for any number $n$, not just 8.

2. (Binomial theorem)
   (a) Find the coefficient of $x^3y^2$ in $(x+y)^5$.
   (b) Find the coefficient of $x^2y^2$ in $(2x+3y)^4$.
   (c) Find the coefficient of $x^4y^6$ in $(3x^2-y^3)^4$.
   (d) Find the coefficient of $x$ in $(x+1/x)^7$.
   (e) Find the coefficient of $x$ in $(2x^2-1/x)^5$.

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   (a) ${}_5C_3 = 10$.
   (b) ${}_4C_2 \cdot 2^2 \cdot 3^2 = 216$.
   (c) ${}_4C_2 \cdot 3^2 \cdot (-1)^2 = 54$.

(d) Binomial theorem gives

$$(x+1/x)^7 = \sum_{k=0}^{7} {}_7C_k x^k (1/x)^{7-k} = \sum_{k=0}^{7} {}_7C_k x^{2k-7},$$

hence $2k-7=1 \Leftrightarrow k=4$ gives the coefficient ${}_7C_4 = 35$.

(e) Binomial theorem gives

$$(2x^2-1/x)^5 = \sum_{k=0}^{5} {}_5C_k (2x^2)^k (-1/x)^{7-k} = \sum_{k=0}^{5} {}_5C_k 2^k (-1)^{7-k} x^{3k-7}.$$

Since $3k-7=1 \Leftrightarrow k=8/3$ is not an integer, there's no $x$ term in the expansion. In other words, the coefficient is 0.

3. Prove that, for $0 \le k \le r \le n$, we have

$$\binom{n}{n-r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k},$$

   (a) by doing algebra, or

   (b) by combinatorial arguments (both sides count the same thing). (Hint: Color $n$ balls with red, green, and blue, $n-r$, $r-k$, $k$-many for each color.)

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(a)

$$\binom{n}{n-r}\binom{r}{k} = \frac{n!}{(n-r)!r!}\frac{r!}{(r-k)!k!} = \frac{n!}{(n-r)!(r-k)!k!} = \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(r-k)!(n-r)!} = \binom{n}{k}\binom{n-k}{r-k}.$$

(b) As given in the hint, we count the number of ways to color $n$ balls with $(n-r)$ reds, $(r-k)$ greens, and $k$ blues. We can color them in the following orders:

   i. First choose $(n-r)$ balls and color them reds ($\binom{n}{n-r}$). Then choose $k$ balls among $r$ remaining balls ans color them blue ($\binom{r}{k}$). The other $(r-k)$ balls automatically becomes green. This gives the left hand side.

   ii. Choose $k$ balls and color them blue ($\binom{n}{k}$). Then choose $(r-k)$ balls among $(n-k)$ remaining balls and color them green ($\binom{n-k}{r-k}$). The other $n-r$ balls automatically becomes red. This gives the right hand side.