Fourier

- A Fourier series represents a periodic signal as a sum of simple sinusoids.
- Instead of treating a waveform as one complicated object, Fourier analysis breaks it down into frequency components called harmonics, then reconstructs the original signal by summing them up.
- Each sinusoid contributes a simple oscillation with a specific frequency and amplitude. Increasing the number of terms N makes the approximation more accurate.
- The process shows how complex systems can be understood by decomposing them into simpler, well defined components. The exact systems-thinking logic used across signal processing, electromagnetics, communications, and controls.
- A. General Fourier Series Formula
  - A periodic function $f(x)$ with period 2π can be expressed as:
    - $$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$
      - Where the coefficients are:
        - $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$
        - $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$
        - $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$
        - Each integral determines how much of each frequency component appears in the signal.
- B. Square Wave Fourier Series
  - A square wave of amplitude $\pm$ 1 on $[-\pi, \pi]$ is an odd function, so all cosine terms vanish:
    - $$a_n = 0$$
  - The sine coefficients become:
    - $$b_{2k} = 0$$

- $$b_{2k} = 0$$
- $$b_{2k+1} = \frac{4}{(2k+1)\pi}$$
- The Fourier approximation using N terms is:
  - $$f_N(x) = \frac{4}{\pi} \sum_{k=0}^{N-1} \sin\left((2k+1)x\right)$$
    - This shows only odd harmonics contribute to reconstructing a square wave.
- C. Sawtooth Wave Fourier Series
  - Define a sawtooth wave on $[-\pi, \pi]$ as:
    - $$f(x) = \frac{x}{\pi}$$
  - This is also odd, so again $a_n = 0$.
  - The sine coefficients are:
    - $$b_n = -\frac{2}{n\pi}$$
  - The N-term Fourier approximation is therefore:
    - $$f_N(x) = -\frac{2}{\pi} \sum_{n=1}^{N} \sin\left(nx\right)$$
      - Unlike the square wave, the sawtooth uses all harmonics.
- D. Convergence and Gibbs Phenomenon
  - Fourier series approximate periodic functions by summing sine and cosine terms. For smooth functions, these partial sums converge rapidly and uniformly to the original function. But for waveforms with jump discontinuities - such as square and sawtooth waves, the behaviour is more subtle.
    - Point-wise convergence:
      - Let $S_N(x)$ be the N-term Fourier partial sum of a periodic function $f(x)$.
      - Classical results/Dirichlet's conditions state:
        - At points where $f(x)$ is continuous,
          - $$S_N(x) \to f(x) \text{ as } N \to \infty$$

        - At a jump discontinuity x_0, where the function jumps from $f(x_0^-)$ to $f(x_0^+)$,
            - $$S_N(x_0) \to \frac{f(x_0^-) + f(x_0^+)}{2}$$
    - So for a square wave jumping from -1 to +1, the Fourier series does not converge to +1 or -1 at the jump, but to 0, the midpoint.
- The Gibbs phenomenon
    - When approximating a function with a discontinuity, the Fourier partial sums produce an oscillatory ripple near the jump. Two key characteristics define this Gibbs phenomenon:
        - Overshoot and undershoot appear near each discontinuity.
            - Just before and just after the jump, $S_N(x)$ rises above or dips below the true function value.
        - The size of the overshoot does not shrink to zero, even as $N \to \infty$.
            - The overshoot approaches a constant fraction of the jump:
                - $$\text{Overshoot} \approx 0.08949\Delta$$
                    - Where $\Delta = f(x_0^+) - f(x_0^-)$ is the jump height.
            - For a square wave, $\Delta = 2$, so limiting overshoot is about:
                - $$0.08949 \times 2 \approx 0.18$$
                    - Meaning approximation spikes about 0.18 above the top level and 0.18 below the bottom level, even with infinitely many Fourier terms.
- How increasing N affects approximation
    - As you increase the number of terms:
        - Away from the discontinuities, the approximation improves rapidly and becomes extremely accurate.
        - Near the discontinuities, the wiggles become narrower but not shorter.
        - The oscillatory region compresses towards the jump, but the overshoot height stays roughly the same.
    - This means:

        - Fourier series converge point-wise but not uniformly near discontinuities.
    - The visualisation makes this phenomenon extremely obvious:
        - At low N, the approximation is smooth and rounded; as N increases, the waveform becomes sharper, but the characteristic "ringing" near the jumps becomes more pronounced.
- Why this matters in engineering
    - The Gibbs phenomenon isn't just mathematical trivia, it shows up everywhere in signal processing:
        - Digital filters
        - Truncated Fourier transforms
        - Sampling and reconstruction
        - Spectral leakage
        - Communication signals
    - Engineers constantly deal with the consequences: overshoot, ringing, aliasing, and nonuniform convergence.
        - The visualisation demonstrates exactly why overshoot appears in real-world systems when you try to approximate discontinuous signals with finite bandwidth.