Sefa Kayraklık
2015401066

Homework 1

Question 2

Part a)

In order to derive the frequency response, the Fourier transform of the $h[n]$ should be taken.

$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n]e^{-jwn} = \sum_{n=-M_1}^{M_2} \frac{1}{M_1+M_2+1}e^{-jwn} \tag{1}$$

Using the finite summation formula, the following expression is obtained.

$$\begin{aligned} H(e^{jw}) &= \frac{1}{M_1+M_2+1}\frac{e^{jwM_1} - e^{-jw(M_2+1)}}{1-e^{-jw}} \\ &= \frac{1}{M_1+M_2+1}\frac{(e^{jw(\frac{M_1+M_2+1}{2})} - e^{-jw(\frac{M_1+M_2+1}{2})})e^{-jw(\frac{-M_1+M_2+1}{2})}}{(e^{j\frac{w}{2}} - e^{-j\frac{w}{2}})e^{-j\frac{w}{2}}} \end{aligned} \tag{2}$$

Using Euler Identity, the following expression is obtained.

$$H(e^{jw}) = \frac{1}{M_1+M_2+1}\frac{sin(w(\frac{M_1+M_2+1}{2}))}{sin(\frac{w}{2})}e^{-jw(\frac{-M_1+M_2}{2})} \tag{3}$$

Part b)

The magnitude response for $M_1 = 0$ and $M_2 = 4$ is given in the Figure 1.

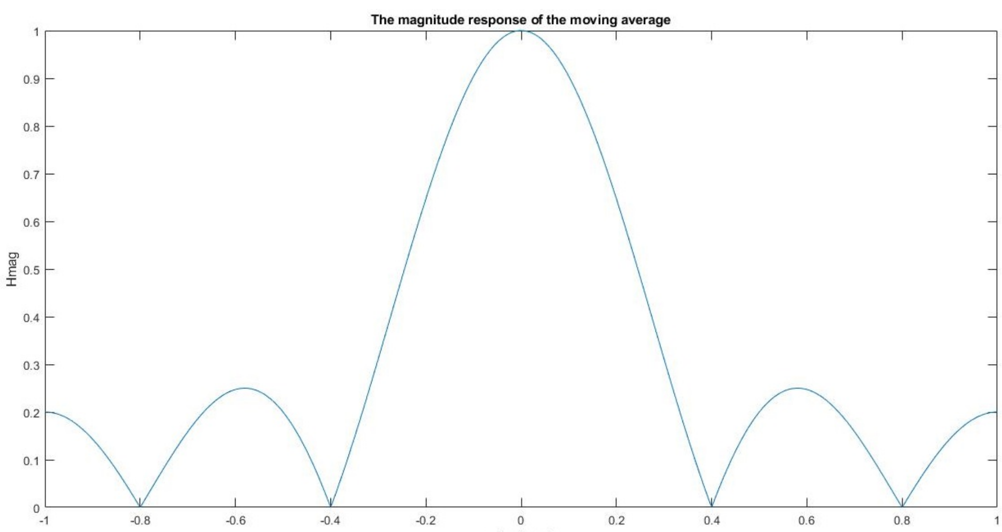

Figure 1: The magnitude response of the given moving average

It can be seen from the figure that the magnitudes of the response at low frequencies are larger than the ones at high frequencies. Namely, it reduces the effects of the high frequency components of the input. Therefore, the moving average can be seen as approximately a low pass filter; however, it isn't able to filter out the high frequency components completely. Observing the impulse response of the moving average, it can be seen that the rapid changes in the input are smoothed by taking the average with its neighbor values. As expected, this effect in the impulse response appears in the magnitude response of the moving average as filtering the high frequency components of the input.

Part c)

The magnitude response for $M_1 = 0$ and $M_2 = 4$ is given in the Figure 2.

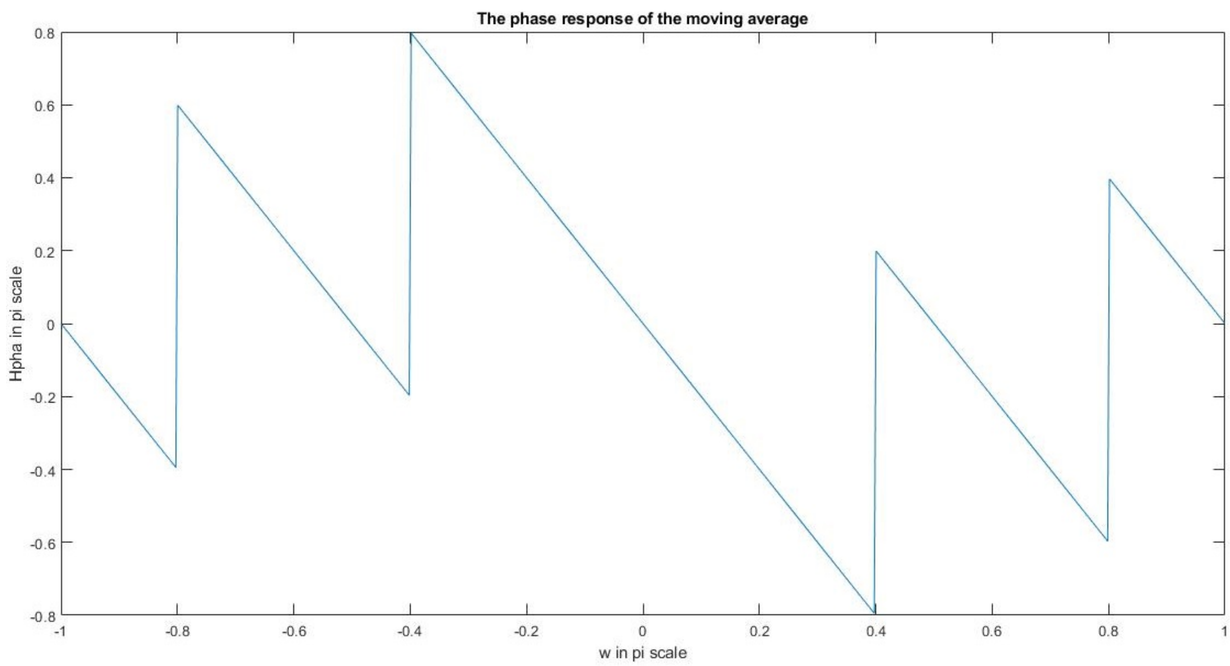

Figure 2: The phase response of the given moving average

It can be seen from the figure that the phase response of the moving average is piecewise linear with respect to the frequency. Since it is not linear in the range $w = [-\pi\pi]$, then this system can be called phase distorting system. However, there is some kind of linearity in the system, the distortion may not be so deleterious change in phase.

Question 3

Part a)

The impulse response of the given difference equation is determined by using $filter$ function in MATLAB. The parameter of the function is given as the coefficients of $x[n]'s$ and $y[n]'s$, and the Dirac delta function in discrete time. The plot of the impulse response is given in the Figure 3.

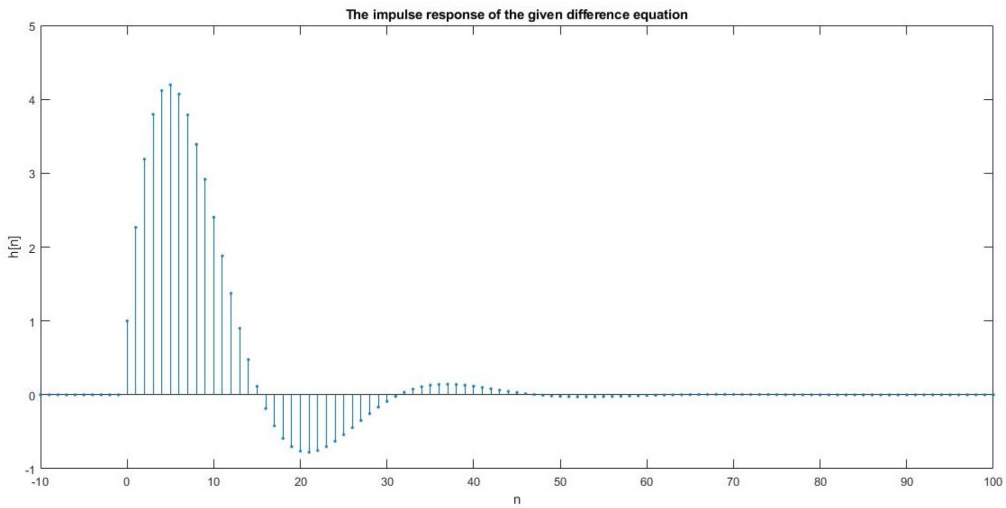

Figure 3: The impulse response of the given difference equation

Part b)

The impulse response of the difference equation can be determined by taking the Fourier transform of the both sides of the equation and calculating $\frac{Y(e^{jw})}{X(e^{jw})}$ $(= H(e^{jw}))$ then taking the inverse Fourier transform of $H(e^{jw})$.
The Fourier transform of the both sides is given in the following expression.

$$Y(e^{jw}) - 1.8cos(\frac{\pi}{16})Y(e^{jw})e^{-jw} + 0.81Y(e^{jw})e^{-j2w} = X(e^{jw}) + \frac{1}{2}X(e^{jw})e^{-jw} \tag{4}$$

The frequency response of the given difference equation is given in the following expression.

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1 + \frac{1}{2}e^{-jw}}{1 - 1.8cos(\frac{\pi}{16})e^{-jw} + 0.81e^{-j2w}} \tag{5}$$

The partial fraction expansion of the equation 5 is given in the following expression.[1]

$$H(e^{jw}) = \frac{0.5 - i3.9375}{1 - (0.8827 + i0.1756)e^{-jw}} + \frac{0.5 + i3.9375}{1 - (0.8827 - i0.1756)e^{-jw}} \tag{6}$$

So, it can be seen in the following form:

$$H(e^{jw}) = \frac{r}{1 - pe^{-jw}} + \frac{r^*}{1 - p^*e^{-jw}},\ r = 0.5 - i3.937\ p = 0.8827 + i0.1756 \tag{7}$$

The inverse discrete time Fourier transform of the equation 7 can be expressed as following:

$$\begin{aligned} h[n] &= rp^n u[n] + r^*(p^*)^n u[n] = |r|e^{jw_r}|p|^n e^{jw_p n}u[n] + |r|e^{-jw_r}|p|^n e^{-jw_p n}u[n] \\ &= |r||p|^n(e^{j(w_p n + w_r)} + e^{-j(w_p n + w_r)})u[n] = 2|r||p|^n(cos(w_p n + w_r))u[n] \end{aligned} \tag{8}$$

After obtaining the last equation 8, the r and p are replaced to the equation. Then the result is given in the following:

$$h[n] = 3.9691(0.9)^n cos(0.1963n - 1.4445)u[n] \tag{9}$$

In order to confirm the both results in part a and part b, the plot of the impulse response can be compared. So, the result in the equation 9 is plotted using MATLAB.[2] The plot is given in the Figure 4.

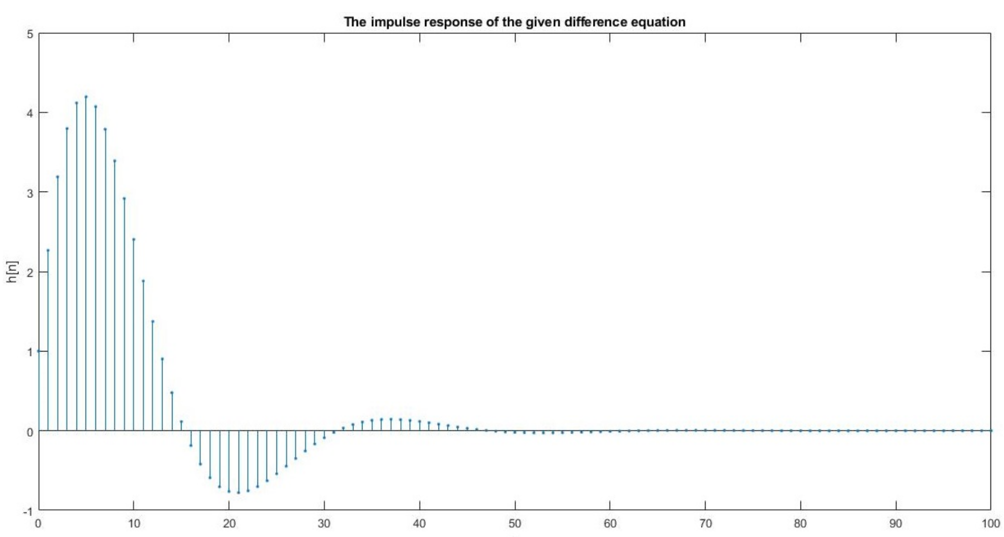

Figure 4: The analytically solved impulse response of the given difference equation

[1] It is calculated using *residuez* function in MATLAB which gives the partial fraction expansion of e^{-jw} or z^{-1}.

[2] The script for generating r and p, and ploting the result is given the file named $HW1_Q3_b.m$

Part c)

The response of the system when $x_1[n] = e^{\frac{j\pi n}{3}}u[n]$ is given as the input is calculated by using the function $filter$ in MATLAB. So, the parameters are the same as the part a except the last parameter being $x_1[n]$.
The plot of the $y_1[n]$ is given in the Figure 5.

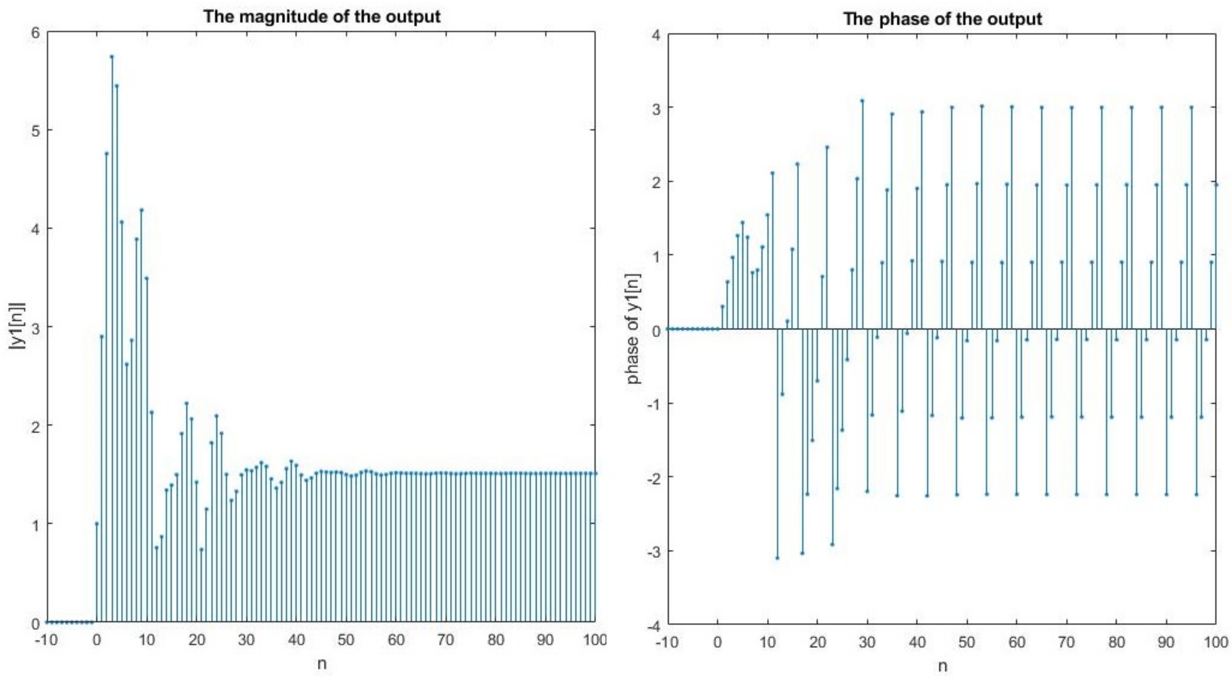

Figure 5: The response of the system to the input $x_1[n]$

It can be seen from the figure that the magnitude of the response goes to 1.51 after around $n = 60$. From looking at the phase of the response, it can said that its phase becomes periodic and its w_y is equal to $w_x - 0.1451$. Since $x_1[n]$ is an eigen-function (at the right side of the time axis), the response of it will approach to its form such as $Hx_1[n]$, where H is a complex number. In this case H can be determined by looking the Figure 5 as $1.51e^{j(w_x - 0.1451)}$. So, the output as $n \to \infty$ is equal to the following expression:

$$y_1[n] = Hx_1[n] = (1.51e^{j(w_x - 0.1451)})e^{\frac{j\pi n}{3}}u[n] \tag{10}$$

Part d)

Since the input is an eigen-function, it is known that the output of the real valued difference equation will be in the same form as the input while the system reaches its steady state. Therefore, as $n \to \infty$, $y_1[n] = Hx_1[n]$. This is the result of the given equation (in the question description, it is the equation (2)) evaluated at $w = \pi/3$. Here, H is being the steady state response of the system when $x_1[n]$ is the input. After obtaining $y_1[n]$, $x_1[n]$, and $y_1[n]$ can be replaced in the difference equation (in the question description, it is the equation (1)). so, the following expression is obtained:

$$\lim_{n \to \infty} \Big(y_1[n] - 1.8cos(\frac{\pi}{16}) y_1[n-1] + 0.81 y_1[n-2] \Big) = \lim_{n \to \infty} \Big(x_1[n] + \frac{1}{2} x_1[n-1] \Big) \tag{11}$$

After substituting $x_1[n] = e^{\frac{j\pi n}{3}} u[n]$ and $y_1[n] = H x_1[n]$, the following expression is obtained:

$$\lim_{n \to \infty} \Big(H e^{\frac{j\pi n}{3}} - 1.8cos(\frac{\pi}{16}) H e^{\frac{j\pi(n-1)}{3}} + 0.81 H e^{\frac{j\pi(n-2)}{3}} \Big) = \lim_{n \to \infty} \Big(e^{\frac{j\pi n}{3}} + \frac{1}{2} e^{\frac{j\pi(n-1)}{3}} \Big) \tag{12}$$

After gathering H to the left side, the following is obtained:

$$H = \lim_{n \to \infty} \frac{e^{\frac{j\pi n}{3}}}{e^{\frac{j\pi n}{3}}} \frac{1 + \frac{1}{2} e^{\frac{-j\pi}{3}}}{1 - 1.8cos(\frac{\pi}{16}) e^{\frac{-j\pi}{3}} + 0.81 e^{\frac{-2j\pi}{3}}} \tag{13}$$

After canceling out the n depended exponential part H is equal to:

$$H = \frac{1 + \frac{1}{2} e^{\frac{-j\pi}{3}}}{1 - 1.8cos(\frac{\pi}{16}) e^{\frac{-j\pi}{3}} + 0.81 e^{\frac{-2j\pi}{3}}} = -0.9355 - 1.1854i = 1.51 e^{j(-2.24)} \tag{14}$$

Part e)

The previous part can be used to determine the frequency response of the system at any given input. It is simply substituting $\pi/3$ with w in the equation 13. Therefore, the following expression is obtained:

$$H(e^{jw}) = \frac{1 + \frac{1}{2} e^{-jw}}{1 - 1.8cos(\frac{\pi}{16}) e^{-jw} + 0.81 e^{-j2w}} \tag{15}$$

Then, this function can be plotted in the MATLAB. So, the frequency response of the system is given in the Figure 6.

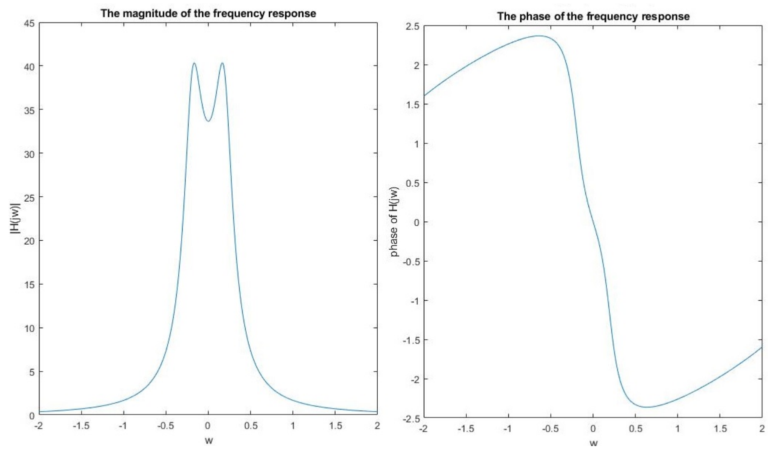

Figure 6: The frequency response of the given difference equation

Part f)

The transient response of the system when its input is $x_1[n]$ can be calculated as the difference between the total response of the system (calculated in the part c) and the steady state response of the system (it can be calculated using the formula given in the question as equation (2)). Namely, transient response can be expressed as following:

$$y_t[n] = y[n] - y_{ss}[n] = y_1[n] - H_{ss}(e^{\frac{j\pi}{3}})e^{\frac{j\pi n}{3}} \tag{16}$$

The plot of the transient response in the range $n = 0, 1, ..., 30$ is given in the Figure 7.

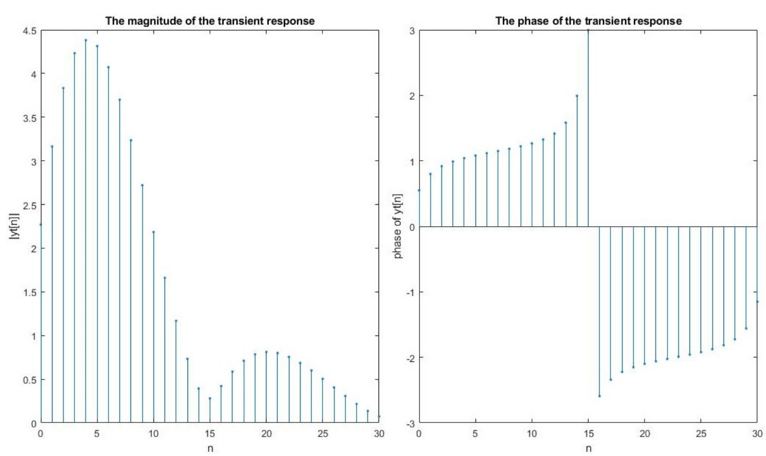

Figure 7: The transient response of the given difference equation with the input $x_1[n]$

It can be seen from the Figure 7, the magnitude of the transient response approaches to 0. However, the first 31 values might not be sufficient, since after some point, it can stop decreasing, and never reach to 0. But, this is not the case in this system.