6) $R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f\tau}\, df = \int_{-B}^{B} N_0\, e^{j2\pi f\tau}\, df = N_0 \frac{\sin(2\pi B\tau)}{\pi\tau}$

$= 2BN_0 \,\mathrm{sinc}\, 2\pi B\tau$

$R_Y(\tau) = E[Y(t)Y(t+\tau)] = E[X^2(t)X^2(t+\tau)]$

$E[X_1^2X_2^2] = E[X_1^2].E[X_2^2] + 2E[X_1.X_2]$ (Eqn. 2.70, from the chapters that instructor sent.)

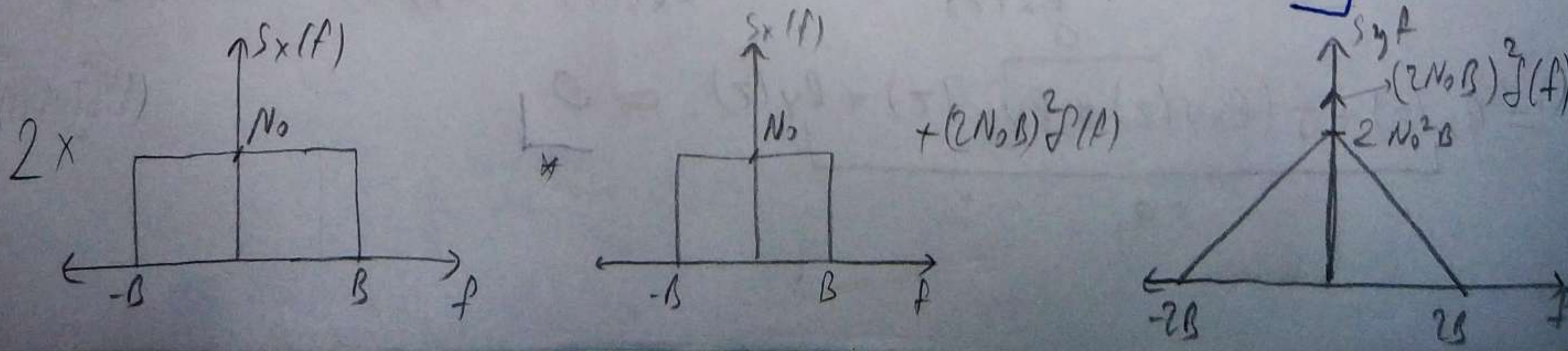
to be able to use the formula, X_1 and X_2 should be zero-mean.

$\Rightarrow$ If $\lim_{\tau\to\infty} R_X(\tau) = C$, then $C = \mu_X^2$ (page 144) $\Rightarrow$ in this case $C = 0 \Rightarrow$ zero-mean

$$E[X^2(t).X^2(t+\tau)] = \underbrace{E[X^2(t)]}_{R_X(0)} . \underbrace{E[X^2(t+\tau)]}_{R_X(0)} + 2\underbrace{E[X(t)X(t+\tau)]^2}_{R_X(\tau)^2}$$

$R_Y(\tau) = 2BN_0 . 2BN_0 + 2 . (2BN_0\, \mathrm{sinc}(2\pi B\tau))^2$

$S_Y(f) = F\{R_X(0)^2 + 2R_X(\tau)^2\} = R_X(0)^2\delta(f) + 2S_X(f) * S_X(f)$



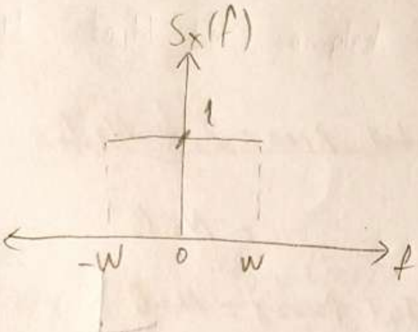
7) a. $R_x(\tau)=\mathcal{F}^{-1}\{S_x(f)\}$, $S_x(f)=\text{rect}\left(\frac{f}{2W}\right) \xrightarrow{\mathcal{F}^{-1}\{\}} R_x(\tau)=2W\,\text{sinc}(2W\tau)$

$$R_{X_s}(\tau)=\sum_{k=-\infty}^{\infty}\frac{1}{T_s}R_x(kT_s)\,\delta(t-kT_s)=\frac{1}{T}R_x(\tau)\sum_{k=-\infty}^{\infty}\delta(\tau-kT)$$

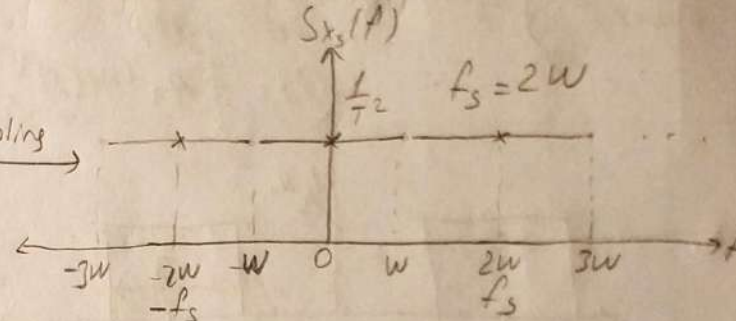
(eqn. from page 195 3rd chapter that the instructer send)

$$R_{X_s}(\tau)=\frac{1}{T}\,2W\,\text{sinc}(2W\tau)\sum_{k=-\infty}^{\infty}\delta(\tau-kT)$$

b.



Sampling



$$\Rightarrow \frac{1}{T_s}=f_s=2W \quad \Rightarrow \quad T_s=\frac{1}{2W}$$

8) a. Since a zero mean stationary narrowband is proper, the following formulas (from Digital Communications; Proakis, Salehi 5th ed.) can be used.

$C_x = C_y \qquad C_{xy}=-C_{yx}$ (eqn. 2.6-19 & 2.6-20)

Since $E[Z(t)]=0 \Rightarrow E[X(t)]=0 \quad E[Y(t)]=0$

$\Rightarrow C_x=R_x-0 \;\&\; C_y=R_y-0 \quad \Rightarrow \quad R_x=R_y$

$\Rightarrow C_{xy}=R_{xy}-0 \;\&\; C_{yx}=R_{yx}-0 \quad \Rightarrow \quad R_{xy}=-R_{yx}$

$$E[Z(t).Z(t+\tau)]=E[\underbrace{X(t).X(t+\tau)}_{R_x(\tau)}+j(\underbrace{X(t)Y(t+\tau)}_{R_{xy}(\tau)}+\underbrace{Y(t)X(t+\tau)}_{R_{yx}(\tau)})\underbrace{-Y(t)Y(t+\tau)}_{R_y(\tau)}]$$

$$=\underbrace{R_x(\tau)+j(\overbrace{R_{xy}(\tau)+R_{yx}}^{0}(\tau)+R_y(\tau)}_{=0}=0$$

b. $E[V^2] = E\left[\int_0^T z(t_1)\, dt_1 \int_0^T z(t_1)\, dt_1 \right] = E\left[\int_0^T \int_0^T z(t_1)\, z(t_2)\, dt_1\, dt_2 \right]$

$= \int_0^T \int_0^T \underbrace{E[z(t_1)\, z(t_2)]}_{= E[z(t) . z(t+\tau)] = 0} dt_1\, dt_2 \quad \Rightarrow \quad = 0$

similarly;

$E[|V|^2] = E[V^* V] = \int_0^T \int_0^T \underbrace{E[z^*(t_1)\, z(t_2)]}_{R_z(t_2 - t_1)} dt_1\, dt_2$

$= \int_0^T \int_0^T R_z(t_2 - t_1)\, dt_1\, dt_2 = \int_0^T \int_0^T N_0\, \delta(t_2 - t_1)\, dt_1\, dt_2 = \int_0^T N_0\, dt_2 = N_0 T$

Sefa Kayraklık
2015401066

Homework 1

EE 477
Fall 2019

Question 2[1]

a) BPSK:

The following expression 1 shows the binary phase shift keying (BPSK) formulation.

$$s_l(t) = g(t)cos(2\pi f_c t + \theta_l),\ where\ \theta_l = 0, \pi \tag{1}$$

In order to simplify the representation of the formulation, the following expression 2 is obtained:

$$s_l(t) = (cos(\theta_l))g(t)cos(2\pi f_c t) - (sin(\theta_l))g(t)sin(2\pi f_c t) \tag{2}$$

Since $sin(\theta_l)$ is equal to 0 when θ_l equals to $0, \pi$, the sin part of the formulation can be ignored.

The constellation diagram of the BPSK that is used in the representation of the given bit stream is shown in the Figure 1. The average symbol energy of BPSK is $E_g/2$, so in order to get unity energy, they should be divided by $\sqrt{E_g/2}$. Since the E_g is selected as 1, so they are divided by $\sqrt{1/2}$.

It can be seen from the constellation diagram that there is only cos part in the waveform of the modulation; and its characteristic is the same as the binary amplitude modulation.

The sinusoidal waveform to represent the symbols in the constellation diagram is given in the Figure 2.

The Figure 2 shows that there is a π-phase difference between the distinct symbols in the modulation.

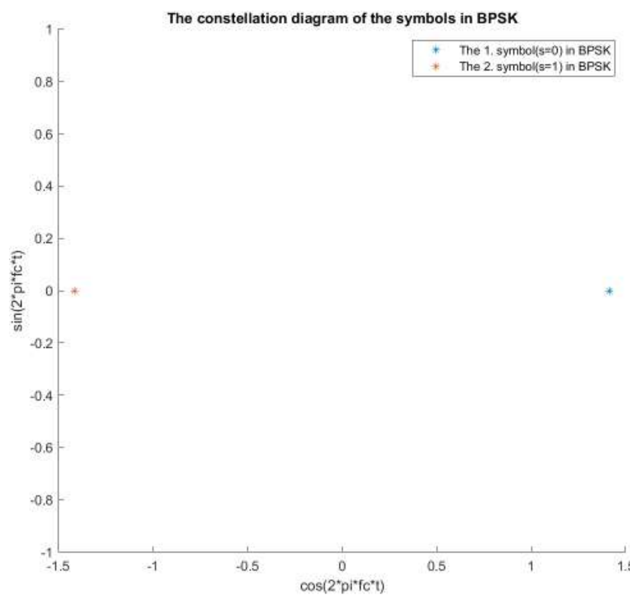


Figure 1: The constellation diagram of the BPSK

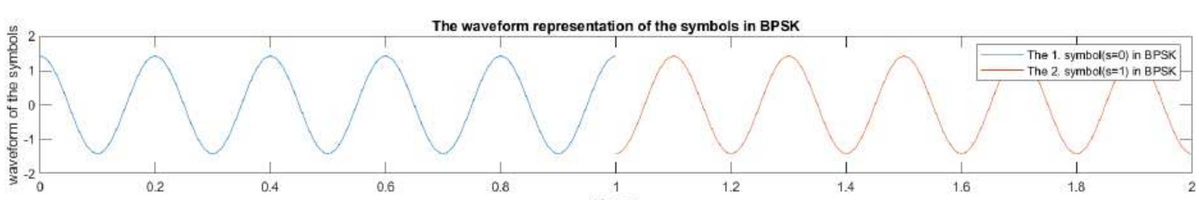


Figure 2: The waveform representation of the symbols in BPSK

[1]The $g(t)$ function is taken as a constant function with gain 1 through the calculations in the modulations.

The given bit stream $s = \{1, 0, 1, 1, 1, 1, 0, 0\}$ is modulated with the modulation formulation that is described above. And the Figure 3 shows the pulse stream that represents the given bit stream.

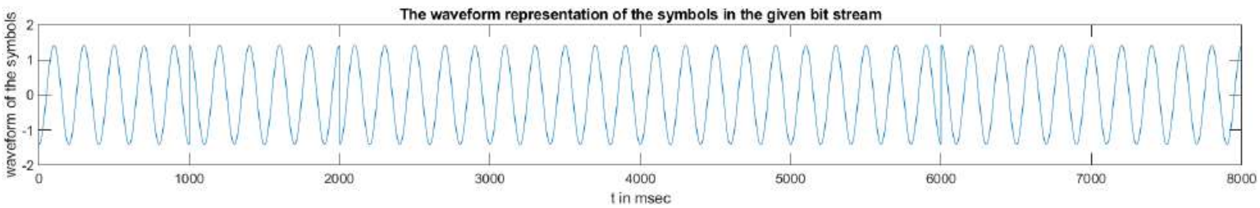


Figure 3: The waveform representation of the symbols in the given bit stream

As expected, there are phase changes at $t = 1, 2$, and 6, and when the pulse stream is mapped onto waveforms that are shown in the Figure 2, the original bit stream is obtained.

b) QPSK:

The following expression 3 shows the quadrature phase shift keying (QPSK) formulation.

$$\begin{aligned} s_l(t) &= g(t)cos(2\pi f_c t + \theta_l) \\ , \ where\ \theta_l &= 0, \pi/2, \pi, 3\pi/2 \end{aligned} \tag{3}$$

In order to simplify the representation of the formulation, the following expression 4 is obtained:

$$\begin{aligned} s_l(t) = &(cos(\theta_l))g(t)cos(2\pi f_c t) \\ &-(sin(\theta_l))g(t)sin(2\pi f_c t) \end{aligned} \tag{4}$$

The constellation diagram of the BPSK that is used in the representation of the given bit stream is shown in the Figure 4. The average symbol energy of QPSK is $E_g/2$, so in order to get unity energy, they should be divided by $\sqrt{E_g/2}$. Since the E_g is selected as 1, so they are divided by $\sqrt{1/2}$.

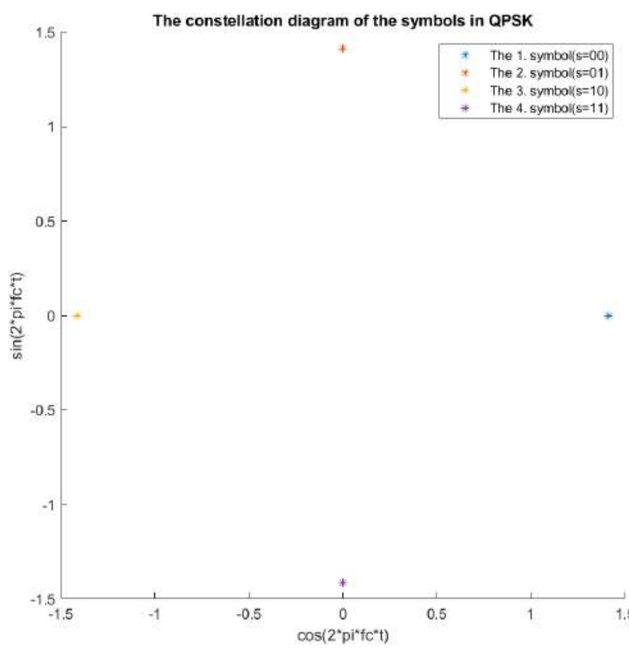


Figure 4: The constellation diagram of the QPSK

It can be seen from the constellation diagram that both *cos* and *sin* part contribute to the waveform of the modulation.

The sinusoidal waveform to represent the symbols in the constellation diagram is given in the Figure 5.

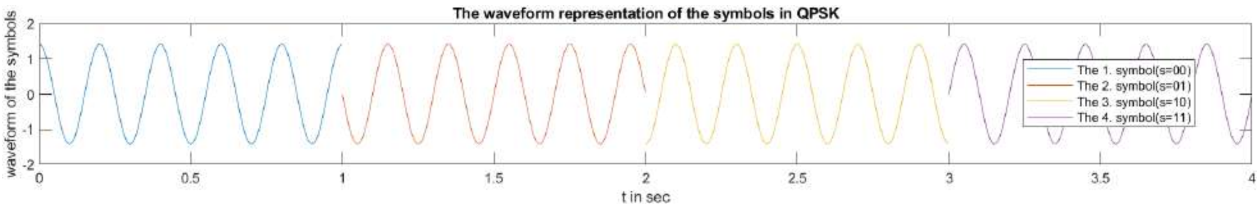


Figure 5: The waveform representation of the symbols in QPSK

The Figure 5 shows that there is a $\pi/2$-phase difference between the distinct symbols in the modulation.

The given bit stream $s_{4-ary} = \{10, 11, 11, 00\}$ is modulated with the modulation formulation that is described above. And the Figure 6 shows the pulse stream that represents the given bit stream.

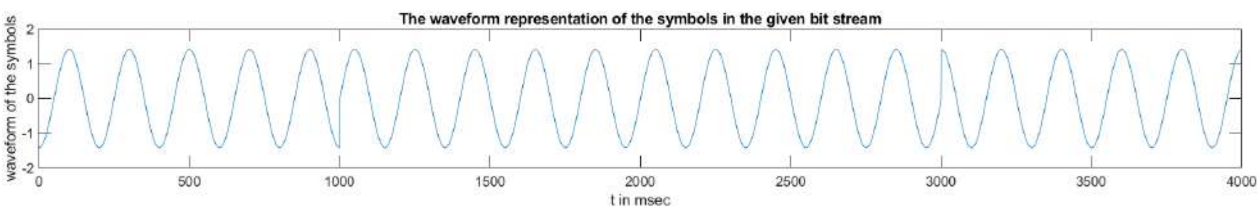


Figure 6: The waveform representation of the symbols in the given bit stream

As expected, there are phase changes at $t = 1$, and 3, and when the pulse stream is mapped onto waveforms that are shown in the Figure 6, the original 4-ary bit stream is obtained.

c) 4-PAM

The following expression 5 shows the quadrature pulse amplitude modulation (4-PAM) formulation.

$$s_l(t) = A_l g(t) cos(2\pi f_c t),\ where\ A_l = -3\Delta, -1\Delta, 1\Delta, 3\Delta \tag{5}$$

The constellation diagram of the 4-PAM that is used in the representation of the given bit stream is shown in the Figure 7. Since 4-PAM has average symbol energy equal to $5\Delta^2 E_g/2$, the constellations should be divided by $\sqrt{5\Delta^2 E_g/2}$; and Δ, and E_g are selected as 1, so they are divided by $\sqrt{5/2}$.

It can be seen from the constellation diagram that the energy of the symbols can vary when different symbols are send. Actually, that is not so much desired attribute for modulations. And the negative amplitude means a π-phase difference between the positive one.

The sinusoidal waveform to represent the symbols in the constellation diagram is given in the Figure 8.

The Figure 8 shows that there is a π-phase difference between the symbols that have negative and positive amplitudes. And, an amplitude of a symbol can be equal to 3 times amplitude of an other one, or vise versa.

The given bit stream $s_{4-ary} = \{10, 11, 11, 00\}$ is modulated with the modulation formulation that is described above. And the Figure 9 shows the pulse stream that represents the given bit stream.

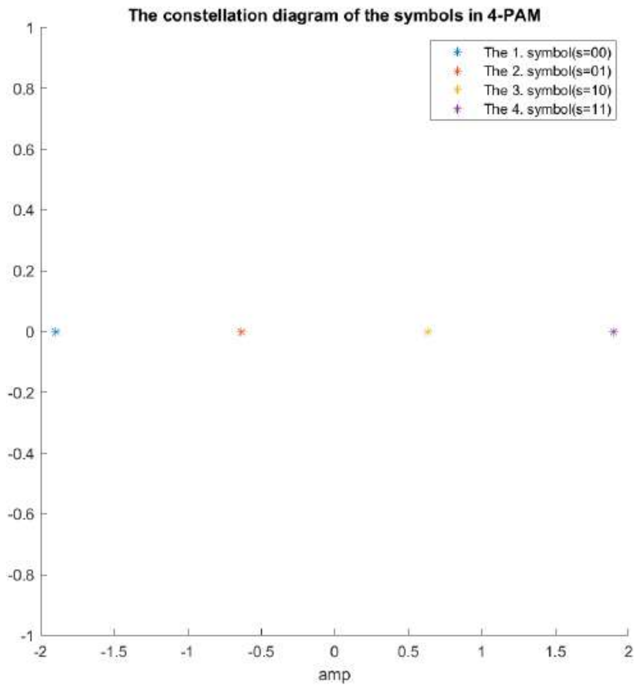


Figure 7: The constellation diagram of the 4-PAM

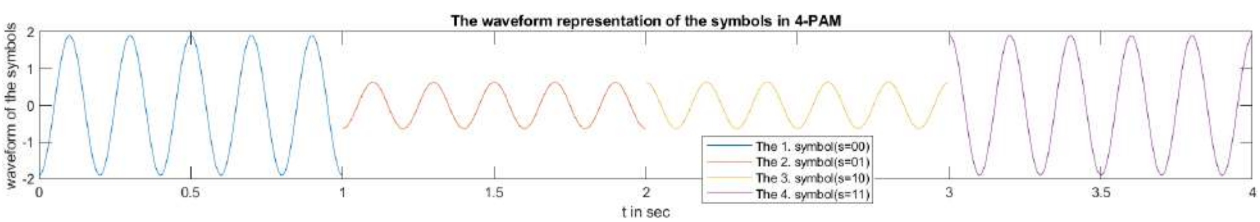


Figure 8: The waveform representation of the symbols in 4-PAM

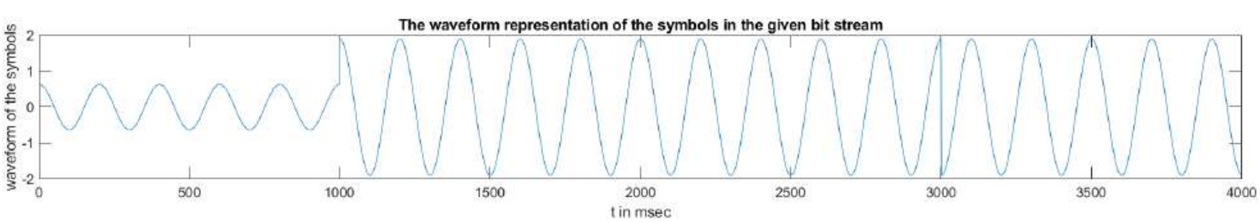


Figure 9: The waveform representation of the symbols in the given bit stream

As expected, there is a amplitude change at $t = 1$, and a phase change at $t = 3$; and when the pulse stream is mapped onto waveforms that are shown in the Figure 8, the original bit stream is obtained.

d) 16-QAM

The following expression 6 shows the quadrature pulse amplitude modulation (4-PAM) formulation.

$$s_{pq}(t) = A_p g(t) cos(2\pi f_c t + \theta_q) = A_p cos(\theta_q) cos(2\pi f_c t) - A_p sin(\theta_q) sin(2\pi f_c t) \tag{6}$$

Since the rectangular QAM is chosen to be implemented, the formulation can be expressed as following 7.

$$s_{lk}(t) = A_l cos(2\pi f_c t) - B_k sin(2\pi f_c t),\ where\ A_l, B_k = -3\Delta, -1\Delta, 1\Delta, 3\Delta \tag{7}$$

The constellation diagram of the 16-QAM that is used in the representation of the given bit stream is shown in the Figure 10.Since 16-QAM has average symbol energy equal to $10\Delta^2 E_g/2$, the constellations should be divided by $\sqrt{10\Delta^2 E_g/2}$; and Δ, and E_g are selected as 1, so they are divided by $\sqrt{10/2}$.

It can be seen from the constellation diagram that the energy of the symbols can vary when different symbols are send. And the negative amplitude means a π-phase difference between the positive one.

The sinusoidal waveform to represent the symbols in the constellation diagram is given in the Figure 11.

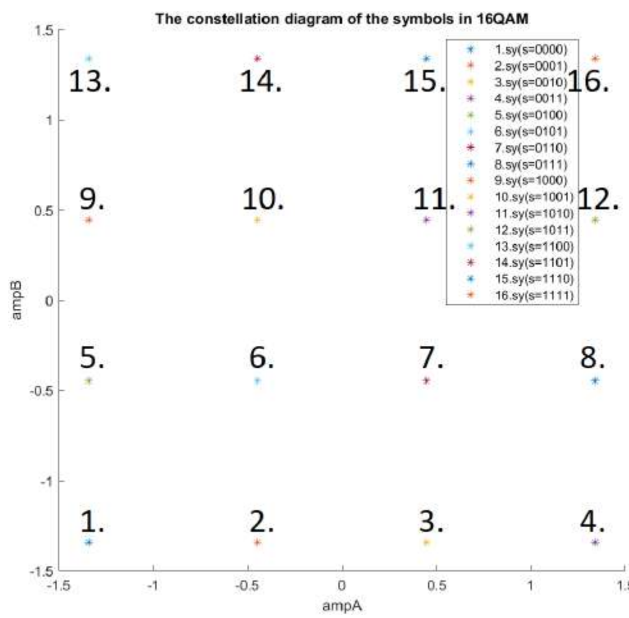


Figure 10: The constellation diagram of the 16-QAM

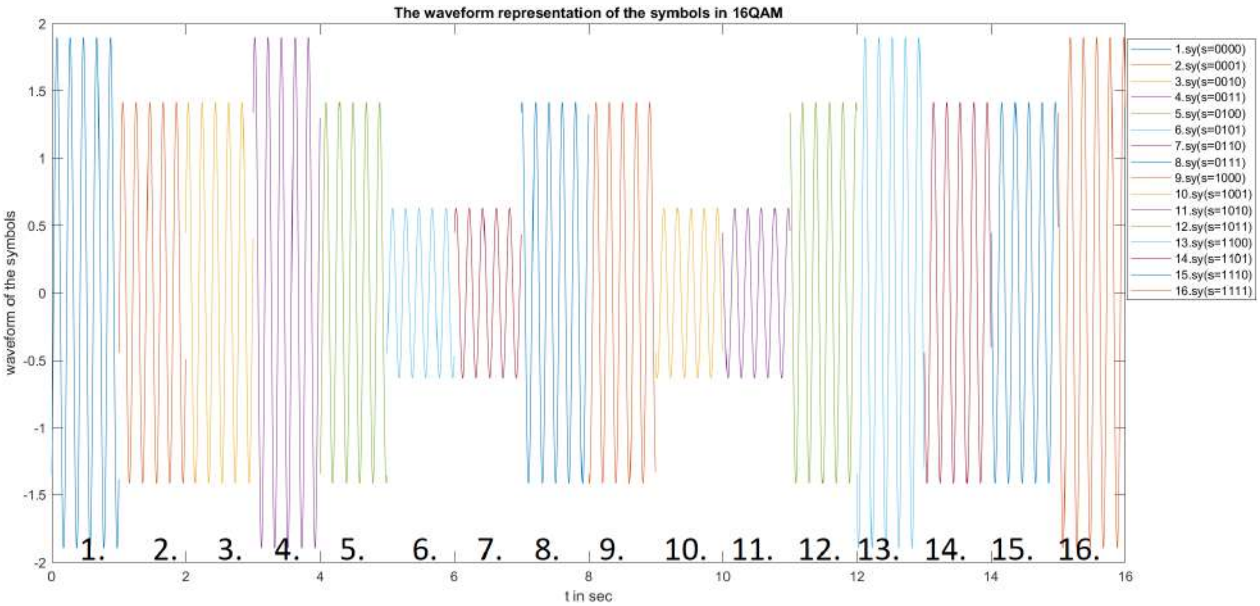


Figure 11: The waveform representation of the symbols in 16-QAM

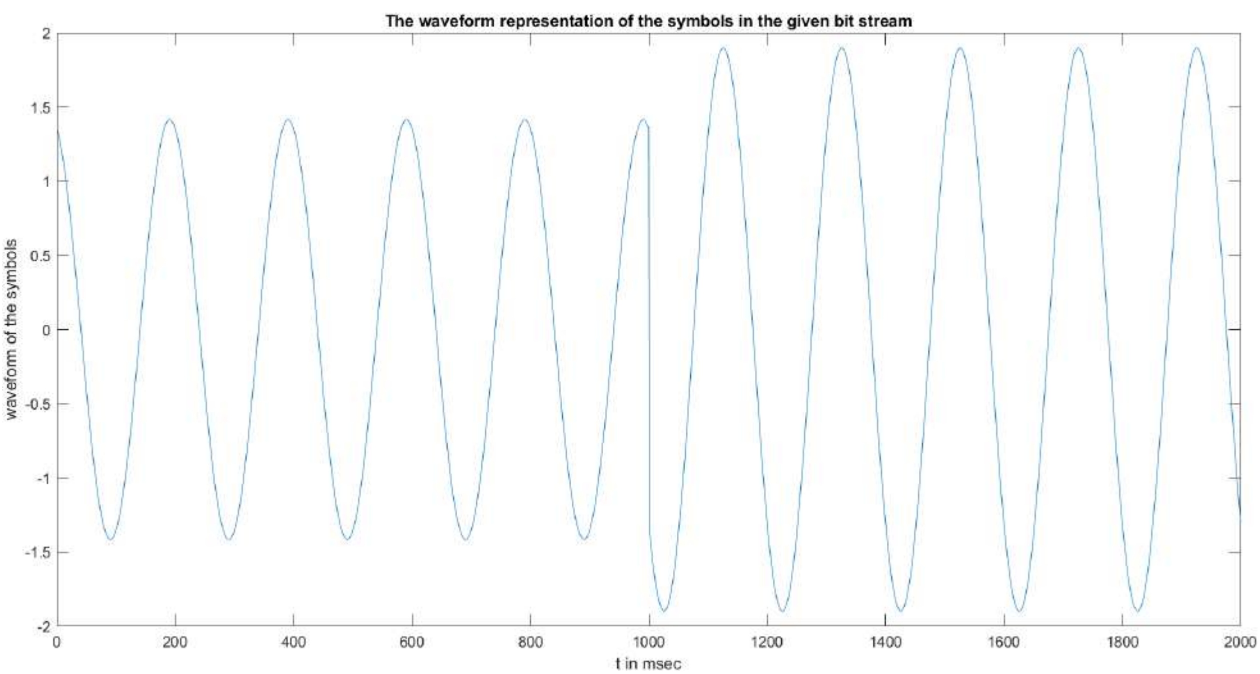


Figure 12: The waveform representation of the symbols in the given bit stream

The Figure 11 shows that there is a π-phase difference between the symbols that have negative and positive amplitudes. And, an amplitude of a symbol can vary along different symbols. The given bit stream $s_{16-ary} = \{1011, 1100\}$ is modulated with the modulation formulation that is described above. And the Figure 12 shows the pulse stream that represents the given bit stream.

As expected, there is a amplitude and phase change at $t = 1$; and when the pulse stream is mapped onto waveforms that are shown in the Figure 11, the original bit stream is obtained.

e) BFSK:

The following expression 8 shows the binary frequency shift keying (BFSK) formulation.

$$s_l(t) = g(t)cos(2\pi(f_c + l\Delta f)t),\ where\ l = 0,1 \tag{8}$$

In order to obtain orthogonal basis functions to represent the $s_l(t)$, the minimum value of the Δf is $1/2T$, where T is the period of one symbol to represent. However, the Δf can be any integer multiples of its minimum value; and it is chosen to be 5, in order to get a clearer frequency change to see in the waveforms.

The constellation diagram of the BFSK that is used in the representation of the given bit stream is shown in the Figure 13. The average symbol energy of BFSK is $E_g/2$, so in order to get unity energy, they should be divided by $\sqrt{E_g/2}$. Since the E_g is selected as 1, so they are divided by $\sqrt{1/2}$.

It can be seen from the constellation diagram that there are 2 different frequencies for the symbols; and 0's have $5Hz$, 1's have $10Hz$.

The sinusoidal waveform to represent the symbols in the constellation diagram is given in the Figure 14.

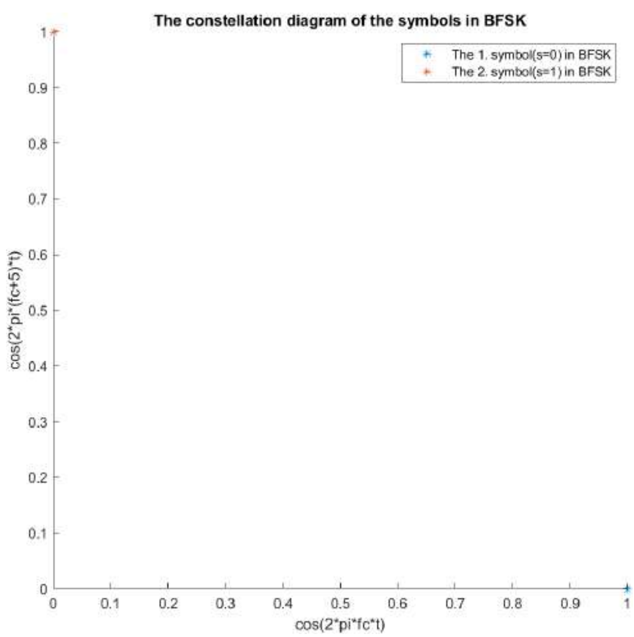


Figure 13: The constellation diagram of the BFSK

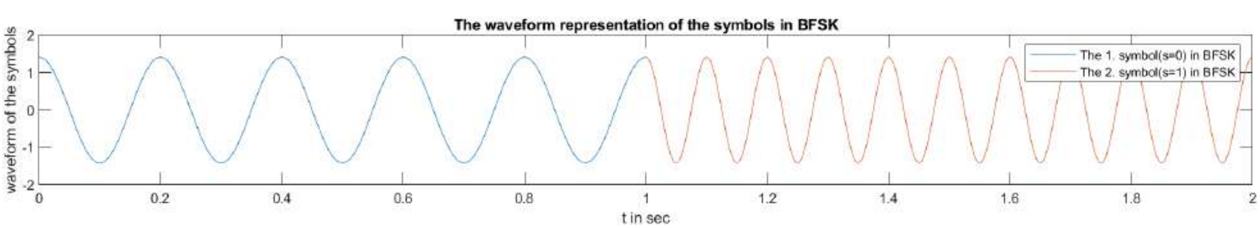


Figure 14: The waveform representation of the symbols in BFSK

It can be seen from the Figure 14 that the distinct symbols in the modulation have different frequencies.

The given bit stream $s = \{1, 0, 1, 1, 1, 1, 0, 0\}$ is modulated with the modulation formulation that is described above. And the Figure 15 shows the pulse stream that represents the given bit stream.

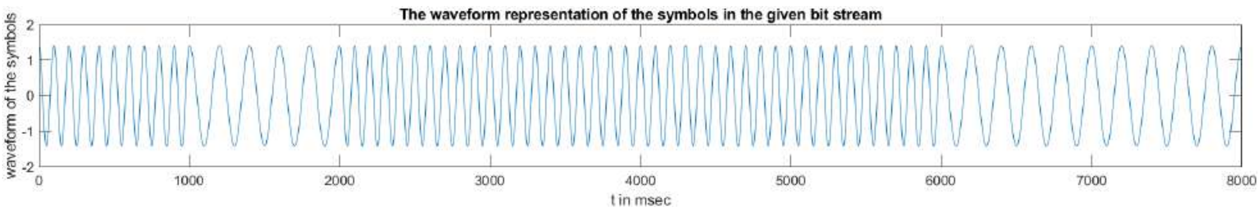


Figure 15: The waveform representation of the symbols in the given bit stream

As expected, there are frequency changes at $t = 1, 2$, and 6, and when the pulse stream is mapped onto waveforms that are shown in the Figure 14, the original bit stream is obtained.

Question 3

a)

Since s_j's are in four-dimensional space, they can be represented with four different unit vectors that are orthogonal to each other.

An orthonormal basis, ρ_i can be the unit vectors of each dimension; and they are shown in the following expression 9.

$$
\begin{aligned}
\rho_1 &= (1, 0, 0, 0) \\
\rho_2 &= (0, 1, 0, 0) \\
\rho_3 &= (0, 0, 1, 0) \\
\rho_4 &= (0, 0, 0, 1)
\end{aligned} \tag{9}
$$

So, s_j's are expressed with the basis that is described above in the expression 10.

$$
\begin{aligned}
s_1 &= (2, -1, -1, -1) = 2\rho_1 - \rho_2 - \rho_3 - \rho_4 \\
s_2 &= (-2, 1, 1, 0) = -2\rho_1 + \rho_2 + \rho_3 \\
s_3 &= (1, -1, 1, -1) = \rho_1 - \rho_2 + \rho_3 - \rho_4 \\
s_4 &= (1, -2, -2, 2) = \rho_1 - 2\rho_2 - 2\rho_3 + 2\rho_4
\end{aligned} \tag{10}
$$

b)

There is also a method, called Gram-Schmidt orthonormalization, to find an orthonormal basis by using the given vectors. It basically generates the basis removing the parts of each vector that are not orthogonal to the others. It starts with any one of the vectors, and normalized it. Then, to find the second basis vector, it uses another vector that is given and subtracts the part that can be expressed by the first vector. And it repeats the procedure to find other basis vectors.

The 1^{st} basis vector is calculated by just normalizing the s_1. It is shown in the following expression 11.

$$\psi_1 = \frac{s_1}{|s_1|} = \frac{1}{\sqrt{7}}(2, -1, -1, -1) \tag{11}$$

The 2^{nd} basis vector is calculated by removing the part that can be expressed by the 1^{st} one, namely removing the projection of the 2^{nd} one onto the 1^{st} one; and normalizing it with its magnitude. It is shown in the following expression 12.

$$\begin{aligned}
\tilde{\psi_2} &= s_2 - \langle s_2, \psi_1 \rangle \psi_1 = s_2 - \frac{-6}{\sqrt{7}}\psi_1 = (-2, 1, 1, 0) + \frac{6}{7}(2, -1, -1, -1) = \frac{1}{7}(-2, 1, 1, -6) \\
\psi_2 &= \frac{\tilde{\psi_2}}{|\tilde{\psi_2}|} = \frac{1}{\sqrt{42}}(-2, 1, 1, -6)
\end{aligned} \tag{12}$$

The 3^{rd} basis vector is calculated by removing the part that can be expressed by the 1^{st} and 2^{nd} ones, namely removing the projection of the 3^{rd} one onto the 1^{st} and 2^{nd} ones; and normalizing it with its magnitude. It is shown in the following expression 13.

$$\begin{aligned}
\tilde{\psi_3} &= s_3 - \langle s_3, \psi_2 \rangle \psi_2 - \langle s_3, \psi_1 \rangle \psi_1 \\
\tilde{\psi_3} &= s_3 - \frac{4}{\sqrt{42}}\psi_2 - \frac{3}{\sqrt{7}}\psi_1 = (1, -1, 1, -1) - \frac{4}{42}(-2, 1, 1, -6) - \frac{3}{7}(2, -1, -1, -1) \\
\tilde{\psi_3} &= \frac{2}{7}(1, -2, 4, 0) \rightarrow \psi_3 = \frac{\tilde{\psi_3}}{|\tilde{\psi_3}|} = \frac{1}{\sqrt{21}}(1, -2, 4, 0)
\end{aligned} \tag{13}$$

The 4^{th} basis vector is calculated by removing the part that can be expressed by the 1^{st}, 2^{nd}, and 3^{rd} ones, namely removing the projection of the 4^{th} one onto the 1^{st}, 2^{nd}, and 3^{rd} ones; and normalizing it with its magnitude. It is shown in the following expression 14.

$$\begin{aligned}
\tilde{\psi_4} &= s_4 - \langle s_4, \psi_3 \rangle \psi_3 - \langle s_4, \psi_2 \rangle \psi_2 - \langle s_4, \psi_1 \rangle \psi_1 \\
\tilde{\psi_4} &= s_4 - \frac{-3}{\sqrt{21}}\psi_3 - \frac{-18}{\sqrt{42}}\psi_2 - \frac{4}{\sqrt{7}}\psi_1 \\
\tilde{\psi_4} &= (1, -2, -2, 2) - \frac{-3}{21}(1, -2, 4, 0) - \frac{-18}{42}(-2, 1, 1, -6) - \frac{4}{7}(2, -1, -1, -1) \\
\tilde{\psi_4} &= \frac{3}{7}(-2, -3, -1, 0) \rightarrow \psi_4 = \frac{\tilde{\psi_4}}{|\tilde{\psi_4}|} = \frac{1}{\sqrt{14}}(-2, -3, -1, 0)
\end{aligned} \tag{14}$$

The orthonormal basis that is found by using Gram-Schmidt method with the given vectors s_j's is shown in the following expression 15

$$\begin{aligned}
\psi_1 &= \frac{1}{\sqrt{7}}(2,-1,-1,-1) \\
\psi_2 &= \frac{1}{\sqrt{42}}(-2,1,1,-6) \\
\psi_3 &= \frac{1}{\sqrt{21}}(1,-2,4,0) \\
\psi_4 &= \frac{1}{\sqrt{14}}(-2,-3,-1,0)
\end{aligned} \tag{15}$$

So, s_j's can be obtained from the 1^{st} line of the expressions 11, 12, 13, 14 with the basis that is described above in the expression 16.

$$\begin{aligned}
s_1 = (2,-1,-1,-1) &= \sqrt{7}\psi_1 \\
s_2 = (-2,1,1,0) &= \frac{-6}{\sqrt{7}}\psi_1 + \frac{6}{\sqrt{42}}\psi_2 \\
s_3 = (1,-1,1,-1) &= \frac{3}{\sqrt{7}}\psi_1 + \frac{4}{\sqrt{42}}\psi_2 + \frac{6}{\sqrt{21}}\psi_3 \\
s_4 = (1,-2,-2,2) &= \frac{4}{\sqrt{7}}\psi_1 + \frac{-18}{\sqrt{42}}\psi_2 + \frac{-3}{\sqrt{21}}\psi_3 + \frac{6}{\sqrt{14}}\psi_4
\end{aligned} \tag{16}$$