Limits to Complex Infinities: Mapping Infinite Numbers to Circles and Spheres

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August 11, 2025

Abstract

This paper develops a framework for mapping infinite numbers onto circles and spheres, extending naturally into the complex plane. The method uses π as a scaling constant to ensure unique mappings at infinity, avoiding degeneracy. By introducing complex numbers as a mapping tool, we leverage their intrinsic curvature to compactify infinity on the Riemann sphere. The implications for limits, continuity, and convergence at $\pm\infty$ are analyzed in both the real and complex cases, with a focus on whether infinity behaves as a single point or distinct endpoints.

1 Introduction

Infinity is traditionally approached linearly along the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. However, compactification techniques allow us to represent infinity as a single point or as part of a closed manifold. This paper explores the geometric mapping of infinite values to circles and spheres, and then extends this concept to the complex plane.

We investigate:

1. Mapping $\mathbb{R}$ onto S^1 (a circle).
2. Mapping $\mathbb{R}^2$ onto S^2 (a sphere).
3. Mapping $\mathbb{C}$ onto the Riemann sphere.
4. How π ensures uniqueness of mapping to infinity.
5. How limits behave when $-\infty$ and $+\infty$ are either continuous or discontinuous in the mapping.

2 Real Line Mapping to a Circle

Let S^1 be the unit circle in $\mathbb{R}^2$. Define:

$$f : \mathbb{R} \to S^1, \quad f(x) = (\cos(\alpha x), \sin(\alpha x)), \quad \alpha = \frac{1}{R}. \tag{1}$$

For $R = 1$, $\alpha = 1$, and the mapping is periodic with period 2π. The irrationality of π prevents rational-step repetition when combined with other irrational scales, ensuring a dense mapping.

2.1 Uniqueness at Infinity

A *unit mapping* is defined such that no two distinct $x, y \in \mathbb{R}$ map to the same point unless they differ by a multiple of $2\pi R$. Spiraling constructions can preserve uniqueness even as $x \to \pm\infty$.

3 Extension to the Sphere

Mapping $\mathbb{R}^2$ to S^2 can be expressed as:

$$g(r, \theta) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{2}$$

where ϕ is determined by $f(x)$ and θ adds an orthogonal degree of variation.

4 Complex Numbers and Curvature

Complex numbers introduce a natural curvature via the *complex plane*. Using stereographic projection, we map $\mathbb{C} \cup \{\infty\}$ onto the Riemann sphere:

$$\Phi(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2 + 1}, \frac{2\,\mathrm{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right). \tag{3}$$

Here, the point at infinity corresponds to the north pole of the sphere. This mapping is conformal—preserving angles—and transforms straight lines and circles in $\mathbb{C}$ into circles on the sphere.

4.1 Implications for Limits

When infinity is represented as a single point in the complex case, limits of functions $f : \mathbb{C} \to \mathbb{C}$ at infinity can be studied via:

$$\lim_{z \to \infty} f(z) \quad \Leftrightarrow \quad \lim_{\Phi(z) \to N} f(\Phi^{-1}(p)),$$

where N is the north pole of the Riemann sphere.

5 Continuity at Infinity

We distinguish two cases:

1. **Continuous Infinity:** $-\infty$ and $+\infty$ map to the same point (as in the Riemann sphere or circle compactification).

2. **Discontinuous Infinity:** $-\infty$ and $+\infty$ map to distinct points (as in the extended real line).

In the continuous case, the limit at infinity requires:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x).$$

In the discontinuous case, these limits may differ.

6 Role of π as a Mapping Constant

Choosing π as the angular scaling constant ensures that mapping from $\mathbb{R}$ to S^1 or $\mathbb{C}$ avoids degenerate periodicities with rational multiples, producing a more uniform angular distribution. This mirrors properties in ergodic theory and Diophantine approximation.

7 Applications

- Complex analysis compactification.
- Number theory in modular arithmetic with irrational moduli.
- Quantum mechanics with periodic boundary conditions.
- Topology of infinity in dynamical systems.

8 Conclusion

Mapping infinity to curved spaces like circles, spheres, and the Riemann sphere transforms the study of limits. Using π ensures unique point mapping, while complex numbers naturally provide curvature for compactification. This approach unifies continuous and discontinuous views of infinity, offering new insights into real and complex analysis.