Closest Approach of Two Parametric Vectors

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Define two parametric equations for straight line tracks $\mathbf{s}(t)$, $\mathbf{r}(t)$ using the following formulas

$$\mathbf{s(t)} = \begin{cases} \text{undefined} & t < t_{s0} \\ \mathbf{s}(t_{s0}) + \mathbf{v}(t - t_{s0}) & t \geq t_{s0} \end{cases}$$

$$\mathbf{r(t)} = \begin{cases} \text{undefined} & t < t_{r0} \\ \mathbf{r}(t_{r0}) + \mathbf{u}(t - t_{r0}) & t \geq t_{r0} \end{cases}$$

Where $\mathbf{s}(t)$ and $\mathbf{r}(t)$ have constant velocity vectors $\mathbf{v}$ and $\mathbf{u}$ respectively. Note that here t is a strictly positive constant measuring global time. In our case, this is the time recorded by Geant. We require both tracks to be defined during the time of cloesest approach (clearly), so we require $t \geq t_M = \max\{t_{s0}, t_{r0}\}$. The distance between these two tracks at a given time $t > t_M$ is

$$\begin{aligned} D(t)^2 &= |\mathbf{s}(t) - \mathbf{r}(t)|^2 \\ &= |\mathbf{r}(t)|^2 + |\mathbf{s}(t)|^2 - 2\mathbf{r}(t) \cdot \mathbf{s}(t) \end{aligned}$$

We note that when seeking the time of closest approach, minimizing the square of the distance is equivalent to minimizing the distance. Thus, we minimize D^2 with respect to t:

$$\frac{dD(t)^2}{dt} = 2\mathbf{r}(t) \cdot \mathbf{u} + 2\mathbf{s}(t) \cdot \mathbf{v} - 2(\mathbf{u} \cdot \mathbf{s}(t) + \mathbf{v} \cdot \mathbf{r}(t)) = 0$$

$$\mathbf{r}(t) \cdot \mathbf{u} + \mathbf{s}(t) \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{s}(t) - \mathbf{v} \cdot \mathbf{r}(t) = 0$$

$$\mathbf{r}(t) \cdot [\mathbf{u} - \mathbf{v}] - \mathbf{s}(t) \cdot [\mathbf{u} - \mathbf{v}] = 0$$

$$[\mathbf{r}(t) - \mathbf{s}(t)] \cdot [\mathbf{u} - \mathbf{v}] = 0$$

Now, we can plug in the explicit forms for the two tracks given above:

$$[(\mathbf{r}(t_{r0}) + \mathbf{u}(t - t_{r0})) - (\mathbf{s}(t_{s0}) + \mathbf{v}(t - t_{s0}))] \cdot [\mathbf{u} - \mathbf{v}] = 0$$

$$[(\mathbf{s}(t_{s0}) - \mathbf{v}t_{s0}) - (\mathbf{r}(t_{r0}) - \mathbf{u}t_{r0}) - (\mathbf{u} - \mathbf{v})t] \cdot [\mathbf{u} - \mathbf{v}] = 0$$

$$[(\mathbf{s}(t_{s0}) - \mathbf{v}t_{s0}) - (\mathbf{r}(t_{r0}) - \mathbf{u}t_{r0})] \cdot [\mathbf{u} - \mathbf{v}] = |\mathbf{u} - \mathbf{v}|^2 t$$

$$t_{CA} = \frac{[(\mathbf{s}(t_{s0}) - \mathbf{v}t_{s0}) - (\mathbf{r}(t_{r0}) - \mathbf{u}t_{r0})] \cdot [\mathbf{u} - \mathbf{v}]}{|\mathbf{u} - \mathbf{v}|^2}$$

Or, in a more illuminating form, we have:

$$t_{CA} = \frac{[\mathbf{s}(t_{s0}) - \mathbf{r}(t_{r0})] \cdot [\mathbf{u} - \mathbf{v}]}{|\mathbf{u} - \mathbf{v}|^2} - \frac{[t_{s0}\mathbf{v} - t_{r0}\mathbf{u}] \cdot [\mathbf{u} - \mathbf{v}]}{|\mathbf{u} - \mathbf{v}|^2}$$

We see that when $t_{s0} = t_{r0} = 0$, the second term is clearly zero, and we see that the first term has the expected behavior: if the tracks are parallel, the closest approach is undefined, and if the tracks have the same starting point, the time of closest approach is 0. Now, recall that we restrict $t_{CA} \geq t_M = \max\{t_{s0}, t_{r0}\}$. Then we have the formula for the distance of closest approach:

$$D_{min} = \begin{cases} |\mathbf{r}(t_{CA}) - \mathbf{s}(t_{CA})| & t_{CA} > t_M \\ |\mathbf{r}(t_M) - \mathbf{s}(t_M)| & t_{CA} \leq t_M \end{cases}$$