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Concrete Mathematics, by Graham, Knuth, and Patashnik
Chapter 1: Recurrent Problems
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Warmups

Exercise 1. All horses are the same color; we can prove this by induction on the number of horses in a given set. Here's how: "If there's just one horse then it's the same color as itself, so the basis is trivial. For the induction step, assume that there are n horses numbered 1 to n. By the induction hypothesis, horses 1 through $n-1$ are the same color, and similarly, horses 2 through n are the same color. But the middle horses, 2 through $n-1$, can't change color when they're in different groups; these are horses, not chameleons. So horses 1 and n must be the same color as well, by transitivity. Thus all n horses are the same color; QED." What, if anything, is wrong with this reasoning?

Solution: It's incorrect because it assumes that the first group of horses numbered 1 through $n-1$ overlaps with the second group numbered 2 through n. However, if $n = 2$, this corresponds to the groups 1 through 1 and 2 through 2, which do not overlap.

Exercise 2. Find the shortest sequence of moves that transfers a tower of n disks from the left peg A to the right peg B, if direct moves between A and B are disallowed (Each move must be to or from the middle peg. As usual, a larger disk must never appear before a smaller one).

Solution: Let T_n be the minimum number of moves that will transfer n disks from one peg to another under the given rules. Note that $T_0 = 0$, $T_1 = 2$, $T_2 = 8$. For three disks, we the pattern emerges:

1. We cannot move disk 3 until disks 1 and 2 are both in peg B, so first we use T_2 moves to do that.
2. Move disk 3 from A to C.
3. Move disks 1 and 2 from B to A by going through C; this is T_2.
4. Move disk 3 from C to B.
5. Move disks 1 and 2 from A to B by going through C; this is T_2.

In other words, $T_3 = 2 + 3T_2$. Generalizing this strategy, we find that we can transfer n disks in at most $2 + 3T_{n-1}$ moves:

$$T_n \leq 2 + 3T_{n-1}, \quad n > 0$$

To prove that equality holds, suppose we have n disks. To move the largest disk, disk n, we must transfer the $n-1$ smallest onto a single peg. If they are in peg C, we still cannot move the largest disk because transfer from A to B are disallowed. Therefore, they must be in B. But this takes precisely T_{n-1} moves. At this point, we transfer the largest disk to C.

By the same logic, we cannot move the largest disk to C until the $n-1$ smallest ones are in a single peg, namely peg A. That takes another T_{n-1} moves. Now the single transfer from C to B places the largest disk at its final position. Now the $n-1$ smallest disks are in peg A. Moving them to B requires another T_{n-1} moves. Hence, the recurrence is:

$$\begin{aligned} T_0 &= 0; \\ T_n &= 3T_{n-1} + 2, \quad \text{for } n > 0 \end{aligned}$$

Let's add 1 to both sides to get:

$$\begin{aligned} T_0 + 1 &= 1; \\ T_n + 1 &= 3T_{n-1} + 3, \quad \text{for } n > 0 \end{aligned}$$

We can substitute $U_n = T_n + 1$ to get

$$\begin{aligned} U_0 &= 1; \\ U_n &= 3U_{n-1}, \quad \text{for } n > 0 \end{aligned}$$

The solution is $U_n = 3^n$, and hence, $T_n = 3^n - 1$.

Exercise 3. Show that, in the process of transferring a tower under the restrictions of the preceding exercise, we will actually encounter every properly stacked arrangement of n disks on three pegs.

Solution: *Proof.* Let S_n be the number of properly stacked arrangements of n disks on the three pegs. The disks begin with a proper arrangement. If a single move is made according to the rules of the previous exercise, then the resulting arrangement is properly stacked. Moreover, it is a *new* arrangement, for if it was an arrangement that we have already seen, then we would be stuck in an endless cycle and never be able to ensure the stack of disks lies entirely on peg B. However, we saw in the previous exercise that it is indeed possible to move them to B after $3^n - 1$ moves. Therefore, each move gives rise to a new arrangement, suggesting that

$$T_n + 1 \leq S_n$$

where $T_n = 3^n - 1$ as in the previous exercise, and the $+1$ accounts for the initial arrangement. Label each disk with an integer from $\{1, \ldots, n\}$, and let D_A, D_B, and D_C denote the set of disks in each of pegs A, B, and C, respectively:

1. D_A corresponds to a subset of $\{1, \ldots, n\}$: This follows because the only way to stack disks on a peg is from largest to smallest. The same is true for D_B and D_C.

2. D_A, D_B, and D_C are disjoint: no two pegs contain the same disk.

3. $D_A \cup D_B \cup D_C = \{1, \ldots, n\}$: Every disk is in one of the three pegs.

In other words, D_A, D_B, and D_C partition $\{1, \ldots, n\}$. The number of arrangements is then equivalent to the number of such partitions. Next, we prove that the number of such partition is 3^n. The basis holds because if $n = 1$, we have:

1. $D_A = \{1\}, D_B = \emptyset, D_C = \emptyset$
2. $D_A = \emptyset, D_B = \{1\}, D_C = \emptyset$
3. $D_A = \emptyset, D_B = \emptyset, D_C = \{1\}$

which is $3 = 3^1$. Suppose that every set of $n-1$ elements has 3^{n-1} partitions, for $n > 1$. Suppose we have a set of n elements. If $1 \leq k \leq n$, let Y_k be the set obtained by removing k from $\{1, \ldots, n\}$. Then Y_k has $n-1$ elements, and it can be partitioned into three distinct subsets in 3^{n-1} distinct ways. Suppose the 3 subsets are called D_A, D_B, D_C. By adding k to any one of them, we obtain a partition of $\{1, \ldots, n\}$ into three sets, one of:

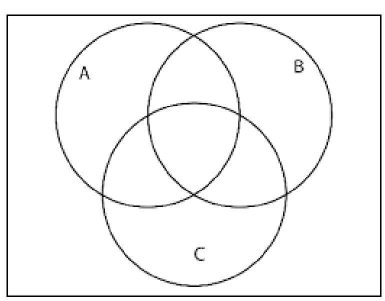
1. $(D_A \cup \{k\}, D_B, D_C)$
2. $(D_A, D_B \cup \{k\}, D_C)$
3. $(D_A, D_B, D_C \cup \{k\})$

Since we have 3 choices for each partition of Y_k, it follows that the number of partitions of $\{1, \ldots, n\}$ is $3 \cdot 3^{n-1} = 3^n$. We have now proved that $S_n = 3^n$, thus, that every arrangement is reachable via a transfer as described in the previous exercise. □

Exercise 4. Are there any starting and ending configurations of n disks on three pegs that are more than $2^n - 1$ moves apart, under Lucas's original rules?

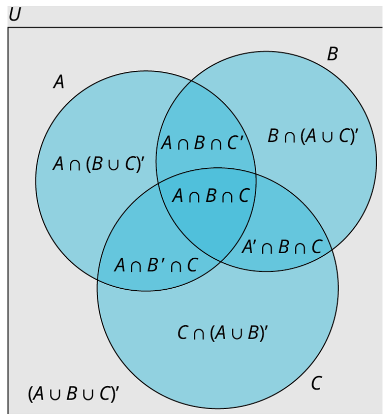
Solution: No. The maximum number of moves between two configurations is 2^{n-1}. For the basis, n_1, we have 3 configurations, and they are all reachable with 1 move. For the inductive hypothesis, assume $n > 1$, we know that $n-1$ disks are at most $2^{n-1}-1$ moves apart. Suppose we have a starting and ending configuration of n disks. For both, the smallest disk is the topmost disk on one of the pegs. Consider the modified start and end configurations without the smallest disk. If the smallest disk were not present, then at most $2^{n-1} - 1$ moves would bring us to the end configuration. If before each move we had to move the smallest out of the way, then we might need twice as many moves, or $2(2^{n-1}-1) = 2^n - 2$. We might require one last move to place the smallest disk in the desired place, for a maximum of $2^n - 2 + 1 = 2^n - 1$ moves.

Exercise 5. A "Venn diagram" with three overlapping circles is often used to illustrate the eight possible subsets associated with three given sets:



Can the sixteen possibilities that arise with four given sets be illustrated by four overlapping circles?

Solution: The 8 sets in questions are given below.



The necessary 16 sets cannot be illustrated by four overlapping circles. Note that any pair of circles intersects in at most 2 points Each intersection increases the number of regions by 1. Therefore, when a fourth circle is added to the Venn diagram above, we get at most 6 intersections (2 for each of the 3 circles), and hence at most 14 regions exist.

Exercise 6. Some of the regions defined by n lines in the plane are infinite, while others are bounded. What's the maximum possible number of bounded regions?

Solution: Note that any bounded region created from intersecting lines has at least 3 vertices. Let B_n be the number of bounded regions defined by n lines in the plane. Since we need at least 3 vertices, $B_0 = 0$, $B_1 = 0$, $B_2 = 0$. Starting with $n = 3$, choosing a non-parallel line, it intersects in 2points (1 for each existing line). It creates a single, triangular region, therefore $B_3 = 3$. A fourth line intersects the remaining 3 in 3 points. If the line intersects the two lines of a bounded region, it splits that region in 2, adding one new region. If, instead, the two lines being intersected define an unbounded region, the intersection creates a new bounded region. Either way, a new region is created. Hence, the intersection with 3 points results in 2 new regions. This also serves as a proof by induction. Therefore,

$$\begin{aligned} B_0 &= 0 \\ B_1 &= 0 \\ B_2 &= 0 \\ B_n &= S_{n-2}, \quad n > 2 \end{aligned}$$

where S_n is the triangular number $n(n+1)/2$.

Exercise 7. Let $H(n) = J(n+1) - J(n)$. Equation (1.8) tells us that $H(2n) = 2$, and $H(2n+1) = J(2n+2) - J(2n+1) = (2J(n+1) - 1) - (2J(n) + 1) = 2H(n) - 2$, for all $n \geq 1$. Therefore it seems possible to prove that $H(n) = 2$ for all n, by induction on n. What's wrong here?

Solution: The basis is not true, since

$$H(1) = J(2) - J(1) = 2J(1) - 1 - J(1) = J(1) - 1 = 1 - 1 = 0.$$

Homework exercises

Exercise 8. Solve the recurrence

$$\begin{aligned} &Q_0 = \alpha; \quad Q_1 = \beta; \\ &Q_n = (1 + Q_{n-1})/Q_{n-2}, \quad \text{for } n > 1. \end{aligned}$$

Assume that $Q_n \neq 0$ for all $n \geq 0$. *Hint*: $Q_4 = (1 + \alpha)/\beta$.

Solution: *Proof.* Computing a few values, we find:

$$\begin{aligned} Q_2 &= \frac{1+\beta}{\alpha} \\ Q_3 &= \frac{1 + \frac{1+\beta}{\alpha}}{\beta} \\ &= \frac{1+\alpha+\beta}{\alpha\beta} \\ Q_4 &= \frac{1+\alpha}{\beta} \\ Q_5 &= \frac{1 + \frac{1+\alpha}{\beta}}{\frac{1+\alpha+\beta}{\alpha\beta}} \\ &= \alpha \\ Q_6 &= \frac{1+\alpha}{\frac{1+\alpha}{\beta}} \\ &= \beta \end{aligned}$$

Note that $Q_5 = Q_0$ and $Q_6 = Q_1$. Since Q_n for $n > 1$ is given in terms of the previous two terms, this implies that Q_n is cyclic with a period of 5, meaning that $Q_n = Q_{n-5}$ for all $n \geq 5$. Hence, the solution is

$$\begin{aligned} Q_0 &= \alpha \\ Q_1 &= \beta \\ Q_2 &= \frac{1+\beta}{\alpha} \\ Q_3 &= \frac{1+\alpha+\beta}{\alpha\beta} \\ Q_4 &= \frac{1+\alpha}{\beta} \\ Q_n &= Q_{n-5}, \quad \text{for } n \geq 5 \end{aligned}$$

□

Exercise 9. Sometimes it's possible to use induction backwards, proving things from n to $n-1$ instead of vice versa! For example, consider the statement

$$P(n): \quad x_1 \cdots x_n \leq \left(\frac{x_1 + \cdots + x_n}{n} \right)^n, \quad \text{if } x_1, \ldots, x_n \geq 0.$$

This is true when $n = 2$ since $(x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2 \geq 0$.

(a) By setting $x_n = (x_1 + \cdots + x_{n-1})/(n-1)$, prove that $P(n)$ implies $P(n-1)$ whenever $n > 1$.

(b) Show that $P(n)$ and $P(2)$ imply $P(2n)$.

(c) Explain why this implies the truth of $P(n)$ for all n.

Solution:

(a) *Proof.* If all x_i are 0, then the statement is definitely true. Suppose that at least one x_i is nonzero, and that $P(n)$ holds. Setting $x_n = (x_1 + \cdots + x_{n-1})/(n-1)$, we find

$$\begin{aligned}
x_1 \cdots x_{n-1} \cdot \left(\frac{x_1 + \cdots + x_{n-1}}{n-1} \right) &\leq \left(\frac{x_1 + \cdots + x_{n-1} + \left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right)}{n} \right)^n \\
&\leq \left(\frac{\frac{(n-1)x_1 + \cdots + (n-1)x_{n-1} + x_1 + \cdots + x_{n-1}}{n-1}}{n} \right)^n \\
&\leq \left(\frac{x_1 + \cdots + x_{n-1}}{n-1} \right)^n
\end{aligned}$$

Since both sides have a factor of $(x_1 + \cdots + x_{n-1})/(n-1)$, and since at least one x_i is nonzero, the term is positive, so we can divide by it without changing the inequality direction:

$$x_1 \cdots x_{n-1} \leq \left(\frac{x_1 + \cdots + x_{n-1}}{n-1} \right)^{n-1}$$

This is precisely $P(n-1)$. □

(b) *Proof.* We were given that $P(2)$ is true. Suppose $P(n)$ is also true. Then

$$\begin{aligned}
x_1 x_2 \cdots x_{2n-1} x_{2n} &= (x_1 x_2) \cdots (x_{2n-1} x_{2n}) \\
&\leq \left(\frac{x_1 + x_2}{2} \right)^2 \cdots \left(\frac{x_{2n-1} + x_{2n}}{2} \right)^2 && \text{(By } P(2)\text{)} \\
&= \left(\frac{1}{2} \right)^{2n} (x_1 + x_2)^2 \cdots (x_{2n-1} + x_{2n})^2 \\
&= \left(\frac{1}{2} \right)^{2n} ((x_1 + x_2) \cdots (x_{2n-1} + x_{2n}))^2 \\
&\leq \left(\frac{1}{2} \right)^{2n} \left(\frac{(x_1 + x_2) + \cdots + (x_{2n-1} + x_{2n})}{n} \right)^{2n} && \text{(By } P(n)\text{)} \\
&= \left(\frac{x_1 + \cdots + x_{2n}}{2n} \right)^{2n}
\end{aligned}$$

□

(c) We know it holds for $P(2)$. Suppose that $n > 2$, and that it holds for 2 through $n-1$.

   (i) If n is even, then $n = 2k$, and since $2 \leq k \leq n-1$, it holds for $P(k)$ by induction, so it must hold for $P(2k) = P(n)$ (since it holds for $P(2)$) by (b).

   (ii) If n is odd, then $n = 2k - 1$ for some $k > 1$. Since $n \geq 3$, it follows that $k = \frac{n+1}{2} \leq n-1$, so $P(k)$ holds, implying that $P(2k)$ holds by (b). Now $P(2k-1)$ holds by (a).

   The result now holds for all n by induction.

Exercise 10. Let Q_n be the minimum number of moves needed to transfer a tower of n disks from A to B if all moves must be made *clockwise* — that is, from A to B, or from B to the other peg, or from the other peg to A. Also, let R_n be the minimum number of moves needed to go from B back to A under this restriction. Prove that

$$Q_n = \begin{cases} 0, & \text{if } n = 0; \\ 2R_{n-1} + 1, & \text{if } n > 0; \end{cases} \quad R_n = \begin{cases} 0, & \text{if } n = 0; \\ Q_n + Q_{n-1} + 1, & \text{if } n > 0. \end{cases}$$

(You need not solve these recurrences; we'll see how to do that in Chapter 7.)

Solution: *Proof.* We begin by proving that

$$Q_n = 2R_{n-1} + 1$$

(a) *Move disks 1 through $n-1$ from A to C*: This is necessary because we want to move disk n to peg B, and that pegs needs to be empty since disk n is the largest move. It takes R_{n-1} moves to do this.

(b) *Move disk n from A to B*: This takes a single move.

(c) *Move disk 1 through $n-1$ from C to B*: This takes R_{n-1}.

The moves made were necessary and sufficient, so the equation is proved. The hints in Appendix A suggest proceeding by proving that

$$R_n = R_{n-1} + 1 + Q_{n-1} + 1 + R_{n-1}, \quad n > 0$$

Suppose disks 1 through n are stacked on peg B, and we wish to move the stack to peg A. We can do so as follows:

(i) *Move disks 1 through $n-1$ to A*: This is necessary because we need to eventually move disk n to peg C (before going to A). Since it is the largest disk, there can be no other disk on peg C, and hence every disk must be on peg A. This takes R_{n-1} moves.

(ii) *Move disk n from B to C*: This takes a single move.

(iii) *Move disks 1 through $n-1$ from A to B*: In order to have disk n on peg A, that peg must be empty, so every disk must be on peg B. This will take Q_{n-1} moves.

(iv) *Move disk n from C to A*: This takes 1 more.

(v) *Move disks 1 through $n-1$ from B to A*: This takes R_{n-1}.

We can combine these to get the desired equation for R_n:

$$\begin{aligned} R_n &= R_{n-1} + 1 + Q_{n-1} + 1 + R_{n-1} \\ &= (2R_{n-1} + 1) + Q_{n-1} + 1 \\ &= Q_n + Q_{n-1} + 1 \end{aligned}$$

as we set out to show. □

Exercise 11. A Double Tower of Hanoi contains $2n$ disks of n different sizes, two of each size. As usual, we're required to move only one disk at a time, without putting a larger one over a smaller one.

(a) How many moves does it take to transfer a double tower from one peg to another, if disks of equal size are indistinguishable from each other?

(b) What if we required to reproduce the original top-to-bottom order of all the equal-size disks in the final arrangement? [*Hint*: This is difficult — it's really a "bonus problem"].

Solution:

(a) Let S_n be the number of moves it takes to move the Double Tower of Hanoi of $2n$ disks. If $n = 1$, then it takes 2 moves, since the two disks are indistinguishable and can be stacked, so $S_1 = 2$. If $n > 1$, suppose we start with $2n$ disks on peg A. To move the tower to peg B, we do the following:

   (a) *Move smallest $2n-2$ disks from A to C*: This is a pre-requisite to moving the largest two disks to B. Since B must be empty before placing the two largest disks, it follows that every other disk must be in C. This takes S_{n-1} moves.

   (b) *Move the two largest disks from A to B*: This takes two moves. This flips their order, which is allowed because the disks are indistinguishable, and disks of the same size can be stacked.

   (c) *Move the remaining disks from C to B*: This takes S_{n-1} moves.

   Hence, $S_n = 2S_{n-1} + 2$, and $S_1 = 2$. We can now solve by first adding 2 to both sides and substituting:

$$\begin{aligned} &S_1 + 2 = 4 \\ &S_n + 2 = 2S_{n-1} + 4 = 2(S_{n-1} + 2) \end{aligned}$$

   Let $T_n = S_n + 2$. Then the recurrence becomes

$$\begin{aligned} T_1 &= 4 \\ T_n &= 2T_n \end{aligned}$$

   The solution is $T_n = 2 \cdot 2^n = 2^{n+1}$, so $S_n = 2^{n+1} - 2$.

(b) Let X_n be the minimum number of moves to achieve the goal. Then $X_1 = 3$, since we move the first disk from A to C, the second disk from A to B, and the first disk from C to A. Call the two largest disks n and n', with n' at the bottom. For $n > 1$, we do the following:

(i) *Move smallest* $2n-2$ *disks from* A *to* B: This is necessary in order to start moving the largest disks. This takes S_{n-1} moves, where S_n is defined in part (a).

(ii) *Move disk* n *from* A *to* C: This takes 1 move.

(iii) *Move smallest* $2n - 2$ *disks from* B *to* C: Since we cannot put n' on top of n (which would flip the disks), we need n on peg B, and hence, that disk must be empty. This takes S_{n-1} moves.

(iv) *Move disk* n *from* A *to* B: This takes 1 move.

(v) *Move smallest* $2n - 2$ *disks from* C *to* A: This is required to move disk n on top of n'. This takes S_{n-1} moves.

(vi) *Move* n *from* C *to* B: This takes 1 move. Now n is on top of n' just like at the start, but they're on peg B.

(vii) *Move smallest* $2n - 2$ *disks from* A *to* B: This takes $2n - 2$ moves.

At this point, we'll have moved the stack of $2n-2$ disks a total of 4 times, so it is back in its original arrangement. Hence,

$$X_n = 4S_{n-1} + 3 = 4(2^n - 2) + 3 = 2^{n+2} - 8 + 3 = 2^{n+2} - 5$$

Exercise 12. Let's generalize exercise 11a even further, by assuming that there are n different sizes of disks and exactly m_k disks of size k. Determine $A(m_1, \ldots, m_n)$, the minimum number of moves needed to transfer a tower when equal-size disks are considered indistinguishable.

Solution: If $n = 1$, then $A(m_1) = m_1$. Suppose $n > 1$, and $m_k > 0$ each k. Suppose all disks start on peg A and want to transfer all disks to B::

(a) *Move all disks of sizes* $1, \ldots, n-1$ *to* C: This is required to be able to move the disks of size m_n. This takes $A(m_1, \ldots, m_{n-1})$ moves.

(b) *Move all disks of size* n: This takes m_n moves since there are m_n such disks.

(c) *Move all disks of size* $1, \ldots, n-2$ *to* B: This takes $A(m_1, \ldots, m_{n-1})$ moves again.

Hence, $A(m_1, \ldots, m_n) = 2A(m_1, \ldots, m_{n-1}) + m_n$, which we can unfold as follows:

$$\begin{aligned}
A(m_1, \ldots, m_n) &= 2A(m_1, \ldots, m_{n-1}) + m_n \\
&= 4A(m_1, \ldots, m_{n-2}) + 2m_{n-1} + m_n \\
&= 2^{n-1}A(m_1) + 2^{n-2}m_2 + \cdots + 2m_{n-1} + m_n \\
&= 2^{n-1}m_1 + 2^{n-2}m_2 + \cdots + 2m_{n-1} + m_n
\end{aligned}$$

Exercise 13. What's the maximum number of regions definable by n zig-zag lines, each of which consists of two parallel infinite half-lines joined by a straight segment? Given: $ZZ_2 = 12$.

Solution: Note $ZZ_1 = 2$, and we are given that $ZZ_2 = 12$. There are 2 unbounded regions and 0 bounded regions for ZZ_1, and there are 4 unbounded regions and 8 bounded regions for ZZ_2. Moreover, there are 9 intersection points between the two zig-zags. Let's consider ZZ_3.

For ZZ_3, we get 2 new unbounded regions, and since there are 2 existing zig-zag lines, the new one intersects each of those in 9 points, and hence, a total of 18 points, so it should create 17 new bounded regions. That is, altogether, we have 19 new regions, so the new total should be 31.

For the general case, suppose we have ZZ_{n-1} regions with $n-1$ zig-zag lines. If we ensure the nth zig-zag intersects every other zig-zag in 9 intersection points, then we will add 2 unbounded regions (due to each infinite segment of the new zig-zag) and if there are k intersections, we get $k-1$ bounded regions. We can ensure the new zig-zag intersects every other zig-zag in 9 points, for a total of $9(n-1)$ new intersections, which creates $9(n-1)-1$ new regions. Hence:

$$ZZ_n = ZZ_{n-1} + 9(n-1) - 1 + 2$$

We can unfold it to get the closed:

$$\begin{aligned}
ZZ_n &= Z_1 + 9(1 + \cdots + n - 1) + 2 \cdot (n-1) - 1(n-1) \\
&= Z_1 + 9 \cdot \frac{(n-1)n}{2} + n - 1 \\
&= Z_1 + \frac{9}{2}n^2 - \frac{9}{2}n + n - 1 \\
&= \frac{9}{2}n^2 - \frac{7}{2}n + 1
\end{aligned}$$

Exercise 14. How many pieces of cheese can you obtain from a single thick piece by making five straight slices? (The cheese must stay in its original position while you do all the cutting, and each slice must correspond to a plane in 3D). Find a recurrence relation for P_n, the maximum number of three-dimensional regions that can be defined by n different planes.

Exercise 15. Josephus had a friend who was saved by getting into the next-to-last position. What is $I(n)$, the number of the penultimate survivor when every second person is executed?

Solution: Note if there are 10 people, the survivor is 5, but the penultimate survivor is 9, so $I(10) = 9$. The following table shows some values:

n	2	3	4	5	6	7	8	9	10	11	12
$I(n)$	2	1	3	5	1	3	5	7	9	11	1

Notice that $I(1)$ is not defined. An analysis similar to the Josephus problem leads to:

$$\begin{aligned}
I(2) &= 2; \\
I(3) &= 1; \\
I(2n) &= 2I(n) - 1, \quad n \geq 2, \\
I(2n+1) &= 2I(n) + 1, \quad n \geq 2
\end{aligned}$$

The sample of table values show that when $n = 3 \cdot 2^m$, we have $I(n) = 1$, marking the start of an increasing group of odd numbers. The next group starts at $3 \cdot 2^{m+1}$, and $I(3 \cdot 2^{m+1}) = 1$ The number of values in each group is $3 \cdot 2^{m+1} - 3 \cdot 2^m = 3 \cdot 2^m$. Therefore, we can write n as the start of each group plus an offset. That is, we can write $n = 3 \cdot 2^m + l$, for some $m \geq 0$ and $0 \leq l < 3 \cdot 2^m$. Then

$$I(n) = I(3 \cdot 2^m + l) = 2l + 1, \quad m \geq 0 \text{ and } 0 \leq l < 3 \cdot 2^m$$

We proceed by induction on m. When $m = 0$, we see that $l = 0$, giving $I(3) = 2 \cdot 0 + 1 = 1$, which is true. For the induction step, first consider the case when $3 \cdot 2^m + l$ is even. This implies that l is even, so

$$I(3 \cdot 2^m + l) = 2I(2^{m-1} + l/2) - 1 = 2(2l/2 + 1) - 1 = 2l + 1$$

The odd case is similar, using $I(2n + 1)$ instead. The equation holds for all n.

Exercise 16. Use the repertoire method to solve the general four-parameter recurrence

$$\begin{aligned} g(1) &= \alpha; \\ g(2n + j) &= 3g(n) + \gamma n + \beta_j, \quad \text{for } j = 0, 1, \text{ and } n \geq 1 \end{aligned}$$

Hint: Try the function $g(n) = n$.

Solution: Let us suppose that the solution has the form

$$g(n) = A(n)\alpha + B(n)\beta_0 + C(n)\beta_1 + D(n)\gamma$$

First let's suppose $g(n) = n$. Then

$$\begin{aligned} 1 &= \alpha \\ 2n &= 3n + \beta_0 \qquad \Longrightarrow \qquad -n = \beta_0 \\ 2n + 1 &= 3n + \gamma n + \beta_1 \qquad \Longrightarrow \qquad -n + 1 = \gamma_n + \beta_1 \end{aligned}$$

By treating the equations as polynomials in n and equating coefficients, we conclude that $\alpha = 1$, $\beta_0 = 0$, $\beta_1 = 1$, and $\gamma = -1$. Now let $g(1) = 1$. We get the equations

$$\begin{aligned} 1 &= \alpha \\ 1 &= 3 + \gamma n + \beta_0 \\ 1 &= 3 + \gamma n + \beta_1 \end{aligned}$$

Similar to before, we conclude that $\alpha = 1$, $\beta_0 = \beta_1 = -2$, and $\gamma = 0$. Hence, we get the equations:

$$\begin{aligned} A(n) + C(n) - D(n) &= n \\ A(n) - 2B(n) - 2C(n) &= 1 \end{aligned}$$

We can solve by determining two of the functions, and expressing the remaining two in terms of those. For example, if we set $\alpha = 1$ and $\gamma = \beta_0 = \beta_1 = 0$, then $g(n) = A(n)$, and

$$\begin{aligned} A(1) &= 1 \\ A(2n) &= 3A(n) \\ A(2n+1) &= 3A(n) \end{aligned}$$

Creating a table reveals that if we let $n = 2^m + l$, where $m \geq 0$ and $0 \leq l < m$, then $A(2^m + l) = 3^m$. We could do this for one more function to and be done. Alternatively, as pointed out in the Appendix for the solution to this problem, setting $\gamma = 0$ leaves us with

$$\begin{aligned} g(1) &= \alpha \\ g(2n+j) &= 3g(n) + \beta_j, \quad 0 \leq j < 1 \text{ and } n \geq 1 \end{aligned}$$

Therefore, this defines $A(n)\alpha + B(n)\beta_0 + C(n)\beta_1$ to be the radix-changing function, from radix 2 to radix 3. Namely, converting $n = (1b_{m-1}b_{m-2} \ldots b_1 b_0)_2$ to $(\alpha\beta_{b_{m-1}}\beta_{b_{m-2}} \ldots \beta_{b_0})_3$. In othet words, $A(n)$, $B(n)$, and $C(n)$ are determined when $\gamma = 0$, and we can express $D(n)$ in terms of them. Hence, all functions are determined.