Sergio Garcia Tapia
Concrete Mathematics, by Graham, Knuth, and Patashnik
Chapter 1: Recurrent Problems
March 5, 2024

Warmups

Exercise 1. What does the notation

$$\sum_{k=4}^{0} q_k$$

mean?

Solution: It is the sum of all terms q_k whose index k is an integer between the lower limit of 4 and the upper limit 0, inclusive. The set of numbers satisfying this property is empty, so this is an empty sum with a value fo 0.

Exercise 2. Simplify the expression $x \cdot ([x > 0] - [x < 0])$.

Solution: Here, $[x > 0]$ is the boolean function that is 1 if $x > 0$ and 0 otherwise; similar for $[x < 0]$. Suppose that $x > 0$. Then the $[x > 0] = 1$ and $[x < 0] = 0$, so the expression simplifies to $x \cdot (1 - 0) = x$. If $x < 0$, it simplifies to $x \cdot (0 - 1) = -x$. If $x = 0$, then it's just 0. Hence

$$x \cdot ([x > 0] - [x < 0]) = |x|.$$

Exercise 3. Demonstrate your understanding of Σ-notation by writing out the sums

$$\sum_{0 \leq k \leq 5} a_k \quad \text{and} \quad \sum_{0 \leq k^2 \leq 5} a_{k^2}$$

in full. (Watch out — the second sum is a bit tricky).

Solution: Recalling that $[0 \leq k \leq 5]$ is the boolean function that is 1 for the statement

$$P(k) : 0 \leq k \leq 5$$

$$\sum_{0 \leq k \leq 5} = a_k = \sum_k a_k [0 \leq k \leq 5] = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$$

Similarly, the only integers $k \in \mathbb{Z}$ satisfying $0 \leq k^2 \leq 5$ are $k = -2$, $k = -1$, $k = 0$, $k = 1$, and $k = 2$, so

$$\sum_{0 \leq k^2 \leq 5} a_{k^2} = a_{(-2)^2} + a_{(-1)^2} + a_{0^2} + a_{1^2} + a_{2^2} = a_2 + a_1 + a_0 + a_1 + a_4$$

Exercise 4. Express the triple sum

$$\sum_{1 \leq i < j < k \leq 4} a_{ijk}$$

as a three-fold summation (with three Σ's),

(a) summing first on k, then j, then i;

(b) summing first on i, then j, then k.

Also write your triple sums out in full without Σ-notation, using parentheses to show what is being added together first.

Solution: The expanded sum fro the expression given is

$$\sum_{1\leq i<j<k\leq 4} a_{ijk} = a_{123} + a_{124} + a_{134} + a_{234}$$

(a) The index condition for the summation can be factored as

$$[1 \leq i < j < k \leq 4] = [1 \leq i < 4][i < j < 4][j \leq k \leq 4]$$

Therefore,

$$\begin{aligned}
\sum_{1\leq i<j<k\leq 4} a_{ijk} &= \sum_{1\leq i<4} \sum_{i<j<4} \sum_{j<k\leq 4} a_{ijk} \\
&= (a_{123} + a_{124}) \quad \text{(terms with } i = 1, j = 2) \\
&+ (a_{134}) \qquad \text{(terms with } i = 1,\ j = 3) \\
&+ (a_{234}) \qquad \text{(terms with } i = 2, j = 3)
\end{aligned}$$

The parentheses show the inner sum k for each fixed i, j. Note that when $i = 3$, we require $i < j$ and $j < k$, which implies $k \geq 5$, and no such terms exist.

(b) A different, but equivalent, factorization is

$$[1 \leq i < j < k \leq 4] = [1 < k \leq 4][1 < j < k][1 \leq i < j]$$

The sum then becomes

$$\begin{aligned}
\sum_{1<k\leq 4} \sum_{1<j<k} \sum_{1\leq i<j} a_{ijk} &= (a_{124}) \qquad \text{(terms with } k = 4,\ j = 2) \\
&+ (a_{134} + a_{234}) \quad \text{(terms with } k = 4,\ j = 3) \\
&+ (a_{123}) \qquad \text{(term with } k = 3)
\end{aligned}$$

Exercise 5. What's wrong with the following derivation?

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_j}{a_k} \qquad = \sum_{k=1}^{n}\sum_{k=1}^{n} \frac{a_k}{a_k} = \sum_{k=1}^{n} n = n^2$$

Solution: The second equality is incorrect. In writing

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_j}{a_k} = \sum_{k=1}^{n}\sum_{k=1}^{n} \frac{a_k}{a_k}$$

we are replacing index j with the index k that is already in use, introducing ambiguity about two otherwise independent indices.

Exercise 6. What is the value of $\sum_k [1 \leq j \leq k \leq n]$, as a function of j and n?

Solution: Factoring the summand condition, we get

$$\begin{aligned}
\sum_k [1 \leq j \leq k \leq n] &= \sum_k [1 \leq j \leq n][j \leq k \leq n] \\
&= [1 \leq j \leq n] \sum_{k=j}^{n} \\
&= [1 \leq j \leq n](n - j + 1)
\end{aligned}$$

because the sum is essentially the count of numbers between j and n for $j \leq n$.

Exercise 7. Let $\nabla f(x) = f(x) - f(x-1)$. What is $\nabla \left(x^{\overline{m}}\right)$?

Solution: Recall that $x^{\overline{m}}$ is "x to the m rising", or rising power, and it is defined as the m-term product

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+(m-1))$$

Then

$$\begin{aligned}
\nabla \left(x^{\overline{m}}\right) &= \left(x^{\overline{m}}\right) - (x-1)^{\overline{m}} \\
&= x(x+1)(x+2)\cdots(x+(m-1)) - (x-1)(x)(x+1)\cdots(x+(m-2)) \\
&= x(x+1)(x+2)\cdots(x+(m-2)) \cdot [x+(m-1)-(x-1)] \\
&= mx(x+1)(x+2)\cdots(x+(m-2)) \\
&= m\left(x^{\overline{m-1}}\right)
\end{aligned}$$

Exercise 8. What is the value of $0^{\underline{m}}$, when m is a given integer?

Solution: It is 0 if $m = 0$. If $m > 0$, then

$$0^{\underline{m}} = 0 \cdot (0-1)(0-2)\cdots(0-(m-1)) = 0$$

If $m < 0$, then

$$0^{\underline{m}} = \frac{1}{(0+1)(0+2)\cdots(0+(-m))} = \frac{1}{1^{\overline{m}}} = \frac{1}{m!}$$

Exercise 9. What is the law of exponents for rising factorial powers, analogous to (2.52)? Use this to define $x^{-\overline{n}}$.

Solution: Note that

$$\begin{aligned}
x^{\overline{3}} &= x(x+1)(x+2) \\
x^{\overline{2}} &= x(x+1) \\
x^{\overline{1}} &= x \\
x^{\overline{0}} &= 1
\end{aligned}$$

In other words, we divide by $(x+2)$, then by $(x+1)$ then by x. Each time, we divide by $x+k$, and we decrease k by 1. It would seem that to continue the pattern we would want to next divide by $(x-1)$, then $(x-2)$, and so on, so that

$$\begin{aligned} x^{\overline{-1}} &= \frac{1}{x-1} \\ x^{\overline{-2}} &= \frac{1}{(x-1)(x-2)} \end{aligned}$$

and so on. Note that

$$x^{\overline{2}} \cdot (x+2)^{\overline{3}} = x(x+1) \cdot (x+2)(x+3)(x+4) = x^{\overline{m}}$$

In general,

$$x^{\overline{m+n}} = x^{\overline{m}}(x+m)^{\overline{n}}$$

Now we can use this to define x^{-n} by letting $m = -n$:

$$\begin{aligned} 1 &= x^{\overline{0}} \\ &= x^{\overline{-n+n}} \\ &= x^{\overline{-n}}(x-n)^{\overline{n}} \end{aligned}$$

Hence

$$x^{\overline{-n}} = \frac{1}{(x-n)^{\overline{n}}} = \frac{1}{(x-1)^{\underline{n}}}$$

Exercise 10. The text derives the formula for the difference of a product

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

How can this formula be correct, when the left-hand side is symmetric with respect to u and v but the right-hand side is not?

Solution: In the derivation of $\Delta(uv)$, we added 0 in the form of $u(x)v(x+1) - u(x)v(x+1)$ to simplify the difference. Here, we arbitrarily chose to apply the shift operator to v. However, we could have equivalently applied it to u, to get

$$\begin{aligned} \Delta(uv) &= u(x+1)v(x+1) - u(x)v(x) \\ &= u(x+1)v(x+1) - u(x+1)v(x) + u(x+1)v(x) - u(x)v(x) \\ &= Eu\Delta v + v\Delta u \end{aligned}$$

Therefore, in the text definition, we could certainly switch what the operators are being applied to, and obtain an equivalent expression.

Basics

Exercise 11. The general rule (2.56):

$$\sum u\Delta v = uv - \sum Ev\Delta u$$

for summation by parts is equivalent to

$$\sum_{0\le k<n}(a_{k+1}-a_k)b_k = a_nb_n - a_0b_0 - \sum_{0\le k<n} a_{k+1}(b_{k+1}-b_k), \qquad \text{for } n \ge 0.$$

Prove this formula directly by using the distributive, pairing, and commutative laws.

Solution: *Proof.*

$$\begin{aligned}\sum_{0\le k<n}(a_{k+1}-a_k)b_k &= \sum_{0\le k<n} a_{k+1}b_k - \sum_{0\le k<n} a_kb_k\\ &= \sum_{0\le k<n} a_{k+1}b_k - \left(a_0b_0 - a_nb_n + \sum_{1\le k\le n} a_kb_k\right)\\ &= \sum_{0\le k<n} a_{k+1}b_k - \left(a_0b_0 - a_nb_n + \sum_{1\le k+1\le n} a_{k+1}b_{k+1}\right)\\ &= \sum_{0\le k<n} a_{k+1}b_k - \left(a_0b_0 - a_nb_n + \sum_{0\le k<n} a_{k+1}b_{k+1}\right)\\ &= a_nb_n - a_0b_0 + \sum_{0\le k<n} a_{k+1}b_k - \sum_{0\le k<n} a_{k+1}b_{k+1}\\ &= a_nb_n - a_0b_0 - \sum_{0\le k<n}(b_{k+1}-b_k)a_{k+1},\end{aligned}$$

where the first equality follows from the pairing law, the second comes from splitting off the sum, which is a consequence of the pairing law and the commutative law (notice $0\le k<n$ changed to $1\le k\le n$), the third from the commutative law because we replaced k by $k+1$, the fourth by simplifying the bounds, the fifth from the distributive law, and the last by the pairing law. □

Exercise 12. Show that the function $p(k) = k + (-1)^k c$ is a permutation of the set of all integers, whenever c is an integer.

Solution: *Proof.* A permutation of the set of all integers is a $1-1$ correspondence. First, we prove that p is 1-1. Suppose that $p(k_1) = p(k_2)$, meaning that

$$k_1 + (-1)^{k_1}c = k_2 + (-1)^{k_2}c \quad \Longrightarrow \quad k_2 - k_1 = (-1)^{k_1}c - (-1)^{k_2}c$$

Suppose that k_1 is even and k_2 is odd. Then $(-1)^{k_1} = c$ and $(-1)^{k_2} = -$, $k_2 - k_1 = 2c$. But this is impossible because the difference of an odd and even number is odd. Hence, k_1 and k_2 are both even, implying that $k_1 = k_2$.

Next, we show that p is onto. Suppose c is even. Let $2m$ be any even integer. If we let $k = 2m - c$, then k is even, so $p(k) = k + (-1)^k c = k + c = (2m - c) + c = 2m$. If $2m + 1$ is an odd integer, choose $k = 2m + 1 + c$. Then k is odd, so $(-1)^k$ is -1, and hence $p(k) = k + (-1)^c m = k - c = (2m + 1 + c) - c = 2m + 1$. Hence, if c is even, then p is onto. A similar argument works when c is odd. Hence, p is onto, and is thus a 1-1 corresponding from $\mathbb{Z}$ onto itself, meaning that it is a permutation. □

Exercise 13. Use the repertoire method to find a closed form for $\sum_{k=0}^{n}(-1)^k k^2$.

Solution: Let $S_n = \sum_{k=0}^{n}(-1)^k k^2$. To leverage the repertoire method, we express it as a recurrence relation, like so:

$$\begin{aligned} S_0 &= 0; \\ S_n &= S_{n-1} + (-1)^n n^2. \end{aligned}$$

Suppose R_n is the more general recurrence

$$\begin{aligned} R_0 &= \alpha; \\ R_n &= R_{n-1} + (-1)^n \left(\beta + \gamma n + \lambda n^2\right). \end{aligned}$$

Then $R(n) = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\lambda$. To solve for the coefficient functions, we choose different solution functions, starting with the simple $R(n) = 1$. Then the recurrence becomes

$$\begin{aligned} &1 = \alpha; \\ &1 = 1 + (-1)^n(\beta + \gamma n + \lambda n^2) \quad\iff\quad 0 = (-1)^n(\beta + \gamma n + \lambda n^2) \end{aligned}$$

Hence $\alpha = 1$. In the second equation, the product is only 0 if the parenthesized expression is 0. This immediately implies that $\beta = \gamma = \lambda = 0$. Hence,

$$1 = R(n) = A(n)$$

Next, we go with $R(n) = (-1)^n$, so the recurrence becomes:

$$\begin{aligned} 1 &= \alpha; \\ (-1)^n &= (-1)^{n-1} + (-1)^n(\beta + \gamma n + \lambda n^2) \quad\iff\quad 2 = \beta + \gamma n + \lambda n^2 \end{aligned}$$

Hence, $\alpha = 1$, $\beta = 2$, and $\gamma = \lambda = 0$. In this case,

$$(-1)^n = R(n) = A(n) + 2B(n)$$

Recalling that $A(n) = 1$, it follows that

$$B(n) = \frac{1}{2}(-1)^n - \frac{1}{2}$$

Next, we consider $R(n) = (-1)^n n$. In this case, the recurrence becomes:

$$
\begin{aligned}
0 &= \alpha; \\
(-1)^n n &= (-1)^{n-1}(n-1) + (-1)^n(\beta + \gamma n + \lambda n^2) \qquad \Longleftrightarrow \qquad 2n - 1 = \beta + \gamma n + \lambda n^2
\end{aligned}
$$

Hence, $\alpha = 0$, $\beta = -1$, $\gamma = 2$, and $\lambda = 0$. This implies that

$$(-1)^n n = R(n) = -B(n) + 2C(n)$$

Since we have already determined $B(n)$, we can solve for $C(n)$:

$$C(n) = \frac{1}{2}(-1)^n n + \frac{1}{4}(-1)^n - \frac{1}{4}$$

We need one last equation to determine $D(n)$. Let $R(n) = (-1)n^2$. The recurrence becomes:

$$
\begin{aligned}
0 &= \alpha; \\
(-1)^n n^2 &= (-1)^{n-1}(n-1)^2 + (-1)^n(\beta + \gamma n + \lambda n^2)
\end{aligned}
$$

We can simply the second equation by dividing by $(-1)^n$, expanding $(n-1)n^2$, and simplifying:

$$
\begin{aligned}
n^2 &= -(n-1)^2 + (\beta + \gamma n + \lambda n^2) \\
n^2 &= (-1 + 2n - n^2) + (\beta + \gamma n + \lambda n^2) \\
2n^2 - 2n + 1 &= \beta + \gamma n + \lambda n^2
\end{aligned}
$$

Equating coefficients, we conclude that $\alpha = 0$, $\beta = 1$, $\gamma = -2$, and $\lambda = 2$. Hence

$$(-1)^n n^2 = R(n) = B(n) - 2C(n) + 2D(n).$$

Substituting our solutions for $B(n)$ and $C(n)$, we get

$$
\begin{aligned}
D(n) &= \frac{1}{2}(-1)^n n^2 + \frac{1}{2}(-1)^n n + \frac{1}{4}(-1)^n - \frac{1}{4} - \frac{1}{4}(-1)^n + \frac{1}{4} \\
&= (-1)^n \frac{(n^2 + n)}{2}.
\end{aligned}
$$

Going back to our series $S_n = \sum_{k=0}^{n}(-1)^k k^2$, recall that its corresponding recurrence is

$$
\begin{aligned}
S_0 &= 0; \\
S_n &= S_{n-1} + (-1)^n n^2.
\end{aligned}
$$

This means $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and $\lambda = 1$. Hence, $S_n = S(n) = D(n)$, so

$$\sum_{k=0}^{n}(-1)^k k^2 = (-1)^n \frac{(n^2 + n)}{2}$$

Exercise 14. Evaluate $\sum_{k=1}^{n} k 2^k$ by rewriting it as the multiple sum $\sum_{1 \le j \le k \le n} 2^k$.

Solution: Recall that

$$[1 \le j \le n][j \le k \le n] = [1 \le j \le k \le n] = [1 \le k \le n][1 \le j \le k].$$

We can use this to re-write the sum

$$\begin{aligned}
\sum_{1\le k\le n} k2^k &= \sum_{1\le k\le n} k2^k \cdot \frac{1}{k} \sum_{1\le j\le k} 1 \\
&= \sum_{1\le k\le n} \sum_{1\le j\le k} k2^k \cdot \frac{1}{k} \\
&= \sum_{1\le j\le k\le n} 2^k \\
&= \sum_{1\le j\le n} \sum_{j\le k\le n} 2^k \\
&= \sum_{1\le j\le n} \sum_{j\le k+j\le n} 2^{k+j} \\
&= \sum_{1\le j\le n} 2^j \sum_{0\le k\le n-j} 2^k \\
&= \sum_{1\le j\le n} 2^j(2^{n-j+1} - 1) \\
&= \sum_{1\le j\le n} 2^{n+1} - \sum_{1\le j\le n} 2^j \\
&= n \cdot 2^{n+1} - \sum_{0\le j+1\le n} 2^{j+1} \\
&= n \cdot 2^{n+1} - \sum_{0\le j\le n-1} 2^{j+1} \\
&= n \cdot 2^{n+1} - 2^{n+1} + 2
\end{aligned}$$

Exercise 15. Evaluate $\text{❒}_n = \sum_{k=1}^{n} k^3$ by using the text's Method 5 (Expand and Contract) as follows: First write $\text{❒}_n + \Box_n = 2\sum_{1\le j\le k\le n} jk$; then apply Equation 1, which is (2.33) in the book, given below:

$$\sum_{1\le j\le k\le n} a_j a_k = \frac{1}{2}\left(\left(\sum_{k=1}^{n} a_k\right)^2 + \sum_{k=1}^{n} a_k^2\right) \tag{1}$$

Solution: Recall that

$$\sum_{1\le j\le k} j = \frac{k(k+1)}{2}$$

We use this equation below:

$$
\begin{aligned}
\text{⧉}_n + \square_n &= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 \\
&= \sum_{1 \le k \le n} k^2(k+1) \\
&= \sum_{1 \le k \le n} k^2(k+1) \cdot \frac{2}{(k+1)k} \sum_{1 \le j \le k} j \\
&= 2 \sum_{1 \le k \le n} \sum_{1 \le j \le k} jk \\
&= 2 \sum_{1 \le j \le k \le n} jk
\end{aligned}
$$

Now applying Equation 1 (2.33 in book) with $a_j = j$ and $a_k = k$:

$$
\begin{aligned}
\text{⧉}_n + \square_n &= 2 \sum_{1 \le j \le k \le n} jk \\
&= \left(\left(\sum_{k=1}^{n} k \right)^2 + \sum_{k=1}^{n} k^2 \right) \\
&= \left(\left(\frac{n(n+1)}{2} \right)^2 + \frac{(n+1)(n+\frac{1}{2})n}{3} \right) \\
&= \left(\left(\frac{n(n+1)}{2} \right)^2 + \square_n \right)
\end{aligned}
$$

Subtracting $\square_n$ on both sides results in

$$
\text{⧉}_n = \left(\frac{n(n+1)}{2} \right)^2
$$

Exercise 16. Prove that $x^{\underline{m}}/(x-n)^{\underline{m}} = x^{\underline{n}}/(x-m)^{\underline{n}}$, unless one of the denominators is zero.

Solution: *Proof.* If $m = n$, then the statement is trivially true. Suppose, without loss generality, that $m > n$. Then

$$
\begin{aligned}
\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} &= \frac{x(x-1)\cdots(x-(n-1))(x-n)(x-(m-1))}{(x-n)(x-n-1))\cdots(x-(m-1))(x-m)\cdots(x-n-(m-1))} \\
&= \frac{x(x-1)\cdots(x-(n-1))}{(x-m)\cdots(x-m-1)\cdots(x-n-(m-1))} \\
&= \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}
\end{aligned}
$$

$\square$

Exercise 17. Show that the following formulas can be used to convert between rising and falling factorial powers, for all integers m:

$$x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = 1/(x-1)^{\underline{-m}};$$
$$x^{\underline{m}} = (-1)^m(-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = 1/(x+1)^{\overline{-m}}.$$

Solution: Starting with $x^{\overline{m}} = 1/(x-1)^{\underline{-m}}$:

$$\begin{aligned}\frac{1}{(x-1)^{\underline{-m}}} &= \frac{1}{\frac{1}{(x-1+1)(x-1+2)\cdots(x-1+m)}} \\ &= x(x+1)(x+2)\cdots(x+(m-1)) \\ &= x^{\overline{m}}\end{aligned}$$

We can prove $x^{\overline{m}} = (x+m-1)^{\underline{m}}$ by reversing the order of the product:

$$\begin{aligned}x^{\overline{m}} = x(x+1)(x+2)\cdots(x+(m-2))(x+(m-1)) &= (x+(m-1))(x+(m-1)-1)\cdots(x+2)(x+1)x \\ &= (x+m-1)^{\underline{m}}\end{aligned}$$

Lastly, we can prove $x^{\overline{m}} = (-1)^m(-x)^{\underline{m}}$ by multiplying by (-1) and changing the additions to subtractions:

$$\begin{aligned}x^{\overline{m}} &= x(x+1)(x+2)\cdots(x+(m-1)) \\ &= [(-1)\cdot(-x)][(-1)\cdot(-x-1)]\cdot[(-1)\cdot(-x-2)]\cdots[(-1)[-x-(m-1)]] \\ &= (-1)^m(-x)(-x-1)(-x-2)\cdots(-x-(m-1)) \\ &= (-1)^m(-x)^{\underline{m}}.\end{aligned}$$

We could use the equations we just proved to prove the $x^{\underline{m}}$ equations, but the same arguments work. First, we can prove $x^{\underline{m}} = (-1)^m(-x)^{\overline{m}}$ by multiplying every term by (-1) and reversing the order of the subtraction:

$$\begin{aligned}x^{\underline{m}} &= x(x-1)(x-2)\cdots(x-(m-1)) \\ &[(-1)\cdot(-x)]\cdot[(-1)\cdot(-x+1)]\cdot[(-1)(-x+2)]\cdots[(-1)(-x+m-1)] \\ &= (-1)^n(-x)(-x+1)\cdots(-x+(m-1)) \\ &= (-1)^n(-x)^{\overline{m}}\end{aligned}$$

Next, we can prove $x^{\underline{m}} = (x-m+1)^{\overline{m}}$ by writing the product in reverse:

$$\begin{aligned}x^{\underline{m}} &= x(x-1)(x-2)\cdots(x-(m-2))(x-(m-1)) \\ &= (x-(m-1))(x-(m-2))\cdots(x-2)(x-1) \\ &= (x-m+1)(x-m+2)\cdots(x-m+1+(m-2))(x-m+1+(m-1)) \\ &= (x-m+1)(x-m+2)\cdots(x-1)x \\ &= (x-m+1)^{\underline{m}}\end{aligned}$$

For the last falling factorial equation, we start from the right-side. From Exercise 9, it follows that

$$x^{\overline{-m}} = \frac{1}{(x-m)^{\overline{m}}} = \frac{1}{(x-1)^{\underline{m}}}$$

Therefore, if we start from the right, replace x with $x+1$, and use the reciprocal:

$$x^{\underline{m}} = ((x+1)-1)^{\underline{m}} = \frac{1}{(x+1)^{\overline{-m}}}$$

Exercise 18. Let $\Re z$ and $\Im z$ be the real and imaginary parts of the complex number z. The absolute value $|z|$ is $\sqrt{(\Re z)^2 + (\Im z)^2}$. A sum $\sum_{k\in K} a_k$ of complex terms ak is said to converge absolutely when the real-valued sums $\sum_{k\in K} \Re a_k$ and $\sum_{k\in K} \Im a_k$ both converge absolutely. Prove that $\sum_{k\in K} a_k$ converges absolutely if and only if there is a boundary constant B such that $\sum_{k\in F} |a_k| \le B$ for all finite subsets $F \subseteq K$.

Solution: *Proof.* (" $\implies$ "): Suppose that $\sum_{k\in K} \Re a_z$ and $\sum_{k\in K} \Im a_z$ both converge absolutely. Recall that

$$\Re a_k = \Re a_k^+ - \Re a_k^-, \quad \Im a_k = \Im a_k^+ - \Im a_k^-,$$

where the operands on the right-hand side are all non-negative. Then the absolute convergence of $\Re a_k$ and $\Im a_k$ imply that the sums

$$\sum_{k\in K} \Re a_k^+, \quad \sum_{k\in K} \Re a_k^-, \quad \sum_{k\in K} \Im a_k^+, \quad \sum_{k\in K} \Im a_k^-,$$

all converge. Therefore, there are constants $B_1, B_2, B_3, B_4 \in \mathbb{R}$ such that for all finite subsets $F \subseteq K$, we have

$$\sum_{k\in F} \Re a_k^+ \le \frac{B_1}{4}, \quad \sum_{k\in F} \Re a_k^- \le \frac{B_2}{4}, \quad \sum_{k\in F} \Im a_k^+ \le \frac{B_3}{4}, \quad \sum_{k\in F} \Im a_k^- \le \frac{B_4}{4}, \quad .$$

Pick $B = \max\{B_1, B_2, B_3, B_4\}$. Then, by the triangle inequality, Recalling that the triangle inequality says that $|x+y| \le |x| + |y|$, we have

$$\begin{aligned}
\sum_{k\in F} |a_k| &= \sum_{k\in F} |\Re a_k + i\Im a_k| \\
&\le \sum_{k\in F} (|\Re a_k| + |\Im a_k|) \\
&= \sum_{k\in F} |\Re a_k| + \sum_{k\in F} |\Im a_k| \\
&= \sum_{k\in F} |\Re a_k^+ - \Re a_k^-| + \sum_{k\in F} |\Im a_k^+ - \Im a_k^-| \\
&\le \sum_{k\in F} |\Re a_k^+| + \sum_{k\in F} |\Re a_k^-| + \sum_{k\in F} |\Im a_k^+| + \sum_{k\in F} |\Im a_k^-| \\
&\le \frac{B_1}{4} + \frac{B_2}{4} + \frac{B_3}{4} + \frac{B_4}{4} \\
&\le \frac{B}{4} + \frac{B}{4} + \frac{B}{4} + \frac{B}{4} \\
&= B
\end{aligned}$$

This concludes the proof of the forward direction.

("$\Longleftarrow$"): Suppose now that there is $B \in \mathbb{R}$ such that $\sum_{k \in F} |a_k| \leq B$ for all finite subsets $F \subseteq K$. Note that $\Re a_k \leq |\Re a_k| \leq |a_k|$ and $\Im a_k \leq |\Im a_k| \leq |a_k|$. Therefore, it follows that

$$\sum_{k \in F} \Re a_k \leq \sum_{k \in F} |\Re a_k| \leq \sum_{k \in F} |a_k| \leq B \quad \text{and} \quad \sum_{k \in F} \Im a_k \leq \sum_{k \in F} |\Im a_k| \leq \sum_{k \in F} |a_k| \leq B.$$

which means that $\sum_{k \in F} \Re a_k$ and $\sum_{k \in F} \Im a_k$ both converge absolutely. $\square$

Homework

Exercise 19. Use a summation factor to solve the recurrence

$$\begin{aligned} T_0 &= 5 \\ 2T_n &= nT_{n-1} + 3 \cdot n!, \qquad \text{for } n > 0. \end{aligned}$$

Solution: Recall that given a sequence of the form

$$a_n T_n = b_n T_{n-1} + c_n, \quad n > 0,$$

we can solve by multiplying by a summation factor s_n, chosen so that

$$s_n b_n = s_{n-1} a_{n-1}.$$

The solution then becomes

$$T_n = \frac{1}{s_n a_n} \left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k \right).$$

For the given recurrence, we have $a_n = 2$, and $b_n = n$. Therefore, we want s_n that satisfies

$$s_n \cdot n = s_{n-1} \cdot 2 \qquad \Longleftrightarrow \qquad s_n = \frac{2}{n} s_{n-1}, \quad n > 0$$

By unfolding the recurrence, we see that the appropriate factor is

$$s_n = \frac{2}{n} s_{n-1} = \frac{2^2}{n(n-1)} s_{n-2} = \frac{2^3}{n(n-1)(n-2)} s_{n-3} = \cdots = \frac{2^{n-1}}{n!} s_1.$$

We pick $s_1 = 1$. Hence, $s_1 b_1 = 1$, and $S_n a_n = \frac{2^n}{n!}$, so

$$
\begin{aligned}
T &= \frac{n!}{2^n}\left(5 + 3\sum_{k=1}^{n} \frac{2^{k-1}}{k!} \cdot k!\right) \\
&= \frac{n!}{2^n}\left(5 + 3\sum_{1\le k\le n} 2^{k-1}\right) \\
&= \frac{n!}{2^n}\left(5 + 3\sum_{1\le k+1\le n} 2^{k}\right) \\
&= \frac{n!}{2^n}\left(5 + 3\sum_{0\le k\le n-1} 2^{k}\right) \\
&= \frac{n!}{2^n}\left(5 + 3(2^n - 1)\right) \\
&= n! \cdot (3 + 2^{n-1}).
\end{aligned}
$$

Exercise 20. Try to evaluate $\sum_{k=0}^{n} k\mathcal{H}_k$ by the perturbation method, but deduce the value of $\sum_{k=0}^{n} \mathcal{H}_k$ instead.

Solution: Recall that if we have a series $S_n = \sum_{0\le k\le n} a_k$, then we can re-write S_{n+1} two different ways by splitting off the first and last term:

$$
\begin{aligned}
S_n + a_{n+1} &= S_{n+1} \\
&= \sum_{0\le k\le n+1} a_k \\
&= a_0 + \sum_{1\le k\le n+1} a_k \\
&= a_0 + \sum_{1\le k+1\le n+1} a_{k+1} \\
&= a_0 + \sum_{0\le k\le n} a_{k+1}.
\end{aligned}
$$

This is the basis of the perturbation method. We apply it to the given series. Let $S_n =$

$\sum_{k=0}^{n} k\mathcal{H}_k$. Then

$$\begin{aligned}
\sum_{0\le k\le n} k\mathcal{H}_k + (n+1)\mathcal{H}_{n+1} &= \sum_{0\le k\le n+1} k\mathcal{H}_k \\
&= 0\cdot\mathcal{H}_0 + \sum_{1\le k\le n+1} k\mathcal{H}_k \\
&= \sum_{1\le k+1\le n+1} (k+1)\mathcal{H}_{k+1} \\
&= \sum_{0\le k\le n} (k+1)\mathcal{H}_{k+1} \\
&= \sum_{0\le k\le n} k\mathcal{H}_{k+1} + \sum_{0\le k\le n} \mathcal{H}_{k+1} \\
&= \sum_{0\le k\le n} k\mathcal{H}_{k+1} + \sum_{1\le k\le n+1} \mathcal{H}_{k} \\
&= \sum_{0\le k\le n} k\mathcal{H}_{k+1} + \sum_{1\le k\le n} \mathcal{H}_{k} + \mathcal{H}_{n+1}
\end{aligned}$$

Re-arranging, and noting that $\mathcal{H}_0 = 0$, we have:

$$\begin{aligned}
\sum_{0\le k\le n} \mathcal{H}_k = \sum_{1\le k\le n} \mathcal{H}_k &= n\mathcal{H}_{n+1} + \sum_{0\le k\le n} k(\mathcal{H}_k - \mathcal{H}_{k+1}) \\
&= n\mathcal{H}_{n+1} - \sum_{0\le k\le n} k\cdot\frac{1}{k+1} \\
&= n\mathcal{H}_{n+1} - \sum_{0\le k-1\le n} (k-1)\cdot\frac{1}{k-1+1} \\
&= n\mathcal{H}_{n+1} + \sum_{1\le k\le n+1} \left(\frac{1}{k} - 1\right) \\
&= n\mathcal{H}_{n+1} + \sum_{1\le k\le n+1} \frac{1}{k} - \sum_{1\le k\le n+1} 1 \\
&= n\mathcal{H}_{n+1} + \mathcal{H}_{n+1} - (n+1) \\
&= (n+1)\mathcal{H}_{n+1} - (n+1).
\end{aligned}$$

Exercise 21. Evaluate the sums $S_n = \sum_{k=0}^{n}(-1)^{n-k}$, $T_n = \sum_{k=0}^{n}(-1)^{n-k}k$, and $U_n = \sum_{k=0}^{n}(-1)^{n-k}k^2$ by the perturbation method, assuming that $n \ge 0$.

Solution: First S_n:

$$\begin{aligned}
\sum_{0\le k\le n} (-1)^{n-k} + (-1)^{n-(n+1)} &= T_{n+1} \\
&= (-1)^{n-0} + \sum_{k=0}^{n}(-1)^{n-(k+1)} \\
&= (-1)^{n} - \sum_{k=0}^{n}(-1)^{n-k}.
\end{aligned}$$

By re-arranging, we get:

$$S_n = \sum_{k=0}^{n} (-1)^{n-k} = \frac{(-1)^n + 1}{2} = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Next we work with T_n:

$$\begin{aligned} \sum_{k=0}^{n} (-1)^{n-k} k + (-1)^{n-(n+1)} (n+1) = S_{n+1} \\ &= (-1)^{n-0} \cdot 0 + \sum_{k=0}^{n} (-1)^{n-(k+1)} (k+1) \\ &= -\sum_{k=0}^{n} (-1)^{n-k} k - \sum_{k=0}^{n} (-1)^{n-k} \\ &= -\sum_{k=0}^{n} (-1)^{n-k} k - S_n. \end{aligned}$$

By re-arranging, we get

$$T_n = \sum_{k=0}^{n} (-1)^{n-k} k = \frac{n+1-S_n}{2}.$$

Finally, we evaluate U_n:

$$\begin{aligned} \sum_{k=0}^{n} (-1)^{n-k} k^2 - (n+1)^2 = U_{n+1} \\ &= 0 + \sum_{k=0}^{n} (-1)^{n-(k+1)} (k+1)^2 \\ &= -\sum_{k=0}^{n} (-1)^{n-k} k^2 - 2\sum_{k=0}^{n} (-1)^{n-k} k - \sum_{k=0}^{n} (-1)^{n-k} \\ &= -\sum_{k=0}^{n} (-1)^{n-k} k^2 - 2S_n - T_n. \end{aligned}$$

This results in

$$U_n = \frac{(n+1)^2 - 2S_n - T_n}{2}$$

Exercise 22. Prove *Lagrange's identity* (without using induction):

$$\sum_{1 \le j < k \le n} (a_j b_k - a_k b_j)^2 = \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right) - \left(\sum_{k=1}^{n} a_k b_k \right)^2.$$

Prove, in fact, an identity for the more general double sum

$$\sum_{1 \le j < k \le n} (a_j b_k - a_k b_j)(A_j B_k - A_k B_j).$$

Solution:

Proof. Let $T_n = \sum_{1\le j<k\le n} (a_j b_k - a_k b_j)(A_j B_k - A_k B_j)$. Recall that

$$[1 \le j < k \le n] + [1 \le k < j \le n] = [1 \le j, k \le n] - [1 \le j = k \le n].$$

Then

$$\begin{aligned}
2T_n &= \sum_{1\le j<k\le n} (a_j b_k - a_k b_j)(A_j B_k - A_k B_j) + \sum_{1\le k<j\le n} (a_k b_j - a_j b_k)(A_k B_j - A_j B_k) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} (a_j b_k - a_k b_j)(A_j B_k - A_k B_j) - \sum_{k=1}^{n} (a_k b_k - a_k b_k)(A_k B_k - A_k B_k) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} (a_j A_j b_k B_k - a_j B_j b_k A_k - b_j A_j a_k B_k + b_j B_j a_k A_k) \\
&= \left(\sum_{j=1}^{n} a_j A_j\right)\left(\sum_{k=1}^{n} b_k B_k\right) - \left(\sum_{j=1}^{n} a_j B_j\right)\left(\sum_{k=1}^{n} b_k A_k\right) \\
&- \left(\sum_{j=1}^{n} b_j A_j\right)\left(\sum_{k=1}^{n} a_k B_k\right) + \left(\sum_{j=1}^{n} b_j B_j\right)\left(\sum_{k=1}^{n} a_k A_k\right) \\
&= 2\left(\sum_{k=1}^{n} a_k A_k\right)\left(\sum_{k=1}^{n} b_k B_k\right) - 2\left(\sum_{k=1}^{n} A_k b_k\right)\left(\sum_{k=1}^{n} a_k B_k\right).
\end{aligned}$$

If we divide by 2, then

$$T_n = \left(\sum_{k=1}^{n} a_k A_k\right)\left(\sum_{k=1}^{n} b_k B_k\right) - \left(\sum_{k=1}^{n} A_k b_k\right)\left(\sum_{k=1}^{n} a_k B_k\right).$$

Lagrange's identity then follows if we let $a_k = A_k$ and $b_k = B_k$. □

,

Exercise 23. Evaluate the sum $\sum_{k=1}^{n}(2k+1)/k(k+1)$ in two ways:

(a) Replace $1/k(k+1)$ by the "partial fractions" $1/k - 1/(k+1)$.

(b) Sum by parts.

Solution:

(a) Since

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1},$$

we get

$$\begin{aligned}
\sum_{1\le k\le n} \frac{2k+1}{k(k+1)} &= \sum_{1\le k\le n} \frac{2k+1}{k} - \sum_{1\le k\le n} \frac{2k+1}{k+1} \\
&= \sum_{1\le k\le n} 2 + \sum_{1\le k\le n} \frac{1}{k} - \sum_{1\le k-1\le n} \frac{2(k-1)+1}{(k-1)+1} \\
&= 2n + \mathcal{H}_n - \sum_{2\le k\le n+1} \frac{2k-1}{k} \\
&= 2n + \mathcal{H}_n - \sum_{2\le k\le n+1} 2 + \sum_{2\le k\le n+1} \frac{1}{k} \\
&= 2n + \mathcal{H}_n - 2n - \frac{1}{1} + \sum_{1\le k\le n+1} \frac{1}{k} \\
&= \mathcal{H}_n - 1 + \mathcal{H}_{n+1} \\
&= 2\mathcal{H}_n - 1 + \frac{1}{n+1} \\
&= 2\mathcal{H}_n - \frac{n}{n+1}.
\end{aligned}$$

(b) The summation by parts formula is

$$\sum u\Delta v = uv - \sum Ev\Delta u,$$

where E is the shift operator and Δ is the difference operator. That is, $Ev(x) = v(x+1)$, and $\Delta u(x) = u(x+1) - u(x)$. Let

$$\begin{aligned}
u = 2x+1, \quad \Delta v &= \frac{1}{x(x+1)} = (x-1)^{\underline{-2}} \\
\Delta u = \Delta(2x^{\underline{1}}+1) = 2, \quad v &= -(x-1)^{\underline{-1}} = -\frac{1}{x}, \quad Ev = -\frac{1}{x+1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum \frac{2x+1}{x(x+1)} &= \sum u\Delta v \\
&= uv - \sum Ev\Delta u \\
&= -\frac{2x+1}{x} + \sum \frac{2}{x+1} \\
&= -\frac{2x+1}{x} + 2\sum \Delta\mathcal{H}_x \\
&= -\frac{2x+1}{x} + 2\mathcal{H}_k
\end{aligned}$$

Now plugging in the bounds of integration:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{2k+1}{k(k+1)} &= -\frac{2k+1}{k}\bigg|_{1}^{n+1} + 2\mathcal{H}_k\bigg|_{1}^{n+1} \\
&= -\frac{2n+3}{n+1} + 3 + 2\mathcal{H}_{n+1} - 2 \\
&= -\frac{2n+3}{n+1} + 1 + 2\mathcal{H}_n + \frac{2}{n+1} \\
&= 2\mathcal{H}_n + \frac{n+1}{n+1} + \frac{2}{n+1} - \frac{2n+3}{n+1} \\
&= 2\mathcal{H}_n - \frac{n}{n+1}.
\end{aligned}$$

Exercise 24. What is $\sum_{0\le k<n} \mathcal{H}_k/(k+1)(k+2)$? *Hint*: Generalize the derivation of (2.57)

Solution: Recall $x^{\underline{-2}} = \frac{1}{(x+1)(x+2)}$. Hence, our sum is

$$\sum \mathcal{H}_x x^{\underline{-2}}\,\delta x$$

With the intent to use summation by parts, let

$$\begin{aligned}
u = \mathcal{H}_x, &\quad \Delta v = \frac{1}{(x+1)(x+2)} = x^{\underline{-2}} \\
\Delta u = x^{\underline{-1}} = \frac{1}{x+1}, &\quad v = -x^{\underline{-1}}, \quad Ev = -\frac{1}{x+2}.
\end{aligned}$$

Hence, using summation by parts:

$$\begin{aligned}
\sum \mathcal{H}_x x^{\underline{-2}}\,\delta x &= -\frac{\mathcal{H}_x}{x+1} + \sum \frac{1}{(x+1)(x+2)} \\
&= -\frac{\mathcal{H}_x}{x+1} + \sum x^{\underline{-2}}\,\delta x
\end{aligned}$$

Now substituting the bounds, we get

$$\begin{aligned}
\sum_{0\le k<n} \frac{\mathcal{H}_k}{(k+1)(k+2)} &= -\frac{\mathcal{H}_k}{k+1}\bigg|_{0}^{n} + (-x^{\underline{-1}})\bigg|_{0}^{n} \\
&= -\frac{\mathcal{H}_n}{n+1} + 1 - \frac{1}{n+1} \\
&= \frac{n-\mathcal{H}_n}{n+1}
\end{aligned}$$

Exercise 25. The notation $\prod_{k\in K} a_k$ means the product of the numbers a_k for all $k \in K$. Assume for simplicity that $a_k \ne 1$ for only finitely many k; hence infinite products need not be defined. What laws does this $\prod$-notation satisfy, analogous to the distributive, pairing, and commutative laws that hold for $\sum$?

Solution: If b_k is another collection of numbers, then

$$\prod_{k\in K}(a_k b_k) = \left(\sum_{k\in K} a_k\right)\left(\prod_{k\in K} b_k\right).$$

The commutative law works in the same way, that is, if p is any permutation of the elements in K, then

$$\prod_{k\in K} a_k = \prod_{p(k)\in K} a_k.$$

Finally,

$$\prod_{k\in K} a_k^c = \left(\prod_{k\in K} a_k\right)^c.$$

Exercise 26. Express the double product $P = \prod_{1\le j\le k\le n} a_j a_k$ in terms of the single product $\prod_{k=1}^n a_k$ by manipulating the $\prod$-notation. (This exercise gives a product analog of the upper-triangle identity (2.33).)

Solution: Using the identity

$$[1\le j\le k\le n][1\le k\le j\le n] = [1\le j,k\le n][1\le j=k\le n],$$

we can write

$$\left(\prod_{1\le j\le k\le n} a_j a_k\right)^2 = \prod_{1\le k\le n}\prod_{1\le j\le n} a_j a_k \cdot \prod_{1\le k\le n} a_k^2,$$

which simplifies to

$$\begin{aligned}\left(\prod_{1\le j\le k\le n} a_j a_k\right)^2 &= \left(\prod_{1\le k\le n} a_k\right)^{2n} \cdot \prod_{1\le k\le n} a_k^2\\ &= \left(\prod_{1\le k\le n} a_k\right)^{2n+2}\end{aligned}$$

Exercise 27. Compute $\Delta(c^{\underline{x}})$, and use it to deduce the value of $\sum_{k=1}^n (-2)^{\underline{k}}/k$.

Solution: If x is only allowed to be a positive integer, then

$$c^{\underline{x}} = c(c-1)\cdots(c-(x-1)).$$

Therefore

$$\begin{aligned}\Delta(c^{\underline{x}}) &= c^{\underline{x+1}} - c^{\underline{x}}\\ &= c(c-1)\cdots(c-(x-1))(c-x) - c(c-1)\cdots(c-(x-1))\\ &= c^{\underline{x}}(c-x-1)\\ &= \frac{c^{\underline{x+2}}}{c-x}.\end{aligned}$$

In our case, $c = -2$, and

$$\Delta((-2)^{\underline{x-2}}) = \frac{(-2)^{\underline{x}}}{-2-(x-2)} = -\frac{(-2)^{\underline{x}}}{x}.$$

Therefore, we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{(-2)^{\underline{k}}}{k} &= -\sum_{k=1}^{n} \Delta\left((-2)^{\underline{k-2}}\right) \\
&= -(-2)^{\underline{k-2}}\Big|_{1}^{n+1} \\
&= (-2)^{\underline{-1}} - (-2)^{\underline{n-1}} \\
&= \frac{1}{-2+1} - (-2)(-2-1)(-2-2)\cdots(-2-(n-2)) \\
&= -1 - (-2)(-3)(-4)\cdots(-n) \\
&= (-1)^n n! - 1.
\end{aligned}$$