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Lecture 5: More on the SVD
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Lecture 5: More on the SVD

Exercise 1. In Example 3.1, we considered the matrix (3.7) and asserted, among other things, that its 2-norm is approximately 2.9208. Using the SVD, work out (on paper) the exact values of $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$.

Solution: The matrix in question is

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

By Theorem 5.4, the nonzero singular values of A are the square roots of the eigenvalues of A^*A and AA^* (these matrices have the same eigenvalues). Thus we proceed to find eigenvalues of A^*A

$$A^*A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$$

Now we find its eigenvalues which will be the values λ such that $A^*A - \lambda I$ is singular. That is, we want $(A^*A - \lambda I)v = 0$ for nonzero $v \in \mathbb{C}^2$:

$$0 = (A^*A - \lambda I)v = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 8-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This is equivalent to the system:

$$\begin{aligned} 0 &= (1-\lambda)x + 2y \\ 0 &= 2x + (8-\lambda)y \end{aligned}$$

We can multiply the second equation by $-\frac{1-\lambda}{2}$ and add it to the first to get:

$$\begin{aligned} 0 &= 0x + \left(2 - \frac{1}{2}(1-\lambda)(8-\lambda)\right) y \\ 0 &= 2x + (8-\lambda)y \end{aligned}$$

This will have nonzero solutions if $2 - \frac{1}{2}(1-\lambda)(8-\lambda) = 0$, which is equivalent to the quadratic equation $\lambda^2 - 9\lambda + 4 = 0$. The solutions can be obtained using the quadratic formula, and they are

$$\lambda_{\pm} = \frac{9 \pm \sqrt{65}}{2}$$

Both are positive, as expected because A^*A is invertible so it has nonzero eigenvalues, and it is also a positive semidefinite operator, so its eigenvalues must be nonnegative. Now

$$\sigma_{\min} = \sqrt{\frac{9-\sqrt{65}}{2}}, \quad \sigma_{\max} = \sqrt{\frac{9+\sqrt{65}}{2}} \approx 2.9208$$

Exercise 5.2. Using the SVD prove that any matrix in $\mathbb{C}^{m\times n}$ is the limit of a sequence of matrices of full rank. In other words, prove that the set of full-rank matrices is a dense subset of $\mathbb{C}^{m\times n}$. Using the 2-norm for your proof. (The norm doesn't matter, since all norms on a finite-dimensional space are equivalent).

Solution:

Proof. Let $A \in \mathbb{C}^{m\times n}$. □

Exercise 5.3. Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$$

1. Determine, on paper, a real SVD of A in the form $A = U\Sigma V^T$. The SVD is not unique, so find one that has the minimal number of minus signs in U and V.
2. List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked.
3. What are the 1-, 2-, ∞- and Frobenius norms of A?
4. Find A^{-1} not directly, but via the SVD.
5. Find the eigenvalues λ_1, λ_2 of A.
6. Verify that $\det A = \lambda_1\lambda_2$ and $|\det A| = \sigma_1\sigma_2$.
7. What is the area of the ellipsoid into which A maps the unit ball of $\mathbb{R}^2$?

Solution:

(a) The matrix A is nonsingular and thus its singular values are all nonzero. By Theorem 5.4, the nonzero singular values of A are the square roots of the nonzero eigenvalues of A^*A or AA^*. Moreover, suppose that $A = U\Sigma V^*$, so that $A^*A = V\Sigma^2V^*$. Then the eigenvectors of A^*A are the right singular vectors of A. Similarly, the eigenvectors of AA^* are the left singular vectors of A. Thus we begin by finding the eigenvalues of A^*A:

$$A^*A = \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix}$$

If λ is an eigenvalue of A^*A, then $A^*A-\lambda I$ is singular, which means that $(A^*A-\lambda I)v = 0$ has a nonzero solution $v \in \mathbb{C}^2$:

$$0 = (A^*v - \lambda I)v = \begin{bmatrix} 104-\lambda & -72 \\ -72 & 146-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Hence we have the system

$$\begin{aligned} 0 &= (104-\lambda)x - 72y \\ 0 &= -72x + (146-\lambda)y \end{aligned}$$

We can multiply the second equation by $\frac{1}{72}(104-\lambda)$ and add it to the first to get

$$\begin{aligned} 0 &= 0x + \left(\frac{1}{72}(104-\lambda)(146-\lambda) - 72\right) y \\ 0 &= -72x + (146-\lambda)y \end{aligned}$$

This equation will have nonzero solutions if

$$\frac{1}{72}(104-\lambda)(146-\lambda) - 72 = 0$$

which simplifies to

$$\lambda^2 - 250\lambda + 10000 = 0$$

The solutions are

$$\sigma_1^2 = 200, \quad \sigma_2^2 = 50$$

Thus $\sigma_1 = \sqrt{200} = 10\sqrt{2}$, and $\sigma_2 == \sqrt{50} = 5\sqrt{2}$. Now to find the eigenvectors, we substitute σ_1^2 and σ_2^2 in our equation

$$\begin{aligned} 0 &= 0x + \left(\frac{1}{72}(104-\lambda)(146-\lambda) - 72\right) y \\ 0 &= -72x + (146-\lambda)y \end{aligned}$$

Since σ_1^2 and $\sigma_2^@$ both make the coefficient of y in the first equation 0, this implies that $72x = (146-\lambda_j)y$. If we let $y = 72$, then

$$\begin{aligned} \sigma_1^2 = 200 \rightarrow \begin{bmatrix} -54 \\ 72 \end{bmatrix} \quad &\Longrightarrow \quad v_1 = \frac{1}{90}\begin{bmatrix} -54 \\ 72 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ \sigma_2^2 = 50 \rightarrow \begin{bmatrix} 96 \\ 72 \end{bmatrix} \quad &\Longrightarrow \quad v_2 = \frac{1}{120}\begin{bmatrix} 96 \\ 72 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} \end{aligned}$$

Here v_1, v_2 is an orthonormal list. Next we find the eigenvectors of AA^*, using the fact that the eigenvalues are the same as those of A^*A. We begin by computing AA^*:

$$AA^* = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix}$$

Now we find eigenvectors by finding the nonzero solutions to:

$$0 = (AA^* - \sigma_1^2 I)v = \begin{bmatrix} -75 & 75 \\ 75 & -75 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$0 = (AA^* - \sigma_2^2 I)v = \begin{bmatrix} 75 & 75 \\ 75 & 75 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The first system has the nontrivial solution $x = y = 1$, and the second system has the nontrivial solution $x = 1$ and $y = -1$. Thus

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is an orthonormal list of eigenvectors of AA^*. Now we let U be the matrix whose columns are u_1 and u_2, let V be the matrix whose columns are v_1 and v_2, and let Σ be the matrix whose diagonal entries are $\sigma_1 = \sqrt{200}$ and $\sigma_2 = \sqrt{50}$, then $U\Sigma V^*$ is an SVD of A. To see this, we can multiply them

$$\begin{aligned} U\Sigma V^* &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{200} & 0 \\ 0 & \sqrt{50} \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \\ &= A \end{aligned}$$

(b) The singular value of A are

$$\sigma_1 = \sqrt{200}, \quad \sigma_2 = \sqrt{50}$$

The left singular vectors are

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The right singular vectors are

$$v_1 = \frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \quad v_2 = \frac{1}{5} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

(c) The 1-norm of A is the largest 1-norm among its columns, by Example 3.3 (Equation 3.9). Since $\|a_1\|_1 = |-2| + |-10| = 12$ and $\|a_2\|_1 = 11 + 5 = 17$, we see that $\|A\|_1 = 17$. By Example 3.4, the infinity norm of A is the largest 1-norm of its rows, which is the largest of $|-10| + 5$ and $|-2| + 11$, and hence $\|A\|_\infty = 15$. By Theorem 5.3, $\|A\|_2 = \sigma_1 = \sqrt{200}$, the largest singular value of A. By the same Theorem, $\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{200 + 50} = \sqrt{250}$.

(d) To find A^{-1} we note that U and V^T are unitary and Σ is diagonal, so $A^{-1} = (V^*)^{-1}\Sigma^{-1}U^{-1}$, and hence $A^{-1} = V\Sigma^{-1}U^T$

$$\begin{aligned}
A^{-1} &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{200}} & 0\\ 0 & \frac{1}{\sqrt{50}}\end{bmatrix}\frac{1}{5}\begin{bmatrix}-3 & 4\\ 4 & 3\end{bmatrix}\\
&= \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{100} & 0\\ 0 & \frac{1}{50}\end{bmatrix}\begin{bmatrix}-3 & 4\\ 4 & 3\end{bmatrix}\\
&= \frac{1}{100}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}-3 & 4\\ 4 & 3\end{bmatrix}\\
&= \frac{1}{100}\begin{bmatrix}5 & -11\\ 10 & -2\end{bmatrix}
\end{aligned}$$

(e) The eigenvalues of A are the values that make $A - \lambda I$ a singular matrix. We want nonzero solutions to

$$(A - \lambda I)v = \begin{bmatrix}-2-\lambda & 11\\ -10 & 5-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

We have the equations

$$\begin{aligned}
0 &= (-2-\lambda)x + 11y\\
0 &= -10x + (5-\lambda)y
\end{aligned}$$

Multiplying the second equation by $\frac{1}{10}(-2-\lambda)$ and adding it to the first equation, we get

$$\begin{aligned}
0 &= 0x + \left(\frac{1}{10}(-2-\lambda)(5-\lambda) + 11\right)y\\
0 &= -10x + (5-\lambda)y
\end{aligned}$$

The system will admit nontrivial solutions if

$$\begin{aligned}
\frac{1}{10}(-2-\lambda)(5-\lambda) + 11 &= 0\\
(-2-\lambda)(5-\lambda) + 110 &= 0\\
-10 - 3\lambda + \lambda^2 + 110 &= 0\\
\lambda^2 - 3\lambda + 100 &= 0
\end{aligned}$$

which has solutions

$$\lambda_1 = \frac{3 + i\sqrt{391}}{2}, \quad \lambda_2 = \frac{3 - i\sqrt{391}}{2}$$

Hence the eigenvalues are $\lambda_1 = 15$ and $\lambda_2 = -8$.

(f) $\det A = a_{11} \cdot a_{22} - a_{21} \cdot a_{12} = -2 \cdot 5 - (-10) \cdot 11 = -10 + 110 = 100$ This also equals $\lambda_1 \cdot \lambda_2$ and $\sigma_1 \cdot \sigma_2$.

(g) The area of an ellipse is $A = \pi ab$, where a and b are the lengths of the ellipse axes. Thus $A = \pi \cdot \sigma_1 \cdot \sigma_2 = \pi \cdot \sqrt{200} \cdot \sqrt{50} = 100\pi$.

Exercise 5.4. Suppose $A \in \mathbb{C}^{m \times m}$ has an SVD $A = U\Sigma V^*$. Find an eigenvalue decomposition of the $2m \times 2m$ hermitian matrix

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$$

Solution: Let $x, y \in \mathbb{C}^m$ and set

$$B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \quad z = \begin{bmatrix} x \\ y \end{bmatrix}$$

That is, z is the vector in $\mathbb{C}^{2m}$ with x stacked on top of y If λ is an eigenvalue of B, then

$$Bz = \lambda z$$
$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} A^*y \\ Ax \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

The equations $A^*y = \lambda x$ and $Ax = \lambda y$ reduce to $A^*Ax = \lambda^2 x$ and $AA^*y = \lambda^2 y$. In other words, every eigenvalue of A^*A is the square of an eigenvalue of B. Thus the eigenvalues are $\sigma_1, -\sigma_1, \ldots, \sigma_n, -\sigma_n$. In particular, the eigenvectors of B are the left and right singular vectors stacked on top of each other, of the form

$$\begin{bmatrix} v_j \\ u_j \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_j \\ -u_j \end{bmatrix}$$

where $j \in \{1, \ldots, m\}$. In particular,

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} v_j \\ u_j \end{bmatrix} = \begin{bmatrix} A^*u_j \\ Av_j \end{bmatrix} = \sigma_1 \begin{bmatrix} v_j \\ u_j \end{bmatrix}$$
$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} v_j \\ -u_j \end{bmatrix} = \begin{bmatrix} -A^*u_j \\ Av_j \end{bmatrix} = -\sigma_1 \begin{bmatrix} v_j \\ -u_j \end{bmatrix}$$