Sergio Garcia Tapia
Numerical Linear Algebra, Lloyd Trefethen and David Bau III
Lecture 1: Matrix-Vector Multiplication
June 19, 2024

Lecture 1: Matrix-Vector Multiplication

Exercise 1. Let B be a 4×4 matrix to which we apply the following operations:

1. double column 1,
2. half row 3,
3. add row 3 to row 1,
4. interchange columns 1 and 4,
5. subtract row 2 from each of the other rows,
6. replace column 4 by column 3,
7. delete column 1 (so that the column dimension is reduced by 1).

(a) Write the result as a product of eight matrices.

(b) Write it again as a product ABC (same B) of three matrices.

Solution:

(a) To perform any operation on the columns, we can multiply by a matrix on the right, thinking of its columns as the weights that will be used to create a linear combination of the columns of the operand matrix. For example, to double column 1 of B, we multiply B on the right by D_1, where

$$D_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We now have BD_1. To perform row operations, we can instead operate with the transpose by relying on the identity

$$(SR)^T = R^T S^T$$

This enables us to think of the row operations as column operations. For example, to halve row 3 of BD_1, we instead think of halving column 3 of its transpose $(BD_1)^T$. Doing this is simply multiplying by D_2^T on the right, where

$$D_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; D_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we have $(BD_1)^T D_2^T$, which is equivalent to $D_2(BD_1)$. To add row 3 to row 1, we perform the equivalent of adding column 3 to column 1 of the transpose. To achieve this, we multiply the transpose on the right of our result so far by D_3^T, where

$$D_3^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D_3 = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating on the transpose results in $(D_2BD_1)^T D_3^T$, which is equivalent to the product $D_3D_2BD_1$. Next, to interchange columns 1 and 4, we multiply on the right by

$$D_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Thus we have $(D_3D_2BD_1)D_4$. To subtract row 2 from each of the other rows, we again think of it in terms of columns by using the transpose. This translates to subtracting column 2 of the transpose from the other transposed columns. To achieve this, we multiply the transpose of our result so far on the right by D_5^T, where

$$D_5^T = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ -2 & 1 & -2 & -2 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & & & 1 \end{bmatrix} \qquad D_5 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

This gives $(D_3D_2BD_1D_4)^T D_5^T$, so using our identity gives $D_5(D_3D_2BD_1D_4)$. To replace column 4 by column 3, we multiply by D_6 on the right:

$$D_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now our product is $(D_5D_3D_2BD_1D_4)D_6$. Finally, to delete column 1, we multiply on the right by D_7, where

$$D_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Our final result is $(D_5D_3D_2BD_1D_4D_6)D_7$, which is a product of 8 matrices.

(b) We let $A = D_5D_3D_2$ and $C = D_1D_4D_7$.

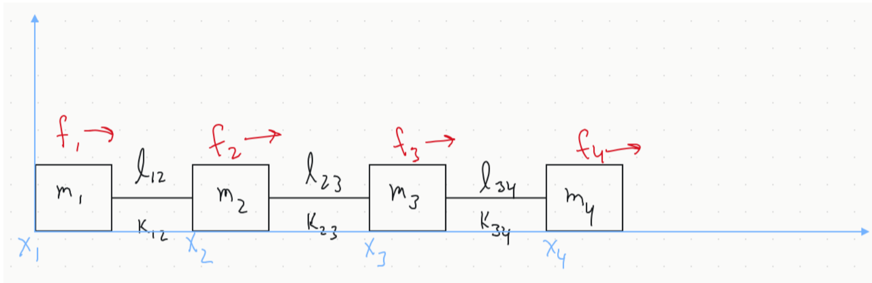

Figure 1: Exercise 1.2: Block masses connected by springs in a line

Exercise 2. Suppose masses m_1, m_2, m_3, m_4 are located at positions x_1, x_2, x_3, x_4 in a line and connected by springs with constants k_{12}, k_{23}, k_{34} whose natural lengths of extension are ℓ_{12}, ℓ_{23}, ℓ_{34}. Let f_1, f_2, f_3, f_4 denote the rightward forces on the masses, e.g., $f_1 = k_{12}(x_2 - x_1 - \ell_{12})$.

(a) Write a 4×4 matrix equation relating the column vectors f and x. Let K denote the matrix in this equation.

(b) What are the dimensions of the entries of K in the physics sense (e.g., mass times time, distance divided by mass, etc.)?

(c) What are the dimensions of $\det(K)$, again in the physics sense?

(d) Suppose K is given numerical values based on the units meters, kilograms, and seconds. Now the system is rewritten with a matrix K' based on centimeters, grams, and seconds. What is the relationship of K' to K? What is the relationship of $\det(K')$ to $\det(K)$?

Solution:

(a) See Figure 1 for a picture I have created of the scenario described. I presume that the force f_1 is calculated according to *Hooke's Law*, giving rightward force on m_1 as the product of the spring constant k_{12} connecting m_1 and m_2 and the displacement of that spring from its relaxed position. When $x_2 - x_1 = \ell_{12}$, the spring is relaxed, and hence there is no force. When mass m_2 is pulled right so that $x_2 - x_1 > \ell_{12}$, the displacement $x_2 - x_1 - \ell_{12}$ is positive, inducing a force f_1 on m_1 that pulls m_1 rightward, and similarly, a leftward force $-f_1$ on m_2 pulling m_2 leftward. Since we are only asked for the rightward forces, and not the net force on each mass, we have

$$
\begin{aligned}
f_1 &= k_{12}(x_2 - x_1 - \ell_{12}) \\
f_2 &= k_{23}(x_3 - x_2 - \ell_{23}) \\
f_3 &= k_{34}(x_4 - x_3 - \ell_{34}) \\
f_4 &= 0
\end{aligned}
$$

We can compactly summarize this with the matrix equation $f = K(x - l)$, where

$$f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \ell = \begin{bmatrix} 0 \\ \ell_{12} \\ \ell_{23} + \ell_{12} \\ \ell_{34} + \ell_{23} + \ell_{12} \end{bmatrix} \quad K = \begin{bmatrix} -k_{12} & k_{12} & 0 & 0 \\ 0 & -k_{23} & k_{23} & 0 \\ 0 & 0 & -k_{34} & k_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) f_i has force dimensions and x and ℓ have length dimensions. Thus each entry of k results from their ratio, which in turn has dimensions of force divided by length.

(c) The dimensions of $\det(K)$ are force divided by length, all raised to the fourth power.

(d) If K is given in units of meters, kilograms, and seconds, then the dimensions of the entry of K are given in the units $\frac{kg \cdot m}{s^2}$, where kg stands for kilograms, m for meters, and s for seconds. Meanwhile, K' is given in dimensions centimeters, grams, and seconds, which is $\frac{g \cdot cm}{s^2}$. Since $1kg = 1000g$, and $1m = 100cm$, we see that $\frac{kg \cdot m}{s^2} = \frac{1000g \cdot 100cm}{s^2} = 10^6 \frac{g \cdot cm}{s^2}$. Thus, the relationship between K and K' is $K' = 10^6 \cdot K$. Since $\det(AB) = \det(A)\det(B)$, it follows that$\det(K') = \det(10^6 I \cdot K) = 10^6 \cdot \det(K)$, where I is the identity matrix.

Exercise 3. Generalizing Example 1.3, we say that a square or rectangular matrix R with entries r_{ij} is *upper-triangular* if $r_{ij} = 0$ for $i > j$. By considering what space is spanned by the first n columns of R using (1.8), show that if R is a nonsingular $m \times m$ upper-triangular matrix, the R^{-1} is also upper-triangular. (The analogous result also holds for lower-triangular matrices).

Solution:

Proof. If R is nonsingular, then it has a unique inverse R^{-1} such that

$$I = R^{-1}R = RR^{-1}$$

where I is the $m \times m$ identity matrix. Denote the j-th column of I by e_j for $j \in \{1, \dots, m\}$, and the i-th column of R^{-1} by z_i. Then since $I = R^{-1}R$, the j-th column of I is a linear combination of the columns of R^{-1} using the entries in the j-th column of R. By (1.8), we have

$$e_j = \sum_{i=1}^{m} r_{ij} z_i = \sum_{i=1}^{j} r_{ij} z_i$$

To show that $z_{ij} = 0$ for $i > j$, we proceed by induction on j, the column index. If $j = 1$, then

$$e_1 = \sum_{i=1}^{1} r_{i1} z_i = r_{11} \cdot z_1$$

The first entry in e_1 is 1 and the rest are 0. Since R is nonsingular and upper-triangular, its diagonal entries are nonzero, so $a_{11\neq 0}$. Dividing by a_{11} shows that $z_1 = \frac{1}{a_{11}}e_1$, and hence, $z_{i1} = 0$ for $i > 1$.

Suppose $j > 1$ and the result holds for all columns of smaller column indices Now

$$e_j = \sum_{i=1}^{j} r_{ij}z_i = r_{jj}z_j + \sum_{i=1}^{j-1} r_{ij}z_i$$
$$z_j = \frac{1}{r_{jj}}e_j + \frac{1}{r_{jj}}\sum_{i=1}^{j-1} r_{ij}z_i$$

Note that $r_{jj} \neq 0$ because R is a nonsingular upper-triangular matrix. By our inductive hypothesis, $z_{ik} = 0$ for $i > k$ whenever $1 \leq i \leq j-1$. Since the scalar multiple $\frac{1}{r_{jj}}e_j$ of e_j only has a nonzero entry in its j-th slot, it satisfies $e_{ij} = 0$ for $i > j$. Since z_{ij} is their sum as seen above, we conclude that it too satisfies $z_{ij} = 0$ for $i > j$. We conclude by induction that R^{-1} is upper-triangular. □

Exercise 4. Let $f_1, \ldots, f_8$ be a set of functions ont he interval $[1, 8]$, with the property that for any numbers $d_1, \ldots, d_8$, there exists a set of coefficients $c_1, \ldots, c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, \quad i = 1, \ldots, 8$$

(a) Show by appealing to the theorems of this lecture that $d_1, \ldots, d_8$ determine $c_1, \ldots, c_8$ uniquely.

(b) Let A be the 8×8 matrix representing the linear mapping from data $d_1, \ldots, d_8$ to coefficients $c_1, \ldots, c_8$. What is the i, j entry of A^{-1}?

Solution:

(a) *Proof.* Let F be the 8×8 matrix whose j-th column f_j represents the column vector $(f_j(1), \ldots, f_j(8))$, where $j \in \{1, \ldots, 8\}$. The given information implies that $\text{rank}(F) = 8$. By Theorem 1.3, this implies that null $(A) = \{0\}$. Thus, $d_1, \ldots, d_8$ determines $c_1, \ldots, c_8$ uniquely. Otherwise, if there is another list of coefficients $c'_1, \ldots, c'_8$ that also yield $d_1, \ldots, d_8$, then the list $c_1 - c'_1, \ldots, c_8 - c'_8$ yields $d_1 - d_1, \ldots, d_8 - d_8$; that is, it yields the 0 vector. Since null $(A) = \{0\}$, this implies that $c_i = c'_i$ for all $i \in \{1, \ldots, 8\}$, and hence the c_i's are uniquely determined. □

(b) Per the description of A and our definition of F, we see that $A = F^{-1}$, and hence $A^{-1} = F$. Thus, the i, j entry of A^{-1} is the value of the j-th function at i.