Sergio Garcia Tapia
Numerical Linear Algebra, Lloyd Trefethen and David Bau III
Lecture 6: Projectors
June 22, 2024

Lecture 6: Projectors

Exercise 1. If P is an orthogonal projector, then $I-2P$ is unitary. Prove this algebraically, and give a geometric interpretation.

Solution:

Proof. A projector satisfies $P^2 = P$ and by Theorem 6.1, a projector P is orthogonal if and only if $P = P^*$, meaning it is hermitian (self-adjoint). To show that $(I-2P)$ is unitary, we need to show that $(I-2P)^* = (I-2P)^{-1}$:

$$\begin{aligned}
(I-2P)^*(I-2P) &= (I^* - 2P^*)(I-2P) \\
&= I^*I - 2I^*P - 2P^*I + 4P^*P \\
&= I - 2P - 2P + 4P^2 \\
&= I - 4P + 4P \\
&= I
\end{aligned}$$

As for the geometric interpretation, it becomes clearer if we re-write $I - 2P = (I-P) - P$. The operator $I-P$ is the complementary projection of P. The operator $I-P$ projects onto null P along range P, whereas $-P$ projects onto range P along null P. However, unlike P, the operator $-P$ reflects every vector in range P. □

Exercise 2. Let E be the $m \times m$ matrix that extracts the "even part" of an m-vector: $Ex = (x + Fx)/2$, where F is the $m \times m$ matrix that flips $(x_1, \dots, x_m)^*$ to $(x_m, \dots, x_1)^*$. Is E an orthogonal projector, an oblique projector, or not a projector at all? What are its entries?

Solution: It is an orthogonal projector. To see this, first note that the matrix F is self-adjoint (hermitian). If $e_1, \dots, e_m$ represent the standard basis of $\mathbb{C}^m$, then

$$F = \begin{bmatrix} \vdots & \cdots & \vdots \\ e_m & \cdots & e_1 \\ \vdots & \cdots & \vdots \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \cdots & 1 & 0 \\ 0 & \iddots & \cdots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

Moreover its columns are orthonormal so F is unitary, and hence $F^* = F^{-1}$. Since $E = \frac{1}{2}(I+F)$, we get

$$\begin{aligned} E^2 &= \frac{1}{4}(I+F)^2 \\ &= \frac{1}{4}(I^2 + I\circ F + F\circ I + F\circ F) \\ &= \frac{1}{4}(I+F+F+I) \\ &= \frac{1}{4}(2I+2F) \\ &= \frac{1}{2}(I+F) \\ &= E \end{aligned}$$

It is orthogonal since

$$\begin{aligned} E^* &= \frac{1}{2}(I+F)^* \\ &= \frac{1}{2}(I^*+F^*) \\ &= \frac{1}{2}(I+F) \\ &= E \end{aligned}$$

Exercise 3. Given $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, show that A^*A is nonsingular if and only if A has full rank.

Solution:

Proof. (" $\Longrightarrow$ "): Suppose that A^*A is nonsingular. By Theorem 1.3, the eigenvalues of A^*A are nonzero. The eigenvalues of A^*A are the singular values of A, and since $A^*A \in \mathbb{C}^{n\times n}$, there are n nonzero singular values. By Theorem 5.1, the rank of A is the number of nonzero singular values, and hence $\operatorname{rank}(A) = n$, meaning A is full rank.

(" $\Longleftarrow$ "): Suppose A has full rank. By Theorem 1.3, 0 is not a singular value of A. The singular values of A are the eigenvalues of A^*A, and hence A^*A does not have any zero eigenvalues. By Theorem 1.3 again, we conclude that A^*A is nonsingular.

Put another way, since A has full rank, its column and null space have dimension n, which means that the column and null space of A^* also has dimension n. In particular, null A = null $A^* = \{0\}$, and hence their product A^*A has null space $\{0\}$, so it has full rank. ☐

Exercise 4. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Answer the following questions by hand calculation.

(a) What is the orthogonal projector P onto range (A), and what is the image under P of the vector $(1,2,3)^*$?

(b) Same questions for B.

Solution:

(a) Matrix A has two columns, and they are linearly independent because they are not a scalar multiple of one another. The vector $v = (1,2,3)^*$ is not in range (A), but we learned in Lecture 6 that if $y \in \text{range}(A)$ is its orthogonal projection onto range (A), then $y - v$ is orthogonal to range(A). Hence all columns of A must be orthogonal to $y - v$, and in particular since $y = Ax$ for some $x \in \mathbb{C}^{3\times 3}$, the solution we expect is

$$x = (A^*A)^{-1}Av$$

where the orthogonal projector is

$$P = A(A^*A)^{-1}A^*$$

Thus,

$$\begin{aligned}
A^* &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
A^*A &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\
P &= A(A^*A)^{-1}A^* \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}
\end{aligned}$$

Note that this satisfies $P = P^*$ as expected, and its rank is 2 just like that of A. Now we can compute the image under P:

$$\begin{aligned} y &= Ax \\ &= A[(A^*A)^{-1}A^*v] \\ &= Pv \\ &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \end{aligned}$$

(b) Similar to before, we compute

$$\begin{aligned} B^* &= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \\ B^*B &= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \\ (B^*B)^{-1} &= \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \\ P &= B(B^*B)^{-1}B \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 1 & 2 \\ -2 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix} \end{aligned}$$

Now we find that

$$\begin{aligned} y &= Pv \\ &= \frac{2}{15} \begin{bmatrix} 15 \\ 0 \\ 14 \end{bmatrix} \end{aligned}$$

Exercise 5. Let $P \in \mathbb{C}^{m \times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with the equality if and only if P is an orthogonal projector.

Solution:

Proof. Since P is a projector, we know that $P^2 = P$. If $v \neq 0 \in \mathbb{C}^m$, then

$$\|Pv\|_2 = \|P^2 v\|_2 = \|P(Pv)\|_2 \leq \|P\|_2 \cdot \|Pv\|_2 \leq \|P\|_2 \cdot \|P\|_2 \cdot \|v\|_2 = \|P\|_2^2 \cdot \|v\|_2$$

Hence $\|P\|_2 \leq \|P\|_2^2$. Since P is nonzero, this implies that $\|P\|_2 \neq 0$. If $x \leq x^2$ for $x \in \mathbb{R}$, then $x \geq 1$, so we conclude $\|P\|_2 \geq 1$, as we set out to show.

If P is an orthogonal projector, then there are subspaces S_1, S_2 of $\mathbb{C}^m$ such that $S_1 \perp S_2$ and $S_1 + S_2 = \mathbb{C}^m$ for which P projects S_1 orthogonally along S_2. It was shown in the proof of Theorem 6.1 that P has a singular value decomposition of the form $P = Q\Sigma Q^*$, where Q is unitary and Σ is a diagonal matrix whose entries are only 1s and 0s. By Theorem 5.3, we learned that $\|P\|_2 = \sigma_1$, where σ_1 is the largest singular value of P. Since Σ only has 1s and 0s on its diagonal, we conclude that $\sigma_1 = 1$, and hence $\|P\|_2 = 1$. $\square$