Sergio Garcia Tapia
Numerical Linear Algebra, Lloyd Trefethen and David Bau III
Lecture 4: The Singular Value Decomposition
June 27, 2024

Lecture 4: The Singular Value Decomposition

Exercise 1. Determine SVDs of the following matrices (by hand calculation):

$$\text{(a)} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{(d)} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{(e)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution:

(a) To determine the SVD, we consider mapping vectors of unit norm to other vectors of unit norm. For example, if we let $v_1 = e_1$, then $Av_1 = 3e_1$, so we can set $u_1 = e_1$. If we set $v_2 = -e_2$, then $Av_2 = A(-e_2) = 2e_2$. Thus we set $u_2 = e_2$. Let $\sigma_1 = 3$, $\sigma_2 = 2$. Then define U to be the 2×2 matrix whose columns are u_1 and u_2, V to be the 2×2 matrix whose columns are v_1 and v_2, and Σ to be the 2×2 diagonal matrix whose two diagonal entries are σ_1 and σ_2.

It's easy to see that U and V are unitary because the columns are orthonormal. Note

$$\begin{aligned} U\Sigma V^* e_1 &= U\Sigma e_1 = U(3e_1) = 3e_1 = Ae_1 \\ U\Sigma V^* e_2 &= U\Sigma(-e_2) = U(-2e_2) = -2e_2 = Ae_2 \end{aligned}$$

Since every vector can be expressed in terms of the standard basis vectors, We conclude that $A = U\Sigma V^*$.

(b) Let $u_1 = v_1 = v_1$, $u_2 = v_2 = e_2$, and $\sigma_1 = 2$, $\sigma_2 = 3$. If U matrix whose columns are u_1, u_2, V is the matrix whose columns are v_1 and v_2, and Σ is the matrix whose diagonal entries are σ_1 and σ_2, then $A = U\Sigma V^*$.

(c) This time $Ae_1 = 0$ and $Ae_2 = 2e_1$. Let $v_1 = e_{2,\mathbb{R}^2}$ and $v_2 = e_{1,\mathbb{R}^2}$. Let $u_1 = e_{1,\mathbb{R}^3}$ and $u_2 = e_{2,\mathbb{R}^3}$, $u_3 = e_{3,\mathbb{R}^3}$, $\sigma_1 = 2$, and $\sigma_2 = 0$. Then $Av_1 = 2u_1$, and $Av_2 = 0$. If we let U be the 3×2 matrix with u_1, u_2, and u_3 as columns, V be the matrix with u_1 and u_2 as columns, and Σ be the diagonal matrix with diagonal entries σ_1 and σ_2, then

$$\begin{aligned} U\Sigma V* &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^* e_{1,\mathbb{R}^2} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} e_{2,\mathbb{R}^2} \\ &= 0 = Ae_{1,\mathbb{R}^2} \end{aligned}$$

and

$$
\begin{aligned}
U\Sigma V* &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^* e_{2,\mathbb{R}^2} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} e_{1,\mathbb{R}^2} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} 2e_{1,\mathbb{R}^3} \\
&= 2e_{1\mathbb{R}^3} \\
&= Ae_{2,\mathbb{R}^2}
\end{aligned}
$$

Hence $A = U\Sigma V^*$.

(d) Let $v_1 = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $u_1 = e_1$ so that $Av_1 = \frac{2}{\sqrt{2}}e_1 = \frac{2}{\sqrt{2}}u_1$. Let $v_2 = \frac{1}{\sqrt{2}}(e_1 - e_2)$ and $u_2 = e_2$. Then $Av_2 = 0$. Set $\sigma_1 = \frac{2}{\sqrt{2}}$, and $\sigma_2 = 0$. Define U to be the matrix whose columns are u_1 and u_2, V to be the matrix whose columns are v_1 and v_2, and Σ to be the diagonal matrix whose diagonal entries are σ_1 and σ_2. Then

$$
\begin{aligned}
U\Sigma V^* e_1 &= \Sigma V^* e_1 = \Sigma\left(\frac{1}{\sqrt{2}}(e_1 + e_2)\right) = \frac{2}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} e_1 = e_1 = Ae_1 \\
U\Sigma V^* e_2 &= \Sigma V^* e_2 = \Sigma\left(\frac{1}{\sqrt{2}}(e_1 - e_2)\right) = \frac{2}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} e_1 = e_1 = Ae_2
\end{aligned}
$$

Hence $A = U\Sigma V^*$.

(e) Let $u_1 = v_1 = \frac{1}{\sqrt{2}}(e_1 + e_2)$, so that $Av_1 = \frac{2}{\sqrt{2}}(e_1 + e_2) = 2u_1$. Similarly let $u_2 = v_2 = \frac{1}{\sqrt{2}}(e_1 - e_2)$, so that $Av_2 = 0$. Define $\sigma_1 = 2$ and $\sigma_2 = 0$. Then if we let U be the matrix whose columns are u_1 and u_2, let V be the matrix whose columns are v_1 and v_2, and Σ be the diagonal matrix whose diagonal entries are σ_1 and σ_2, we get $A = U\Sigma V^*$.

Exercise 4.2. Suppose A is an $m \times n$ matrix and B is the $n \times m$ matrix obtained by rotating A ninety degrees clockwise on paper (not exactly a standard mathematical transformation!). Do A and B have the same singular values? Prove that the answer is yes, or give a counter example.

Solution:

Proof. Suppose $A \in \mathbb{C}^{m \times n}$. Then B and A^T are both $n \times m$, and they are mirror images of one another. That is, given row b_i^* of B, the entries in the i-th row of A^T has the same entries listed in the reverse order. As a result, A^T and B contain the same columns in reverse. In other words, if we let J be the matrix whose columns are $e_n, \dots, e_1$, meaning the standard basis vectors in reverse, then $B = A^T J$.

First, note that if Q is unitary, then Q^T is unitary because $(Q^*)^T = (Q^T)^*$, so

$$\begin{aligned} Q^*Q &= I \\ (Q^*Q)^T &= I^T \\ Q^T(Q^*)^T &= I \end{aligned}$$

Thus, if A has an singular value decomposition $A = U\Sigma V^*$, then $A^T = (V^*)^T\Sigma^T U^T$, which is a singular value decomposition of A^T, so

$$\begin{aligned} B &= (V^*)^T\Sigma^T U^T J \\ &= (V^*)^T\Sigma^T (J^T U)^T \end{aligned}$$

Since the product of unitary matrices is unitary, we see that $J^T U$ is unitary, and thus $(J^T U)^T$ is unitary. Hence the equation above is a singular value decomposition of B. Since Σ is diagonal, its entries are the same as Σ^T, and hence A and B have the same singular values. $\square$

Exercise 4.3. Write a MATLAB program (see Lecture 9) which, given a real 2×2 matrix A, plots the right singular vectors v_1 and v_2 in the unit circle and also the left singular vectors u_1 and u_2 in the appropriate ellipse, as in Figure 4.1. Apply your program to the matrix (3.7) and also to the 2×2 matrices of Exercise 4.1.

Solution: I have implemented the program in Python, consisting of the files `main.py` and and `plot_singular_vectors.py` under the directory `./03-svd-vector-plot`. See Figure 1, Figure 2, Figure 3, Figure 4, Figure 5, and Figure 6

Exercise 4.4. Two matrices $A, B \in \mathbb{C}^{m\times m}$ are *unitarily equivalent* if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m\times m}$. Is it true or false that A and B are unitarily equivalent if and only if they have the same eigenvalues?

Solution: This is true.

Proof. (" $\implies$ "): Suppose that A and B are unitarily equivalent, meaning there is a unitary matrix $Q \in \mathbb{C}^{m\times m}$ such that $A = QBQ^*$. By Theorem 4.1, every matrix in $\mathbb{C}^{m\times m}$ has an SVD, so let U, V be unitary matrices of the left and right singular vectors of B, and Σ be the diagonal matrices consisting of the singular values of B, so that $B = U\Sigma V^*$, where V^* denotes the adjoint of V (and hence its inverse since V is unitary). Then

$$\begin{aligned} A &= QBQ^* \\ A &= QU\Sigma V^*Q^* \\ A &= (QU)\Sigma(QV)^* \end{aligned}$$

The product of two unitary matrices is unitary. To see this, note that if $x \in \mathbb{C}^m$, then

$$\|QUx\| = \|Q(Ux)\| = \|Ux\| = \|x\|$$

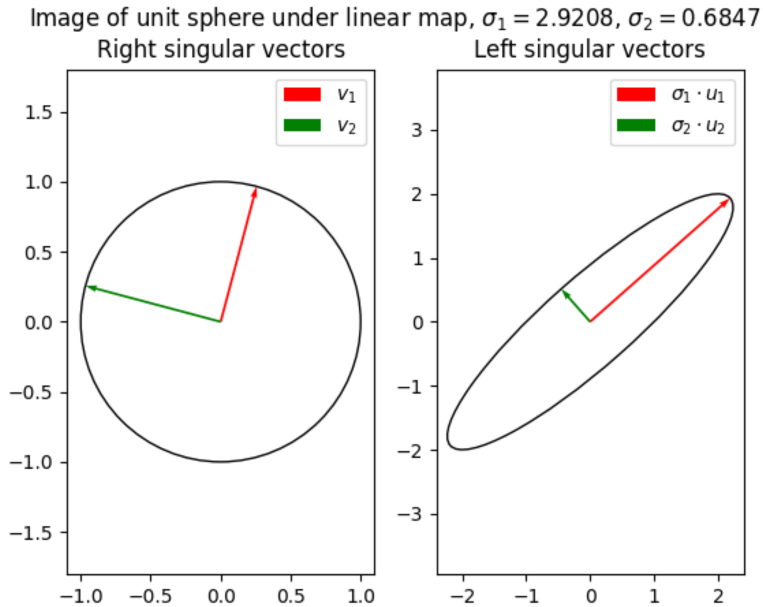

Figure 1: Exercise 4.3: Plot of singular vectors of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ from Equation 3.7 in Example 3.1 of the book

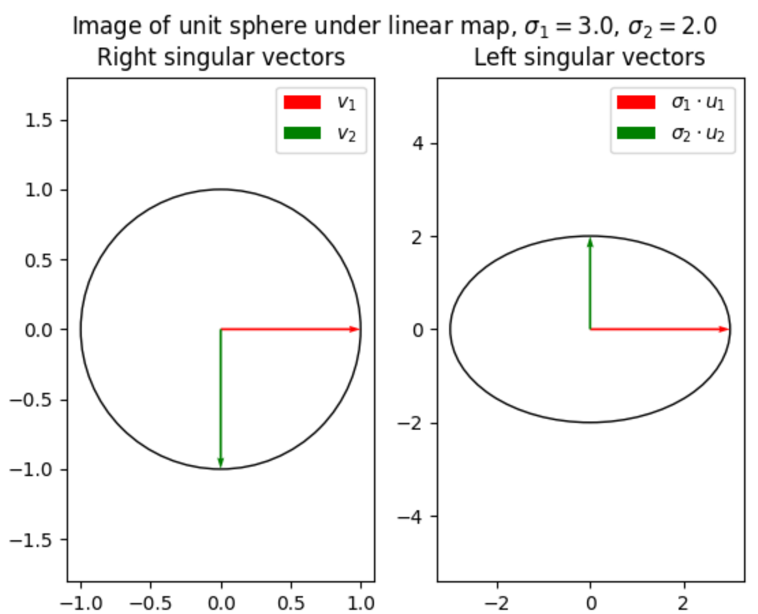

Figure 2: Exercise 4.3a: Plot of singular vectors of $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ from Exercise 4.1(a)

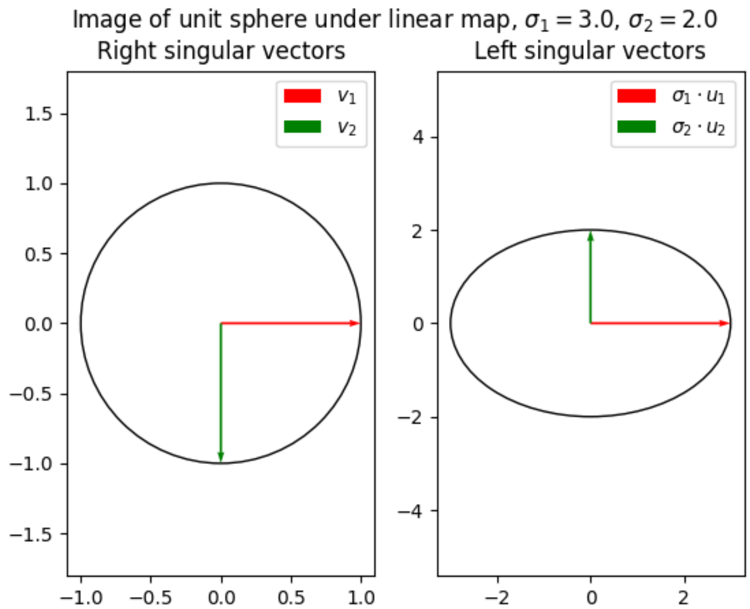

Figure 3: Exercise 4.3(b): Plot of singular vectors of $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ from Exercise 4.1(b)

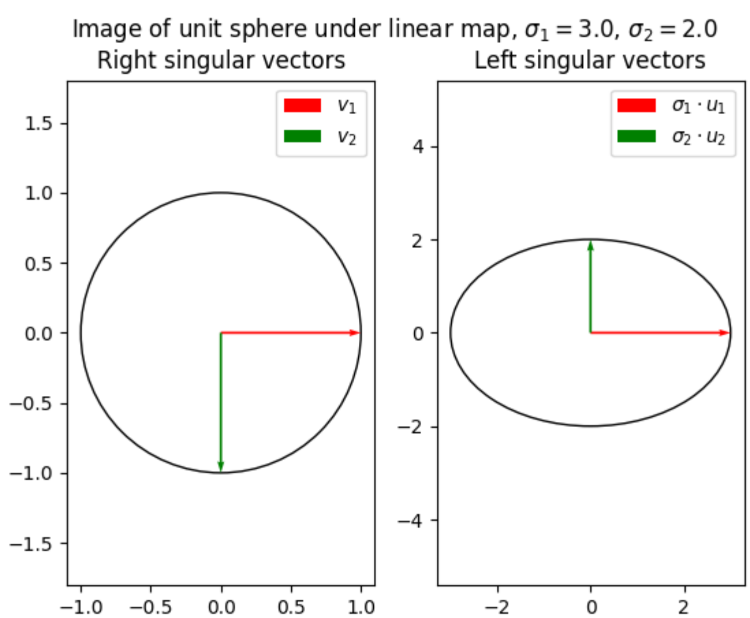

Figure 4: Exercise 4.3(c): Plot of singular vectors of $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ from Exercise 4.1(c)

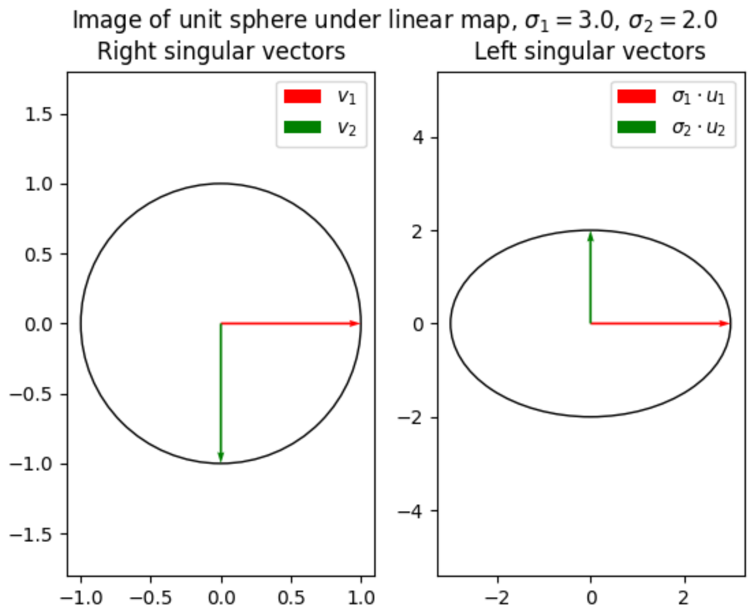

Figure 5: Exercise 4.3(d): Plot of singular vectors of $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ from Exercise 4.1(d)

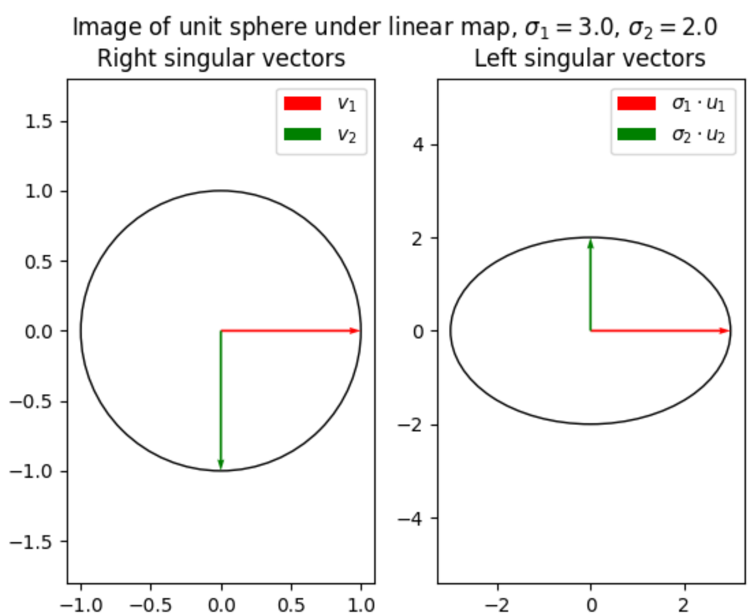

Figure 6: Exercise 4.3: Plot of singular vectors of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ from Exercise 4.1(e)

because both Q and U are unitary. Similarly QV is unitary. Thus, the equation above for A is a singular value decomposition of A. By Theorem 4.1, the singular values of A are uniquely determined, so the diagonal entries in Σ must also be singular values of A.

(" $\Longleftarrow$ "): Suppose A and B have the singular values. Then their SVD consists of the same diagonal matrix Σ. Let S, R, U, V be unitary matrices composing the SVDs of A and B, respectively:

$$A = S\Sigma R^* \quad B = U\Sigma V^*$$

Then $\Sigma = U^*BV$, and hence

$$A = S(U^*BV)R^* = (SU^*)B(RV^*)^*$$

Just like before the product of unitary matrices is unitary, so A and B are unitarily equivalent. □

Exercise 4.5. Theorem 4.1 asserts that every $A \in \mathbb{C}^{m\times n}$ has an SVD $A = U\Sigma V^*$. Show that if A is real, then it has a real SVD ($U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$).

Solution:

Proof. Since $A \in \mathbb{R}^{m\times n}$, we can consider it also as a matrix over the vector space $\mathbb{C}^{m\times n}$. Thus A has an SVD, with $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ unitary matrices, and $\Sigma \in \mathbb{R}^{m\times n}$ a diagonal matrix, since the singular values of A are nonnegative real numbers. We have

$$A = U\Sigma V^*$$

Then $AV = U\Sigma$, so $Av_j = \sigma_j u_j$ for each j. If v_j has complex entries, split it into real and imaginary parts as $v_j = \Re v_j + i\Im v_j$, and similarly $u_j = \Re u_j + i\Im u_j$. Then

$$A\Re v_j + iA\Im v_j = Av_j = \sigma_j u_j = \sigma_j \Re u_j + i\sigma_j \Im u_j$$

Hence, since A is real, we have

$$A\Re v_j = \sigma_j \Re u_j \qquad A\Im v_j = \sigma_j \Im u_j$$

Note that $A^*A = V\Sigma^2 V^*$, so $A^*V = V\Sigma^2$. This implies that $A^*Av_j = \sigma_j^2 v_j$ for each j, and hence

$$A^*A\Re v_j = \sigma_j^2 \Re v_j, \quad A^*A\Im v_j = \sigma_j^2 \Im v_j$$

Since v_j is an eigenvector, it is nonzero, so either $\Im v_j$ or $\Re v_j$ must be nonzero, and hence is an eigenvector. (Proof incomplete). □