Sergio Garcia Tapia
Numerical Linear Algebra, Lloyd Trefethen and David Bau III
Lecture 3: Norms
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Lecture 3: Norms

Exercise 1. Prove that if W is an arbitrary nonsingular matrix, the function $\|\cdot\|_W$ defined by (3.3) is a vector norm.

Solution:

Proof. Suppose $\|\cdot\|$ be any norm, and let $x \in \mathbb{C}^m$. Since W is nonsingular, we know that $Wx = 0$ if and only if $x = 0$. Thus, $\|x\|_w = \|Wx\|$ is 0 if and only $x = 0$, and otherwise x is positive.

Suppose y is another vector in $\mathbb{C}^m$. Then

$$\begin{aligned} \|x+y\|_W &= \|W(x+y)\| \\ &= \|Wx + Wy\| \\ &\leq \|Wx\| + \|Wy\| \\ &= \|x\|_W + \|y\|_W \end{aligned}$$

where the second equality follows from the fact that W, and the inequality follows because $\|\cdot\|$ is a norm so it satisfies the triangle inequality. We conclude that $\|\cdot\|_W$ also satisfies the triangle inequality. Finally, if $\alpha \in \mathbb{C}$, then

$$\|\alpha x\|_W = \|\alpha W x\| = |\alpha| \cdot \|Wx\| = |\alpha| \cdot \|x\|_W$$

so that scaling a vector scales a norm by the same amount. Thus, the weighted norm $\|\cdot\|_W$ is indeed a norm for every nonsingular W. $\square$

Exercise 3.2. Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq \|A\|$, where $\rho(A)$ is the *spectral radius* of A, i.e., the largest absolute value $|\lambda|$ of an eigenvalue λ of A.

Proof. Let λ be the eigenvalue of A with largest absolute value, and let u be an associated eigenvector of unit length, i.e., $\|u\| = 1$. By definition of the induced matrix norm, we have

$$\|A\| = \sup_{x\in\mathbb{C}^n, \|x\|=1} \|Ax\|$$

Thus, if $\|x\| = 1$, we have $\|Ax\| \leq \|A\|$. Since u is a unit vector, we have

$$\begin{aligned} \rho(A) &= |\lambda| \cdot 1 \\ &= |\lambda| \cdot \|u\| \\ &= \|Au\| \\ &\leq \|A\| \end{aligned}$$

$\square$

Exercise 3.3. Vector and matrix p-norms are related by various inequalities, often involving the dimensions of m and n. For each of the following, verify the inequality and give an example of a nonzero vector or matrix (for general m, n) for which equality is achieved. In this problem, x is an m-vector and A is an $m \times n$ matrix.

(a) $\|x\|_\infty \leq \|x\|_2$,

(b) $\|x\|_2 \leq \sqrt{m}\|x\|_\infty$,

(c) $\|A\|_\infty \leq \sqrt{n}\|A\|_2$,

(d) $\|A\|_2 \leq \sqrt{m}\|A\|_\infty$.

Solution: Credit to OSU for parts (c) and (d).

(a) Suppose that $x \in \mathbb{C}^m$. Then $\|x\|_\infty = x_k$ for some $k \in \{1, \ldots, m\}$, where x_k denotes its k-th entry. Thus,

$$\|x\|_\infty^2 = |x_k|^2 \leq \sum_{j=1}^{m} |x_j|^2 = \|x\|_2^2$$

Now we can take the square root, finishing verification. If e_j is the j-th standard basis vector of $\mathbb{C}^m$ and $c \in \mathbb{C}$, then we can achieve equality by letting $x = ce_j$.

(b) Let x_j denote the j-th entry in $x \in \mathbb{C}^m$. Since $|x_j| \leq \|x\|_\infty$ for all $j \in \{1, \ldots, m\}$, we have

$$\|x\|_2^2 = \sum_{j=1}^{m} |x_j|^2 \leq \sum_{j=1}^{m} \|x\|_\infty^2 = m\|x\|_\infty^2$$

After taking square roots, the verification is complete. Let v be a vector whose every entry is 1, and let $c \in \mathbb{C}$. Then equality is achieved for $x = cv$, since

$$\|x\|_2 = \sqrt{\sum_{j=1}^{m} |c \cdot 1|^2} = \sqrt{m \cdot |c|^2} = \sqrt{m} \cdot |c| = \sqrt{m} \cdot \|x\|_\infty$$

(c) Assuming that the 2-norm and ∞-norm here refer to induced matrix norms. Let $u \in \mathbb{C}^n$. By (a), we know that $\|Ax\|_\infty \leq \|Ax\|_2$. By (b), we know that $\|u\|_2 \leq \sqrt{n}\|u\|_\infty$, which if $u \neq 0$ is equivalent to $\frac{1}{\|u\|_\infty} \leq \frac{\sqrt{n}}{\|u\|_2}$. Thus, using the definition of the induced matrix norms, we get

$$\begin{aligned}
\|A\|_\infty &= \sup_{u \in \mathbb{C}^n, u \neq 0} \frac{\|Au\|_\infty}{\|u\|_\infty} \\
&\leq \sup_{u \in \mathbb{C}^n, u \neq 0} \frac{\|Au\|_2}{\|u\|_\infty} \\
&\leq \sup_{u \in \mathbb{C}^n, u \neq 0} \sqrt{n} \cdot \frac{\|Au\|_2}{\|u\|_2} \\
&= \sqrt{n} \cdot \|A\|_2
\end{aligned}$$

If we let A be a matrix whose first row is all 1s, and every other entry is 0. Then the maximum row sum of A is n, so $\|A\|_\infty = \|a_1^*\|_1 = n$. On the other hand, if $u \in \mathbb{C}^n$, we have

$$\|Au\|_2 = |a_1^* u| \leq \|a_1^*\|_2 \cdot \|u\|_2$$

where a_1^* denotes the first row of A. Hence, $\|A\|_2 \leq \|a_1^*\|_2 = \sqrt{n}$. If u is the n-vector of all 1s, then $\|Au\|_2 = n = \|u\|_2^2$, which implies that the bound is attainable, and hence $\|A\|_2 = \|a_1^*\|_2$, so

$$\sqrt{n} \cdot \|A\|_2 = \sqrt{n} \cdot \|a_1^*\|_2 = \sqrt{n} \cdot \sqrt{n} = n = \|a_1^*\|_1 = \|A\|_\infty$$

(d) If $u \in \mathbb{C}^n$ is nonzero, by (b) we have $\|Ax\|_2 \leq \sqrt{m}\|Ax\|_\infty$, and by (a) we have $\frac{1}{\|u\|_2} \leq \frac{1}{\|u\|_\infty}$, so similar to (c), we have

$$\begin{aligned}
\|A\|_2 &= \sup_{u \in \mathbb{C}^n, u \neq 0} \frac{\|Au\|_2}{\|u\|_2} \\
&\leq \sup_{u \in \mathbb{C}^n, u \neq 0} \frac{\sqrt{m}\|Au\|_\infty}{\|u\|_2} \\
&\leq \sup_{u \in \mathbb{C}^n, u \neq 0} \frac{\sqrt{m}\|Au\|_\infty}{\|u\|_\infty} \\
&= \sqrt{m} \cdot \|A\|_\infty
\end{aligned}$$

Now to see an example where equality is achieved, let A be a matrix of all 1s in the first column and 0 otherwise. For such a matrix, the maximum matrix sum is 1, so $\|A\|_\infty = 1$. If $u \in \mathbb{C}^n$ and u_1 is its first entry, then $Au = u_1 \cdot a_1$, where a_1 is the first column of A. Since $a_1 \in \mathbb{C}^m$, we have $\|a_1\|_2 = \sqrt{m}$. Thus

$$\|Au\|_2 = |u_1| \cdot \sqrt{m} \leq \sqrt{m} \cdot \|u\|_2$$

Thus $\|A\|_2 \leq \sqrt{m}$. Since we can achieve equality by letting $u = e_1$, we conclude that $\|A\|_2 = \sqrt{m}$. Thus,

$$\|A\|_2 = \sqrt{m} \cdot 1 = \sqrt{m} \cdot \|A\|_\infty$$

Exercise 4. Let A be an $m \times n$ matrix and let B be a submatrix of A, that is, a $\mu \times \nu$ matrix ($\mu \leq m$, $\nu \leq n$) obtained by selecting certain rows and columns of A.

(a) Explain how B can be obtained by multiplying A by a certain row and column "deletion matrices" as in step 7 of Exercise 1.1?

(b) Using this product, show that $\|B\|_p \leq \|A\|_p$ for any p with $1 \leq p \leq \infty$.

Solution:

(a) Suppose $j_1, \ldots, j_\nu$ are the indices of the columns of the submatrix B of A. Let C be the $n \times n$ matrix whose entries are all 0 except for the diagonal entries $c_{j_q j_q}$, whose value is 1. where $j_q\{j_1, \ldots, j_\nu\}$. Then the matrix product AC deletes the columns of interest all but columns $j_1, \ldots, j_\nu$. To delete rows of AC we simply delete columns of $(AC)^T$. Let $i_1, \ldots, i_\mu$ are the rows we ought to keep. Define D to be the $m \times m$ matrix the matrix whose entries are all 0 except the diagonal entries d_{i_p, i_p}, where $i_p \in \{i_1, \ldots, i_\mu\}$. Then the matrix product $(AC)^T D$ deletes these columns from the transpose, and hence the product $D^T(AC)$ retains only the rows of interest. Hence $B = D^T AC$.

(b) *Proof.* By equation (3.14), we have

$$\begin{aligned}\|B\|_p &= \|D^T AC\|_p \\ &\leq \|D^T|_p \cdot \|AC\|_p \\ &\leq \|D^T\|_p \cdot \|A\|_p \cdot \|C\|_p\end{aligned}$$

If $u \in \mathbb{C}^n$, then

$$\|Cu\|_p = \|u_{j_1} e_{j_1} + \cdots + u_{j_\mu} e_{j_\mu}\|_p \leq \|u\|_p$$

Thus $\|C\|_p \leq 1$, and similarly, $\|D^T\|_p \leq 1$. The result follows. □

Exercise 5. Example 3.6 shows that if E is an outer product $E = uv^*$, then $\|E\|_2 = \|u\|_2 \cdot \|v\|_2$. Is the same true for the Frobenius norm, i.e., $\|E\|_F = \|u\|_F \|v\|_F$? Prove it or give a counter example.

Solution:

Proof. The Frobenius norm for a vector in $\mathbb{C}^m$ is equivalent to the vector 2-norm. If $E = uv^*$ is an outer product, then every column is a multiple of u of the form $v_j u$, where v_j is the j-th entry in the vector v. Thus,

$$\begin{aligned}\|E\|_F &= \left(\sum_{j=1}^{n} \|v_j \cdot u\|_2^2 \right)^{1/2} \\ &= \left(\|u\|_2^2 \cdot \sum_{j=1}^{n} |v_j|^2 \right)^{1/2} \\ &= \left(\|u\|_2^2 \cdot \|v_j\|_2^2 \right)^{1/2} \\ &= \|u\|_2 \cdot \|v\|_2 \\ &= \|u\|_F \cdot \|v\|_F\end{aligned}$$

□

Exercise 6. Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$. The corresponding *dual norm* $\|\cdot\|'$ is defined by the formula $\|x\|' = \sup_{\|y\|=1} |y^* x|$.

(a) Prove that $\|\cdot\|'$ is a norm.

(b) Let $x, y \in \mathbb{C}^m$ with $\|x\| = \|y\| = 1$ be given. Show that there exists a rank-one matrix $B = yz^*$ such that $Bx = y$ and $\|B\| = 1$, where $\|B\|$ is the matrix norm of B induced by the vector norm $\|\cdot\|$. You may use the following lemma, without proof: given $x \in \mathbb{C}^m$, there exists a nonzero $z \in \mathbb{C}^m$ such that $|z^*x\|| = \|z\|' \cdot \|x\|$.

Solution:

(a) *Proof.* Suppose that $x, y \in \mathbb{C}^m$ with $\|y\| = 1$. Then by definition of the supremum, we have $|y^*x| \leq \|x\|'$. If $x = 0$, then the supremum is 0. If $x \neq 0$, then picking $y = \frac{x}{\|x\|}$, we get a positive quantity again since $|x^*x| > 0$.

Suppose $z \in \mathbb{C}^m$. Then

$$\begin{aligned}
\|x + z\|' &= \sup_{\|y\|=1} |y^*(x + z)| \\
&= \sup_{\|y\|=1} |y^*x + y^*z| \\
&\leq \sup_{\|y\|=1} (|y^*x| + |y^*z|) \\
&\leq \sup_{\|y\|=1} |y^*x| + \sup_{\|y\|=1} |y^*z| \\
&= \|x\|' + \|z\|'
\end{aligned}$$

where the first inequality follows from triangle inequality (which is satisfied by the absolute value), and the last inequality follows from the fact that the supremum of a sum is the sum of the supremums. Finally, if $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}
\|\alpha x\|' &= \sup_{\|y\|=1} |y^*(\alpha x)| \\
&= \sup_{\|y\|=1} |(\alpha| \cdot |y^*x|) \\
&= \alpha \cdot \sup_{\|y\|=1} |y^*x| \\
&= \alpha \cdot \|x\|'
\end{aligned}$$

We conclude that $\|\cdot\|'$ is a norm. $\square$

(b) *Proof.* Let $z = x$. Then

$$Bx = (yz^*)x = y(z^*x) = y(x^*x) = y \cdot \|x\|^2 = y \cdot 1 = y$$

Moreover, if $u \in \mathbb{C}^m$ is nonzero, then there is $v \in \mathbb{C}^m$ such that $|v^*u| = \|v\|' \cdot \|u\|$.

Thus,

$$\begin{aligned}
\|Bu\| &= \|yz^*u\| \\
&= |z^*u| \cdot \|y\| \\
&= \frac{|z^*u|}{|v^*u|} \cdot |v^*u| \cdot \|y\| \\
&= \frac{|z^*u|}{|v^*u|} \cdot \|v\|' \cdot \|u\| \cdot \|y\| \\
&= \left(\frac{|z^*u|}{|v^*u|} \cdot \|v\|' \cdot \|y\| \right) \cdot \|u\| \\
&= \left(\frac{|z^*u|}{|v^*u|} \cdot \|v\|' \right) \cdot \|u\|
\end{aligned}$$

Hence,

$$\|B\| \leq \frac{|z^*u|}{|v^*u|} \cdot \|v\|'$$

for all u. In particular we attain equality if we let $u = x$, since

$$\|Bx\| = \|y\| = \|x\|$$
$$\frac{x^*x}{|v^* \cdot x|} \cdot \|v\|' = \frac{x^*x}{\|v\|' \cdot \|x\|} \cdot \|v\|' = \frac{x^*x}{\|x\|} = 1$$

Hence we deduce that $\|B\| = 1$. $\square$