Report regarding the ongoing research

1 Main

Definition 1 (Extended social argumentation frameworks). *An* extended social argumentation framework *is a 4-tuple $F = \langle \mathcal{A}, \mathcal{R}, V_{\mathcal{A}}, V_{\mathcal{R}} \rangle$, where*

- *$\mathcal{A}$ is the set of arguments,*
- *$\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ is a binary attack relation between arguments,*
- *$V_{\mathcal{A}} : \mathcal{A} \to \mathbb{N} \times \mathbb{N}$ stores the crowd's pro and con votes for each argument.*
- *$V_{\mathcal{R}} : \mathcal{R} \to \mathbb{N} \times \mathbb{N}$ stores the crowd's pro and con votes for each attack.*

Definition 2. *[Vote Aggregation Function] Given a totally ordered set L with top and bottom elements $\top$, $\bot$, a vote aggregation function τ over L is any function such that $\tau : \mathbb{N}^2 \times \mathbb{N} \to L$.*

Definition 3 (Semantic Framework). *A semantic framework is a 6-tuple $\langle L, \curlywedge_1, \curlywedge_2, \curlyvee, \neg, \tau, \rangle$ where:*

- *L is a totally ordered set with top and bottom elements $\top$, $\bot$, containing all possible valuations of an argument.*
- *$\curlywedge_{\mathcal{A}}, \curlywedge_{\mathcal{R}} : L \times L \to L$, are two binary algebraic operations used to restrict strengths to given values.*
- *$\curlyvee : L \times L \to L$, is a binary algebraic operation on argument valuations used to combine or aggregate valuations and strengths.*
- *$\neg : L \to L$ is a unary algebraic operation for computing a restricting value corresponding to a given valuation or strength.*
- *τ is a vote aggregation function, which aggregates positive and negative votes and the maximum number of total votes into a social support value.*

Notation 1. *Let $F = \langle \mathcal{A}, \mathcal{R}, V_{\mathcal{A}}, V_{\mathcal{R}} \rangle$ be an ESAF, $\mathcal{S} = \langle L, \curlywedge_1, \curlywedge_2, \curlyvee, \neg, \tau, \rangle$ a semantic framework. Then, let*

- *$\mathcal{R}^{-}(a) \triangleq \{a_i \in \mathcal{A} : (a_i, a) \in \mathcal{R}\}$ be the set of direct attackers of an argument $a \in \mathcal{A}$,*

- - $V_{\mathcal{A}}^{+}(a) \triangleq x$ *denote the number of positive votes for argument* a,
  - $V_{\mathcal{A}}^{-}(a) \triangleq y$ *denote the number of negative votes for argument* a,

  whenever $V_{\mathcal{A}}(a) = (x, y)$,
- $v^r : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *be a function s.t.* $v^r(v^+, v^-) \triangleq \frac{v^+}{v^-}$ *where* $v^+, v^- \in \mathbb{N}$, *that denotes the ratio of positive votes to negative votes for some argument or attack relation,*
- $v^t : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *be a function s.t.* $v^t(v^+, v^-) \triangleq v^+ + v^-$ *where* $v^+, v^- \in \mathbb{N}$, *that denotes the total number of votes for some argument or attack relation,*
- $v_{emax} \triangleq v^t(V_{\mathcal{A}}(b))$ *where* $b \in \mathcal{A}$ *and* $\forall a \in \mathcal{A}$, $v^t(V_{\mathcal{A}}(b)) \geq v^t(V_{\mathcal{A}}(a))$ *denote the maximum total number of votes for an argument,*
- $\tau(a) \triangleq \tau(V_{\mathcal{A}}(a), v_{emax})$ *denote the social support for an argument* a *via utilizing a vote aggregation function* τ,
- - $V_{\mathcal{R}}^{+}((a, b)) \triangleq x$ *denote the number of positive votes for an attack relation between arguments* a *and* b,
  - $V_{\mathcal{R}}^{-}((a, b)) \triangleq y$ *denote the number of negative votes for an attack relation between arguments* a *and* b,

  whenever $V_{\mathcal{R}}((a, b)) = (x, y)$,
- $v_{rmax} \triangleq v^t(V_{\mathcal{R}}(y))$ *where* $y \in \mathcal{R}$ *and* $\forall x \in \mathcal{R}$, $v^t(V_{\mathcal{R}}(y)) \geq v^t(V_{\mathcal{R}}(x))$ *denote the maximum total number of votes for an attack relation,*
- $\tau((a, b)) \triangleq \tau(V_{\mathcal{R}}((a, b)), v_{rmax})$ *denote the social support for an attack relation between arguments* a *and* b *via utilizing a vote aggregation function* τ,
- $$\bigcurlyvee_{x \in R} x \triangleq (((x_1 \curlyvee x_2) \curlyvee ...) \curlyvee x_n)$$

  $R = \{x_1, x_2, ..., x_n\}$ *denote the aggregation of a multiset of elements of* L.

Definition 4 (Model). *Let* $F = \langle \mathcal{A}, \mathcal{R}, V_{\mathcal{A}}, V_{\mathcal{R}} \rangle$ *be a social argumentation framework,* $\mathcal{S} = \langle L, \curlywedge_1, \curlywedge_2, \curlyvee, \neg, \tau, \rangle$ *be a semantic framework. A* $\mathcal{S}$*-model of* F *is a total mapping* $M : \mathcal{A} \to L$ *such that for all* $a \in \mathcal{A}$,

$$M(a) = \tau(a) \curlywedge_{\mathcal{A}} \neg \bigcurlyvee_{a_i \in \mathcal{R}^{-}(a)} (\tau((a_i, a)) \curlywedge_{\mathcal{R}} M(a_i))$$

Notation 2. *Whenever a single spesific system* $F = \langle \mathcal{A}, \mathcal{R}, V_{\mathcal{A}}, V_{\mathcal{R}} \rangle$ *with some semantic framework* $\mathcal{S} = \langle L, \curlywedge_1, \curlywedge_2, \curlyvee, \neg, \tau, \rangle$ *is considered, we implicitly consider* v_{emax} *and* v_{rmax} *are constants and drop them from the respective function's argument list. Thus we update the two function definitions from Notation 1 as follows:* $\tau(a) \triangleq \tau(V_{\mathcal{A}}(a))$ *and* $\tau((a, b)) \triangleq \tau(V_{\mathcal{R}}((a, b)))$, *where* $a \in \mathbb{A}$.

Definition 5. *[Enhanced Vote Aggregation]*
Enhanced vote aggregation function $\tau_e : \mathbb{N} \times \mathbb{N} \to [0,1]$ *is the vote aggregation function such that*

$$\tau_e(v^+, v^-) = \begin{cases} 0 & v_{max} = 0 \\ \frac{v^+}{v^+ + v^- + \frac{1}{v_{max}}} & \textit{otherwise} \end{cases}$$

Property 1. *[Absolute argument freeness]*
Let τ *be a a vote aggregation function given* L*. We say that* τ *is 'absolute argument free' if*

$$\forall n_1, n_2 \in \mathbb{N},\ \tau(n_1, n_2) \neq \top.$$

The gist of the property can be captured in a verbal context with the following sentence: *No argument enjoys perfect Social Support.*

Property 2. *[Precedence of the vote ratio]*
Let τ *be a a vote aggregation function given* L*. We say that* τ *is 'vote ratio precedent' if*

$$\forall n_1, n_2, n_3, n_4 \in \mathbb{N},$$
$$(v^r(n_1, n_2) \geq v^r(n_3, n_4)) \implies (\tau(n_1, n_2) \geq \tau(n_3, n_4)).$$
just for convenience:
$$(v^r(a_1) \geq v^r(a_2)) \implies (\tau(a_1) \geq \tau(a_2)).$$

The gist of the property can be captured in a verbal context with the following sentence: *If the value of the ratio of positive votes to the negative votes is higher for an argument a than an argument b; a's social support value exceeds the social support of b.*

Property 3. *[Precedence of the total number of votes]*
Let τ *be a a vote aggregation function given* L*. We say that* τ *is 'total votes precedent' if*

$$\forall n_1, n_2, n_3, n_4 \in \mathbb{N},$$
$$((v^r(n_1, n_2) = v^r(n_3, n_4)) \wedge (v^t(n_1, n_2) \geq (v^t(n_3, n_4))) \implies$$
$$(\tau(n_1, n_2) \geq \tau(n_3, n_4)).$$
just for convenience:
$$((v^r(a_1) = v^r(a_2)) \wedge (v^t(a_1) \geq (v^t(a_2))) \implies (\tau(a_1) \geq \tau(a_2)).$$

The gist of the property can be captured in a verbal context with the following sentence: *When the ratios are equal, the function should return a higher social support value for the one with the higher number of total votes.*

Proposition 1. *Enhanced Vote Aggregation function enjoys Property 1.*

Proof. Trivial.
Let $\mathcal{F}$ be an extended social argumentation framework with a semantic framework $\mathcal{S} = \langle L, \curlywedge_1, \curlywedge_2, \curlyvee, \neg, \tau, \rangle$ and $a \in \mathcal{A}$.

There are mainly two cases for a vote aggregation with respect to the value the constant v_{max} takes.

Firstly, if $v_{max} = 0$ then $\tau(a) = \tau(v^+, v^-) = 0 \leq 1$.
Else if $v^+ \neq 0$, then $v_{max} \neq 0$ and since $v_{max} \in \mathcal{Z}^+$, then $\frac{1}{v_{max}} > 0$. Since the denominator of the sole term in the social support function is simply the sum of the numerator and some $r \in R^+$, the denominator is bigger than the numerator and hence:

$$\frac{v^+}{v^+ + v^- + \frac{1}{v_{max}}} = \tau(v^+, v^-) \leq 1.$$

□

2 Ongoing misc proofs

Theorem 1 (Relation of ESAF models and stable extensions)**.** *Let $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ be an Abstract Argumentation Framework and $F = \langle \mathcal{A}, \mathcal{R}, V_{\mathcal{A}}, V_{\mathcal{R}} \rangle$ be an Extended Social Abstract Argumentation Framework. Let $\mathcal{S} = \langle L, \curlywedge_1, \curlywedge_2, \curlyvee, \neg, \tau \rangle$ and $V_{\mathcal{A}}, V_{\mathcal{R}}$ such that*

- $\curlywedge = \curlywedge_1 = \curlywedge_2$
- $\top \curlyvee \top = \top$ *and* $\top \curlyvee \bot = \top$
- $\forall_{x \in \mathcal{A}}\ \tau(x) = \top$
- $\forall_{(x,y) \in \mathcal{R}}\ \tau((x, y)) = \top$
- $l \curlywedge \neg\bot = l$*, where* $l \in L$
- $l \curlywedge \neg\top = \bot$*, where* $l \in L$
- $\top \curlyvee \top = \top$ *and* $\top \curlyvee \bot = \top$

Then, $\exists S = \{x_1, ..., x_n\}$ s.t. S is a stable extension of AF iff $\exists M \in \mathcal{M}$ in F and $a \in S \Leftrightarrow M(a) = \top$ and $a \notin S \Leftrightarrow M(a) = \bot$.

Proof. We start by $\Leftarrow$ direction via using proof by contradiction. Suppose there is an argument $a \in \mathcal{A}$ such that $M(a) = \top$ for some $M \in \mathcal{M}$ and for all stable extensions S of AF, $a \notin S$. The definition of stable semantics dictates $\exists a_i \in S$

s.t. $a_i \mathcal{R} a$. The model for a is thus

$$
\begin{aligned}
M(a) &= \tau(a) \curlywedge \neg \curlyvee_{a_i \in \mathcal{R}^{-}(a)} ((\tau((a_i, a)) \curlywedge M(a_i)) \\
&= \top \curlywedge \neg ((\tau((a_1, a)) \curlywedge M(a_1)) \curlyvee ... \curlyvee ((\tau((a_i, a)) \curlywedge M(a_i)) \curlyvee ... \curlyvee ((\tau((a_n, a)) \curlywedge M(a_n))) \\
&= \top \curlywedge \neg (\top \curlywedge M(a_1)) \curlyvee ... \curlyvee \top \curlyvee ... \curlyvee (\top \curlywedge M(a_n))) \\
&= \top \curlywedge \neg (M(a_1) \curlyvee ... \curlyvee \top \curlyvee ... \curlyvee M(a_n)) \\
&= \top \curlywedge \neg \top \\
&= \bot
\end{aligned}
$$

Contradiction! Therefore, a must be contained in S.

We again use proof by contradiction for the $\Rightarrow$ direction. Assume $\exists a \in S$ and $\forall M \in \mathcal{M}$, $M(a) = \bot$.

Let us inspect the model function of a:

$$
\begin{aligned}
M(a) &= \tau(a) \curlywedge \neg \curlyvee_{a_i \in \mathcal{R}^{-}(a)} ((\tau((a_i, a)) \curlywedge M(a_i)) \\
\bot &= \top \curlywedge \neg ((\tau((a_1, a)) \curlywedge M(a_1)) \curlyvee ... \curlyvee ((\tau((a_n, a)) \curlywedge M(a_n))) \\
\bot &= \top \curlywedge \neg (\top \curlywedge M(a_1)) \curlyvee ... \curlyvee (\top \curlywedge M(a_n))) \\
\bot &= \top \curlywedge \neg (M(a_1) \curlyvee ... \curlyvee M(a_n)) \\
\bot &= \neg (M(a_1) \curlyvee ... \curlyvee M(a_n)) \\
\top &= M(a_1) \curlyvee ... \curlyvee M(a_n)
\end{aligned}
$$

Thus for some $i \in [1..n]$, $M(a_i) = \top$. And so it follows that $\exists a_i \in \mathcal{A}$ s.t. $a_i \mathcal{R} a$. But since $a \in S$, $\exists_{b \in S}\, b \mathcal{R} a_i$. Which results into $M(a_i) = \bot$. Contradiction!

$\square$